

# Blaschke–Santaló inequality for many functions and geodesic barycenters of measures <sup>\*</sup>

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## Abstract

Motivated by the geodesic barycenter problem from optimal transportation theory, we prove a natural generalization of the Blaschke–Santaló inequality and the affine isoperimetric inequalities for many sets and many functions. We derive from it an entropy bound for the total Kantorovich cost appearing in the barycenter problem. We also establish a “pointwise Prékopa–Leindler inequality” and show a monotonicity property of the multimarginal Blaschke–Santaló functional.

## 1 Introduction

The Blaschke–Santaló inequality, see [4, 37], states that every 0-symmetric convex body  $K$  in  $\mathbb{R}^n$  satisfies

$$\operatorname{vol}_n(K)\operatorname{vol}_n(K^\circ) \leq (\operatorname{vol}_n(B_2^n))^2,$$

where  $K^\circ = \{y \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \ \forall x \in K\}$  is the polar body of  $K$ ,  $B_2^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$  is the Euclidean unit ball and  $|\cdot|$  denotes the Euclidean norm on  $\mathbb{R}^n$ . The left-hand side of this inequality is called the Mahler volume. The sharp lower bound for the Mahler volume is still open in dimensions 4 and higher. The famous Mahler conjecture suggests that this functional is minimized by the couple  $(B_1^n, B_\infty^n)$ . Partial results can be found in, e.g., [23, 27, 34, 35].

Here we ask: What is a natural generalization of the bounds for the Mahler volume for multiple sets? While this is not obvious from the geometric viewpoint, we suggest in this paper a reasonable extension, which is naturally related to a functional counterpart of the Blaschke–Santaló inequality.

The functional Blaschke–Santaló inequality was discovered by K. Ball [6] and later extended and generalized in [3], [20], [30]. In its simplest form it states that for every two measurable even functions  $V, W$  on  $\mathbb{R}^n$  we have that

$$\int e^{-V(x)} dx \int e^{-W(y)} dy \leq (2\pi)^n,$$

provided that  $V(x) + W(y) \geq \langle x, y \rangle$  and either  $0 < \int e^{-V(x)} < \infty$  or  $0 < \int e^{-W(y)} < \infty$ . Equality is attained if and only if  $V(x) = |Tx|^2 + c$ ,  $W(y) = |T^{-1}y|^2 - c$ , where  $T$  is a positive definite matrix and  $c > 0$  is a constant. Interesting links to optimal transportation

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theory were noted in [3] and more recently in [17]. There, it is shown that for probability measures  $\mu = f \cdot \gamma$ ,  $\nu = g \cdot \gamma$ , where  $\gamma$  is the standard Gaussian measure, such that  $\int x f d\gamma = 0$ , the following inequality holds,

$$\frac{1}{2} W_2^2(\mu, \nu) \leq \text{Ent}_\gamma(\mu) + \text{Ent}_\gamma(\nu) \quad (1.1)$$

and that this inequality is equivalent to the functional Blaschke–Santaló inequality. Here,  $W_2^2(\mu, \nu)$  is the  $L^2$  Kantorovich distance (see Section 2 for the definition) and

$$\text{Ent}_\gamma(\mu) = \int f \log f d\gamma$$

is the relative entropy with respect to Gaussian measure. Inequality (1.1) is a remarkable strengthening of the Talagrand transportation inequality and the starting point of our paper. We refer to, e.g., [5] for Talagrand’s inequality and its fundamental importance in probability theory. In this context, please also note a very recent result of N. Gozlan about a transportational approach to the lower bound for the functional Blaschke–Santaló inequality [26].

We would like to point out an important connection of the Blaschke–Santaló inequality to the Kähler–Einstein equation. Inequality (1.1) implies, in particular, that the functional  $\mu \rightarrow \frac{1}{2} W_2^2(\mu, \nu) - \text{Ent}_\gamma(\mu)$  is bounded from above. The minimum of this functional solves the so-called Kähler–Einstein equation. This was established by F. Santambrogio [36]. The form of the functional presented here was considered in [29]. The well-posedness of the Kähler–Einstein equation was proved by D. Cordero-Erausquin and B. Klartag [14]. Generalization to the sphere and relations to the logarithmic Minkowski problem were established in [28]. Other related transportation inequalities can be found in [19].

To analyze the case of  $k$  functions with  $k > 2$  we consider the cost functional

$$c(x_1, \dots, x_n) = \sum_{i,j=1, i < j}^n |x_i - x_j|^2 \quad (1.2)$$

and the corresponding multimarginal Monge–Kantorovich problem, i.e., the minimization problem

$$P \rightarrow \int c dP, \quad P \in \mathcal{P}(\mu_1, \dots, \mu_k)$$

among the measures  $\mathcal{P}(\mu_1, \dots, \mu_k)$  with fixed projections  $\mu_1, \dots, \mu_k$ . This problem has been studied by Gangbo and Świech [24]. Agueh and Carlier realized in [1] that this problem is naturally related to the barycenter problem for  $\mu_1, \dots, \mu_k$ . A measure  $\mu$  is called geodesic (or Wasserstein) barycenter of  $\mu_1, \dots, \mu_k$  with coefficients  $\frac{1}{k}$ , if it gives the minimum to the functional  $\nu \rightarrow \sum_{i=1}^k \frac{1}{2k} W_2^2(\mu_i, \nu)$ . Barycenters of measures have attracted much attention, also among applied scientists. We refer to the recent book of Peyrè and Cuturi [15] and the references therein for more information.

Motivated by these results we conjecture that

$$\prod_{i=1}^k \int_{\mathbb{R}^n} f_i(x_i) dx_i \leq \left( \int_{\mathbb{R}^n} \rho^{\frac{1}{k}} \left( \frac{k(k-1)}{2} |u|^2 \right) du \right)^k, \quad (1.3)$$

where  $f_i: \mathbb{R}^n \rightarrow \mathbb{R}_+$ ,  $1 \leq i \leq k$ , are even, measurable, integrable functions satisfying

$$\prod_{i=1}^k f_i(x_i) \leq \rho \left( \sum_{i,j=1, i < j}^k \langle x_i, x_j \rangle \right)$$

and  $\rho$  is a positive non-increasing function. We verify this conjecture in several cases. Some of our main results are stated next.

## 1.1 The main results

In Section 2 we discuss some preliminary facts about Kantorovich duality theory for many functions and prove that our integral functional is bounded for the case of quadratic cost (1.2). We also show that for  $k > 2$  our functional has a trivial (zero) lower bound, unlike the case of two functions.

In Section 3 we verify the above conjecture in the unconditional case (see Section 3 for the definition) and prove the following theorem.

**Theorem 3.1** *Let  $f_i: \mathbb{R}^n \rightarrow \mathbb{R}_+$ ,  $1 \leq i \leq k$ , be unconditional integrable functions satisfying*

$$\prod_{i=1}^k f_i(x_i) \leq \rho \left( \sum_{\substack{i,j=1 \\ i < j}}^k \langle x_i, x_j \rangle \right) \text{ for every } x_i, x_j \in \mathbb{R}_+^n,$$

where  $\rho$  is a positive non-increasing function on  $[0, \infty)$  such that  $\int_{\mathbb{R}} \rho^{\frac{1}{k}}(t^2) dt < \infty$ . Then

$$\prod_{i=1}^k \int_{\mathbb{R}^n} f_i(x_i) dx_i \leq \left( \int_{\mathbb{R}^n} \rho^{\frac{1}{k}} \left( \frac{k(k-1)}{2} |u|^2 \right) du \right)^k.$$

For  $k > 2$ , equality holds in this inequality if and only if there exist positive constants  $c_i$ ,  $1 \leq i \leq k$ , such that  $\prod_{i=1}^k c_i = 1$ , and such that for all  $1 \leq i \leq k$ ,

1.

$$f_i(x) = c_i \rho^{\frac{1}{k}} \left( \frac{k(k-1)}{2} |x|^2 \right)$$

almost everywhere on  $\mathbb{R}^n$ .

