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Anosov Actions of Isometry Groups on Lorentzian 2-Orbifolds

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Abstract—We prove a criterion of an Anosov action of the isometry group for compact Lorentzian 2-orbifolds. It is proved also that a non-compact complete flat Lorentzian 2-orbifold has an Anosov isometry if and only if its isometry group Lie acts improperly. The existence of chaotic behavior of such Anosov actions is investigated. It is shown that among smooth 2-orbifolds other than manifolds, only the "pillow" and the $\mathbb{Z}_2$-cone admit the specified Lorentz metric with an Anosov action of the isometry group. The corresponding Lorentzian metrics and groups of isometries are indicated.

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1. INTRODUCTION. MAIN RESULTS

Anosov diffeomorphisms were introduced by D.V. Anosov [2] under the name U-systems, and he proved that their behaviour was in an appropriate sense generic (when they exist at all). Anosov diffeomorphisms play an important role in dynamics as the notion represents the most perfect kind of global hyperbolic behavior, giving examples of structurally stable dynamical systems. In his seminal paper [13], S. Smale gave the first example of a non-toral Anosov diffeomorphism and raised the problem of classifying the closed manifolds admitting an Anosov diffeomorphism.

In this paper, the concept of an Anosov diffeomorphism of a manifold is generalized to smooth orbifolds. The original definition of an orbifold was due to Satake [12], who called them V-manifolds. We recall the main notions on orbifolds in Section 2.2.1, more detailed information about orbifolds can be found in [1]. A smooth group action of a diffeomorphism group of G of an orbifold $\mathcal{N}$ is called Anosov if there exists an element of G which is an Anosov orbifold diffeomorphism.

Orbifolds are used in various fields of mathematics and physics. In modern theoretical physics orbifolds are used as string propagation spaces. Orbifolds have also proved useful in conformal field theory. The theory of deformation quantization is developed on the symplectic orbispaces which include symplectic orbifolds. An overview of orbifold applications can be found in [1]. Beyond various applications in mathematics and physics, orbifolds have been applied to the music theory: a musical chord can be represented as a point in an orbifold: $\mathbb{T}^n/S_n$, where $\mathbb{T}^n$ is the n-torus and S_n is the symmetric group [15].

Orbifolds arise in foliation theory as Hausdorff leaf spaces. Thurston's well-known results on the classification of closed three-dimensional manifolds use the classification of two-dimensional compact orbifolds [14].

Automorphism groups of geometric structures on orbifolds were studied in [3, 4, 19]. Riemannian geometry on orbifolds was studied in [7]. The classification two-dimensional compact Lorentzian orbifolds with non-compact isometry groups was obtained in [17]. Let $(\mathcal{N}, g)$ be a Lorentzian orbifold. The Lie group of all its isometries is denoted by $Iso(\mathcal{N}, g)$.

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The aim of this work is to investigate the structure of two-dimensional compact Lorentzian orbifolds $(\mathcal{N}, g)$ admitting an Anosov action of the isometry group $Iso(\mathcal{N}, g)$ and to show that behaivior Anosov isometries of non-compact Lorentzian orbifolds $(\mathcal{N}, g)$ radically differs from their behavior on compact orbifolds. Emphasize that we are investigating orbifolds other than manifolds.

An Anosov diffeomorphism of a Lorentzian orbifold $(\mathcal{N}, g)$, which is an isometry of $(\mathcal{N}, g)$, is referred to as an *Anosov isometry of* $(\mathcal{N}, g)$. An isometry $\varphi \in Iso(\mathcal{N}, g)$ of a compact orbifold is called *essential* if it generates a non-compact group $\Phi = \langle\varphi\rangle$. Anosov actions of essential isometries on compact Lorentzian orbifolds are closely related to chaotic actions. By an analogy to chaotic behaviour of diffeomorphism groups [5], we give the following definition.

Definition 1. A diffeomorphism group G of a smooth orbifold $\mathcal{N}$ is chaotic (or chaotically acting on $\mathcal{N}$) if it satisfies the following:

(i) there exists a dense orbit of G in $\mathcal{N}$ (topological transitivity);

(ii) the union of all closed orbits of G is dense in $\mathcal{N}$ (density of closed orbits).

A diffeomorphism h of a smooth orbifold $\mathcal{N}$ is chaotic if it generates a chaotic group $H = \langle h\rangle$.

For two-dimensional compact Lorentzian orbifolds, we prove the following criterion of an Anosov action of their isometry groups.

Theorem 1. *Let $(\mathcal{N}, g)$ be a two-dimensional compact Lorentzian orbifold and $Iso(\mathcal{N}, g)$ be the Lie group of all its isometries. Then the following conditions are equivalent:*

(1) *the action of the isometry group $Iso(\mathcal{N}, g)$ on $\mathcal{N}$ is Anosov;*

(2) *the isometry group $Iso(\mathcal{N}, g)$ is not compact;*

(3) *the isometry group $Iso(\mathcal{N}, g)$ acts improperly on $\mathcal{N}$;*

(4) *there is a chaotic isometry $h \in Iso(\mathcal{N}, g)$.*

Throughout the rest of this paper, we consider the standard two-dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and we call a pair of vectors $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ standard basis of the tangent vector the space $T_x\mathbb{T}^2$, $x \in \mathbb{T}^2$. Through $\Omega : \mathbb{R}^2 \to \mathbb{T}^2$ is denoted the quotient map which is the universal covering of the torus. Anosov torus automorphism $\mathbb{T}^2$, given by a matrix $A \in SL(2, \mathbb{Z})$, is denoted by A, and E is the symbol the identity matrix 2×2.

Theorem 1 and the results of the work of N.I. Zhukova and E.A. Rogozhina [17] implies the following statement describing the structure of two-dimensional compact Lorentz orbifolds with Anosov actions of isometry groups.

