ISSN 1560-3547, Regular and Chaotic Dynamics, 2022, Vol. 27, No. 1, pp. 77–97. © Pleiades Publishing, Ltd., 2022.

Components of Stable Isotopy Connectedness of Morse – Smale Diffeomorphisms

Timur V. Medvedev[1*], Elena V. Nozdrinova[2], and Olga V. Pochinka[2***]**

[1] *Laboratory of Algorithms and Technologies for Network Analysis, HSE University, ul. Rodionova 136, 603093 Nizhny Novgorod, Russia*

[2] *International Laboratory of Dynamical Systems and Applications, HSE University, ul. Bolshaya Pecherckaya 25/12, 603155 Nizhny Novgorod, Russia*

Received October 23, 2021; revised January 10, 2022; accepted January 14, 2022

Abstract—In 1976 S. Newhouse, J. Palis and F. Takens introduced a stable arc joining two structurally stable systems on a manifold. Later in 1983 they proved that all points of a regular stable arc are structurally stable diffeomorphisms except for a finite number of bifurcation diffeomorphisms which have no cycles, no heteroclinic tangencies and which have a unique nonhyperbolic periodic orbit, this orbit being the orbit of a noncritical saddle-node or a flip which unfolds generically on the arc. There are examples of Morse – Smale diffeomorphisms on manifolds of any dimension which cannot be joined by a stable arc. There naturally arises the problem of finding an invariant defining the equivalence classes of Morse – Smale diffeomorphisms with respect to connectedness by a stable arc. In the present review we present the classification results for Morse – Smale diffeomorphisms with respect to stable isotopic connectedness and obstructions to existence of stable arcs including the authors' recent results.

MSC2010 numbers: 37C15, 37D15

DOI: 10.1134/S1560354722010087

Keywords: stable arc, Morse – Smale diffeomorphism

1. INTRODUCTION

The problem of existence of an arc which connects two structurally stable systems (Morse – Smale systems) on a manifold and which has only finitely many (or countably many) bifurcations was the 33d in the list of fifty problems in dynamical systems suggested by J. Palis and C. Pugh in [34].

In 1976 S. Newhouse, J. Palis and F. Takens in [26] introduced a *stable arc* connecting two structurally stable systems on a manifold. This arc preserves its qualitative properties under small disturbances. In the same year S. Newhouse and M. Peixoto [28] proved the existence of a simple arc (i. e., an arc having elementary bifurcations only) joining any two Morse – Smale flows. From the paper [8] by G. Fleitas it follows that the simple arc constructed by S. Newhouse and M. Peixoto can always be replaced by a stable one. There are examples of Morse – Smale diffeomorphisms on manifolds of any dimension which cannot be connected by a stable arc (see Section 3). Therefore, there naturally arises the problem of finding an invariant defining the equivalence class of Morse – Smale diffeomorphisms with respect to connectedness by a stable arc (*stable isotopy connected component*).

In the present paper we review the classification results for Morse – Smale systems with respect to stable isotopy connectedness including the recent research by the authors.

*E-mail: tmedvedev@hse.ru
**E-mail: maati@mail.ru
***E-mail: olga-pochinka@yandex.ru

2. STABLE ARCS IN THE SPACE OF DYNAMICAL SYSTEMS

In this section we give an exact definition of the stable arc and the criterion of stability for arcs composed of diffeomorphisms with finite limit sets. Then we outline the idea of the proof of existence of a stable arc joining any two Morse – Smale flows on a given manifold.

2.1. Main Concepts of the Theory of Dynamical Systems

Let $f : M^n \to M^n$ be a diffeomorphism on a smooth closed (compact without a border) n-manifold ($n \geqslant 1$) M^n with a metric d.

Two diffeomorphisms $f, f' : M^n \to M^n$ are said to be *topologically conjugate* if there is a homeomorphism $h : M^n \to M^n$ such that $fh = hf'$.

A point $x \in M^n$ is *wandering* for f if there exists an open neighborhood U_x of x for which $f^n(U_x) \cap U_x = \varnothing$ for every $n \in \mathbb{N}$. Otherwise x is said to be *nonwandering*. The set of the nonwandering points of a diffeomorphism f is the *nonwandering set* denoted by Ω_f.

For example, every limit point of a diffeomorphism is nonwandering. Recall that a point $y \in M^n$ is an ω*-limit* point for a point $x \in M^n$ if there is a sequence $t_k \to +\infty$, $t_k \in \mathbb{Z}$ such that $\lim\limits_{t_k \to +\infty} d(f^{t_k}(x), y) = 0$. The set $\omega(x)$ of all ω-limit points for x is its ω*-limit set*. By changing $+\infty$ to $-\infty$ one defines the α*-limit set* $\alpha(x)$ of the point x. The set $L_f = cl\,(\bigcup\limits_{x \in M^n} \omega(x) \cup \alpha(x))$ is called the *limit set* of the diffeomorphism f.

If the set Ω_f is finite, then each point $p \in \Omega_f$ is periodic. Denote by $m_p \in \mathbb{N}$ its period. Each periodic point p has the corresponding *stable* and *unstable* manifolds defined by

$W^s_p = \{x \in M^n : \lim\limits_{k \to +\infty} d(f^{km_p}(x), p) = 0\}$,

$W^u_p = \{x \in M^n : \lim\limits_{k \to +\infty} d(f^{-km_p}(x), p) = 0\}$.

Both the stable and the unstable manifolds are the *invariant manifolds*.

On the set of periodic orbits there is a partial order relation (*Smale order*) defined by

$$\mathcal{O}_p \prec \mathcal{O}_r \Longleftrightarrow W^s_{\mathcal{O}_p} \cap W^u_{\mathcal{O}_r} \neq \varnothing.$$

The periodic orbits $\mathcal{O}_1, \ldots, \mathcal{O}_k$ compose *a cycle* if $W^s_{\mathcal{O}_i} \cap W^u_{\mathcal{O}_{i+1}} \neq \varnothing$ for $i \in \{1, \ldots, k\}$ and $\mathcal{O}_{k+1} = \mathcal{O}_1$.

A periodic point $p \in \Omega_f$ is said to be *hyperbolic* if the absolute value of each eigenvalue of the Jacobian matrix $\left(\frac{\partial f^{m_p}}{\partial x}\right)|_p$ is not equal to 1. If the absolute value of each eigenvalue is less (greater) than 1, then p is a *sink* (*a source*). Sinks and sources are called *nodes*. If a hyperbolic point is not a node, then it is called a *saddle*.

It follows from the hyperbolic structure of a periodic point p that its stable W^s_p and unstable W^u_p manifolds are the images of the spaces $\mathbb{R}^{q_p}$ and $\mathbb{R}^{n-q_p}$ by injective immersions where q_p is the number of the eigenvalues of the Jacobian matrix whose absolute value is greater than 1. The number ν_p is called the *orientation type* of a point p and it equals $+1(-1)$ if the map $f^{m_p}|_{W^u_p}$ preserves (reverses) the orientation of W^u_p. A linear connected component of the set $W^u_p \setminus p$ ($W^s_p \setminus p$) is called an *unstable* (*stable*) *separatrix* of the point p.

A closed f-invariant set $A \subset M^n$ is called an *attractor* of the dynamical system f if A has a compact neighborhood U_A such that $f(U_A) \subset int\, U_A$ and $A = \bigcap\limits_{k \geqslant 0} f^k(U_A)$. In this case U_A is called a *trapping* or an *isolating* neighborhood. A *repeller* for f is the attractor for f^{-1}. Let A be an attractor for f with a trapping neighborhood U_A, $V = M^n \setminus int(U_A)$ and let $R = \bigcap\limits_{k \geqslant 0} f^{-k}(V)$ be a repeller. Then the pair A, R is called *dual*.

A diffeomorphism $f : M^n \to M^n$ is said to be a *Morse – Smale diffeomorphism* if

1) the nonwandering set Ω_f consists of a finite number of hyperbolic orbits;

2) for any two nonwandering points p, q the manifolds W^s_p, W^u_q transect transversally.

A Morse – Smale diffeomorphism is said to be *gradient-like* if from $W^s_{\sigma_1} \cap W^u_{\sigma_2} \neq \varnothing$ for any two distinct points $\sigma_1, \sigma_2 \in \Omega_f$ it follows that $dim\, W^u_{\sigma_1} < dim\, W^u_{\sigma_2}$.

A *Morse – Smale flow* on a manifold M^n is defined in a similar way, and if such a flow has no periodic trajectories, it is called *gradient-like*.

A detailed explanation of the facts of this section can be found in [13, 16, 39].

2.2. Stability of an Arc of Diffeomorphisms

Consider a single parameter family of diffeomorphisms (an *arc*) $\varphi_t : M^n \to M^n, t \in [0,1]$. The arc φ_t is called *smooth* if the map $F : M^n \times [0,1] \to M^n$ defined by $F(x,t) = \varphi_t(x)$ is a *diffeotopy*, i. e., a smooth map which is a diffeomorphism for each fixed t. This map is called the *isotopy* in the topological category.

The smooth arc φ_t is called the *smooth composition* of the smooth arcs ϕ_t and ψ_t if $\phi_1 = \psi_0$ and $\varphi_t = \begin{cases} \phi_{\tau(2t)}, 0 \leqslant t \leqslant \frac{1}{2}, \\ \psi_{\tau(2t-1)}, \frac{1}{2} \leqslant t \leqslant 1, \end{cases}$ where $\tau : [0,1] \to [0,1]$ is a smooth monotonic map such that $\tau(t) = 0$ for $0 \leqslant t \leqslant \frac{1}{3}$ and $\tau(t) = 1$ for $\frac{2}{3} \leqslant t \leqslant 1$. We write

$$\varphi_t = \phi_t * \psi_t.$$

Following [27] a smooth arc φ_t is called *stable* if it is an inner point of the equivalence class with respect to the following relation: two arcs φ_t, φ'_t are said to be *conjugate* if there are homeomorphisms $h : [0,1] \to [0,1]$, $H_t : M \to M$ such that $H_t \varphi_t = \varphi'_{h(t)} H_t, t \in [0,1]$ and H_t continuously depends on t.

Denote by $\mathcal{Q}$ the set of smooth arcs $\varphi_t, t \in [0,1]$ such that the starting point and the ending point of arcs of $\mathcal{Q}$ is a Morse – Smale diffeomorphism and every diffeomorphism φ_t has a finite limit set.

It is shown in [27] that an arc $\varphi_t \in \mathcal{Q}$ for $t \in [0,1]$ is stable if and only if its every point is a stable diffeomorphism except for a finite number of bifurcation points $\varphi_{b_i}, i = 1, \ldots, q$ for which

1) the limit set of the diffeomorphism φ_{b_i} contains a unique nonhyperbolic periodic orbit which is a saddle-node or a flip;
2) the diffeomorphism φ_{b_i} has no cycles;
3) the invariant manifolds of all periodic points of φ_{b_i} transect transversally;
4) φ_{b_i} has a unique nonhyperbolic periodic orbit which is the orbit of a noncritical saddle-node or of a flip which unfolds generically (Fig. 1).