2. The function  $\rho$  satisfies the inequality

$$\prod_{i=1}^k \rho^{\frac{1}{k}} \left( \frac{k(k-1)}{2} |x_i|^2 \right) \leq \rho \left( \sum_{i,j=1, i < j}^k \langle x_i, x_j \rangle \right)$$

for all  $x_i, x_j$  in  $\mathbb{R}_+^n$ .

Our proof uses the Prékopa–Leindler inequality for many functions and an exponential change of variables as an intermediate step.

The above theorem and the affine isoperimetric inequality of affine surface area for log-concave functions of [12] lead to multi-functional affine isoperimetric inequalities for log-concave functions, which we also prove in this section.

In Section 4 we study equality cases for unconditional functions and prove the above stated equality characterizations. To do so, we need equality characterizations in the Prékopa–Leindler inequality. We could not find such characterizations in the literature and therefore give a proof of those.

In Section 5 we prove a generalization of the Blaschke–Santaló inequality which involves more than two convex bodies. There,  $\|\cdot\|_K$  denotes the norm with the convex body  $K$  as unit ball.

**Theorem 5.1** Let  $K_i$ ,  $1 \leq i \leq k$ , be unconditional convex bodies in  $\mathbb{R}^n$  such that

$$\prod_{i=1}^k e^{-\frac{1}{2}\|x_i\|_{K_i}^2} \leq \rho \left( \sum_{i,j=1,i < j}^k \langle x_i, x_j \rangle \right) \text{ for every } x_i, x_j \in \mathbb{R}_+^n,$$

where  $\rho$  is a positive non-increasing function  $[0, \infty)$  such that  $\int_{\mathbb{R}} \rho^{\frac{1}{k}}(t^2) dt < \infty$ . Then

$$\prod_{i=1}^k \text{vol}_n(K_i) \leq \left( \frac{\text{vol}_n(B_2^n)}{(2\pi)^{\frac{n}{2}}} \right)^k \left( \int_{\mathbb{R}^n} \rho^{\frac{1}{k}} \left( \frac{k(k-1)}{2} |x|^2 \right) dx \right)^k.$$

For  $k > 2$ , equality holds if and only if  $K_i = r B_2^n$  and  $\rho(t) = e^{-\frac{t}{(k-1)r^2}}$  for some  $r > 0$ . In particular, if  $\rho(t) = e^{-\frac{t}{k-1}}$ , then, if  $\sum_{i=1, i < j}^k \langle x_i, x_j \rangle \leq \frac{k-1}{2} \sum_{i=1}^k \|x_i\|_{K_i}^2$ , we have that

$$\prod_{i=1}^k \text{vol}_n(K_i) \leq (\text{vol}_n(B_2^n))^k$$

and for  $k > 2$  equality holds if and only if  $K_i = B_2^n$  for all  $1 \leq i \leq k$ .

Proposition 5.3 of this section gives a version of the  $L_p$ -affine isoperimetric inequalities for many sets.

In Section 6 we prove several strengthenings of classical inequalities using barycenters. Among them is the following ‘‘pointwise Prékopa–Leindler inequality’’.

**Theorem 6.1** Let  $\mu$  be the barycenter of measures  $\mu_i = \frac{f_i}{\int f_i dx_i} dx_i$  with weights  $\lambda_i$ ,  $1 \leq i \leq k$ , where  $f_i$  are nonnegative integrable functions. Then it has density  $p$  satisfying

$$\prod_{i=1}^k \left( \int f_i dx_i \right)^{\lambda_i} p(x) \leq \sup_{x = \sum_{i=1}^k \lambda_i y_i} \prod_{i=1}^k f_i^{\lambda_i}(y_i), \text{ for } p - a.e. x. \quad (1.4)$$

In Section 7 we study applications of our results to transportation inequalities for the barycenter problem. We obtain the following bound which generalizes (1.1) and, in particular, a classical estimate of Talagrand.

**Theorem 7.1** Assume that  $\mu_i = \rho_i \cdot \gamma$ , where  $\gamma$  is the standard Gaussian measure and the  $\rho_i$  are unconditional, then

$$\mathcal{F}(\mu) \leq \frac{k-1}{k^2} \sum_{i=1}^k \int \rho_i \log \rho_i d\gamma,$$

where  $\mathcal{F}(\mu) = \frac{1}{2k} \sum_{i=1}^k W^2(\mu, \mu_i)$  and  $\mu$  is the barycenter of  $\{\mu_i\}$  with weights  $\frac{1}{k}$ .

Moreover, from our refinement of the Prékopa–Leindler inequality, we deduce some new inequalities related to displacement convexity of the Gaussian entropy.

In Section 8 we prove a monotonicity property of the multimarginal Blaschke–Santaló functional. A simplified version of the result is stated next.

**Theorem 8.2** Assume that for  $1 \leq i \leq k$ ,  $V_i(x_i)$  are measurable functions such that  $e^{-V_i}$  are integrable, satisfying

$$\sum_{i=1}^k V_i(x_i) \geq \sum_{i,j=1,i < j}^k \frac{1}{k-1} \langle x_i, x_j \rangle.$$

Let the tuple of functions  $U_i(x_i)$  be the solution to the dual multimarginal maximization problem with marginals  $\frac{e^{-V_i dx_i}}{\int e^{-V_i dx_i}}$  and the cost function  $\frac{1}{k-1} \sum_{i,j=1, i < j}^k \langle x_i, x_j \rangle$ . Then

$$\prod_{i=1}^k \int e^{-V_i} dx_i \leq \prod_{i=1}^k \int e^{-U_i} dx_i.$$

## 2 Integral bounds and facts about barycenters

We start this section with the proof that the Blaschke–Santaló functional is bounded on the set of even functions. We will need the definition of the Legendre conjugate  $V^*$ , which for a proper (not identically equal to  $+\infty$ ) function  $V : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  is defined as

$$V^*(y) = \sup_{x \in \mathbb{R}^n} (\langle x, y \rangle - V(x)).$$

**Proposition 2.1.** *Let  $V_i$ ,  $1 \leq i \leq k$ , be a family of Borel functions on  $\mathbb{R}^n$  such that  $e^{-V_i}$  is integrable for all  $1 \leq i \leq k$ . Then the functional*

$$\mathcal{S}(V_1, \dots, V_k) = \prod_{i=1}^k \int e^{-V_i(x_i)} dx_i$$

is bounded on the set

$$L_{n,k} =$$

$$\left\{ (V_1, \dots, V_k) : \forall i \in \{1, \dots, k\}, V_i \text{ is even, } \int e^{-V_i(x)} dx < \infty, \sum_{i=1}^k V_i(x_i) \geq \sum_{i,j=1, i < j}^k \langle x_i, x_j \rangle \right\}.$$

*Proof.* Let us fix arbitrary finite  $(V_1, \dots, V_k) \in L_{n,k}$  and estimate  $\mathcal{S}(V_1, \dots, V_k)$ . First we note that the functions  $V_i$  can be assumed to be convex. Indeed, if  $V_1$  is not convex, replace it by the following convex function

$$\tilde{V}_1(x_1) = \sup_{x_i, i \neq 1} \left( \sum_{i,j=1, i < j}^k \langle x_i, x_j \rangle - \sum_{i \neq 1} V_i(x_i) \right).$$

The tuple  $(\tilde{V}_1, \dots, V_k)$  belongs to  $L_{n,k}$ . Note that all the desired properties can be easily checked except of integrability of  $e^{-\tilde{V}_1}$ . We will show below that  $\tilde{V}_1$  is integrable. Since  $V_1 \geq \tilde{V}_1$ , we get  $\mathcal{S}(\tilde{V}_1, \dots, V_k) \geq \mathcal{S}(V_1, \dots, V_k)$ . Next we apply the same procedure to the tuple  $(\tilde{V}_1, \dots, V_k)$  and the function  $V_2$ . Repeating this procedure, we finally obtain a tuple  $(\tilde{V}_1, \dots, \tilde{V}_k)$  consisting of only convex functions such that  $\mathcal{S}(\tilde{V}_1, \dots, \tilde{V}_k) \geq \mathcal{S}(V_1, \dots, V_k)$ . Let us denote this new tuple again by  $(V_1, \dots, V_k)$ .