Theorem 2. *If the action of the isometry group of a two-dimensional compact Lorentz orbifold $(\mathcal{N}, g)$ is Anosov, then:*

1) there is a finite-sheeted regular covering map $p : \mathbb{T}^2 \to \mathcal{N}$ by a flat Lorentz torus $(\mathbb{T}^2, g_0)$, where the Lorentzian metric is g_0 in the standard basis has the form

$$(g_{ij}) = \eta \begin{pmatrix} -2c & a-d \\ a-d & 2b \end{pmatrix}, \tag{1}$$

where η is some nonzero real number, a, b, c, d are integers satisfying the following conditions:

$$\begin{cases} ad - bc = 1 \\ a + d > 2 \end{cases}; \tag{2}$$

2) the orbifold $\mathcal{N}$ is isomorphic (in the category $\mathfrak{Orb}$) "pillow" $\mathcal{P} \cong \mathbb{T}^2/\{\pm E\}$, *3) there exists an Anosov torus automorphism $f \in Iso(\mathbb{T}^2, g_0)$ given by the matrix*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}), \tag{3}$$

which projects onto some Anosov diffeomorphism $f_A \in Iso(\mathcal{N}, g)$ respectively to $p : \mathbb{T}^2 \to \mathcal{N}$, and the stable and unstable foliations of f_A are formed by isotropic geodesics dense in $\mathcal{N}$;

4) the group of orientation-preserving isometries of the Lorentz orbifold $(\mathcal{N}, g)$ is isomorphic to the semidirect product of the groups $\mathbb{Z} \ltimes (\mathbb{Z} \times \mathbb{Z})$.

Moreover, up to a factor, there is a countable family of Lorentzian metrics g on the "pillow" $\mathcal{P}$ admitting an Anosov action of the isometry group $Iso(\mathcal{P}, g)$.

Corollary 1. *The "pillow" is a unique (up to isomorphism in the category $\mathfrak{Orb}$) compact orbifold admitting a Lorentzian metric with an Anosov action of the isometry group.*

Corollary 2. *The "pillow" $\mathcal{P} = \mathbb{T}^2/\{\pm E\}$ with the flat Lorentz metric g given by the matrix*

$$(g_{ij}) = \eta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

does not allow Anosov isometries.

Remark 1. For more complete information about the non-compact isometry groups of compact Lorentzian orbifolds, see [17, Theorem 5].

Remark 2. The fact that the group of isometries of a closed Lorentzian 2-manifold (M, g) is non-compact only if $(M, g) = (\mathbb{T}^2, g)$ is a flat Lorentzian torus, was proved by P. Mounoud [11].

Remark 3. In contrast to our Definition 5, in [16] and [9] in the definition of an Anosov diffeomorphism on the plane, only the continuity of the respective Riemannian metric is required. With this definition, W. White [16] proved that a translation of the plane is an Anosov diffeomorphism, and S. Matsumoto [9] constructed an example of an Anosov diffeomorphism of a plane without fixed points, which is not topological conjugate to a translation. This shows that there are Anosov actions of groups of diffeomorphisms of the plane in the sense of the specified definition that are proper actions.

As compact Lorentzian orbifolds with an Anosov action of their isometry groups are complete and flat, we also investigate dynamics of the isometry groups of non-compact complete flat Lorentzian two-dimensional orbifolds and prove the following theorem.

Theorem 3. *Let $(\mathcal{N}, g)$ be a non-compact complete flat Lorentzian 2-orbifold. Then the following conditions are equivalent:*

(1) *the action of the isometry group $Iso(\mathcal{N}, g)$ on $\mathcal{N}$ is Anosov;*

(2) *the isometry group $Iso(\mathcal{N}, g)$ acts improperly on $\mathcal{N}$.*

Moreover, every Anosov isometry of $(\mathcal{N}, g)$ is not chaotic.

Theorem 3 shows a significant difference in the behavior of isometries of compact and non-compact Lorentzian orbifolds.

The following two statements follows from the proof of Theorem 3.

Corollary 3. *The stable and unstable foliations of every Anosov isometry of a complete flat Lorentian orbifold are formed by its isotropic geodesics.*

Corollary 4. *Every Anosov isometry of a complete flat Lorentian orbifold has a fixed point.*

C. Counsillas [8] introduces the concept of a topological Anosov homeomorphism and proves that such a homeomorphism of the plane $\mathbb{R}^2$ has a fixed point.

Theorem 3 and the results of the work of the authors [6] implies the following statement describing the structure of two-dimensional non-compact complete flat Lorentz orbifolds with Anosov actions of their isometry groups.

Theorem 4. *Let $(\mathcal{N}, g)$ be a non-compact complete flat Lorentzian 2-orbifold. If the action of the isometry group of a two-dimensional compact Lorentz orbifold $(\mathcal{N}, g)$ is Anosov, then:*

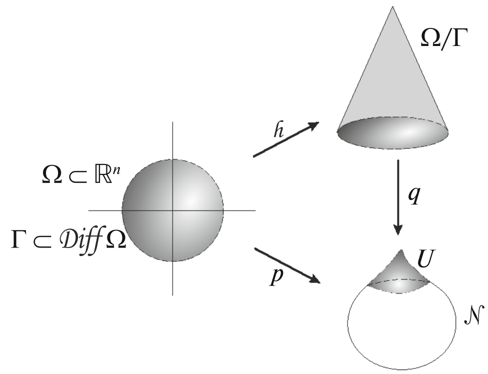

Fig. 1. An orbifold chart on $\mathcal{N}$.

(1) *the orbifold* $\mathcal{N}$ *is isomorphic to the* $\mathbb{Z}_2$*-cone* $\mathbb{R}^2/\{\pm E\}$;

(2) *the isometry group* $Iso(\mathcal{N}, g)$ *is isomorphic to the quotient group* $O(1,1)/\{\pm E\}$ *where* $O(1,1)$ *is the pseudo-orthogonal group.*

Corollary 5. *Up to isomorphism in the category* $\mathfrak{Orb}$*, there exist only two proper* 2*-orbifolds admitting complete flat Lorentz metrics with Anosov actions of their isometry groups, they are:*

(1) *the "pillow"* $\mathcal{P} = \mathbb{T}^2/\{\pm E\}$;

(2) *the cone* $\mathcal{N} = \mathbb{R}^2/\{\pm E\}$.

Remark 4. In contrast to compact orbifolds, for any complete pseudo-Euclidean metric g on a $\mathbb{Z}_2$-cone $\mathcal{N}$, the action of the isometry group $Iso(\mathcal{N}, g)$ is Anosov.

Remark 5. All Anosov diffeomorphisms considered here are isometries of Lorentzian orbifolds, so they preserve the volume form induced by the Lorentzian metric.

Assumption. By smoothness (of orbifolds, maps and other objects) we mean the smoothness of the class C^r where $r \geq 2$.

2. BASIC CONCEPTS

2.1. Category of Orbifolds

Let $\mathcal{N}$ be a connected Hausdorff topological space with a countable base, U be an open subset of $\mathcal{N}$. Assume that Ω is a connected open subset of the n-dimensional arithmetic space $\mathbb{R}^n$ and Γ is a finite diffeomorphism group of Ω. The triple (Ω, Γ, p) is called a chart on $\mathcal{N}$, if there exists a homeomorphism $h : \Omega/\Gamma \to U$ for some subset of $U \subset \mathcal{N}$ such that $p = q \circ h$ where $q : \Omega \to \Omega/\Gamma$ is a quotient map (see Figure 1).

