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A New Proof of a Result Concerning a Complete Description of $(n, n+2)$-Graphs with Maximum Value of the Hosoya Index

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Abstract—The Hosoya index is an important topological index of graphs defined as the number of their matchings. At present, for any n and $k \in \{-1, 0, 1, 2\}$, all connected graphs with n vertices and $n + k$ edges that have a maximum value of the Hosoya index among all such graphs have been described (in the case $k = 2$ for $n \geq 15$). This paper proposes a new proof for the case $k = 2$ for $n \geq 17$ based on a decomposition of the Hosoya index by subsets of separating vertices and local graph transformations induced by them. This approach is new in the search for graphs with extreme value of the Hosoya index, where many standard techniques are usually employed. The new proof is more combinatorial, shorter, and less technical than the original proof.

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INTRODUCTION

Chemical compounds are often expressed in the form of *molecular graphs* in which atoms correspond to vertices of the graph and connections between atoms correspond to edges of the graph. The properties of chemical compounds are described in terms of *topological indices* that represent certain graph invariants with respect to the re-notation of vertices and make analytical studies of certain aspects of the chemical structure of substances possible.

In [1], the Japanese chemist Haruo Hosoya proposed a topological index, now called the *Hosoya index*. By a *graph matching* we mean any set of pairwise nonadjacent edges of the graph. The Hosoya index of a graph G is defined as the number of its matchings and is denoted by $z(G)$. The empty set of matchings is also regarded as a matching, and hence the Hosoya index of any graph (including the *null-graph*, i.e., the graph with an empty set of vertices) is always positive. It was shown in [1] that certain physico-chemical properties of alkanes (in particular, their boiling point) are related to the value of the Hosoya index of their molecular graphs. Subsequently, it was discovered that there is a connection between the Hosoya index and other physico-chemical properties of alkanes, and also the energy of conjugate π-electronic systems; see the surveys [2]–[4].

Since topological indices determine the energy of chemical compounds, it is of interest to study the problem of identifying graphs from given classes with the extreme (minimum or maximum) value of some topological index. In the present paper, unless otherwise stated, only *simple graphs*, i.e., undirected, unmarked graphs without loops and multiple edges, are considered.

A connected graph with n vertices and m edges is called an (n, m)-*graph*. It was proved in [5] that, among $(n, n-1)$-graphs, only the n-cycle has the maximal Hosoya index. It was shown in [6] that,

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among (n, n)-graphs, only the n-cycle has the maximal Hosoya index. In [7], [8], $(n, n+1)$-graphs were considered, and there, for any n, all $(n, n+1)$-graphs with maximum value of the Hosoya index were described. It was discovered that, for any $n \geq 10$, the corresponding maximal graph is unique and is obtained by connecting 4-cycles and $(n-4)$-cycles by an edge. In [8], $(n, n+2)$-graphs were also considered, and it was shown that, for $n \geq 15$, the maximal graph is unique and isomorphic to the graph shown in Fig. 1, where the dotted segment indicates a simple path of length $n - 11$.

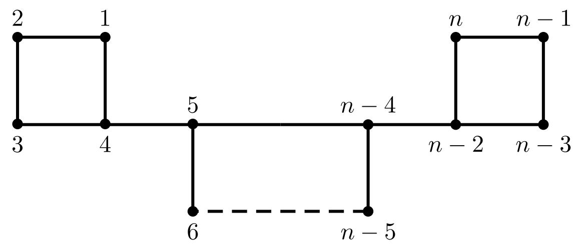

Fig. 1. The unique extremal graph with $n \geq 15$ vertices.

In this paper, we also consider the case of $(n, n+2)$-graphs for $n \geq 17$. We propose a new proof based on the decomposition of the Hosaya index by subsets of separating vertices and local graph transformations induced by them. This approach is new in the search for graphs with extreme values of the Hosoya index, where a number of standard techniques are usually employed. Among these techniques, we note a proof of the absence of pendant vertices by using standard graph transformations, the establishment of the form of the resulting graphs, their parametrization and the tuning of their parameters by using standard cycle splitting, cycle rolling along a path or cycle, and the properties of Fibonacci numbers. The new proof is more combinatorial, shorter, and less technical than the original proof.

1. THE SYMBOLS USED

By P_n and C_n we denote a simple path and a cycle with n vertices, respectively. By P_0 we denote the null graph. Let us also indicate that by C_2 we mean a multigraph with 2 vertices and an edge of multiplicity two and by C_1 we mean the pseudograph with 1 vertex and a self-loop. In particular,

$$z(C_1) = z(P_1) = z(P_0) = 1, \qquad z(C_2) = 3.$$

Let G be a graph, and let $v \in V(G)$, $V' \subseteq V(G)$, $E' \subseteq E(G)$. The degree of a vertex v is denoted by $\deg_G(v)$, and its neighborhood by $N_G(v)$. The graphs $G \setminus V'$ and $G \setminus E'$ are obtained from G by removing all the vertices from V' and all the edges from E', respectively.

Let G be a graph, and let $e \in E(G)$. The equality

$$z(G) = z_+(G, e) + z_-(G, e),$$

where $z_+(G, e)$ and $z_-(G, e)$ is the number of matchings of G containing or not containing e, respectively, will be called the *decomposition of $z(G)$ along the edge e*. For example, using the decomposition of the Hosoya index along the pendant edge of a simple path and along an arbitrary edge of the cycle, it is not difficult to establish that, for any $i \geq 2$,

$$z(P_i) = z(P_{i-1}) + z(P_{i-2}), \qquad z(C_{i+1}) = z(C_i) + z(C_{i-1}).$$

It readily follows that, for all $i \geq 0$,

$$z(P_i) = F_{i+1}, z(C_{i+3}) = F_{i+4} + F_{i+2},$$

where $F_0 = 0$, $F_1 = F_2 = 1$, $F_i = F_{i-1} + F_{i-2}$, $i \geq 3$, is the sequence of Fibonacci numbers. The use of the edge decomposition of the Hosoya index is a known standard procedure.

Let $A, B \subseteq V(G)$, and let $A \cap B = \varnothing$. By $z(G, A, B)$ we denote the number of matchings of the graph G, covering all vertices from A and not covering any vertices from B.

2. CONTRACTIONS OF $(n, n+2)$-GRAPHS

A pseudo-graph G' is called a *contraction* of an simple graph G if G is obtained by subdividing the edges of G', and G' contains a minimum number of vertices. It is clear that there exists a unique contraction of any simple graph. It is also clear that, for each connected graph G, other than a simple cycle, we have

$$V(G') = \{v \in V(G) : \deg_G(v) \neq 2\} \qquad \forall\, v \in V(G')[\deg_{G'}(v) = \deg_G(v)].$$

The concept of graph contraction turns out to be useful for describing all (n, m)-graphs without pendant vertices for small values of $m - n$. Indeed, if $(d_1, \ldots, d_n)$ is the sequence of degrees of an (n, m)-graph without pendant vertices, where $d_1 \leq \cdots \leq d_n$, then $d_1 \geq 2$ and $d_1 + \cdots + d_n = 2m$. For small values of $m - n$, it is easy to enumerate all the relevant sequence of degreess. In particular, if $m = n + 2$, then only the following sequences are such sequences:

$$(2, \ldots, 2, 6), \qquad (2, \ldots, 2, 3, 5), \qquad (2, \ldots, 2, 4, 4),$$
$$(2, \ldots, 2, 3, 3, 4), \qquad (2, \ldots, 2, 3, 3, 3, 3).$$

This makes it possible to enumerate all the contractions of $(n, n+2)$-graphs without pendant vertices (see Fig. 2).