Next, note that without loss of generality we can restrict ourself to the case of convex functions satisfying  $V_i(0) = 0$ . Indeed, one can replace  $V_i$  by  $\tilde{V}_i(x_i) = V_i(x_i) - V_i(0)$ ,  $1 \leq i \leq k-1$ , and  $V_k$  by  $\tilde{V}_k(x_k) = V_k(x_k) + V_1(0) + \dots + V_{k-1}(0)$  and this replacement does not influence the value of the integral functional. One has  $\tilde{V}_i(0) = 0$ ,  $1 \leq i \leq k-1$ . Next we note that

$$\sum_{i=1}^{k-1} \tilde{V}_i(x_i) \geq \sum_{i < j}^{k-1} \langle x_i, x_j \rangle + \left\langle \sum_{i=1}^{k-1} x_i, x_k \right\rangle - \tilde{V}_k(x_k), \text{ for all } x_k,$$

is equivalent to

$$\sum_{i=1}^{k-1} \tilde{V}_i(x_i) \geq \sum_{i < j}^{k-1} \langle x_i, x_j \rangle + (\tilde{V}_k)^* \left( \sum_{i=1}^{k-1} x_i \right),$$

which in turn is equivalent to

$$\sum_{i=1}^{k-1} \frac{|x_i|^2}{2} + \tilde{V}_i(x_i) \geq \frac{1}{2} \left| \sum_{i=1}^{k-1} x_i \right|^2 + (\tilde{V}_k)^* \left( \sum_{i=1}^{k-1} x_i \right).$$

We now define a function  $F$  by the following relation

$$\frac{|t|^2}{2} + F(t) = \inf_{t = \sum_{i=1}^{k-1} x_i} \sum_{i=1}^{k-1} \frac{|x_i|^2}{2} + \tilde{V}_i(x_i).$$

Clearly  $(\tilde{V}_k)^* \leq F$ , hence  $\tilde{V}_k \geq F^*$ . Thus  $\mathcal{S}(\tilde{V}_1, \dots, \tilde{V}_k) \leq \mathcal{S}(\tilde{V}_1, \dots, F^*)$ . Moreover, it follows immediately from the definition of  $F$  and the above inequalities that  $(\tilde{V}_1, \dots, F^*) \in L_{n,k}$ . Since  $\tilde{V}_i \geq 0$  and  $\tilde{V}_i(0) = 0$ , we immediately get  $F(0) = 0$ . Hence,  $F^*(0) = 0$ . Thus the tuple  $(\tilde{V}_1, \dots, F^*)$  satisfies  $\tilde{V}_1(0) = \dots = F^*(0) = 0$  and gives a larger value to  $\mathcal{S}$ .

Finally, it is sufficient to show that  $\mathcal{S}$  is bounded for finite convex even functions  $V_i$  satisfying  $V_i(0) = 0$  and  $\sum_{i=1}^k V_i(x_i) \geq \sum_{i,j=1, i < j}^k \langle x_i, x_j \rangle$ . We observe that for every  $j \neq m$

$$\begin{aligned} V_m(x_m) &\geq \sup_{x_i, x_s, i, s \neq m} \left( \sum_{i, s=1, i < s}^k \langle x_i, x_s \rangle - \sum_{i \neq m} V_i(x_i) \right) \geq \\ &\sup_{x_j} \left( \left[ \sum_{i, s=1, i < s}^k \langle x_i, x_s \rangle - \sum_{i \neq m} V_i(x_i) \right]_{x_i=0, i \neq m, i \neq j} \right) = \sup_{x_j} (\langle x_m, x_j \rangle - V_j(x_j)) = V_j^*(x_m). \end{aligned}$$

If  $e^{-V_j}$  is integrable, then by the functional Blaschke–Santaló inequality

$$\int e^{-V_m} dx_m \int e^{-V_j} dx_j \leq \int e^{-V_j^*} dx_j \int e^{-V_j} dx_j \leq (2\pi)^n.$$

Hence

$$\prod_{i=1}^k \int e^{-V_i} dx_i = \left( \prod_{i, j=1, i < j}^k \int e^{-V_i} dx_i \int e^{-V_j} dx_j \right)^{\frac{1}{k-1}} \leq (2\pi)^{\frac{n}{k-1}}.$$

If  $e^{-V_j}$  is not integrable, then again by the Blaschke–Santaló inequality  $\int e^{-V_j^*} dx_j = 0$ , hence  $\int e^{-V_m} dx_m = 0$ , but this contradicts to finiteness of  $V_m$ .  $\square$

A related natural question is whether there is a non-trivial lower bound for  $\mathcal{S}$ ? For the case of two functions this is a functional variant of the well-known open problem, known as Mahler’s conjecture. More precisely, for  $k = 2$  we are looking for the lower bound of the functional

$$\int e^{-V} dx \int e^{-V^*} dy.$$

It is conjectured that the minimum is reached, in particular, when  $V(x) = \|x\|_1 = \sum_{i=1}^n |x_i|$  or  $V(x) = \|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$ , or their Legendre transform. See e.g., M. Fradelizi and M. Meyer [21], [22], where the conjecture was proved in dimension 1.

The natural generalization of this problem for the case of  $k > 2$  functions however has a trivial solution.

**Proposition 2.2.** *There exist even functions  $V_1, V_2, V_3$  such that the triple  $(V_1, V_2, V_3)$  satisfies*

$$V_m(x_m) = \sup_{x_i, i \neq m} \left( \sum_{i,j=1, i < j}^k \langle x_i, x_j \rangle - \sum_{i \neq m} V_i(x_i) \right) \quad (2.1)$$

and  $\mathcal{S}(V_1, V_2, V_3) = 0$ .