Consider two orbifold charts (Ω, Γ, p) and $(\widetilde{\Omega}, \widetilde{\Gamma}, \widetilde{p})$ with neighborhoods U and $\widetilde{U}$, where $U \subset \widetilde{U}$ is a smooth embedding of $\phi_{\widetilde{U}U} : \Omega \to \widetilde{\Omega}$ such that $p = \widetilde{p} \circ \phi_{\widetilde{U}U}$ is called an embedding of the chart (Ω, Γ, p) to the chart $(\widetilde{\Omega}, \widetilde{\Gamma}, \widetilde{p})$.

It is said that two cards (Ω, Γ, p) and $(\widehat{\Omega}, \widehat{\Gamma}, \widehat{p})$ with neighborhoods U and $\widehat{U}$, respectively, satisfy the coherent condition if for any points $x \in U \cap \widehat{U}$ there is a chart $(\widetilde{\Omega}, \widetilde{\Gamma}, \widetilde{p})$ with the coordinate neighborhood $W = \widetilde{p}(\widetilde{\Omega})$ such that $x \in W \subset U \cap \widehat{U}$, for which there are embeddings $\phi_{UW} : \widetilde{\Omega} \to \Omega$ and $\phi_{\widehat{U}W} : \widetilde{\Omega} \to \widehat{\Omega}$.

Definition 2. A family of charts $\mathcal{A} = \{(\Omega_\alpha, \Gamma_\alpha, p_\alpha) \mid \alpha \in \mathfrak{J}\}$ is called a smooth atlas if the following two conditions hold true:

1) the set $\{U_\alpha = p_\alpha(\Omega_\alpha) \mid \alpha \in \mathfrak{J}\}$ is an open covering of the topological space $\mathcal{N}$;

2) any two charts from the atlas $\mathcal{A}$ satisfy the coherent condition.

Recall that an atlas $\mathcal{A}$ is called *maximal* with respect to inclusion if it coincides with any atlas containing it.

Definition 3. The pair $(\mathcal{N}, \mathcal{A})$, where $\mathcal{A}$ is the maximal atlas on $\mathcal{N}$, is called a smooth n-dimensional orbifold and is still denoted by $\mathcal{N}$.

Smooth orbifold maps. Let $\mathcal{N}$ and $\mathcal{N}'$ be smooth orbifolds with atlases $\mathcal{A}$ and $\mathcal{A}'$, respectively. The continuous mapping $f\colon \mathcal{N} \to \mathcal{N}'$ is called smooth if for any point $x \in \mathcal{N}$ there are such charts $(\Omega, \Gamma, p) \in \mathcal{A}$ and $(\Omega', \Gamma', p') \in \mathcal{A}'$ that $x \in U = p(\Omega)$, $f(U) \subset U' = p'(\Omega')$, and there is a smooth mapping $f_{U'U}\colon \Omega \to \Omega'$ satisfying the equality $p' \circ f_{U'U} = f|_U \circ p$. A map $f_{U'U}$ is called a representative of a map f in charts (Ω, Γ, p) and (Ω', Γ', p'), and $f_{U'U}$ is defined up to the composition with elements of the groups Γ_U and $\Gamma_{U'}$, respectively.

Denote by $\mathfrak{Orb}$ the category of orbifolds, whose objects are smooth orbifolds and whose morphisms are smooth maps of orbifolds. Note that the category of smooth manifolds is a complete subcategory in $\mathfrak{Orb}$.

The stratification of orbifolds. For any point x of an orbifold $\mathcal{N}$, there is a chart (Ω, Γ, p) such that the neighborhood $U = p(\Omega)$ contains x. Take $y \in p^{-1}(x)$ and denote by $(\Gamma)_y$ the stationary subgroup of the group Γ at the point y. We emphasize that for a given point x an abstract group $(\Gamma)_y$ does not depend on the choice of the chart (Ω, Γ, p) and $y \in p^{-1}(x)$. The abstract group $(\Gamma)_y$ is referred to as the orbifold group at the point x [3]. A point x is called regular if its orbifold group is trivial. A point x is referred to as orbifold point if its orbifold group is not trivial.

Let $\mathcal{N}$ be an n-dimensional smooth orbifold. It is said that points x and y from $\mathcal{N}$ have the same orbifold type if there exist neighborhoods of these points that are isomorphic in the category of orbifolds $\mathfrak{Orb}$. A subset of points of the same orbifold type with an induced topology has a natural structure of a smooth manifold, which, generally speaking, is not connected. Manifolds of points of different types can have the same dimension. Denote by Δ^k the union of the specified manifolds of dimension k. Possible $\Delta_k = \emptyset$ for $k \in \{0, \ldots, n-1\}$. As is known, every connected component Δ^k_i of Δ^k is formed by points of the same orbifold type. The set

$$\Delta(\mathcal{N}) = \{\Delta^k_i \mid k \in \{0, \ldots, n\}, i \in J(k)\}$$

is called *the stratification* of the orbifold $\mathcal{N}$, and every Δ^k_i is called its *k-dimensional stratum*. This definition is equivalent to the definition of the stratification in [1].

The stratum Δ^n is formed by regular points, and Δ^n is a connected, open and dense subset in $\mathcal{N}$. Moreover, Δ^n with an induced smooth structure is an n-dimensional manifold.

By a diffeomorphism of a smooth n-dimensional orbifold $\mathcal{N}$ we mean an automorphism f of $\mathcal{N}$ in the category of smooth orbifolds $\mathfrak{Orb}$. Denote by $\mathrm{Diff}(\mathcal{N})$ the group of all diffeomorphisms of the orbifold $\mathcal{N}$. We emphasize that the orbifold stratification of $\mathcal{N}$ is invariant with respect to the group $\mathrm{Diff}(\mathcal{N})$.

Note that the topological space of an n-dimensional orbifold is $\mathcal{N}$ for $n \geq 3$, generally speaking, it is not locally Euclidean in contrast to $n = 2$ ([4], Example 1).