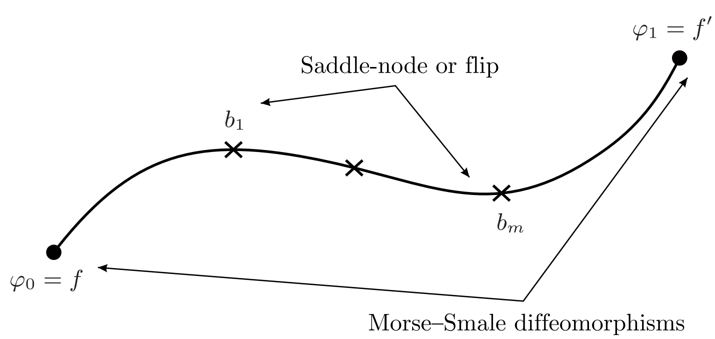

Fig. 1. A stable arc.

Recall the definition of a fixed saddle-node or a flip generic unfolding when the nonhyperbolic point is of period $k > 1$. For the arc φ^k_t the definition is similar.

A saddle-node p *unfolds generically* on the arc $\varphi_t \in \mathcal{Q}$ where $t \in [0,1]$ (Fig. 2) if in some neighborhood of the point $(p, b_i) \in M^n \times [0,1]$ the arc φ_t (or the arc φ_{1-t}) is conjugate to the arc

$$\tilde{\varphi}_{\tilde{t}}(x_1, x_2, \ldots, x_{1+n_u}, x_{2+n_u}, \ldots, x_n)$$
$$= \left(x_1 + \frac{x_1^2}{2} + \tilde{t}, \pm 2x_2, \ldots, \pm 2x_{1+n_u}, \frac{\pm x_{2+n_u}}{2}, \ldots, \frac{\pm x_n}{2}\right),$$

where $(x_1, \ldots, x_n) \in \mathbb{R}^n$, $|x_i| < 1/2$, $|\tilde{t}| < 1/10$.

In the local coordinates $(x_1, \ldots, x_n, \tilde{t})$ the bifurcation occurs at the time $\tilde{t} = 0$, the coordinate origin $O \in \mathbb{R}^n$ being the saddle-node (Fig. 3). Here the axis Ox_1 is the *central manifold*, the set $\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1 \geqslant 0,\ x_{2+n_u} = \ldots = x_n = 0\}$ is the *unstable manifold* of O, the set $\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1 \leqslant 0,\ x_2 = \ldots = x_{1+n_u} = 0\}$ is the *stable manifold* of O.

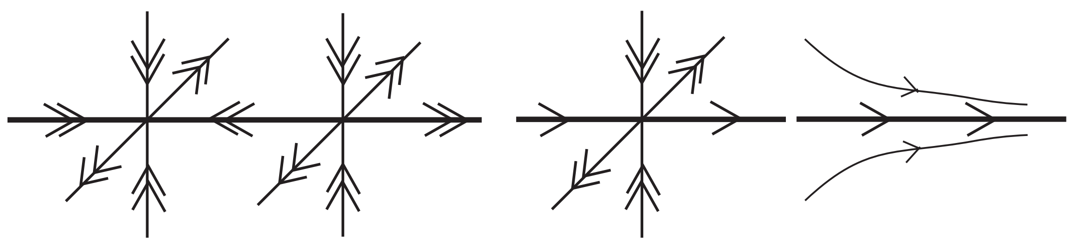

Fig. 2. The saddle-node bifurcation.

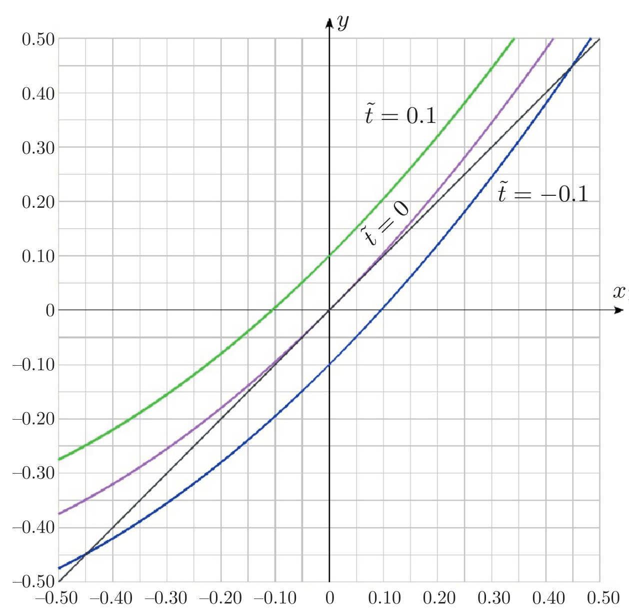

Fig. 3. The graphs of the map $x_1 + \frac{x_1^2}{2} + \tilde{t}$ for $\tilde{t} = -0,1; \tilde{t} = 0$ and $\tilde{t} = 0,1$.

If p is a saddle-node of the diffeomorphism φ_{b_i}, then there is a unique φ_{b_i}-invariant foliation F_p^{ss} with smooth leaves such that ∂W_p^s is a leaf of F_p^{ss} [18]. F_p^{ss} is called the *strongly stable foliation* (Fig. 4). The analogous *strongly unstable foliation* is denoted by F_p^{uu}. A point p is called s-*critical* if there exists a hyperbolic periodic point q such that W_q^u intersects some leaf of F_p^{ss} nontransversally; the u-*criticality* is defined analogously. The point p is called

- *semicritical* if it is either s- or u-critical;
- *bicritical* if it is both s- and u-critical;
- *noncritical* if it is neither s- nor u-critical.

For the first time instability of an arc in the neighborhood of a critical saddle-node was found in 1974 by V. S. Afraimovich and L. P. Shilnikov [36, 37]. The existence of invariant foliations F_p^{ss}, F_p^{uu} was previously proved by V. I. Lukyanov and L. P. Shilnikov in [20].

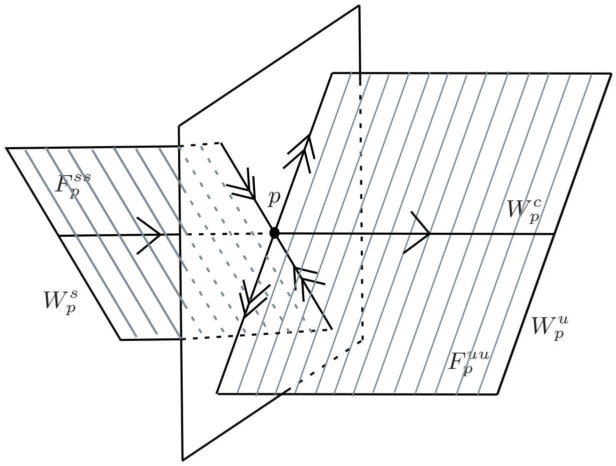

Fig. 4. The strongly stable and unstable foliations.

Remark 1. For surface diffeomorphisms the stable and the unstable manifolds of a saddle-node p are of dimension one and two. If p is noncritical, then its invariant 1-manifold does not intersect separatrices of saddles. The 1-dimensional foliation of the 2-manifold of the saddle-node must transversally intersect the separatrices of the saddles (Fig. 5).

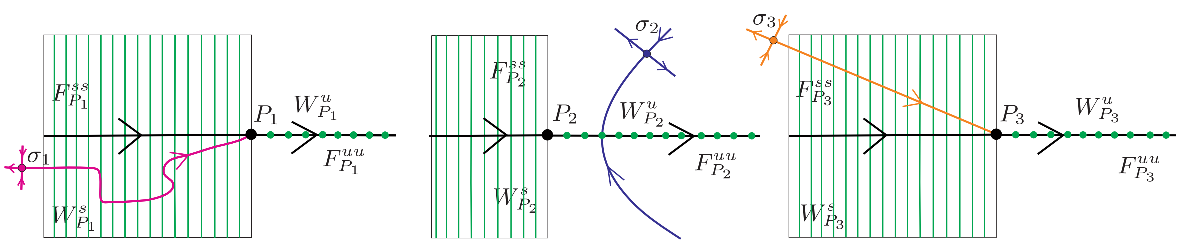

Fig. 5. p_1 is a s-critical saddle-node, p_2 is an u-critical saddle-node, p_3 is a noncritical saddle-node.

A flip p *unfolds generically* on the arc $\varphi_t \in \mathcal{Q}$ where $t \in [0, 1]$ (Fig. 6) if in some neighborhood of the point (p, b_i) the arc φ_t (or the arc φ_{1-t}) is conjugate to the arc

$$\begin{aligned}&\tilde{\varphi}_{\tilde{t}}(x_1, x_2, \ldots, x_{1+n_u}, x_{2+n_u}, \ldots, x_n)\\ &= \left(-x_1(1 \pm \tilde{t}) + x_1^3, \pm 2x_2, \ldots, \pm 2x_{1+n_u}, \frac{\pm x_{2+n_u}}{2}, \ldots, \frac{\pm x_n}{2}\right),\end{aligned}$$

where $(x_1, \ldots, x_n) \in \mathbb{R}^n$, $|x_i| < 1/2$, $|\tilde{t}| < 1/10$ (Fig. 7).

We say two diffeomorphisms f_0, f_1 are of the same class of *stable isotopy connectedness* if in the space of diffeomorphisms they can be joined by an arc satisfying the previously described properties 1–4.

2.3. A Simple Arc Joining Two Morse – Smale Flows

The problem of existence of a “good” arc joining two Morse – Smale flows on a given manifold was solved by S. Newhouse and M. Peixoto in [28].

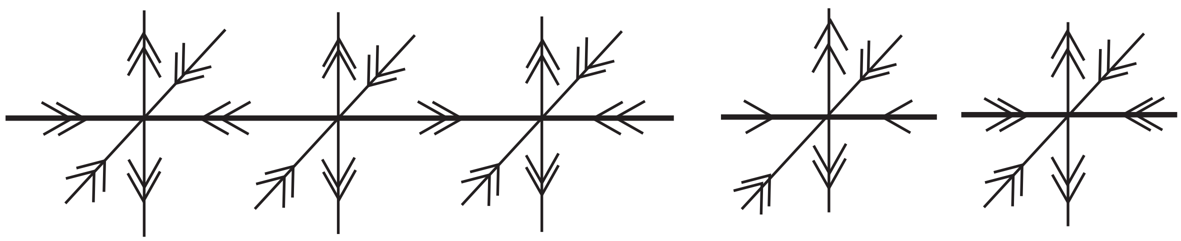

Fig. 6. The period-doubling bifurcation (flip).

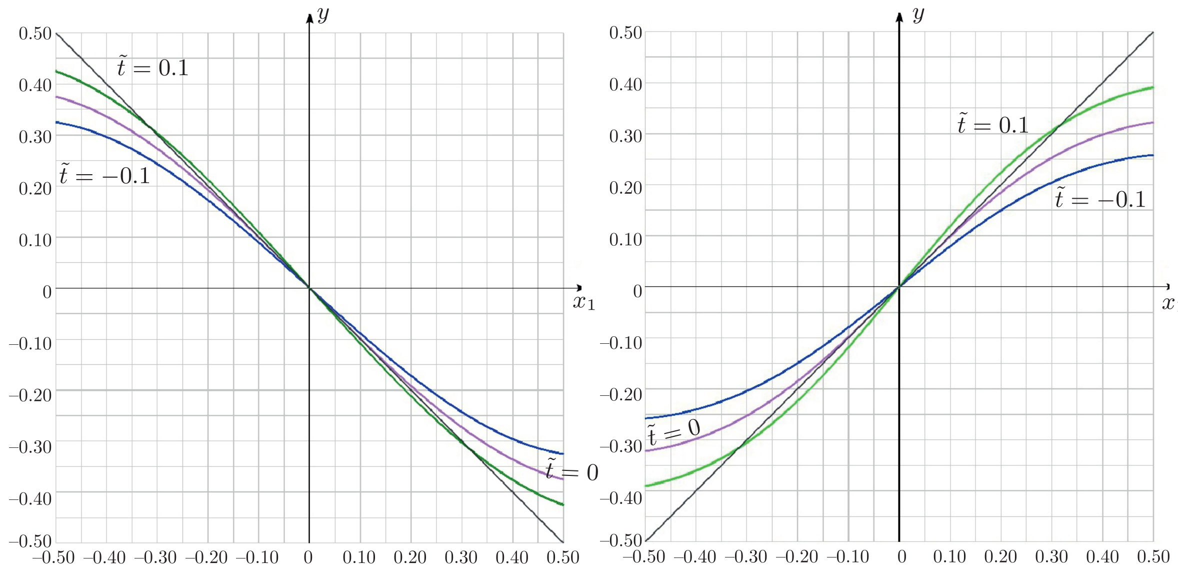

Fig. 7. The graphs of the map $-x_1(1 \pm \tilde t) + x_1^3$ and its square for $\tilde t = -0,1; \tilde t = 0$ and $\tilde t = 0,1$.