**Remark 2.3.** *Assumption (2.1) seems to be a natural generalization for  $k > 2$  functions of the condition that two convex functions are related by the Legendre transform.*

*Proof.* The desired functions are

$$V_1(x_1) = \begin{cases} 0 & \text{if } x = 0 \\ +\infty & \text{else} \end{cases}$$

$$V_2(x_2) = \frac{|x_2|^2}{2}, \quad V_3(x_3) = \frac{|x_3|^2}{2}.$$

The reader can easily check the claim. □

At the end of this section we recall basic facts on duality relations for the transportation cost appearing in the theory of barycenters of measures. Recall that for a given family of probability measures  $\mu_1, \dots, \mu_k$  and weights  $\lambda_i \in [0, 1]$  satisfying  $\sum_{i=1}^k \lambda_i = 1$  its barycenter  $\mu$  is the minimum point of the functional

$$\mathcal{F}(\nu) = \frac{1}{2} \sum_{i=1}^k \lambda_i W_2^2(\mu_i, \nu).$$

Here,

$$W_2^2(\nu_1, \nu_2) = \inf \left\{ \int |x - y|^2 dP(x, y) : P \in \mathcal{P}(\mathbb{R}^n \times \mathbb{R}^n), P(\cdot, \mathbb{R}^n) = \nu_1, P(\mathbb{R}^n, \cdot) = \nu_2 \right\}$$

is the  $L^2$  Kantorovich distance of probability measures  $\nu_1, \nu_2$ . It is well-known that the barycenter problem is closely related to the multimarginal (maximization) Kantorovich problem with the cost function

$$(x_1, \dots, x_k) \mapsto \sum_{i,j=1, i \neq j}^k \lambda_i \lambda_j \langle x_i, x_j \rangle$$

and marginals  $\mu_i$ . Let  $\pi$  be the solution to this problem, i.e. a measure that gives a maximum to the functional

$$P \rightarrow \int \sum_{i,j=1, i \neq j}^k \lambda_i \lambda_j \langle x_i, x_j \rangle dP \quad (2.2)$$

among the measures on  $(\mathbb{R}^n)^k$  having  $\mu_1, \dots, \mu_k$  as marginals.

The following facts are collected from [1] and [24].

**Theorem 2.4.** *[1], [24] Assume that  $\mu_i$  are absolutely continuous measures with finite second moments and  $\lambda_i \in (0, 1)$  are numbers satisfying  $\sum_{i=1}^k \lambda_i = 1$ . Then the following facts hold.*

1. There exists a unique absolutely continuous solution  $\mu$  to the barycenter problem and a unique solution  $\pi$  to the problem (2.2).
2. The measure  $\mu$  is the push-forward measure of  $\pi$  under the mapping  $T(x_1, \dots, x_k) = \sum_{i=1}^k \lambda_i x_i$  and the following relation holds:

$$\sum_{i=1}^k \lambda_i W_2^2(\mu_i, \mu) = \int \sum_{i=1}^k \lambda_i |x_i - T(x)|^2 d\pi.$$

3. The optimal transportation mappings  $\nabla\Phi_i$  of  $\mu$  onto  $\mu_i$  satisfy

$$\sum_{i=1}^k \lambda_i \nabla\Phi_i(x) = x$$

for  $\mu$ -a.e.  $x$ . and  $\pi$  is supported on the set  $\{(\nabla\Phi_1(x), \dots, \nabla\Phi_k(x)) : x \in \mathbb{R}^n\}$ .

4. There exists a tuple of convex functions  $(v_i)$  solving the problem dual to (2.2), which is unique up to addition of constants and modification of sets of zero measure, i.e. a  $k$ -tuple of functions satisfying

$$\sum_{i=1}^k v_i(x_i) \geq \sum_{i,j,i \neq j}^k \lambda_i \lambda_j \langle x_i, x_j \rangle$$

with equality  $\pi$ -a.e. The following relation holds between  $v_i$  and  $\Phi_i$ :

$$\Phi_i^*(x_i) = \lambda_i \frac{|x_i|^2}{2} + \frac{v_i(x_i)}{\lambda_i} + C_i \quad (2.3)$$

for  $\mu_i$ -almost all  $x_i$ .

**Remark 2.5.** The results of item 1. are obtained in Section 3 of [1], item 2. is contained in Proposition 4.2 [1], item 3. corresponds to Proposition 3.8 of [1]. Formula (2.3) needs some explanations. It corresponds to formula (4.8) in [1], but in the presentation in [1] there is no direct link to the optimal transportation of the barycenter  $\mu$  onto  $\mu_i$ . Let us give some informal explanations.

By the Kantorovich duality  $\pi$  is concentrated on the zero set of the positive function

$$\sum_{i=1}^k v_i(x_i) - \sum_{i,j,i \neq j}^k \lambda_i \lambda_j \langle x_i, x_j \rangle.$$

Thus, for  $\pi$ -a.e.  $(x_1, \dots, x_k)$  and all  $1 \leq i \leq k$  one has  $\nabla v_i(x_i) = \sum_{j \neq i}^k \lambda_i \lambda_j x_j$ . Equivalently,

$$\lambda_i x_i + \frac{\nabla v_i(x_i)}{\lambda_i} = \sum_{j=1}^k \lambda_j x_j, \quad \pi - \text{a.e.} \quad (2.4)$$

It remains to note that  $\mu$  is the image of  $\pi$  under  $T = \sum_{j=1}^k \lambda_j x_j$  and  $\mu_i$  is the projection of  $\pi$  onto the  $i$ -th factor. Thus relation (2.4) immediately implies that  $\mu$  is the image of  $\mu_i$  under the mapping  $x_i \rightarrow \lambda_i x_i + \frac{\nabla v_i(x_i)}{\lambda_i}$ . Since the latter is the gradient of the convex function  $\lambda_i \frac{|x_i|^2}{2} + \frac{v_i(x_i)}{\lambda_i}$ , we conclude by uniqueness of the optimal transportation mapping that  $\nabla\Phi_i^* = \lambda_i x_i + \frac{\nabla v_i(x_i)}{\lambda_i}$ .



### 3 The unconditional case

In this section we verify our conjecture (inequality part) for the unconditional functions. A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is called unconditional, if

$$f(\varepsilon_1 x_1, \dots, \varepsilon_n x_n) = f(x_1, x_2, \dots, x_n),$$

for every  $(\varepsilon_1, \dots, \varepsilon_n) \in \{-1, 1\}^n$  and every  $(x_1, \dots, x_n) \in \mathbb{R}^n$ .

**Theorem 3.1.** *Let  $f_i : \mathbb{R}^n \rightarrow \mathbb{R}_+$ ,  $1 \leq i \leq k$ , be measurable unconditional integrable functions satisfying*

$$\prod_{i=1}^k f_i(x_i) \leq \rho \left( \sum_{i,j=1, i < j}^k \langle x_i, x_j \rangle \right) \text{ for every } x_i, x_j \in \mathbb{R}_+^n,$$

where  $\rho$  is a positive non-increasing function on  $[0, \infty)$  such that  $\int_{\mathbb{R}} \rho^{\frac{1}{k}}(t^2) dt < \infty$ . Then

$$\prod_{i=1}^k \int_{\mathbb{R}^n} f_i(x_i) dx_i \leq \left( \int_{\mathbb{R}^n} \rho^{\frac{1}{k}} \left( \frac{k(k-1)}{2} |u|^2 \right) du \right)^k. \quad (3.1)$$

In particular, if

$$\prod_{i=1}^k f_i(x_i) \leq e^{-\alpha \sum_{i,j=1, i < j}^k \langle x_i, x_j \rangle}, \quad \alpha \in \mathbb{R}_+,$$

then

$$\prod_{i=1}^k \int_{\mathbb{R}^n} f_i(x_i) dx_i \leq \left( \int_{\mathbb{R}^n} e^{-\alpha \frac{k-1}{2} |u|^2} du \right)^k.$$

*Proof.* Clearly, for unconditional functions it is sufficient to check that

$$\prod_{i=1}^k \int_{\mathbb{R}_+^n} f_i(x_i) dx_i \leq \left[ \int_{\mathbb{R}_+^n} \rho^{\frac{1}{k}} \left( \frac{k(k-1)}{2} |u|^2 \right) du \right]^k,$$

provided that on  $\mathbb{R}_+^n$ ,

$$\prod_{i=1}^k f_i(x_i) \leq \rho \left( \sum_{i,j=1, i < j}^k \langle x_i, x_j \rangle \right).$$

We prove this using the Prékopa–Leindler inequality and a trick involving a change of variables formula (see, for instance, [25] or [30], Lemma 5).