2.2. Lorentzian Metrics on Orbifolds

It is said that on a smooth n-dimensional orbifold $\mathcal{N}$, a Lorentzian metric g is given if for every chart (Ω, Γ, p), Γ-invariant Lorentzian metric g_Ω is defined on Ω, and for every injection $f_{\widetilde{\Omega}\Omega} : \Omega \to \widetilde{\Omega}$ of the chart (Ω, Γ, p) into the chart $(\widetilde{\Omega}, \widetilde{\Gamma}, \widetilde{p})$ the equality holds $(f_{\widetilde{\Omega}\Omega})^* g_{\widetilde{\Omega}} = g_\Omega$.

Similarly, the concept of a Riemannian metric on orbifolds is introduced.

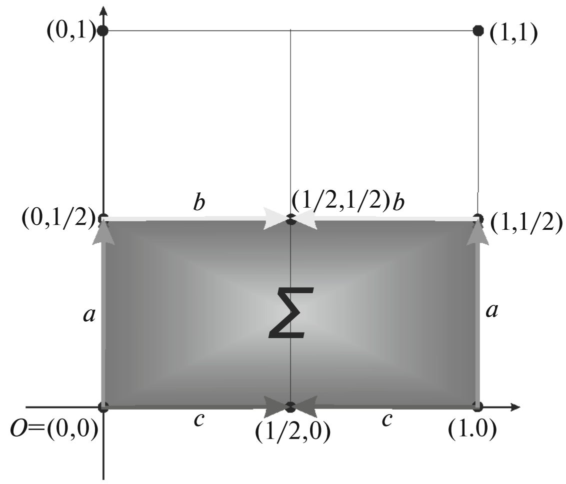

Fig. 2. The orbifold "pillow."

2.3. The Orbifolds "Pillow" and $\mathbb{Z}_2$-Cone

The orbifold "pillow". Let $\mathbb{T}^2$ be the standard two-dimensional torus. Denote by γ the diffeomorphism of $\mathbb{T}^2$ defined by the matrix $-E$ where E is the unit 2×2 matrix. Let Γ be equal to the group $\langle\gamma\rangle \cong \mathbb{Z}_2$. Then $\mathcal{N} = \mathbb{T}^2/\Gamma$ is very good orbifold which is called the "pillow".

Let the torus $\mathbb{T}^2$ be represented by the square $[0,1] \times [0,1]$ with glued in the corresponding directions by the parties. The fundamental polygon of the orbifold $\mathcal{N}$ is a curvilinear polygon Σ in the square $[0,1] \times [0,1]$ such that $\mathcal{N} = p(\Sigma)$, and restriction of $p|_{\Sigma^0}$ to the interior of Σ^0 of the polygon Σ is a homeomorphism onto the two-dimensional stratum Δ^2. As a fundamental polygon Σ in the torus $\mathbb{T}^2$ for orbifold $\mathcal{N}$, we can take the rectangle $[0,1] \times [0,1/2]$, the rule of gluing the sides of which is shown in Figure 2.

Therefore, the orbifold $\mathcal{N}$ has a zero-dimensional strata $\{\Delta_i^0 = w_i \mid i = \overline{1,4}\}$ consisting of four points $w_i = p(z_i)$, where $z_1 = (0,0)$, $z_2 = (1/2,0)$, $z_3 = (0,1/2)$, $z_4 = (1/2,1/2)$ belong to Σ_0. Thus, the "pillow" has the following stratification $\{\Delta^2, \Delta_i^0 \mid i = \overline{1,4}\}$.

The orbifold $\mathbb{Z}_2$-cone. Let γ be the diffeomorphism of the plane $\mathbb{R}^2$ defined by the equality $\gamma(x) = (-x)\ \forall x \in \mathbb{R}^2$. Denote by $\Gamma = \langle\gamma\rangle \cong \mathbb{Z}_2$ a diffeomorphism group generated by γ. Then a very good orbifold $\mathcal{N} = \mathbb{R}^2/\Gamma$ is defined, and it is also denoted by $\mathcal{N} = \mathbb{R}^2/\{\pm E\}$. This orbifold is referred to as $\mathbb{Z}_2$-cone. Denote by the $p : \mathbb{R}^2 \to \mathcal{N}$ the quotient map. The orbifold has a single orbifold point $p(O)$, where $O = (0,0)$ is zero in $\mathbb{R}^2$, and its stratification has the form $\Delta = \{\Delta^2, \Delta^0\}$, $\Delta^0 = p(O)$.

2.4. Improper Actions of Groups on Orbifolds

Recall that a continuous mapping $f\colon X \to Y$ of topological spaces is called a proper if the preimage of any compact subset of Y is a compact set of X.

Proposition 1. *Let $\Phi\colon G \times \mathcal{N} \to \mathcal{N}$ be a smooth action of a Lie group G on an orbifold $\mathcal{N}$. Then the following three conditions are equivalent:*

(i) *the inducted mapping*

$$(\Phi, \mathrm{id}_{\mathcal{N}})\colon G \times \mathcal{N} \to \mathcal{N} \times \mathcal{N}\colon (g,x) \mapsto (g(x), x) \quad \forall (g,x) \in G \times \mathcal{N}$$

is proper;

(ii) *if there exist consequences $\{g_n\} \subset G$ and $\{x_n\} \subset \mathcal{N}$ such that $x_n \to x$ and $g_n(x_n) \to y$, $x, y \in \mathcal{N}$, then $\{g_n\}$ has a convergence subsequence in G;*

(iii) *for all compact sets K and L in $\mathcal{N}$ the set $\{g \in G \mid K \cap g(L) \neq \varnothing\}$ is compact in G.*

Proof. The proof of Proposition 1 is analogously to the proof of the respective assertion for manifold [10, p. 41]. □

Definition 4. A smooth action of a Lie group G on an orbifold $\mathcal{N}$ is called *proper* if it satisfies at least one of the three conditions of Proposition 1, otherwise the action is called *improper*.

Corollary 6. *If a smooth action of a Lie group G on an orbifold $\mathcal{N}$ is proper, then the stationary subgroup G_z is compact at every point $z \in M$.*

Proof. Assume that a Lie group G acts on an orbifold $\mathcal{N}$ properly. Pick a point $x \in \mathcal{N}$. Put $K = L = \{x\}$. In this case, the statement (iii) of Proposition 1 implies that the stationary subgroup G_x of G at the point x is compact. □

2.5. Anosov Actions of Diffeomorphism Groups on Orbifolds

Definition 5. A diffeomorphism $f : M \to M$ of a smooth manifold M is called *Anosov* if the tangent bundle TM admits a continuous invariant splitting $T(M) = E^s \oplus E^u$ such that the differential f_* expands E^u and contracts E^s exponentially, that is, there exist constants $c > 0$ and $0 < \lambda < 1$ satisfying the following inequalities

$$||f_*^n(X)|| \leq c\lambda^n ||X|| \quad \forall X \in E^s, \quad ||f_*^n(Y)|| \geq c\lambda^{-n}||Y|| \quad \forall Y \in E^u, \tag{4}$$

for all $n \in \mathbb{N}$, where $||\cdot||$ is the norm defined by some complete Riemannian metric on M.