Theorem 1 ([28], Theorem B). *Any two Morse – Smale flows on a given manifold can be joined by a simple arc.*

The simplicity means that the entire arc consists of Morse – Smale systems except for a finite set of points at which the vector field deviates from the Morse – Smale field in the least way in some sense. More exactly, at these points the vector field either has a unique nonhyperbolic saddle-node or it has a unique curve which is the nontransversal intersection of the invariant manifolds of the saddles (*heteroclinic tangency*).

The idea of the proof is to construct an arc joining the initial flow with the gradient flow of some Morse function.

Let M^n be a smooth n-manifold and $\Phi : M^n \to \mathbb{R}$ be a C^r-smooth ($r \geqslant 2$) function. The point $p \in M^n$ is called the *critical* point of Φ if $grad\,\Phi(p) = 0$, i. e., $\frac{\partial\Phi}{\partial x_1}(p) = \ldots = \frac{\partial\Phi}{\partial x_n}(p) = 0$ in the local coordinates $x_1, \ldots, x_n$ of p. A critical point p is *nondegenerate* if the matrix of the second-order partial derivatives (*Hessian matrix*) $\left(\frac{\partial^2\Phi}{\partial x_i \partial x_j}\right)|_p$ is nonsingular. Otherwise p is said to be *degenerate*. The function Φ is called the *Morse function* if it has no degenerate critical points.

A continuous function $\Phi : M^n \to \mathbb{R}$ is called a *Lyapunov function* for a Morse – Smale system (a diffeomorphism or a flow) on the manifold M^n if it strictly decreases along the wandering trajectories and is constant on the periodic orbits.

A smooth Lyapunov function Φ is called an *energy function* for a Morse – Smale system if the set of the critical points of Φ coincides with the nonwandering set of the system.

When constructing the desired arc the first step would be to destroy the closed orbits by sequentially creating a saddle-node on each closed trajectory (Fig. 8). Then when passing through the saddle-node there appear two hyperbolic points of neighboring indices, while the end point of the arc is the gradient-like flow.

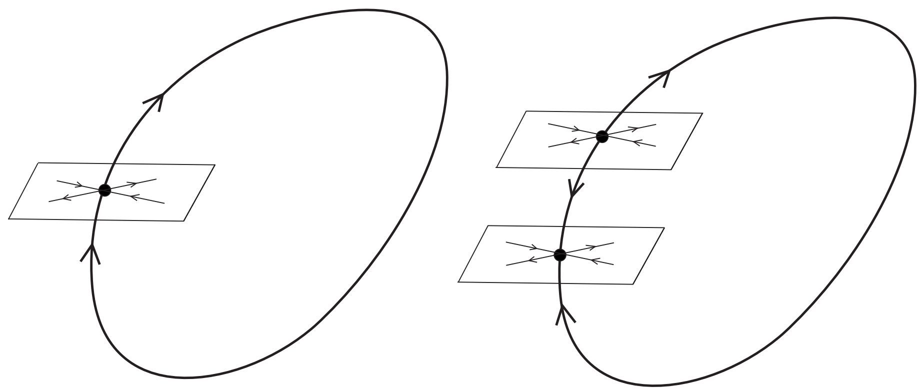

Fig. 8. The destruction of periodic orbits.

By Franks' lemma in [9] a gradient-like flow can be connected by an arc without bifurcations to a gradient-like flow which is locally gradient in the neighborhood of its fixed points. S. Smale in [38] proved that any such flow allows a Morse energy function. Then using the level curves of this function one constructs an arc joining the gradient-like flow to the gradient flow of its Morse function (see Section 4.2 where this idea is used for Morse – Smale diffeomorphisms). The last step is to join the two Morse functions by an arc which typically generates the simple arc of gradient flows.

Unfortunately, the results of S. Newhouse and M. Peixoto cannot be directly used to construct a stable arc between two Morse – Smale diffeomorphisms. The first reason is that typically Morse – Smale diffeomorphisms cannot be included in Morse – Smale flows (see, for example, [10, 11] and [12]). The other reason is that *the discretization* (i. e., replacement of the flows by their 1-time shifts) of an arc with a heteroclinic tangency is not a stable arc. The last problem is solved by the following theorem due to G. Fleitas [8].

Theorem 2 ([8], Theorem). *If two gradient-like flows on the manifold M^n are joined by an arc with the unique bifurcation point of heteroclinic tangency, then they can be joined by an arc with two saddle-node bifurcations* (*Fig.* 9). *The discretization of this arc is a stable arc that joins* 1-*time shifts of the initial gradient-like flows.*

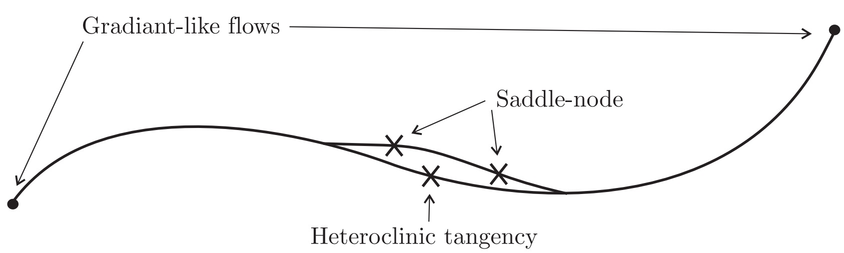

Fig. 9. Replacing heteroclinic tangencies by saddle-nodes.

3. THE OBSTRUCTIONS TO THE EXISTENCE OF A STABLE ARC BETWEEN TWO ISOTOPIC MORSE – SMALE DIFFEOMORPHISMS

In this section we review the well-known obstructions to the existence of a stable arc between two isotopic Morse – Smale diffeomorphisms.

3.1. The Discrepancy Between Rotation Numbers of Two Rough Transformations of a Circle

The rotation number was introduced by H. Poincaré for homeomorphisms of a circle while studying flows on the torus without fixed points.

Consider the circle $S^1 = \mathbb{R}/\mathbb{Z}$, an orientation-preserving homeomorphism $f : S^1 \to S^1$ and its lift $\bar{f} : \mathbb{R} \to \mathbb{R}$ with respect to the covering map

$$\pi(x) = \{x\},$$

where $\{x\}$ is the fractional part of $x \in \mathbb{R}$. Then for every $x \in \mathbb{R}$ there exists the limit

$$\lim_{n\to\infty} \frac{\bar{f}^n(x) - x}{n}$$

whose fractional part $r(f)$ is independent of the choice of the lift of f as well as of the point x. It is called the *Poincaré rotation number.*

Morse–Smale diffeomorphisms of the circle were thoroughly studied by A. G. Mayer [22]. He showed that, if a circle transformation is rough, then it is a Morse–Smale diffeomorphism (a diffeomorphism with a finite number of hyperbolic periodic points). Any such diffeomorphism is a composition of the 1-time shift of a gradient flow of some Morse function and either an orientation-changing involution for an orientation-changing diffeomorphism or a rational rotation for an orientation-preserving diffeomorphism.

Since the rotation number continuously depends on the homeomorphism in the C^0 topology (see, for example, [19]), any arc joining two orientation-preserving Morse–Smale diffeomorphisms with distinct rotation numbers contains a continuum of bifurcations and, therefore, it is not stable.

Theorem 3 ([30], Theorem 1). *All rough orientation-reversing diffeomorphisms of the circle lie in the same component of the stable isotopy connectedness, whereas the stable isotopy class of the rough orientation-preserving transformation of the circle is completely determined by the Poincaré rotation number.*

The idea of the proof is to construct a bifurcation-free arc joining an arbitrary diffeomorphism of a given topological conjugacy class to the corresponding *model diffeomorphism* $\Phi_{n,m,k}$ or $\Psi_{q,\nu}$ (Fig. 10) constructed as follows.

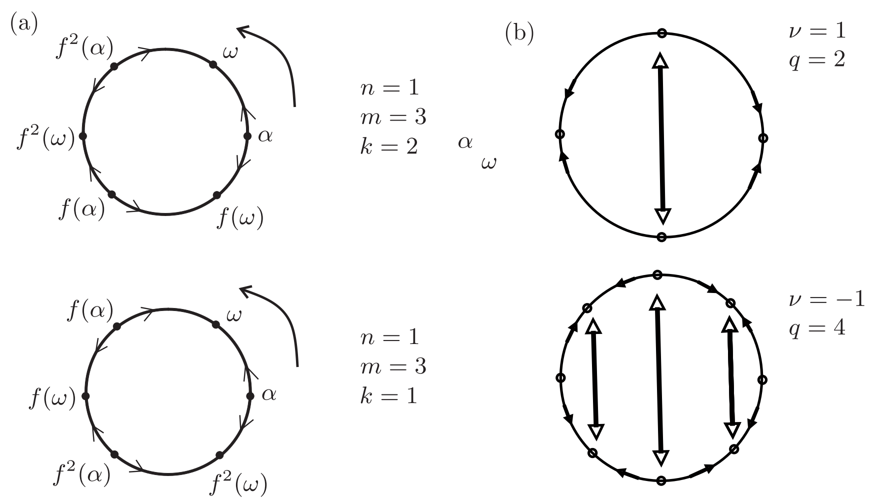

Fig. 10. (a) Orientation-preserving diffeomorphisms of the circle; (b) orientation-reversing diffeomorphisms of the circle.

Let (n, m, k) be a triple of integers such that either $n \in \mathbb{N}$ and $k = 0$ for $m = 1$ or $k \in \{1, \ldots, m-1\}$ when $m > 1$ and (m, k) are mutually prime. Consider the map $\bar{\Phi}_{n,m,k} : \mathbb{R} \to \mathbb{R}$ defined by (Fig. 11):

$$\bar{\Phi}_{n,m,k}(x) = x + \frac{1}{4nm\pi}\sin(2nm\pi x) + \frac{k}{m}.$$

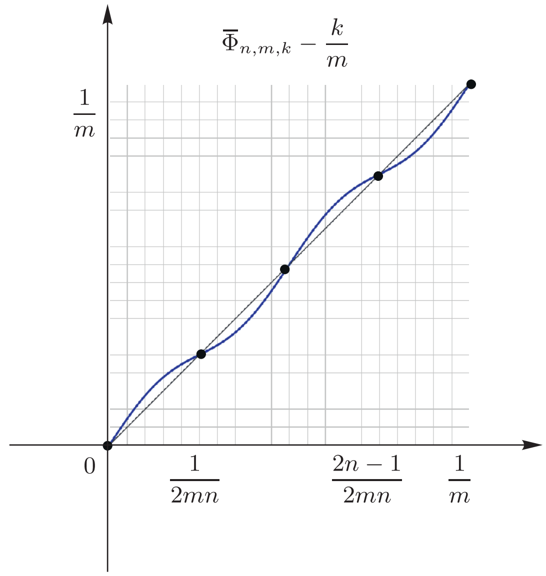

Fig. 11. The map $\bar{\Phi}_{n,m,k}(x) - \frac{k}{m}$.

