On rationally integrable planar dual and projective billiards

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Abstract

A *caustic* of a strictly convex planar bounded billiard is a smooth curve whose tangent lines are reflected from the billiard boundary to its tangent lines. The famous Birkhoff Conjecture states that if the billiard boundary has an inner neighborhood foliated by closed caustics, then the billiard is an ellipse. It was studied by many mathematicians, including H.Poritsky, M.Bialy, S.Bolotin, A.Mironov, V.Kaloshin, A.Sorrentino and others. In the paper we study its following generalized *dual* version stated by S.Tabachnikov. Consider a closed smooth strictly convex curve $\gamma \subset \mathbb{RP}^2$ equipped with a *dual billiard structure:* a family of non-trivial projective involutions acting on its projective tangent lines and fixing the tangency points. *Suppose that its outer neighborhood admits a foliation by closed curves (including γ) such that the involution of each tangent line permutes its intersection points with every leaf. Then γ and the leaves are conics forming a pencil.* We prove positive answer in the case, when the curve γ is C^4-smooth and the foliation admits a rational first integral. To this end, we show that each C^4-smooth germ of curve carrying a rationally integrable dual billiard structure is a conic and classify rationally integrable dual billiards on (punctured) conic. They include the dual billiards induced by pencils of conics, two infinite series of exotic dual billiards and five more exotic ones.

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Contents

1 Introduction

1.1 Main results: classification of rationally integrable dual planar billiards

The famous Birkhoff Conjecture deals with a billiard in a bounded planar domain $\Omega \subset \mathbb{R}^2$ with smooth strictly convex boundary. Recall that its *caustic* is a curve $S \subset \mathbb{R}^2$ such that each tangent line to S is reflected from the boundary $\partial\Omega$ to a line tangent to S. A billiard Ω is called *Birkhoff caustic-integrable,* if a neighborhood of its boundary in Ω is foliated by closed caustics, and the boundary $\partial\Omega$ is a leaf of this foliation. It is well-known that each elliptic billiard is integrable: ellipses confocal to the boundary are caustics, see [44, section 4]. The **Birkhoff Conjecture** states the converse: *the only Birkhoff caustic-integrable convex bounded planar billiards with smooth boundary are ellipses.*[1] See its brief survey in Subsection 1.5.

S.Tabachnikov suggested its generalization to projective billiards introduced by himself in 1997 in [43]. See the following definition and conjecture.

Definition 1.1 [43] A *projective billiard* is a smooth planar curve $C \subset \mathbb{R}^2$ equipped with a transversal line field $\mathcal{N}$. For every $Q \in C$ the *projective billiard reflection involution* at Q acts on the space of lines through Q as the affine involution $\mathbb{R}^2 \to \mathbb{R}^2$ that fixes the points of the tangent line to C at Q, preserves the line $\mathcal{N}(Q)$ and acts on $\mathcal{N}(Q)$ as central symmetry with respect to the point[2] Q. In the case, when C is a strictly convex closed curve, the *projective billiard map* acts on the *phase cylinder:* the space of oriented lines intersecting C. It sends an oriented line to its image under the above reflection involution at its last point of intersection with C in the sense of orientation. See Fig. 1.

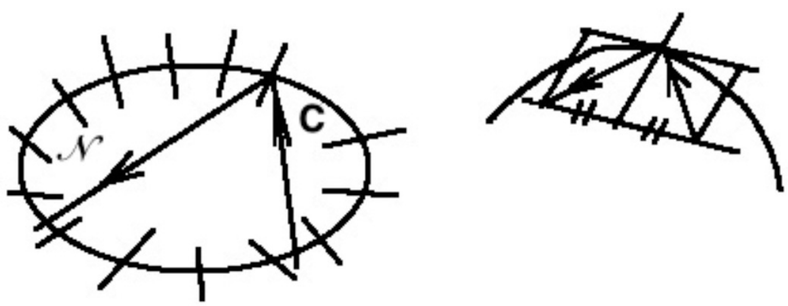

Figure 1: The projective billiard reflection.

[1]This conjecture, attributed to G.Birkhoff, was first mentioned in print in the paper [41] by H. Poritsky, who worked with Birkhoff as a post-doctoral fellow in late 1920-ths.

[2]In other words, two lines a, b through Q are permuted by reflection at Q, if and only if the quadruple of lines T_QC, $\mathcal{N}(Q)$, a, b is harmonic: there exists a projective involution of the space $\mathbb{RP}^1$ of lines through Q that fixes T_QC, $\mathcal{N}(Q)$ and permutes a, b.

Example 1.2 A usual Euclidean planar billiard is a projective billiard with transversal line field being normal line field.

Example 1.3 Each simply connected complete Riemannian surface of constant curvature is isometric (up to constant factor) to one of the two-dimensional space forms: the Euclidean plane, the unit sphere, the hyperbolic plane. *Any billiard in the hyperbolic plane (hemisphere) is isomorphic to a projective billiard.* Namely, each space form is represented by a hypersurface Σ in the space $\mathbb{R}^3$ equipped with appropriate quadratic form

$$< Ax, x >, \quad < x, x >:= x_1^2 + x_2^2 + x_3^2,$$

A is a symmetric 3x3-matrix called *space form matrix*:

Euclidean plane: $A = \mathrm{diag}(1, 1, 0)$, $\Sigma = \{x_3 = 1\}$.

Sphere: $A = Id$, $\Sigma = \{< x, x >= 1\}$ is the unit sphere.

Hyperbolic plane: $A = \mathrm{diag}(1, 1, -1)$, $\Sigma = \{< Ax, x >= -1, \ x_3 > 0\}$.

The metric of the surface Σ is induced by the quadratic form $< Ax, x >$. Its geodesics are the sections of the surface Σ by two-dimensional vector subspaces in $\mathbb{R}^3$. The billiard in a domain $\Omega \subset \Sigma_+ := \Sigma \cap \{x_3 > 0\}$ is defined by reflection of geodesics from its boundary. The tautological projection $\pi : \mathbb{R}^3 \setminus \{0\} \to \mathbb{RP}^2$ sends Ω diffeomorphically to a domain in the affine chart $\{x_3 = 1\}$. It sends billiard orbits in Ω to orbits of the projective billiard on $C = \pi(\partial\Omega)$ with the transversal line field $\mathcal{N}$ on C being the image of the normal line field to $\partial\Omega$ under the differential $d\pi$. The projective billiard on C is a space form billiard, see the next definition.

Definition 1.4 Let A be a space form matrix. Let C be a curve in an affine chart in $\mathbb{RP}^2$. Let $\mathcal{N}$ be the transversal line field on C defined as follows.

a) Case, when $A = \mathrm{diag}(1, 1, 0)$. Then $\mathcal{N}$ is the normal line field to C in the affine chart $\{x_3 \neq 0\}$.

b) Case, when $\det A \neq 0$, i.e., $A = \mathrm{diag}(1, 1, \pm 1)$. Then for every $Q \in C$ the two-dimensional subspaces in $\mathbb{R}^3$ projected to the lines tangent to T_QC and $\mathcal{N}(Q)$ are orthogonal with respect to the scalar product $< Ax, x >$.

Then the projective billiard defined by $\mathcal{N}$ is called a *space form billiard*[3].

The definitions of caustic and Birkhoff integrability for projective billiards repeat the above definitions given for classical billiards.

[3]A space form projective billiard with matrix $A = \mathrm{diag}(1, 1, -1)$ is not necessarily the projection of a billiard in the hyperbolic plane $\Sigma = \Sigma_+$. Some its part may lie in the projection to $\mathbb{RP}^2$ of the de Sitter cylinder $\{< Ax, x >= 1\}$, where the quadratic form $< Ax, x >$ defines a pseudo-Riemannian metric of constant curvature.

Conjecture 1.5 (S.Tabachnikov) In every Birkhoff integrable projective billiard its boundary and closed caustics forming a foliation are ellipses whose projective-dual conics form a pencil.

Below we state the dual version of the Tabachnikov's Conjecture (2008, [45]) and present partial positive results. To do this, consider $\mathbb{R}^2_{x_1,x_2}$ as the plane $\{x_3 = 1\} \subset \mathbb{R}^3_{x_1,x_2,x_3}$ identified with the corresponding affine chart in $\mathbb{RP}^2_{[x_1:x_2:x_3]}$. The *orthogonal polarity* sends a two-dimensional vector subspace $W \subset \mathbb{R}^2$ to its Euclidean-orthogonal subspace $W^{\perp}$. The corresponding *projective duality* (also called orthogonal polarity) is the map $\mathbb{RP}^{2*} \to \mathbb{RP}^2$ sending lines to points so that the tautological projection of each punctured two-dimensional subspace $W \setminus \{0\} \subset \mathbb{R}^3$ (a line L) is sent to the projection of its punctured orthogonal complement $W^{\perp} \setminus \{0\}$ (called its dual point and denoted by L^*). The line dual to a point P will be denoted by P^*. To each curve $C \subset \mathbb{R}^2$ we associate the *dual curve* $\gamma = C^* \subset \mathbb{RP}^2$ consisting of those points that are dual to the tangent lines to C.

Let now a planar curve C be equipped with a projective billiard structure: a transversal line field $\mathcal{N}$. For every point $Q \in C$ let L_Q denote the projective tangent line to C at Q in the ambient projective plane $\mathbb{RP}^2 \supset \mathbb{R}^2$. The projective duality sends the space $\mathbb{RP}^1_Q$ of lines through Q to the projective line Q^* dual to Q. The line Q^* is tangent to γ at the point $P = L_Q^*$ dual to L_Q. *The duality "line $\mapsto$ point" conjugates the projective billiard involution acting on $\mathbb{RP}^1_Q$ with a non-trivial projective involution $\sigma_P : L_P \to L_P$ fixing P and the point dual to $\mathcal{N}(Q)$.* Thus, *the duality transforms a projective billiard on C to a dual billiard on $\gamma = C^*$,* see the next definition.

Definition 1.6 A *dual billiard structure* on a smooth curve $\gamma \subset \mathbb{RP}^2$ is a family of non-trivial projective involutions $\sigma_P : L_P \to L_P$ fixing P.

Remark 1.7 Let a projective billiard on C have a strictly convex closed caustic S. Then its dual curve S^* is also strictly convex and closed, and for every $P \in \gamma = C^*$ the dual billiard involution $\sigma_P : L_P \to L_P$ permutes the two points of intersection $L_P \cap S^*$. See Fig. 2. A curve S^* satisfying the latter statement is called an *invariant curve for the dual billiard.*

Definition 1.8 A dual billiard on a strictly convex closed curve γ is *integrable,* if there exists a C^0-foliation by closed strictly convex invariant curves on a neighborhood of γ on its concave side, with γ being a leaf. See Fig. 3.

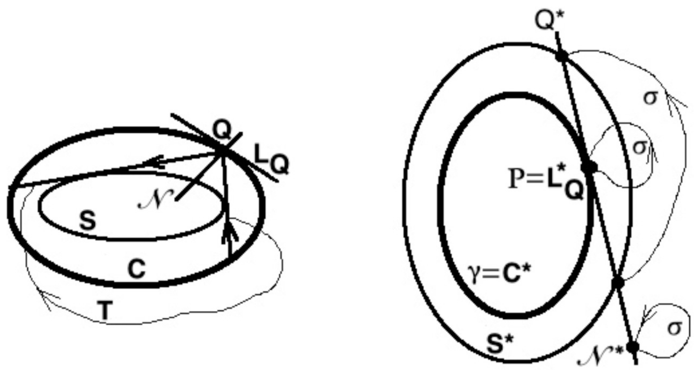

Figure 2: The projective billiard reflection involution T acting on lines through a point $Q \in C$ and the dual involution $\sigma = \sigma_P$ acting on the dual line Q^* tangent to the dual curve $\gamma = C^*$ at the point $P = L_Q^*$.

Conjecture 1.9 (S.Tabachnikov [45]); dual to Conjecture 1.5). For every integrable dual billiard the underlying curve and the corresponding invariant curves forming foliation are conics forming a pencil.

Remark 1.10 A projective billiard on a strictly convex closed curve is integrable, if and only if so is its dual billiard. The *outer* dual billiard in $\mathbb{R}^2$, with $\sigma_P : L_P \to L_P$ being the central symmetry with respect to the tangency point P, is dual to the centrally-projective billiard, whose transversal field consists of lines passing through the origin [43]. Thus, *Conjecture 1.9 would imply the Birkhoff Conjecture and its versions on surfaces of constant curvature and for outer billiards* (as observed in [45]), see Examples 1.2, 1.3.

One of the main results of the present paper is the following theorem.

Theorem 1.11 *Let $\gamma \subset \mathbb{RP}^2$ be a C^4-smooth strictly convex closed planar curve equipped with an integrable dual billiard structure. Let the corresponding foliation by invariant curves admit a rational first integral. Then its leaves, including γ, are conics forming a pencil.*

Below we state a more general result for γ being a germ. To do this, let us introduce the following definition.

Definition 1.12 A dual billiard on a (germ of) curve $\gamma \subset \mathbb{RP}^2$ given by involution family $\sigma_P : L_P \to L_P$ is called *rationally integrable,* if there

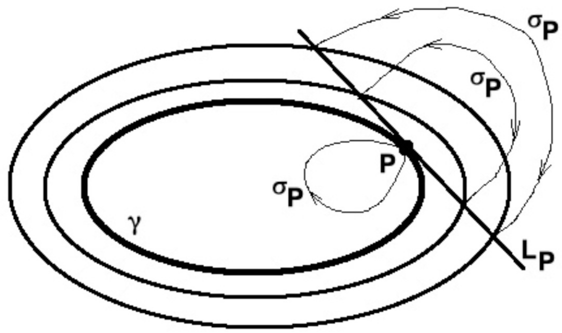

Figure 3: An integrable dual billiard structure

exists a non-constant rational function R on $\mathbb{RP}^2$ whose restriction to L_P is σ_P-invariant for every $P \in \gamma$: $R \circ \sigma_P = R$ on L_P.

Example 1.13 Let a dual billiard on γ be *polynomially* integrable: the above integral R is polynomial in some affine chart $\mathbb{R}^2$. Then for every $P \in \gamma \cap \mathbb{R}^2$ the involution σ_P fixes the intersection point of the line L_P with the infinity line, and hence, is the central symmetry $L_P \to L_P$ with respect to the tangency point P. Thus, the dual billiard in question is a polynomially integrable outer billiard. It is known that in this case the underlying curve is a conic: stated as a conjecture and proved in [45] under some non-degeneracy assumption; proved in full generality in [28].

Example 1.14 (S.Tabachnikov's observation). Let A, B be real symmetric 3×3-matrices, B be non-degenerate. Consider the pencil of conics $\mathcal{C}_\lambda := \{< (B - \lambda A)x, x >= 0\}$; set $\gamma := \mathcal{C}_0 = \{< Bx, x >= 0\}$. The set of those points in $\mathbb{CP}^2$ that lie in complexifications of all $\mathcal{C}_\lambda$ simultaneously will be called the *basic set* of the pencil and denoted by $\mathcal{B}(\mathcal{C})$. *For every $P \in \gamma^o := \gamma \setminus \mathcal{B}(\mathcal{C})$ the involution permuting the two complex points of intersection $\mathcal{C}_\lambda \cap L_P$ for each λ is a well-defined real projective involution $\sigma_P : L_P \to L_P$.* This yields a dual billiard on γ^o, which will be called *dual billiard of conical pencil type.* It is known to be rationally integrable with a quadratic integral: the ratio of quadratic polynomials vanishing on some two given conics of the pencil.

Definition 1.15 Two dual billiard structures on two (germs of) curves γ_1, γ_2 in $\mathbb{RP}^2$ are *real-projective equivalent,* if there exists a projective transformation $\mathbb{RP}^2 \to \mathbb{RP}^2$ sending γ_1 to γ_2 and transforming one structure to the other one. (Projective equivalence preserves rational integrability.) Real-projective equivalence of projective billiards is defined analogously.

The main result of the paper is the next theorem stating that for every rationally integrable dual billiard the underlying curve is a conic, the dual billiard structure extends to the conic punctured in at most four points, and classifying rationally integrable dual billiards on punctured conic. Unexpectedly, *there are infinitely many exotic, non-pencil rationally integrable dual billiards on punctured conic, with integrals of arbitrarily high degrees.*

Theorem 1.16 *Let $\gamma \subset \mathbb{R}^2 \subset \mathbb{RP}^2$ be a C^4-smooth non-linear germ of curve equipped with a rationally integrable dual billiard structure. Then γ is a conic, and the dual billiard structure has one of the three following types (up to real-projective equivalence):*

1) The dual billiard is of conical pencil type and has a quadratic integral.

2) There exists an affine chart $\mathbb{R}^2_{z,w} \subset \mathbb{RP}^2$ in which $\gamma = \{w = z^2\}$ and such that for every $P = (z_0, w_0) \in \gamma$ the involution $\sigma_P : L_P \to L_P$ is given by one of the following formulas:

a) In the coordinate

$$\zeta := \frac{z}{z_0}$$

$$\sigma_P : \zeta \mapsto \eta_\rho(\zeta) := \frac{(\rho-1)\zeta - (\rho-2)}{\rho\zeta - (\rho-1)},$$

$$\rho = 2 - \frac{2}{2N+1}, \quad \text{or} \quad \rho = 2 - \frac{1}{N+1} \quad \text{for some } N \in \mathbb{N}. \tag{1.1}$$

b) In the coordinate

$$u := z - z_0$$

$$\sigma_P : u \mapsto -\frac{u}{1 + f(z_0)u}, \tag{1.2}$$

$$f = f_{b1}(z) := \frac{5z-3}{2z(z-1)} \text{ (type 2b1)}, \quad \text{or} \quad f = f_{b2}(z) := \frac{3z}{z^2+1} \text{ (type 2b2)}. \tag{1.3}$$

c) In the above coordinate u the involution σ_P takes the form (1.2) with

$$f = f_{c1}(z) := \frac{4z^2}{z^3-1} \text{ (type 2c1)}, \quad \text{or} \quad f = f_{c2}(z) := \frac{8z-4}{3z(z-1)} \text{ (type 2c2)}. \tag{1.4}$$

d) In the above coordinate u the involution σ_P takes the form (1.2) with

$$f = f_d(z) = \frac{4}{3z} + \frac{1}{z-1} = \frac{7z-4}{3z(z-1)} \quad \text{(type 2d)}. \tag{1.5}$$

Addendum to Theorem 1.16. *Every dual billiard structure on γ of type 2a) has a rational first integral $R(z,w)$ of the form*

$$R(z,w) = \frac{(w-z^2)^{2N+1}}{\prod_{j=1}^{N}(w-c_jz^2)^2}, \quad c_j = -\frac{4j(2N+1-j)}{(2N+1-2j)^2}, \quad \text{for } \rho = 2 - \frac{2}{2N+1}; \tag{1.6}$$

$$R(z,w) = \frac{(w-z^2)^{N+1}}{z\prod_{j=1}^{N}(w-c_jz^2)}, \quad c_j = -\frac{j(2N+2-j)}{(N+1-j)^2}, \quad \text{for } \rho = 2 - \frac{1}{N+1}. \tag{1.7}$$

The dual billiards of types 2b1) and 2b2) have respectively the integrals

$$R_{b1}(z,w) = \frac{(w-z^2)^2}{(w+3z^2)(z-1)(z-w)}, \tag{1.8}$$

$$R_{b2}(z,w) = \frac{(w-z^2)^2}{(z^2+w^2+w+1)(z^2+1)}. \tag{1.9}$$

The dual billiards of types 2c1), 2c2) have respectively the integrals

$$R_{c1}(z,w) = \frac{(w-z^2)^3}{(1+w^3-2zw)^2}, \tag{1.10}$$

$$R_{c2}(z,w) = \frac{(w-z^2)^3}{(8z^3-8z^2w-8z^2-w^2-w+10zw)^2}. \tag{1.11}$$

The dual billiard of type 2d) has the integral

$$R_d(z,w) = \frac{(w-z^2)^3}{(w+8z^2)(z-1)(w+8z^2+4w^2+5wz^2-14zw-4z^3)}. \tag{1.12}$$

We prove the following theorem, which is a unifying complex version of Theorems 1.11, 1.16. To state it, let us introduce the following definition.

Definition 1.17 Consider a regular germ of holomorphic curve $\gamma \subset \mathbb{CP}^2$ at a point O. A *complex (holomorphic or not) dual billiard* on γ is a germ of (holomorphic or not) family of complex projective involutions $\sigma_P : L_P \to L_P$, $P \in \gamma$, acting on complex projective tangent lines L_P to γ at P and fixing P. A complex dual billiard on γ is said to be *rationally integrable,* if there exists a non-constant complex rational function R on $\mathbb{CP}^2$ such that for every $P \in \gamma$ the restriction $R|_{L_P}$ is σ_P-invariant: $R \circ \sigma_P = R$ on L_P. The definition of complex-projective equivalent complex dual billiards repeats the definition of real-projective equivalent ones with change of real projective transformations $\mathbb{RP}^2 \to \mathbb{RP}^2$ to complex ones acting on $\mathbb{CP}^2$.

Theorem 1.18 *Every regular germ of holomorphic curve in $\mathbb{CP}^2$ (different from a straight line) equipped with a rationally integrable complex dual billiard structure is a conic. Up to complex-projective equivalence, the corresponding billiard structure has one of the types 1), 2a), 2b1), 2c1), 2d) listed in Theorem 1.16, with a rational integral as in its addendum. (Here the coordinates (z, w) as in the addendum are complex affine coordinates.)*

Addendum to Theorem 1.18 *The billiards of types 2b1), 2b2), see (1.3), are complex-projectively equivalent, and so are the billiards of types 2c1) and 2c2). For every $g = b, c$ there exists a complex projective equivalence between the billiards 2g1), 2g2) that sends the integral R_{g1} of the former, see (1.8), (1.10) (treated as a rational function on $\mathbb{CP}^2_{[z:w:t]} \supset \mathbb{C}^2_{z,w} = \{t = 1\}$), to the integral R_{g2} of the latter, see (1.9), (1.11), up to constant factor.*

1.2 Classification of rationally 0-homogeneously integrable projective billiards with smooth connected boundary

Let $\Omega \subset \mathbb{R}^2_{x_1,x_2}$ be a domain with smooth boundary $\partial\Omega$ equipped with a projective billiard structure (transverse line field). The *projective billiard flow* (introduced in [43]) acts on $T\mathbb{R}^2|_\Omega$ analogously to the classical case of Euclidean billiards. Given a point $(Q, v) \in T\mathbb{R}^2$, $Q \in \Omega$, $v = (v_1, v_2) \in T_Q\mathbb{R}^2$, the flow moves the point Q along the straight line directed by v with the fixed uniform velocity v, until it hits the boundary $\partial\Omega$ at some point H. Let $v^* \in T_H\mathbb{R}^2$ denote the image of the velocity vector v (translated to H) under the projective billiard reflection from the tangent line $T_H\partial\Omega$. Afterwards the flow moves the point H with the new uniform velocity v^* until its trajectory hits the boundary again etc. See Fig. 4 below.

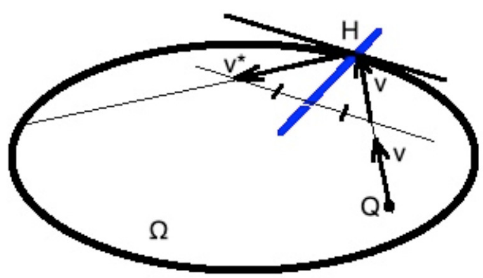

Figure 4: Projective billiard flow

Remark 1.19 The flow in a Euclidean planar billiard always has a trivial first integral $||v||^2$. But it is not a first integral in a generic projective billiard. It is a well-known folklore fact that *Birkhoff integrability of a Euclidean planar billiard with strictly convex closed boundary is equivalent to the existence of a non-trivial first integral of the billiard flow independent with* $||v||^2$ *on a neighborhood of the unit tangent bundle to* $\partial\Omega$ *in* $T\mathbb{R}^2|_\Omega$.

Billiard flows in space forms of constant curvature and their integrability were studied by many mathematicians, including A.P.Veselov [49, 50], S.V.Bolotin [14, 15] (both in any dimension), M.Bialy and A.E.Mironov [10, 11, 8, 9, 12], the author [29, 31] and others. A Euclidean planar billiard is called *polynomially integrable,* if its flow admits a first integral that is polynomial in the velocity v whose restriction to the unit velocity hypersurface $\{||v|| = 1\}$ is non-constant [14, 15, 37, 10], [31, definition 1.1]. S.V.Bolotin suggested the *polynomial version* of Birkhoff Conjecture stating that *if a billiard in a strictly convex bounded planar domain with* C^2*-smooth boundary is polynomially integrable, then the billiard boundary is an ellipse,* together with its versions on the sphere and on the hyperbolic plane. Now this is a theorem: a joint result of M.Bialy, A.E.Mironov and the author of the present paper [10, 11, 29, 31]. Here we present a version of this result for rationally integrable projective billiard flows, see the following definition.

All the results of this subsection will be proved in Section 9.

Definition 1.20 A planar projective billiard is *rationally 0-homogeneously integrable,* if its flow admits a non-constant first integral I that is a rational homogeneous function of the velocity with numerator and denominator having the same degrees (called a *rational 0-homogeneous integral*):

$$I(Q,v) = \frac{I_{1,Q}(v)}{I_{2,Q}(v)}; \quad I_{1,Q}(v),\ I_{2,Q}(v) \text{ are homogeneous polynomials,}$$

$$\deg I_{1,Q} = \deg I_{2,Q}.$$

Here we consider that the degrees $\deg I_{j,Q}(v)$ are uniformly bounded.

Example 1.21 It is known that for every polynomially integrable planar billiard the polynomial integral $I_Q(v)$ can be chosen homogeneous of even degree $2n$, see [14], [15, p.118; proposition 2 and its proof on p.119], [37, chapter 5, section 3, proposition 5]. Then the rational function

$$\Psi(Q,v) := \frac{I_Q(v)}{||v||^{2n}}$$

is a rational 0-homogeneous integral of the billiard. Thus, *every polynomially integrable Euclidean planar billiard is rationally 0-homogeneously integrable.* This also holds for billiards on the sphere and the hyperbolic plane.

Theorem 1.22 *Let a projective billiard in a strictly convex bounded domain $\Omega \subset \mathbb{R}^2$ with C^4-smooth boundary be defined by a continuous transversal line field on $\partial\Omega$ and be rationally 0-homogeneously integrable. Then its boundary is a conic, and the projective billiard is a space form billiard (see Definition 1.4).*

Theorem 1.26 stated below extends Theorem 1.22 to germs of planar projective billiards. Each of them is a germ of C^4-smooth curve C equipped with a transversal line field $\mathcal{N}$. Here C is not necessarily convex. We choose a side from the curve C and a simply connected domain U adjacent to C from the chosen side. Let $Q \in U$ and $v \in T_Q\mathbb{R}^2$ be such that the ray issued from the point Q in the direction of the vector v intersects C, and the distance of the point Q to their first intersection point be equal to $\tau_0||v||$, $\tau_0 > 0$. Then for $t_0 > \tau_0$ close enough to τ_0 the projective billiard flow maps in times $\tau \in (0, t_0)$ are well-defined on (Q, P). As before, we say that a *germ of projective billiard* thus defined is *rationally* 0*-homogeneously integrable,* if it admits a first integral rational and 0-homogeneous in v on $T\mathbb{R}^2|_U$ for some U (small enough) whose degree is uniformly bounded in $Q \in U$.

Before the statement of Theorem 1.26 let us state two preparatory propositions: the first saying that integrability is independent on choice of side; the second reducing classification of germs of integrable projective billiards to classification of germs of integrable dual billiards given by Theorem 1.16. To do this, following S.V.Bolotin [15], let us identify the ambient plane $\mathbb{R}^2$ of a projective billiard with the plane $\{x_3 = 1\}$ in the Euclidean space $\mathbb{R}^3_{x_1,x_2,x_3}$ and represent a point $x = (x_1, x_2) \in \mathbb{R}^2$ and a vector $v = (v_1, v_2) \in T_x\mathbb{R}^2$ by the vectors

$$r = (x_1, x_2, 1), \quad v = (v_1, v_2, 0) \in \mathbb{R}^3.$$

Proposition 1.23 *1) Let a germ of projective billiard in $\mathbb{R}^2_{x_1,x_2}$ with reflection from a C^2-smooth germ of curve C (or a global planar projective billiard in a connected domain with C^2-smooth boundary) be rationally* 0*-homogeneously integrable. Then the rational* 0*-homogeneous integral can be chosen as a rational* 0*-homogeneous function of the moment vector M:*

$$M = M(r, v) := [r, v] = (-v_2, v_1, \Delta), \;\; \Delta = \Delta(x, v) := x_1v_2 - x_2v_1. \quad (1.13)$$

2) The property of a projective billiard germ to be rationally 0*-homogeneously integrable depends only on the germ of curve with transversal line field and does not depend on the choice of side.*

Proposition 1.24 *A planar projective billiard with C^2-smooth boundary is rationally 0-homogeneously integrable, if and only if its dual billiard is rationally integrable. If R is a rational integral of the dual billiard, written as a* 0*-homogeneous rational function in homogeneous coordinates on the ambient projective plane, then $R([r,v])$ is a* 0*-homogeneous rational integral of the projective billiard.*

Remark 1.25 Versions of Propositions 1.23, 1.24 for polynomially integrable billiards on surfaces of constant curvature were earlier proved respectively in the paper [15] by S.V.Bolotin (Proposition 1.23) and in two joint papers by M.Bialy and A.E.Mironov [10, 11] (Proposition 1.24, see also [31, theorem 2.8]).

Theorem 1.26 *Let $C \subset \mathbb{R}^2_{x_1,x_2}$ be a non-linear C^4-smooth germ of curve equipped with a transversal line field $\mathcal{N}$. Let the corresponding germ of projective billiard be* 0*-homogeneously rationally integrable. Then C is a conic; the line field $\mathcal{N}$ extends to a global analytic transversal line field on the whole conic C punctured in at most four points; the corresponding projective billiard has one of the following types up to projective equivalence.*

1) A space form billiard whose matrix can be chosen $A = \operatorname{diag}(1,1,-1)$.

2) $C = \{x_2 = x_1^2\} \subset \mathbb{R}^2_{x_1,x_2} \subset \mathbb{RP}^2$, and the line field $\mathcal{N}$ is directed by one of the following vector fields at points of the conic C:

2a) $\quad (\dot x_1, \dot x_2) = (\rho, 2(\rho-2)x_1),$

$$\rho = 2 - \frac{2}{2N+1} \text{ (case 2a1), or } \quad \rho = 2 - \frac{1}{N+1} \text{ (case 2a2)}, \quad N \in \mathbb{N},$$

the vector field 2a) has quadratic first integral $\mathcal{Q}_\rho(x_1,x_2) := \rho x_2 - (\rho-2)x_1^2$.

2b1) $(\dot x_1, \dot x_2) = (5x_1+3, 2(x_2-x_1)),$ *2b2)* $(\dot x_1, \dot x_2) = (3x_1, 2x_2-4),$

2c1) $(\dot x_1, \dot x_2) = (x_2, x_1x_2-1),$ *2c2)* $(\dot x_1, \dot x_2) = (2x_1+1, x_2-x_1).$

2d) $(\dot x_1, \dot x_2) = (7x_1+4, 2x_2-4x_1).$

Addendum to Theorem 1.26. *The projective billiards from Theorem 1.26 have the following* 0*-homogeneous rational integrals:*

Case 1): A ratio of two homogeneous quadratic polynomials in (v_1, v_2, Δ),

$$\Delta := x_1 v_2 - x_2 v_1.$$

Case 2a1), $\rho = 2 - \frac{2}{2N+1}$*:*

$$\Psi_{2a1}(x_1, x_2, v_1, v_2) := \frac{(4v_1\Delta - v_2^2)^{2N+1}}{v_1^2 \prod_{j=1}^{N}(4v_1\Delta - c_j v_2^2)^2}. \tag{1.14}$$

Case 2a2), $\rho = 2 - \frac{1}{N+1}$*:*

$$\Psi_{2a2}(x_1, x_2, v_1, v_2) = \frac{(4v_1\Delta - v_2^2)^{N+1}}{v_1 v_2 \prod_{j=1}^{N}(4v_1\Delta - c_j v_2^2)}. \tag{1.15}$$

The c_j *in (1.14), (1.15) are the same, as in (1.6) and (1.7) respectively.*

Case 2b1):

$$\Psi_{2b1}(x_1, x_2, v_1, v_2) = \frac{(4v_1\Delta - v_2^2)^2}{(4v_1\Delta + 3v_2^2)(2v_1 + v_2)(2\Delta + v_2)}. \tag{1.16}$$

Case 2b2):

$$\Psi_{2b2}(x_1, x_2, v_1, v_2) = \frac{(4v_1\Delta - v_2^2)^2}{(v_2^2 + 4\Delta^2 + 4v_1\Delta + 4v_1^2)(v_2^2 + 4v_1^2)}. \tag{1.17}$$

Case 2c1): $$\Psi_{2c1}(x_1, x_2, v_1, v_2) = \frac{(4v_1\Delta - v_2^2)^3}{(v_1^3 + \Delta^3 + v_1 v_2 \Delta)^2}. \tag{1.18}$$

Case 2c2):

$$\Psi_{2c2}(x_1, x_2, v_1, v_2) = \frac{(4v_1\Delta - v_2^2)^3}{(v_2^3 + 2v_2^2 v_1 + (v_1^2 + 2v_2^2 + 5v_1 v_2)\Delta + v_1\Delta^2)^2}. \tag{1.19}$$

Case 2d): $\Psi_{2d}(x_1, x_2, v_1, v_2)$

$$= \frac{(4v_1\Delta - v_2^2)^3}{(v_1\Delta + 2v_2^2)(2v_1 + v_2)(8v_1 v_2^2 + 2v_2^3 + (4v_1^2 + 5v_2^2 + 28v_1 v_2)\Delta + 16v_1\Delta^2)}. \tag{1.20}$$

In Subsection 9.3 we prove the following characterization of space form billiards on conics as projective billiards on conics with conical caustics.