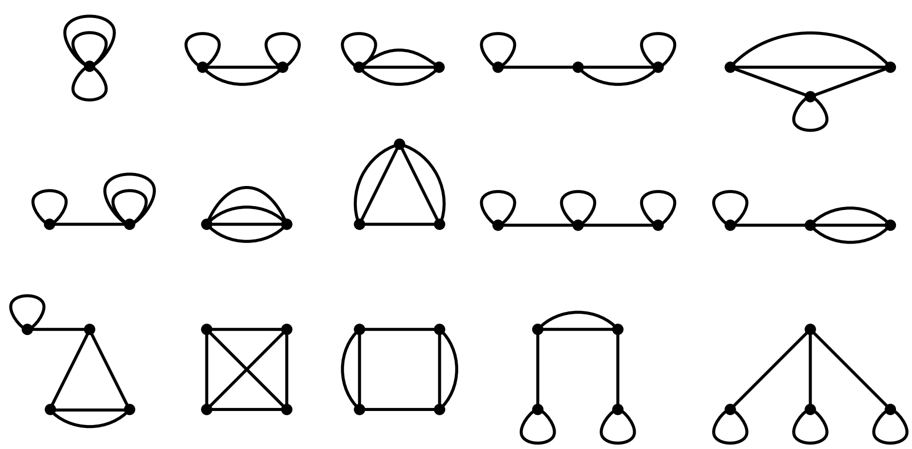

Fig. 2. All types of contractions of $(n, n+2)$-graphs without pendant vertices.

It will be shown below that, for any $m \geq n$, each maximal (n, m)-graph does not contain pendant vertices, and some other properties of maximal (n, m)-graphs will also be formulated. This, in particular, will make it possible to identify the contractions of precisely maximal $(n, n+2)$-graphs and adjust the parameters (i.e., the number of subdivisions of edges) in them.

3. A LEMMA ON THE DECOMPOSITION OF THE HOSOYA INDEX AND ITS COROLLARIES

Let G be a graph, and let H be its induced subgraph. Any subset $S \subseteq V(H)$ such that no vertex from $V(G) \setminus V(H)$ is adjacent to any vertex from $V(H) \setminus S$, is called *H-separating*. Let S be an H-separating set. By G_S we denote the result of removing from G all elements of the set $V(H) \setminus S$ as well as all edges simultaneously incident to two vertices from S. The following statement is valid.

Lemma 1. *The following equality holds*:

$$z(G) = \sum_{S' \subseteq S} z(G_S, S', S \setminus S') \cdot z(H \setminus S').$$

Proof. Each matching of the graph G can be split into two parts, the matching of the graph G_S and the matching of the graph H. In the graph G_S, the matching covers some subset $S' \subseteq S$, but does not cover the subset $S \setminus S'$. For any $S' \subseteq S$, the number of such matchings of the graph G is equal to $z(G_S, S', S \setminus S') \cdot z(H \setminus S')$. Summing over all subsets of the set S, we obtain the number of all matchings of the graph G. □

Suppose that, in a graph G, vertices u and w, not necessarily distinct, are chosen. For $k \geq 1$, by $G^{(k)}$ we denote the graph obtained by adding to G a simple path $(v_1, \ldots, v_k)$ and edges $v_1 u$, $v_k w$ (see Fig. 3). Let us point out that, in the case $u = w$, $k = 1$, the graph $G^{(1)}$ will be a multigraph with edge uv_1 of multiplicity 2.

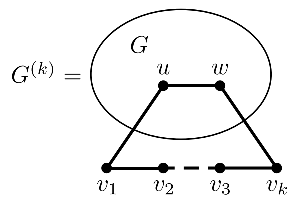

Fig. 3. The graph $G^{(k)}$.

Let us rename G as $G^{(0)}$. The following statement is valid.

Corollary 1. *For any $k \geq 2$, the following equality holds*:

$$z(G^{(k)}) = z(G^{(k-1)}) + z(G^{(k-2)}).$$

Proof. Suppose that $u \neq w$. For $S = \{u, w\}$ and any $p \geq 1$, by Lemma 1, we have

$$z(G^{(p)}) = \sum_{S' \subseteq S} z(G_S^{(p)}, S', S \setminus S') \cdot F_{k-|S'|+1}.$$

Obviously, the graphs $G_S^{(0)}, G_S^{(1)}, G_S^{(2)}, \ldots$ are isomorphic to each other. Thus, for any $S' \subseteq S$, we can write

$$z(G_S^{(0)}, S', S \setminus S') = z(G_S^{(1)}, S', S \setminus S') = z(G_S^{(2)}, S', S \setminus S') = \cdots .$$

For any $i \geq 2$, we have $z(P_i) = z(P_{i-1}) + z(P_{i-2})$. Therefore, the assertion of the corollary holds if $u \neq w$.

If $u = w$, then, for $S = \{u\}$, the proof is carried out by analogy. The case $k = 2$ is considered separately. For $k \geq 3$, we use the equality

$$z(C_i) = z(C_{i-1}) + z(C_{i-2}),$$

which holds for all $i \geq 3$. The proof of Corollary 1 is complete. □

Suppose that graphs G^1 and G^2 contain induced subgraphs H_1 and H_2, and the following two conditions hold:

1) some subset $S \subseteq V(G^1) \cap V(G^2)$ is simultaneously H_1-separating and H_2-separating;

2) the subgraphs G_S^1 and G_S^2 are isomorphic.

Thus, the equality

$$z(G_S^1, S', S \setminus S') = z(G_S^2, S', S \setminus S')$$

holds for any $S' \subseteq S$. This fact and Lemma 1 imply the following statement.

Corollary 2. *Suppose that, for any subset $S' \subseteq S$, the inequality $z(H_1 \setminus S') \le z(H_2 \setminus S')$ holds and, for some $\widetilde{S}' \subseteq S$,*

$$z(H_1 \setminus S') < z(H_2 \setminus S'), \qquad z(G^1_S, \widetilde{S}', S \setminus \widetilde{S}') \neq 0.$$

Then the following inequality holds:

$$z(G^1) < z(G^2).$$

Corollary 2 turns out to be useful for proving the fact that a certain transformation increases the Hosoya index. It will be used in the next section.

4. GRAPH THE TRANSFORMATIONS THAT INCREASE THE HOSOYA INDEX

We define several transformations that preserve connectivity, the number of vertices and edges, and prove that, under certain conditions, they increase the Hosoya index.

4.1. Transformations of Induced Paths and Cycles

Let G be a graph with a simple path $(v_1, \dots, v_a)$, where

$$\deg_G(v_a) \ge 3, \qquad \deg_G(v_{a-1}) = \dots = \deg_G(v_2) = 2, \qquad \deg_G(v_1) = 1.$$

The transformation I taking G to the graph G' removes the edge $v_a u$, where u is an arbitrary element of the set $N_G(v_a) \setminus \{v_{a-1}\}$, and adds the edge uv_1 (see Fig. 4).

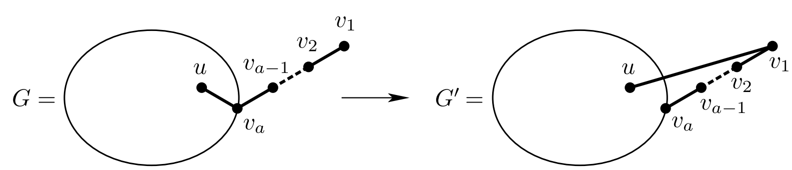

Fig. 4. The transformation I.