Let $\Phi_{n,m,k} = \pi\bar{\Phi}_{n,m,k}\pi^{-1} : S^1 \to S^1$. Then $\Phi_{n,m,k}$ is an orientation-preserving diffeomorphism with the rotation number $\frac{k}{m}$ and $2n$ periodic orbits (half of which are attracting and the other half are repelling) of period m.

Let (q, ν) be a pair of numbers $q \in \mathbb{N}$, $\nu \in \{-1, 0, +1\}$ and let $\nu = 0$ if and only if q is odd. Consider the maps $\bar{\Psi}_{q,\nu} : \mathbb{R} \to \mathbb{R}$ (Fig. 12) defined by

$$\Psi_{q,\nu}(x) = -x - \frac{1}{4\pi q}\sin(2\pi q x),\ \nu = 0;\ -1;$$

$$\Psi_{q,+1}(x) = \Psi^{-1}_{q,-1}\left(x - \frac{1}{2q}\right) + \frac{1}{2q},\ q = 2\kappa,\ \kappa \in \mathbb{N}.$$

Let $\Psi_{q,\nu} = \pi\bar{\Psi}_{q,\nu}\pi^{-1} : S^1 \to S^1$. Then $\Psi_{q,\nu}$ is an orientation-reversing diffeomorphism with $2q$ periodic orbits, two of which $(1, 0)$, $(-1, 0)$ are fixed points and the other $2(q-1)$ are periodic orbits with period 2. Thus, $\Psi_{2\kappa-1,0}$ corresponds to the case when the fixed points have opposite stability, and $\Psi_{2\kappa,\pm 1}$ corresponds to the case where the fixed points are both unstable (-1) or both stable (+1), respectively.

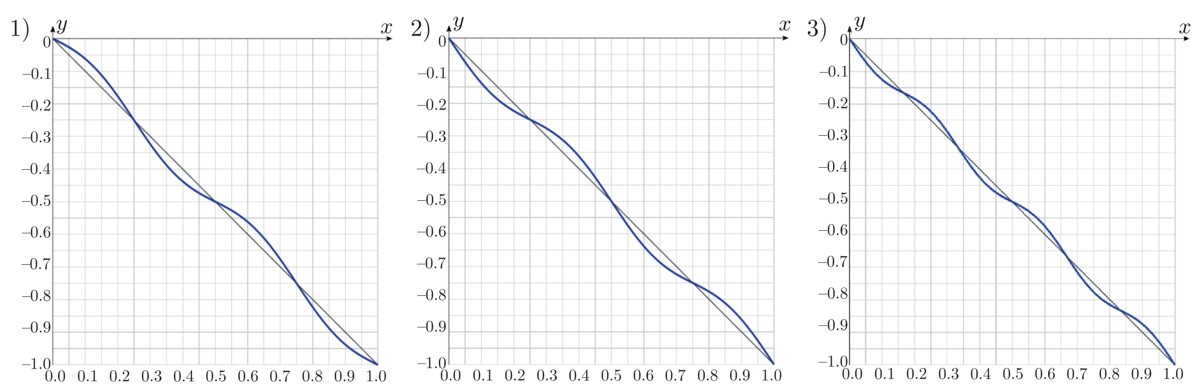

Fig. 12. The maps $\Psi_{q,\nu}$: 1) $\bar{\Psi}_{2,1}$, 2) $\bar{\Psi}_{2,-1}$, 3) $\bar{\Psi}_{3,0}$.

Thus, the problem is reduced to the problem of finding the classes of the stable isotopy connectedness for the model diffeomorphisms.

For the orientation-preserving diffeomorphism $\Phi_{n,m,k}$, $n > 1$ the number of periodic orbits can be reduced by one pair if one constructs the arc passing through the noncritical saddle-node which unfolds generically. Therefore, the diffeomorphism $\Phi_{n,m,k}$ can be joined by a stable arc to the diffeomorphism $\Phi_{1,m,k}$ with the same rotation number. Since the rotation number is a topological invariant of circle homeomorphism and since it continuously depends on the parameter of the arc, any arc joining two orientation-preserving circle diffeomorphisms f, f' with distinct rotation numbers is not stable because it contains a continuum of bifurcations, and that contradicts the definition of the stable arc.

The orientation-reversing diffeomorphism $\Psi_{2\kappa-1,0}$, $\kappa > 1$ (as in the case of orientation-preserving diffeomorphism) can be joined to the "source-sink" diffeomorphism $\Psi_{1,0}$ by an arc with $(2\kappa - 3)$ noncritical saddle-nodes unfolding generically (this applies to period-2 orbits). For the diffeomorphism $\Psi_{q,\pm 1}$ the number q is odd and $q > 2$. The described technique allows one to join the diffeomorphism $\Psi_{2\kappa,\pm 1}$, $\kappa > 1$ to the diffeomorphism $\Psi_{2,\pm 1}$ which, in turn, is joined to the "source-sink" diffeomorphism $\Psi_{1,0}$ by an arc with period-doubling bifurcations.

3.2. The Nontrivial Relatedness of Periodic Points

In [21] S. Matsumoto showed that the 2-torus $\mathbb{T}^2$ allows Morse–Smale diffeomorphisms that are isotopy equivalent to the identity, but cannot be joined by a stable arc.

Two periodic points p, q of a diffeomorphism $f : M^n \to M^n$ are said to be *trivially related* if there is an arc $c \subset M^n$ for which $\partial c = \{q\} - \{p\}$ and for some integer N the closed curve $f^N(c) - c$ is null-homotopic, $f^N(p) = p$ and $f^N(q) = q$. Otherwise the points p, q are *nontrivially related*. Notice that this property is independent of the choice of the curve c if f is isotopy equivalent to the identity. If all periodic points of f are trivially related, then f is *trivial*. Otherwise f is *nontrivial*.

S. Matsumoto constructed two Morse–Smale diffeomorphisms $f_0, f_1 : \mathbb{T}^2 \to \mathbb{T}^2$, both being isotopy equivalent to the identity and such that f_0 is the 1-time shift of the gradient flow of a typical Morse function with 4 critical points, while f_1 is the composition of f_0 with the two oppositely directed Dehn twists (Fig. 13). One can easily see that f_0 is trivial, while f_1 is nontrivial.

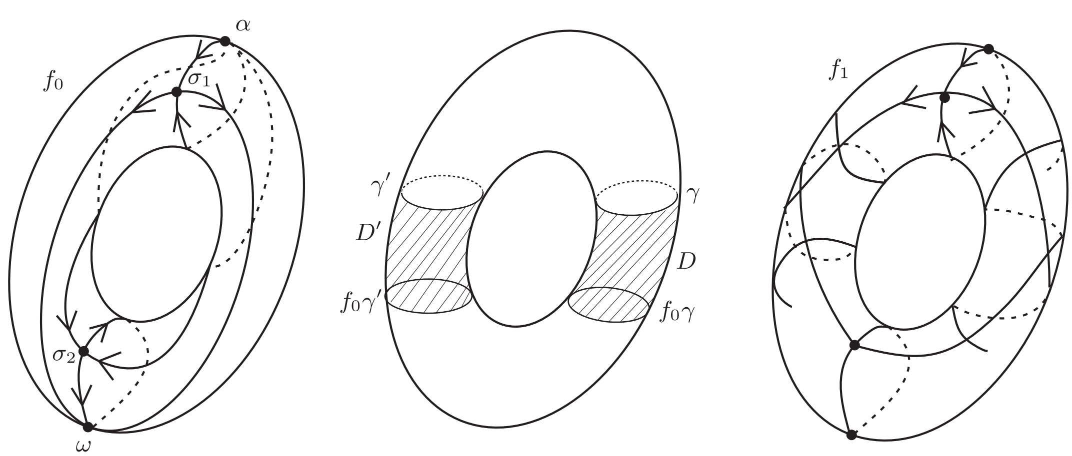

Fig. 13. S. Matsumoto's example.

Theorem 4 ([21], Theorem 1.3). *The diffeomorphisms f_0, f_1 of the 2-torus $\mathbb{T}^2$ cannot be joined by a stable arc*[1].

[1] Indeed, S. Matsumoto's proved Theorem 4 for the so-called simple arcs which include the stable arc as a special case.

To explain this result, suppose the contrary, i. e., that there is a stable arc that joins trivial and nontrivial diffeomorphisms. Then there is a stable arc ϕ_t with the unique bifurcation $g = \phi_{\frac{1}{2}}$ and such that ϕ_0 is trivial, while ϕ_1 is nontrivial. Since the period-doubling bifurcation does not produce nontrivial relations between the periodic points, the diffeomorphism g must have a saddle-node periodic point; denote it by p. Denote by $\mathcal{O}_p$ its orbit and denote by $\mathcal{O}_q, \mathcal{O}_r$ the nearest with respect to the Smale order neighboring orbits (possibly not unique). Since g has no cycles, these orbits are in the partial Smale order

$$\mathcal{O}_q \preceq \mathcal{O}_p \preceq \mathcal{O}_r,$$

i.e., $W^s_{\mathcal{O}_q} \cap W^u_{\mathcal{O}_p} \neq \varnothing$, $W^s_{\mathcal{O}_p} \cap W^u_{\mathcal{O}_r} \neq \varnothing$ and there is no such periodic point x for which either $W^s_{\mathcal{O}_q} \cap W^u_{\mathcal{O}_x} \neq \varnothing$, $W^s_{\mathcal{O}_x} \cap W^u_{\mathcal{O}_p} \neq \varnothing$ or $W^s_{\mathcal{O}_p} \cap W^u_{\mathcal{O}_x} \neq \varnothing$, $W^s_{\mathcal{O}_x} \cap W^u_{\mathcal{O}_r} \neq \varnothing$.

Since the saddle-node p is noncritical, either q or r is a node. Suppose that an unstable separatrix l of a saddle-node point p is attracted by the sink q. Then consider a curve $c = cl(l)$ which is invariant under some power N of f such that $f^N(p) = p$ and $f^N(q) = q$ and, hence, $f^N(c) - c$ is null-homotopic. As the trivial relatedness is transitive, we find that g is trivial. On the other hand, the saddle-node bifurcation creates new periodic points that are trivially related to the old ones and, hence, ϕ_1 is trivial and we come to a contradiction.