Proposition 1.27 *A transversal line field* $\mathcal{N}$ *on a punctured planar regular conic* C *defines a projective billiard that is projectively equivalent to a space form billiard, if and only if there exists a regular conic* $S \neq C$ *such that for every* $Q \in C$ *the complexified projective billiard reflection at* Q *permutes the complex lines through* Q *tangent to the complexified conic* S*.*

Remark 1.28 The latter permutation condition determines $\mathcal{N}$ by S in a unique way. The corresponding projective billiard has a family of conical caustics whose dual conics form a pencil, see [17] and [24, subsection 2.3].

1.3 Applications to billiards with complex algebraic caustics

Definition 1.29 Let $C \subset \mathbb{R}^2 \subset \mathbb{RP}^2$ be a planar curve equipped with a projective billiard structure. For every $Q \in C$ consider the complexification of the billiard reflection involution acting on the space of complex lines through Q. Let $S \subset \mathbb{CP}^2$ be an algebraic curve in the complexified ambient projective plane that contains no straight line. We say that S is a *complex caustic* of the *real billiard* on C, if for every $Q \in C$ each *complex* projective line tangent to S and passing through Q is reflected by the complexified reflection at Q to a line tangent to S.

Remark 1.30 The usual Euclidean billiard on a strictly convex planar curve C has C as a real caustic: through each its point Q passes the unique tangent line to C, and it is fixed by the reflection. If C is a conic, then its complexification $C_{\mathbb{C}}$ is a complex caustic for C for the same reason. But if C is a higher degree algebraic curve, then a priori its complexification $C_{\mathbb{C}}$ is not necessarily a complex caustic. In this case through a generic point $Q \in C$ passes at least one complex line tangent to $C_{\mathbb{C}}$ that does not coincide with its tangent line at Q. To check, whether $C_{\mathbb{C}}$ is a complex caustic, one has to check whether the collection of all the complex lines through Q tangent to $C_{\mathbb{C}}$ is invariant under the reflection at Q. This is a non-trivial condition on the algebraic curve C.

Open problem. Consider Euclidean billiard on a strictly convex closed planar curve C. Let C be *contained in an algebraic curve* and have a *real caustic contained in an algebraic curve.* Is it true that C is a conic?

A positive answer would imply the particular case of the Birkhoff Conjecture, when the billiard boundary is contained in an algebraic curve.

We prove the following theorem as an application of results of [31].

Theorem 1.31 *Let $C \subset \mathbb{R}^2$ be a non-linear C^2-smooth connected embedded (not necessarily closed, convex or algebraic) curve equipped with the structure of standard Euclidean billiard (with the usual reflection law). Let the latter billiard have a* **complex** *algebraic caustic. Then C is a conic.*

The next theorem is its analogue for projective billiards. It will be deduced from main results of the present paper.

Theorem 1.32 *Let C be a non-linear C^4-smooth connected embedded planar curve. Let C be equipped with a projective billiard structure having at least two different complex algebraic caustics. Then C is a conic.*

Theorems 1.31 and 1.32 will be proved in Section 10.

1.4 Sketch of proof of Theorem 1.18 and plan of the paper

First we prove algebraicity of a rationally integrable dual billiard.

Definition 1.33 A *singular* holomorphic dual billiard on a holomorphic curve $\gamma \subset \mathbb{CP}^2$ is a holomorphic dual billiard structure on the complement of the curve γ to a discrete subset of points where the corresponding family of involutions $\sigma_P : L_P \to L_P$ does not extend holomorphically.

Proposition 1.34 *Let a regular non-linear germ of holomorphic curve $\gamma \subset \mathbb{CP}^2$ carry a complex (not necessarily holomorphic) rationally integrable dual billiard structure with a rational integral R. Then 1) $R|_\gamma \equiv const$, and thus, γ is contained in an irreducible algebraic curve, which will be also denoted by γ; 2) the involution family σ_P extends to a singular holomorphic dual billiard structure on the algebraic curve γ with the same rational integral R.*

Proposition 1.35 *Let $\gamma \subset \mathbb{R}^2$ be a regular non-linear C^2-smooth germ of curve equipped with a dual billiard structure having a rational integral R. Then 1) $R|_\gamma \equiv const$, and thus, the complex Zariski closure of the curve γ is an algebraic curve in $\mathbb{CP}^2$; 2) the dual billiard structure extends to a singular holomorphic dual billiard structure on each its non-linear irreducible component, with the same integral R.*

Parts 1), 2) of these propositions will be proved in Subsections 2.1, 2.2.

Recall that each (may be singular) germ of analytic curve in $\mathbb{CP}^2$ is a finite union of its irreducible components, which are locally bijectively holomorphically parametrized germs called *local branches.*

Definition 1.36 [31, definition 3.3] Let b be an irreducible (i.e., parametrized) non-linear germ of analytic curve at a point $O \in \mathbb{CP}^2$. An affine chart (z, w) centered at O such that the z-axis is tangent to b at O is called *adapted* to b. In an adapted chart the germ b can be holomorphically bijectively parametrized by a complex parameter t from a disk centered at 0 as follows:

$$t \mapsto (t^{q_b}, c_b t^{p_b}(1 + o(1))), \text{ as } t \to 0; \quad q_b, p_b \in \mathbb{N}, \ 1 \le q_b < p_b, \ c_b \neq 0,$$

$$q_b = 1, \text{ if and only if } b \text{ is a regular germ.}$$

The *projective Puiseux exponent* [27, p. 250, definition 2.9] of the germ b is the number

$$r = r_b := \frac{p_b}{q_b}.$$

The germ b is called *quadratic,* if $r = 2$ [28, definition 3.5]. When b is a germ of line, it is parametrized by $t \mapsto (t, 0)$: then we set $q_b = 1$, $p_b = r_b = \infty$.

The main part of the proof of Theorem 1.18 is the proof of the following theorem on possible types of singularities and local branches of the curve γ.

Theorem 1.37 *Let an irreducible complex algebraic curve $\gamma \subset \mathbb{CP}^2$ carry a structure of rationally integrable singular holomorphic dual billiard. Then the following statements hold:*

(i) the curve γ has no inflection points, and at each its singular point (if any) all its local branches are quadratic;

(ii) there exists at most unique singular point of the curve γ where there exists at least one singular local branch.

Theorem 1.38 *Every complex irreducible projective planar algebraic curve satisfying the above statements (i) and (ii) is a conic.*

The proof of Theorem 1.38 will be given in Section 6. It is based on E.Shustin's generalized Plucker formula [42], dealing with intersection of an irreducible algebraic curve with its Hessian curve. It gives formula for the contributions of singular and inflection points to their intersection index.

Theorems 1.37, 1.38 together with Proposition 1.34 immediately imply that every germ of holomorphic curve γ carrying a rationally integrable complex dual billiard structure is a germ of a conic. Afterwards in Section 7 we classify the rationally integrable dual billiard structures on a conic. This will finish the proof of Theorem 1.18. Then in Section 8 we classify the real forms of the complex dual billiards from Theorem 1.18 and finish the proof of Theorems 1.16, 1.11.

The proof of Theorem 1.37 is based on studying the Hessian of appropriately normalized rational integral: the Hessian introduced by S.Tabachnikov, who used it to study polynomially integrable outer billiards [45]. This idea was further elaborated and used by M.Bialy and A.Mironov in a series of papers on Bolotin's Polynomial Birkhoff Conjecture and its analogues for magnetic billiards [10, 11, 9]. It was also used in the previous paper by the author and E.Shustin on polynomially integrable outer billiards [28] and in the author's recent paper on S.V.Bolotin's Polynomial Birkhoff Conjecture [31]. The rational integral R being constant along the curve γ (Proposition 1.34), we normalize it to vanish identically on γ. Let f be the irreducible polynomial vanishing on γ in an affine chart $\mathbb{C}^2_{x_1,x_2} \subset \mathbb{CP}^2$. Then

$$R = f^k g_1, \quad g_1|_\gamma \not\equiv 0.$$

Replacing R by its k-th root $G := fg$, $g := g_1^{\frac{1}{k}}$, we consider its Hessian

$$H(G) := \frac{\partial^2 G}{\partial x_1^2}\left(\frac{\partial G}{\partial x_2}\right)^2 - 2\frac{\partial^2 G}{\partial x_1 \partial x_2}\frac{\partial G}{\partial x_2}\frac{\partial G}{\partial x_1} + \frac{\partial^2 G}{\partial x_2^2}\left(\frac{\partial G}{\partial x_1}\right)^2.$$

Its key property is that $H(G)|_\gamma \neq 0$ *outside singular and inflection points of the curve* γ *and zeros (poles) of the function* $g_1|_\gamma$.

Plan of proof of Theorem 1.37.

Step 1 (Subsection 2.2): differential equation on $H(G)$. Given a point $P_0 \in \gamma$, consider an affine chart (z, w) in which the tangent line L_{P_0} is not parallel to the w-axis. Then for every $P \in \gamma$ close enough to P_0 the line L_P is parametrized by affine coordinate z. The involution $\sigma_P : L_P \to L_P$ is conjugated to the standard involution $\overline{\mathbb{C}}_\theta \to \overline{\mathbb{C}}_\theta$, $\theta \mapsto -\theta$, via a mapping $\mathcal{F}_P : \theta \mapsto \mathcal{F}_P(\theta)$ that sends θ to the point in the tangent line L_P with z-coordinate $z(P) + \frac{\theta}{1+\psi(P)\theta}$; $\psi(P) \in \mathbb{C}$ is uniquely determined by σ_P. Invariance of the function $R|_{L_P}$ under the involution σ_P is equivalent to statement that the function $R \circ \mathcal{F}_P(\theta)$ is even. Writing the condition that its cubic Taylor coefficient vanishes (analogously to [45, 10, 11]), we get the differential equation

$$\frac{dH(G)|_\gamma}{dz}(P) = 6\psi(P)H(G). \tag{1.21}$$

We prove equation (1.21) in a more general situation, for an irreducible germ of analytic curve b at a point B equipped with a family of projective involutions $\sigma_P : L_P \to L_P$, $P \in b\setminus\{B\}$, admitting a germ R of meromorphic (not necessarily rational) integral. Here f is a local defining function of the germ b, and G, $H(G)$ are defined as above. Relation between meromorphic and rational integrability will be explained in Subsection 2.4.

Step 2 (Subsection 2.3): formula relating asymptotics of $H(G)$ and σ_P. We fix a potential singular (inflection) point $B \in \gamma$, a local branch b of the curve γ at B (whose quadraticity we have to prove) and affine coordinates (z, w) centered at B and adapted to b. The function $H(G)$ being a linear combination of products of rational powers of holomorphic functions at B, we get that $H(G)|_b = \alpha z^d(1 + o(1))$, as $z \to 0$, for some $d \in \mathbb{Q}$ and $\alpha \neq 0$. This together with equation (1.21) implies that

$$\psi|_{b\setminus\{B\}} = \frac{1}{z}\left(\frac{d}{6} + o(1)\right) = -\frac{1}{z}\left(\frac{\rho}{2} + o(1)\right), \text{ as } z \to 0; \ \rho = -\frac{d}{3} \in \mathbb{Q}.$$

We then say that the involution family σ_P is *meromorphic* with pole at B of order at most one with *residue* ρ. This means exactly that

$$\text{in the coordinate} \quad \zeta := \frac{z}{z(P)} \quad \text{on} \ \ L_P$$

the involution σ_P converges to $\eta_\rho(\zeta) := \dfrac{(\rho-1)\zeta-(\rho-2)}{\rho z-(\rho-1)}$, as $P\to B$.

Thus, the above number ρ in the limit and $H(G)$ are related by the formulas

$$H(G)|_b = \alpha z^d(1+o(1)), \quad \alpha\neq 0, \quad \rho = -\frac{d}{3}. \tag{1.22}$$

Step 3 (Section 3) Consider the projective Puiseux exponent r of the local branch b, and let us represent it as an irreducible fraction:

$$r = \frac{p}{q},\ p,q\in\mathbb{N},\ (p,q)=1:\ \ p_b = ps_b,\ q_b = qs_b,\ s_b = G.C.D(p_b,q_b). \tag{1.23}$$

For given p, q as above and $\rho\in\mathbb{C}$ we introduce the $(p,q;\rho)$*-billiard:* the curve

$$\gamma_{p,q} := \{w^q = z^p\}\subset\mathbb{C}^2$$

equipped with the dual billiard structure given by the family of involutions

$$\sigma_P: L_P\to L_P, \quad P\in\gamma_{p,q}; \quad \sigma_P:\zeta\to\eta_\rho(\zeta) \quad \text{in the coordinate } \zeta \text{ for all } P.$$

We show (Theorem 3.3) that *if a germ of family of involutions $\sigma_P: L_P\to L_P$ defined on a punctured irreducible germ of holomorphic curve b with Puiseux exponent $r=\frac{p}{q}$ admits a meromorphic first integral, then the corresponding $(p,q;\rho)$-billiard (with ρ given by (1.22)) admits a (p,q)-quasihomogeneous rational first integral.* To do this, we consider the lower (p,q)-quasihomogeneous parts in the numerator and the denominator of the meromorphic integral. We show that their ratio is an integral of the $(p,q;\rho)$-billiard. If it is non-constant, then we get a non-trivial integral. The opposite case, when the latter lower (p,q)-quasihomogeneous parts are the same up to constant factor, will be reduced to the previous one by replacing the denominator by appropriate linear combination of the numerator and denominator.

Step 4 (Section 4). Classification of quasihomogeneously rationally integrable $(p,q;\rho)$-billiards (Theorem 4.1). Our first goal is to show that the underlying curve $\gamma_{p,q}$ is a conic: $p=2$, $q=1$. In Subsection 4.1 we treat the case of polynomial integral. To treat the case of non-polynomial rational integral, we first show (in Subsection 4.2) that one can normalize it to be a so-called η_ρ-primitive quasihomogeneous rational function $R=\frac{\mathcal{P}_1^{m_1}}{\mathcal{P}_2^{m_2}}$ vanishing on $\gamma_{p,q}$. In particular, this means that each $\mathcal{P}_j$ is a product of prime factors $w^q-c_jz^p$, $c_j\neq 0$ (and may be z, w) in power 1, including the factor w^q-z^p. Then in Subsection 4.3 we prove two formulas (4.12), (4.14) expressing ρ via the powers m_1, m_2, the number of factors $w^q-c_jz^p$

and the powers of z, w in $\mathcal{P}_j$. The first formula (4.12) will be deduced from (1.22). The second formula (4.14) is obtained as follows. Restricting the polynomial $\mathcal{P}_1$ from the numerator to the line L_P, $P = (1,1)$, and dividing it by appropriate power $(z - \frac{\rho-2}{\rho})^d$ yields a η_ρ-invariant rational function in the coordinate z with numerator divisible by $(z-1)^2$. Existence of such a power d will follow from the fact that $\frac{\rho-2}{\rho}$ is a fixed point of the involution η_ρ. Afterwards we replace the numerator $\mathcal{P}_1$ by the difference $\mathcal{P}_1 - \lambda(z - \frac{\rho-2}{\rho})^d$ with small λ; we get a family of η_ρ-invariant rational functions depending on the parameter λ, which has a pair of roots $\zeta_\pm(\lambda)$ converging to 1, as $\lambda \to 0$. Comparing the asymptotics of the roots $\zeta_\pm(\lambda)$ and taking into account that they should be permuted by the involution η_ρ, we get formula (4.14). The *main miracle* in the proof of Theorem 4.1 (Subsection 4.4) is that *combining First and Second Formulas (4.12), (4.14) yields that $p = 2$, $q = 1$ and the curve $\gamma_{p,q}$ is the conic $\{w = z^2\}$*, and it also yields the constraints on ρ given by Theorem 4.1: the necessary condition for quasihomogeneous integrability. Then we prove its sufficience by constructing integrals (Subsection 4.5).

Steps 3 and 4 together imply Statement (i) of Theorem 1.37: each local branch of the curve γ is quadratic. They also yield a list of a priori possible values of the residue ρ.

Step 5. Proof of statement (ii) of Theorem 1.37: uniqueness of singular point of the curve γ with a singular local branch. To do this, we prove Theorem 5.1 stating that if a quadratic local branch b at a point O is singular, then the integral R is constant along its projective tangent line L_O, and the punctured line $L_O \setminus \{O\}$ is a regular leaf of the foliation $R = const$. This implies that if γ had two distinct points with singular local branches, then the corresponding tangent lines would intersect, and we get a contradiction with regularity of foliation at the intersection point. The proof of Theorem 5.1 given in Subsection 5.2 is partly based on Theorem 5.6 (Subsection 5.1), which implies that if there exists a singular quadratic local branch, then its self-contact order is expressed via the corresponding residue ρ by an explicit formula (5.3) implying that $\rho > r = 2$. Once having inequality $\rho > r$, we deduce the statements of Theorem 5.1 analogously to [31, subsection 4.6, proof of theorem 4.24]. Step 5 finishes the proof of Theorem 1.37.

In Section 6 we prove Theorem 1.38. Theorems 1.37 and 1.38 together imply that γ is a conic. Afterwards in Section 7 we classify singular holomorphic rationally integrable dual billiards on a complex conic. The list of a priori possible residues ρ at singularities is given by Theorem 4.1, Step 4. In Subsection 7.1 we show that the sum of residues should be equal to four (a version of residue formula for singular holomorphic dual billiard structures).

Afterwards we consider all the a priori possible residue configurations given by these constraints and show that all of them are indeed realized by rationally integrable dual billiards. This will finish the proof of Theorem 1.18. Then Theorems 1.16, 1.11 are proved in Section 8 by describing different real forms of thus classified complex integrable dual billiards. Theorems 1.22 and 1.26 classifying integrable projective billiards (which are dual to the latter real forms) will be proved in Section 9. Theorems 1.31 and 1.32 on billiards with complex caustics will be proved in Section 10.

1.5 Historical remarks

In 1973 V.Lazutkin [38] proved that every strictly convex bounded planar billiard with sufficiently smooth boundary has an infinite number (continuum) of closed caustics. The Birkhoff Conjecture was studied by many mathematicians. In 1950 H.Poritsky [41] (and later E.Amiran [3] in 1988) proved it under the additional assumption that the billiard in each closed caustic near the boundary has the same closed caustics, as the initial billiard. In 1993 M.Bialy [5] proved that if the phase cylinder of the billiard in a domain Ω is foliated by non-contractible continuous closed curves which are invariant under the billiard map, then the boundary $\partial\Omega$ is a circle. (Another proof of the same result was later obtained in [51].) In 2012 Bialy proved a similar result for billiards on the constant curvature surfaces [7] and also for magnetic billiards [6]. In 1995 A.Delshams and R.Ramirez-Ros suggested an approach to prove splitting of separatrices for generic perturbation of ellipse [18]. D.V.Treschev [46] made a numerical experience indicating that there should exist analytic *locally integrable* billiards, with the billiard reflection map having a two-periodic point where the germ of its second iterate is analytically conjugated to a disk rotation. See also [47] for more detail and [48] for a multi-dimensional version. A similar effect for a ball rolling on a vertical cylinder under the gravitation force was discovered in [1]. Recently V.Kaloshin and A.Sorrentino have proved a *local version* of the Birkhoff Conjecture [35]: *an integrable deformation of an ellipse is an ellipse.* Very recently M.Bialy and A.E.Mironov [13] proved the Birkhoff Conjecture for centrally-symmetric billiards having a family of closed caustics that extends up to a caustic tangent to four-periodic orbits. For a dynamical entropic version of the Birkhoff Conjecture and related results see [39]. For a survey on the Birkhoff Conjecture and results see [35, 36, 13] and references therein.

Recently it was shown by the author [32] that every strictly convex C^∞-smooth non-closed planar curve has an adjacent domain from the convex side that admits an infinite number (continuum) of distinct C^∞-smooth foliations by non-closed caustics (with the boundary being a leaf).

A.P.Veselov proved a series of complete integrability results for billiards bounded by confocal quadrics in space forms of any dimension and described billiard orbits there in terms of a shift of the Jacobi variety corresponding to an appropriate hyperelliptic curve [49, 50]. Dynamics in (not necessarily convex) billiards of this type was also studied in [19, 20, 21, 22, 23].

The Polynomial Birkhoff Conjecture together with its generalization to surfaces of constant curvature was stated by S.V.Bolotin and partially studied by himself, see [14], [15, section 4], and by M.Bialy and A.E.Mironov [8]. Its complete solution is a joint result of M.Bialy, A.E.Mironov and the author given in the series of papers [10, 11, 29, 31].

For a survey on the Polynomial Birkhoff Conjecture, its version for magnetic billiards and related results see the above-mentioned papers [10, 11] by M.Bialy and A.E.Mironov, [9, 12] and references therein.

The analogues of the Birkhoff Conjecture for outer and dual billiards was stated by S.Tabachnikov [45] in 2008. Its polynomial version for outer billiards was stated by Tabachnikov and proved by himself under genericity assumptions in the same paper [45], and solved completely in the joint work of the author of the present paper with E.I.Shustin [28].

In 1995 M.Berger have shown that in Euclidean space $\mathbb{R}^n$ with $n \geq 3$ the only hypersurfaces admitting caustics are quadrics [4]. In 2020 this result was extended to space forms of constant curvature of dimension greater than two by the author of the present paper [30].

In 1997 S.Tabachnikov [43] introduced projective billiards and proved a criterium and a necessary condition for a planar projective billiard to preserve an area form. He had shown that if a projective billiard on circle preserves an area form that is smooth up to the boundary of the phase cylinder, then the billiard is integrable.

A series of results on projective billiards with open sets of n-periodic orbits (classification for $n = 3$ and new examples for higher n) were obtained by C.Fierobe [25, 26, 24].

2 Preliminaries

2.1 Algebraicity of underlying curve. Proof of Propositions 1.34 and 1.35, parts 1)

Let $\gamma \subset \mathbb{CP}^2$ be a regular germ of holomorphic curve. For every $P \in \gamma$ the restriction $R|_{L_P}$ is invariant under an involution σ_P fixing P. In appropriate affine coordinate θ on L_P centered at P the latter involution takes the form

$\theta \mapsto -\theta$. Therefore, the restriction $R|_{L_P}$ has zero derivative at P, since an even function has zero derivative at the origin. Finally, the rational function R has zero derivative along any vector tangent to γ. Hence, it is constant on γ, and the germ of curve γ is algebraic. This proof remains valid in the case, when γ is a real germ. Parts 1) of Propositions 1.34 and 1.35 are proved.

For completeness of presentation (to state some results in full generality), we will deal with the following notion of meromorphically integrable dual billiard structure and meromorphic version of Proposition 1.34.

Definition 2.1 Let b be a non-linear (may be singular) irreducible germ of analytic curve in $\mathbb{C}^2$ at a point B, and let $\sigma_P : L_P \to L_P$ be a family of projective involutions parametrized by $P \in b \setminus \{B\}$. The family σ_P is called a *meromorphically integrable (singular) dual billiard structure,* if there exists a germ of meromorphic function R at B (defined on a neighborhood of the point B in $\mathbb{C}^2$), $R \not\equiv const$, such that the restrictions $R|_{L_P}$ are σ_P-invariant: there exists a neighborhood $U = U(B) \subset \mathbb{C}^2$ such that for every $P \in b \cap U$, $P \neq B$, and every $x, y \in L_P \cap U$ such that $\sigma_P(x) = y$ one has $R(x) = R(y)$.

Proposition 2.2 *In the conditions of the above definition 1) $R|_b \equiv const$; 2) the family σ_P is holomorphic in $P \in b \setminus \{B\}$ close enough to B.*

The proof of the first part of Proposition 2.2 repeats that of Proposition 1.34, part 1). Its second part will be proved in the next subsection.

Later on, in Subsection 2.4 we will show that in many cases meromorphic integrability implies rational integrability.

2.2 The Hessian of integral and its differential equation. Singular holomorphic extension of dual billiard structure

Let b be an irreducible germ of holomorphic curve in $\mathbb{C}^2_{x_1,x_2}$ at a point B. Let $b \setminus \{B\}$ be equipped with a germ of dual billiard structure having a non-constant meromorphic integral R, see the above definition. Recall that $R|_b \equiv const$, by Proposition 2.2. Without loss of generality we consider that

$$R|_b \equiv 0,$$

adding a constant to R (if $R|_b \not\equiv \infty$), or replacing R by R^{-1} (if $R|_b \equiv \infty$).

Let f be an irreducible germ of holomorphic function defining b:

$$b = \{f = 0\}.$$

One has

$$R = g_1 f^k, \quad g_1 \quad \text{is meromorphic}, \quad g_1|_b \not\equiv 0, \ k \in \mathbb{N}.$$

From now on we will work with the k-th root

$$G = R^{\frac{1}{k}} = gf, \ g = g_1^{\frac{1}{k}}. \tag{2.1}$$

For every $P \in b \setminus \{B\}$ close enough to B each holomorphic branch of the function G on L_P is σ_P-invariant, since any two its holomorphic branches are obtained one from the other by multiplication by a root of unity.

Recall that the *skew gradient* of the function G is the vector field

$$\nabla_{skew} G := (\frac{\partial G}{\partial x_2}, -\frac{\partial G}{\partial x_1}),$$

which is tangent to its level curves.

The involution σ_P is conjugated to the standard involution $\mathbb{C}_\tau \to \mathbb{C}_\tau$, $\tau \mapsto -\tau$, via a transformation

$$\Phi_P : \tau \mapsto P + \frac{\tau}{1 + \phi(P)\tau} \nabla_{skew} G(P), \ \Phi_P(0) = P. \tag{2.2}$$

The conjugacy is unique up to its pre-composition with a multiplication by constant $\tau \to \lambda\tau$; we can normalize it to be of type (2.2) in unique way. The σ_P-invariance of the function G is equivalent to the statement that

$$\text{the function } \xi(\tau) := G(P + \frac{\tau}{1 + \phi(P)\tau} \nabla_{skew} G(P)) \text{ is even}, \tag{2.3}$$

which holds if and only if the function $\xi(\tau)$ has zero Taylor coefficients at odd powers. The first coefficient vanishes for trivial reason, being derivative of a function G along a vector tangent to its zero level curve.

Recall that the *Hessian* of the function G is the function

$$H(G) := \frac{\partial^2 G}{\partial x_1^2} \left(\frac{\partial G}{\partial x_2} \right)^2 - 2 \frac{\partial^2 G}{\partial x_1 \partial x_2} \frac{\partial G}{\partial x_2} \frac{\partial G}{\partial x_1} + \frac{\partial^2 G}{\partial x_2^2} \left(\frac{\partial G}{\partial x_1} \right)^2. \tag{2.4}$$

It coincides with the value of its Hessian quadratic form on its skew gradient and also with the second derivative $\xi''(0)$, see [45, 10, 11].

Remark 2.3 The Hessian function $H(G)$ was introduced by S.Tabachnikov [45] and used in [45, 28, 10, 11, 31] to classify polynomially integrable Birkhoff and outer planar billiards; see results mentioned in Subsection 1.5.

Theorem 2.4 *For a given $P \in b \setminus \{B\}$ the cubic Taylor coefficient of the function ξ from (2.3) at 0 vanishes, if and only if*

$$\frac{dH(G)}{d\nabla_{skew}G}(P) = 6\phi(P)H(G)(P). \tag{2.5}$$

Remark 2.5 Theorems analogous to Theorem 2.4 were stated and proved in [45, 10, 11], and the proofs from loc. cit. remain valid in our case. The proof of Theorem 2.4 given below follows similar arguments.

Proof of Theorem 2.4. The third derivative $\xi'''(\tau)$ is equal to the third derivative in τ of the function

$$G(P+\omega(\tau)W),\ W := \nabla_{skew}G(P),\ \omega(\tau) := \frac{\tau}{1+\phi(P)\tau} = \tau - \phi(P)\tau^2 + O(\tau^3),$$

as $\tau \to 0$. The first derivative equals

$$\xi'(\tau) = \frac{dG}{dW}(P+\omega(\tau)W)(1-2\phi(P)\tau + O(\tau^2)).$$

For simplicity, in what follows we omit the argument $P + \omega(\tau)W$ at the derivatives. One has

$$\xi''(\tau) = (-2\phi(P) + O(\tau))\frac{dG}{dW} + (1-2\phi(P)\tau + O(\tau^2))^2\frac{d^2G}{dW^2},$$

$$\begin{aligned}\xi'''(\tau) = {} & O(1)\frac{dG}{dW} - (2\phi(P)+O(\tau))(1-2\phi(P)\tau+O(\tau^2))\frac{d^2G}{dW^2} \\ & -(4\phi(P)+O(\tau))\frac{d^2G}{dW^2} + (1-2\phi(P)\tau+O(\tau^2))^3\frac{d^3G}{dW^3}.\end{aligned}$$

The value of the third derivative at zero is thus equal to

$$\xi'''(0) = \frac{d^3G}{dW^3}(P) - 6\phi(P)\frac{d^2G}{dW^2}(P), \tag{2.6}$$

since $\frac{dG}{dW}(P) = 0$. One has

$$\frac{d^3G}{dW^3} = \frac{d}{dW}\left(\frac{d^2G}{dW^2}\right), \quad \frac{d^2G}{dW^2}(P) = H(G)(P).$$

This together with (2.6) implies the statement of the theorem. □

Let us consider affine coordinates (z, w) such that the tangent line T_Bb is not parallel to the w-axis. For every $P \in b$ close to B the restriction to L_P of the coordinate z is an affine coordinate on the projective line L_P.

We will deal with the following normalizations of mapping (2.2) and equation (2.5) with respect to the coordinate z. Set

$$V(P) = (1, \beta(P)) := \text{ the vector in } T_P b \text{ with unit } z-\text{component}. \quad (2.7)$$

The vectors $V(P)$ form a holomorphic vector field V on $b \setminus \{B\}$. One has

$$\nabla_{skew} G = hV, \ h : b \setminus \{B\} \to \mathbb{C} \text{ is a non-zero function}; \quad (2.8)$$

a priori the function h is multivalued holomorphic on $b \setminus \{B\}$ (with a priori possible branching at B). Set

$$\theta := h(P)\tau, \ \psi(P) := \phi(P)h^{-1}(P).$$

Let $\Phi_P(\tau)$ be the mapping from (2.2). Set

$$\mathcal{F}_P(\theta) := \Phi_P(h^{-1}(P)\theta) = P + \frac{h^{-1}(P)\theta}{1 + \phi(P)h^{-1}(P)\theta} h(P)V = P + \frac{\theta}{1 + \psi(P)\theta} V. \quad (2.9)$$

Proposition 2.6 *The map $\mathcal{F}_P(\theta)$ conjugates the involution σ_P with the standard symmetry $\theta \mapsto -\theta$, and its differential at 0 sends the unit vector $\frac{\partial}{\partial \theta}$ to $V(P)$. One has*

$$\frac{dH(G)}{dV}(P) = 6\psi(P)H(G)(P). \quad (2.10)$$

Proof The statements on conjugacy and differential follow by construction. Equation (2.10) is obtained from (2.5) by multiplication by $h^{-1}(P)$. □

We use the following formula for Hessian of product, see [10, theorem 6.1], [11, formulas (16) and (32)]:

$$H(fg) = g^3 H(f) \text{ on the set } \{f = 0\}. \quad (2.11)$$

Proof of parts 2) of Propositions 1.34, 1.35, 2.2. Let us prove part 2) of Proposition 1.34: for the other propositions the proof is analogous. Let γ denote the irreducible algebraic curve containing the initial germ γ. Let $\phi(P)$ denote the function (2.2) defined by the involutions σ_P. Equation (2.5) extends the function $\phi(P)$ holomorphically along paths in γ avoiding a finite collection of points where some branch of the multivalued function $H(G)$ either vanishes, or is not holomorphic, or its derivative in the left-hand side in (2.5) is not holomorphic. It defines a holomorphic extension of the involution family σ_P. The relation $R \circ \sigma_P|_{L_P} = R$ remains valid for the extended dual billiard structure, by uniqueness of analytic extension. Let us show that this yields a well-defined singular holomorphic dual billiard structure on γ. Suppose the contrary: thus extended family σ_P is multivalued, i.e., its extensions along two different paths arriving to one and the same point A are two different involutions σ_A and $\widetilde{\sigma}_A$. Then their composition $\sigma_A \circ \widetilde{\sigma}_A : L_A \to L_A$ is a parabolic transformation with unique fixed point A, leaving invariant the restriction $R|_{L_A}$. Its orbits (except for the fixed point A) being infinite and accumulating to A, one has $R|_{L_A} \equiv const$. The involutions σ_A, $\widetilde{\sigma}_A$ are well-defined and satisfy the above statements on an open subset of points A in γ, by local analyticity. Therefore, $R|_{L_A} \equiv const$ for an open subset of points $A \in \gamma$, which is impossible. The contradiction thus obtained implies that the extended dual billiard structure is singular holomorphic. $\square$

2.3 Asymptotics of degenerating involutions

Here we deal with an irreducible germ b at a point B of analytic curve in $\mathbb{C}^2$ equipped with a meromorphically integrable singular holomorphic dual billiard structure. We study asymptotics of involutions σ_P, as $P \to B$.