For  $u = (u_1, \dots, u_n)$ , we denote  $e^u = (e^{u_1}, \dots, e^{u_n})$ . We apply the change of variables formula

$$x_i = e^{t_i}, \quad t_i \in \mathbb{R}^n,$$

and get

$$\prod_{i=1}^k \int_{\mathbb{R}_+^n} f_i(x_i) dx_i = \prod_{i=1}^k \int_{\mathbb{R}^n} f_i(e^{t_i}) e^{\sum_{m=1}^n (t_i)_m} dt_i,$$

where we write  $t_i = ((t_i)_1, (t_i)_2, \dots, (t_i)_n)$ . Next we apply the Prékopa–Leindler inequality (see, e.g., [25], formula (21) or (27)),

$$\prod_{i=1}^k \left( \int_{\mathbb{R}^n} g_i dt_i \right)^{\frac{1}{k}} \leq \int_{\mathbb{R}^n} \sup_{t = \frac{1}{k} \sum_{i=1}^k t_i} \prod_{i=1}^k g_i^{\frac{1}{k}}(t_i) dt.$$

After the change of variables and the application of the Prékopa–Leindler inequality, we use the assumptions of the theorem in the second inequality below. We also use the arithmetic-geometric mean inequality and the fact that  $\rho$  is non-increasing in the third inequality below. We get

$$\begin{aligned} & \left( \prod_{i=1}^k \int_{\mathbb{R}_+^n} f_i(x_i) dx_i \right)^{\frac{1}{k}} \leq \int_{\mathbb{R}^n} \sup_{t = \frac{1}{k} \sum_{i=1}^k t_i} \prod_{i=1}^k \left( f_i^{\frac{1}{k}}(e^{t_i}) e^{\frac{1}{k} \sum_{m=1}^n (t_i)_m} \right) dt \\ & \leq \int_{\mathbb{R}^n} \sup_{t = \frac{1}{k} \sum_{i=1}^k t_i} \left[ \rho^{\frac{1}{k}} \left( \sum_{i,j=1, i < j}^k \sum_{m=1}^n e^{(t_i+t_j)_m} \right) e^{\frac{1}{k} \sum_{i=1}^k \sum_{m=1}^n (t_i)_m} \right] dt \\ & = \int_{\mathbb{R}^n} \sup_{t = \frac{1}{k} \sum_{i=1}^k t_i} \left[ \rho^{\frac{1}{k}} \left( \sum_{i,j=1, i < j}^k \sum_{m=1}^n e^{(t_i+t_j)_m} \right) \right] e^{\sum_{m=1}^n (t)_m} dt \\ & \leq \int_{\mathbb{R}^n} \sup_{t = \frac{1}{k} \sum_{i=1}^k t_i} \left[ \rho^{\frac{1}{k}} \left( \sum_{m=1}^n \frac{k(k-1)}{2} e^{\frac{2}{k(k-1)} \sum_{i,j=1, i < j}^k (t_i+t_j)_m} \right) \right] e^{\sum_{m=1}^n (t)_m} dt \\ & = \int_{\mathbb{R}^n} \sup_{t = \frac{1}{k} \sum_{i=1}^k t_i} \left[ \rho^{\frac{1}{k}} \left( \frac{k(k-1)}{2} \sum_{m=1}^n e^{\frac{2}{k} \sum_{i=1}^k (t_i)_m} \right) \right] e^{\sum_{m=1}^n (t)_m} dt \\ & = \int_{\mathbb{R}^n} \rho^{\frac{1}{k}} \left( \frac{k(k-1)}{2} \sum_{m=1}^n e^{2(t)_m} \right) e^{\sum_{m=1}^n (t)_m} dt. \end{aligned}$$

Changing variables  $u_m = e^{(t)_m}$  one gets

$$\left( \prod_{i=1}^k \int_{\mathbb{R}_+^n} f_i(x_i) dx_i \right)^{\frac{1}{k}} \leq \int_{\mathbb{R}_+^n} \rho^{\frac{1}{k}} \left( \frac{k(k-1)}{2} \sum_{m=1}^n u_m^2 \right) du = \int_{\mathbb{R}_+^n} \rho^{\frac{1}{k}} \left( \frac{k(k-1)}{2} |u|^2 \right) du.$$

□

The above theorem and the affine isoperimetric inequalities of affine surface area for log-concave functions of [12] lead to multi-functional affine isoperimetric inequalities for log-concave functions.

We first recall that for  $\lambda \in \mathbb{R}$ , the  $\lambda$ -affine surface area of a convex function  $V$  was introduced in [12] as

$$as_\lambda(V) = \int_{\Omega_V} e^{(2\lambda-1)V(x) - \lambda \langle x, \nabla V(x) \rangle} (\det D^2V(x))^\lambda dx, \quad (3.2)$$

where  $\Omega_V = \text{int}(\{x \in \mathbb{R}^n : V(x) < +\infty\})$  is the interior of the convex domain of  $V$  and  $D^2V$  is the Hessian of  $V$ . The gradient of  $V$ , denoted by  $\nabla V$ , exists almost everywhere by Rademacher's theorem (see, e.g., [9]), and a theorem of Alexandrov [2] and Busemann and Feller [11] guarantees the existence of the Hessian, denoted by  $D^2V$ , almost everywhere in  $\Omega_V$ .

In the next theorem we collect several results that were shown in [12].

**Theorem 3.2.** [12] Let  $V : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$  be convex.

(i) For any linear invertible map  $A$  on  $\mathbb{R}^n$ ,  $as_\lambda(V \circ A) = |\det A|^{2\lambda-1} as_\lambda(V)$ .

(ii) For all  $\lambda \in \mathbb{R}$ ,  $as_\lambda(V) = as_{1-\lambda}(V^*)$ .

(iii)  $as_{\frac{1}{2}}(V) \leq \left( \int e^{-V} dx \right)^{\frac{1}{2}} \left( \int e^{-V^*} dx \right)^{\frac{1}{2}}$ .

(iv) Let  $V$  in addition be such that  $\int_{\mathbb{R}^n} x e^{-V(x)} dx = 0$ , and let  $\lambda \in [0, 1]$ . Then

$$as_\lambda(V) \leq (2\pi)^{n\lambda} \left( \int_{\mathbb{R}^n} e^{-V} dx \right)^{1-2\lambda}, \quad (3.3)$$

and equality holds for  $\lambda \neq 0$ , if and only if there exists  $a \in \mathbb{R}$  and a positive definite matrix  $A$  such that  $V(x) = \langle Ax, x \rangle + a$ , for every  $x \in \mathbb{R}^n$ . For  $\lambda = 0$ , equality holds trivially.

**Remark.** Theorem 3.2 (iii) is just a special case for  $\lambda = \frac{1}{2}$  of a more general statement proved in [12].

We then get the following Proposition.

**Proposition 3.3.** Let  $V_i : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$ ,  $1 \leq i \leq k$ , be convex unconditional functions and let  $\rho$  be a positive non-increasing function on  $[0, \infty)$  such that  $\int_{\mathbb{R}} \rho^{\frac{1}{k}}(t^2) dt < \infty$ .