Lemma 2. *The following inequality holds*: $z(G') > z(G)$.

Proof. To prove this lemma, we use Corollary 2 and its notation. We put

$$G^1 = G, \qquad G^2 = G', \qquad V(H_1) = V(H_2) = \{u, v_1, \dots, v_a\}, \qquad S = \{v_a, u\}.$$

The graphs G^1_S and G^2_S are isomorphic. Obviously, for any $S' \subseteq S$, except $S' = \{v_a\}$, the graph $H_1 \setminus S'$ is isomorphic to the graph $H_2 \setminus S'$. It is also obvious that

$$z(H_2 \setminus \{v_a\}) = F_{a+1} > F_a = z(H_1 \setminus \{v_a\}).$$

Under the assumptions of the lemma, it is obvious that $z(G^1_S, \{v_a\}, S \setminus \{v_a\}) \neq 0$. Thus, by Corollary 2, we have the inequality $z(G') > z(G)$. □

If there is a leaf in some maximal (n, m)-graph, where $m \ge n$,, then it contains an induced path from the given leaf to a vertex of degree at least 3; in this leaf, all the interior vertices are of degree 2. Thus, the following statement follows from Lemma 2.

Corollary 3. *If $m \ge n$, then each maximal (n, m)-graph does not contain pendant vertices.*

Suppose that, in some graph G, there is an induced cycle $(v_1, \dots, v_a)$ in which all vertices, except v_1, have degree 2, $\deg_G(v_1) \ge 4$, $u \in N_G(v_1) \setminus \{v_2, v_a\}$. The transformation II takes G to the graph G', removing the edge uv_1 and adding the edge uv_2 (see Fig. 5).

Lemma 3. *The inequality $z(G') > z(G)$ holds.*

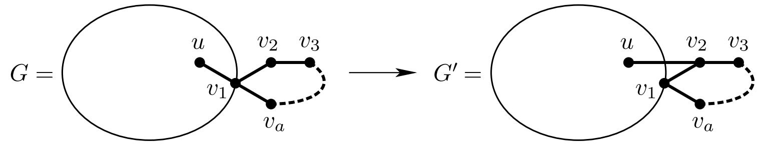

Fig. 5. The transformation II.

Proof. To prove this lemma, we use Corollary 2 and its notation. We put $G^1 = G$, $G^2 = G'$, and let H_1 and H_2 be the subgraphs of the graphs G^1 and G^2 induced by the vertices $u, v_1, \dots, v_a$, respectively, $S = \{v_1, u\}$. The graphs G^1_S and G^2_S are isomorphic. Obviously, for any $S' \subseteq S$, except $S' = \{v_1\}$, the graph $H_1 \setminus S'$ is isomorphic to the graph $H_2 \setminus S'$. It is also obvious that

$$z(H_1 \setminus \{v_1\}) = F_a < F_{a+1} = z(H_2 \setminus \{v_1\}).$$

Under the assumptions of the lemma, it is obvious that $z(G^1_S, \{v_1\}, S \setminus \{v_1\}) \neq 0$. Therefore, by Corollary 2, we have the inequality $z(G') > z(G)$. □

Corollary 4. *Each maximal (n, m)-graph does not contain induced cycles in which only one vertex of the cycle has degree at least 4 in the graph.*

Note that some special cases of the transformations I and II are already known (for example, when G has a special form); see, for example, transformations 1 and 3 from [8].

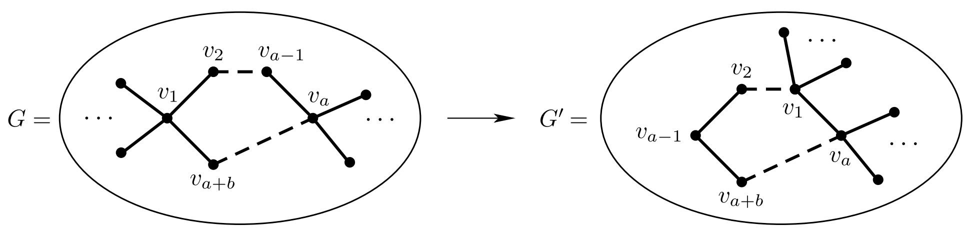

Fig. 6. The transformation III.

Let $a \geq 3$ and $b \geq 1$. Let G be an arbitrary graph in which there is an induced cycle $(v_1, \dots, v_{a+b})$, and all of its vertices, except v_1 and v_a, have degree 2 in G. The transformation III takes G to the graph G', removing all the edges v_1u, where $u \in N_G(v_1) \setminus \{v_2, v_{a+b}\}$, adding edges $v_{a-1}u$, where $u \in N_G(v_1) \setminus \{v_2, v_{a+b}\}$, and also interchanging the names of the vertices v_1 and v_{a-1} (see Fig. 6).

To formulate the next lemma, we use the notation of Corollary 2. Put $G^1 = G$, $G^2 = G'$, H_1 and H_2 is the induced cycle $(v_1, \dots, v_{a+b})$, $S = \{v_1, v_a\}$.

Lemma 4. *The following relation holds*:

$$z(G') - z(G) = F_{a-2} \cdot F_b \cdot z(G^1_S, S, \varnothing).$$

Proof. The graphs G^1_S and G^2_S are isomorphic. Obviously, for any $S' \subset S$, the graphs $H_1 \setminus S'$ and $H_2 \setminus S'$ are isomorphic. It is also obvious that

$$z(H_1 \setminus S) = F_{a-1} \cdot F_{b+1}, \qquad z(H_2 \setminus S) = F_{a+b-1}.$$

Using the decomposition of the Hosoya index of the simple path $(w_1, \dots, w_{a+b-2})$ along the edge $w_{a-2}w_{a-1}$, we obtain

$$F_{a+b-1} = F_{a-1} \cdot F_{b+1} + F_{a-2} \cdot F_b.$$

Therefore, by Lemma 1, we have

$$z(G') - z(G) = F_{a-2} \cdot F_b \cdot z(G^1_S, S, \varnothing).$$

□

If, under the assumptions of Lemma 4, the inequality $z(G_S^1, S, \varnothing) \neq 0$ holds, then $z(G') > z(G)$.

Let $a \geq 2$, $b \geq 2$, and let G be an arbitrary graph in which there is an induced cycle $(v_1, \dots, v_{a+b+1})$, and all of its vertices, except v_1, v_2, v_{a+2}, have degree 2 in G. The transformation IV takes the graph G to the graph G', removing all the edges $v_{a+2}u$, where $u \in N_G(v_{a+2}) \setminus \{v_{a+1}, v_{a+3}\}$, adding all the edges v_3u, where $u \in N_G(v_{a+2}) \setminus \{v_{a+1}, v_{a+3}\}$, and also interchanging the names of the vertices v_3 and v_{a+2} (see Fig. 7).

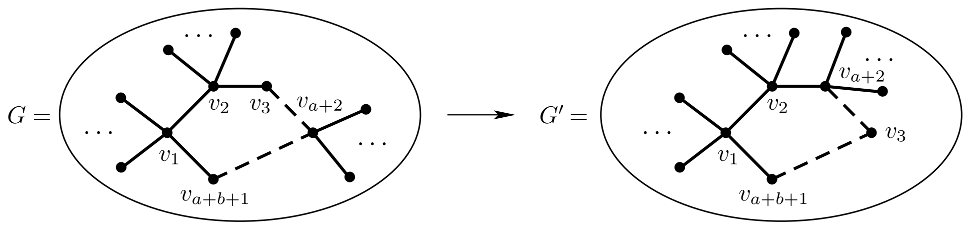

Fig. 7. The transformation IV.