For every $\rho \in \mathbb{C}$ we denote by $\eta_\rho \in \mathrm{PSL}_2(\mathbb{C})$ the projective involution

$$\eta_\rho : \overline{\mathbb{C}}_\zeta \to \overline{\mathbb{C}}_\zeta, \quad \eta_\rho(\zeta) := \frac{(\rho-1)\zeta - (\rho-2)}{\rho\zeta - (\rho-1)}. \tag{2.12}$$

Remark 2.7 Every projective involution $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$ fixing 1 coincides with η_ρ for some $\rho \in \mathbb{C}$ and vice versa.

Definition 2.8 Let b be an irreducible germ of holomorphic curve at a point $B \in \mathbb{C}^2$. Let (z, w) be an affine chart adapted to b. A germ of singular holomorphic dual billiard structure on b given by a holomorphic family of involutions $\sigma_P : L_P \to L_P$, $P \in b \setminus \{B\}$ is said to be *meromorphic with pole of order at most one* at B, if the involutions σ_P written in the coordinate

$$\zeta := \frac{z}{z(P)}$$

on L_P converge in $\mathrm{PSL}_2(\mathbb{C})$ to some involution $\overline{\mathbb{C}}_\zeta \to \overline{\mathbb{C}}_\zeta$. Then the limit involution is equal to η_ρ for some ρ, by the above remark. The latter number ρ is called the *residue* of the billiard structure at B. In the case, when $\rho \neq 0$, we say that σ_P has *pole of order exacly one* at B.

Remark 2.9 For every meromorphic billiard structure of order at most one the above residue is independent on choice of adapted chart.

Example 2.10 1) In the case, when σ_P limits to a well-defined projective involution $L_B \to L_B$, as $P \to B$ (e.g., if σ_P extends holomorphically to $P = B$), we say that the dual billiard structure is *regular* at B. In this case the involutions σ_P written in the above coordinate ζ converge to the symmetry $\eta_0 : \zeta \mapsto 2 - \zeta$, and the billiard structure has residue $\rho = 0$ at B.

2) Consider now the case, when there exists a conic Γ passing through B such that each involution σ_P permutes the points of intersection $L_P \cap \Gamma$. Let Γ be transversal to b at B. Then σ_P converges to the unique involution $\eta_1 : \zeta \mapsto \frac{1}{\zeta}$ fixing 1 and permuting the origin and the infinity: $\rho = 1$.

One of the key statements used in the proof of main results is the following proposition.

Proposition 2.11 *Let $b \subset \mathbb{CP}^2$ be an irreducible germ of holomorphic curve at a point B equipped with a singular holomorphic dual billiard structure admitting a meromorphic integral R. Let f be an irreducible germ of holomorphic function defining b, i.e., $b = \{f = 0\}$, and let k and $G = R^{\frac{1}{k}}$ be the same, as in (2.1). Let (z, w) be affine coordinates centered at B and adapted to b. Let us equip the germ b with the coordinate z. Consider the restriction $H(G)|_b$ as a multivalued function of z. Let $d \in \mathbb{Q}$ be the minimal number such that the monomial z^d is contained in its Laurent Puiseux series:*

$$H(G)|_b = \alpha z^d (1 + o(1)), \quad \alpha \neq 0.$$

Then the involution family $\sigma_P : L_P \to L_P$ defining the dual billiard is meromorphic with pole B of order at most one, and its residue ρ is equal to

$$\rho = -\frac{d}{3}. \tag{2.13}$$

Remark 2.12 The above asymptotic exponent d is well-defined, since $H(G)$ is a finite sum of products of rational powers of holomorphic functions, see (2.4). It is independent on the affine chart containing B chosen to define $\nabla_{skew} G$ and $H(G)$, see the statement after formula (2.4) above and the discussion in [31, p. 1022, proof of proposition 3.6]. Therefore, we can calculate the exponent d writing $H(G)$ in the adapted coordinates (z, w).

Proof of Proposition 2.11. Consider a line L_P equipped with the coordinate $\zeta = \frac{z}{z(P)}$ and its parametrization by the parameter θ:

$$\zeta = 1 + \frac{\theta}{z(P)(1 + \psi(P)\theta)}.$$

Set $\rho = -\frac{d}{3}$, see (2.13). One has

$$\psi(P) = \frac{1}{z(P)}\left(\frac{d}{6} + o(1)\right) = \frac{1}{z(P)}\left(-\frac{\rho}{2} + o(1)\right), \text{ as } P \to B,$$

by equation (2.10). Therefore,

$$\zeta = 1 + \frac{2\theta}{2z(P)(1+o(1)) - \rho\theta} = \frac{2z(P)(1+o(1)) - (\rho-2)\theta}{2z(P)(1+o(1)) - \rho\theta}. \tag{2.14}$$

In the coordinate θ the involution σ_P is standard: $\theta \mapsto -\theta$. Therefore, its matrix in the coordinate ζ treated as an element in $\mathrm{PSL}_2(\mathbb{C})$ is the conjugate of the matrix $\mathrm{diag}(1,-1)$ by the matrix of transformation (2.14). Up to a scalar factor, this is the matrix

$$\begin{pmatrix} 2-\rho & 2z(P)(1+o(1)) \\ -\rho & 2z(P)(1+o(1)) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 2z(P)(1+o(1)) & -2z(P)(1+o(1)) \\ \rho & 2-\rho \end{pmatrix}$$

$$= -4z(P)\left(\begin{pmatrix} \rho-1 & -(\rho-2) \\ \rho & -(\rho-1) \end{pmatrix} + o(1)\right).$$

Hence, $\sigma_P \to \eta_\rho$ in the coordinate ζ. This proves the proposition. □

The number ρ is called "residue" due to the following proposition.

Proposition 2.13 *Let b be a regular germ at B equipped with a meromorphic dual billiard structure with pole of order at most one with residue ρ. Then in the coordinate*

$$u := z - z(P)$$

the family of involutions $\sigma_P : L_P \to L_P$, $P \in b \setminus \{B\}$, takes the form

$$\sigma_P : u \mapsto -\frac{u}{1 + f(z(P))u}, \tag{2.15}$$

$$f(z) = \frac{\rho}{z} + g(z), \ g(z) \text{ is a holomorphic function at } 0.$$

Conversely, an involution family holomorphic in $P \in b \setminus \{B\}$ and having form (2.15) is meromorphic with pole of order at most one at B with residue ρ. In particular, σ_P is regular at B, if and only if it has zero residue at B.

Proof The family σ_P is meromorphic of order at most one at B with residue ρ, if and only if in the coordinate

$$\widetilde{u} := \zeta - 1$$

the involutions σ_P take the form

$$\sigma_P : \widetilde{u} \mapsto -\frac{\widetilde{u}}{1+(\rho+o(1))\widetilde{u}}, \text{ as } P \to B, \tag{2.16}$$

by definition and since η_ρ sends $\widetilde{u}$ to $-\frac{\widetilde{u}}{1+\rho\widetilde{u}}$. Rescaling $\widetilde{u}$ to $u = \widetilde{u}z(P)$ yields (2.15) with $f(z) = \frac{\rho}{z} + g(z)$, $g(z) = o(\frac{1}{z})$. Conversely, rescaling u to $\widetilde{u}$ transforms (2.15) to (2.16). The family of involution σ_P depends holomorphically on $P \in b \setminus \{B\}$, and hence, on $z = z(P)$, by regularity of the germ b. Therefore, if (2.16) holds, then the function $zf(z)$, and hence, $h(z) := zg(z)$ extends holomorphically to 0. One has $h(0) = 0$, since $g(z) = o(\frac{1}{z})$. Hence, $g(z) = \frac{h(z)}{z}$ is holomorphic at 0. Statement (2.15) is proved, and it immediately implies the last statement of the proposition. □

2.4 Meromorphic integrability versus rational

Here we prove the following proposition.

Proposition 2.14 *Let b be a non-linear irreducible germ of holomorphic curve at $O \in \mathbb{C}^2$ equipped with a meromorphically integrable singular dual billiard structure with integral R. Let ρ be its residue at O (see Proposition 2.11). If $\rho \neq 0$, then R is rational, and b lies in an algebraic curve.*

Proof Let (z, w) be coordinates adapted to b. Let $U = U_z \times U_w$, $U_z = \{|z| < \varepsilon\}$, $U_w = \{|w| < \delta\}$, be a polydisk such that the meromorphic integral R is well-defined on a bigger polydisk containing its closure. For every $P \in b$ let P_ρ denote the point in L_P with the ζ-coordinate $\theta_\rho := \frac{\rho-1}{\rho} = \eta_\rho(\infty)$; $\zeta = \frac{z}{z(P)}$. The involution $\sigma_P : L_P \to L_P$ sends the neighborhood of infinity $V_P(\varepsilon) := L_P \cap \{|z| > \frac{\varepsilon}{2}\}$ to a $o(z(P))$-neighborhood of the point P_ρ (thus, contained in U, if P is close enough to O), since $\sigma_P \to \eta_\rho$ in the coordinate ζ. The pullback of the integral R under the map $\sigma_P|_{V_P(\varepsilon)}$ is a meromorphic function on $V_P(\varepsilon)$ whose restriction to the open subset $L_P \cap \{\frac{\varepsilon}{2} < |z| < \varepsilon\} \subset V_P(\varepsilon)$ coincides with R, by σ_P-invariance. This extends R to a meromorphic (and hence, rational) function on all of L_P for every $P \in b \setminus \{O\}$ close enough to O. The domains $V_P(\frac{\varepsilon}{2}) \subset L_P$ corresponding to P close enough to O foliate a neighborhood of the complement $L_O \setminus \{|z| < \frac{\varepsilon}{2}\}$ in $\mathbb{CP}^2$. The function R thus extended is meromorphic on the union of the latter neighborhood and the bidisk U, which covers a neighborhood of the line L_O in $\mathbb{CP}^2$.

Proposition 2.15 *A function meromorphic on a neighborhood of a projective line in $\mathbb{CP}^2$ is rational.*

Proof Take an affine chart $\mathbb{C}^2_{z,w}$ on the complement of the projective line in question. We choose the center of coordinates close to the infinity line and the axes also close to the infinity line. The function in question is rational in z with fixed small w and vice versa. Each function rational in two separate variables is rational (Proposition 9.2). This proves Proposition 2.15. $\square$

Proposition 2.14 follows from Proposition 2.15 and the above discussion. $\square$

3 Reduction to quasihomogeneously integrable $(p, q; \rho)$-billiards

Definition 3.1 Let $p, q \in \mathbb{N}$, $1 \leq q < p$, be coprime numbers. The curve

$$\gamma_{p,q} := \{w^q = z^p\} \subset \mathbb{C}^2 \subset \mathbb{CP}^2$$

will be called the (p, q)-*curve.* (It is injectively holomorphically parametrized by $\mathbb{C}^*$ via the mapping $t \mapsto (t^q, t^p)$.) Let $\rho \in \mathbb{C}$. The $(p, q; \rho)$-*billiard* is the structure of singular holomorphic dual billiard on the (p, q)-curve $\gamma_{p,q}$ defined by the family of involutions $\sigma_P : L_P \to L_P$, $P \in \gamma_{p,q} \setminus \{(0, 0)\}$, all of them acting as the involution η_ρ in the coordinate $\zeta = \frac{z}{z(P)}$ on L_P.

Definition 3.2 Recall that a polynomial $P(z, w)$ is (p, q)-*quasihomogeneous,* if it contains only monomials $z^k w^m$ with (k, m) lying on the same line parallel to the segment $[(p, 0), (0, q)]$. That is, a polynomial that becomes homogeneous after the substitution $z = t^q$, $w = t^p$, i.e., after restriction to the curve $\gamma_{p,q}$. A ratio of two (p, q)-quasihomogeneous polynomials will be called a (p, q)-*quasihomogeneous rational function.* A $(p, q; \rho)$-billiard is said to be *quasihomogeneously integrable,* if it admits a non-constant (p, q)-quasihomogeneous rational integral.

The main result of the present section is the following theorem.

Theorem 3.3 *Let b be a non-linear irreducible germ of analytic curve at a point $B \in \mathbb{C}^2$. Let $r = \frac{p}{q}$ be its projective Puiseux exponent, $(p, q) = 1$, see (1.23). Let b admit a structure of meromorphically integrable singular dual billiard, ρ be its residue at B. Then the $(p, q; \rho)$-billiard is quasihomogeneously integrable.*

3.1 Preparatory material. Newton diagrams

Let us recall the well-known notion of Newton diagram of a germ of holomorphic function $f(z, w)$ at the origin. We consider that $f(0, 0) = 0$. To each monomial $z^m w^n$ entering its Taylor series we put into correspondence the quadrant $K_{m,n} := (m, n) + (\mathbb{R}_{\geq 0})^2$. Let $K(f)$ denote the convex hull of the union of the quadrants $K_{m,n}$ through all the Taylor monomials of the function f; it is an unbounded polygon with a finite number of sides. The *Newton diagram* $\mathbf{N}_f$ is the union of those edges of the boundary $\partial K(f)$ that do not lie in the coordinate axes.

Fix a coprime pair of numbers $p, q \in \mathbb{N}$, $(p, q) = 1$. For every monomial $z^k w^m$ define its (p, q)*-quasihomogeneous degree:*

$$\deg_{p,q} z^k w^m := kq + mp.$$

Let $M_{p,q}(f)$ denote the minimal (p, q)-quasihomogeneous degree of a Taylor monomial of the function f. The sum of its monomials $f_{km} z^k w^m$ with $\deg_{p,q} = M_{p,q}(f)$ is a (p, q)-quasihomogeneous polynomial called the *lower* (p, q)*-quasihomogeneous part* of the function f; it will be denoted by $\widetilde{f}_{p,q}(z.w)$.

Remark 3.4 In the case, when the Newton diagram $\mathbf{N}_f$ contains an edge parallel to the segment $[(p, 0), (0, q)]$, the collection of bidegrees of monomials entering the lower (p, q)-quasihomogeneous part $\widetilde{f}_{p,q}$ lies in the latter edge and contains its vertices. In the opposite case $\widetilde{f}_{p,q}$ is a monomial whose bidegree is the unique vertex V of the Newton diagram such that the line through V parallel to the above segment intersects $K(f)$ only at V. One has

$$\varepsilon^{-M_{p,q}(f)} f(\varepsilon^q z, \varepsilon^p w) = \widetilde{f}_{p,q}(z, w) + o(1), \text{ as } \varepsilon \to 0, \tag{3.1}$$

uniformly on compact subsets in $\mathbb{C}^2$.

Example 3.5 Let a germ of holomorphic function f at the origin be irreducible (not a product of holomorphic germs vanishing at 0). If $f(z, 0) \equiv 0$, then the Newton diagram $\mathbf{N}_f$ consists of just one, horizontal edge of height one. Let now $f(z, 0), f(0, w) \not\equiv 0$. It is well-known that then the Newton diagram of the germ f consists of one edge $[(ps_b, 0), (0, qs_b)]$ with some $s_b \in \mathbb{N}$ and coprime $p, q \in \mathbb{N}$. Let b be its zero locus. Then b is a germ of curve injectively parametrized by a germ at 0 of holomorphic map of the type

$$t \mapsto (t^{qs_b}, c_b t^{ps_b}(1 + O(t))), \quad c_b \neq 0; \tag{3.2}$$

$$\widetilde{f}_{p,q}(z, w) = (w^q - C_b z^p)^{s_b}, \quad C_b = c_b^q, \tag{3.3}$$

up to constant factor. The proof of formula (3.3) repeats the proof of [31, proposition 3.5] with minor changes.

Proposition 3.6 *Let a, b be two irreducible germs of holomorphic curves at O. Let $r = r_b = \frac{p}{q}$ be the projective Puiseux exponent of the germ b, $(p,q) = 1$, r_a be that of the germ a. Let (z,w) be affine coordinates centered at O adapted to b; the coordinate w being rescaled so that $c_b = 1$ in (3.2).*

1) Let $\widetilde{f}_a(z,w)$ be the lower (p,q)-quasihomogeneous part of the germ of function f_a defining a. Up to constant factor, the polynomial $\widetilde{f}_a(z,w)$ has one of the following types:

a) z^m, if either a is transversal to b, or a, b are tangent and $r_a < r_b$;

b) w^m, if a, b are tangent and $r_a > r_b$;

c) $(w^q - C_a z^p)^m$, if a, b are tangent and $r_a = r_b$; C_a is given by (3.3).

2) For every $P \in b \setminus \{O\}$ consider the coordinate $\zeta := \frac{z}{z(P)}$ on the line L_P. Let $L = L_{(1,1)}$ denote the tangent line to $\gamma_{p,q} := \{y^q - \zeta^p = 0\} \subset \mathbb{C}^2_{\zeta,y}$ at the point $(1,1)$. As $P \to O$, the ζ-coordinates of points of the intersection $a \cap L_P$ tend to some (finite or infinite) limits in $\overline{\mathbb{C}}_\zeta$. The set of their finite limits coincides with the set of zeros of the restriction to L of the polynomial $\widetilde{f}_a(\zeta,y)$. In the above cases a), b), c) it coincides respectively with the sets $\{0\}$, $\{\frac{r-1}{r}\}$ and the collection of roots of the polynomial

$$\mathcal{R}_{p,q,C_a}(\zeta) := (1 - r + r\zeta)^q - C_a \zeta^p.$$

Proof Cases a) and b) correspond exactly to the cases, when the unique edge of the Newton diagram of the function f_a is not parallel to the segment $[(p,0),(0,q)]$; then the polynomial $\widetilde{f}_a$ corresponds to one of its two vertices, and hence, is a power of either z, or w. In Case c) Statement 1) of the proposition follows from (3.3). Statement 2) follows from [27, p.268, Proposition 2.50] and can be proved directly as follows. Let $P \in b \setminus \{O\}$, $z_0 := z(P)$. Consider the variable change $(z,w) = (z_0\zeta, z_0^r y)$ (for some chosen value of fractional power z_0^r). As $P \to O$, i.e., as $z_0 \to 0$, the curve b written in the coordinates (ζ, y) tends to the curve $\gamma_{p,q}$, $(\zeta(P), y(P)) \to (1,1)$, and $L_P \to L$. The function $z_0^{-\frac{1}{q}M_{p,q}(f_a)} f_a(z_0\zeta, z_0^r y)$ tends to $\widetilde{f}_a(\zeta,y)$, by (3.1). This implies that each point of intersection $a \cap L_P$, whose ζ-coordinate converges to a finite limit after passing to a subsequence, does converge to a zero of the restriction $\widetilde{f}_a|_L$, and each zero is realized as a limit. The ζ-coordinates of the other intersection points (if any) converge to infinity, by construction. The polynomial $\widetilde{f}_a(\zeta,y)$ is a power of the polynomial ζ, y, $y^q - C_a\zeta^p$ respectively up to constant factor, by Statement 1). The restrictions of the latter polynomials to the line L are equal respectively to ζ, $1-r+r\zeta$ and $\mathcal{R}_{p,q,C_a}$. This together with the above convergence implies Statement 2). $\Box$

3.2 Proof of Theorem 3.3.

Let $R(z,w)=\frac{f(z,w)}{g(z,w)}$ be a non-constant meromorphic first integral of the dual billiard on b. Here f and g are coprime germs of holomorphic functions at B written in affine coordinates (z,w) adapted to b. Let $r=r_b=\frac{p}{q}$ be the irreducible fraction representation of the projective Puiseux exponent r of the germ b. Without loss of generality we consider that the corresponding constant c_b in (3.2) is equal to one, rescaling the coordinate w. Then the function $f_b(z,w)$ defining the curve b is equal to $(w^q-z^p)^{s_b}$ plus higher (p,q)-quasihomogeneous terms, by (3.3). For a point $P\in b\setminus\{B\}$ set $z_0=z(P)$. In the above rescaled coordinates $(\zeta,y)=(z_0^{-1}z, z_0^{-r}w)$ one has $P\to(1,1)$, $L_P\to L$ ($L=L_{(1,1)}$ is the same, as Proposition 3.6), and the functions $z_0^{-\frac{1}{q}M_{p,q}(f)}f(z_0\zeta, z_0^r y)$, $z_0^{-\frac{1}{q}M_{p,q}(g)}g(z_0\zeta, z_0^r y)$ tends to $\widetilde{f}_{p,q}(\zeta,y)$ and $\widetilde{g}_{p,q}(\zeta,y)$ respectively, by (3.1). The restriction $R|_{L_P}$ is σ_P-invariant, and $\sigma_P\to\eta_\rho$ in the coordinate ζ on L_P. Therefore, the restriction to the line L of the ratio

$$\widetilde{R}(\zeta,y):=\frac{\widetilde{f}_{p,q}(\zeta,y)}{\widetilde{g}_{p,q}(\zeta,y)}$$

is η_ρ-invariant. Consider the action of group $\mathbb{C}^*$ on $\mathbb{C}^2$ by rescalings $(\zeta,y)\mapsto(\tau^q\zeta,\tau^p y)$. It preserves the curve $\gamma_{p,q}$ punctured at the origin and at infinity and acts transitively on it. These rescalings multiply the quasihomogeneous rational function $\widetilde{R}$ by constants. This together with η_ρ-invariance of its restriction to the tangent line L implies invariance of its restriction to tangent line at any other point $Q\in\gamma_{p,q}$ under the involution η_ρ acting in the coordinate $\frac{\zeta}{\zeta(Q)}$. Therefore, $\widetilde{R}$ is a quasihomogeneous integral of the $(p,q;\rho)$-billiard. A priori it may be constant. This occurs exactly in the case, when $\widetilde{g}_{p,q}\equiv\lambda\widetilde{f}_{p,q}$, $\lambda\in\mathbb{C}$. But then replacing g by $g-\lambda f$ cancels $\widetilde{g}_{p,q}$, and the lower (p,q)-quasihomogeneous part of the function $g-\lambda f$ is not constant-proportional to $\widetilde{f}_{p,q}$. The ratio $\frac{f}{g-\lambda f}$ being a meromorphic integral of the billiard on b, the above construction applied to it yields a non-constant quasihomogeneous integral of the $(p,q;\rho)$-billiard. Theorem 3.3 is proved.

4 Classification of quasihomogeneously integrable $(p, q; \rho)$-billiards

The main result of the present section is the following theorem.

Theorem 4.1 *A $(p, q; \rho)$-billiard is quasihomogeneously integrable, if and only if $p = 2$, $q = 1$ (i.e., the underlying curve $\gamma_{p,q}$ is a conic) and*

$$\rho \in \mathcal{M} := \{0, 1, 2, 3, 4\} \cup_{k \in \mathbb{N}_{\geq 3}} \{2 \pm \frac{2}{k}\}. \tag{4.1}$$

Then the following quasihomogeneous functions $R_\rho(z, w)$ are integrals.

$\rho = 0$	$\rho = 1$	$\rho = 2$	$\rho = 3$	$\rho = 4$
$R_0 = w - z^2$	$R_1 = \frac{w-z^2}{z}$	$R_2 = \frac{w-z^2}{w}$	$R_3 = \frac{w-z^2}{zw}$	$R_4 = \frac{w-z^2}{w^2}$

$\rho = 2 - \frac{2}{2N+1}$	$R_\rho(z, w) = \frac{(w-z^2)^{2N+1}}{\prod_{j=1}^{N}(w-c_j z^2)^2}, \quad c_j = -\frac{4j(2N+1-j)}{(2N+1-2j)^2}$
$\rho = 2 + \frac{2}{2N+1}$	$R_\rho(z, w) = \frac{(w-z^2)^{2N+1}}{w^2 \prod_{j=1}^{N}(w-c_j z^2)^2}, \quad c_j = -\frac{4j(2N+1-j)}{(2N+1-2j)^2}$
$\rho = 2 - \frac{1}{N+1}$	$R_\rho(z, w) = \frac{(w-z^2)^{N+1}}{z \prod_{j=1}^{N}(w-c_j z^2)}, \quad c_j = -\frac{j(2N+2-j)}{(N+1-j)^2}$
$\rho = 2 + \frac{1}{N+1}$	$R_\rho(z, w) = \frac{(w-z^2)^{N+1}}{zw \prod_{j=1}^{N}(w-c_j z^2)}, \quad c_j = -\frac{j(2N+2-j)}{(N+1-j)^2}$

Addendum to Theorem 4.1. *The variable change $(\widetilde{z}, \widetilde{w}) = (\frac{z}{w}, \frac{1}{w})$ transforms a $(2, 1, \rho)$-billiard to a $(2, 1, 4-\rho)$-billiard. It interchanges the integrals $R_\rho(\widetilde{z}, \widetilde{w})$ and $R_{4-\rho}(z, w)$ given by the above formulas for every $\rho \in \mathcal{M}$.*

Everywhere below by $L = L_{(1,1)}$ we denote the projective tangent line to the curve $\gamma_{p,q}$ at the point $(1, 1)$.

Remark 4.2 A (p, q)-quasihomogeneous rational function is an integral of the $(p, q; \rho)$-billiard, if and only if its restriction to L written in the coordinate z is η_ρ-invariant, see the above proof of Theorem 3.3.

Remark 4.3 It is well-known that each (p, q)-quasihomogeneous polynomial is a product of powers of *prime* quasihomogeneous polynomials z, w, $w^q - c_j z^p$ with $c_j \in \mathbb{C} \setminus \{0\}$.

The proof of Theorem 3.3 is based on the following formula for the Hessian (calculated in the coordinates (z, w)) of a product

$$G(z, w) = (w^q - z^p) z^\alpha w^\beta \prod_{j=2}^{M} (w^q - c_j z^p)^{\mu_j}, \ \alpha, \beta, \mu_j \in \mathbb{R}, \ c_j \neq 0, 1. \quad (4.2)$$

Proposition 4.4 *Let G be the same, as in (4.2), with $c_j \in \mathbb{C} \setminus \{0, 1\}$. Set*

$$N := 1 + \sum_{j=2}^{M} \mu_j, \ \rho_0 = \frac{2}{3}(r + 1), \ r = \frac{p}{q}.$$

There exists a $c \in \mathbb{C} \setminus \{0\}$ such that

$$H(G)|_{\gamma_{p,q}} = c z^d; \ d = 3(pN + \alpha + \beta r - \rho_0). \quad (4.3)$$

Formula (4.3) holds for $c = qp(q - p) \left(\prod_{j=2}^{p} (1 - c_j) \right)^3$.

Proof The Hessian of the defining polynomial $w^q - z^p$ of the curve $\gamma_{p,q}$ calculated in the coordinates (z, w) is equal to

$$H(w^q - z^p) = q(q - 1)p^2 w^{q-2} z^{2(p-1)} - p(p - 1)q^2 z^{p-2} w^{2q-2}.$$

Its restriction to $\gamma_{p,q}$ is equal to the same expression with w replaced by z^r, which yields $qp(q-p)z^{3p-2(r+1)}$. Each polynomial $w^q - c_j z^p$ being restricted to $\gamma_{p,q}$ is equal to $(1 - c_j)z^p$. This together with (2.11) implies (4.3). □

4.1 Case of (p, q)-quasihomogeneous polynomial integral

Proposition 4.5 *Let a $(p, q; \rho)$-billiard admit a (p, q)-quasihomogeneous polynomial integral. Then $\rho = 0$, $p = 2$, $q = 1$, and the polynomial $w - z^2$ is an integral.*

Proof The restriction to L of a (p, q)-quasihomogeneous polynomial integral $\mathcal{P}$ is η_ρ-invariant and has one pole, at infinity. Hence, $\eta_\rho(\infty) = \infty$, thus, $\rho = 0$. Its restriction to $\gamma_{p,q}$ should be constant, see Proposition 1.34.

On the other hand, the latter restriction written in the coordinate z is a monomial cz^{ϕ}. Therefore, $c=0$ and $\mathcal{P}|_{\gamma_{p,q}}\equiv 0$. Hence,

$$\mathcal{P}(z,w)=z^{\alpha}w^{\beta}\prod_{j=1}^{M}(w^q-c_jz^p)^{n_j},\ c_j\neq 0,\ c_1=1,\ c_j \text{ are distinct},\ \alpha,\beta\in\mathbb{Z}_{\geq 0},$$

see Remark 4.3. Set $k=n_1$,

$$G(z,w):=\mathcal{P}^{\frac{1}{k}}(z,w)=z^{\widetilde{\alpha}}w^{\widetilde{\beta}}(w^q-z^p)\prod_{j=2}^{M}(w^q-c_jz^p)^{\mu_j},$$

$$\mu_j:=\frac{n_j}{k},\ \widetilde{\alpha}:=\frac{\alpha}{k},\ \widetilde{\beta}:=\frac{\beta}{k},\ N:=1+\sum_{j=2}^{M}\mu_j.$$

The restriction to $\gamma_{p,q}$ of the Hessian $H(G)$ is given by formula (4.3). Hence,

$$\rho=-\frac{d}{3}=\rho_0-(pN+\alpha+\beta r)=\frac{2}{3}(r+1)-qNr-(\alpha+\beta r),$$

by (2.13). The latter right-hand side should vanish, since $\rho=0$. Therefore, $\frac{2}{3}(r+1)\geq qNr$, hence $r\leq\frac{2}{3Nq-2}$. But $r>1$, and $N,q\geq 1$. Therefore, $q=1$, and $r\leq 2$. Hence, $p=r=2$.

Let us now show that the polynomial $w-z^2$ is an integral of the $(2,1;0)$-billiard. Indeed, its restriction to the tangent line $L=L_{(1,1)}$ is the polynomial $-1+2z-z^2=-(z-1)^2$; here $\zeta=z$. The latter polynomial is clearly invariant under the involution $\eta_0: z\mapsto 2-z$. Hence, $w-z^2$ is an integral, by Remark 4.2. Proposition 4.5 is proved. □

4.2 Normalization of rational integral to primitive one

Here we consider a quasihomogeneously integrable $(p,q;\rho)$-billiard. We prove that its integral (if it cannot be reduced to a polynomial) can be normalized to a primitive integral, see definitions and Lemma 4.10 below.

The restriction of a (p,q)-quasihomogeneous polynomial $\mathcal{P}$ to the tangent line $L=L_{(1,1)}$ to the curve $\gamma_{p,q}$ at the point $(1,1)$ is a polynomial in the coordinate z on L. One has

$$w|_L=1-r+rz,\ (w^q-cz^p)|_L=\mathcal{R}_{p,q,c}(z):=(1-r+rz)^q-cz^p. \quad (4.4)$$

Definition 4.6 The roots of the restriction $\mathcal{P}|_L$ of a (p,q)-quasihomogeneous polynomial $\mathcal{P}$ will be called its *tangent line roots*. The linear combination of points representing roots with coefficients equal to their multiplicities is a divisor on $L \simeq \mathbb{C}_z$. It will be called the *root divisor* of the polynomial $\mathcal{P}$ and denoted by $\chi(\mathcal{P})$. Sometimes we will deal with $\chi(\mathcal{P})$ as with a collection of roots, e.g., when we write inclusions that some points belongs to $\chi(\mathcal{P})$.

Definition 4.7 Recall that the *complement of a divisor* χ *to a point* θ is the divisor χ with the term corresponding to the point θ deleted. Set

$$\theta_\rho := \frac{\rho - 1}{\rho} = \eta_\rho(\infty).$$

A (p,q)-quasihomogeneous polynomial $\mathcal{P}$ is called *η_ρ-quasi-invariant,* if the complement $\chi(\mathcal{P}) \setminus \{\theta_\rho\}$ is η_ρ-invariant. A η_ρ-quasi-invariant polynomial $\mathcal{P}$ is *η_ρ-primitive*, if it is not a product of two η_ρ-quasi-invariant polynomials.

Proposition 4.8 *1) A primitive η_ρ-quasi-invariant polynomial $\mathcal{P}$ is (up to constant factor) a product $\prod_j Q_j$ of some distinct prime (p,q)-quasihomogeneous polynomials Q_j equal to $w^q - c_j z^p$, z or w.*

2) Any two prime factors Q_k, Q_ℓ are **equivalent** *in the following sense: there exists a finite sequence $k = j_1, j_2, \dots, j_m = \ell$ such that for every $s = 1, \dots, m-1$ there exist tangent line roots z_s, z_{s+1} of the polynomials Q_{j_s} and $Q_{j_{s+1}}$ respectively such that $z_{s+1} = \eta_\rho(z_s)$.*

3) If $\rho = r$, then either $\mathcal{P} = cw$, $c \in \mathbb{C} \setminus \{0\}$, or $\mathcal{P}$ contains no w-factor.