Obviously, similar effects are intrinsic to all surfaces with the nontrivial fundamental group. Moreover, they are naturally generalized to manifolds of greater dimensions. In [7] the trivial f_0 and the nontrivial f_1 Morse–Smale diffeomorphisms are constructed on the manifold $S^{n-1} \times S^1, n \geqslant 3$, both being isotopy equivalent to the identity (Fig. 14). As in Matsumoto's example, the diffeomorphism f_0 is the Cartesian product of the "source-sink" diffeomorphisms on the sphere S^{n-1} and on the circle S^1. A diffeomorphism f_1 diffeotopic to the identity is constructed from f_0 by composition with the following multidimensional Dehn twist Γ. Namely, consider an $(n-1)$-annulus $K = S^{n-2} \times S^1$ in $S^{n-1} \times S^1$. Then the set $W \subset S^{n-1} \times S^1$ bounded by K and $f_0(K)$ is a fundamental domain of f_0 on $S^{n-1} \times S^1 \setminus (cl(W^u_{\sigma_1} \cup W^s_{\sigma_2}))$ and $W \cong S^{n-2} \times [0,1] \times S^1$. Let $\Gamma : S^{n-1} \times S^1 \to S^{n-1} \times S^1$ be the identity outside W, and on W let it be defined by $\Gamma(s, t, e^{i2\pi\varphi}) = (s, t, e^{i2\pi(\varphi+t)})$.

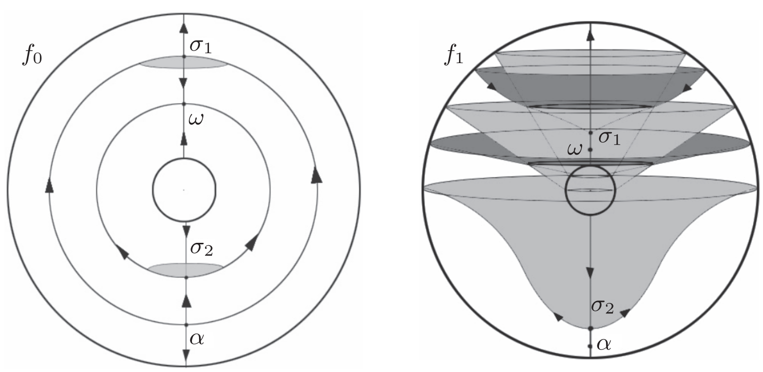

Fig. 14. An example of diffeomorphisms on $S^{n-1} \times S^1, n \geqslant 3$ which cannot be joined by a stable arc.

Theorem 5 ([7], Theorem 1). *The diffeomorphisms f_0, f_1 on the manifold $S^{n-1} \times S^1, n \geqslant 3$ cannot be joined by a stable arc.*

3.3. The Inconsistency of the Periodic Decompositions

D. Pixton in [35] proved that an energy Morse function $\Phi_f : M^2 \to \mathbb{R}$ exists for any Morse–Smale diffeomorphism f on a surface M^2. P. Blanchard [1] constructed the special decomposition of M^2 by the level curves of Φ_f using the notion of the oddness of a periodic orbit. He proved that the consistency of these decompositions for distinct diffeomorphisms is the necessary condition of existence of a stable arc between them.

Let M^2 be an orientable surface and $f : M^2 \to M^2$ be an orientation-preserving Morse – Smale diffeomorphism with the nonwandering set Ω_f. The *oddness of the periodic orbit* $\mathcal{O}_p \subset \Omega_f$ of period m is the odd integer l for which $m = 2^k l$ for some nonnegative k. A decomposition

$$M^2 = M_1 \cup \cdots \cup M_n$$

of M^2 is said to be a *periodic decomposition* if it satisfies:

- each element M_i is a surface whose boundary is either empty or consists of the regular level curves of Φ_f; distinct submanifolds M_i and M_j may intersect only along their common boundary component;
- $\Omega_f \cap M_i \neq \varnothing$ for each i; all the periodic orbits $\mathcal{O}_p \subset M_i$ are of the same oddness l_i, called the *oddness of* M_i; if $M_i \cap M_j \neq \varnothing$ for $i \neq j$, then their oddnesses are different.

An element M_i of the periodic decomposition is called *inessential* if it is homeomorphic to a union of annuli and for each annulus the vector field Φ_f points out along one boundary component and it points in along the other boundary component.

The *elementary cancellation* of a periodic decomposition is the elimination of an inessential element M_i from the decomposition by uniting it with the element M_j which has the common boundary with the annulus and such that $grad\,\Phi_f$ points out of the annulus along this boundary. Then the new element is assigned the oddness l_j and is united with the elements of the same oddness with which it has the common border. Notice that the result of an elementary cancellation is not a periodic decomposition.

Denote by $\mathcal{D}_f$ the decomposition of the surface M^2 obtained from a periodic decomposition by all elementary cancellations. Two decompositions $\mathcal{D}_f$, $\mathcal{D}_{f'}$ of the surface M^2 are *equivalent* if there is a homeomorphism of M^2 which sends the elements of one decomposition to the elements of the other and preserves the oddness. Notice that decompositions corresponding to different energy functions for the same diffeomorphism f are equivalent.

Theorem 6 ([1], Theorem 2.13). *If two Morse – Smale diffeomorphisms f, $f' : M^2 \to M^2$ can be joined by a stable arc, then their respective decompositions $\mathcal{D}_f$, $\mathcal{D}_{f'}$ are equivalent.*

The idea of the proof is based on the fact that for the saddle-node bifurcation and the flip bifurcation the periodic decomposition of the diffeomorphism after the bifurcation is equivalent to the elementary cancellation of the periodic decomposition of the diffeomorphism before the bifurcation.

One can easily construct infinitely many diffeomorphisms with pairwise nonequivalent decompositions. Therefore, in every isotopy class containing an orientation-preserving Morse – Smale diffeomorphism there are infinitely many classes of stable isotopic connectedness.

For illustration consider the following diffeomorphisms of the sphere S^2:

1) h is the "source-sink" diffeomorphism (Fig. 15.1);

2) g is the Morse – Smale diffeomorphism whose nonwandering set Ω_g consists of the fixed sink ω_1, the fixed source α, the sink orbit of period m: $\{\omega_2, f(\omega_2), \ldots, f^{m-1}(\omega_2)\}$ and the saddle orbit of period m: $\{\sigma, f(\sigma), \ldots, f^{m-1}(\sigma)\}$ (Fig. 15.2 for $m = 3$);

3) f is the diffeomorphism with two fixed sources at the poles, which on the equator of the sphere coincides with the rough transformation of the circle with two periodic orbits of period m (Fig. 15.3 for $m = 3$).

Consider the respective periodic decompositions for these diffeomorphisms (Fig. 15):

1) $M_1 = S^2$;

2), 3) $S^2 = D_1 \cup C \cup D_2$ where D_1, D_2 are the 2-disks and C is the 2-annulus.

One can check that, though the periodic decompositions of the diffeomorphisms h and g are different, the annulus C in the decomposition of g is inessential. The vector field $grad\,\Phi_g$ enters from one side of C and leaves from the other. After the elementary cancellation the annulus is united with the disk D_1 and the two remaining disks are united along the border producing the

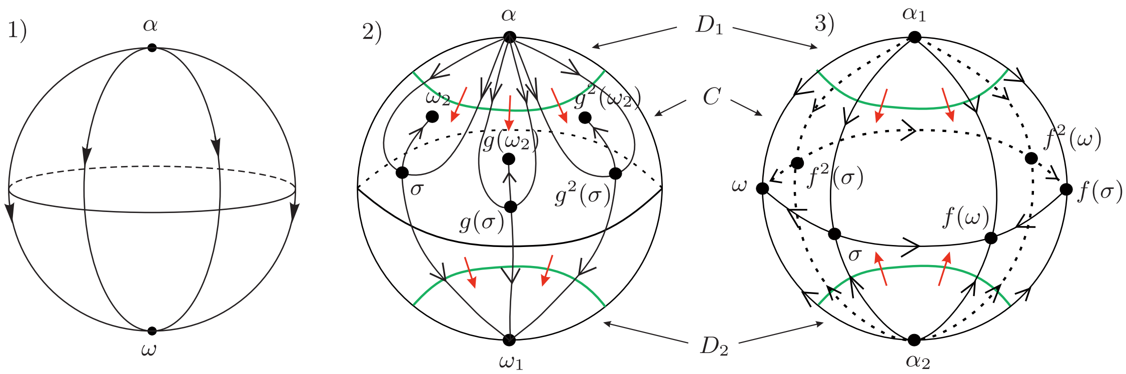

Fig. 15. Examples of diffeomorphisms on the sphere.

sphere S^2. Thus, the decompositions $\mathcal{D}_h$ and $\mathcal{D}_g$ are equivalent, which is not surprising because h and g can be joined by a stable arc with one saddle-node bifurcation.

The annulus C in the decomposition of f is not inessential, its periodic decomposition cannot be reduced. Therefore, the decompositions $\mathcal{D}_f$ and $\mathcal{D}_g$ are not equivalent and the diffeomorphisms f and g are in different classes of stable connectedness.

Notice that diffeomorphisms like f with the distinct oddnesses of periodic orbits cannot be joined by a stable arc, and this produces infinitely many classes of stable isotopy connectedness. Nevertheless, diffeomorphisms with the same oddness but with distinct periods of periodic orbits (for example, $m = 3$ and $m = 6$) have equivalent periodic decompositions. We will see in Section 4.1 that such diffeomorphisms cannot be joined by a stable arc, therefore, the inverse of Theorem 6 does not hold true.

3.4. The Nonhomeomorphic Characteristic Spaces

V. Grines and O. Pochinka found ([14]) a new obstruction to the existence of a stable arc. To explain it let us represent the dynamics of any Morse – Smale diffeomorphism in the following way (see, for example, [2, 5, 13] and Chapter 2.2 of [16]).

Denote by Ω^q_f, $q = 0, 1, 2, 3$ the set of periodic points p such that $\dim W^u_p = q$. Then $A_f = W^u_{\Omega^0_f \cup \Omega^1_f}$ is the connected attractor and $R_f = W^s_{\Omega^3_f \cup \Omega^2_f}$ is the connected repeller, their topology dimensions being less or equal to 1. The sets A_f and R_f do not intersect; every point of the set $V_f = M^3 \setminus (A_f \cup R_f)$ is wandering and moves under f from R_f to A_f. The orbit space $\hat{V}_f = V_f / f$ is called the *characteristic space*. It is proved (see, for example, Theorem 1.2 of [17]) that a characteristic space is a simple manifold[2)]. Denote by $p_f : V_f \to \hat{V}_f$ the natural projection.

A *heteroclinic intersection* of a Morse – Smale diffeomorphism is an intersection of the invariant manifolds of distinct saddle points of this diffeomorphism.

Theorem 7 ([14], Lemma 1). *Let diffeomorphisms $f, f' : M^3 \to M^3$ have no heteroclinic points and let them be joined by a stable arc. Then their characteristic spaces $\hat{V}_f, \hat{V}_{f'}$ are homeomorphic.*

The idea of the proof is based on the fact that the dynamics in the trapping neighborhood of either attractor or repeller R_f does not change when passing though a bifurcation point of the stable arc if this bifurcation is not the appearance or the disappearance of a heteroclinic curve. Therefore, the topology of the characteristic space remains unchanged.

To illustrate Theorem 7, we first consider the simplest Morse – Smale diffeomorphism *"source-sink"*. Its nonwandering set consists of two points: the source and the sink. The ambient manifold is homeomorphic to the sphere.

Theorem 8 ([4], Theorem 1). *Any two source-sink diffeomorphisms on S^3 can be joined by a bifurcation-free arc.*

---

2) A smooth 3-manifold is *simple* if it is either *irreducible* (every smooth 2-sphere bounds the 3-ball in it) or it is homeomorphic to $S^2 \times S^1$.