To formulate the next lemma, we use the notation of Corollary 2. Put $G^1 = G$, $G^2 = G'$, H_1, and H_2 is the induced cycle $(v_1, \dots, v_{a+b+1})$, $S = \{v_1, v_2, v_{a+2}\}$. By analogy with the proof of Lemma 4, it is easy to establish the validity of the following statement.

Lemma 5. *The following relation holds*:

$$\begin{aligned} z(G') - z(G) = {} & F_{a-1} \cdot F_b \cdot z(G_S^1, \{v_2, v_{a+2}\}, S \setminus \{v_2, v_{a+2}\}) \\ & - F_{a-1} \cdot F_{b-2} \cdot z(G_S^1, \{v_1, v_{a+2}\}, S \setminus \{v_1, v_{a+2}\}) \\ & + F_{a-1} \cdot F_{b-1} \cdot z(G_S^1, S, \varnothing). \end{aligned}$$

From symmetry considerations, we can assume that

$$z(G_S^1, \{v_2, v_{a+2}\}, S \setminus \{v_2, v_{a+2}\}) \geq z(G_S^1, \{v_1, v_{a+2}\}, S \setminus \{v_1, v_{a+2}\}),$$

since, otherwise, we can interchange the names of the vertices v_1 and v_2. Hence we have $z(G') \geq z(G)$ and the equality is achieved only if

$$z(G_S^1, \{v_2, v_{a+2}\}, S \setminus \{v_2, v_{a+2}\}) = z(G_S^1, \{v_1, v_{a+2}\}, S \setminus \{v_1, v_{a+2}\}) = 0.$$

4.2. The transformations of the Induced Subgraphs $H_{a,b,c}^*$

The graph $H_{a,b,c}^*$ with $a \geq 3$, $b \geq 3$, $c \geq 2$, is shown in Fig. 8.

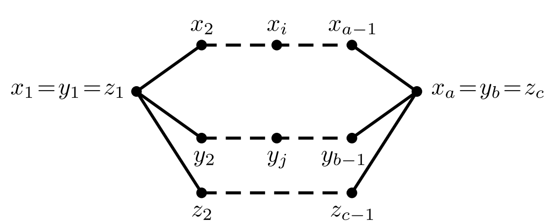

Fig. 8. The graph $H_{a,b,c}^*$.

Recall that the notation $\operatorname{Arg\,max}_{x \in D} f(x)$ indicates the set of points of the domain D at which the function $f(x)$ takes its maximum value in this domain.

Lemma 6. *The following assertions hold*:

1) $\operatorname{Arg\,max}_{\{i,j: x_i \neq y_j\}} z(H_{a,b,c}^* \setminus \{x_i, y_j\}) = \{(2, b-1), (a-1, 2)\}$;

2) $\operatorname{Arg\,max}_i z(H^*_{a,b,c} \setminus \{x_i\}) = \{2, a-1\}$.

Proof. Let us prove assertion 1). Obviously, after deleting x_i and y_j, a forest is formed. By Lemma 2, all the connected components of the given forest must be paths. Using the decomposition of the Hosoya index of the path along the corresponding edge (possibly several times), it is not difficult to conclude that the component must be unique. Hence $H^*_{a,b,c} \setminus \{x_i, y_j\}$ must be a simple path, i.e.,

$$\operatorname{Arg} \max_{\{i,j:x_i \neq y_j\}} z(H^*_{a,b,c} \setminus \{x_i, y_j\}) = \{(2, b-1), (a-1, 2)\}.$$

Let us prove assertion 2). From symmetry considerations, we assume that x_i is removed. We can assume that $i \notin \{1, a\}$, because, otherwise, $H^*_{a,b,c} \setminus \{x_i\}$ is a tree and

$$z(H^* \setminus \{x_i\}) < z(P_{a+b+c-5}), \qquad z(P_{a+b+c-5}) < z(H^*_{a,b,c} \setminus \{x_2\}),$$

which can be verified by decomposing $z(H \setminus \{x_2\})$ along the edge $x_{a-1}x_a$. Let us check that if $i \notin \{2, a-1\}$, then

$$z(H^*_{a,b,c} \setminus \{x_i\}) < z(H^*_{a,b,c} \setminus \{x_2\}) = z(H^*_{a,b,c} \setminus \{x_{a-1}\}).$$

We can assume that $|i-2| \le |i-(a-1)|$. Let us decompose $z(H^*_{a,b,c} \setminus \{x_i\})$ along the edge x_1x_2 and $z(H^*_{a,b,c} \setminus \{x_2\})$ along the edge x_ix_{i+1}:

$$z(H^*_{a,b,c} \setminus \{x_i\}) = z(H^*_{a,b,c} \setminus \{x_2, \dots, x_i\}) \cdot z(P_{i-2}) + z(H^*_{a,b,c} \setminus \{x_1, \dots, x_i\}) \cdot z(P_{i-3}),$$
$$z(H^*_{a,b,c} \setminus \{x_2\}) = z(H^*_{a,b,c} \setminus \{x_2, \dots, x_i\}) \cdot z(P_{i-2}) + z(H^*_{a,b,c} \setminus \{x_2, \dots, x_{i+1}\}) \cdot z(P_{i-3}).$$

The graph $H^*_{a,b,c} \setminus \{x_1, \dots, x_i\}$ is a tree and, therefore,

$$z(H^*_{a,b,c} \setminus \{x_1, \dots, x_i\}) < z(P_{a+b+c-i-4}).$$

Decomposing $z(H^*_{a,b,c} \setminus \{x_2, \dots, x_{i+1}\})$ along the edge $y_{b-1}y_b$, we obtain

$$z(H^*_{a,b,c} \setminus \{x_2, \dots, x_{i+1}\}) > z(P_{a+b+c-i-4}).$$

Therefore, $z(H^*_{a,b,c} \setminus \{x_2\}) > z(H^*_{a,b,c} \setminus \{x_i\})$. Thus,

$$\operatorname{Arg\,max}_i z(H^*_{a,b,c} \setminus \{x_i\}) = \{2, a-1\}. \quad \square$$

Suppose that, in a graph G, there is an induced subgraph $H^*_{a,b,c}$ and

$$A = N_G(x_i) \setminus V(H^*_{a,b,c}) \neq \varnothing.$$

Also suppose that, in $V(H^*_{a,b,c}) \setminus \{x_i\}$, there is at most one vertex with neighbors outside $V(H^*_{a,b,c})$, and if such a vertex exists, then it is the vertex $y_j \neq x_i$. We put $B = N_G(y_j) \setminus V(H^*_{a,b,c})$. Also suppose that $i \notin \{2, a-1\}$, if y_j does not exist and that $(i, j) \notin \{(2, b-1), (a-1, 2)\}$.

The transformation V takes G to the graph G', removing all the edges $x_iv, v \in A$ and adding all the edges $x_2v, v \in A$, and also if y_j exists, removing all the edges $y_jv, v \in B$ and adding all the edges $y_{b-1}v, v \in B$, and also interchanging the names of the vertices x_i and x_2, y_j and y_{b-1} (see Fig. 9).