4) For any two distinct primitive η_ρ-quasi-invariant polynomials their tangent line root collections do not intersect.

5) Every η_ρ-quasi-invariant polynomial is a product of powers of primitive ones.

The proposition follows from definition and the fact that the polynomial $w|_L = 1 - r + rz$ has one root $\frac{r-1}{r}$.

Definition 4.9 A (p,q)-quasihomogeneous *rational integral* of the $(p,q;\rho)$-billiard is *η_ρ-primitive,* if it is a ratio of nonzero powers of two non-trivial primitive η_ρ-quasi-invariant (p,q)-quasihomogeneous polynomials.

Lemma 4.10 *Let a $(p,q;\rho)$-billiard be quasihomogeneously integrable and admit no polynomial (p,q)-quasihomogeneous integral. Then it admits a η_ρ-primitive rational integral vanishing identically on $\gamma_{p,q}$, and $\rho \neq 0$.*

Proof Let R be a quasihomogeneous integral of the $(p,q;\rho)$-billiard represented as an irreducible ratio of two quasihomogeneous polynomials: numerator and denominator, both being non-constant (absence of polynomial integral). Its restriction to the curve $\gamma_{p,q}$ is constant, by Proposition 1.34. If the latter constant is finite non-zero, then the numerator and the denominator have equal (p,q)-quasihomogeneous degrees. Therefore, replacing the numerator by its linear combination with denominator one can get another quasihomogeneous integral that vanishes identically on $\gamma_{p,q}$. If the above constant is infinity, we replace R by R^{-1} and get an integral vanishing on $\gamma_{p,q}$. Thus, we can and will consider that $R\equiv 0$ on $\gamma_{p,q}$. Both numerator and denominator are η_ρ-quasi-invariant, which follows from η_ρ-invariance of the restriction of the integral to the tangent line $L=L_{(1,1)}$. Therefore, they are products of powers of primitive η_ρ-quasi-invariant polynomials. Among all the η_ρ-quasi-invariant primitive factors in the numerator and the denominator there are at least two distinct ones, by irreducibility and non-polynomiality of the ratio R. Take one of them $\mathcal{P}_1$, vanishing identically on $\gamma_{p,q}$ (hence, divisible by w^q-z^p) and another one $\mathcal{P}_2$. For every $i=1,2$ one has

$$\mathcal{P}_i(z,w)=z^{\alpha_i}w^{\beta_i}\prod_{j=1}^{N_i}(w^q-c_{ij}z^p), \quad c_{ij}\neq 0; \tag{4.5}$$

$$\alpha_i,\beta_i\in\{0,1\}, \ \alpha_1\alpha_2=\beta_1\beta_2=0, \ c_{11}=1, \quad \text{all } c_{ij} \text{ are distinct},$$

by Proposition 4.8. Set now

$$R(z,w):=\frac{\mathcal{P}_1^{m_1}}{\mathcal{P}_2^{m_2}}, \quad d_i:=\deg\mathcal{P}_i=N_ip+\alpha_i+\beta_i. \tag{4.6}$$

Proposition 4.11 *The ratio (4.6) of powers of two non-trivial primitive η_ρ-quasi-invariant polynomials $\mathcal{P}_1$, $\mathcal{P}_2$ is an integral of the $(p,q;\rho)$-billiard, if the following relation holds:*

$$\textit{Case 1), } \theta_\rho\notin\chi(\mathcal{P}_1)\cup\chi(\mathcal{P}_2): \qquad d_1m_1=d_2m_2. \tag{4.7}$$

$$\textit{Case 2), } \theta_\rho\in\chi(\mathcal{P}_1): \qquad (d_1+1)m_1=d_2m_2. \tag{4.8}$$

$$\textit{Case 3), } \theta_\rho\in\chi(\mathcal{P}_2): \qquad d_1m_1=(d_2+1)m_2. \tag{4.9}$$

As is shown below, Proposition 4.11 is implied by the following obvious

Proposition 4.12 *Let a rational function $R(\zeta)$ either do not vanish at 1, or have 1 as a root of even degree. Then it is η_ρ-invariant, if and only if its zero locus and its pole locus are both η_ρ-invariant.*

Proof A rational function R is uniquely determined by its zero and pole loci up to constant factor. Therefore, if the latter loci are invariant under a conformal involution η_ρ, then $R \circ \eta_\rho = \pm R$. The sign $\pm$ is in fact $+$, taking into account the condition at the point 1, which is fixed by η_ρ. $\Box$

Proof of Proposition 4.11. Let us show, case by case, that if the corresponding relation (4.7), (4.8) or (4.9) holds, then the zero and pole divizors of the restriction $R|_L$ are η_ρ-invariant. This together with Proposition 4.12 implies that $R|_L$ is η_ρ-invariant, and hence, R is an integral (Remark 4.2).

Case 1): $\theta_\rho \notin \chi(\mathcal{P}_1) \cup \chi(\mathcal{P}_2)$ and $m_1 d_1 = m_2 d_2$. Then the infinity in L is not a pole of the restriction $R|_L$. Therefore, its zeros (poles) are zeros of the polynomial $\mathcal{P}_1|_L$ (respectively, $\mathcal{P}_2|_L$). Their divisors are η_ρ-invariant, by η_ρ-quasi-invariance of the polynomials $\mathcal{P}_i$, and since the root collections of their restrictions to L do not contain $\theta_\rho = \eta_\rho(\infty)$. Hence, $R|_L$ is η_ρ-invariant.

Case 2): $\theta_\rho \in \chi(\mathcal{P}_1)$ and $m_1(d_1 + 1) = m_2 d_2$. Then the infinity in L is a zero of multiplicity m_1 of the restriction $R|_L$. The point θ_ρ is a simple root of the polynomial $\mathcal{P}_1|_L$, by assumption and primitivity. This together with its η_ρ-quasi-invariance implies that the zero divizor of the function $R|_L$ is η_ρ-invariant. Its pole divisor, i.e., the zero divisor of the function $\mathcal{P}_2^{m_2}|_L$ is also η_ρ-invariant, as in the above discussion.

Case 3) is treated analogously to Case 2). $\Box$

Proposition 4.11 immediately implies the statement of Lemma 4.10, except for the statement that $\rho \neq 0$. Suppose the contrary: $\rho = 0$. Then $\eta_\rho(\infty) = \infty \notin \chi(\mathcal{P}_1)$. Therefore, the restriction to L of the η_ρ-quasi-invariant polynomial $\mathcal{P}_1$ is η_ρ-invariant (Proposition 4.12). Hence, $\mathcal{P}_1$ is a polynomial integral of the $(p, q; \rho)$-billiard. The contradiction thus obtained proves that $\rho \neq 0$ and finishes the proof of Lemma 4.10. $\Box$

4.3 Case of rational integral. Two formulas for ρ

Here we treat the case, when the $(p, q; \rho)$-billiard in question admits a rational quasihomogeous integral and does not admit a polynomial one: thus, $\rho \neq 0$ (Lemma 4.10). Everywhere below we consider that the integral $R(z, w)$ is η_ρ-primitive, vanishes on $\gamma_{p,q}$ (Lemma 4.10) and is given by formula (4.6) with $m_1, m_2 \neq 0$ satisfying some of relations (4.7)-(4.9). Set

$$G(z, w) := (R(z, w))^{\frac{1}{m_1}} = \frac{\mathcal{P}_1}{\mathcal{P}_2^{\nu}}, \ \nu := \frac{m_2}{m_1}. \tag{4.10}$$

We prove two different formulas for the residue ρ, deduced

- on one hand, from formula (4.3) for the Hessian $H(G)$ and formula (2.13) expressing ρ via the asymptotic exponent d;

- on the other hand, by applying a similar argument to a special η_ρ-invariant function on $L_P \simeq \mathbb{C}_z$: the ratio of the numerator of the integral and a power of $z - \frac{\rho-2}{\rho}$. Combining the two formulas for ρ thus obtained, we will show in Subsection 4.4 that $p = 2$, $q = 1$ and $\rho \in \mathcal{M}$.

In our case formulas (4.3) and (2.13) yield

$$H(G)|_{w^q=z^p} = cz^d, \ d = 3((N_1 - \nu N_2)p + \alpha_1 - \nu\alpha_2 + r(\beta_1 - \nu\beta_2) - \rho_0), \quad (4.11)$$

$$r = \frac{p}{q}, \quad \rho_0 = \frac{2}{3}(r+1),$$

$$\rho = -\frac{d}{3} = \rho_0 - (d_1 - \nu d_2) - (r-1)(\beta_1 - \nu\beta_2), \ d_i = N_i p + \alpha_i + \beta_i. \quad (4.12)$$

This is the First Formula for ρ. Substituting to (4.12) the relations between the degrees d_1 and d_2 given by Proposition 4.11 and taking into account that $\beta_j = 1$ if and only if $\theta_r := \frac{r-1}{r} \in \chi(\mathcal{P}_j)$, we get

Proposition 4.13 *Let d_1, d_2, G be as above. Then one has the following formulas for the residue ρ dependently on whether or not some of the numbers $\theta_\rho = \eta_\rho(\infty) = \frac{\rho-1}{\rho}$, $\theta_r = \frac{r-1}{r}$ lie in some of $\chi(\mathcal{P}_{1,2})$:*

	$\theta_\rho \notin \chi(\mathcal{P}_1) \cup \chi(\mathcal{P}_2)$	$\theta_\rho \in \chi(\mathcal{P}_1)$	$\theta_\rho \in \chi(\mathcal{P}_2)$
$\theta_r \notin \chi(\mathcal{P}_1) \cup \chi(\mathcal{P}_2)$	$\rho = \rho_0$	$\rho = \rho_0 + 1$	$\rho = \rho_0 - \nu$
$\theta_r \in \chi(\mathcal{P}_1)$	$\rho = \rho_1 := \rho_0 + 1 - r$	$\rho = \rho_1 + 1$	$\rho = \rho_1 - \nu$
$\theta_r \in \chi(\mathcal{P}_2)$	$\rho = \rho_2 := \rho_0 - \nu(1-r)$	$\rho = \rho_2 + 1$	$\rho = \rho_2 - \nu$

The Second Formula for the residue ρ is given by the next lemma.

Lemma 4.14 *Let $\mathcal{P}(z,w) = z^\alpha w^\beta \prod_{j=1}^N (w^q - c_j z^p)$ be a primitive η_ρ-quasi-invariant (p,q)-quasihomogeneous polynomial vanishing on $\gamma_{p,q}$: $c_1 = 1$. Set*

$$d_{\mathcal{P}} := \deg \mathcal{P} = Np + \alpha + \beta, \quad \hat{d}_{\mathcal{P}} := \left[\begin{array}{l} d_{\mathcal{P}}, \ \text{if } \theta_\rho \notin \chi(\mathcal{P}) \\ d_{\mathcal{P}} + 1, \ \text{if } \theta_\rho \in \chi(\mathcal{P}). \end{array} \right. \quad (4.13)$$

Then the residue ρ is expressed by the formula

$$\rho(\hat{d}_{\mathcal{P}} - 2) = 2(Np + \alpha + \beta r - \rho_0); \quad \text{here } \rho_0 = \frac{2}{3}(r+1). \quad (4.14)$$

Proof The restriction of the polynomial $\mathcal{P}$ to the tangent line $L = \mathbb{C}_z$ is

$$H(z) := z^\alpha(1-r+rz)^\beta \prod_{i=1}^{N}((1-r+rz)^q - c_i z^p), \ \deg H = d_{\mathcal{P}} = Np+\alpha+\beta.$$

The roots of the latter polynomial are exactly points of $\chi(\mathcal{P})$. The involution $\eta_\rho : L \to L$ has two fixed points: those with z-coordinates 1 and $\frac{\rho-2}{\rho}$. We consider the following auxiliary rational function

$$G(z) := \frac{H(z)}{(z - \frac{\rho-2}{\rho})^{\hat{d}_{\mathcal{P}}}}. \tag{4.15}$$

Claim 1. *The rational function G is η_ρ-invariant.*
Proof The zero divisor of the function $G|_L$ is η_ρ-invariant. Indeed, the complement of the root divisor $\chi(\mathcal{P})$ of the polynomial H to θ_ρ is η_ρ-invariant (η_ρ-quasi-invariance of the polynomial $\mathcal{P}$). In the case, when $H(\theta_\rho) = 0$, one has $\hat{d}_{\mathcal{P}} = \deg H + 1$, and hence, ∞ is a simple zero of the function G. The pole divisor of the function H is the fixed point $\frac{\rho-2}{\rho}$ of the involution η_ρ. This together with Proposition 4.12 implies that G is η_ρ-invariant. □

Corollary 4.15 *For every $\lambda \in \mathbb{C}$ the polynomial*

$$H_\lambda(z) := H(z) - \lambda \left(z - \frac{\rho-2}{\rho}\right)^{\hat{d}_{\mathcal{P}}}$$

has exactly two roots $\zeta_\pm(\lambda)$ converging to 1, as $\lambda \to 0$. These roots are permuted by the involution η_ρ.

We will deduce formula (4.14) by comparing asymptotics of the numbers $\zeta_\pm(\lambda)$ as roots of the polynomial H_λ and writing the condition that they should be permuted by the involution η_ρ with known Taylor series. To this end, we write the polynomials H_λ and their roots in the new coordinate

$$u := z-1; \quad u_\pm := u(\zeta_\pm(\lambda)) = \zeta_\pm(\lambda) - 1.$$

Claim 2. *There exists a constant $A \in \mathbb{C}^*$ such that as $u \to 0$, one has*

$$H(z) = H(1+u) = A(1 + (Np+\alpha+\beta r - \rho_0)u + O(u^2))u^2. \tag{4.16}$$

Proof One has $p = qr$,

$$z^\alpha(1-r+rz)^\beta = (1+u)^\alpha(1+ru)^\beta = 1 + (\alpha + r\beta)u + O(u^2), \tag{4.17}$$

$$(1-r+rz)^q - z^p = (1+ru)^q - (1+u)^p$$
$$= \frac{q(q-1)r^2 - p(p-1)}{2}u^2 + \frac{q(q-1)(q-2)r^3 - p(p-1)(p-2)}{6}u^3 + O(u^4)$$
$$= \frac{p(1-r)}{2}u^2 + \frac{p}{6}((p-r)(p-2r) - (p-1)(p-2))u^3 + O(u^4)$$
$$= \frac{p(1-r)}{2}u^2(1 + (p-\rho_0)u + O(u^2)). \tag{4.18}$$

For $c_i \neq 1$ one has the equality

$$(1-r+rz)^q - c_i z^p = ((1-r+rz)^q - z^p) + (1-c_i)(1+u)^p = (1-c_i)(1+pu) + O(u^2).$$

Multiplying it with (4.17), (4.18) yields (4.16) with $A = \frac{p(1-r)}{2}\prod_{i\geq 2}(1-c_i)$.
□

Corollary 4.16 *One has $u_- = -u_+(1+o(1))$, as $\lambda \to 0$, and*

$$A(1 + (Np + \alpha + \beta r - \rho_0)u_\pm + O(u_+^2))u_\pm^2 = B(1 + \frac{\hat{d}_{\mathcal{P}}\rho}{2}u_\pm + O(u_\pm^2)), \tag{4.19}$$

$$B = B(\lambda) = \lambda\left(\frac{2}{\rho}\right)^{\hat{d}_{\mathcal{P}}}.$$

Proof One has $H(\zeta_\pm) = \lambda(\zeta_\pm - \frac{\rho-2}{\rho})^{\hat{d}_{\mathcal{P}}}$, by definition. Substituting (4.16) to the latter formula yields (4.19). □

The involution $\eta_\rho(z)$ written in the coordinate $u = z-1$ takes the form

$$\eta_\rho : u \mapsto -\frac{u}{1+\rho u}. \tag{4.20}$$

Therefore, $u_- = -u_+ + (\rho + O(u_+))u_+^2$, since $u_\pm$ converge to 0 and are permuted by η_ρ. Dividing equations (4.19) for u_+ and for u_- and substituting the latter asymptotic formula for u_- yields

$$\frac{1 + (Np + \alpha + \beta r - \rho_0)u_+ + O(u_+^2)}{(1 - (Np + \alpha + \beta r - \rho_0)u_+ + O(u_+^2))(1-\rho u_+)^2} = 1 + \hat{d}_{\mathcal{P}}\rho u_+ + O(u_+^2),$$

$$2(Np + \alpha + \beta r - \rho_0) + 2\rho = \hat{d}_{\mathcal{P}}\rho.$$

This proves (4.14). □

4.4 Proof of the main part of Theorem 4.1: necessity

The main part of Theorem 4.1 is given by the next lemma.

Lemma 4.17 *Let a $(p,q;\rho)$-billiard be quasihomogeneously integrable. Then $p=2$, $q=1$ and $\rho\in\mathcal{M}$.*

Proof The case of polynomial integrability, was already treated in Subsection 4.1. Let us treat the case, when there are no polynomial integral. Then there exists a primitive integral R vanishing on $\gamma_{p,q}$; let us fix it. One has

$$R(z,w)=\frac{\mathcal{P}_1^{m_1}}{\mathcal{P}_2^{m_2}}(z,w),\ \mathcal{P}_i=z^{\alpha_i}w^{\beta_i}\prod_{j=1}^{N_i}(w^q-c_{ij}z^p),$$

$$\alpha_j,\beta_j\in\{0,1\},\ \alpha_1\alpha_2=\beta_1\beta_2=0,\ c_{11}=1,\quad \text{all}\quad c_{ij}\quad \text{are distinct.}$$

The statement of the lemma will be deduced by equating the two formulas for the residue ρ given by Proposition 4.13 and Lemma 4.14 (applied to $\mathcal{P}_1$).

Case 1): $\theta_\rho=\eta_\rho(\infty)\notin\chi(\mathcal{P}_1)\cup\chi(\mathcal{P}_2)$. Then

$$\rho=\rho_0+(1-r)(\beta_1-\nu\beta_2)=\frac{2(N_1p+\alpha_1+\beta_1r-\rho_0)}{N_1p+\alpha_1+\beta_1-2}, \tag{4.21}$$

by Proposition 4.13 and Lemma 4.14 applied to $\mathcal{P}_1$.

Subcase 1a): $\beta_1=\beta_2=0$. Then (4.21) yields

$$(\rho_0-2)(N_1p+\alpha_1)=0,\quad \rho_0=\frac{2}{3}(r+1).$$

Hence, $\rho_0=r=2$, since $N_1p>0$, $\alpha_1\geq 0$. This together with (4.21) yields

$$p=2,\ q=1,\ \rho=\rho_0=2.$$

Subcase 1b): $\beta_1=1$, $\beta_2=0$. Then (4.21) yields

$$\begin{aligned}2N_1p+2\alpha_1+2r&=\rho_0(N_1p+\alpha_1+1)+(1-r)(N_1p+\alpha_1-1)\\&=\frac{2}{3}(r+1)(N_1p+\alpha_1+1)+(1-r)(N_1p+\alpha_1-1).\end{aligned}$$

Substituting $r=\frac{p}{q}$ and multiplying the latter equation by $3q$ yields

$$6N_1pq+6\alpha_1q+6p=2(p+q)(N_1p+\alpha_1+1)+3(q-p)(N_1p+\alpha_1-1). \tag{4.22}$$

Writing equation (4.22) modulo p and dividing it by $q(\operatorname{mod}p)$ yields

$$6\alpha_1=2(\alpha_1+1)+3(\alpha_1-1)=5\alpha_1-1(\operatorname{mod}p),\quad \alpha_1\equiv-1(\operatorname{mod}p).$$

Thus, $\alpha_1 \in \{0, 1\}$ and $\alpha_1 \equiv -1(\mathrm{mod}\, p)$, $p \in \mathbb{N}$, $p \geq 2$. Therefore,

$$p = 2,\ q = 1,\ \alpha_1 = 1,\ \rho_0 = 2,$$

$$\rho = \rho_0 + (1 - 2) = 1 = \frac{2(2N_1 + 1 + 2 - \rho_0)}{2N_1},$$

by (4.21). Hence, $2(2N_1 + 1) = 2N_1$ and $N_1 < 0$. The contradiction thus obtained shows that Subcase 1b) is impossible.

Subcase 1c): $\beta_1 = 0$, $\beta_2 = 1$. Then

$$\rho = \frac{2(N_1 p + \alpha_1 - \rho_0)}{N_1 p + \alpha_1 - 2}$$

$$= \rho_0 + (r - 1)\nu = \rho_0 + (r - 1)\frac{N_1 p + \alpha_1}{N_2 p + \alpha_2 + 1}, \tag{4.23}$$

by (4.21) and since in our case $\nu = \frac{m_2}{m_1} = \frac{d_1}{d_2}$, see (4.7). Moving ρ_0 from the right- to the left-hand side, dividing both sides by $N_1 p + \alpha_1$ and multiplying them by the product of denominators in (4.23) yields

$$(2 - \rho_0)(N_2 p + \alpha_2 + 1) = (r - 1)(N_1 p + \alpha_1 - 2).$$

Substituting the value of ρ_0 and multiplying the latter equation by $3q$ yields

$$(4q - 2p)(N_2 p + \alpha_2 + 1) = 3(p - q)(N_1 p + \alpha_1 - 2).$$

Reducing the latter equation modulo p and dividing it by $q(\mathrm{mod}\, p)$ yields

$$4\alpha_2 + 3\alpha_1 \equiv 2(\mathrm{mod}\, p). \tag{4.24}$$

In the case, when $\alpha_1 = 1$, one has $\alpha_2 = 0$ and $3 \equiv 2(\mathrm{mod}\, p)$, which is impossible, since $p \geq 2$. Hence, $\alpha_1 = 0$. In this case $\alpha_2 \in \{0, 1\}$, $4\alpha_2 - 2 = \pm 2 \equiv 0(\mathrm{mod}\, p)$. Hence, $p = 2$, $q = 1$, $\rho_0 = 2$. This together with the first equality in (4.23) implies that either $\rho = 2$, or $N_1 p + \alpha_1 - 2 = 0$. If $\rho = 2$, then $\rho = \rho_0$, which contradicts the second equality in (4.23), Therefore,

$$N_1 p + \alpha_1 - 2 = 2N_1 + \alpha_1 - 2 = 0,\ N_1 = 1,\ \alpha_1 = 0,$$

$$\rho = \rho_0 + (r - 1)\frac{N_1 p + \alpha_1}{N_2 p + \alpha_2 + 1} = 2 + \frac{2}{2N_2 + \alpha_2 + 1} \in \mathcal{M}. \tag{4.25}$$

Case 2): $\theta_\rho = \eta_\rho(\infty) \in \chi(\mathcal{P}_1)$. Then

$$\rho = \rho_0 + 1 + (1 - r)(\beta_1 - \nu\beta_2) = \frac{2(N_1 p + \alpha_1 + \beta_1 r - \rho_0)}{N_1 p + \alpha_1 + \beta_1 - 1},$$

by (4.14) (applied to $\mathcal{P}_1$) and Proposition 4.13. Multiplying by the denominator yields

$$((r-1)\nu\beta_2 - s)(N_1 p + \alpha_1 + \beta_1 - 1) = 2s, \ s := 1 + \beta_1(r-1) - \rho_0. \quad (4.26)$$

Subcase 2a): $\beta_2 = 0$. Then (4.26) yields $(N_1 p + \alpha_1 + \beta_1 + 1)s = 0$, hence $s = 0$ and $\rho_0 = \frac{2}{3}(r+1) = \beta_1(r-1) + 1$. Thus, $\beta_1 = 1$, since $\rho_0 > \frac{4}{3} > 1$,

$$\frac{2}{3}(r+1) = r, \ r = p = 2, \ q = 1, \ \rho = \rho_0 + 1 + 1 - r = \rho_0 = 2.$$

Subcase 2b): $\beta_2 = 1$. Then $\beta_1 = 0$, and (4.26) yields

$$(-s + \nu(r-1))(N_1 p + \alpha_1 - 1) = 2s, \ s = 1 - \rho_0 < 0.$$

The right-hand side of the latter equation is negative, while the first factor in the left-hand side is positive. Therefore, the second factor should be negative, which is obviously impossible. Hence, Subcase 2b) is impossible.

Case 3): $\theta_\rho = \eta_\rho(\infty) \in \chi(\mathcal{P}_2)$. Then

$$\rho = \rho_0 - \nu + (1-r)(\beta_1 - \nu\beta_2) = \frac{2(N_1 p + \alpha_1 + \beta_1 r - \rho_0)}{N_1 p + \alpha_1 + \beta_1 - 2}, \quad (4.27)$$

$$\nu = \frac{d_1}{d_2 + 1} = \frac{N_1 p + \alpha_1 + \beta_1}{N_2 p + \alpha_2 + \beta_2 + 1}, \quad (4.28)$$

by (4.14), Proposition 4.13 and (4.9).

Claim 3. *If $\beta_2 = 0$, then one has $\beta_1 = \alpha_1 = 0$, $p = 2$, $N_1 = 1$,*

$$\nu = \frac{2}{2N_2 + \alpha_2 + 1}, \quad \rho = \rho_0 - \nu = 2 - \frac{2}{2N_2 + \alpha_2 + 1} \in \mathcal{M}. \quad (4.29)$$

Proof Multiplying (4.27) with $\beta_2 = 0$ by its denominator yields

$$(s + 2 - \nu)t = 2t - 2s, \ s := \rho_0 + \beta_1(1-r) - 2, \ t = N_1 p + \alpha_1 + \beta_1 - 2,$$

$$(s - \nu)t = -2s. \quad (4.30)$$

Note that one has always $t \geq 0$.

Case $t > 0$. Then $s - \nu$ and s either have different signs, or both vanish (which is impossible, since $\nu > 0$). Thus, $s - \nu < 0 < s$. But $s = \frac{1}{3}(2(r + 1) + 3\beta_1(1-r) - 6)$. If $\beta_1 = 1$, then the latter expression in the brackets is $2(r+1) - 3r + 3 - 6 = -1 - r < 0$, hence $s < 0$. The contradiction thus obtained shows that $\beta_1 = 0$. Hence, $s = \rho_0 - 2$,

$$(\rho_0 - 2 - \nu)(N_1 p + \alpha_1 - 2) = -2(\rho_0 - 2), \quad \nu = \frac{N_1 p + \alpha_1}{N_2 p + \alpha_2 + 1},$$

$$(\rho_0 - 2)(N_1 p + \alpha_1) = \nu(N_1 p + \alpha_1 - 2).$$

Substituting the above formula for the number ν, multiplying by its denominator and by $3q$ and dividing by $N_1 p + \alpha_1$ yields

$$(2p - 4q)(N_2 p + \alpha_2 + 1) = 3q(N_1 p + \alpha_1 - 2). \tag{4.31}$$

Reducing (4.31) modulo p and dividing by $q(\operatorname{mod} p)$ yields $-4(\alpha_2 + 1) \equiv 3(\alpha_1 - 2)(\operatorname{mod} p)$,

$$2 - 4\alpha_2 - 3\alpha_1 \equiv 0(\operatorname{mod} p).$$

In the case, when $\alpha_1 = 1$, one has $\alpha_2 = 0$. Hence, $-1 \equiv 0(\operatorname{mod} p)$, which is impossible. Thus, $\alpha_1 = 0$. Then $2 - 4\alpha_2 = \pm 2 \equiv 0(\operatorname{mod} p)$. Hence, $p = 2$, $q = 1$, $\rho_0 = 2$, $s = 0$. This together with (4.30) implies that $t = N_1 p + \alpha_1 - 2 = 0$. Hence, $N_1 = 1$, $\alpha_1 = 0$. The first statement of the claim is proved. Together with (4.27) and (4.28), it implies (4.29). □

Claim 4. *In the case, when $\beta_2 = 1$, one has $p = r = \rho = 2$, $q = 1$.*

Proof In this case $\beta_1 = 0$, and (4.27) yields

$$\rho = \rho_0 + \nu(r - 2) = \frac{2(N_1 + \alpha_1 - \rho_0)}{N_1 p + \alpha_1 - 2},$$

$$(s + 2 + \nu(r - 2))t = 2t - 2s, \ s := \rho_0 - 2 = \frac{2}{3}(r - 2), \ t := N_1 p + \alpha_1 - 2,$$

$$(s + \nu(r - 2))t = -2s, \quad (\frac{2}{3} + \nu)(r - 2)t = -\frac{4}{3}(r - 2), \quad t \geq 0, \ \nu > 0.$$

The latter equality implies that $r = 2$. Thus, $p = r = \rho_0 = \rho = 2$, $q = 1$. □

Claims 3, 4 together with the previous discussion imply the statement of Lemma 4.17. □

4.5 Sufficience and integrals. End of proof of Theorem 4.1. Proof of the addendum

Lemma 4.17 reduces Theorem 4.1 to the following lemma.

Lemma 4.18 A) *The following statements are equivalent:*

1) The $(2, 1; \rho)$-billiard is quasihomogeneously integrable.

2) One has $\rho \in \mathcal{M}$.

3) Either the mapping $T := \eta_2 \circ \eta_\rho$ is the identity, or the points ∞ and $\frac{1}{2}$ lie in the same T-orbit, i.e., $T^m(\frac{1}{2}) = \infty$ for some $m \in \mathbb{Z}$.

B) *For every $\rho \in \mathcal{M}$ the corresponding function $R_\rho(z, w)$ from the table in Theorem 4.1 is an integral of the $(2, 1; \rho)$-billiard.*

Proof The implication 1) => 2) is given by Lemma 4.17.

Proof of the equivalence 2) <=> 3). The map T is identity, if and only if $\rho = 2 \in \mathcal{M}$. Let us consider the case, when $\rho \neq 2$. Let us write the map $\eta_\rho : L \to L$, $L = \mathbb{C}_z$, in the chart

$$y := \frac{1}{z-1}; \ \ y(1) = \infty, \ y(\infty) = 0, \ y\left(\frac{1}{2}\right) = -2, \ y(0) = -1.$$

The involution η_ρ fixes 1, $\frac{\rho-2}{\rho}$, and $y(\frac{\rho-2}{\rho}) = -\frac{\rho}{2}$. Therefore, in the chart y

$$\eta_\rho : y \mapsto -y - \rho, \ T = \eta_2 \circ \eta_\rho : y \mapsto y + \rho - 2. \tag{4.32}$$

The condition that $\rho \in \mathcal{M} \setminus \{2\}$ is equivalent to the condition saying that $\rho - 2 \in \{\frac{2}{m} \mid m \in \mathbb{Z} \setminus \{0\}\}$. The latter in its turn is equivalent to the condition that in the chart y the point $y(\infty) = 0$ is the T^m-image of the point $y(\frac{1}{2}) = -2$. This proves equivalence of Statements 2) and 3). $\square$

Proof of the implication 3) => 1). Case $\rho = 0$ was already treated in Subsection 4.1; in this case the polynomial $R_0(z, w) = w - z^2$ is an integral.

Case 1): $\rho = 2$. Then $\eta_\rho(z) = \frac{z}{2z-1}$ fixes 1 and permutes ∞, $\frac{1}{2}$. The restriction to $L = L_{(1,1)}$ of the function $R_2(z, w) = \frac{w-z^2}{w}$ written in the coordinate z is $\frac{(z-1)^2}{z-\frac{1}{2}}$ up to constant factor. It is η_ρ-invariant, by Proposition 4.12 and invariance of its zero and pole divisors: double zero 1 and the pair of simple poles $\frac{1}{2}$, ∞. Hence, R_2 is an integral of the $(2, 1; 2)$-billiard.

Case 2): $\rho = 1$. Then the involution $\eta_\rho(z) = \frac{1}{z}$ fixes 1 and permutes 0, ∞. The restriction to L of the function $R_1(z, w) = \frac{w-z^2}{z}$ is equal to $\frac{(z-1)^2}{z}$ up to constant factor. It is η_ρ-invariant, by Proposition 4.12 and invariance of its zero and pole divisors, and R_1 is an integral, as in the above case.

Case 3): $\rho = 3$. Then $\eta_\rho(z) = \frac{2z-1}{3z-2}$ fixes 1 and permutes 0, $\frac{1}{2}$. The restriction to L of the function $R_3(z, w) = \frac{w-z^2}{zw}$ has double zero 1 and simple poles 0, $\frac{1}{2}$. Hence, it is η_ρ-invariant, and R_3 is an integral, as above.

Case 4): $\rho = 4$. Then $\eta_\rho(z) = \frac{3z-2}{4z-3}$ fixes the points 1 and $\frac{1}{2}$. The latter points taken twice are respectively zero and pole divisors of the function $R_4|_L$. Hence, the latter function is invariant, and R_4 is an integral.