(i) Let  $\lambda \in [0, \frac{1}{2}]$  and suppose the  $V_i$  satisfy

$$\prod_{i=1}^k e^{-V_i(x_i)} \leq \rho \left( \sum_{i,j=1, i < j}^k \langle x_i, x_j \rangle \right), \text{ for every } x_i, x_j \in \mathbb{R}_+^n.$$

Then

$$\prod_{i=1}^k as_\lambda(V_i) \leq (2\pi)^{kn\lambda} \left( \int_{\mathbb{R}^n} \rho^{\frac{1}{k}} \left( \frac{k(k-1)}{2} |u|^2 \right) du \right)^{k(1-2\lambda)}. \quad (3.4)$$

In particular, if  $\rho(t) = e^{-\frac{t}{k-1}}$ , then

$$\prod_{i=1}^k as_\lambda(V_i) \leq \left( as_\lambda \left( \frac{|\cdot|^2}{2} \right) \right)^k. \quad (3.5)$$

(ii) Let  $\lambda \in [\frac{1}{2}, 1]$  and suppose the  $V_i$  are such that

$$\prod_{i=1}^k e^{-V_i^*(x_i)} \leq \rho \left( \sum_{i,j=1, i < j}^k \langle x_i, x_j \rangle \right), \text{ for every } x_i, x_j \in \mathbb{R}_+^n.$$

Then

$$\prod_{i=1}^k as_\lambda(V_i) \leq (2\pi)^{kn(1-\lambda)} \left( \int_{\mathbb{R}^n} \rho^{\frac{1}{k}} \left( \frac{k(k-1)}{2} |u|^2 \right) du \right)^{k(2\lambda-1)}. \quad (3.6)$$

And again, if  $\rho(t) = e^{-\frac{t}{k-1}}$ , then

$$\prod_{i=1}^k as_\lambda(V_i) \leq \left( as_\lambda \left( \frac{|\cdot|^2}{2} \right) \right)^k. \quad (3.7)$$

*Proof.* (i) We get immediately from Theorem 3.1 and inequality (3.3) that for  $\lambda \in [0, \frac{1}{2}]$ ,

$$\prod_{i=1}^k as_\lambda(V_i) \leq (2\pi)^{kn\lambda} \left( \prod_{i=1}^k \int e^{-V_i} \right)^{(1-2\lambda)} \leq (2\pi)^{kn\lambda} \left( \int_{\mathbb{R}^n} \rho^{\frac{1}{k}} \left( \frac{k(k-1)}{2} |u|^2 \right) du \right)^{k(1-2\lambda)}.$$

If  $\rho(t) = e^{-\frac{t}{k-1}}$ , then

$$(2\pi)^{kn\lambda} \left( \int_{\mathbb{R}^n} \rho^{\frac{1}{k}} \left( \frac{k(k-1)}{2} |u|^2 \right) du \right)^{k(1-2\lambda)} = (2\pi)^{\frac{kn}{2}} = \left( as_\lambda \left( \frac{|\cdot|^2}{2} \right) \right)^k,$$

which shows the second part of (i).

(ii) We use Theorem 3.2 (iii) and (iv) and Theorem 3.1 and get that for  $\lambda \in [\frac{1}{2}, 1]$ ,

$$\begin{aligned} \prod_{i=1}^k as_\lambda(V_i) &= \prod_{i=1}^k as_{1-\lambda}(V_i^*) \leq (2\pi)^{n(1-\lambda)} \left( \prod_{i=1}^k \int e^{-V_i^*} \right)^{k(2\lambda-1)} \\ &\leq (2\pi)^{kn(1-\lambda)} \left( \int_{\mathbb{R}^n} \rho^{\frac{1}{k}} \left( \frac{k(k-1)}{2} |u|^2 \right) du \right)^{k(2\lambda-1)}. \end{aligned}$$

The second part for  $\rho(t) = e^{-\frac{t}{k-1}}$  follows.  $\square$

**Remark 3.4.** (i) Please note that for  $\lambda = 0$ , inequalities (3.4) and (3.5) are just the inequalities of Theorem 3.1. For  $\lambda = \frac{1}{2}$ , we do not need that the  $V_i$  are unconditional and the inequalities are just the inequalities of (3.3),

$$\prod_{i=1}^k as_{\frac{1}{2}}(V_i) \leq (2\pi)^{kn\lambda}.$$

See also Section 8 for more on  $as_{\frac{1}{2}}$ .

(ii) For  $\lambda > 1$ , we get an estimate from below with an absolute constant  $c$ , see [12],

$$\prod_{i=1}^k as_\lambda(V_i) \geq c^{kn\lambda} \left( \int_{\mathbb{R}^n} \rho^{\frac{1}{k}} \left( \frac{k(k-1)}{2} |u|^2 \right) du \right)^{k(1-2\lambda)}.$$

## 4 Characterization of the equality cases

In the proof of Theorem 3.1 we have used the Prékopa–Leindler inequality which is a particular case of the more general Brascamp–Lieb inequality (see [10], [7]). To analyze the equality case we need the equality characterizations of the Prékopa–Leindler inequality. We could not find those in the literature, except in the case of two functions, established by Dubuc [16]. We therefore give a proof of the equality characterization.

**Theorem 4.1** (Prékopa–Leindler). *Let  $f_i$ ,  $1 \leq i \leq k$ , and  $h$  be nonnegative integrable real functions on  $\mathbb{R}^n$  such that for all  $x_i$  and for all  $\lambda_i \geq 0$ ,  $1 \leq i \leq k$ , with  $\sum_{i=1}^k \lambda_i = 1$ ,*

$$h \left( \sum_{i=1}^k \lambda_i x_i \right) \geq \prod_{i=1}^k f_i^{\lambda_i}(x_i).$$

Then

$$\prod_{i=1}^k \left( \int_{\mathbb{R}^n} f_i dx_i \right)^{\lambda_i} \leq \int_{\mathbb{R}^n} h dx. \quad (4.1)$$

Equality holds in the Prékopa–Leindler inequality if and only if there exist vectors  $y_1, \dots, y_k$  in  $\mathbb{R}^n$  such that, after modification on a set of measure zero, the functions  $f_i$  satisfy

$$\frac{f_1(x - y_1)}{\int_{\mathbb{R}^n} f_1 dx} = \frac{f_2(x - y_2)}{\int_{\mathbb{R}^n} f_2 dx} = \dots = \frac{f_k(x - y_k)}{\int_{\mathbb{R}^n} f_k dx} = e^{-\psi(x)}, \quad (4.2)$$

where  $\psi$  is a convex function such that  $\int_{\mathbb{R}^n} e^{-\psi(x)} dx = 1$ . In addition, after modification on a set of zero measure, the function  $h$  can be chosen to satisfy

$$h(x) = \sup_{x = \sum_{i=1}^k \lambda_i x_i} \prod_{i=1}^k f_i^{\lambda_i}(x_i) = \prod_{i=1}^k \left( \int_{\mathbb{R}^n} f_i dx_i \right)^{\lambda_i} e^{-\psi(x + \sum_{i=1}^k \lambda_i y_i)}$$

for all  $x$ .

*Proof.* It is clear that equality holds in inequality (4.1), if the functions satisfy the condition (4.2).