Lemma 7. *The inequality $z(G') > z(G)$ holds.*

Proof. We will consider the more general case of the existence of a vertex y_j, while the proof for the first case is carried out by analogy. From symmetry considerations, we can assume that $i \notin \{2, a-1\}$. To prove this lemma, we use Corollary 2 and its notation. We put $H_1 = H_2 = H^*_{a,b,c}$. The set of $S = \{x_i, y_j\}$ is H_1- and H_2-separating in the graphs $G^1 = G$ and $G^2 = G'$, and G^1_S and G^2_S are isomorphic. It is not difficult to see that, for any $S' \in \{\{x_i\}, \{y_j\}\}$, we have $z(G^1_S, S', S \setminus S') \neq 0$. Hence, by virtue of Corollary 2 and Lemma 6, we have the inequality $z(G') > z(G)$. $\square$

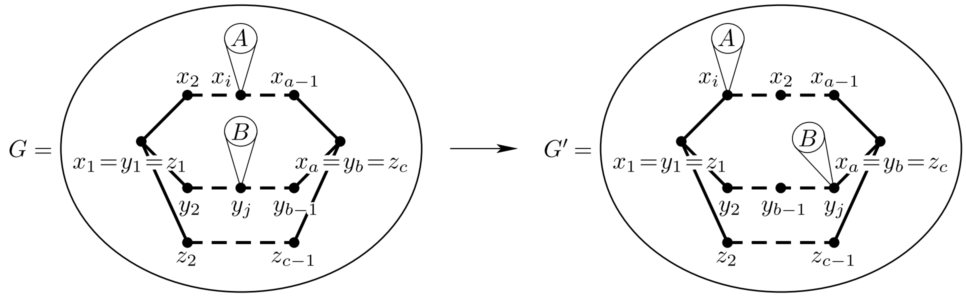

Fig. 9. The transformation V.

Suppose that the graph G contains the induced subgraph $H^*_{4,b,2}$, where $b \neq 4$, in which

$$N_G(x_2) = \{v, x_1, x_3\}, \qquad vx_3 \notin E(G), \qquad \deg_G(v) \geq 2,$$

and no element from $V(H^*_{4,b,2}) \setminus \{x_2, x_3\}$ has any neighbor outside $V(H^*_{4,b,2})$. The transformation VI takes the graph G' to the graph G, removing the edges x_2x_3, x_1x_4 and adding the edges x_1x_3, x_2x_4 (see Fig. 10).

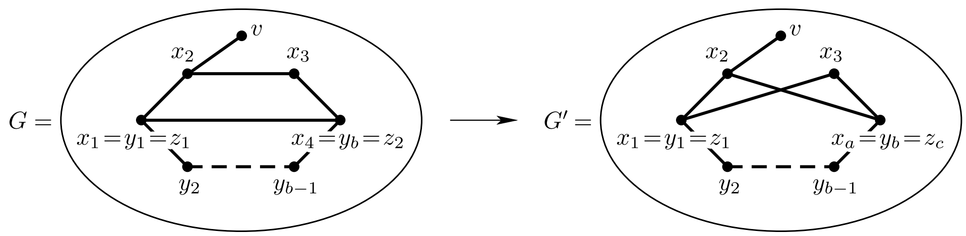

Fig. 10. The transformation VI.

Lemma 8. *The inequality $z(G') > z(G)$ holds.*

Proof. To prove this lemma, we use Corollary 2 and its notation. We put

$$V(H_1) = V(H_2) = V(H^*_{4,b,2}) \cup \{v\}.$$

The set of $S = \{v, x_3\}$ is H_1- and H_2-separating in the graphs $G^1 = G$ and $G^2 = G'$ and, further, G^1_S and G^2_S are isomorphic. Decomposing the Hosoya indices along the edges $x_2v, x_1x_4 \in E(H_1)$ and $x_2v, x_2x_4 \in E(H_2)$, we obtain

$$\begin{aligned}
z(H_2) &= z(H^*_{4,b,2}) + z(C_{b+1}) = z(H^*_{4,b,2}) + z(P_{b+1}) + z(P_{b-1}),\\
z(H_1) &= z(H^*_{4,b,2}) + z(C_b) + z(P_{b-1}) = z(H^*_{4,b,2}) + z(P_b) + z(P_{b-1}) + z(P_{b-2}),\\
z(H_2 \setminus \{v\}) &= z(H_1 \setminus \{v\}) = z(H^*_{4,b,2}),\\
z(H_2 \setminus \{x_3\}) &= z(P_{b+2}) + z(P_{b-1}),\\
z(H_1 \setminus \{x_3\}) &= z(P_{b+2}) + 2z(P_{b-2}),\\
z(H_2 \setminus S) &= z(P_{b+1}) + z(P_{b-1}),\\
z(H_1 \setminus S) &= z(P_{b+1}) + z(P_{b-2}).
\end{aligned}$$

Thus, the following equalities hold:

$$\begin{aligned}
z(H_2) - z(H_1) &= F_{b-2},\\
z(H_2 \setminus \{x_3\}) - z(H_1 \setminus \{x_3\}) &= -F_{b-3},\\
z(H_2 \setminus \{v\}) - z(H_1 \setminus \{v\}) &= 0,\\
z(H_2 \setminus S) - z(H_1 \setminus S) &= F_{b-2}.
\end{aligned}$$

It is clear that $F_{b-2} > F_{b-3}$. Since $vx_3 \notin E(G)$, $\deg_G(v) \geq 2$, it follows that $z(G^1_S, \varnothing, S) \neq 0$. Therefore, by Corollary 2, we have $z(G') > z(G)$. □

Apparently, the transformations III–VI are new.

4.3. The the transformations VII–IX and Their Meaning

Let a graph G contain an induced loop $C = (v_1, \ldots, v_a)$, where $a \geq 5$, $N_G(v_2) = \{v, v_1, v_3\}$, and no vertex C, except v_1, v_2, v_3, has any neighbor outside C. All four cases of the presence of the edges vv_1 and vv_3 in G are allowed. The transformation VII takes the graph G to the graph G', removing the edges vv_2, v_1v_a and adding the edges v_1v_4, vv_a (see Fig. 11).

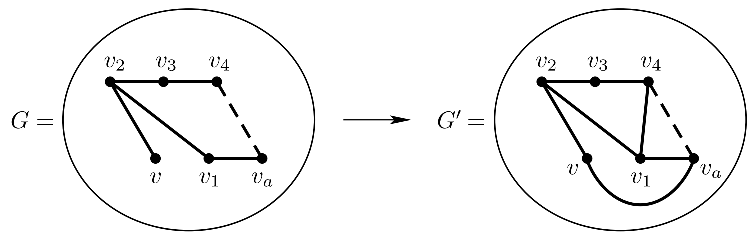

Fig. 11. The transformation VII when $vv_1 \notin E(G)$, $vv_3 \notin E(G)$.

To formulate the next lemma, we use the notation of Corollary 2. We put

$$G^1 = G, \qquad G^2 = G', \qquad V(H_1) = V(H_2) = \{v, v_1, \ldots, v_a\}, \qquad S = \{v, v_1, v_3\}.$$

Lemma 9. *The following relation holds*:

$$z(G') - z(G) = F_{a-4} \cdot (z(G^1_S, \varnothing, S) - z(G^1_S, \{v\}, S \setminus \{v\}) - z(G^1_S, \{v_1, v_3\}, S \setminus \{v_1, v_3\})).$$

Proof. The graphs G^1_S and G^2_S are isomorphic. Obviously, for any $S' \subseteq S$, where $\{v, v_1\} \subseteq S'$ or $\{v, v_3\} \subseteq S'$, the graph $H_1 \setminus S'$ is isomorphic to the graph $H_2 \setminus S'$. Also note that if $S' \subseteq S$ and vv_1 or if vv_3 is an edge of one of the graphs $H_1 \setminus S'$ or $H_2 \setminus S'$ (and, consequently, of the other one), then the following equality holds:

$$z(H_2 \setminus S') - z(H_1 \setminus S') = z((H_2 \setminus S') \setminus \{vv_1, vv_3\}) - z((H_1 \setminus S') \setminus \{vv_1, vv_3\}).$$

This is easy to verify by decomposing the Hosoya index along the edges vv_1, vv_3. Therefore, we can assume that $vv_1 \notin E(G)$ and $vv_3 \notin E(G)$.