Case 5): $\rho - 2 = \frac{2}{m}$, $m \in \mathbb{Z} \setminus \{0\}$, $|m| \geq 3$. Note that the integer number m has the same sign, as the number $\rho - 2$. Set

$$\zeta_0 = \frac{1}{2}, \zeta_j = T^j(\zeta_0), \ j = 0, \dots, m; \ \ \zeta_m = \infty,$$

$$\chi := \{\zeta_0, \dots, \zeta_{m-1}\}, \text{ if } \rho > 2, \ \text{ i.e., } m > 0,$$

$$\chi := \{\zeta_{m+1}, \dots, \zeta_{-1}\}, \text{ if } \rho < 2 \text{ i.e., } m < 0.$$

Claim 5. *The set χ is a collection $\chi(\mathcal{P})$ of roots of restriction to L of a primitive η_ρ-quasi-invariant $(2,1)$-quasihomogeneous polynomial $\mathcal{P}$. The polynomial $\mathcal{P}$ does not vanish identically on $\gamma = \gamma_{2,1} = \{w = z^2\}$.*

Proof The restriction to L of a prime quasihomogeneous polynomial $w - cz^2$ is $\mathcal{R}_c(z) := -cz^2 + 2z - 1$. The map η_2 permutes roots of the polynomial $\mathcal{R}_c$ for every c, since the sum of inverses of roots is equal to 2 and η_2 acts as $v \mapsto 2 - v$ in the chart $v = \frac{1}{z}$. It permutes ζ_j and ζ_{m-j} for every $j = 0, \dots, m$, since $y(\zeta_j)$ form an arithmetic progression, see (4.32), $y(\zeta_0) = -2$, $y(\zeta_m) = 0$, and $\eta_2 : y \mapsto -y - 2$. Therefore, the numbers ζ_j and ζ_{m-j} are roots of a quadratic polynomial $\mathcal{R}_{c_j}(z)$, unless $\zeta_j \in \{0, \frac{1}{2}, \infty\}$. The middle point -1 of the segment $[-2, 0] \in \mathbb{R}_y$ corresponds to $z = 0$. One has $\zeta_j = 0$ for some j, if and only if $m \in 2\mathbb{Z}$, and then $j = \frac{m}{2}$. Therefore, χ consists of the union of roots of the quadratic polynomials $\mathcal{R}_{c_j}$, $|j| = 1, \dots, [\frac{|m|-1}{2}]$, the point $\frac{1}{2}$ (if $\rho > 2$) and zero (if $m \in 2\mathbb{Z}$). The complement $\chi \setminus \{\frac{1}{2}\}$ is η_2-invariant, by construction. One has $m \geq 0$, if and only if $\rho - 2 \geq 0$. The map $\eta_\rho = \eta_2 \circ T$ sends each $\zeta_j \in \chi$ to ζ_{m-j-1}, since $T(\zeta_j) = \zeta_{j+1}$, by construction. The latter image $\zeta_{m-j-1} = \eta_\rho(\zeta_j)$ lies in χ, except for the case, when

$$\rho < 2, \ m < 0, \ \zeta_{m-j-1} = \zeta_m = \infty, \ j = -1, \ \zeta_{-1} = \eta_\rho(\infty) = \theta_\rho \in \chi.$$

Indeed, if $\rho > 2$, then $\zeta_j \in \chi$ exactly for $j \in [0, m-1]$, and in this case $m - 1 - j$ lies there as well. If $\rho < 2$, then $\zeta_j \in \chi$ exactly for $j \in [m+1, -1]$, and in this case $\zeta_{m-1-j} \in \chi$, unless $j = -1$. Thus, the complement $\chi \setminus \{\theta_\rho\}$ is η_ρ-invariant. Any two points $\zeta_j, \zeta_k \in \chi$ can be obtained one from the other by the map $T^{j-k} = (\eta_2 \circ \eta_\rho)^{j-k}$ so that the latter map considered as a composition of $2|j-k|$ involutions is well-defined at ζ_k and $T^{j-k}(\zeta_k) = \zeta_j$. This follows from the above discussion. Thus, $\chi = \chi(\mathcal{P})$ for some primitive η_ρ-quasi-invariant $(2,1)$-quasihomogeneous polynomial $\mathcal{P}$ that is the product of the polynomials $w - c_j z^2$ and may be some of the monomials z, w. One has $1 \notin \chi$, since $y(1) = \infty$. Hence, $\mathcal{P}|_\gamma \not\equiv 0$. Claim 5 is proved. □

Thus, if Statement 3) of the lemma holds, then there exist at least two distinct primitive quasihomogeneous η_ρ-quasi-invariant polynomials: $w - z^2$ and the above polynomial $\mathcal{P}$. Hence, the $(2,1;\rho)$-billiard admits a quasihomogeneous rational first integral

$$R = \frac{(w - z^2)^{m_1}}{(\mathcal{P}(z,w))^{m_2}},$$

by Proposition 4.11. Implication 3) => 1) is proved. □

Equivalence of Statements 1)–3) is proved. Now for the proof of Lemma 4.18 it remains to calculate the integrals. To do this, let us calculate the c_j from the proof of Claim 5 for $j \neq \frac{m}{2}$. One has

$$\rho - 2 = \frac{2}{m}, \ y_j := y(\zeta_j) = -2 + j(\rho - 2) = -2 + \frac{2j}{m}, \ \zeta_j = \frac{1}{y_j} + 1 = \frac{2j - m}{2(j - m)},$$

$$c_j = (\zeta_j \zeta_{m-j})^{-1} = -\frac{4j(m-j)}{(2j-m)^2}, \ m = \frac{2}{\rho - 2} \in \mathbb{Z} \setminus \{0\}. \tag{4.33}$$

Therefore, in the case, when $\rho > 2$, the polynomial $\mathcal{P}(z, w)$ is the product of the quadratic polynomials $w - c_j z^2$, $j = 1, \dots, [\frac{m-1}{2}]$, the polynomial w, and also the polynomial z (which enters $\mathcal{P}$ if and only if $m \in 2\mathbb{Z}$). In the case, when $\rho < 2$, the polynomial $\mathcal{P}(z, w)$ is the product of the polynomials $w - c_j z^2$, $j = 1, \dots, [-\frac{m+1}{2}]$, and also the polynomial z (if $m \in 2\mathbb{Z}$).

Subcase 5a): $\rho = 2 + \frac{2}{m}$, $m = 2N + 1$, $N \in \mathbb{N}$. Then $\mathcal{P}$ is the product of the polynomial w and N above polynomials $w - c_j z^2$. Its degree is equal to $2N + 1 = m$. The divisor $\chi(\mathcal{P})$ contains $\frac{1}{2}$ and does not contain $\eta_\rho(\infty)$, by construction and the above discussion. Substituting $m = 2N + 1$ to formula (4.33) yields the formula for the coefficients c_j given by the table in Theorem 4.1. This together with Proposition 4.11 implies that the corresponding function R_ρ from the same table is an integral of the $(2, 1; \rho)$-billiard.

Subcase 5b): $\rho = 2 - \frac{2}{2N+1}$. It is treated analogously. In this case $\mathcal{P}$ is just the product of the above N polynomials $w - c_j z^2$. The divisor $\chi(\mathcal{P})$ contains $\theta_\rho = \eta_\rho(\infty)$. This together with Proposition 4.11 implies that the corresponding function R_ρ from the table is an integral.

Subcase 5c): $\rho = 2 + \frac{1}{N+1}$, $m = 2N + 2$. Treated analogously to Subcase 5a). But now the polynomial $\mathcal{P}$ contains the additional factor z, and substituting $m = 2N + 2$ to (4.33) yields the formula for c_j from the corresponding line of the table in Theorem 4.1.

Subcase 5d): $\rho = 2 - \frac{1}{N+1}$. Treated analogously. Lemma 4.18 is proved. The proof of Theorem 4.1 is complete. □

Proof of the addendum to Theorem 4.1. The equation for the curve $\gamma = \{w = z^2\}$ in the new coordinates $(\widetilde{z}, \widetilde{w})$ is the same: $\widetilde{w} = \widetilde{z}^2$. The variable change $(z, w) \mapsto (\widetilde{z}, \widetilde{w})$ preserves the point $(1, 1)$, and hence, the corresponding tangent line $L = L_{(1,1)}$ to γ, along which one has

$$w = 2z - 1, \quad \widetilde{z} = \frac{z}{2z - 1} = \eta_2(z).$$

Therefore, in the coordinate $\widetilde{z}$ the involution η_ρ takes the form

$$\eta_2 \circ \eta_\rho \circ \eta_2 = \eta_{\widetilde{\rho}}, \quad \widetilde{\rho} := 4 - \rho; \tag{4.34}$$

the latter formula follows from (4.32). Thus, the variable change in question transforms the $(2,1;\rho)$-billiard to the $(2,1;4-\rho)$-billiard.

In new coordinates one has $R_\rho(z,w)=R_{4-\rho}(\widetilde{z},\widetilde{w})$. Indeed,

$$R_0(z,w)=w-z^2=\frac{1}{\widetilde{w}^2}(\widetilde{w}-\widetilde{z}^2)=R_4(\widetilde{z},\widetilde{w}),$$

$$R_1(z,w)=\frac{1}{\widetilde{w}\widetilde{z}}(\widetilde{w}-\widetilde{z}^2)=R_3(\widetilde{z},\widetilde{w}),\ R_2(z,w)=R_2(\widetilde{z},\widetilde{w}).$$

For the other integrals R_ρ from the table in Theorem 4.1 the proof is analogous. The addendum is proved. □

5 Local branches. Proof of Theorem 1.37

The main result of this section is the following theorem.

Theorem 5.1 *Let a non-linear irreducible germ of analytic curve $b\subset\mathbb{CP}^2$ at a point O admit a germ of singular holomorphic dual billiard structure with a meromophic integral R. Then the germ b is quadratic and one of the two following statements holds:*

a) either b is regular;

b) or b is singular, the integral $R(z,w)$ is a rational function that is constant along the projective tangent line L_O to b at O, and the punctured line $L_O\setminus\{O\}$ is a regular leaf of the foliation $R=const$ on $\mathbb{CP}^2$.

Theorem 5.1 will be proved in Subsection 5.2. Theorem 1.37 will be deduced from it in Subsection 5.3. Quadraticity of germ in Theorem 5.1 follows from Theorems 3.3 and 4.1. The proof of Statement b) of Theorem 5.1 for a singular germ is based on Theorem 5.6 (stated and proved in Subsection 5.1), which yields a formula for residue of a meromorphically integrable singular dual billiard on a singular quadratic germ in terms of its self-contact order.

5.1 Meromorphically integrable dual billiard structure on singular germ: formula for residue

Recall that each non-linear irreducible germ b of analytic curve at $O\in\mathbb{C}^2$ in adapted coordinates centered at O admits an injective holomorphic parametrization

$$t\mapsto(t^{qs},\phi(t)),\quad \phi(t)=ct^{ps}(1+O(t)),\ 1\leq q<p,\ s,q,p\in\mathbb{N},\qquad(5.1)$$

$c\in\mathbb{C}\setminus\{0\};\quad (q,p)=1,\quad r=\dfrac{p}{q}$ is the projective Puiseux exponent.

Definition 5.2 A non-linear irreducible germ b will be called *primitive*, if $s = 1$ in (5.1) (following a suggestion of E.Shustin).

Remark 5.3 A quadratic germ is primitive, if and only if it is regular. It is well-known that if $s \geq 2$, then the function $\phi(t)$ in (5.1) satisfies one of the two following statements:

a) either there exist a $\delta \in \mathbb{Q}_{>0}$ and $\nu_1, \dots, \nu_{s-1} \in \mathbb{C} \setminus \{0\}$, such that

$$\phi(te^{\frac{2\pi ij}{s}}) - \phi(t) = \nu_j t^{qs(r+\delta)}(1 + o(1)), \text{ for } j = 1, \dots, s-1, \text{ as } t \to 0; \quad (5.2)$$

b) or there exit at least two distinct $\delta_1, \delta_2 \in \mathbb{Q}_{>0}$ and distinct $j_1, j_2 \in \{1, \dots, s-1\}$ for which (5.2) holds with δ replaced by δ_1 and δ_2 respectively.

Definition 5.4 If b is not primitive and (5.2) holds for a unique δ, then b will be called *uniformly (δ-) folded.*

Remark 5.5 This definition is equivalent to the well-known definition of a germ having *two Puiseux pairs.*

Theorem 5.6 *Let an irreducible quadratic germ b of analytic curve at $O \in \mathbb{C}^2$ admit a structure of meromorphically integrable singular dual billiard with residue ρ at O. Then b is either regular, or uniformly δ-folded with δ related to ρ by the formula*

$$\rho = 2 + \delta. \quad (5.3)$$

Proof Below we prove a more general theorem. To state it, let us introduce the following definition.

Definition 5.7 Let a and b be two irreducible germs of analytic curves at $O \in \mathbb{C}^2$ tangent to each other. Let b be non-linear; let $r = r_b$ be its projective Puiseux exponent. Let $\delta \in \mathbb{Q}_{>0}$. We say that a is a *δ-satellite for b,* if a, b are graphs of two multivalued functions $\{w = g_a(z)\}$, $\{w = g_b(z)\}$ (represented by Puiseux series in z) satisfying the following statement: there exist a sector S with vertex at 0, a $c \in \mathbb{C} \setminus \{0\}$, and holomorphic branches of the functions $g_a(z)$, $g_b(z)$ over S (near 0) for which

$$g_a(z) - g_b(z) = cz^{r+\delta}(1 + o(1)), \text{ as } z \to 0, \ z \in S. \quad (5.4)$$

Remark 5.8 Any two satellites have the same Puiseux exponent. A uniformly δ-folded germ b is a δ-satellite for itself.

Remark 5.9 If a and b are δ-satellites, then the above sector S can be chosen with angle arbitrarily large and containing an arbitrary given ray. A priori it may happen that a and b are δ_1- and δ_2-satellites with different $\delta_1, \delta_2 > 0$ corresponding to two different pairs of holomorphic branches. (This holds, e.g., for $a = b$, if b is neither primitive, nor uniformly folded.) Two germs that are δ-satellites for a unique $\delta > 0$ are called *pure δ-satellites.*

Theorem 5.10 *Let an irreducible quadratic germ b at $O \in \mathbb{C}^2$ admit a structure of meromorphically integrable singular dual billiard. Let the corresponding involution family have residue ρ at O. Let a be an irreducible germ with the same base point O that lies in a level curve of the meromorphic integral. Let a be a δ-satellite of the germ b. Then they are pure δ-satellites, and the corresponding number δ is given by formula (5.3).*

Proof Let S, g_a and g_b be the same, as in (5.4). Fix a smaller sector S', $\overline{S'} \setminus \{0\} \subset S$. The graphs of the functions g_a, g_b over the sector S will be denoted by Γ_a, Γ_b respectively. Fix a $z_0 \in S'$, set $P = (z_0, g_b(z_0)) \in \Gamma_b \subset b$. Let L_P denote the line tangent to b at P. We introduce the coordinate

$$u := \zeta - 1 = \frac{z}{z_0} - 1$$

on the tangent line L_P. Let us find asymptotics of those u-coordinates of points of the intersection $\Gamma_a \cap L_P$, that tend to zero, as $z_0 \in S'$ tends to 0.

Proposition 5.11 *Let a and b be non-linear irreducible δ-satellite germs at a point $O \in \mathbb{C}^2$ with Puiseux exponent r. Let S, g_a, g_b be the same, as in (5.4). Let S' and $\Gamma_{a,b}$ be as above. As $P = (z_0, g_b(z_0)) \to O$, $z_0 \in S'$, the intersection $\Gamma_a \cap L_P$ contains exactly two points whose u-coordinates converge to zero. Their u-coordinates $u_\pm$ are related by the asymptotic formula*

$$u_- = -u_+ + (\rho_0 + \delta)u_+^2 + o(u_+^2), \quad \rho_0 = \frac{2}{3}(r+1). \tag{5.5}$$

Proof Without loss of generality we consider that $g_b(z) \simeq z^r(1 + o(1))$, as $z \to 0$, rescaling the coordinate w. Let us work in the further rescaled coordinates (ζ, y), $(z, w) = (z_0\zeta, z_0^r y)$, in which $\Gamma_{a,b}$ are graphs of functions $h_{a,b}(\zeta)$, $h_{a,b}(\zeta)$ converging to ζ^r together with derivatives uniformly on compact subsets in S, as $z_0 \to 0$. In the new coordinates

$$\zeta(P) = 1, \ h_a(\zeta) - h_b(\zeta) = cz_0^\delta(1+u)^{r+\delta}(1 + \theta(z_0, u)),$$

$\theta(z_0, u) \to 0$, as $z_0 \to 0$, uniformly with derivatives in u lying in a disk centered at 0, by (5.4). Hence,

$$\theta(z_0, u) = \chi(z_0) + o(u), \ \chi(z_0) \to 0, \text{ as } z_0 \to 0.$$

The u-coordinates of points of the intersection $\Gamma_a \cap L_P$ are found from the equation

$$h_b(1) + h_b'(1)u = h_a(1+u) = h_b(1+u) + cz_0^\delta(1+u)^{r+\delta}(1 + \chi(z_0) + o(u)).$$

Moving $h_b(1+u)$ to the left-hand side and expressing the new left-hand side by Taylor formula with base point 1 we get:

$$-\frac{1}{2}h_b''(1)u^2 - \frac{1}{6}h_b'''(1)u^3 + o(u^3) = cz_0^\delta(1+u)^{1+\delta}(1 + \chi(z_0) + o(u)). \quad (5.6)$$

Since $h_b(\zeta) \to \zeta^r$, we get $h_b''(1) = r(r-1)(1+\phi(z_0))$, $\phi(z_0) \to 0$, $h_b'''(1) \to r(r-1)(r-2)$, as $z_0 \to 0$. Substituting these expressions to (5.6) yields

$$-u^2(1+\frac{r-2}{3}u+o(u)+\phi(z_0)) = \frac{2c}{r(r-1)}z_0^\delta(1+u)^{r+\delta}(1+\chi(z_0)+o(u)), \quad (5.7)$$

$\phi(z_0), \chi(z_0) \to 0$. Equation (5.7) has exactly two solutions $u_\pm$ that tend to zero, as $z_0 \to 0$: they have asymptotics

$$u_+ \simeq -u_- \simeq \sqrt{-\frac{2c}{r(r-1)}}z_0^{\frac{\delta}{2}}(1 + o(1)). \quad (5.8)$$

Let us now prove (5.5). Dividing equations (5.7) written for $u+$ and u_- and taking into account that $u_+ \simeq -u_-$, we get

$$\left(\frac{u_+}{u_-}\right)^2 \frac{1 + \phi(z_0) + \frac{r-2}{3}u_+ + o(u_+)}{1 + \phi(z_0) - \frac{r-2}{3}u_+ + o(u_+)} = \left(\frac{1+u_+}{1-u_+}\right)^{r+\delta}(1 + \chi(z_0) + o(u_+)),$$

$$\frac{u_+}{u_-} = -(1 + (r + \delta - \frac{r-2}{3})u_+ + o(u_+) = -(1 + (\rho_0 + \delta)u_+ + o(u_+)).$$

The latter formula implies (5.5). Proposition 5.11 is proved. □

Let now b be a singular quadratic germ equipped with a meromorphically integrable singular dual billiard structure. Fix two graphs $\Gamma_a \subset a$, $\Gamma_b \subset b$ satisfying (5.4) over S with some $\delta > 0$. For the proof of Theorem 5.10 it suffices to show that $\rho = 2 + \delta$. Suppose the contrary:

$$\Theta := \rho - 2 - \delta \neq 0. \quad (5.9)$$

The curve a should lie in a level curve α of the meromorphic integral, which is a one-dimensional analytic subset in a neighborhood of the origin O and hence, has finite intersection index with the tangent line L_O at O. Therefore, only finite and uniformly bounded number of points of intersection $\alpha \cap L_P$ converge to O, as $P \to O$, i.e., as $z_0 \to 0$. Using the next proposition, we show that for every $N \in \mathbb{N}$ and z_0 small enough (dependently on N) there are at least N above intersection points that converge to O. The contradiction thus obtained will prove Theorem 5.10.

We use the following characterization of satellite germs.

Proposition 5.12 *Let a, b be two irreducible germs of holomorphic curves at $O \in \mathbb{C}^2$. Let b be non-linear, $r = \frac{p}{q}$ be its Puiseux exponent, $(p,q) = 1$.*

1) The germs a, b are satellites, if and only the germ a has the same Puiseux exponent r, and the lower (p,q)-quasihomogeneous parts of their defining functions are powers of one and the same prime (p,q)-quasihomogeneous polynomial $w^q - cz^p$.

2) Let a and b be quadratic germs. They are satellites if and only if all the points of intersection $a \cap L_P$ have ζ-coordinates, $\zeta = \frac{z}{z(P)}$, that tend to one. This holds if and only if **some** *point of the above intersection has ζ-coordinate that tends to one.*

Proof Clearly the germs a, b cannot be satellites, if they have different Puiseux exponents. Let f_a, f_b be the functions defining a and b, and let $\widetilde{f}_a$, $\widetilde{f}_b$ be their lower (p,q)-quasihomogeneous parts. Then up to constant factor, $\widetilde{f}_g(z,w) = (w^q - C_g z^p)^{s_g}$, $g = a, b$, $s_g \in \mathbb{N}$, $C_g \in \mathbb{C} \setminus \{0\}$, see (3.3). Without loss of generality we can and will consider that $C_b = 1$, rescaling w. The curves b and a are parametrized respectively by $t \mapsto (t^{qs_b}, t^{ps_b}(1+o(1)))$ and $\tau \mapsto (\tau^{qs_a}, c_a \tau^{ps_a}(1+o(1)))$, $c_a^q = C_a$, see the discussion in Example 3.5. Therefore, they are satellites, if and only if $C_a = 1$. This proves Statement 1). The equality $C_a = 1$ is equivalent to the statement that the restrictions to the line $L = L_{(1,1)}$ (tangent to the curve $\{y = \zeta^2\}$ at $(1,1)$) of the quasihomogeneous polynomials $\widetilde{f}_a(\zeta, y)$ and $\widetilde{f}_b(\zeta, y)$ have the same roots. The latter roots are exactly the finite limits of the ζ-coordinates of points of the intersections $a \cap L_P$ and $b \cap L_P$ respectively (Proposition 3.6). In the case of quadratic germs the polynomial $\widetilde{f}_b|_L = -1 + 2\zeta - \zeta^2 = -(1-\zeta)^2$ has just one, double root 1. If $\widetilde{f}_a \neq \widetilde{f}_b$, i.e., $C_a \neq 1$, then the polynomial $\widetilde{f}_a|_L = -1 + 2\zeta - C_a \zeta^2$ does not vanish at 1. This together with Statement 1) and the above discussion proves Statement 2). □

Recall that in the chart ζ the dual billiard involution σ_P converges to η_ρ. Therefore, in the chart u it converges to the involution $u \mapsto -\frac{u}{1+\rho u}$, see

(4.20). Hence, the germ of the involution σ_P at $u = 0$ acts as

$$\sigma_P : u \mapsto -u + (\rho + \phi(z_0))u^2 + \dots, \ \phi(z_0) \to 0, \text{ as } z_0 \to 0. \tag{5.10}$$

The intersection points from Proposition 5.11 with u-coordinates $u_\pm$ will be denoted by $A_{0\pm}$. The point $A_{1+} = \sigma_P(A_{0-}) \in L_P$ should lie in the same level curve α of the integral, as A_{0-}. Therefore, it lies in some irreducible germ a_1 of holomorphic curve at O, since the germ of α is analytic. One has

$$u_{1+} := u(A_{1+}) = -u_- + (\rho + o(1))u_-^2 = u_+ + \Theta u_+^2 + o((u_+)^2), \tag{5.11}$$

by (5.5), (5.9), (5.10), and since $r = \rho_0 = 2$. In particular, $\zeta(A_{1+}) = 1 + u_{1+} \to 1$, as $z_0 \to 1$, and $u_{1+} \simeq u_+$. Hence, u_{1+} has asymptotics (5.8), and a_1 is a δ-satellite of the germ b (Proposition 5.12 and (5.8)). Therefore, a_1 intersects L_P at another point A_{1-} with

$$u_{1-} := u(A_{1-}) = -u_{1+} + (2 + \delta)u_{1+}^2,$$

by (5.5). The point $A_{2+} := \sigma_P(A_{1-})$ also lies in the intersection $\alpha \cap L_P$, and

$$u_{2+} := u(A_{2+}) = -u_{1-} + (\rho + o(1))u_{1-}^2$$
$$= u_{1+}(1 + \Theta u_{1+} + o(u_{1+})) = u_+ + 2\Theta u_+^2 + o(u_+^2).$$

Repeating this procedure we get a sequence of distinct points $A_{k+} \in \alpha \cap L_P$ with coordinates asymptotic to $u_+ + k\Theta u_+^2 + o(u_+^2)$, $k \in \mathbb{N}$. Passing to limit we get that the level curve α, which is a one-dimensional analytic subset in a neighborhood of the point O, has infinite intersection index with the tangent line L_O at O. This is obviously impossible. The contradiction thus obtained proves Theorem 5.10. □

Theorem 5.10 together with Remarks 5.8, 5.9 imply the statement of Theorem 5.6. □

5.2 Singular quadratic germs. Proof of Theorem 5.1

In the proof of Theorem 5.1 we use Theorem 5.6 and the following proposition. To state it, let us recall the following definition.

Definition 5.13 [28, definition 3.3] Let $L \subset \mathbb{CP}^2$ be a line, and let $O \in L$. A (L, O)-*local multigerm (divisor)* is respectively a finite union (linear combination $\sum_j k_j b_j$ with $k_j \in \mathbb{R} \setminus \{0\}$) of distinct irreducible germs of analytic curves b_j (called *components*) at base points $B_j \in L$ such that each germ at $B_j \neq O$ is different from the line L. (A germ at O can be arbitrary, in particular, it may coincide with the line germ (L, O).) The (L, O)-*localization* of an algebraic curve (divisor) in $\mathbb{CP}^2$ is the (L, O)-local multigerm (divisor) formed by all its local branches b_j of the above type.

Proposition 5.14 *Let b be an irreducible germ of holomorphic curve at $O \in \mathbb{C}^2$, and let (z, w) be coordinates adapted to b in which the corresponding constant C_b from (3.3) is equal to one. Let L_O be the projective tangent line to b at O. Let Γ be a (L_O, O)-local multigerm. Let $\sigma_P : L_P \to L_P$ be a family of projective involutions, $P \in b \setminus \{O\}$, such that the intersections $\Gamma \cap L_P$ are σ_P-invariant for all $P \in b$ close enough to O. Let σ_P converge to η_ρ in the coordinate $\zeta := \frac{z}{z(P)}$ on L_P, as $P \to O$. Let the corresponding number ρ be greater than the projective Puiseux exponent $r = r_b$ of the germ b. Then all the germs in Γ are based at the point O, and the ζ-coordinate of each point of the intersection $\Gamma \cap L_P$ has a finite limit, as $P \to O$.*

Proof The proof of Proposition 5.14 is analogous to the proof of theorem 4.24 in [31, p. 1037]. The intersection points with those germs in Γ that are based at points different from O (if any) have ζ-coordinates that tend to infinity, since their z-coordinates tend to either infinity, or non-zero finite limits, as $z(P) \to 0$. Suppose the contrary to the statement of the proposition: the ζ-coordinate of some point of the intersection $\Gamma \cap L_P$ tends to infinity. Its σ_P-image also lies in $\Gamma \cap L_P$, by invariance, and has ζ-coordinate converging to $\theta_\rho := \frac{\rho-1}{\rho} = \eta_\rho(\infty)$, since $\sigma_P(\zeta) \to \eta_\rho(\zeta)$. This implies that there exists a germ $b_1 \subset \Gamma$ based at O whose intersection point with L_P has ζ-coordinate converging to θ_ρ. One has $\theta_\rho \in (\theta_r, 1)$, $\theta_r = \frac{r-1}{r}$, since $\rho > r$. Therefore, $\zeta_{1-} := \theta_\rho$ is a root of a polynomial $\mathcal{R}_{p,q,C_{b_1}} = (1-r+r\zeta)^q - C_{b_1}\zeta^p$, by Proposition 3.6. Hence, $0 < C_{b_1} < 1$, and the same polynomial $\mathcal{R}_{p,q,C_{b_1}}$ has a unique root $\zeta_{1+} \in (1, +\infty)$, due to the following proposition.

Proposition 5.15 *Let $p, q \in \mathbb{N}$, $1 \leq q < p$, $r = \frac{p}{q}$. The following statements are equivalent:*

1) The polynomial $\mathcal{R}_{p,q,C}$ has a real root in the interval $(\theta_r, 1)$.

2) It has a real root greater than 1.

3) $0 < C < 1$.

In this case the above roots are unique, and the correspondence between them for all $C \in (0, 1)$ is a decreasing homeomorphism $(\theta_r, 1) \to (1, +\infty)$.

Proof The complex roots of the polynomial $\mathcal{R}_{p,q,C}(\zeta)$ are q-th powers of roots of a polynomial

$$H_{p,q,c}(\theta) = c\theta^p - r\theta^q + r - 1, \quad c^q = C,$$

since $\mathcal{R}_{p,q,c}(\theta^q) = \prod_{j=0}^{q-1}((1-r+r\theta^q) - ce^{\frac{2\pi j}{q}}\theta^p)$. The statement of Proposition 5.15 for q-th powers of roots of the polynomial $H_{p,q,c}$ is given by [31, proposition 4.25], and it implies Proposition 5.15. $\Box$

The root ζ_{1+} is the limit of the ζ-coordinate of some point of intersection $b_1 \cap L_P$ (Proposition 3.6). Hence, its η_ρ-image, which will be denoted by ζ_{2-}, is the limit of the ζ-coordinate of an intersection point of the line L_P with a germ $b_2 \subset \Gamma$ based at O. One has $\theta_r < \zeta_{1-} < \zeta_{2-} < 1$, by monotonicity of the map $\eta_\rho|_{\mathbb{R}}$. Again ζ_{2-} is a root of a polynomial $\mathcal{R}_{p,q,C_{b_2}}$, $C_{b_2} > 1$, and the latter polynomial has another root $\zeta_{2+} \in (1, \zeta_{1+})$, as in [31, proof of theorem 4.24]. Continuing this procedure we get an infinite decreasing sequence of roots ζ_{j+}, all of them being limits of ζ-coordinates of points of the intersection $\Gamma \cap L_P$. Hence, the cardinality of the latter intersection is unbounded, as $P \to O$, while the intersection index of the multigerm Γ with L_O is finite. The contradiction thus obtained proves the proposition. $\Box$

Proof of Theorem 5.1. Quadraticity of the germ b follows from Theorems 3.3 and 4.1. If b is regular, then there is nothing to prove. Let b be singular. Let ρ denote the residue at O of the dual billiard structure. Then b is uniformly δ-folded for some $\delta > 0$, and $\rho = 2 + \delta > 2$, by Theorem 5.6. Therefore, the meromorphic integral R is rational and b lies in an algebraic curve, by Proposition 2.14.

Suppose the contrary to the constance statement: $R \not\equiv const$ along the line L_O tangent to b at O, i.e.,, the z-axis. Fix a point $A \in L_O \setminus \{O\}$ with finite z-coordinate $z_1 = z(A)$ that is not an indeterminacy point for the integral R. For every $P \in b \setminus \{O\}$ the intersection of the line L_P with the level curve $\Gamma := \{R = R(A)\}$ is σ_P-invariant. The (L_O, O)-localization of the algebraic curve Γ is a (L_O, O)-local multigerm satisfying the conditions of Proposition 5.14, by construction. Hence, all its curves are based at one point O, by Proposition 5.14. On the other hand, it contains a germ of analytic curve based at the point A, by construction. The contradiction thus obtained proves that $R|_{L_O} \equiv const$.

Suppose now the contrary to the last statement of Theorem 5.1: the punctured line $L_O \setminus \{O\}$ contains a singular point A for the foliation $R = const$. Without loss of generality we consider that $R|_{L_O} \equiv 0$. Let us consider the germ of the integral R at A and write it as the product $w^k f(z,w)$ with $k \in \mathbb{N}$; $f(z,w)$ being a germ of meromorphic function with $f(z,0) \not\equiv 0, \infty$. The point A is singular for the foliation, if and only if at least one of the two following statements holds: either A is an indeterminacy point for the function f, or A is its pole (zero). In both cases at least one of the level curves $\{R = 0\}$ or $\{R = \infty\}$ contains a local branch a based at A that does not lie in the z-axis L_O. Let us denote the latter level curve by Γ. Its (L_O, O)-localization is a multigerm satisfying the conditions, and hence, the statement of Proposition 5.14. Therefore, it consists of germs of curves based at the unique point O, while, by assumption, some of its germs has base point $A \neq O$. The contradiction thus obtained proves that $L_O \setminus \{O\}$ is a regular leaf of the foliation $R = const$ and proves Theorem 5.1. $\square$

5.3 Uniqueness of singular point with singular branch. Proof of Theorem 1.37

Here we prove Theorem 1.37. Quadraticity of local branches is already proved (Theorem 5.1). Let us prove uniqueness of point $O \in \gamma$ at which some local branch of the curve γ is singular. Suppose the contrary: there exist at least two distinct points $O_1, O_2 \in \gamma$ with singular local branches b_1 and b_2 respectively. Let L^1, L^2 denote their projective tangent lines at O_1, O_2. The rational integral is constant along both lines L^1 and L^2, by Theorem 5.1. One has $L^1 \neq L^2$. Indeed, if $L^1 = L^2$, then the punctured line $L^1 \setminus \{O_1\}$ would contain a singular point O_2 of foliation by level curves of the integral, which is forbidden by Theorem 5.1. Thus, L^1 and L^2 intersect at some point A distinct from some of the points O_j, say, O_1. But then the punctured line $L^1 \setminus \{O_1\}$ contains a singular point A of foliation by level curves, – a contradiction to Theorem 5.1. Theorem 1.37 is proved.