Notice that for a source-sink diffeomorphism on the 3-sphere the attractor A_f consists of the unique sink ω, while the repeller R_f consists of the unique source α. Since the diffeomorphism $f|_{W^s_\omega}$ is topologically conjugate in $\mathbb{R}^3$ to a homothety, the manifold $\hat{V}_f = \hat{V}_\omega = (W^s_\omega \setminus \omega)/f$ is homeomorphic to the manifold $S^2 \times S^1$. So, the attractor and the repeller of a source-sink diffeomorphism are separated by a 2-sphere in the following sense.

We say that the attractor A_f and a repeller R_f of a Morse – Smale diffeomorphism $f : M^3 \to M^3$ are *separated by a 2-sphere* if there is a smooth 2-sphere $\Sigma_f \subset V_f$ such that A_f and R_f are in the different connected components of the set $M^3 \setminus \Sigma_f$ (Fig. 16).

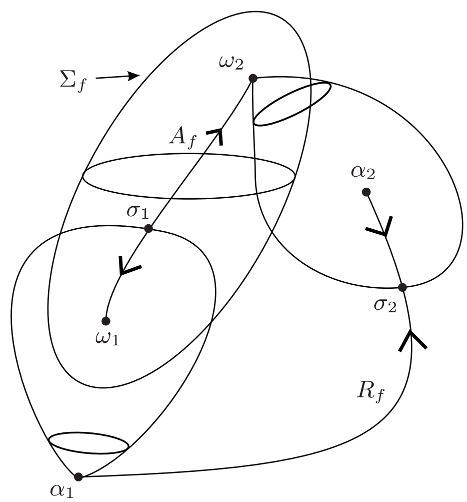

Fig. 16. A Morse – Smale diffeomorphism $f : M^3 \to M^3$ whose attractor A_f and repeller R_f are separated by a 2-sphere.

Theorem 9 ([14], Theorem 1). *A Morse – Smale diffeomorphism $f : M^3 \to M^3$ without heteroclinic intersections can be joined to the source-sink diffeomorphism by a stable arc if and only if its attractor A_f and repeller R_f are separated by a 2-sphere.*

The key technical point for the proof of Theorem 9 is the following fact. Any diffeomorphism of the considered class (distinct from source-sink diffeomorphism) has a node point whose basin contains a unique saddle separatrix. This separatrix is one-dimensional and has the same period as the node point. Moreover, it is tamely embedded to the basin in the sense of the following definition.

Consider a Morse – Smale diffeomorphism $f : M^3 \to M^3$ with a fixed saddle point σ whose 1-dimensional unstable saddle separatrix ℓ^u_σ satisfies

$$cl(\ell^u_\sigma) \setminus (\ell^u_\sigma \cup \sigma) = \{\omega\},$$

where ω is a sink. If $\dim W^u_\sigma = 1\,(2)$, then $cl(\ell^u_\sigma)$ is the arc (the 2-sphere) topologically embedded[3)] to M^3 (see, for example, Proposition 2.3 [16]). The set $\ell^u_\sigma \cup \sigma$ is the smooth submanifold of M^3, but the manifold $cl(\ell^u_\sigma)$ may be wild at the point ω. In this case the separatrix ℓ^u_σ is called *wild*, while otherwise it is called *tame*. The wildness and the tameness of a stable separatrix is defined in the same way.

The first "wild" example was constructed by D. Pixton in [35]. This was a diffeomorphism of Pixton class which is the class of 3-dimensional Morse – Smale diffeomorphisms whose nonwandering

[3)] A C^0-map $g : B \to X$ is called a *topological embedding* of the topological manifold B to the manifold X if it homeomorphically sends B to the subspace $g(B)$ with the topology induced from X. The image $A = g(B)$ is called the *topologically embedded manifold*. Notice that a topologically embedded manifold is not a topological submanifold in general. If A is a submanifold, then it is said to be *tame or tamely embedded*. Otherwise A is called *wild or wildly embedded*; the points in which the conditions of the topological manifold are not satisfied are called the *points of wildness*.

sets consist of exactly four points: three nodes and one saddle (Fig. 17). This class allows diffeomorphisms with wild separatrices, but either the attractor A_f or the repeller R_f of any diffeomorphism of Pixton class consists of the unique node. Therefore, the attractor and the repeller are separated by a 2-sphere. C. Bonatti and V. Grines [3] gave the topological classification of Pixton diffeomorphisms. Theorem 9 for Pixton diffeomorphisms was proved in [4].

Now consider a diffeomorphism f constructed as the connected sum of two 2-spheres on each of which a Pixton diffeomorphism is defined (Fig. 17. The 3-balls for the connected sum are marked).

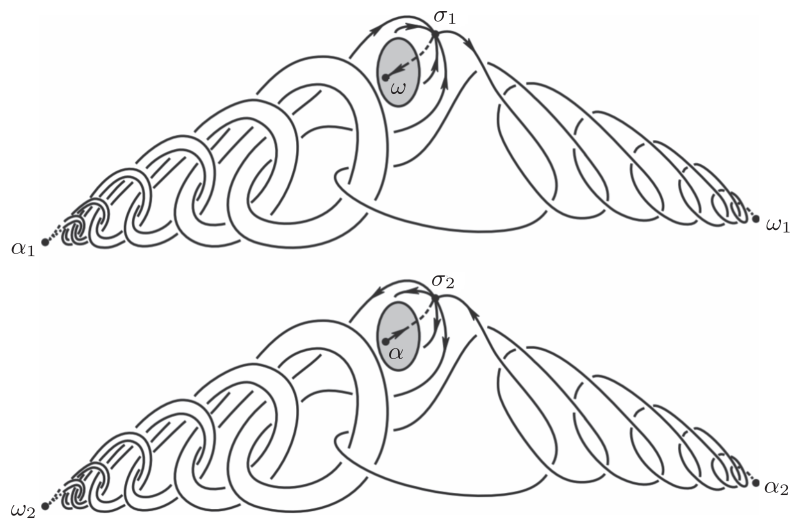

Fig. 17. The connected sum of two Pixton diffeomorphisms.

All separatrices of such a diffeomorphism are wild by construction, while the attractor A_f (the repeller R_f) is the arc with two points of wildness. Therefore, the trapping neighborhood of the attractor is not homeomorphic to the 3-ball and the orbit space $\hat{V}_f$ is not homeomorphic to the manifold $S^2 \times S^1$. Thus, the constructed diffeomorphism cannot be joined to the source-sink diffeomorphism by a stable arc.

3.5. The Existence of Exotic Spheres

At the end of the 1950s J. Milnor [24] proved the surprising fact that the 7-sphere allows 28 pairwise nondiffeomorphic smooth structures. One of them coincides with the standard one, i. e., the one produced on $S^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \dots x_{n+1}^2 = 1\}$ by two charts (U_-, ϑ_-), (U_+, ϑ_+) where $N(\underbrace{0, \dots, 0}_{n}, 1)$, $S(\underbrace{0, \dots, 0}_{n}, -1)$, $U_- = S^n \setminus \{S\}$, $U_+ = S^n \setminus \{N\}$ and $\vartheta_- : S^n \setminus \{S\} \to \mathbb{R}^n$, $\vartheta_+ : S^n \setminus \{N\} \to \mathbb{R}^n$ (called *stereographic projections*, Fig. 18) and defined by

$$\vartheta_-(x_1, \dots, x_{n+1}) = \left(\frac{x_1}{1 + x_{n+1}}, \dots, \frac{x_{n-1}}{1 + x_{n+1}}, \frac{x_n}{1 + x_{n+1}}\right),$$

$$\vartheta_+(x_1, \dots, x_{n+1}) = \left(\frac{x_1}{1 - x_{n+1}}, \dots, \frac{x_{n-1}}{1 - x_{n+1}}, \frac{x_n}{1 - x_{n+1}}\right).$$

The others define the so-called *exotic spheres*. One of the first examples of exotic spheres found by J. Milnor [24, Chapter 3] was the following. Consider two copies of $D^4 \times S^3$ with the boundary $S^3 \times S^3$ and identify the point (a, b) on the boundary with the point (a, a^2ba^{-1})[4]. The resulting

[4] Here the sphere S^3 is identified with a specific unitary group

$$SU(2) = \left\{ \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1 \right\} \text{ where the dash means complex conjugate.}$$

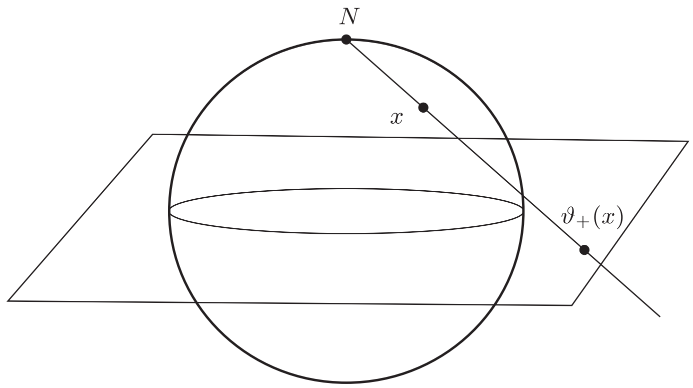

Fig. 18. The stereographic projection.

manifold has the natural smooth structure of the 7-manifold. Milnor showed that this manifold allows the Morse function with exactly two critical points, and this means that topologically it is the 7-sphere. He also showed that this manifold allows no orientation-reversing diffeomorphism, and this means that this is not the standard 7-sphere.

It was shown that the exotic structures are impossible in dimensions $1, 2, 3, 5, 6, 12, 56, 61$. Some multidimensional spheres have exactly two differential structures, while others have thousands of them. The problem of existence of exotic 4-spheres is still unsolved.

It follows from Smale's h-cobordism theorem (see, for example, [25]) that for $n > 5$ each exotic n-sphere is diffeomorphic to the *tangled sphere*, i. e., the sphere obtained from two copies of the standard n-balls by identification of their borders by an orientation-preserving diffeomorphism of the sphere S^{n-1}. Though any gluing diffeomorphism is isotopic to the identity, J. Milnor showed [25] that it can be nondiffeotopic to it. Finally, it follows from the result by J. Cerf [6] that two diffeomorphisms of the sphere S^{n-1}, $n > 6$, define diffeomorphic smooth structures on S^n if and only if these diffeomorphisms are diffeotopic.

Hence, nondiffeotopic orientation-preserving diffeomorphisms exist on the n-sphere ($n \geqslant 6$) if and only if there exist exotic $(n + 1)$-spheres and the number of diffeotopy classes coincides with the number of different exotic spheres. Moreover, from Theorem 10 it follows that there is a Morse–Smale diffeomorphism in every diffeotopy class of diffeomorphisms of the n-sphere. Therefore, diffeomorphisms of distinct classes cannot be joined by a smooth arc, in particular, by a stable arc.

Theorem 10 ([4], Proposition 3.6). *Any orientation-preserving diffeomorphism of the n-sphere is diffeotopic to a source-sink diffeomorphism.*

The idea of the proof is the following. At first the initial diffeomorphism is replaced with such a diffeomorphism that the polar points of the sphere are its fixed points. Then the fixed points are turned into the hyperbolic source and the hyperbolic sink. After that the resulting map is composed with a source-sink map whose expansion at the source is great. Since all the changes are the compositions with identity diffeotopic diffeomorphisms, the resulting diffeomorphism is diffeotopic to the initial one.