The subbundles E^s and E^u of TM are called stable and unstable, respectively.

As is known, the set of all regular points of a n-dimensional smooth orbifold $\mathcal{N}$ forms the stratum Δ^n, which is a connected smooth n-dimensional manifold, and Δ^n is dense in $\mathcal{N}$. Let $f : \mathcal{N} \to \mathcal{N}$ be an arbitrary diffeomorphism of $\mathcal{N}$ in the category $\mathfrak{Orb}$. It follows from the definition of the category $\mathfrak{Orb}$, that f maps a regular point of an orbifold to a regular one. Consequently, the stratum Δ^n is invariant under f. Therefore the following definition is correct.

Definition 6. Let $\mathcal{N}$ be an n-dimensional smooth orbifold. A diffeomorphism f of $\mathcal{N}$ is called *Anosov* if its restriction $f|_{\Delta^n}$ onto the n-dimensional stratum Δ^n is an Anosov diffeomorphism of the manifold Δ^n with respect to the restriction onto of Δ^n some complete Riemannian metric on $\mathcal{N}$.

3. THE PROOF OF THEOREM 1

(1) $\Rightarrow$ (2). Suppose that the action of the isometry group $Iso(\mathcal{N}, g)$ of a compact two-dimensional Lorentzian orbifold $(\mathcal{N}, g)$ is Anosov. By definition, this means that there is an isometry $h \in Iso(\mathcal{N}, g)$ such that the restriction of $h|_{\Delta^2}$ onto the manifold of regular points Δ^2 is an Anosov diffeomorphism with respect to some Riemannian metric d on Δ^2 which is induced by a complete Riemannian metric d on $\mathcal{N}$. We will denote the induced metric also by d.

According to [17, Theorem 4], every two-dimensional Loretzian orbifold is very good. Hence there exists some compact Loretzian manifold (M, g_0) and its isometry group Ψ isomorphic to either $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_2$ such that $\mathcal{N} = M/\Psi$. Denote by $p : M \to \mathcal{N} = M/\Psi$ the quotient map. According to [17, Prop. 1], there exists an isometry $\widetilde{h} \in Iso(M, g_0)$ lying over h relatively p, i.e., $p \circ \widetilde{h} = h \circ p$. If the Lorentzian orbifold $\mathcal{N}$ is not oriented, there exists a two-sheeted pseudo-Riemannian covering $k : \hat{\mathcal{N}} \to \mathcal{N}$ of $\mathcal{N}$ by an oriented Lorentzian orbifold $(\hat{\mathcal{N}}, \hat{g})$ and every isometry h of $(\mathcal{N}, g)$ there exists $\hat{h} \in Iso(\hat{\mathcal{N}}, \hat{g})$ such that $k \circ \hat{h} = h \circ k$. We say that $\hat{h}$ is a lift of h to $\hat{\mathcal{N}}$. Since both orbifolds $\mathcal{N}$ and $\hat{\mathcal{N}}$ are compact, it follows that both groups of isometries $Iso(\mathcal{N}, g)$ and $Iso(\hat{\mathcal{N}}, k^*g)$ are simultaneously either compact or not compact. Thus, without loss generality, we assume that the Lorentzian orbifold $(\mathcal{N}, g)$ is oriented. In this case, $\Psi = \langle\psi\rangle \cong \mathbb{Z}_2$. It follows from [17, Theorem 4] and the specifics of the pseudo-orthogonal group $O(1, 1)$, that at every fixed point $u \in M$ the matrix of the differential ψ_{*u} is equal to $-E$, where E is the unit matrix. Hence, every fixed point of ψ is isolated in M. Note that orbifold points of $\mathcal{N}$ are images of fixed points of ψ. Therefore, every orbifold point $\mathcal{N}$ is isolated in $\mathcal{N}$. Since the set of all orbifold points of $\mathcal{N}$ is closed in $\mathcal{N}$, then compactness of $\mathcal{N}$ implies finiteness of the set of orbifold points. Let $H = \langle h\rangle$ be the isometry group generated by h. Pick an orbifold point $v = p(u)$. As orbifold points have the same orbifold type, the orbit $H.v = \{h^n(v) \mid n \in \mathbb{Z}\}$ is equal to the set of orbifold points, hence it is finite. Therefore the stationary subgroup H_v of H at v contains an Anosov isometry h^m, $m \in \mathbb{Z}$, so

H_v is a non-compact subgroup of the stationary isometry subgroup $Iso_v(\mathcal{N}, g)$ at v. This means that the Lie group $Iso_v(\mathcal{N}, g)$ is also non-compact. Corollary 6 implies that the action of $Iso(\mathcal{N}, g)$ on the orbifold is not proper. Thus, the implication (1) $\Rightarrow$ (2) is proved.

The equivalence (2) $\Leftrightarrow$ (3) for compact two-dimensional Lorentzian orbifolds is known [17].

(3) $\Rightarrow$ (4) Assume that $(\mathcal{N}, g_0)$ is a compact two-dimensional Loretzian orbifold, and the Lie group $Iso(\mathcal{N}, g)$ acts improper on $\mathcal{N}$, that is equivalent to non-compactness of $Iso(\mathcal{N}, g)$. As known ([17, Theorem 1]), there exists an Anosov automorphism $\widetilde{h}$ of the torus $\mathbb{T}^2$ which is an isometry of $(\mathcal{N}, g)$, and $\widetilde{h}$ lies over some isometry $h \in Iso(\mathcal{N}, g)$ with respect to the projection $p : M \to \mathcal{N} = M/\Psi = \mathcal{P}$ indicated above. It is well known [2], that $\widetilde{h}$ is chaotic. Pick $u \in \mathbb{T}^2$, let $v = p(u) \in \mathcal{N}$. Denote by $\mathcal{O}.u = \{\widetilde{h}^n(u) \mid n \in \mathbb{Z}\}$ and $\mathcal{O}.v = \{h^n(u) \mid n \in \mathbb{Z}\}$ the respective orbits of the points u and v. Since $p \circ \widetilde{h} = h \circ p$, then $p(\mathcal{O}.u) = \mathcal{O}.v$. Therefore, every periodic orbit of the group $\langle\widetilde{h}\rangle$ projects onto periodic orbit of the group $\langle h\rangle$, a dense orbit of the group $\langle\widetilde{h}\rangle$ projects onto dense orbit of the group $\langle h\rangle$. Hence, chaotic behaviour of $\widetilde{h}$ implies chaotic behaviour of h. Thus, h is a chaotic isometry of $(\mathcal{N}, g)$.