6 Plane curve invariants. Proof of Theorem 1.38

Here we prove Theorem 1.38 stating that every irreducible algebraic curve $\gamma \subset \mathbb{CP}^2$ satisfying the statements of Theorem 1.37 is a conic. The proof given in Subsection 6.2 is based on Bézout Theorem applied to the intersection of the curve γ with its Hessian curve and Shustin's formula [42] for Hessians of singular points. The corresponding background material is recalled in Subsection 6.1.

6.1 Invariants of plane curve singularities

Hereby we recall the material from [16, Chapter III], [40, §10], [42], see also a modern exposition in [33, Section I.3]. This material in a brief form needed here is presented in [28, subsection 4.1].

Let $\gamma \subset \mathbb{CP}^2$ be a non-linear irreducible algebraic curve. Let d denote its degree. Let H_γ denote its Hessian curve: the zero locus of the Hessian determinant of the defining homogeneous polynomial of γ. It is an algebraic curve of degree $3(d-2)$. The set of all singular and inflection points of the curve γ coincides with the intersection $\gamma \cap H_\gamma$. The intersection index of these curves is equal to $3d(d-2)$, by Bézout Theorem. On the other hand, it is equal to the sum of the contributions $h(\gamma, Q)$, which are called the *Hessians of the germs* (γ, Q), through all the singular and inflection points Q of the curve γ:

$$3d(d-2) = \sum_{Q \in \gamma} h(\gamma, Q). \tag{6.1}$$

Let us recall an explicit formula for the Hessians $h(\gamma, Q)$ [42, formula (2) and theorem 1]. To do this, let us introduce the following notations. For every local branch b of the curve γ at Q let $s(b)$ denote its multiplicity: its intersection index with a generic line through Q. Let $s^*(b)$ denote the analogous multiplicity of the dual germ. Note that $s(b) = q_b$, $s^*(b) = p_b - q_b$, where p_b and q_b are the exponents in the parametrization $t \mapsto (t^{q_b}, c_b t^{p_b}(1 + o(1)))$ of the local branch b in adapted coordinates. One has

$$s(b) = s^*(b) = q_b \text{ for every quadratic branch } b. \tag{6.2}$$

Let $b_{Q1}, \dots, b_{Qn(Q)}$ denote the local branches of the curve γ at Q; here $n(Q)$ denotes their number. The above-mentioned formula for $h(\gamma, Q)$ from [42] has the form

$$h(\gamma, Q) = 3\kappa(\gamma, Q) + \sum_{j=1}^{n(Q)} (s^*(b_{Qj}) - s(b_{Qj})), \tag{6.3}$$

where $\kappa(\gamma, Q)$ is the κ-invariant, the class of the singular point. Namely, consider the germ of function f defining the germ (γ, Q); $(\gamma, Q) = \{f = 0\}$. Fix a line L through Q that is transversal to all the local branches of the curve γ at Q. Fix a small ball $U = U(Q)$ centered at Q and consider a level curve $\gamma_\varepsilon = \{f = \varepsilon\} \cap U$ with small $\varepsilon \neq 0$, which is non-singular. The number $\kappa(Q) = \kappa(\gamma, Q)$ is the number of points of the curve γ_ε where its tangent line is parallel to L. It is well-known that

$$\kappa(\gamma, Q) = 2\delta(\gamma, Q) + \sum_{j=1}^{n(Q)} (s(b_{Qj}) - 1), \tag{6.4}$$

see, for example, [33, propositions I.3.35 and I.3.38], where $\delta(\gamma, Q) = \delta(Q)$ is the δ-invariant. Namely, consider the curve γ_ε, which is a Riemann surface whose boundary is a finite collection of closed curves: their number equals to $n(Q)$. Let us take the 2-sphere with $n(Q)$ deleted disks. Let us paste it to γ_ε:

this yields to a compact surface. By definition, its genus is the δ-invariant $\delta(Q)$. One has $\delta(Q)\geq 0$, and $\delta(Q)=0$ whenever Q is a non-singular point. Hironaka's genus formula [34] implies that

$$\sum_{Q\in\mathrm{Sing}(\gamma)}\delta(\gamma,Q)\leq\frac{(d-1)(d-2)}{2}. \tag{6.5}$$

Formulas (6.1), (6.3) and (6.4) together imply the formula

$$3d(d-2)=6\sum_{Q}\delta(\gamma,Q)+3\sum_{Q}\sum_{j=1}^{n(Q)}(s(b_{Qj})-1) \tag{6.6}$$

$$+\sum_{Q}\sum_{j=1}^{n(Q)}(s^*(b_{Qj})-s(b_{Qj})).$$

6.2 Proof of Theorem 1.38

All the local branches of the curve γ are quadratic. All of them are regular, except maybe for some branches at a unique singular point O (if any). Therefore, the third sum in the right-hand side in (6.6) vanishes. All the terms in the second sum vanish except for those corresponding to the singular branches based at the point O. The first sum is no greater than $\frac{(d-1)(d-2)}{2}$, by (6.5). Therefore,

$$3d(d-2)\leq 3(d-1)(d-2)+\sum_{j=1}^{n(O)}(s(b_{Oj})-1). \tag{6.7}$$

If all the local branches at O are regular, then the latter sum vanishes, and we get $3d(d-2)\leq 3(d-1)(d-2)$, hence $d=2$. Let now there exist at least one singular branch, say b_{Ot}: $s(b_{Ot})\geq 2$. The intersection index of the curve γ with a line through O tangent to b_{Ot} is no less than $2s(b_{Ot})+\sum_{j\neq t}s(b_{Oj})$. The latter intersection index should be no greater than d, by Bézout Theorem. Therefore,

$$2s(b_{Ot})+\sum_{j\neq t}s(b_{Oj})\leq d,\ \sum_{j=1}^{n(O)}(s(b_{Oj})-1)<d-2,$$

$$3d(d-2)<3(d-1)(d-2)+d-2=3d(d-2).$$

The contradiction thus obtained proves Theorem 1.38.

7 Classification of complex rationally integrable dual billiards. Proof of Theorem 1.18

Let $\gamma \subset \mathbb{CP}^2$ be a non-linear irreducible algebraic curve equipped with a rationally integrable singular dual billiard structure. The curve γ is a conic, by Theorems 1.37 and 1.38. Thus, for the proof of Theorem 1.18 it suffices to classify rationally integrable singular dual billiard structures on the conic

$$\gamma = \{wt = z^2\} \subset \mathbb{CP}^2_{[z:w:t]}.$$

To do this, we first classify the a priori possible residue configurations of the corresponding involution family. In Subsection 7.1 we show that the billiard structure in question may have at most four singularities, the corresponding residues lie in $\mathcal{M} \setminus \{0\}$ and their sum is equal to 4. This implies that the a priori possible residue configurations are 4, $(1,1,1,1)$, $(2,1,1)$, $(\rho, 4-\rho)$ with $\rho \in \mathcal{M} \setminus \{0\}$, $(\frac{4}{3}, \frac{4}{3}, \frac{4}{3})$, $(\frac{3}{2}, \frac{3}{2}, 1)$, $(\frac{4}{3}, \frac{5}{3}, 1)$. We prove that each residue configuration is realized by a rationally integrable dual billiard, and we find the corresponding integrals. We show that the cases of integer residues correspond to the dual billiard structures of conical pencil type.

7.1 Residues of singular dual billiard structures on conic

Proposition 7.1 *Let $\gamma \subset \mathbb{CP}^2$ be a regular conic equipped with a singular holomorphic dual billiard structure with isolated singularities that are its poles of order at most one. Then the sum of their residues is equal to 4.*

Proof Let us take an affine chart $\mathbb{C}^2_{z,w} = \{t = 1\} \subset \mathbb{CP}^2_{[z:w:t]}$ in which $\gamma \cap \mathbb{C}^2 = \{w = z^2\}$. For the $(2,1;2)$-billiard on the latter conic the sum of residues is equal to four: the residues at 0, ∞ are both equal to 2, since the projective symmetry $(z,w) \mapsto (\frac{z}{w}, \frac{1}{w})$ of the billiard structure (see the addendum to Theorem 4.1) permutes 0 and ∞ and preserves the residues. To treat the general case, we use the following proposition.

Proposition 7.2 *Let γ and the affine chart $\mathbb{C}^2_{z,w}$ be as above. A singular holomorphic dual billiard structure on γ is meromorphic (i.e., has order at most one at each singular point), if and only if the corresponding involutions $\sigma_P : L_P \to L_P$, $P = (z_0, z_0^2) \in \mathbb{C}^2$, written in the coordinate $u = z - z_0$ on L_P, have the form*

$$\sigma_P : u \mapsto -\frac{u}{1 + f(z_0)u}, \quad f(z) \text{ is a rational function with simple poles,}$$

$$f(z) = \frac{1}{z}(\lambda + o(1)), \quad as\ z \to \infty; \quad \lambda \in \mathbb{C}. \tag{7.1}$$

The residue at infinity is equal to $4 - \lambda$. The above dual billiard structure has regular point at infinity, if and only if $\lambda = 4$.

Proof A finite singular point of σ_P is of order one, if and only if the corresponding function $f(z)$ has simple pole there (Proposition 2.13). Let $E = [0 : 1 : 0]$ be the infinite point of the conic γ. Consider the affine coordinates $(\widetilde{z}, \widetilde{w}) = (\frac{z}{w}, \frac{1}{w})$ centered at E. Let ρ denote the residue at E: then in the coordinate $\widetilde{\zeta} := \frac{\widetilde{z}}{\widetilde{z}(P)}$ on L_P the involution σ_P converges to $\eta_\rho(\widetilde{\zeta})$, as $P \to E$. Therefore, in the coordinate ζ the involution σ_P converges to $\eta_{4-\rho}$, see statement (4.34) and discussion before it. Hence, in the coordinate $\hat{u} := \zeta - 1$ the involution σ_P takes the form $\hat{u} \mapsto -\frac{\hat{u}}{1+g(z_0)u}$, $g(z)$ is a rational function, $g(z) \to 4 - \rho$, as $z \to \infty$. Rescaling to the coordinate $u = z_0\hat{u}$ yields (7.1) with $f(z) = \frac{g(z)}{z}$, $\lambda = 4 - \rho$. The converse is proved by converse argument. The last statement of Proposition 7.2 (regularity at E) follows from Proposition 2.13. Proposition 7.2 is proved. □

Consider now an arbitrary dual billiard structure on a conic whose singularities are of order at most one. Let us choose an affine chart $\mathbb{C}^2_{z,w}$ in which $\gamma \cap \mathbb{C}^2 = \{w = z^2\}$ and so that the above point E at infinity be regular for the dual billiard structure. Then the corresponding function $f(z)$ from (7.1) is rational with simple poles, let us denote them a_j (Proposition 2.13). Hence, $f(z) = \sum_j \frac{\lambda_j}{z-a_j}$, λ_j being residues, and thus, $\sum_j \lambda_j = 4$, by the last statement of Proposition 7.2. This proves Proposition 7.1. □

Proposition 7.3 *Let γ be a regular conic equipped with a rationally integrable singular dual billiard structure. Then each singular point of the structure is its pole of order 1, and its residue lies in $\mathcal{M} \setminus \{0\}$.*

Proof Well-definedness of residues follows from integrability and Proposition 2.11. Their non-vanishing follows from Proposition 2.13. Let O be a singular point, and let ρ be the corresponding residue. Then the $(2, 1; \rho)$-billiard is quasihomogeneously integrable, by Theorem 3.3. Therefore, $\rho \in \mathcal{M}$, by Theorem 4.1. Proposition 7.3 is proved. □

Corollary 7.4 *Let γ be a regular conic equipped with a rationally integrable singular dual billiard structure. Then it has at least one and most four singular points, with residue collections being of one of the following types:*

$$4,\ (2,2),\ (1,3),\ (2,1,1),\ (1,1,1,1); \quad (\rho, 4-\rho)\ \mathit{with}\ \rho \in \mathcal{M} \setminus \mathbb{Z}; \tag{7.2}$$

$$\left(\frac{3}{2}, \frac{3}{2}, 1\right), \ \left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right), \ \left(\frac{4}{3}, \frac{5}{3}, 1\right).$$

Proof The residues lie in $\mathcal{M} \setminus \{0\}$ (Proposition 7.3), and hence, are greater or equal to one. Their sum is equal to 4 (Proposition 7.1). Therefore, the number of singularities is between one and four. The cases of one and two singularities are obviously given by the first, second, third and sixth collections in (7.2). The case of three singularities with natural residues is the collection $(2, 1, 1)$. The case of four singularities is $(1, 1, 1, 1)$. Cases of three singularities with some of residues being non-integer correspond to the three last residue collections in (7.2). Indeed, each non-integer number in $\mathcal{M} \setminus \{0\}$ takes the form $2 \pm \frac{2}{k}$, $k \in \mathbb{N}_{\geq 3}$. Therefore, if the number of singularities is three, then non-integer residues are of the type $2 - \frac{2}{k}$, $k \geq 3$. Finally, all possible configurations have one of the two following types: $(2 - \frac{2}{k_1}, 2 - \frac{2}{k_2}, 2 - \frac{2}{k_3})$, $(2 - \frac{2}{k_1}, 2 - \frac{2}{k_2}, 1)$. For the first type, writing the condition that the sum of residues is equal to 4 yields

$$\frac{2}{k_1} + \frac{2}{k_2} + \frac{2}{k_3} = 2, \qquad k_1, k_2, k_3 \geq 3.$$

Therefore, $k_1 = k_2 = k_3 = 3$, and we get the residue collection $(\frac{4}{3}, \frac{4}{3}, \frac{4}{3})$. For the second type we get $\frac{2}{k_1} + \frac{2}{k_2} = 1$, $k_1, k_2 \in \mathbb{N}_{\geq 3}$. The only solutions of the latter equation are $\{k_1, k_2\} = \{4, 4\}, \{3, 6\}$, which correspond to the residue configurations $(\frac{3}{2}, \frac{3}{2}, 1)$ and $(\frac{4}{3}, \frac{5}{3}, 1)$ respectively. The corollary is proved. □

Proposition 7.5 *Let $\gamma \subset \mathbb{CP}^2$ be a regular conic. For any two collections of distinct points $a_1, \dots, a_n \in \gamma$ and non-zero numbers $(x_1, \dots, x_n)$ with $\sum_{j=1}^{n} x_j = 4$ there exists a unique singular holomorphic dual billiard structure on γ with singular points a_j being poles of order one with residues x_j.*

Proof The proposition follows from (7.1) and uniqueness of a rational function $f(z)$ vanishing at infinity as $\frac{\lambda}{z}(1 + o(1))$ with given λ and simple poles with given positions and residues (see the proof of Proposition 7.1). □

7.2 Case of integer residues: pencil of conics

Proposition 7.6 *Let a singular holomorphic dual billiard structure on a regular conic γ have singularities $a_1, \dots, a_m$, $m \in \{1, 2, 3, 4\}$, with residues $\lambda_j \in \mathbb{N}$, $j = 1, \dots, m$. Then it is realized by the pencil of conics passing through a_j and having contact with γ of order λ_j at a_j.*

Proof Case 1): four distinct points $a_1, \dots, a_4$ with residues $\lambda_j = 1$. Consider the pencil of conics passing through them. It defines the projective involutions $\sigma_P : L_P \to L_P$, $P \in \gamma \setminus \{a_1, \dots, a_4\}$, permuting the intersection points of the lines L_P with each conic of the pencil. This yields a singular holomorphic dual billiard structure on γ with singularities at a_j.

Claim 6. *The involution family σ_P is holomorphic with singularities a_j of order one and residue one.*

Proof For every a_j each conic C of the pencil, $C \neq \gamma$, intersects the line L_{a_j} transversally at two distinct points: a_j and some point b_j. For every $P \in \gamma$ close to a_j the line L_P intersects C at two points $Q(P)$ and $Y(P)$ converging to a_j and b_j respectively, as $P \to a_j$. They are permuted by the involution σ_P, and their ζ-coordinates tend to 0 and ∞ respectively. Therefore, in the coordinate ζ the involution σ_P converges to $\eta_1(\zeta) = \frac{1}{\zeta}$. Hence, σ_P has simple pole with residue one at a_j. The claim is proved. $\Box$

Claim 6 together with Proposition 7.5 imply that the initial dual billiard coincides with the one defined by the above pencil.

Case 2): three singular points a_1, a_2, a_3 with residues 1, 1, 2 respectively. Consider the pencil of conics passing through these points and tangent to γ at a_3. The above involution family σ_P defined by this pencil has a rational quadratic integral (Example 1.14). Therefore, its singularities a_1, a_2, a_3 are poles of order one, by Proposition 2.11. Its residues at a_1, a_2 are equal to 1, see Claim 6 and its proof. Therefore, its residue at the third point a_3 is equal to $4-2=2$ (Proposition 7.1). Hence, the initial dual billiard structure coincides with the one defined by the pencil, by Propositon 7.5.

The remaining cases of residue configurations $(1, 3)$, 4 are treated analogously. Proposition 7.6 is proved. $\Box$

7.3 Case of two singularities: a quasihomogeneously integrable $(2, 1; \rho)$-billiard

Proposition 7.7 *Every singular holomorphic dual billiard structure on a regular conic with two singularities of order one is projectively equivalent to a $(2, 1; \rho)$-billiard. It is rationally integrable, if and only if the latter billiard is quasihomogeneously integrable; this holds if and only if $\rho \in \mathcal{M}$.*

Proof The first statement of Proposition 7.7, with ρ being the residue at some singularity, follows from Propositions 7.1 and 7.5. The condition that $\rho \in \mathcal{M}$ is necessary for rational integrability, by Proposition 7.3. Conversely, it $\rho \in \mathcal{M}$, then the billiard, which is equivalent to the $(2, 1; \rho)$-billiard, is rationally integrable, by Theorem 4.1. This proves Proposition 7.7. $\Box$

7.4 Integrability of residue configuration $(\frac{3}{2}, \frac{3}{2}, 1)$

Take an affine chart (z, w) in which the conic γ is given by the equation $w = z^2$, the singular points of the billiard structure with residue $\frac{3}{2}$ are $(0, 0)$ and the infinite point, and the singular point with residue 1 is $(1, 1)$. The corresponding involution family $\sigma_P : L_P \to L_P$ written in the coordinate $u = z - z_0$, $z_0 := z(P)$, takes the form (1.3), by Proposition 2.13, 7.2, 7.5:

$$u \mapsto -\frac{u}{1 + f(z_0)u}, \ f(z) = \frac{3}{2z} + \frac{1}{z-1} = \frac{5z-3}{2z(z-1)}. \tag{7.3}$$

Lemma 7.8 *The billiard structure on γ defined by the above involution family σ_P admits the rational integral*

$$R(z, w) = \frac{(w - z^2)^2}{(w + 3z^2)(z - w)(z - 1)}. \tag{7.4}$$

Motivation of construction of integral. The singular point at the origin has residue $\frac{3}{2}$. The corresponding $(2, 1; \frac{3}{2})$-billiard has a quasihomogeneous integral $R_{\frac{3}{2}}(z, w) = \frac{(w-z^2)^2}{z(w+3z^2)}$, see Theorem 4.1. Let us try to construct a rational integral of our non-quasihomogeneous dual billiard in the form $\frac{(w-z^2)^2}{Q(z,w)}$ so that the lower $(2, 1)$-quasihomogeneous part at $(0, 0)$ of the denominator $Q(z, w)$ be equal to the denominator in $R_{\frac{3}{2}}$ up to constant factor (see the proof of Theorem 3.3). Then the zero locus $\{Q = 0\}$ should contain an irreducible quadratic germ of analytic curve at $(0, 0)$ having a contact bigger than two with the conic $C := \{w+3z^2 = 0\}$. Let us look for a polynomial Q of degree four vanishing on the conic C: $Q(z, w) = (w+3z^2)H(z, w)$. To find the zero locus of the polynomial H, we have to find the images of the points of intersection $L_P \cap C$ under the involutions σ_P. We show that the latter images lie in the union of lines $\{w = z\}$ and $\{z = 1\}$. This together with Proposition 4.12 will imply that the function (7.4) is an integral.

Proof of Lemma 7.8. Fix a $P = (z_0, z_0^2) \in \gamma$. The line L_P intersects the conic C at two points A and D with coordinates $z = -z_0$ and $z = \frac{1}{3}z_0$ respectively, since the ζ-coordinates of the intersection points are roots of the polynomial $\mathcal{R}_{2,1,-3}(\zeta) = 3\zeta^2 + 2\zeta - 1$, see (4.4); its roots are -1 and $\frac{1}{3}$. Let B and F denote respectively the intersection points of the line L_P with the lines $\{z = 1\}$ and $\{w = z\}$ respectively.

Claim 7. *One has $\sigma_P(D) = B$, $\sigma_P(A) = F$.*

Proof The u-coordinates of the points A, D, B, F, $u = z - z_0$, are

$$u(A) = -2z_0, \ u(D) = -\frac{2}{3}z_0, \ u(B) = 1 - z_0, \ u(F) = \frac{z_0(1 - z_0)}{2z_0 - 1}. \tag{7.5}$$

For A, D, B the formulas are obvious. The z-coordinate of the point F is found from the equation on F as the point of intersection of the lines $\{w = z\}$ and $L_P = \{w = 2zz_0 - z_0^2\}$:

$$w(F) = 2z_0z(F) - z_0^2 = z(F), \quad z(F) = \frac{z_0^2}{2z_0 - 1}.$$

The latter formula implies the last formula in (7.5). One has

$$u(\sigma_P(D)) = -\frac{u(D)}{1 + f(z_0)u(D)} = \frac{2z_0}{3(1 + \frac{5z_0-3}{2z_0(z_0-1)}(-\frac{2}{3}z_0))}$$

$$= -\frac{2z_0(z_0 - 1)}{2z_0} = 1 - z_0 = u(B),$$

$$u(\sigma_P(A)) = -\frac{u(A)}{1 + f(z_0)u(A)} = \frac{2z_0}{1 - \frac{5z_0-3}{2z_0(z_0-1)}2z_0} = \frac{2z_0(z_0 - 1)}{2 - 4z_0} = u(F).$$

The claim is proved. □

The claim together with the above discussion implies the statement of Lemma 7.8. □

7.5 Integrability of residue configuration $(\frac{4}{3}, \frac{4}{3}, \frac{4}{3})$

Lemma 7.9 *The singular holomorphic dual billiard structure on a regular conic with three poles of order one and residues equal to $\frac{4}{3}$ is rationally integrable. In the affine chart $\mathbb{C}^2_{z,w}$, where the conic is given by the equation $w = z^2$ and the singularities are*

$$a_j := (\varepsilon^j, \varepsilon^{2j}), \ \varepsilon = e^{\frac{2\pi i}{3}}, \ j = 0, 1, 2,$$

the corresponding involution family $\sigma_P : L_P \to L_P$ written in the coordinate $u := z - z_0$, $z_0 = z(P)$, takes the form (1.4):

$$\sigma_P : u \mapsto -\frac{u}{1 + f(z_0)u}, \ f(z) = \frac{4z^2}{z^3 - 1}. \tag{7.6}$$

The rational function given by (1.10):

$$R(z, w) = \frac{(w - z^2)^3}{(1 + w^3 - 2zw)^2}, \tag{7.7}$$

is an integral of the billiard.

Proof Take an affine chart as above. The involution family σ_P given by (7.6) has first order poles at a_j with residues $\frac{4}{3}$ and is regular at infinity, by Propositions 2.13 and 7.2. For the proof of invariance of the restrictions $R|_{L_P}$ under the involutions σ_P, it suffices to show that the intersection of each line L_P with the zero locus of the denominator, the cubic

$$C := \{1 + w^3 - 2zw = 0\},$$

is σ_P-invariant. We will do this in the two following propositions.

Proposition 7.10 *For every $P \in \gamma \setminus \{a_0, a_1, a_2\}$ let $S(P) \in L_P$ denote the fixed point distinct from P of the involution σ_P. The fixed point family $S(P)$ coincides with the triple punctured cubic $C \setminus \{a_0, a_1, a_2\}$. In particular, the cubic C is rational.*

Proof Solving the fixed point equation $u = -\frac{u}{1+f(z_0)u}$ in non-zero u yields

$$u(S(P)) = -\frac{2}{f(z_0)} = \frac{1 - z_0^3}{2z_0^2},\ z(S(P)) = \frac{1 + z_0^3}{2z_0^2},$$

$$w(S(P)) = w(P) + 2z_0 u(S(P)) = z_0^2 + \frac{1 - z_0^3}{z_0} = \frac{1}{z_0}.$$

Therefore, the fixed point family $S(P)$ runs along the parametrized rational curve

$$K := \left(t \mapsto \left(\frac{1 + t^3}{2t^2}, \frac{1}{t} \right),\ |\ t \in \overline{\mathbb{C}} \right). \tag{7.8}$$

The curve K obviously satisfies the equation $1 + w^3 - 2zw = 0$ of the cubic C, and hence, coincides with C. $\square$

Proposition 7.11 *For every $P \in \gamma \setminus \{a_0, a_1, a_2\}$ the intersection $L_P \cap C$ is σ_P-invariant.*

Proof One of the points of the intersection $L_P \cap C$ is the fixed point $S(P)$. Let us show that the other intersection points are permuted by σ_P. To do this, let us find explicitly their t-parameters, see (7.8). Along the line L_P one has $w = z_0^2 + 2z_0(z - z_0) = 2z_0 z - z_0^2$. Substituting $z = \frac{1+t^3}{2t^2}$ and $w = \frac{1}{t}$ to the latter equation yields the equation

$$(t - z_0)(t^2 - \frac{1}{z_0}) = 0.$$

Its solution $t = z_0$ corresponds to the fixed point $S(P)$. The other two solutions are $t = \pm\frac{1}{\sqrt{z_0}}$. Here we fix some value of square root and denote it $\sqrt{z_0}$; the other value is $-\sqrt{z_0}$. The corresponding values z and u are equal respectively to

$$z_\pm = \frac{z_0 \pm \frac{1}{\sqrt{z_0}}}{2}, \ u_\pm = z_\pm - z_0 = \frac{-z_0 \pm \frac{1}{\sqrt{z_0}}}{2}.$$

The involution σ_P sends the point with the u-coordinate u_+ to the point with the u-coordinate

$$-\frac{u_+}{1+f(z_0)u_+} = -\frac{\frac{1}{\sqrt{z_0}} - z_0}{2(1 + 2\frac{z_0^{\frac{3}{2}}}{z_0^3-1}(1 - z_0^{\frac{3}{2}}))}.$$

Writing $z_0^3 - 1 = (z_0^{\frac{3}{2}} - 1)(z_0^{\frac{3}{2}} + 1)$ in the denominator and cancelling the former factor yields

$$-\frac{u_+}{1+f(z_0)u_+} = -\frac{(1-z_0^{\frac{3}{2}})(1+z_0^{\frac{3}{2}})}{2\sqrt{z_0}(1+z_0^{\frac{3}{2}} - 2z_0^{\frac{3}{2}})} = -\frac{z_0 + \frac{1}{\sqrt{z_0}}}{2} = u_-.$$

This implies that the involution σ_P permutes the intersection points with u-coordinates $u_\pm$. The proposition is proved. □

Lemma 7.9 follows from Propositions 7.11 and 4.12. □

7.6 Integrability of the configuration $(\frac{4}{3}, \frac{5}{3}, 1)$. End of proof of Theorem 1.18

Lemma 7.12 *The singular holomorphic dual billiard structure on a regular conic γ with three singularities of order one and residues $\frac{4}{3}$, $\frac{5}{3}$, 1 is rationally integrable. In the affine chart $\mathbb{C}^2_{z,w}$ where $\gamma = \{w = z^2\}$ and the corresponding singularities are $(0,0)$, infinity and $(1,1)$ respectively the involutions $\sigma_P : L_P \to L_P$ defining the dual billiard structure have the following form in the coordinate $u = z - z_0$, $z_0 = z(P)$:*

$$\sigma_P : u \mapsto -\frac{u}{1+f(z_0)u}, \quad f(z) = \frac{4}{3z} + \frac{1}{z-1} = \frac{7z-4}{3z(z-1)}. \qquad (7.9)$$

The function

$$R(z,w) = \frac{(w-z^2)^3}{(w+8z^2)(z-1)(w+8z^2+4w^2+5z^2w-14zw-4z^3)} \qquad (7.10)$$

is an integral of the dual billiard.

Proof Formula (7.9) follows from Propositions 7.2 and 7.5.

Motivation of the construction of the integral R**.** The residue of the dual billiard at $(0,0)$ is equal to $\frac{4}{3}$. The corresponding $(2,1;\frac{4}{3})$-billiard has quasihomogeneous integral $R_{\frac{4}{3}}(z,w) = \frac{(w-z^2)^3}{(w+8z^2)^2}$, by Theorem 4.1. Let E denote the infinity point of the conic γ. Its residue is equal to $\frac{5}{3}$. In the coordinates $(\widetilde{z},\widetilde{w}) = (\frac{z}{w},\frac{1}{w})$ the corresponding $(2,1;\frac{5}{3})$-billiard has quasihomogeneous integral $R_{\frac{5}{3}}(\widetilde{z},\widetilde{w}) = \frac{(\widetilde{w}-\widetilde{z}^2)^3}{(\widetilde{w}+8\widetilde{z}^2)(\widetilde{w}+\frac{5}{4}\widetilde{z}^2)}$ (Theorem 4.1). Note that the denominators in both $R_{\frac{4}{3}}$ and $R_{\frac{5}{3}}$ vanish on the same conic

$$C := \{w = -8z^2\} = \{\widetilde{w} = -8\widetilde{z}^2\}.$$

We would like to construct an integral of the billiard from the lemma as a ratio $R(z,w) = \frac{(w-z^2)^3}{(w+8z^2)Y(z,w)}$. To find the polynomial Y, we find the images of points of the intersection $L_P \cap C$ under the involution σ_P. We show that their families parametrized by P form the union of the line $\{z = 1\}$ and a cubic. The latter union will be the zero locus of the polynomial Y.

Proposition 7.13 *For every $P = (z_0, z_0^2) \in \gamma \setminus \{(0,0),(1,1),E\}$ the intersection $L_P \cap C$ consists of two points $A = A(P)$ and $D = D(P)$: $z(A) = \frac{z_0}{4}$, $z(D) = -\frac{z_0}{2}$. One has*

$$z(\sigma_P(A)) = 1, \ \sigma_P(D) = \left(-\frac{z_0(2z_0+1)}{2-5z_0}, \frac{z_0^2(z_0-4)}{2-5z_0}\right). \tag{7.11}$$

Proof The ζ-coordinates of points of the intersection $L_P \cap C$, $\zeta = \frac{z}{z_0}$, are roots of the polynomial $\mathcal{R}_{2,1,-8}(\zeta) = 8\zeta^2 + 2\zeta - 1$. Its roots are $\frac{1}{4}$ and $-\frac{1}{2}$, and the corresponding intersection points will be denoted by A and D respectively. This proves the first statement of the proposition. Let us find their σ_P-images in the coordinate $u = z - z_0$. One has

$$u(A) = -\frac{3z_0}{4}, \ u(\sigma_P(A)) = -\frac{u(A)}{1+f(z_0)u(A)} = \frac{\frac{3z_0}{4}}{1 - \frac{7z_0-4}{3z_0(z_0-1)}\frac{3z_0}{4}} = 1 - z_0,$$

$$z(\sigma_P(A)) = 1, \qquad u(D) = -\frac{3z_0}{2},$$

$$u(\sigma_P(D)) = -\frac{u(D)}{1+f(z_0)u(D)} = \frac{\frac{3z_0}{2}}{1 - \frac{7z_0-4}{3z_0(z_0-1)}\frac{3z_0}{2}} = \frac{3z_0(z_0-1)}{2-5z_0},$$

$$z(\sigma_P(D)) = -\frac{z_0(2z_0+1)}{2-5z_0}, \ w(\sigma_P(D)) = 2z_0 z(\sigma_P(D)) - z_0^2 = \frac{z_0^2(z_0-4)}{2-5z_0}.$$

Proposition 7.11 is proved. $\square$

Corollary 7.14 *The families of images $\sigma_P(A(P))$ and $\sigma_P(D(P))$ are respectively the line $\{z = 1\}$ and the rational cubic*

$$S := \{\left(-\frac{t(2t+1)}{2-5t}, \frac{t^2(t-4)}{2-5t}\right) \mid t \in \overline{\mathbb{C}}\} \tag{7.12}$$

Proposition 7.15 *The cubic S is the zero locus of the polynomial*

$$K(z, w) = w + 8z^2 + 4w^2 + 5z^2w - 14zw - 4z^3 \tag{7.13}$$

Proof One can prove the proposition directly by substituting the parametrization (7.12) to (7.13). But we will give a geometric proof explaining how formula (7.13) was found. First let us show that

$$\gamma \cap S = \Sigma := \{(0, 0), (1, 1), E\}, \tag{7.14}$$

$$t = 0, 1, \infty \ \text{ at } (0, 0), (1, 1), E \ \text{ respectively.}$$

Indeed, for every $P \in \gamma \setminus \Sigma$ the line L_P intersects γ only at P, and its intersection points A and D with C do not coincide with P, since γ and C intersect only at two points: $(0, 0)$ and E. Therefore, $\sigma_P(D) \in L_P \setminus \{P\}$ lies outside γ. On the other hand, as P tends to a point $X \in \Sigma$, one has $\sigma_D(P) \to X$. Indeed, this holds exactly when $z_0 = z(P)$ tends to some of the points 0, 1 or ∞, and in this case $\sigma_D(P) \to X$: both latter statements follow from (7.11). This proves (7.14).