Decomposing the Hosoya indices along the edges $v_1v_2 \in E(H_1)$ and $v_1v_4 \in E(H_2)$, we obtain

$$\begin{aligned} z(H_2) &= z(P_{a+1}) + 2z(P_{a-3}), \\ z(H_1) &= z(P_{a+1}) + z(P_{a-2}), \\ z(H_2 \setminus \{v\}) &= z(P_a) + 2z(P_{a-4}), \\ z(H_1 \setminus \{v\}) &= z(P_a) + z(P_{a-2}), \\ z(H_2 \setminus \{v_1, v_3\}) &= z(P_{a-2}), \\ z(H_1 \setminus \{v_1, v_3\}) &= 2z(P_{a-3}). \end{aligned}$$

Therefore, the following equalities are valid:

$$\begin{aligned} z(H_2) - z(H_1) &= 2F_{a-2} - F_{a-1} = F_{a-4}, \\ z(H_2 \setminus \{v\}) - z(H_1 \setminus \{v\}) &= 2F_{a-3} - F_{a-1} = -F_{a-4}, \\ z(H_2 \setminus \{v_1, v_3\}) - z(H_1 \setminus \{v_1, v_3\}) &= 2F_{a-2} - F_{a-1} = -F_{a-4}. \end{aligned}$$

Hence, by Lemma 1, we have

$$z(G') - z(G) = F_{a-4} \cdot (z(G^1_S, \varnothing, S) - z(G^1_S, \{v\}, S \setminus \{v\}) - z(G^1_S, \{v_1, v_3\}, S \setminus \{v_1, v_3\})).$$

□

It is not difficult to see that if $N_G(v) = \{v_2, v'\}$ and $\deg_G(v') \geq 2$, then

$$z(G_S^1, \varnothing, S) > z(G_S^1, \{v\}, S \setminus \{v\}).$$

It is also clear that $z(G_S^1, \{v_1, v_3\}, S \setminus \{v_1, v_3\}) = 0$ if

$$\min(\deg_G(v_1), \deg_G(v_3)) = 2 \vee N_G(v_1) = \{v_2, v_a, u\}, \qquad N_G(v_3) = \{v_2, v_4, u\}.$$

Therefore, if $N_G(v) = \{v_2, v'\}$, $\deg_G(v') \geq 2$, and

$$\min(\deg_G(v_1), \deg_G(v_3)) = 2 \vee N_G(v_1) = \{v_2, v_a, u\}, \qquad N_G(v_3) = \{v_2, v_4, u\},$$

then $z(G') > z(G)$.

Note that the transformation VII and the accompanying Lemma 9 are already known (see Lemma 2.1 from [8]), but the lemma was proved there by a purely technical method.

Suppose that the graph G contains the triangle (v_1, v_2, v_3), in which

$$N_G(v_1) = \{u, v_2, v_3\}, \qquad \deg_G(v_2) = 2$$

and either $N_G(u) = \{v_1, u_1\}$, $u_1 v_3 \notin E(G)$, $\deg_G(u_1) \geq 2$ or

$$\deg_G(v_3) = 2, \qquad N_G(u) = \{v_1, u_1, u_2\}, \qquad N_G(u_1) \setminus \{u, u_2\} \neq \varnothing.$$

The transformation VIII takes the graph G to the graph G', removing the edge $v_1 v_3$ and the edge uu_2 (if $uu_2 \in E(G)$), and also adding the edges uv_3 and $v_3 u_2$ (if $uu_2 \in E(G)$) (see Fig. 12).

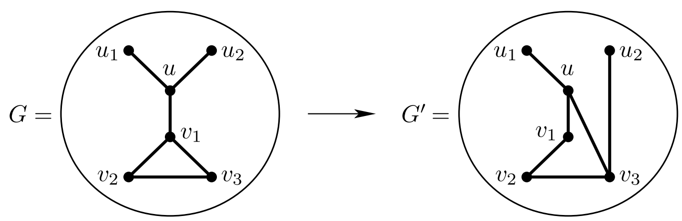

Fig. 12. The transformation VIII when $uu_2 \in E(G)$, $u_1u_2 \notin E(G)$.

Lemma 10. *The inequality $z(G') > z(G)$ holds.*

Proof. To prove this lemma, we use Corollary 2 and its notation. First, consider the case where $N_G(u) = \{v_1, u_1\}$, $u_1 v_3 \notin E(G)$, $\deg_G(u_1) \geq 2$. We put

$$V(H_1) = V(H_2) = \{u, v_1, v_2, v_3\}, \qquad S = \{u, v_3\}.$$

The graphs G_S^1 and G_S^2 are isomorphic. The following relations hold:

$$z(H_2) = 7, \qquad z(H_1) = 6, \qquad z(H_2 \setminus \{u\}) = 3, \qquad z(H_1 \setminus \{u\}) = 4,$$
$$z(H_2 \setminus \{v_3\}) = z(H_1 \setminus \{v_3\}) = 3, \qquad z(H_2 \setminus S) = z(H_1 \setminus S) = 2.$$

Since $\deg_G(u) = 2$, $u_1 v_3 \notin E(G)$, $\deg_G(u_1) \geq 2$, it follows that

$$z(G_S^1, \varnothing, S) > z(G_S^1, \{u\}, S \setminus \{u\}).$$

By Corollary 2, we have $z(G') > z(G)$.

Now consider the case where

$$\deg_G(v_3) = 2, \qquad N_G(u) = \{v_1, u_1, u_2\}, \qquad N_G(u_1) \setminus \{u, u_2\} \neq \varnothing.$$

We put

$$V(H_1) = V(H_2) = \{u, u_1, u_2, v_1, v_2, v_3\}, \qquad S = \{u_1, u_2\}.$$

The graphs G_S^1 and G_S^2 are isomorphic. The following relations hold:

$$z(H_2) = 15, \quad z(H_1) = 14 \qquad (u_1 u_2 \notin E(G)),$$

$$z(H_2) = 22, \quad z(H_1) = 20 \qquad (u_1u_2 \in E(G)),$$
$$z(H_2 \setminus \{u_1\}) = z(H_1 \setminus \{u_1\}) = z(H_2 \setminus \{u_2\}) = z(H_1 \setminus \{u_2\}) = 10,$$
$$z(H_2 \setminus S) = 7, \qquad z(H_1 \setminus S) = 6.$$

Since $N_G(u_1) \setminus \{u, u_2\} \neq \varnothing$, it follows that $z(G^1_S, \varnothing, S) \neq 0$. Combining this result with Corollary 2, we obtain $z(G') > z(G)$. $\square$

Suppose that the graph G contains an induced cycle $C = (v_1, v_2, v_3, v_4)$, where $N_G(v_1) = \{u, v_2, v_4\}$, $N_G(v_2) = \{w, v_1, v_3\}$, and no vertex of C, except v_1, v_2,, has any neighbor outside C. The transformation IX takes the graph G to the graph G', removing the edge v_2w and adding the edge uw (see Fig. 13).