(4) $\Rightarrow$ (1) Assume that there exists a chaotic isometry h of a compact Lorentzian orbifold $(\mathcal{N}, g)$. According to [17, Theorem 4], every two-dimensional Loretzian orbifold is very good. Hence there exists some compact Loretzian manifold (M, g_0) and its isometry group Ψ isomorphic to either $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_2$ such that $\mathcal{N} = M/\Psi$. Denote by $p : M \to \mathcal{N} = M/\Psi$ the quotient map. According to [17, Prop. 1], there exists an isometry $\widetilde{h} \in Iso(M, g_0)$ lying over h relatively p, i.e., $\widetilde{h} \circ p = p \circ h$. We use the notations entered above. The last equality implies that $p(\mathcal{O}.u) = \mathcal{O}.v$ for every $u \in M$ and $v = p(u)$. Since Ψ is a finite group, we get that chaotic behaviour of h implies chaotic behaviour of $\widetilde{h}$. It is well known that there are no isometries of complete Riemannian manifolds that have chaotic behavior. Therefore, the group $Iso(M, g_0)$ of a compact manifold M is essential. This means that the action of $Iso(M, g_0)$ on M is improper and the group $Iso(M, g_0)$ is not compact. According to [17, Theorem 1], the manifold M is $\mathbb{T}^2$ (up to a diffeomorphism), and the Lorentzian metric g_0 on $\mathbb{T}^2$ in the standard basis has the following form

$$(g_{ij}) = \eta \begin{pmatrix} -2c & a-d \\ a-d & 2b \end{pmatrix}, \tag{5}$$

where η is some nonzero real number, a, b, c, d are integers satisfying the following conditions:

$$\begin{cases} ad - bc = 1 \\ a + d > 2 \end{cases}, \tag{6}$$

and there exists an algebraic Anosov diffeomorphism f of the torus $\mathbb{T}^2$ given by the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$$

with a trace satisfying the inequality $trA > 2$, which is an isometry of the Lorentzian manifold $(\mathbb{T}^2, g_0)$, and f is projected with respect to covering $p : \mathbb{T}^2 \to \mathcal{N}$ onto some isometry f_A of the orbifold $(\mathcal{N}, g)$. The projectability means that the following equality holds

$$p \circ f = f_A \circ p. \tag{7}$$

Let us check that f_A is the Anosov automorphism of the orbifold $\mathcal{N}$. Eigenvalues λ_1 and λ_2 of the matrix A are the roots of a quadratic equation

$$\lambda^2 - (a + d)\lambda + 1 = 0. \tag{8}$$

Let us denote by X_i, $i = 1, 2$, eigenvectors of the matrix A, corresponding to eigenvalues λ_i, then $AX_i = \lambda_i X_i$. Since f is an isometry of the Lorentzian metric g_0, its differential f_* preserves two isotropic directions at every point $u \in \mathbb{T}^2$. This means that the eigenvectors of the matrix A define two isotropic directions at every point of the Lorentz torus $(\mathbb{T}^2, g_0)$. Note that the torus $\mathbb{T}^2$ has the standard locally Euclidean metric d_0. According to (8), $\lambda_1 \cdot \lambda_2 = 1$, and $|\lambda_i| \neq 1$, without loss generality, we assume that $|\lambda_1| < 1$ and $|\lambda_2| = |\lambda_1|^{-1} > 1$. Consider one-dimensional distributions $E^s = span\{X_1\}$ and $E^u = span\{X_2\}$ on $\mathbb{T}^2$. Therefore we have

$$||f_*^n(X)|| = |\lambda_1|^n ||X|| \quad \forall X \in E^s, \quad ||f_*^n(Y)|| = |\lambda_1|^{-n} ||Y|| \quad \forall Y \in E^u \tag{9}$$

for all $n \in \mathbb{N}$, where $||\cdot||$ is the norm defined by the complete locally Euclidean metric d_0 on M. Since $0 < |\lambda_1| < 1$, the equality (9) implies that the one-dimensional distributions E^s and E^u are stable and unstable distributions of the Anosov diffeomorphism f, respectively. Emphasize that E^s and E^u are tangent to isotropic geodesics of the Lorentzian torus $(\mathbb{T}^2, g_0)$.

Let x be any point from Δ^2 and $z \in p^{-1}(x)$. Put $y = f_A(x)$ and $w = f(z) \in \mathbb{T}^2$. The equality (7) implies that there exist neighborhoods U_x, U_y of points x, y in Δ^2 and V_z, V_w of points z, w in $\mathbb{T}^2$, respectively, such that all restrictions $f_A|_{U_x} : U_x \to U_y$, $f|_{V_z} : V_z \to V_w$, $p|_{V_z} : V_z \to U_x$ and $p|_{V_w} : V_w \to U_y$ are diffeomorphisms satisfying the equality

$$p|_{V_w} \circ f|_{V_z} = f_A|_{U_x} \circ p|_{V_z}. \tag{10}$$

Since $p : \mathbb{T}^2 \to \mathcal{N}$ is a pseudo-Riemannian covering of the Lorentzian orbifold $(\mathcal{N}, g)$ by the torus $(\mathbb{T}^2, g_0)$, then $p|_{V_z}$ and $p|_{V_w}$ are local isometries, therefore they map isotropic geodesics to isotropic geodesics. The invariance of the distributions E^s and E^u of the torus $(\mathbb{T}^2, g_0)$ with respect to the group Ψ implies that on the manifold Δ^2, two distributions $\widehat{E}^s$ and $\widehat{E}^u$ are induced, which are the projections distributions E^s and E^u with respect to $p : \mathbb{T}^2 \to \mathcal{N}$. Thus, there is a decomposition of the tangent bundle $T\Delta^2 = \widehat{E}^s \oplus \widehat{E}^u$ to the stratum Δ^2, invariant with respect to the differential of f_A. Since $\Psi = \langle -E \rangle$ is an isometry group of the locally Euclidean torus $(\mathbb{T}^2, d_0)$, a locally Euclidean metric d is induced on the orbifold $\mathcal{N}$. Emphasize that $p : \mathbb{T}^2 \to \mathcal{N}$ is also a Riemannian covering of the Euclidean orbifold $(\mathcal{N}, d)$ by the torus $(\mathbb{T}^2, d_0)$. Therefore, due to (9), we get

$$||f_{A*}^n(X)|| = |\lambda_1|^n ||X|| \quad \forall X \in \widehat{E}^s, \quad ||f_{A*}^n(Y)|| = |\lambda_1|^{-n} ||Y|| \quad \forall Y \in \widehat{E}^u \tag{11}$$

for all $n \in \mathbb{N}$, where $||\cdot||$ is the norm defined by the locally Euclidean metric d on Δ^2. This means that f_A is an Anosov diffeomorphism of the orbifold $\mathcal{N}$. As $f_A \in Iso(\mathcal{N}, g)$, the action of $Iso(\mathcal{N}, g)$ on $\mathcal{N}$ is Anosov. $\square$