The proof of the inequality is well known and can be found in, e.g., [25, 37]. We give a proof of the inequality by induction on the number of functions. This allows to establish the equality characterizations, as for two functions, those were established by Dubuc [16]. We have

$$\sup_{x = \sum_{i=1}^k \lambda_i x_i} \prod_{i=1}^k f_i^{\lambda_i}(x_i) = \sup_{x = \lambda_1 x_1 + (1 - \lambda_1) y} f_1^{\lambda_1}(x_1) g^{1 - \lambda_1}(y),$$

where

$$g(y) = \sup_{y = \frac{1}{1 - \lambda_1} \sum_{i=2}^k \lambda_i x_i} \prod_{i=2}^k f_i^{\frac{\lambda_i}{1 - \lambda_1}}(x_i).$$

Applying the Prékopa–Leindler inequality for two functions gives

$$\int \sup_{x = \sum_{i=1}^k \lambda_i x_i} \prod_{i=1}^k f_i^{\lambda_i}(x_i) \geq \left( \int f_1 dx_1 \right)^{\lambda_1} \left( \int g dy \right)^{1 - \lambda_1}.$$

Applying the induction step, one gets

$$\int g dy \geq \prod_{i=2}^k \left( \int f_i(x_i) dx_i \right)^{\frac{\lambda_i}{1 - \lambda_1}}.$$

This completes the proof of the inequality. The equality characterization follows from the equality characterization for two functions.  $\square$

**Theorem 4.2.** Let  $f_i: \mathbb{R}^n \rightarrow \mathbb{R}_+$ ,  $1 \leq i \leq k$ , be measurable unconditional integrable functions satisfying

$$\prod_{i=1}^k f_i(x_i) \leq \rho \left( \sum_{i,j=1, i < j}^k \langle x_i, x_j \rangle \right) \text{ for every } x_i, x_j \in \mathbb{R}_+^n, \quad (4.3)$$

where  $\rho$  is a positive non-increasing function on  $[0, \infty)$ . Then for  $k > 2$  equality holds in inequality (3.1), i.e.,

$$\prod_{i=1}^k \int_{\mathbb{R}^n} f_i(x_i) dx_i = \left( \int_{\mathbb{R}^n} \rho^{\frac{1}{k}} \left( \frac{k(k-1)}{2} |x|^2 \right) dx \right)^k$$

if and only if there exist positive constants  $c_i$ ,  $1 \leq i \leq k$ , such that  $\prod_{i=1}^k c_i = 1$ , and such that for all  $1 \leq i \leq k$ ,

1.

$$f_i(x_i) = c_i \rho^{\frac{1}{k}} \left( \frac{k(k-1)}{2} |x_i|^2 \right). \quad (4.4)$$

for almost all  $x \in \mathbb{R}^n$ ,

2. The function  $\rho$  satisfies the inequality

$$\prod_{i=1}^k \rho^{\frac{1}{k}} \left( \frac{k(k-1)}{2} |x_i|^2 \right) \leq \rho \left( \sum_{i,j=1, i < j}^k \langle x_i, x_j \rangle \right) \quad (4.5)$$

for all  $x_i, x_j$  in  $\mathbb{R}_+^n$ .

*Proof.* Obviously, if (4.4) and (4.5) hold, then one has equality in (3.1) and the assumption (4.3) is satisfied.

If equality holds in Theorem 3.1, then we have equality everywhere in the proof of Theorem 3.1. We have equality in the Prékopa–Leindler inequality. Note that the Prékopa–Leindler inequality is applied to the functions

$$f_i(e^{t_i}) e^{\sum_{m=1}^n (t_i)_m}.$$

Hence by the above equality characterizations in the Prékopa–Leindler inequality one can modify the functions  $f_i$  on a set of zero measure in such a way, that there exist  $y_1, \dots, y_k$  such that and all  $1 \leq i \leq k$

$$f_i(e^{t_i}) = \left( \int_{\mathbb{R}^n} f_i dx \right) e^{-\sum_{m=1}^n (t_i)_m} e^{-\psi(t_i + y_i)} \quad (4.6)$$

where  $\psi$  is a convex function such that  $\int_{\mathbb{R}^n} e^{-\psi(x)} dx = 1$ . In addition, the following equality must hold for almost all  $t$

$$\begin{aligned} \sup_{t = \frac{1}{k} \sum_{i=1}^k t_i} \prod_{i=1}^k \left( f_i^{\frac{1}{k}}(e^{t_i}) e^{\frac{1}{k} \sum_{m=1}^n (t_i)_m} \right) &= \sup_{t = \frac{1}{k} \sum_{i=1}^k t_i} \left[ \rho^{\frac{1}{k}} \left( \sum_{i,j=1, i < j}^k \sum_{m=1}^n e^{(t_i + t_j)_m} \right) \right] e^{\sum_{m=1}^n (t)_m} \\ &= \rho^{\frac{1}{k}} \left( \frac{k(k-1)}{2} \sum_{m=1}^n e^{2(t)_m} \right) e^{\sum_{m=1}^n (t)_m}. \end{aligned}$$

In particular, changing variables  $x_i = e^{t_i}$  one gets

$$\sup_{(x)_m = \prod_{i=1}^k (x_i)^{\frac{1}{k}}} \left[ \rho^{\frac{1}{k}} \left( \sum_{i,j=1, i < j}^k \langle x_i, x_j \rangle \right) \right] = \rho^{\frac{1}{k}} \left( \frac{k(k-1)}{2} |x|^2 \right). \quad (4.7)$$

Further, substituting (4.6), one gets that for a.e.  $t$

$$\prod_{i=1}^k \left( \int_{\mathbb{R}^n} f_i dx \right)^{\frac{1}{k}} \sup_{t=\frac{1}{k} \sum_{i=1}^k t_i} \prod_{i=1}^k e^{-\frac{1}{k} \psi(t_i+y_i)} = \rho^{\frac{1}{k}} \left( \frac{k(k-1)}{2} \sum_{m=1}^n e^{2(t)_m} \right) e^{\sum_{m=1}^n (t)_m}.$$

Applying convexity of  $\psi$ , one gets that  $\sup_{t=\frac{1}{k} \sum_{i=1}^k t_i} \prod_{i=1}^k e^{-\frac{1}{k} \psi(t_i+y_i)} = e^{-\psi(t+y)}$ , where  $y = \frac{1}{k} \sum_{i=1}^k y_i$ . Finally,

$$\prod_{i=1}^k \left( \int_{\mathbb{R}^n} f_i dx \right)^{\frac{1}{k}} e^{-\psi(t+y)} = \rho^{\frac{1}{k}} \left( \frac{k(k-1)}{2} \sum_{m=1}^n e^{2(t)_m} \right) e^{\sum_{m=1}^n (t)_m}$$

almost everywhere. Note that, in fact, equality holds pointwise, because  $e^{-\psi}$  is a continuous function on  $\{\psi < \infty\}$  and  $\rho$  is non-increasing. Substitute  $t = t_i + y_i - y$  into this identity. One gets

$$\prod_{i=1}^k \left( \int_{\mathbb{R}^n} f_i dx \right)^{\frac{1}{k}} e^{-\psi(t_i+y_i)} = \rho^{\frac{1}{k}} \left( \frac{k(k-1)}{2} \sum_{m=1}^n e^{2(t_i+y_i-y)_m} \right) e^{\sum_{m=1}^n (t_i+y_i-y)_m}.$$

Hence (4.6) implies that for all  $t_i$ ,

$$f_i(e^{t_i}) = \frac{\left( \int_{\mathbb{R}^n} f_i dx \right)}{\prod_{i=1}^k \left( \int_{\mathbb{R}^n} f_i dx \right)^{\frac{1}{k}}} \rho^{\frac{1}{k}} \left( \frac{k(k-1)}{2} \sum_{m=1}^n e^{2(t_i+y_i-y)_m} \right) e^{\sum_{m=1}^n (y_i-y)_m}.$$

We make a change of variables  $x = e^{t_i}$  and get

$$f_i(x) = c_i \rho^{\frac{1}{k}} \left( \frac{k(k-1)}{2} |e^{y_i-y} x|^2 \right),$$

where  $e^{y_i-y} x \in \mathbb{R}^n$  is defined by  $(e^{y_i-y} x)_m = e^{(y_i-y)_m} (x)_m$  and where

$$c_i = \frac{\left( \int_{\mathbb{R}^n} f_i dx \right)}{\prod_{i=1}^k \left( \int_{\mathbb{R}^n} f_i dx \right)^{\frac{1}{k}}} e^{\sum_{m=1}^n (y_i-y)_m}.$$