Claim 8. *The germs of the curve S at $(0, 0)$ and E are regular and tangent to the conic C and to the conic $\{w = -\frac{5}{4}z^2\}$ respectively with contact of order at least three. The curve S is bijectively parametrized by the parameter t, see (7.12), except maybe for possible self-intersections.*

Proof The coordinates of a point of the curve S with a parameter $t \to 0$ are asymptotic to $-\frac{t}{2}$ and $-2t^2$ respectively. This implies the statement of the claim for the germ at $(0, 0)$. The proof for the germ at infinity is analogous. The parametrization (7.12) is either bijective (up to self-intersections), or a covering of degree at least two. The latter case is clearly impossible, since the germ of the curve S at $(0, 0)$ is injectively parametrized by a neighborhood of the point $t_0 = 0$ and no other parameter value is sent to $(0, 0)$. $\square$

Claim 9. *The germ of the curve S at $(1, 1)$ is a cusp.*

Proof The germ of the curve S at $(1, 1)$ is irreducible, since it is a germ of curve parametrized by the parameter t at the base point $t_0 = 1$ in the parameter line, see (7.14). It is a singular germ, since the derivative of the map (7.12) at $t = 1$ is zero and by the last statement of Claim 8. Therefore, the projective line $L = L_{(1,1)}$ tangent to S at the point $(1,1)$ is tangent to S with contact at least three. On the other hand, the tangency order cannot be bigger than three, since S is a cubic. Hence, it is equal to three, and $(1,1)$ is a cusp. The claim is proved. $\Box$

Claim 10. *There exists a cubic polynomial vanishing on S of the form*

$$K(z,w) = w + 8z^2 + \alpha w(w + \frac{5}{4}z^2) + \beta zw + \psi z^3. \tag{7.15}$$

Proof Let $K(z,w)$ be a cubic polynomial vanishing on S. Its homogeneous cubic part should contain no w^3 and w^2z terms, since S contains the point $E = [0:1:0] \in \mathbb{CP}^2$ and is tangent to the infinity line there. Therefore, it is a linear combination of monomials from (7.15) (and may be z). Its lower $(2,1)$-quasihomogeneous part at $(0,0)$ is $w + 8z^2$ up to constant factor, since the germ of the curve S at the origin is regular and tangent to the conic $C = \{w + 8z^2 = 0\}$ with contact of order at least three (Claim 8). Hence normalizing K by constant factor, we can and will consider that K is equal to $w + 8z^2$ plus a linear combination of monomials w^2, wz, wz^2, z^3. Passing to the affine coordinates $(\widetilde{z}, \widetilde{w}) = (\frac{z}{w}, \frac{1}{w})$ centered at E we get that $K(z,w)$ is equal to $\frac{1}{\widetilde{w}^3}H(\widetilde{z}, \widetilde{w})$, where $H(\widetilde{z}, \widetilde{w})$ is a polynomial vanishing on the germ of the curve S at E. The latter germ being regular and tangent to the conic $\{w + \frac{5}{4}z^2 = 0\} = \{\widetilde{w} + \frac{5}{4}\widetilde{z}^2 = 0\}$ with contact of order at least three (Claim 8), the $(2,1)$-quasihomogeneous part of the polynomial $H(\widetilde{z}, \widetilde{w})$ is equal to $\widetilde{w} + \frac{5}{4}\widetilde{z}^2$ up to constant factor; hence $H(\widetilde{z}, \widetilde{w})$ is equal to $\alpha(\widetilde{w} + \frac{5}{4}\widetilde{z}^2) + \psi\widetilde{z}^3$ plus a polynomial of degree at most three whose monomials are divisible by either $\widetilde{w}^2$, or $\widetilde{w}\widetilde{z}$. One has

$$\frac{1}{\widetilde{w}^3}(\widetilde{w} + \frac{5}{4}\widetilde{z}^2) = w(w + \frac{5}{4}z^2), \ \frac{\widetilde{z}^3}{\widetilde{w}^3} = z^3, \ \frac{1}{\widetilde{w}^3}\widetilde{w}\widetilde{z} = zw.$$

This together with the above discussion implies that the polynomial K has the type (7.15). The claim is proved. $\Box$

Finding unknown coefficients α, β, ψ from the linear equation saying that $K(z,w)$ vanishes at $(1,1)$ with its first derivatives yields (7.13). One can also check directly that the polynomial K given by (7.13) vanishes at $(1,1)$ with its first partial derivatives. Proposition 7.15 is proved. $\Box$

Proposition 7.16 *Set*

$$\Gamma := C \cup \{z = 1\} \cup S, \quad C = \{w + 8z^2 = 0\}, \ S \textit{ is given by (7.12).}$$

For every $P \in \gamma \setminus \Sigma$ the intersection $L_P \cap \Gamma$ is σ_P-invariant.

Proof For every $P \in \gamma \setminus \Sigma$ the line L_P intersects Γ at six points: $A, D \in C \cap L_P$, the point $\sigma_P(A) \in \{z = 1\} \cap L_P$, the point $\sigma_P(D) \in S \cap L_P$ and two more points $B_1, B_2 \in S \cap L_P$; $B_j = B_j(P)$. It suffices to show that

$$\text{the involution } \sigma_P \text{ permutes } B_1 \text{ and } B_2 \text{ for every } P \in \gamma \setminus \Sigma. \tag{7.16}$$

The proof of (7.16) will be split into the two following claims. Set

$$u_j = u(B_j(P)) = z(B_j(P)) - z(P) = z(B_j(P)) - z_0, \quad j = 1, 2.$$

Claim 11. *One has*

$$u_1 + u_2 = \frac{4 - 7z_0}{2}, \quad u_1 u_2 = \frac{3z_0(z_0 - 1)}{2}. \tag{7.17}$$

Proof One has $z = z_0 + u$, $w = z_0^2 + 2z_0 u$ on L_P. In the coordinate u on L_P the restriction to L_P of the polynomial $K(z, w)$ takes the form

$$K|_{L_P} = (10z_0 - 4)u^3 + (41z_0^2 - 40z_0 + 8)u^2 + \chi u + 9z_0^2(z_0 - 1)^2, \ \ \chi \in \mathbb{C}. \tag{7.18}$$

Indeed, the cubic term in $K|_{L_P}$ coincides with that of the sum

$$-4z^3 + 5z^2 w = -4(u + z_0)^3 + 5(u + z_0)^2(z_0^2 + 2z_0 u). \tag{7.19}$$

Thus, the coefficient at u^3 equals $10z_0 - 4$. The coefficient at u^2 in $K|_{L_P}$ is the sum of similar coefficients in (7.19) and in the expression

$$8z^2 + 4w^2 - 14zw = (8 + 16z_0^2 - 28z_0)u^2 + \text{lower terms}.$$

The coefficient at u^2 in (7.19) is equal to $-12z_0 + 25z_0^2$. Hence, the coefficient at u^2 in $K|_{L_P}$ is equal to $41z_0^2 - 40z_0 + 8$. The free term of the polynomial $K|_{L_P}$ is equal to its value at the point $P = (z_0, z_0^2)$:

$$K(z_0, z_0^2) = z_0^2 + 8z_0^2 + 4z_0^4 + 5z_0^4 - 14z_0^3 - 4z_0^3 = 9z_0^2(z_0 - 1)^2.$$

This proves (7.18). The roots of the restriction $K|_{L_P}$ are u_1, u_2 and the u-coordinate u_3 of the point $\sigma_P(D)$:

$$u_3 := z(\sigma_P(D)) - z_0 = -\frac{z_0(2z_0 + 1)}{2 - 5z_0} - z_0 = \frac{3z_0(z_0 - 1)}{2 - 5z_0}.$$

Formula (7.18) together with Vieta's formulas imply that

$$u_1 + u_2 = \frac{41z_0^2 - 40z_0 + 8}{4 - 10z_0} - u_3 = \frac{41z_0^2 - 40z_0 + 8 - 6z_0^2 + 6z_0}{4 - 10z_0} = \frac{4 - 7z_0}{2},$$

$$u_1u_2 = u_3^{-1}\frac{9z_0^2(z_0-1)^2}{4-10z_0} = \frac{3z_0(z_0-1)}{2}.$$

This proves (7.17). □

Thus, by (7.17), the numbers u_1, u_2 are roots of the quadratic polynomial

$$Q(u) := 2u^2 + (7z_0 - 4)u + 3z_0(z_0 - 1).$$

Claim 12. *One has*

$$Q \circ \sigma_P(u) = \frac{Q(u)}{(1 + f(z_0)u)^2}, \ f(z) = \frac{7z-4}{3z(z-1)}. \qquad (7.20)$$

Proof Recall that $\sigma_P(u) = -\frac{u}{1+f(z_0)u}$. Therefore,

$$Q \circ \sigma_P(u) = \frac{2u^2 - (1 + f(z_0)u)(7z_0 - 4)u + 3z_0(z_0 - 1)(1 + f(z_0)u)^2}{(1 + f(z_0)u)^2}.$$

The numerator in the latter ratio is equal to

$$(2 - f(z_0)(7z_0 - 4) + 3z_0(z_0 - 1)f^2(z_0))u^2$$
$$+(-(7z_0 - 4) + 6z_0(z_0 - 1)f(z_0))u + 3z_0(z_0 - 1) = Q(u).$$

This proves Claim 12. □

The involution σ_P sends the collection of roots of the polynomial Q to itself: their images are zeros of the pullback $Q \circ \sigma_P$, which are roots of Q, by Claim 12. Therefore, σ_P permutes the roots: otherwise, it would fix three points, two roots and $0 = u(P)$, which is impossible, since $\sigma_P \neq Id$. Thus, σ_P permutes the points $B_1, B_2 \in S \cap L_P$. This proves Proposition 7.16. □

The zero locus of the rational function $R(z, w)$ given by (7.10) is the conic γ. Its polar locus is the curve Γ from Proposition 7.16. For every $P \in \gamma \setminus \Sigma$ the intersections of the latter loci with L_P are respectively the point P and $\Gamma \cap L_P$. They are σ_P-invariant, by Proposition 7.16. Therefore, the function $R|_{L_P}$ is also σ_P-invariant, by Proposition 4.12. Hence, R is an integral of the dual billiard in question. This proves Lemma 7.12. □

Proof of Theorem 1.18. Let an irreducible germ of analytic curve $\gamma \subset \mathbb{CP}^2$ admit a structure of rationally integrable dual billiard. Then the curve γ is a conic, and the billiard structure extends to a singular holomorphic one with poles of order at most one and residues lying in $\mathcal{M}\setminus\{0\}$, by Proposition 1.34, Theorems 1.37, 1.38 and Proposition 7.3. The sum of residues should be equal to four, by Proposition 7.1. All the collections of residue values lying in $\mathcal{M}\setminus\{0\}$ with sum equal to 4 are described above. The corresponding billiard structures are rationally integrable with integrals given in this and previous subsections. This proves Theorem 1.18. □

8 Real integrable dual billiards. Proof of Theorems 1.16, 1.11 and the addendums to Theorems 1.18, 1.16

8.1 Real germs: proof of Theorem 1.16 and the addendums to Theorems 1.18, 1.16

Let a germ of real C^4-smooth curve $\gamma \subset \mathbb{R}^2$ carry a rationally integrable dual billiard structure. Then its complex Zariski closure $\overline{\gamma} \subset \mathbb{CP}^2_{[z:w:t]} \supset \mathbb{C}^2_{z,w} = \{t = 1\}$ is an algebraic curve, and the billiard structure extends to a rationally integrable singular holomorphic dual billiard structure on every its non-linear irreducible component (Proposition 1.35). Therefore, each non-linear irreducible component is a conic (Theorem 1.18). Thus, the germ γ is a chain of adjacent arcs of a finite collection of non-linear conics and maybe lines. It contains at least one conical arc, being non-linear. A conical arc cannot be adjacent to a straightline segment, since γ is C^2-smooth. There are no adjacent arcs of distinct conics: they would have contact of order at most 4 (Bézout Theorem), and hence, would not paste together in C^4-smooth way, while γ is C^4-smooth. This is the place where we use the condition that γ is C^4-smooth. Thus, γ is a germ of real conic, which will be also denoted γ, and the complexified billiard structure on its complexification is one of those given by Theorem 1.18. Let us find real forms of the complex dual billiards on conic given by Theorem 1.18.

Case 1): The complexified dual billiard on the complexified conic γ is given by a complex pencil of conics $\mathcal{C}_\lambda$; $\gamma = \mathcal{C}_0$.

Proposition 8.1 *The real dual billiard on the real conic γ is defined by a real pencil of conics whose complexification is the pencil $\mathcal{C}_\lambda$.*

Claim 13. *The pencil $\mathcal{C}_\lambda$ contains a complexified real conic $\mathcal{C}_{\lambda_0} \neq \gamma$.*

Proof Fix a real point $P_0 \in \gamma$ where the dual billiard involution $\sigma_{P_0} : L_{P_0} \to L_{P_0}$ is well-defined. Fix a point $E_1 \in \mathbb{RP}^2 \setminus \gamma$ close to P_0 and lying on the concave side from the conic γ. Consider the right real tangent line to γ through E_1, let P_1 be its tangency point with γ. Set $E_2 = \sigma_{P_1}(E_1)$. Take now the right real tangent line to γ through E_2, let P_2 be the corresponding tangency point. Similarly we construct E_3, E_4, E_5. If E_1 is close enough to P_0, then the points $E_2, \dots, E_5$, $P_2, \dots, P_4$ are well-defined and close to P_0. The five real points $E_1, \dots, E_5$ lie in the same complex conic $\mathcal{C}_{\lambda_0} \neq \gamma$, since the complex dual billiard is defined by the pencil $\mathcal{C}_\lambda$. The conic $\mathcal{C}_{\lambda_0}$ is the complexification of a real conic. Indeed, otherwise $\mathcal{C}_{\lambda_0}$ and its complex conjugate conic would be distinct and would intersect at five distinct points $E_1, \dots, E_5$, which is impossible. The claim is proved. $\Box$

Proof of Proposition 8.1. Let $\mathcal{C}_{\lambda_0}$ be the real conic from the claim. The complex pencil $\mathcal{C}_\lambda$ of complex conics is the complexification of the real pencil of real conics containing γ and $\mathcal{C}_{\lambda_0}$, by construction and since a pencil of conics is uniquely determined by its two conics. Hence, the involution $\sigma_P : L_P \to L_P$ permutes the points of intersection of the line L_P with each conic from the real pencil, since this is true for the pencil $\mathcal{C}_\lambda$. Thus, the real dual billiard on γ is defined by a real pencil. This proves Proposition 8.1. $\Box$

Case 2a): the billiard structure has two singular points with residues $2 \pm \frac{2}{k}$, $k \in \mathbb{N}_{\geq 3}$. The coordinatewise complex conjugation cannot permute them, since it should preserve the residue. Therefore, both singular points lie in the real part of the conic, and the dual billiard structure has the type 2a) from Theorem 1.16 and has the corresponding integral (1.6) or (1.7).

Case 2b): the billiard structure has three singular points with residues $\frac{3}{2}$, $\frac{3}{2}$, 1. Then the coordinatewise complex involution fixes the singular point with residue 1 and may either fix, or permute the two other singular points.

Subcase 2b1): the coordinatewise complex conjugation fixes all the three singular points. Then all of them lie in the real conic. Let us choose real homogeneous coordinates $[z : w : t]$ so that the singular points with residue $\frac{3}{2}$ are $[0 : 0 : 1]$ and $[0 : 1 : 0]$, the singular point with residue 1 is $[1 : 1 : 1]$ and the lines $w = 0$, $z = 0$ are tangent to γ at the two former points. Then we get that in the affine chart $\mathbb{C}^2_{z,w} = \{t = 1\}$ the real conic γ is given by the equation $w = z^2$, the dual billiard structure has type 2b1) from Theorem 1.16 and has integral (1.8) (Lemma 7.8). The fact that γ indeed coincides with the conic $\{wt = z^2\} \subset \mathbb{CP}^2_{[z:w:t]}$ follows from Bézout Theorem and the fact that the conics in question are tangent to each other at two points $[0 : 0 : 1]$ and $[0 : 1 : 0]$ and have yet another common point $[1 : 1 : 1]$.

Subcase 2b2): the coordinatewise complex conjugation permutes the singular points with residue $\frac{3}{2}$. Applying a real projective transformation, we can and will consider that the singular point with residue 1 (which lies in the real conic) has coordinates $[0 : 1 : 0]$, the line $\{t = 0\}$ is tangent to γ at the latter point, the line $\{w = 0\}$ is tangent to γ at the point $[0 : 0 : 1]$, the points with residue $\frac{3}{2}$ have coordinates $[\pm i : -1 : 1]$. Then γ is given by the equation $wt = z^2$, as in the above discussion. Passing to the affine chart $\mathbb{C}^2_{z,w} = \{t = 1\}$ we get that the dual billiard structure has type 2b2) from Theorem 1.16. This argument implies that the dual billiard structures 2b1) and 2b2) are complex-projective equivalent and proves the corresponding statement of the addendum to Theorem 1.18.

Let us calculate the integral of the dual billiard structure 2b2) in the above affine chart (z, w). To do this, we find explicitly the projective equivalence $F : \mathbb{CP}^2 \to \mathbb{CP}^2$ between the structures 2b1) and 2b2). It should send singular points $[0:0:1]$, $[0:1:0]$, $[1:1:1]$ of the structure 2b1) to the singular points of the structure 2b2) with the same residues. Let us construct an F sending them to $[i, -1:1]$, $[-i:-1:1]$, $[0:1:0]$. It should send the projective tangent lines to the conic at the points $[0:0:1]$, $[0:1:0]$ to its tangent lines at their images $[i, -1:1]$, $[-i:-1:1]$. The two former tangent lines are the z-axis and the infinity line $\{t=0\}$; their intersection point is $[1:0:0]$. The two latter tangent lines at $[i, -1:1]$, $[-i:-1:1]$ intersect at the point $(0,1) = [0:1:1]$, by symmetry and since in the affine chart $\mathbb{C}^2_{z,w} = \{t=1\}$ they pass through $(\pm i, -1)$ and have slopes $\pm 2i$. Therefore, F should also send $[1:0:0]$ to $[0:1:1]$. Finally, it sends $[1:0:0]$, $[0:1:0]$, $[0:0:1]$ to $[0:1:1]$, $[-i:-1:1]$, $[i,-1:1]$. The matrix of the projective transformation F is uniquely defined up to scalar factor. Its columns are $\lambda_1(0,1,1)$, $\lambda_2(-i,-1,1)$, $\lambda_3(i,-1,1)$, by the above statement; $\lambda_j \in \mathbb{C}^*$. Let us choose the normalizing scalar factor so that $\lambda_1 = 1$. Then the coefficients λ_2 and λ_3 are found from the linear equation saying that $F([1:1:1]) = [0:1:0]$: $\lambda_2 = \lambda_3 = -\frac{1}{2}$. We get that

$$F \text{ is given by } M := \begin{pmatrix} 0 & \frac{i}{2} & -\frac{i}{2} \\ 1 & \frac{1}{2} & \frac{1}{2} \\ 1 & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}, \quad M^{-1} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ -i & \frac{1}{2} & -\frac{1}{2} \\ i & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}. \tag{8.1}$$

The transformation F thus constructed preserves the conic $\gamma = \{wt = z^2\}$. Indeed, its image is a conic tangent to γ at two points $[\pm i : -1 : 1]$ and intersecting γ in yet another point $[0:1:0]$ (by construction). Hence, it has intersection index at least 5 with γ and thus, coincides with γ, by Bézout Theorem. The map F sends the billiard structure 2b1) to 2b2), by construction. Let us check that it sends the integral R_{b1} of the structure 2b1) to the integral R_{b2} of the structure 2b2). Indeed, the integrals written in the homogeneous coordinates $[z:w:t]$ take the form

$$R_{b1}(z,w,t) = \frac{(wt - z^2)^2}{(wt + 3z^2)(z-t)(z-w)},$$

$$R_{b2}(z,w,t) = \frac{(wt - z^2)^2}{(z^2 + w^2 + t^2 + wt)(z^2 + t^2)}.$$

The variable change given by the inverse matrix in (8.1),

$$\begin{pmatrix} \widetilde{z} \\ \widetilde{w} \\ \widetilde{t} \end{pmatrix} = M^{-1} \begin{pmatrix} z \\ w \\ t \end{pmatrix},$$

transforms $R_{b1}(\widetilde{z}, \widetilde{w}, \widetilde{t})$ to $R_{b2}(z, w, t)$.

Case 2c): the complexified dual billiard structure has three singularities with residues equal to $\frac{4}{3}$. The coordinatewise complex conjugation permutes them. Hence, it should fix some of them, being an involution. Thus, it either fixes one singularity and permutes the two other ones (Subcase c1)), or fixes all the three singularities (Subcase c2)).

Subcase c1). The singular point fixed by coordinatewise complex conjugation is a real point of the conic γ, and the two other (permuted) singularities are complex-conjugated. Applying a projective transformation with a real matrix, we can and will consider that $\gamma = \{wt = z^2\}$, the fixed singularity is the point $[1 : 1 : 1]$ and the permuted singularities are $[e^{\pm\frac{2\pi i}{3}} : e^{\mp\frac{2\pi i}{3}} : 1]$. Then in the affine chart $\mathbb{C}^2_{z,w} = \{t = 1\}$ the dual billiard structure in question takes the form 2c1) as in Theorem 1.16. Hence, it has first integral R_{c1} given by (1.10), see Lemma 7.9.

Subcase c2). Then all the singularities of the billiard structure are real points in γ. Applying a real projective transformation, we can and will consider that $\gamma = \{wt = z^2\}$ and the singularities are $[0 : 0 : 1]$, $[0 : 1 : 0]$ and $[1 : 1 : 1]$. Then in the affine chart $\mathbb{C}^2_{z,w} = \{t = 1\}$ the dual billiard takes the form 2c2). This together with Proposition 7.5 implies that there exists a complex projective transformation F fixing the complexified conic γ and sending the dual billiard of type 2c2) to that of type 2c1). This proves the corresponding statement of the addendum to Theorem 1.18. Let us show that the real dual billiard of type 2c2) has the integral R_{c2} given by (1.11), and $R_{c2} = R_{c1} \circ F$ up to constant factor. To do this, we find F. Set

$$\varepsilon := e^{-\frac{2\pi i}{3}}.$$

We choose F so that it sends the singular points $[0 : 0 : 1]$, $[0 : 1 : 0]$, $[1 : 1 : 1]$ of the structure 2c2) respectively to the singular points $[\varepsilon : \bar{\varepsilon} : 1]$, $[\bar{\varepsilon} : \varepsilon : 1]$, $[1 : 1 : 1]$ of the structure 2c1). It should preserve the conic γ. Hence, it sends its projective tangent lines at $[0 : 0 : 1]$, $[0 : 1 : 0]$ to those at the points $[\varepsilon : \bar{\varepsilon} : 1]$, $[\bar{\varepsilon} : \varepsilon : 1]$. Therefore, the intersection point $[1 : 0 : 0]$ of the two former tangent lines should be sent to the intersection point of the two latter tangent lines. In the above affine chart $\mathbb{C}^2_{z,w}$ the z-coordinate of the latter intersection point is found from the equation

$$\bar{\varepsilon} + 2\varepsilon(z-\varepsilon) = \varepsilon + 2\bar{\varepsilon}(z-\bar{\varepsilon}) :$$

$$z = -\frac{1}{2}, \ w = -\bar{\varepsilon} - \varepsilon = 1.$$

Finally, the projective transformation F should send the points $[1:0:0]$, $[0:1:0]$, $[0:0:1]$ to $[-\frac{1}{2}:1:1]$, $[\bar{\varepsilon}:\varepsilon:1]$, $[\varepsilon:\bar{\varepsilon}:1]$. Hence, its matrix (normalized by appropriate scalar factor) takes the form

$$\begin{pmatrix} -\frac{1}{2} & \lambda_2\bar{\varepsilon} & \lambda_3\varepsilon \\ 1 & \lambda_2\varepsilon & \lambda_3\bar{\varepsilon} \\ 1 & \lambda_2 & \lambda_3 \end{pmatrix}, \quad \lambda_2, \lambda_3 \in \mathbb{C}^*. \tag{8.2}$$

The coefficients λ_2 and λ_3 are found from the following system of equations saying that the transformation F should fix the point $[1:1:1]$:

$$\begin{cases} \lambda_2(1-\varepsilon) + \lambda_3(1-\bar{\varepsilon}) = 0 \\ \frac{3}{2} + \lambda_2(1-\bar{\varepsilon}) + \lambda_3(1-\varepsilon) = 0. \end{cases}$$

We get that $\lambda_3 = \varepsilon\lambda_2$, $\frac{3}{2} + \lambda_2(1+\varepsilon-2\bar{\varepsilon}) = 0$, $\lambda_2 = \frac{\varepsilon}{2}$, $\lambda_3 = \frac{\bar{\varepsilon}}{2}$, and the transformation F is given by the matrix

$$M := \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & \frac{\bar{\varepsilon}}{2} & \frac{\varepsilon}{2} \\ 1 & \frac{\varepsilon}{2} & \frac{\bar{\varepsilon}}{2} \end{pmatrix} \tag{8.3}$$

Let us now calculate the pullback of the integral R_{c1} under the projective transformation F. Writing R_{c1} in the homogeneous coordinates $[z:w:t]$, we get

$$R_{c1}([z:w:t]) = \frac{(wt-z^2)^3}{(t^3+w^3-2zwt)^2}.$$

Applying the linear transformation given by the matrix M to the polynomial $Q_1(z,w,t) = wt - z^2$ in the numerator yields

$$Q_1 \circ M(z,w,t) = \frac{1}{4}((2z+\bar{\varepsilon}w+\varepsilon t)(2z+\varepsilon w+\bar{\varepsilon}t) - (w+t-z)^2) = -\frac{3}{4}(wt-z^2).$$

Applying M to the polynomial $Q_2(z,w,t) = t^3 + w^3 - 2zwt$ in the denominator yields

$$8Q_2 \circ M(z,w,t) = (2z+\varepsilon w+\bar{\varepsilon}t)^3 + (2z+\bar{\varepsilon}w+\varepsilon t)^3$$

$$
\begin{aligned}
&-2(w+t-z)(2z+\varepsilon w+\bar{\varepsilon}t)(2z+\bar{\varepsilon}w+\varepsilon t)\\
&=16z^3+2w^3+2t^3-12z^2w-12z^2t-6zw^2-6zt^2-3w^2t-3wt^2\\
&+24zwt+2(z-w-t)(4z^2+w^2+t^2-2zw-2zt-wt)\\
&=24(z^3-z^2w-z^2t)-3(w^2t+wt^2)+30zwt\\
&=3(8z^3-8z^2w-8z^2t-w^2t-wt^2+10zwt).
\end{aligned}
$$

In the affine chart $\mathbb{C}^2_{z,w}=\{t=1\}$ we get $Q_1\circ M(z,w,1)=-\frac{3}{4}(w-z^2)$,

$$Q_2\circ M(z,w,1)=\frac{3}{8}(8z^3-8z^2w-8z^2-w^2-w+10zw).$$

Therefore, $R_{c1}\circ F=R_{c2}$ up to constant factor.

Case 2d): the complexified dual billiard on the conic has three singularities with residues $\frac{4}{3}$, 1, $\frac{5}{3}$. The complex conjugation, which preserves the dual billiard, should fix them, since their residues are distinct. Therefore, applying a real projective transformation, we can and will consider that the underlying real conic is the parabola $\{w=z^2\}$, and the singularities are respectively the points $(0,0)$, $(1,1)$ and its infinite point. The involution family defining the dual billiard is of the type 2d), by construction and Proposition 7.5. It has integral R_{2d} given by (1.12), due to Lemma 7.12. Theorem 1.16 and the addendums to Theorems 1.16 and 1.18 are proved.

8.2 Case of closed curve. Proof of Theorem 1.11

Let now γ be a C^4-smooth closed curve equipped with an integrable dual billiard structure. The involutions σ_P can be defined by just one convex closed invariant curve, and they depend continuously on $P\in\gamma$. Let R be a non-trivial rational first integral of the foliation by invariant curves. For every $P\in\gamma$ the restriction $R|_{L_P}$ is σ_P-invariant, since this holds in a neighborhood of the point P in L_P (by definition), and by analyticity. Therefore, γ is a conic, by Theorem 1.16, and it contains no singularity of the dual billiard. Hence, the dual billiard is given by a pencil of conics containing γ, since all the other rationally integrable dual billiards on conic listed in Theorem 1.16 have real singularities. Those conics of the pencil that are close enough to γ and lie on its concave side are disjoint and form a foliation of a topological annulus adjacent to γ. Indeed, otherwise the pencil would consists of conics intersecting at some point $P_0\in\gamma$. But then P_0 would be a singular point of the dual billiard, see the proof of Proposition 7.6. The contradiction thus obtained proves Theorem 1.11.

The basic set of the corresponding pencil lies in $\mathbb{CP}^2\setminus\mathbb{RP}^2$ and is described by the following obvious proposition.

Proposition 8.2 *Let $\gamma \subset \mathbb{RP}^2$ be a regular conic equipped with a dual billiard structure given by a real pencil of conics. Let the basic set of the pencil contain no real points. Then it consists of either four distinct points, or two distinct points. In the latter case the regular complexified conics of the pencil are tangent to each other at the two points of the basic set.*

9 Integrable projective billiards. Proof of Theorem 1.26 and its addendum

Here we prove Theorem 1.26 classifying rationally 0-homogeneously integrable projective billiards and its addendum providing formulas for integrals (in Subsection 9.4). To do this, in Subsection 9.1 we prove Proposition 1.23 stating that such a billiard admits an integral that is a 0-homogeneous rational function in the moment vector. Afterwards in Subsection 9.2 we prove Proposition 1.24 stating that rational 0-homogeneously integrability of a projective billiard is equivalent to rational integrability of its dual billiard. In Subsection 9.3 we prove Proposition 1.27.

9.1 The moment map and normalization of integral. Proof of Proposition 1.23

Recall that we identify the ambient Euclidean plane $\mathbb{R}^2_{x_1,x_2}$ of a projective billiard with the plane $\{x_3 = 1\} \subset \mathbb{R}^3_{x_1,x_2,x_3}$, and we denote $r = (x_1, x_2, 1)$. The geodesic flow has an universal invariant: the moment vector

$$M := [r, v] = (-v_2, v_1, \Delta(x, v)), \quad \Delta(x, v) := x_1 v_2 - x_2 v_1,$$

which separates any two orbits of the geodesic flow [15]. This implies that, *every integral of the projective billiard is a reflection-invariant function of M and vice versa,* as in [15].

Consider now a C^2-germ of planar curve equipped with a transversal line field, a connected domain U adjacent to it and the projective billiard in U. (Or a (global) projective billiard in some connected domain U in $\mathbb{R}^2$.) Let it have a first integral $R(x, v)$ that is a rational 0-homogeneous function in v of degree uniformly bounded by some constant d. Let $W \subset \mathbb{R}^3_{M_1,M_2,M_3}$ denote the image of the moment map $T\mathbb{R}^2|_U \to \mathbb{R}^3$, $(x, v) \mapsto [r, v]$, $x = (x_1, x_2)$.

Proposition 9.1 *Let us represent the above integral R as a function of the moment M. The function R is 0-homogeneous in M: $R(\lambda M) = R(M)$ for every $\lambda \in \mathbb{R}$; thus, it is well-defined on the tautological projection image*

$\mathbb{P}(W) = \pi(W \setminus \{0\})$. *The image* $\mathbb{P}(W)$ *is a union of projective lines along which the function* R *is rational of degree at most* d. *The family of the latter lines forms an open subset in the space* $\mathbb{RP}^{2*}$ *of lines.*

Proof As $x = (x_1, x_2)$ is fixed, the restricted moment map $v \mapsto M = [r, v]$ is a linear isomorphism of the tangent plane $T_x\mathbb{R}^2$ and the plane $r^\perp$ orthogonal to r. Therefore, the restriction to $r^\perp$ of the function R is rational 0-homogeneous of degree no greater than d. This proves 0-homogeneity of the function $R(M)$ and well-definedness of the function R on $\mathbb{P}(W)$. Let $\ell(x) \subset \mathbb{RP}^2$ denote the projective line that is the projectivization of the subspace $r^\perp$. One has $\mathbb{P}(W) = \cup_{x \in U}\ell(x)$. The function $R|_{\ell(x)}$ is rational for every $x \in U$, by rationality on $r^\perp$. The map $\mathbb{R}^2 \to \mathbb{RP}^{2*}$, $x \mapsto \ell(x)$ is a diffeomorphism onto the open subset of those projective lines that do not pass through the origin in the affine chart $\mathbb{R}^2_{x_1,x_2} = \{x_3 = 1\}$. Hence, it maps U onto an open subset in $\mathbb{RP}^{2*}$. This proves the proposition. □

As is shown below, the statement of Proposition 1.23 is implied by Proposition 9.1 and the next proposition.