4. ANOSOV ACTIONS OF ISOMETRY GROUPS OF NON-COMPACT FLAT LORENTZIAN 2-ORBIFOLDS

4.1. Anosov Actions Respectively Locally Euclidean Metric are Improper

Proposition 2. *Let $(\mathcal{N}, g)$ be a complete non-compact flat 2-orbifold. If the action of its isometry group $Iso(\mathcal{N}, g)$ is Anosov, then the group $Iso(\mathcal{N}, g)$ acts improperly on $\mathcal{N}$.*

Proof. Let $(\mathcal{N}, g)$ be a non-compact complete flat 2-orbifold having an Anosov isometry h. According to [18, Theorem 1.4], since $(\mathcal{N}, g)$ is a complete flat 2-orbifold, it is a very good orbifold, hence its universal covering space is the pseudo-Euclidean plane $\mathbb{E}_1^2$. Denote by $p : \mathbb{E}_1^2 \to \mathcal{N}$ the universal covering map. If Ψ is the group of deck transformations of p, then $\mathcal{N} = \mathbb{E}_1^2/\Psi$, and $\Psi \subset Iso(\mathbb{E}_1^2)$.

According condition, the action of the isometry group $Iso(\mathcal{N}, g)$ is Anosov, i.e. there exists an Anosov isometry $h \in Iso(\mathcal{N}, g)$. It is well known that there exists an isometry $f \in Iso(\mathbb{R}^2, g_0)$ lying over h, i.e. satisfying the equality $p \circ f = h \circ p$. Let Δ^2 be the 2-dimensional stratum of the orbifold $\mathcal{N}$ formed by regular points. According to the Definition 5, there exist a Riemannian metric d on $\mathcal{N}$ and two one-dimentional continuous distributions E^s and E^u on Δ^2 satisfying the conditions (5) with respect to d. Pick $x \in \Delta^2$ and $z \in p^{-1}(x) \in \mathbb{R}^2$. Let $U = p^{-1}(\Delta^2)$. Note that $p|_U : U \to \Delta^2$ is a covering map, hence $p|_U$ is a local diffeomorphism. Therefore, two one-dimentional distributions $\widetilde{E^s}$ and $\widetilde{E^u}$ are induced on U, and they are lying over E^s and E^u, respevtively. Using this and the equality $p \circ f = h \circ p$, we will show that $f|_U$ is an Anosov diffeomorphism with respect to the Euclidean metric $d_0 = p^* d|_{\Delta^2}$.

Without loss generality, we assume that f preserves the orientation, in the opposite case, instead of f, consider f^2. In every basis of parallel vectors, the matrix elements g_{ij} of the pseudo-Euclidean metric

g_0 of $\mathbb{E}^2_1 = (\mathbb{R}^2, g_0)$ are constant functions. This implies that that isometries of $\mathbb{E}^2_1$ in this basis also do not depend on the choice of a point $x \in \mathbb{R}^2$. Denote by A the matrix given by the differential $f_{*z} : T_zU \to T_{f(z)}U$ at a point $z \in U$. Consider f as isometry of the pseudo-Euclidean space $\mathbb{E}^2_1 = (\mathbb{R}^2, g_0)$ defined by the matrix A. Since f is an isometry of $\mathbb{E}^2_1$ preserving the orientation, then $det(A) = 1$, A has two different eigenvalues λ_1, λ_2, and $\lambda_1 \cdot \lambda_2 = 1$. Let $\lambda = \lambda_1$, $0 < |\lambda| < 1$ and $\lambda_2 = \frac{1}{\lambda}$. Therefore, in some basis of eigenvectors $A = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}$. As f is a linear (affine) transformation of the plane $\mathbb{R}^2$, it may be represented in the form $f = \langle A, a\rangle \in Aff(\mathbb{R}^2)$, where a is a vector from the associated vector space $\mathbb{R}^2$. Since

$$u = \langle A, a\rangle u = Au + a \quad \Leftrightarrow \quad u = (A - E)^{-1} a = \begin{pmatrix} (\lambda - 1)^{-1} & 0 \\ 0 & \lambda(1 - \lambda)^{-1} \end{pmatrix} a,$$

then there exists a unique point $u \in \mathbb{E}^2_1$ which is fixed relative to f. Due to the equation $h \circ p = p \circ f$, we get that $v = p(u)$ is a fixed point of h, and v is a fixed point of the group $H = \langle h\rangle$. Note that may be $v \notin \Delta^2$. As h is an Anosov diffeomorphism of $\mathcal{N}$, the group $H = \langle h\rangle$ is non-compact. Denote by $Iso_v(\mathcal{N}, g)$ the stationary subgroup of $Iso(\mathcal{N}, g)$ at the point $v \in \mathcal{N}$. Emphasize that $Iso_v(\mathcal{N}, g)$ is closed subgroup in $Iso(\mathcal{N}, g)$. Since H is a discrete subgroup of $Iso_v(\mathcal{N}, g)$, then $Iso_v(\mathcal{N}, g)$ is also non-compact. It follows from Corollary 6 that the action of the group of isometries $Iso(\mathcal{N}, g)$ on $\mathcal{N}$ is improper. □

4.2. The Proof of Theorem 3

The implication (1) $\Rightarrow$ (2) follows from Proposition 2.

Show that (2) $\Rightarrow$ (1). Let $(\mathcal{N}, g)$ be a complete non-compact flat 2-orbifold. Assume that the isometry group $Iso(\mathcal{N}, g)$ acts improperly on $\mathcal{N}$. As it was proved by the authors in [6], in this case, $\mathcal{N} = \mathbb{R}^2/\{\pm E\} \cong \mathbb{R}^2/\mathbb{Z}_2$ is the $\mathbb{Z}_2$-cone and $Iso(\mathcal{N}, g) = O(1, 1)/\{\pm E\}$. Show that the action of $Iso(\mathcal{N}, g)$ is Anosov.