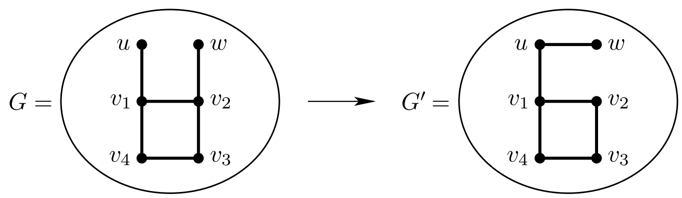

Fig. 13. Conversion IX.

To formulate the next lemma, we use the notation of Corollary 2. We put

$$G^1 = G, \qquad G^2 = G', \qquad V(H_1) = V(H_2) = \{u, w, v_1, v_2, v_3, v_4\}, \qquad S = \{u, w\}.$$

Lemma 11. *The following relation holds*:

$$z(G') - z(G) = 2z(G^1_S, \varnothing, S) - 3z(G^1_S, \{u\}, S \setminus \{u\}).$$

Proof. The graphs G^1_S and G^2_S are isomorphic. For $S' \in \{\{w\}, \{u, w\}\}$, the graphs $H_1 \setminus S'$ and $H_2 \setminus S'$ are isomorphic. The following relations hold:

$$z(H_2) = 17, \qquad z(H_1) = 15, \qquad z(H_2 \setminus \{u\}) = 7, \qquad z(H_1 \setminus \{u\}) = 10.$$

Therefore, by Lemma 1, we have the equality

$$z(G') - z(G) = 2z(G^1_S, \varnothing, S) - 3z(G^1_S, \{u\}, S \setminus \{u\}). \quad \square$$

If, under the assumptions of Lemma 11, it also turns out that G contains a simple path $(u_1 = u, u_2, u_3, u_4)$ in which

$$\deg_G(u_1) = \deg_G(u_2) = \deg_G(u_3) = 2, \qquad \deg_G(u_4) \geq 2, \qquad w \notin \{u_1, u_2, u_3, u_4\},$$

then

$$\begin{aligned}2z(G^1_S, \varnothing, S) &- 3z(G^1_S, \{u\}, S \setminus \{u\}) \\ &= G \setminus (V(H_1) \cup \{u_1, u_2, u_3\}) - G \setminus (V(H_1) \cup \{u_1, u_2, u_3, u_4\}) > 0.\end{aligned}$$

Apparently, the transformations VIII and IX are new.

5. AUXILIARY STATEMENT

The graph H^*_a, where $a \geq 1$, is shown in Fig. 14.

Lemma 12. *For any $a \geq 1$, $b \geq 1$, the following inequality holds*:

$$z(H^*_1) \cdot z(H^*_{a+b-1}) \geq z(H^*_a) \cdot z(H^*_b),$$

where the equality is attained only for $a = 1$ or $b = 1$.

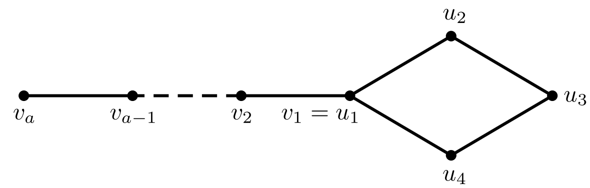

Fig. 14. The graph H_a^*.

Proof. Let us prove the statement by induction on the sum $k = a + b$. The cases $a = 1$ and $b = 1$, obviously imply the equality. From symmetry considerations, we can assume that $b \geq a$. The induction base is $k = 4$, i.e., $a = b = 2$, $k = 5$, i.e., $a = 2, b = 3$, and $k = 6$, i.e., $a = 2, b = 4$ and $a = 3, b = 3$. In this case,

$$z(H_1^*) = 7, \qquad z(H_2^*) = 10, \qquad z(H_3^*) = 17, \qquad z(H_4^*) = 27, \qquad z(H_5^*) = 44$$

and the inequalities hold. Suppose that the inequality holds for all a and b such that $a + b = k$, and prove that it will also hold for all a and b such that $a + b = k + 1$.

Since $a + b = k + 1 \geq 7$, it follows that $b \geq 4$. By the induction hypothesis, the following inequalities hold:

$$z(H_1^*) \cdot z(H_{a+b-2}^*) > z(H_a^*) \cdot z(H_{b-1}^*),$$
$$z(H_1^*) \cdot z(H_{a+b-3}^*) > z(H_a^*) \cdot z(H_{b-2}^*).$$

Adding together these inequalities and decomposing $z(H_{a+b-1}^*)$ and $z(H_b^*)$ along the pendant edges, we obtain

$$\begin{aligned} z(H_1^*) \cdot z(H_{a+b-1}^*) &= z(H_1^*) \cdot z(H_{a+b-2}^*) + z(H_1^*) \cdot z(H_{a+b-3}^*) \\ &> z(H_a^*) \cdot z(H_b^*) = z(H_a^*) \cdot z(H_{b-1}^*) + z(H_a^*) \cdot z(H_{b-2}^*). \quad \square \end{aligned}$$

6. MAIN RESULT

Theorem 1. *For any $n \geq 17$, the only maximal $(n, n + 2)$-graph is the graph shown in Fig.* 1.

Proof. Recall that all types of contractions of $(n, n + 2)$-graphs without pendant vertices were shown in Fig. 1. Hence it follows from Corollaries 3 and 4, as well as from Lemma 7, that the cases from the upper and middle rows are impossible. It follows from Lemmas 9 and 10 that the pendant cycles in maximal $(n, n + 2)$-graphs must have exactly 4 vertices. Let us denote by G an arbitrary maximal $(n, n + 2)$-graph.

Suppose that G contains an induced subgraph $H_{a,b,c}^*$ and does not contain bridges. Then $\deg_G(x_1) = \deg_G(x_a) = 3$ by Lemma 7. It is not difficult to see that $G \setminus V(H_{a,b,c}^*)$ is a simple path or the null graph. Therefore, by the same Lemma 7, we can assume that G is obtained from $G \setminus V(H_{a,b,c}^*)$ by adding the simple path $(v_1, \ldots, v_d)$ between x_2 and y_{b-1} (i.e., $v_1 = x_2, v_d = y_{b-1}$) or between x_i and x_j (i.e., $v_1 = x_i, v_d = x_j$), where $i < j$.

Let us consider the first case. Then, for each $b \geq 5$, by Lemma 5, we have $c = d = 2$. If $b = 4$, then, for $d \geq 3$, $a = 3$, $c = 2$ for the same reason. By Lemma 5, we have

$$z((G \setminus \{y_3y_4\}) \cup \{y_2y_4\}) = z(G),$$

which means, in view of Lemma 9, that G is not a maximal $(n, n + 2)$-graph. The case $b = 4$, $d = 2$ is considered similarly. If $b = 3$, then, by analogy, it can be proved that, for $a \geq 4$, we have $c = d = 2$. In this way, G is obtained by subdivision of one or two nonadjacent edges of the complete graph with 4 vertices.

In the second case, by Lemma 4, we have $j = i + 1$. Then $a \geq 4$ and, by Lemma 4, $c = 2$. By Lemmas 9 and 10, we see that if $d \neq 4$, then $i = 2$, $a = 4$. Similarly, if $b \neq 4$, then $i = 2$, $a = 4$. If $b = d = 4$, then either $(x_1, \ldots, x_i)$ or $(x_{i+1}, \ldots, x_a)$ contains at least 7 vertices, because $n \geq 17$. This case is impossible in view of Lemma 11 and in view of the maximality of G. Hence G is obtained from two cycles $(v_1, \ldots, v_d)$ and $(y_1, \ldots, y_b)$ by adding the edges v_1u_1 and v_dy_b.