Proposition 9.2 *Let* $d \in \mathbb{N}$. *Let a function* $f(z, w)$ *be defined on a neighborhood of the origin in* $\mathbb{R}^2_{z,w}$. *Let it be rational in each variable, and let its degree in the variable* w *be no greater than* d. *Then it is a rational function of two variables.*

Proof Let us write

$$f(z, w) = \frac{a_0(z) + a_1(z)w + \cdots + a_d(z)w^d}{b_0(z) + b_1(z)w + \cdots + b_d(z)w^d}.$$

Fix $2d + 1$ distinct points $w_0, \ldots, w_{2d}$ close to zero. The functions $R_j(z) := f(z, w_j)$ are rational. The system of $2d + 1$ equations

$$f(z, w_j) = \frac{a_0(z) + a_1(z)w_j + \cdots + a_d(z)w_j^d}{b_0(z) + b_1(z)w_j + \cdots + b_d(z)w_j^d} = R_j(z)$$

in $2d + 2$ unknown coefficients $a_s(z)$, $b_s(z)$ can be rewritten as a system of $2d + 1$ linear equations on them (multiplying by denominator). For every z it has a unique solution up to constant factor depending on z, since two rational functions in w of degree at most d cannot coincide at $2d+1$ distinct points. This follows from the fact that their difference, which is a rational function in w of degree at most $2d$, cannot have more than $2d$ zeros. The solution $(a_0(z), \ldots, a_d(z), b_0(z), \ldots, b_d(z))$ of the above linear systems can be normalized by constant factor so that its components be expressed as rational functions of the parameters w_j and $R_j(z)$ of the system. Therefore, $a_s(z)$ and $b_s(z)$ are rational functions in z. This proves the proposition. $\Box$

Let $V \subset \mathbb{RP}^{2*}$ denote the open set of lines from the last statement of Proposition 9.1. Fix two distinct lines $\Lambda_1, \Lambda_2 \in V$ and two distinct points $y_j \in \Lambda_j$, $j = 1, 2$. Consider two pencils $\mathcal{P}_j$ of lines through y_j. The function R is rational along each line in $\mathcal{P}_j$ close to Λ_j. Choosing affine chart $\mathbb{R}^2_{z,w} \subset \mathbb{RP}^2$ so that y_1, y_2 be the intersection points of the infinity line with the coordinate axes we get that R is locally a rational function in each separate variable z, w. Therefore, it is locally rational in two variables, by Proposition 9.2. Hence, it is globally rational on all of $\mathbb{P}(W)$, by connectivity of the domain U, and hence, of the open subset $\mathbb{P}(W)$. Therefore, $R(M)$ is a 0-homogeneous rational function in M. The first part of Proposition 1.23 is proved. Let us prove its second part: independence of integrability on choise of side. Let a C^2-smooth germ of curve C equipped with a transversal line field define a rationally 0-homogeneously integrable projective billiard on one side from C. Then it admits an integral that is a rational 0-homogeneous function $R(M)$ of the moment vector M (the first part of Proposition 1.23). The moment vector (and hence, the integral) extends as a constant function along straight lines crossing C (treated as orbits of geodesic flow) from one side of the curve C to the other side. Invariance of the integral $R(M)$ under the billiard flow is equivalent to its reflection invariance. But reflection invariance depends only on the transversal line field and not on the choice of side. Therefore, if R is an integral on one side, it will be automatically an integral on the other side. Proposition 1.23 is proved.

9.2 Integrability and duality. Proof of Proposition 1.24

The proof of Proposition 1.24 is analogous to the arguments from [15, 10, 31]. On the ambient projective plane $\mathbb{RP}^2_{[x_1:x_2:x_3]} \supset \mathbb{R}^2$ we deal with the projective duality $\mathbb{RP}^{2*} \to \mathbb{RP}^2$ given by the orthogonal polarity. We use the following

Remark 9.3 [15, 10, 31]. For every $r = (x_1, x_2, 1) \in \mathbb{R}^3$ and $v \in T_{(x_1,x_2)}\mathbb{R}^2$ consider the two-dimensional vector subspace in $\mathbb{R}^3$ generated by r and v (punctured at the origin). Let $L(r, v) \subset \mathbb{RP}^2$ denote the corresponding projective line (its projectivization). The composition of the moment map $(r, v) \mapsto M = [r, v]$ and the tautological projection $\mathbb{R}^3_{M_1,M_2,M_3} \setminus \{0\} \to \mathbb{RP}^2_{[M_1,M_2,M_3]}$ sends each pair (r, v) to the point $L^*(r, v)$ dual to $L(r, v)$.

Consider a projective billiard on a curve $C \subset \mathbb{R}^2_{x_1,x_2}$. Its dual curve is identified with a curve $\gamma = C^* \subset \mathbb{RP}^2_{[M_1:M_2:M_3]}$, see the above remark. Let the dual billiard on γ have a rational integral. It can be written as a 0-homogeneous rational function $R(M_1, M_2, M_3)$. The corresponding function $R([r, v])$ is a rational 0-homogeneous integral of the projective billiard. Indeed, its invariance under reflections acting on $v \in T_Q\mathbb{R}^2$, $Q \in C$, follows from the above remark and the fact that duality conjugates the billiard reflection acting on lines through Q to the dual billiard involution acting on the dual line Q^*. Conversely, let the projective billiard have a rational 0-homogeneous integral. Then it can be written as $R[r, v]$, where $R(M)$ is a rational 0-homogeneous function, by Proposition 1.23. The function $R(M)$ is an integral of the dual billiard, since $R[r, v]$ is an integral of the projective billiard and by the above conjugacy. Proposition 1.24 is proved.

9.3 Space form billiards on conics. Proof of Proposition 1.27

Let a projective billiard on a finitely punctured conic C be a space form billiard with matrix A. In the case, when $A = \mathrm{diag}(1, 1, 0)$, the billiard is Euclidean, and each conic confocal to C is a caustic. Analogous statement holds in the case of non-zero constant curvature, when $A = \mathrm{diag}(1, 1, \pm 1)$. This implies the second statement of Proposition 1.27.

Let us prove the converse. Let a transversal line field $\mathcal{N}$ on a punctured conic C define a projective billiard having a complex conical caustic S. Let us show that it is projectively equivalent to a space form billiard with matrix $\mathrm{diag}(1, 1, -1)$. Let $\mathcal{D} \subset C$ denote the finite set of those points $Q \in C$ for which the line L_Q tangent to C at Q is also tangent to S at some point. For every $Q \in C^o := C \setminus (\mathcal{D} \cup S)$ the line $\mathcal{N}(Q)$ is well-defined by harmonicity condition on the tuple of four distinct lines through Q: L_Q, $\mathcal{N}(Q)$ and the complex lines Λ_1, Λ_2 through Q tangent to S. It says that there exists a projective involution of the space $\mathbb{CP}^1$ of complex lines through Q that fixes L_Q, $\mathcal{N}(Q)$ and permutes Λ_1, Λ_2. Let $E_j = E_j(Q)$, $j = 1, 2$, denote the tangency points of the lines Λ_j with S. Fix coordinates (x_1, x_2, x_3) on $\mathbb{R}^3$ (homogeneous coordinates on $\mathbb{RP}^2 \supset C$) in which $S = \{< Ax, x >= 0\}$, $A = \mathrm{diag}(1, 1, -1)$. Let us show that the projective billiard on C^o is the space form billiard with the matrix A: for every $Q \in C^o$ the two-dimensional subspaces $H_T(Q), H_{\mathcal{N}}(Q) \subset \mathbb{R}^3$ projected to the lines L_Q and $\mathcal{N}(Q)$ respectively are orthogonal in the scalar product $< Ax, x >$.

Fix a point $B \in \mathcal{N}(Q) \cap S$. The four points E_1, E_2, Q, B are distinct, and no three of them are collinear, since $Q \in C^o$. There exists a projective involution $\mathbf{I} : \mathbb{CP}^2 \to \mathbb{CP}^2$ fixing the points of the line QB and permuting E_1, E_2 (and hence, Λ_1, Λ_2). It fixes $\mathcal{N}(Q)$, and hence, L_Q, by harmonicity. It preserves S: the conic $\mathbf{I}(S)$ is tangent to S at E_1 and E_2 and intersects S at $B \neq E_{1,2}$; hence, $\mathbf{I}(S) = S$. Thus, $\mathbf{I}$ is the projectivization of a non-trivial linear involution $\mathbb{R}^3 \to \mathbb{R}^3$ preserving the quadratic form $< Ax, x >$ and transversal two-dimensional subspaces $H_T(Q)$, $H_{\mathcal{N}}(Q)$ and acting trivially on $H_{\mathcal{N}}(Q)$. This implies orthogonality of the latter subspaces in the scalar product $< Ax, x >$. Proposition 1.27 is proved.

9.4 Proof of Theorem 1.26 and its addendum

Proof of Theorem 1.26. Let a nonlinear germ of C^4-smooth curve $C \subset \mathbb{R}^2$ carry a transversal line field $\mathcal{N}$ defining a 0-homogeneously rationally integrable projective billiard. Then the dual billiard on the dual curve $\gamma = C^*$ is rationally integrable (Proposition 1.24). Let $C' \subset C$ denote the complement of the curve C to the set of its inflection points, i.e., points where the geodesic curvature vanishes. (A priori the set of inflection points may contain a straightline interval.) The dual to C' is a union of C^4-smooth arcs of the curve γ. The latter arcs are conics, by Theorem 1.16. Hence, C' is a union of conical arcs. The curve C being C^4-smooth, the boundary points of the set C' are not inflection points, and adjacent conical arcs paste C^4-smoothly. This implies that $C = C'$ is a conic.

The rationally 0-homogeneously integrable projective billiards on a (punctured) conic C are exactly those dual to the rationally integrable dual billiards on (punctured) conic γ (Proposition 1.24). Thus, it suffices to find the projective billiards dual to all the integrable dual billiards in Theorem 1.16. In each of these projective billiards the transversal line field $\mathcal{N}$ is defined on

$$C^o = C \setminus (\text{at most four points}).$$

Case 1): the dual billiard structure on a (punctured) conic γ is given by a pencil of conics. Then the complexified conic dual to any regular conic from the pencil is a complex caustic of the projective billiard on C. This together with Proposition 1.27 implies that the projective billiard is a space form billiard, whose space form matrix can be chosen $\mathrm{diag}(1, 1, -1)$.

To treate the other cases, let us introduce the next notations. For $Q \in C^o$ set

$$P = L_Q^* := \text{ the point dual to the line } L_Q; \quad P \in \gamma = C^*; \qquad (9.1)$$

$$\widetilde{P} = \mathcal{N}^*(Q) := \text{ the point dual to the projective line tangent to } \mathcal{N}(Q);$$

$$\widetilde{P} \text{ lies in the line } Q^* = L_P \text{ tangent to } \gamma \text{ at } P.$$

Proposition 9.4 *Consider the dual billiard on γ for the projective billiard defined by the line field $\mathcal{N}$. For every $Q \in C^o$ the point $\widetilde{P}$ is the unique fixed point distinct from P of the dual billiard involution $\sigma_P : Q^* \to Q^*$.*

The proposition follows from definition.

In what follows for every rationally integrable dual billiard from Theorem 1.16, cases 2a)–2d), we find the above fixed points $\widetilde{P}$ of the corresponding involutions. Their dual lines $\mathcal{N}(Q) = \widetilde{P}^*$ form the line field defining the corresponding projective billiard. To do this, we work in homogeneous coordinates $[z : w : t]$ in the ambient projective plane $\mathbb{RP}^2 \supset \gamma$ in which

$$\gamma = \{wt - z^2 = 0\}; \quad \gamma = \{w = z^2\} \text{ in the affine chart } \mathbb{R}^2_{z,w} = \{t = 1\}.$$

The curve C is projective dual to γ with respect to the duality $\mathbb{RP}^{2*} \to \mathbb{RP}^2_{[z:w:t]}$ given by the orthogonal polarity. We will work with the curve C in the new homogeneous coordinates $[x_1, x_2, x_3]$ given by the projectivization $[F] : \mathbb{RP}^2_{[z:w:t]} \to \mathbb{RP}^2_{[x_1,x_2,x_3]}$ of the linear map

$$F : (z, w, t) \mapsto (x_1, x_2, x_3) := (\frac{z}{2}, t, w). \tag{9.2}$$

For every point $Q \in C$ let $P \in \gamma$ be the corresponding point in (9.1). Set

$$z_0 := z(P).$$

Claim 14. *In the coordinates $[x_1 : x_2 : x_3]$ given by (9.2) one has*

$$Q = [-z_0 : z_0^2 : 1], \quad C = \{x_2 x_3 = x_1^2\}; \quad C \cap \{x_3 = 1\} = \{x_2 = x_1^2\}. \tag{9.3}$$

Proof The projective tangent line Q^* to γ at the point P and its orthogonal-polar-dual point $Q \in \mathbb{RP}^2_{[z:w:t]}$ are given by the equations

$$Q^* = \{-2z_0 z + w + z_0^2 t = 0\}, \ Q = [-2z_0 : 1 : z_0^2] \in C.$$

In the coordinates $[x_1 : x_2 : x_3]$ one has $Q = [-z_0 : z_0^2 : 1]$. □

Claim 15. *Let $Q = (x_1, x_2) \in C$ in the affine chart $\mathbb{R}^2_{x_1,x_2} = \{x_3 = 1\}$. Consider a Q-parametrized family of points $B(Q) \in Q^*$. Let $z(B(Q)) = g(z_0)$; $g(z_0)$ is a function of z_0. For every $Q \in C$ the dual to the point $B(Q)$ is the line through Q directed by the vector $(\dot{x}_1, \dot{x}_2) = (1, -2g(-x_1))$ at Q.*

Proof In the affine chart $\mathbb{R}^2_{z,w} = \{t = 1\} \subset \mathbb{RP}^2_{[z:w:t]}$ one has

$$w(B(Q)) = 2z(B(Q))z(P) - z^2(P) = 2g(z_0)z_0 - z_0^2.$$

Thus, $[z:w:t](B(Q)) = [g(z_0) : 2g(z_0)z_0 - z_0^2 : 1]$. Hence, the dual line $B^*(Q) \subset \mathbb{RP}^2_{[z:w:t]}$ is given by the equation $g(z_0)z + (2g(z_0)z_0 - z_0^2)w + t = 0$. Writing it in the coordinates (x_1, x_2, x_3), see (9.2), yields $2g(z_0)x_1 + (2g(z_0)z_0 - z_0^2)x_3 + x_2 = 0$. Thus, in the affine chart $\mathbb{R}^2_{x_1,x_2}$ the latter line is directed by the vector $(1, -2g(z_0)) = (1, -2g(-x_1(Q))$, by (9.3). $\square$

Case 2a): the dual billiard structure on γ is given by the family of involutions $\sigma_P : L_P \to L_P$ taking the form

$$\sigma_P : \zeta \mapsto \frac{(\rho-1)\zeta - (\rho-2)}{\rho\zeta - (\rho-1)}, \ \zeta = \frac{z}{z_0}, \ \rho = 2 - \frac{2}{2N+1}, \ \text{or } \rho = 2 - \frac{1}{N+1}.$$

The fixed point $\widetilde{P} \in L_P$ of the involution σ_P has ζ-coordinate $\frac{\rho-2}{\rho}$, hence

$$z(\widetilde{P}) = g(z_0), \ g(\theta) = \frac{\rho-2}{\rho}\theta.$$

Therefore, the dual line $\widetilde{P}^*$ is directed by the vector $(1, -2g(-x_1(Q))) = (1, \frac{2(\rho-2)}{\rho}x_1(Q))$ at Q. Thus, the line field $\mathcal{N}$ defining the projective billiard on C is directed by the vector field $(\rho, 2(\rho-2)x_1)$ on C. The latter field is tangent to the level curves of the quadratic polynomial $\mathcal{Q}_\rho(x_1, x_2) := \rho x_2 - (\rho-2)x_1^2$. Thus, it has type 2a) from Theorem 1.26.

Cases 2b), 2c), 2d): in the coordinate $u := z - z_0$, $z_0 = z(P)$, the involutions $\sigma_P : Q^* \to Q^*$ take the form

$$\sigma_P : u \mapsto -\frac{u}{1 + f(z_0)u}$$

$$f = f_{b1}(z) := \frac{5z-3}{2z(z-1)} \ \text{(type 2b1)), or} \ \ f = f_{b2}(z) := \frac{3z}{z^2+1} \ \text{(type 2b2)),}$$

$$f = f_{c1}(z) := \frac{4z^2}{z^3-1} \ \text{(type 2c1)), or} \ \ f = f_{c2}(z) := \frac{8z-4}{3z(z-1)} \ \text{(type 2c2)),}$$

$$f = f_d(z) := \frac{7z-4}{3z(z-1)}.$$

The u- and z-coordinates of the fixed point $\widetilde{P}$ of the involution σ_P are

$$u(\widetilde{P}) = -\frac{2}{f(z_0)}, \ z(\widetilde{P}) = z_0 - \frac{2}{f(z_0)}.$$

Subcase 2b1). One has

$$z(\widetilde{P}) = z_0 - \frac{4z_0(z_0-1)}{5z_0-3} = g(z_0), \quad g(\theta) = \frac{\theta(\theta+1)}{5\theta-3}.$$

Therefore, the dual line $\widetilde{P}^*$ is directed by the vector

$$(1, -2g(-x_1)) = (1, \frac{2x_1(x_1-1)}{5x_1+3}) = (1, \frac{2(x_2-x_1)}{5x_1+3}), \ x_j = x_j(Q).$$

Here we have substituted $x_1^2 = x_2$, since $Q \in C$. Hence, the line field $\mathcal{N}$ is directed by the vector field $(5x_1+3, 2(x_2-x_1))$ and thus, has type 2b1).

Subcase 2b2). One has

$$z(\widetilde{P}) = z_0 - \frac{2(z_0^2+1)}{3z_0} = g(z_0), \quad g(\theta) = \frac{\theta^2-2}{3\theta},$$

and the line field $\mathcal{N}$ on C is directed by the vector field $(1, -2g(-x_1)) = (1, \frac{2(x_1^2-2)}{3x_1})$. Or equivalently, by the vector field $(3x_1, 2x_2-4)$, since $x_1^2 = x_2$ on C. Thus, it has type 2b2).

Subcase 2c1). One has

$$z(\widetilde{P}) = z_0 + \frac{1-z_0^3}{2z_0^2} = g(z_0), \quad g(\theta) = \frac{\theta^3+1}{2\theta^2},$$

the field $\mathcal{N}$ is directed by the vector field $(1, -2g(-x_1)) = (1, \frac{x_1^3-1}{x_1^2}) = (1, \frac{x_1x_2-1}{x_2})$ on C. Or equivalently, by (x_2, x_1x_2-1). We get type 2c1).

Subcase 2c2). One has

$$z(\widetilde{P}) = z_0 - \frac{3z_0(z_0-1)}{4z_0-2} = g(z_0), \quad g(\theta) = \frac{\theta(\theta+1)}{4\theta-2},$$

the line field $\mathcal{N}$ is directed by the vector field $(1, \frac{x_1(x_1-1)}{2x_1+1}) = (1, \frac{x_2-x_1}{2x_1+1})$ on C. Or equivalently, by the field $(2x_1+1, x_2-x_1)$. Hence, it has type 2c2).

Subcase 2d). One has

$$z(\widetilde{P}) = z_0 - \frac{6z_0(z_0-1)}{7z_0-4} = g(z_0), \quad g(\theta) = \frac{\theta(\theta+2)}{7\theta-4},$$

the line field $\mathcal{N}$ is directed by the vector field $(1, \frac{2x_1(x_1-2)}{7x_1+4}) = (1, \frac{2x_2-4x_1}{7x_1+4})$. Or equivalently, by the field $(7x_1+4, 2x_2-4x_1)$. Hence, it has type 2d). This proves Theorem 1.26. $\square$

Proof of the addendum to Theorem 1.26. Consider the real conic $C = \{x_2x_3 = x_1^2\} \subset \mathbb{RP}^2_{[x_1:x_2:x_3]}$ equipped with a projective billiard structure from Theorem 1.26. Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be the transformation from (9.2):

$$(z, w, t) := F^{-1}(x_1, x_2, x_3) = (2x_1, x_3, x_2).$$

Let $[F]^{-1} : \mathbb{RP}^2 \to \mathbb{RP}^2$ denote the projectivization of the transformation F^{-1}. Recall that the orthogonal-polar-dual to the conic $[F]^{-1}(C)$ is the conic $\gamma = C^* = \{wt = z^2\}$, see Claim 14 above, and the post-composition of $[F]^{-1}$ with the duality sends the projective billiard on C to the corresponding dual billiard on γ given by Theorem 1.16. See the above proof of Theorem 1.26.

Proposition 9.5 *Let R be a rational integral of the dual billiard on γ written as a 0-homogeneous rational function $R(z, w, t)$. Then the function*

$$\widetilde{R}(x, v) := R(v_2, -2\Delta, -2v_1), \quad \Delta := x_1v_2 - x_2v_1, \tag{9.4}$$

is a 0-homogeneous rational integral of the projective billiard on C.

Proof In the affine charts $\mathbb{R}^2_{x_1,x_2} = \{x_3 = 1\} \subset \mathbb{RP}^2$, $\mathbb{R}^2_{z,w} = \{t = 1\} \subset \mathbb{RP}^2$ in the source and image the map $[F]^{-1}$ and its differential take the form

$$[F]^{-1} : (x_1, x_2) \mapsto (z, w) := \left(\frac{2x_1}{x_2}, \frac{1}{x_2}\right),$$

$$d[F]^{-1}(x_1, x_2)(v) = \hat{v} := \left(-\frac{2\Delta}{x_2^2}, -\frac{v_2}{x_2^2}\right), \ \Delta := x_1v_2 - x_2v_1.$$

Set $r := (z, w, 1) = (\frac{2x_1}{x_2}, \frac{1}{x_2}, 1)$, and let us identify $\hat{v} = (\hat{v}_1, \hat{v}_2)$ with

$$\hat{v} = (\hat{v}_1, \hat{v}_2, 0) = \left(-\frac{2\Delta}{x_2^2}, -\frac{v_2}{x_2^2}, 0\right) \in \mathbb{R}^3_{z,w,t}.$$

The function $R([r, \hat{v}])$ is an integral of the $[F]^{-1}$-pushforward of the projective billiard on C, which is a projective billiard on $[F]^{-1}(C)$; see Proposition 1.24. Therefore, $R([r, \hat{v}])$ written as a function of $x = (x_1, x_2)$ and $v = (v_1, v_2)$ is an integral of the projective billiard on C. One has

$$[r, \hat{v}] = \frac{1}{x_2^2}(v_2, -2\Delta, -2v_1).$$

Hence, $R([r, \hat{v}])$ takes the form (9.4), by 0-homogeneity. Proposition 9.5 is proved. □

In what follows we calculate the integral (9.4) explicitly for the integrals R listed in the addendum to Theorem 1.16.

Case 1): the dual billiard structure on γ is given by a pencil of conics containing γ. Then it admits a quadratic rational integral R, which is a ratio of two quadratic forms in (M_1, M_2, M_3). The corresponding integral (9.4) is a ratio of two quadratic forms in the vector $(v_2, -2\Delta, -2v_1)$.

Case 2a1): $\rho = 2 - \frac{2}{2N+1}$, the integral $R(z,w)$ written in the affine chart $\mathbb{R}^2_{z,w} = \{t = 1\}$ has type (1.6). In the homogeneous coordinates $[z : w : t]$ it takes the form

$$R(z,w,t) = \frac{(wt - z^2)^{2N+1}}{t^2 \prod_{j=1}^{N} (wt - c_j z^2)^2}, \quad c_j = -\frac{4j(2N+1-j)}{(2N+1-2j)^2}.$$

Substituting

$$(z,w,t) = (v_2, -2\Delta, -2v_1), \;\; \Delta = x_1 v_2 - x_2 v_1, \tag{9.5}$$

to R, see (9.4), and multiplying by 4 yields integral (1.14):

$$\Psi = \Psi_{2a1}(x_1, x_2, v_1, v_2) := \frac{(4v_1\Delta - v_2^2)^{2N+1}}{v_1^2 \prod_{j=1}^{N} (4v_1\Delta - c_j v_2^2)^2}.$$

Case 2a2): $\rho = 2 - \frac{1}{N+1}$, the integral $R(z,w)$ has type (1.7), and in the homogeneous coordinates

$$R(z,w,t) = \frac{(wt - z^2)^{N+1}}{zt \prod_{j=1}^{N} (wt - c_j z^2)}, \quad c_j = -\frac{j(2N+2-j)}{(N+1-j)^2}.$$

Substitution (9.5) and multiplication by -2 yield (1.15):

$$\Psi = \Psi_{2a2}(x_1, x_2, v_1, v_2) = \frac{(4v_1\Delta - v_2^2)^{N+1}}{v_1 v_2 \prod_{j=1}^{N} (4v_1\Delta - c_j v_2^2)}.$$

Case 2b1): $R(z,w)$ has type (1.8), and in the homogeneous coordinates

$$R(z,w,t) = \frac{(wt - z^2)^2}{(wt + 3z^2)(z - t)(z - w)}.$$

Substitution (9.5) yields (1.16):

$$\Psi = \Psi_{2b1}(x_1, x_2, v_1, v_2) = \frac{(4v_1\Delta - v_2^2)^2}{(4v_1\Delta + 3v_2^2)(2v_1 + v_2)(2\Delta + v_2)}.$$

Case 2b2): $R(z,w)$ has type (1.9), and in the homogeneous coordinates

$$R(z,w,t) = \frac{(wt - z^2)^2}{(z^2 + w^2 + wt + t^2)(z^2 + t^2)}.$$

Substitution (9.5) yields (1.17):

$$\Psi = \Psi_{2b2}(x_1, x_2, v_1, v_2) = \frac{(4v_1\Delta - v_2^2)^2}{(v_2^2 + 4\Delta^2 + 4v_1\Delta + 4v_1^2)(v_2^2 + 4v_1^2)}.$$

Case 2c1): $R(z, w)$ has type (1.10), and in the homogeneous coordinates

$$R(z, w, t) = \frac{(wt - z^2)^3}{(t^3 + w^3 - 2zwt)^2}.$$

Substitution (9.5) and multiplication by 64 yield (1.18):

$$\Psi = \Psi_{2c1}(x_1, x_2, v_1, v_2) = \frac{(4v_1\Delta - v_2^2)^3}{(v_1^3 + \Delta^3 + v_1v_2\Delta)^2}$$

Case 2c2): $R(z, w)$ has type (1.11), and in the homogeneous coordinates

$$R(z, w, t) = \frac{(wt - z^2)^3}{(8z^3 - 8z^2w - 8z^2t - w^2t - wt^2 + 10zwt)^2}.$$

Substituting (9.5) and multiplying by 64 yields (1.19):

$$\Psi = \Psi_{2c2}(x_1, x_2, v_1, v_2) = \frac{(4v_1\Delta - v_2^2)^3}{(v_2^3 + 2v_2^2v_1 + (v_1^2 + 2v_2^2 + 5v_1v_2)\Delta + v_1\Delta^2)^2}.$$

Case 2d): $R(z, w)$ is as in (1.12), and in the homogeneous coordinates

$$R(z, w, t) = \frac{(wt - z^2)^3}{(wt + 8z^2)(z - t)(wt^2 + 8z^2t + 4w^2t + 5wz^2 - 14zwt - 4z^3)}.$$

Substituting (9.5) and multiplying by -8 yields (1.20):

$$\Psi = \Psi_{2d}(x_1, x_2, v_1, v_2)$$

$$= \frac{(4v_1\Delta - v_2^2)^3}{(v_1\Delta + 2v_2^2)(2v_1 + v_2)(8v_1v_2^2 + 2v_2^3 + (4v_1^2 + 5v_2^2 + 28v_1v_2)\Delta + 16v_1\Delta^2)}.$$

The addendum to Theorem 1.26 is proved. $\square$

10 Billiards with complex algebraic caustics. Proof of Theorems 1.31 and 1.32

10.1 Case of Euclidean billiard. Proof of Theorem 1.31

Let $C \subset \mathbb{R}^2_{x_1,x_2} = \{x_3 = 1\} \subset \mathbb{R}^3_{x_1,x_2,x_3}$ be a C^2-smooth connected curve. We identify its ambient plane with the affine chart $\{x_3 = 1\} \subset \mathbb{RP}^2_{[x_1:x_2:x_3]}$. Let $\gamma = C^* \subset \mathbb{RP}^2_{[x_1:x_2:x_3]}$ be its orthogonal-polar dual curve. Consider the usual billiard on C. Its dual billiard on γ is given by *Bialy–Mironov angular symmetries* $\sigma_P : L_P \to L_P$, $P \in \gamma$, defined as follows: $\sigma_P(P) = P$; σ_P permutes points $a^*, b^* \in L_P$, if and only if the lines Oa^*, Ob^* are symmetric with respect to the line OP. See Fig. 5 below. Here $O = (0,0) \in \mathbb{R}^2$.

Remark 10.1 The Bialy – Mironov angular billiard was used in the solution of Bolotin's polynomial version of Birkhoff Conjecture [10, 11, 31].

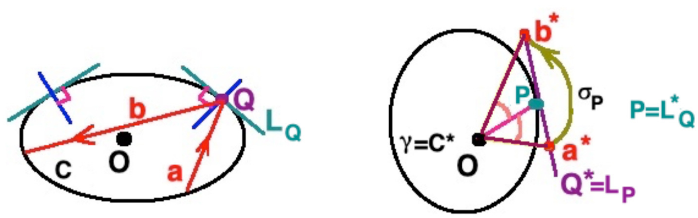

Figure 5: Euclidean billiard and its dual: Bialy – Mironov angular billiard

Proposition 10.2 *Let C, γ, σ_P be as above. Let the billiard in C have a complex algebraic caustic S. Let S^* be its complex projective dual, and let $H(x_1, x_2, x_3)$ be its defining homogeneous polynomial: $S^* = \{H = 0\}$, and H has the minimal possible degree. Set*

$$d := \deg H, \quad R(x_1, x_2, x_3) := \frac{H^2(x_1, x_2, x_3)}{(x_1^2 + x_2^2)^d}.$$

The function R is an integral of the Bialy–Mironov angual billiard on γ.

Proof Set

$$\mathbb{I} := \{x_1^2 + x_2^2 = 0\} \subset \mathbb{CP}^2_{[x_1:x_2:x_3]}.$$

For every $P \in \gamma$ the complexification of the angular symmetry $\sigma_P : L_P \to L_P$ is the projective involution of the complexified line L_P that permutes its intersection points with $\mathbb{I}$, see [10], [31, proposition 2.18]. Thus, it leaves invariant polar and zero loci $L_P \cap \mathbb{I}$, $L_P \cap S^*$ of the rational function $R|_{L_P}$. One has $S^* \not\subset \mathbb{I}$, since a caustic contains no straight line. This together with Proposition 4.12 applied to the involution σ_P implies non-constance and σ_P-invariance of the restriction $R|_{L_P}$ and proves Proposition 10.2. $\square$

Thus, the dual billiard structure on γ is rationally integrable. Therefore, γ (and hence, C) lies in a conic. In more detail, if C were C^4-smooth, then this would follow from Theorem 1.16. Let us treat the case, when C is C^2-smooth. The polar locus of the integral R lies in $\mathbb{I}$, and $R|_{L_P}$ is invariant under the angular symmetry. Therefore, the billiard on C is polynomially integrable, see the discussion on p. 1004 in [31], and hence, C is a conic, by [31, theorem 1.6]. Here is a more detailed explanation. The complex Zariski closure of the curve γ is an algebraic curve (Proposition 1.35). The family σ_P extends to a singular dual billiard structure on each its non-linear irreducible component, with integral R having polar locus in $\mathbb{I}$. Hence, each component is a conic, by [31, theorem 1.25]. Thus, C is a union of conical arcs. Different conical arcs (if any) should be confocal, see the discussion in [31, subsection 6.2]. Any two intersecting confocal conics are orthogonal. This together with C^2-smoothness of the curve C implies that C lies in a conic; see [31, subsection 6.3] for more details. Theorem 1.31 is proved.

10.2 Case of projective billiard. Proof of Theorem 1.32

Let S_1 and S_2 be two complex algebraic caustics. Let S_1^*, S_2^* be their dual curves. Let d_j denote the degrees of the curves S_j^*, and let $\mathcal{P}_j$ be their defining polynomials of degrees d_j. The dual curve $\gamma = C^*$ equipped with the corresponding dual billiard structure has a non-constant rational integral

$$R := \frac{\mathcal{P}_1^{2d_2}}{\mathcal{P}_2^{2d_1}},$$

as in the above proof of Proposition 10.2. Therefore, γ, and hence, C is a conic, by Theorem 1.16. Theorem 1.32 is proved.

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