4. COMPONENTS OF STABLE CONNECTEDNESS OF GRADIENT-LIKE SURFACE DIFFEOMORPHISMS

In this chapter we discuss the classification results with respect to the stable isotopy connectedness for some classes of surface gradient-like diffeomorphisms.

4.1. Gradient-like Diffeomorphisms of the 2-sphere

Consider S^1 as the equator of the sphere S^2. Then a structurally stable diffeomorphism of the circle with exactly two periodic orbits of the period $m \in \mathbb{N}$ and the rotation number $\frac{k}{m}$ can be extended to a diffeomorphism $\phi_{k,m} : S^2 \to S^2$ with the two fixed sources in the north pole and in the south pole. We call this diffeomorphism the *model* diffeomorphism. Denote by $C_{k,m}$ the component of stable isotopy connectedness of $\phi_{k,m}$ and denote by $C^-_{k,m}$ the component of stable isotopy connectedness of $\phi^{-1}_{k,m}$. Denote by C_0 the component of stable isotopy connectedness of the source-sink diffeomorphism $\phi_0 \in G$ whose nonwandering set consists of exactly one source and one sink.

The following theorem provides a complete classification of gradient-like diffeomorphisms with respect to the stable connectivity.

Theorem 11 ([32], Theorem 1.1). *Any orientation-preserving gradient-like diffeomorphism of the* 2*-sphere* S^2 *lies in one of the components* C_0, $C_{k,m}$, $C^-_{k,m}$, k, $m \in \mathbb{N}$, $k < m/2$, $(k, m) = 1$ *and*

- *the components* C_0, $C_{k,m}$, $C^-_{k,m}$, k, $m \in \mathbb{N}$, $k < m/2$, $(k, m) = 1$ *are pairwise disjoint;*
- $C_{k,m} = C_{m-k,m}$, $C^-_{k,m} = C^-_{m-k,m}$, $C_{1,2} = C^-_{1,2} = C_{0,1} = C^-_{0,1} = C_0$.

Notice that from Theorem 6 it follows that the diffeomorphisms $\phi_{k,m}, \phi_{k',m'} : S^2 \to S^2$ for $m = 2^r \cdot q, m' = 2^{r'} \cdot q'$ for integer $r, r' \geqslant 0$ and natural $q \neq q'$ lie in the different components of stable isotopy connectedness.

The key point of the proof of Theorem 11 is the fact that the dynamics of any orientation-preserving gradient-like diffeomorphism $f : M^2 \to M^2$ can be represented as the dynamics of the global dual pair attractor-repeller whose space of the wandering orbits is connected. Denote by Ω^0_f, Ω^1_f, Ω^2_f the sets of the sinks, the saddles and the sources of f. For any (possibly empty) f-invariant set $\Sigma \subset \Omega^1_f$ let

$$A_\Sigma = \Omega^0_f \cup W^u_\Sigma,\ R_\Sigma = \Omega^2_f \cup W^s_{\Omega^1_f \setminus \Sigma}.$$

From [17] it follows that A_Σ and R_Σ are the *dual* attractor and repeller. We say

$$V_\Sigma = M^2 \setminus (A_\Sigma \cup R_\Sigma)$$

are the *characteristic space*. Denote by $\hat{V}_\Sigma$ the orbit space of action of the diffeomorphism f on V_Σ. According to [15], each connected component of the manifold $\hat{V}_\Sigma$ is homeomorphic to the 2-torus. Moreover, due to [29] there is such a set Σ that the orbit space $\hat{V}_\Sigma$ is connected. Let Σ be this set and let

$$A_f = A_\Sigma,\ R_f = R_\Sigma,\ V_f = V_\Sigma.$$

For the class G of gradient-like diffeomorphisms on the 2-sphere S^2 the attractor A_f and the repeller R_f can be described more precisely (Fig. 19). Notice that V_f consists of m_f mutually disjoint cylinders. A collection of noncontractible curves, one curve on each component, divides the sphere S^2 into two disjoint parts U and V such that

$$f(U) \subset U,\ A_f = \bigcap_{j\in\mathbb{N}} f^j(U);\ f^{-1}(V) \subset V,\ R_f = \bigcap_{j\in\mathbb{N}} f^{-j}(V).$$

Then for any diffeomorphism $f \in G$ (or for f^{-1}) the following holds:

1) the set U consists of $m_f \in \mathbb{N}$ pairwise disjoint disks $D_f, f(D_f), \ldots, f^{m_f-1}(D_f)$ such that $f^{m_f}(cl\, D_f) \subset int\, D_f$;

2) the attractor A_f consists of m_f connected components $A, f(A), \ldots, f^{m_f-1}(A)$ such that $A = \bigcap\limits_{j\in\mathbb{N}} f^{jm_f}(D_f)$ and $f^{m_f}(A) = A$;

3) the repeller R_f is connected.

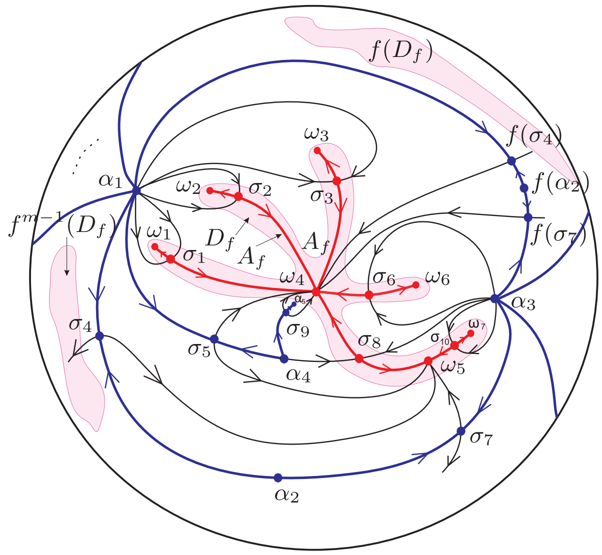

Fig. 19. An example of the attractor A_f and the repeller R_f of a gradient-like diffeomorphism of the 2-sphere. The set U is shown in rose, the part of the attractor A_f lying in D_f is shown in red, and the connected repeller R_f is shown in blue.

The reason for this is that the m_f cylinders are mapped into each other by f cyclically, and thus so do the boundary circles of U and V. This implies that U is either a union of m_f disjoint discs or a connected set, and the same alternative holds for V. Since $U \cup V$ is a sphere, we must have that either U is a union of discs and V is connected, or the other way around.

Denote by G^+ the subset of the set G consisting of diffeomorphisms with all saddle points of positive orientation type. Let $G^- = G \setminus G^+$ and denote by G_1 the subset of G consisting of diffeomorphisms f such that there exists a fixed pair A_f, R_f for which $m_f = 1$. Due to the topology of the 2-sphere one can prove that

- $G^- \subset G_1$;
- for any diffeomorphism $f \in G^+$ the number m_f is well-defined, i.e., it is independent of the choice of the pair A_f, R_f.

Thus, the set $G^+ \setminus G_1$ is the union of pairwise disjoint subsets

$$G^+ \setminus G_1 = G_2 \cdots \cup G_m \cup \ldots$$

and $m_f = m$ for any diffeomorphism $f \in G_m$, $m > 1$. Since the attractor A_f and the repeller R_f for $f \in G_1$ are connected, it is possible to construct an arc joining f to ϕ_0. Due to connectedness of A_f for $f \in G_m$, $m > 1$, it is possible to reduce A_f, i.e., to join it by a stable arc to a diffeomorphism g of the class $H_m \subset G_m$ where H_m consists of diffeomorphisms g whose attractor A_g is the unique sink orbit of period m (Fig. 20).

Then one proves that for the diffeomorphism g there exists a saddle orbit $\mathcal{O}_\sigma$ of period m such that $cl\, W^u_{\mathcal{O}_\sigma}$ is the g-invariant closed curve C_σ and the map $g|_{C_\sigma}$ is topologically conjugate to the rough circle transformation with the rotation number $\frac{k}{m}$ (Fig. 21). The rotation numbers for all such circles are equal. This makes it possible to join the diffeomorphism g by a stable arc to the diffeomorphism whose nonwandering set consists of one saddle orbit $\mathcal{O}_\sigma = \{\sigma, f(\sigma), \ldots, f^{m-1}(\sigma)\}$, one sink orbit $\mathcal{O}_\omega = \{\omega, f(\omega), \ldots, f^{m-1}(\omega)\}$ and fixed sources α_1, α_2. The resulting diffeomorphism can be joined to the model diffeomorphism $\phi_{k,m}$ by a stable arc. Then one proves that distinct model diffeomorphisms cannot be joined by a stable arc.

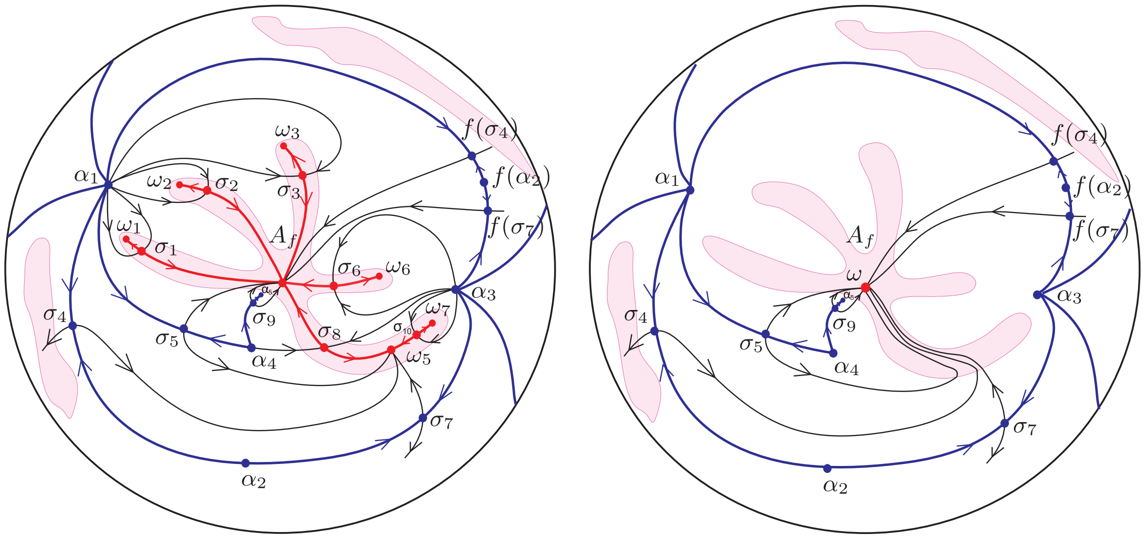

Fig. 20. Transfer from the diffeomorphism $f \in G_m$ to the diffeomorphism $g \in H_m$. The connected component of the attractor A_f lying in the disk D_f is shown in red (left). It is the union of the closures of the unstable manifolds of the saddle points σ_1, σ_2, σ_3, σ_6, σ_8, σ_{10}. The arc from f to g has six saddle-node bifurcations resulting in reduction of the attractor as shown on the right. There the attractor A_g of the diffeomorphism g consists of the sink orbit ω.