Suppose that G contains the induced subgraph $H^*_{a,b,c}$ and the bridge incident to the vertex x_i. Then $i = 2$ by Lemma 7. If $a \geq 4$, then $b = 4, c = 2$ by Lemmas 4, 9, and 10. Therefore, $a \geq 11$ (because $n \geq 17$), which is impossible by Lemma 11 and in view of the maximality of G. Hence $a = 3$.

Suppose that G is obtained from the graphs H^*_a and H^*_b and the cycle $(x_1, \ldots, x_k)$ by adding the edges $x_i v_a$, where $v_a \in V(H^*_a)$, and $x_j v_b$, where $v_b \in V(H^*_b)$. From symmetry considerations, we can assume that $a \geq b$. Then $k + a + b = n - 6 \geq 11$. By Lemma 10, we have $k \neq 3$. Hence $j = i + 1$ by Lemma 4. If $k = 4$, then $a \geq 4$ and this case is impossible by Lemma 11. To show this, we note that

$$2z(H^*_{a-1}) - 3z(H^*_{a-2}) = 2z(H^*_{a-3}) - z(H^*_{a-2}) > 0, \qquad a \geq 4.$$

If $k \geq 5$, then $a = b = 1$ by Lemma 9.

It follows from our arguments that each maximal $(n, n+2)$-graph has one of the types A–E (see Fig. 15) and only marked edges can be subdivided.

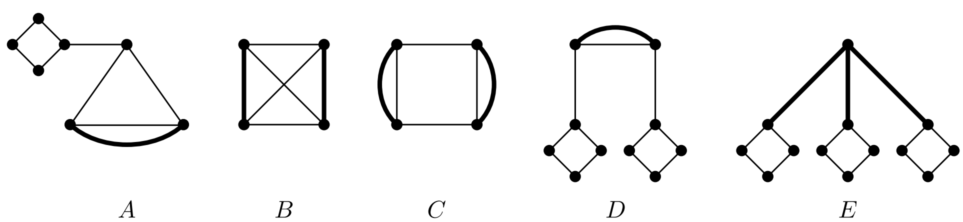

Fig. 15. The pseudographs A- E.

Denote by $z_n(A), \ldots, z_n(E)$ the maximum value of the Hosoya index in n-vertex type graphs $A, \ldots, E$, respectively. By Lemma 8, we have $z_n(B) > z_n(C)$. Note that graphs of types A and D are unique. We have

$$\begin{aligned} z_n(A) &= 3z(C_{n-5}) + 7(2z(P_{n-6}) + z(C_{n-5})) = 10z(C_{n-5}) + 14z(P_{n-6}) \\ &= 10z(P_{n-5}) + 14z(P_{n-6}) + 10z(P_{n-7}), \\ z_n(D) &= 3z(H^*_{n-8}) + 7(3z(P_{n-9}) + 7z(C_{n-8})) \\ &= 3(z(P_{n-5}) + 2z(P_{n-9})) + 21z(P_{n-9}) + 49(z(P_{n-8}) + z(P_{n-10})) \\ &= 3z(P_{n-5}) + 49z(P_{n-8}) + 27z(P_{n-9}) + 49z(P_{n-10}). \end{aligned}$$

Hence the following equalities hold:

$$\begin{aligned} z_{14}(A) &= 10 \cdot 55 + 14 \cdot 34 + 10 \cdot 21 = 1236, \\ z_{15}(A) &= 10 \cdot 89 + 14 \cdot 55 + 10 \cdot 34 = 2000, \\ z_{14}(D) &= 3 \cdot 55 + 49 \cdot 13 + 27 \cdot 8 + 49 \cdot 5 = 1263, \\ z_{15}(D) &= 3 \cdot 89 + 49 \cdot 21 + 27 \cdot 13 + 49 \cdot 8 = 2039. \end{aligned}$$

Each n-vertex graph of type E is obtained by identifying the vertices $v_{a_1}, v_{a_2}, v_{a_3}$ of the graphs $H^*_{a_1}$, $H^*_{a_2}$, $H^*_{a_3}$, where $a_1 + a_2 + a_3 = n - 7$. We denote it by E_{a_1,a_2,a_3}. From symmetry considerations, we assume that $a_1 \geq a_2 \geq a_3 \geq 2$. If $a_2 > 2$, then we consider the graph $E_{a_1+a_2-2,2,a_3}$. Lemma 11 implies that

$$z(E_{a_1+a_2-2,2,a_3}) > z(E_{a_1,a_2,a_3});$$

To show this, it suffices to consider the decompositions $z(E_{a_1+a_2-2,2,a_3})$ and $z(E_{a_1,a_2,a_3})$ along the pendant edge of the subgraphs $H^*_{a_3}$. Thus, among n-vertex graphs of type E, the graph $E_{n-11,2,2}$ has a maximum value of the Hosoya index,.

We have

$$\begin{aligned} z_n(E) = z(E_{n-11,2,2}) &= 21z(H^*_{n-12}) + 7(3z(H^*_{n-12}) + 7z(H^*_{n-11})) \\ &= 49z(H^*_{n-11}) + 42z(H^*_{n-12}) \\ &= 49(z(P_{n-8}) + 2z(P_{n-12})) + 42(z(P_{n-9}) + 2z(P_{n-13})) \end{aligned}$$

$$= 49z(P_{n-8}) + 42z(P_{n-9}) + 98z(P_{n-12}) + 84z(P_{n-13}).$$

Hence the following equalities hold:

$$z_{14}(E) = 49 \cdot 13 + 42 \cdot 8 + 98 \cdot 2 + 84 \cdot 1 = 1253,$$
$$z_{15}(E) = 49 \cdot 21 + 42 \cdot 13 + 98 \cdot 3 + 84 \cdot 2 = 2037.$$

By Corollary 1, we see that, for all $k \geq 0$, the following identities hold:

$$z_{14+k+2}(A) = z_{14+k+1}(A) + z_{14+k}(A),$$
$$z_{14+k+2}(D) = z_{14+k+1}(D) + z_{14+k}(D),$$
$$z_{14+k+2}(E) = z_{14+k+1}(E) + z_{14+k}(E).$$

In view of these identities and the inequalities

$$z_{14}(D) > z_{14}(E) > z_{13}(A),$$
$$z_{15}(D) > z_{15}(E) > z_{15}(A),$$

using Corollary 1, we can easily prove by induction the inequality

$$z_n(D) > z_n(E) > z_n(A),$$

which holds for any $n \geq 14$. □

7. CONCLUSIONS

In this paper, we consider and solve the problem of finding all connected graphs with n vertices and $n+2$ edges with maximum number of matchings among graphs with such parameters. Before this work, the solution of this problem was already known, but it was obtained by using standard techniques. In the present paper, the solution is obtained by using a new approach, which is based on the decomposition by subsets of separating vertices. The new proof is shorter and more combinatorial than the original proof.

The corresponding problem for connected n-vertex graphs with $n+3$ edges is more complicated than for $(n, n+2)$-graphs. Without computer calculations, even to obtain a complete enumeration of their contractions seems to be a difficult problem. Therefore, it would be advisable to consider the restriction of the problem to, for example, subcubic $(n, n+3)$-graphs, where hopes for obtaining complete solutions are much higher. This is a possible topic for future research.

The problem of enumerating perfect matchings was also investigated for other important classes of graphs in, for example, [9], where the class of grid (lattice) graphs was studied..

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