Recall that in there is a coordinate system on the plane $\mathbb{R}^2$ in which the pseudo-Euclidean metric g_0 has the form

$$g_0 = -dx^1 \otimes dx^1 + dx^2 \otimes dx^2.$$

The isometry group $Iso(\mathbb{E}^2_1) = Iso(\mathbb{R}^2, g_0)$ is the pseudo-orthogonal group $O(1, 1)$. Use the following notations: $A^{++}_t = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$, $E^{+-} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $E^{-+} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad \forall t \in \mathbb{R}$.

Note that $A \in O(1, 1)$ defines an Anosov isometry f of the pseudo-Euclidean plane $\mathbb{E}^2_1$ with respect to the standard Euclidean metric d_0 if and only if $|trA| > 2$. The unit component $O_e(1, 1)$ is formed by matrices A^{++}_t, and $O(1, 1) = O_e(1, 1) \sqcup E^{+-} \cdot O_e(1, 1) \sqcup (-E) \cdot O_e(1, 1) \sqcup E^{-+} \cdot O_e(1, 1)$. Therefore if $f \in O_e(1, 1)$, i.e. when f is defined by the matrix $A = A^{++}_t$, where $t \in (0, +\infty)$ is fixed, then f is an Anosov isometry of $\mathbb{E}^2_1$ respectively d_0. The eigenvalues of the matrix A^{++}_t are equal to $\lambda_1 = e^{-t}$ and $\lambda_2 = e^t$. Its eigenvectors are equal to $X_1 = \begin{pmatrix} x^1 \\ -x^1 \end{pmatrix}$ and $X_2 = \begin{pmatrix} x^1 \\ x^1 \end{pmatrix}$, respectively, and they define two isotropic directions at each point of the plane $\mathbb{R}^2$, invariant with respect to f. Hence the stable and unstable foliations F^s, F^u are formed by isotropic geodesics given by the equalities $x^2 = -x^1$ and $x^2 = +x^1$ of the pseudo-Euclidean plane $\mathbb{E}^2_1$, respectively. Let $\mathcal{O}.z = \{f^n(z) \mid n \in \mathbb{Z}\}$ be the orbit of a point z respectively the group $H = \langle f\rangle$. Then: 1) $\mathcal{O}.0 = 0$ for zero $0 = (0, 0) \in \mathbb{R}^2$; 2) for $z = (x^1, x^2)$ where $|x^1| > |x^2|$, the orbit $\mathcal{O}.z$ belongs to the hyperbole $(x^1)^2 - (x^2)^2 = a^2$ for some $a \in \mathbb{R}^1 \setminus \{0\}$; 3) for $z = (x^1, x^2)$ where $|x^1| < |x^2|$, the orbit $\mathcal{O}.z$ belongs to the hyperbole $(x^2)^2 - (x^1)^2 = a^2$ for some

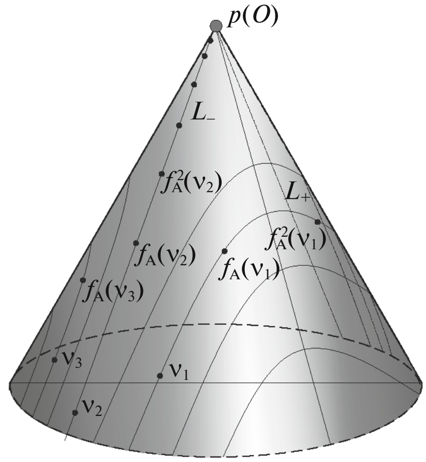

Fig. 3. Orbits of an Anosov isometry f_A on $\mathbb{Z}_2$-cone $\mathcal{N}$.

$a \in \mathbb{R}^1 \setminus \{0\}$; 4) if $|x^1| = |x^2| \neq 0$, the orbit $\mathcal{O}.z$ lies on one of the four open rays formed by straight lines $x^2 = \pm x^1$. Emphasize that $\mathcal{O}.z$ is not closed in $\mathbb{R}^2$ only in case 4).

A locally Euclidean metric d is induced on $\mathcal{N}$ such that the quotient map $p : \mathbb{R}^2 \to \mathcal{N} = \mathbb{R}^2/\{\pm E\}$ is a Riemannian covering. Therefore an isometry $h \in Iso(\mathcal{N}, g)$ is Anosov respectively d, if and only if $h = f_A$ is induced by an Anosov isometry f of $\mathbb{E}^2_1$, i.e. iff the equality $f_A \circ p = p \circ f$ is fulfilled. Hence f_A is an Anosov isometry iff f is defined by the matrix A_t^{++}, $t \neq 0$. The stable and unstable foliations $\mathcal{F}^s$, $\mathcal{F}^u$ of f_A are the projections of the foliations F^s and F^u with respect to the map $p : \mathbb{R}^2 \to \mathcal{N}$. As p is preudo-Riemannian covering, $\mathcal{F}^s$ and $\mathcal{F}^u$ are formed by isotropic geodesics of the Lorentzian orbifold $(\mathcal{N}, g)$. Denote by L_-, L_+ the images in $\mathcal{N}$ of the rays indicated above in 4), they are examples of isotropic geodesics in $(\mathcal{N}, g)$, and L_- is a leaf of the foliation $\mathcal{F}^s$, L_+ is a leaf of the foliation $\mathcal{F}^u$. Let $\mathcal{O}.v = \{(f_A)^n(v) \mid n \in \mathbb{Z}\}$ be the orbit of a point $v \in \mathcal{N}$ respectively the group $\mathcal{H} = \langle f_A \rangle$. Note that $\mathcal{O}.v = p(\mathcal{O}.z)$ where $z \in p^{-1}(v)$ and $\mathcal{O}.z$ is the orbit of z respectively the group H. Therefore, there the union of all closed orbits of the Anosov isometry f_A is equal to $\mathcal{N} \setminus (L_- \cup L_+)$, and it is dense in $\mathcal{N}$, see Figure 3.

The Anosov isometry f_A of $(\mathcal{N}, g)$ satisfies the condition 2) of Definition 1, but does not satisfy condition 1) of this definition, since f_A does not have a dense orbit.

Thus, every Anosov isometry of the Lorentzian orbifold $(\mathcal{N}, g)$ is not chaotic. □

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