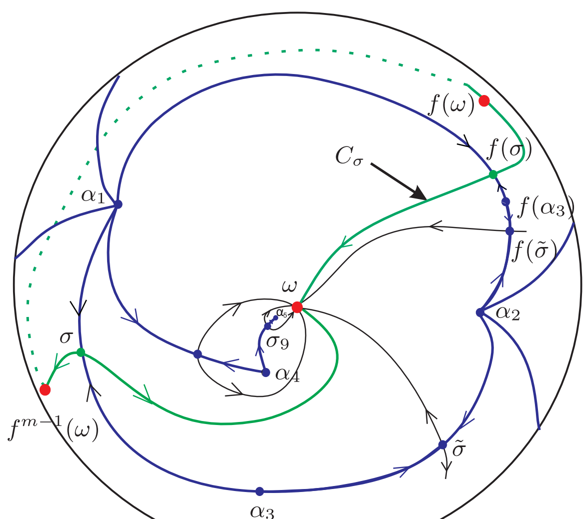

Fig. 21. The curve C_σ.

4.2. Palis Diffeomorphisms

In this chapter we consider the class P of orientation-preserving gradient-like diffeomorphisms f on the orientable surface M^2 such that all nonwandering points of f are fixed and are of positive orientation type. This class of diffeomorphisms was singled out by J. Palis in [33] as the class of Morse – Smale surface diffeomorphisms which can be included in a topological flow.

Theorem 12 ([31], Theorem 1). *Any two diffeomorphisms $f, f' \in P$ defined on the same surface M^2 can be joined by a stable arc with a finite number of noncritical saddle-nodes which unfold generically.*

First one constructs a bifurcation-free arc joining the diffeomorphism $f \in P$ to a certain diffeomorphism $\phi_f \in P$ which is the 1-time shift of the gradient flow of some Morse function.

Then, due to Theorems 1 and 2, the diffeomorphisms $\phi_f, \phi_{f'}$ are joined by an arc with a finite number of saddle-node bifurcations.

FUNDING

The research on the obstructions to existence of a stable arc between isotopic Morse – Smale diffeomorphisms is supported by RSF (Grant No. 21-11-00010), and the research on components of the stable connection of gradient-like diffeomorphisms of surfaces is supported by the Laboratory of Dynamical Systems and Applications NRU HSE, by the Ministry of Science and Higher Education of the Russian Federation (ag. 075-15-2019-1931) and by the Foundation for the Advancement of Theoretical Physics and Mathematics "BASIS" (project 19-7-1-15-1).

CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

REFERENCES

1. Blanchard, P. R., Invariants of the NPT Isotopy Classes of Morse – Smale Diffeomorphisms of Surfaces, *Duke Math. J.,* 1980, vol. 47, no. 1, pp. 33–46.
2. Bonatti, Ch., Grines, V., Laudenbach, F., and Pochinka, O., Topological Classification of Morse – Smale Diffeomorphisms without Heteroclinic Curves on 3-Manifolds, *Ergodic Theory Dynam. Systems,* 2019, vol. 39, no. 9, pp. 2403–2432.
3. Bonatti, Ch. and Grines, V., Knots As Topological Invariants for Gradient-Like Diffeomorphisms of the Sphere S^3, *J. Dynam. Control Systems,* 2000, vol. 6, no. 4, pp. 579–602.
4. Bonatti, C., Grines, V. Z., Medvedev, V. S., and Pochinka, O. V., Bifurcations of Morse – Smale Diffeomorphisms with Wildly Embedded Separatrices, *Proc. Steklov Inst. Math.,* 2007, vol. 256, no. 1, pp. 47–61; see also: *Tr. Mat. Inst. Steklova,* 2007, vol. 256, pp. 54–69.
5. Bonatti, C., Grines, V., and Pochinka, O., Topological Classification of Morse – Smale Diffeomorphisms on 3-Manifolds, *Duke Math. J.,* 2019, vol. 168, no. 13, pp. 2507–2558.
6. Cerf, J., *Sur les difféomorphismes de la sphère de dimension trois* $(\Gamma_4 = 0)$*,* Lecture Notes in Math., vol. 53, Berlin: Springer, 1968.
7. Pochinka, O., Nozdrinova, E., and Dolgonosova, A., On the Obstructions to the Existence of a Simple Arc Joining the Multidimensional Morse – Smale Diffeomorphisms, *Dinam. Sist.,* 2017, vol. 7(35), no. 2, pp. 103–111.
8. Fleitas, G., Replacing Tangencies by Saddle–Nodes, *Bol. Soc. Brasil. Mat.,* 1977, vol. 8, no. 1, pp. 47–51.
9. Franks, J., Necessary Conditions for the Stability of Diffeomorphisms, *Trans. Amer. Math. Soc.,* 1971, vol. 158, no. 2, pp. 301–308.
10. Grines, V. Z., Gurevich, E. Ya., Medvedev, V. S., and Pochinka, O. V., Embedding in a Flow of Morse – Smale Diffeomorphisms on Manifolds of Dimension Higher Than Two, *Math. Notes,* 2012, vol. 91, nos. 5–6, pp. 742–745; see also: *Mat. Zametki,* 2012, vol. 91, no. 5, pp. 791–794.
11. Grines, V., Gurevich, E., and Pochinka, O., On Embedding of the Morse – Smale Diffeomorphisms into Topological Flows, *Mosc. Math. J.,* 2019, vol. 19, no. 4, pp. 739–760.
12. Grines, V., Gurevich, E., and Pochinka, O., On Embedding of the Morse – Smale Diffeomorphisms in a Topological Flows, *SMFD,* 2020, vol. 66, no. 2, pp. 160–181 (Russian).
13. Grines, V. Z., Gurevich, E. Ya., Zhuzhoma, E. V., and Pochinka, O. V., Classification of Morse – Smale Systems and the Topological Structure of Underlying Manifolds, *Russian Math. Surveys,* 2019, vol. 74, no. 1, pp. 37–110; see also: *Uspekhi Mat. Nauk,* 2019, vol. 74, no. 1(445), pp. 41–116.
14. Grines, V. Z. and Pochinka, O. V., On the Simple Isotopy Class of a Source-Sink Diffeomorphism on the 3-Sphere, *Math. Notes,* 2013, vol. 94, nos. 5–6, pp. 862–875; see also: *Mat. Zametki,* 2013, vol. 94, no. 6, pp. 828–845.
15. Grines, V. Z., Pochinka, O. V., and Van Strien, S., On 2-Diffeomorphisms with One-Dimensional Basic Sets and a Finite Number of Moduli, *Mosc. Math. J.,* 2016, vol. 16, no. 4, pp. 727–749.
16. Grines, V., Medvedev, T., and Pochinka, O., *Dynamical Systems on* 2- *and* 3-*Manifolds,* Dev. Math., vol. 46, New York: Springer, 2016.
17. Grines, V. Z., Zhuzhoma, E. V., Medvedev, V. S., and Pochinka, O. V., Global Attractor and Repeller of Morse – Smale Diffeomorphisms, *Proc. Steklov Inst. Math.,* 2010, vol. 271, no. 1, pp. 103–124; see also: *Tr. Mat. Inst. Steklova,* 2010, vol. 271, pp. 111–133.
18. Hirsch, M. W., Pugh, C. C., and Shub, M., *Invariant Manifolds,* Lecture Notes in Math., vol. 583, New York: Springer, 1977.

19. Katok, A. and Hasselblatt, B., *Introduction to the Modern Theory of Dynamical Systems,* Encyclopedia Math. Appl., vol. 54, Cambridge: Cambridge Univ. Press, 1995.
20. Lukyanov, V. I. and Shilnikov, L. P., On Some Bifurcations of Dynamical Systems with Homoclinic Structures, *Soviet Math. Dokl.,* 1978, vol. 19, pp. 1314–1318; see also: *Dokl. Akad. Nauk SSSR,* 1978, vol. 243, no. 1, pp. 26–29.
21. Matsumoto, Sh., There Are Two Isotopic Morse – Smale Diffeomorphisms Which Cannot Be Joined by Simple Arcs, *Invent. Math.,* 1979, vol. 51, no. 1, pp. 1–7.
22. Maier, A. G., Rough Transform Circle into a Circle, *Uchen. Zap. Gorkov. Gos. Univ.,* 1939, no. 12, pp. 215–229 (Russian).
23. Meyer, K. R., Energy Function for Morse – Smale Systems, *Am. J. Math.,* 1968, vol. 90, pp. 1031–1040.
24. Milnor, J., On Manifolds Homeomorphic to the 7-Sphere, *Ann. of Math. (2),* 1956, vol. 64, no. 2, pp. 399–405.
25. Milnor, J., *Lectures on the h-Cobordism Theorem,* Princeton, N.J.: Princeton Univ. Press, 1965.
26. Newhouse, S., Palis, J., and Takens, F., Stable Arcs of Diffeomorphisms, *Bull. Amer. Math. Soc.,* 1976, vol. 82, no. 3, pp. 499–502.
27. Newhouse, S., Palis, J., and Takens, F., Bifurcations and Stability of Families of Diffeomorphisms, *Inst. Hautes Études Sci. Publ. Math.,* 1983, No. 57, pp. 5–71.
28. Newhouse, S. and Peixoto, M. M., There Is a Simple Arc Joining Any Two Morse – Smale Flows, in *Trois études en dynamique qualitative,* Astérisque, vol. 31, Paris: Soc. Math. France, 1976, pp. 15–41.
29. Nozdrinova, E., The Existence Connected Characteristic Space at the Gradient-Like Diffeomorphisms of Surfaces, *Zh. SVMO,* 2017, vol. 19, no. 2, pp. 91–97.
30. Nozdrinova, E. V., Rotation Number As a Complete Topological Invariant of a Simple Isotopic Class of Rough Transformations of a Circle, *Russian J. Nonlinear Dyn.,* 2018, vol. 14, no. 4, pp. 543–551.
31. Nozdrinova, E., On a Stable Arc Connecting Palis Diffeomorphisms on a Surface, *Dynamical Systems,* 2020, vol. 10(38), no. 2, pp. 139–148.
32. Nozdrinova, E. and Pochinka, O., Solution of the 33rd Palis – Pugh Problem for Gradient-Like Diffeomorphisms of a Two-Dimensional Sphere, *Discrete Contin. Dyn. Syst.,* 2021, vol. 41, no. 3, pp. 1101–1131.
33. Palis, J., On Morse – Smale Dynamical Systems, *Topology,* 1968, vol. 8, no. 4, pp. 385–404.
34. Palis, J. and Pugh, C. C., Fifty Problems in Dynamical Systems, in *Dynamical Systems: Proc. Sympos. Appl. Topology and Dynamical Systems (Univ. Warwick, Coventry, 1973/1974): Presented to E. C. Zeeman on His Fiftieth Birthday,* A. Manning (Ed.), Lecture Notes in Math., vol. 468, Berlin: Springer, 1975, pp. 345–353.
35. Pixton, D., Wild Unstable Manifolds, *Topology,* 1977, vol. 16, no. 2, pp. 167–172.
36. Afraimovich, V. S. and Shilnikov, L. P., Small Periodic Perturbations of Autonomous Systems, *Soviet Math. Dokl.,* 1974, vol. 15, pp. 734–742; see also: *Dokl. Akad. Nauk SSSR,* 1974, vol. 214, pp. 739–742.
37. Afraimovich, V. S. and Shilnikov, L. P., Certain Global Bifurcations Connected with the Disappearance of a Fixed Point of Saddle-Node Type, *Dokl. Akad. Nauk SSSR,* 1974, vol. 219, pp. 1281–1284 (Russian).
38. Smale, S., On Gradient Dynamical Systems, *Ann. of Math. (2),* 1961, vol. 74, pp. 199–206.
39. Smale, S., Differentiable Dynamical Systems, *Bull. Amer. Math. Soc.,* 1967, vol. 73, no. 6, pp. 747–817.