Note that  $\prod_{i=1}^k c_i = 1$ . Then we have by assumption (4.3) for all  $x_i, x_j \in \mathbb{R}_+^n$ ,

$$\rho \left( \sum_{i,j=1, i < j}^k \langle x_i, x_j \rangle \right) \geq \prod_{i=1}^k f_i(x_i) = \prod_{i=1}^k \rho^{\frac{1}{k}} \left( \frac{k(k-1)}{2} |e^{y_i-y} x_i|^2 \right).$$

However, inequality

$$\rho \left( \sum_{i,j=1, i < j}^k \langle x_i, x_j \rangle \right) \geq \prod_{i=1}^k \rho^{\frac{1}{k}} \left( \frac{k(k-1)}{2} |e^{y_i-y} x_i|^2 \right) \quad (4.8)$$

only holds if  $y_i = y$  for all  $i$ . To see that, note that (4.8) holds in particular for  $x_i = e^{-y_i}$  which leads to

$$\rho \left( \sum_{i,j=1, i < j}^k \langle e^{-y_i}, e^{-y_j} \rangle \right) \geq \prod_{i=1}^k \rho^{\frac{1}{k}} \left( \frac{k(k-1)}{2} |e^{-y}|^2 \right) = \rho \left( \frac{k(k-1)}{2} |e^{-y}|^2 \right)$$

and, as  $\rho$  is decreasing, to

$$\sum_{i,j=1,i<j}^k \langle e^{-y_i}, e^{-y_j} \rangle \leq \frac{k(k-1)}{2} |e^{-y}|^2. \quad (4.9)$$

Note that for  $k > 2$  inequality (4.9) only holds if  $y_i = y$  for all  $i$ . Indeed, by Jensen's inequality,

$$\begin{aligned} \frac{1}{\frac{k(k-1)}{2}} \sum_{i,j=1,i<j}^k \langle e^{-y_i}, e^{-y_j} \rangle &= \sum_{m=1}^n \frac{1}{\frac{k(k-1)}{2}} \sum_{i,j=1,i<j}^k e^{-(y_i+y_j)m} \geq \sum_{m=1}^n e^{-\frac{1}{\frac{k(k-1)}{2}} \sum_{i,j=1,i<j}^k (y_i+y_j)m} \\ &= \sum_{m=1}^n e^{-\frac{2}{k} \sum_{i=1}^k (y_i)m} = |e^{-y}|^2. \end{aligned}$$

Equality in Jensen's inequality shows that thus  $y_i = y$  for all  $i$ .

Consequently, equality in (3.1) is equivalent to

1.  $f_i(x) = c_i \rho^{\frac{1}{k}} \left( \frac{k(k-1)}{2} |x|^2 \right)$ , almost everywhere and
2.  $\prod_{i=1}^k \rho^{\frac{1}{k}} \left( \frac{k(k-1)}{2} |x_i|^2 \right) \leq \rho \left( \sum_{i,j=1,i<j}^k \langle x_i, x_j \rangle \right)$ . □

Equation (4.4) says in particular that if equality holds, then all  $f_i$  are equal modulo normalization.

Under some natural assumptions on the function  $\rho$ , one can show that inequality (4.5) always holds.

**Remark 4.3.** Let  $\rho(t) = e^{-W(t)}$ , where  $W$  is convex and increasing. Then (4.5) holds.

*Proof.* If  $\rho(t) = e^{-W(t)}$ , inequality (4.5) is equivalent to

$$\frac{1}{k} \sum_{i=1}^k W \left( \frac{k(k-1)}{2} |x_i|^2 \right) \geq W \left( \sum_{i,j=1,i<j}^k \langle x_i, x_j \rangle \right).$$

By convexity of  $W$ ,  $\frac{1}{k} \sum_{i=1}^k W \left( \frac{k(k-1)}{2} |x_i|^2 \right) \geq W \left( \frac{(k-1)}{2} \sum_{i=1}^k |x_i|^2 \right)$ . Therefore it is enough to have that

$$W \left( \frac{(k-1)}{2} \sum_{i=1}^k |x_i|^2 \right) \geq W \left( \sum_{i,j=1,i<j}^k \langle x_i, x_j \rangle \right)$$

or, as  $W$  is increasing,

$$\frac{(k-1)}{2} \sum_{i=1}^k |x_i|^2 \geq \sum_{i,j=1,i<j}^k \langle x_i, x_j \rangle,$$

which holds, because

$$\sum_{i,j=1,i<j}^k \langle x_i, x_j \rangle \leq \frac{1}{2} \sum_{i,j=1,i<j}^k (|x_i|^2 + |x_j|^2) = \frac{(k-1)}{2} \sum_{i=1}^k |x_i|^2.$$

□

Theorems 3.1 and 4.2 and Remark 4.3 immediately yield the following corollary.



**Corollary 4.4.** Let  $\rho(t) = e^{-\frac{t}{k-1}}$ . Let  $f_i: \mathbb{R}^n \rightarrow \mathbb{R}_+$ ,  $1 \leq i \leq k$ , be measurable unconditional integrable functions satisfying

$$\prod_{i=1}^k f_i(x_i) \leq e^{-\frac{1}{k-1} (\sum_{i,j=1, i < j}^k \langle x_i, x_j \rangle)} \text{ for every } x_i, x_j \in \mathbb{R}_+^n.$$

Then

$$\prod_{i=1}^k \int_{\mathbb{R}^n} f_i(x_i) dx_i \leq \left( \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{k}} dx \right)^k = (2\pi)^{k \frac{n}{2}}$$

and for  $k > 2$  equality holds if and only if there exist positive constants  $c_i$ ,  $1 \leq i \leq k$ , such that  $\prod_{i=1}^k c_i = 1$  and such that for almost all  $x \in \mathbb{R}^n$ , for all  $1 \leq i \leq k$ ,

$$f_i(x_i) = c_i e^{-\frac{|x_i|^2}{2}}.$$

The next proposition addresses the equality characterizations of Proposition 3.3.

**Proposition 4.5.** Let  $V_i: \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$ ,  $1 \leq i \leq k$ , be convex unconditional functions and let  $\rho$  be a positive non-increasing function on  $[0, \infty)$  such that  $\int_{\mathbb{R}} \rho^{\frac{1}{k}}(t^2) dt < \infty$ .

(i) Let  $\lambda \in [0, \frac{1}{2}]$  and suppose the  $V_i$  satisfy

$$\prod_{i=1}^k e^{-V_i(x_i)} \leq \rho \left( \sum_{i,j=1, i < j}^k \langle x_i, x_j \rangle \right), \text{ for every } x_i, x_j \in \mathbb{R}^n \text{ satisfying } \langle x_i, x_j \rangle \geq 0. \quad (4.10)$$

Then equality holds in inequality (3.4), i.e.,

$$\prod_{i=1}^k as_{\lambda}(V_i) = (2\pi)^{kn\lambda} \left( \int_{\mathbb{R}^n} \rho^{\frac{1}{k}} \left( \frac{k(k-1)}{2} |u|^2 \right) du \right)^{k(1-2\lambda)},$$

if and only if for all  $i$ , there are  $a_i \in \mathbb{R}$  such that for almost all  $x \in \mathbb{R}^n$ ,

$$V_i(x_i) = c \frac{|x_i|^2}{2} + a_i, \quad (4.11)$$

$$\rho(t) = e^{-\frac{c}{k-1} t - \sum_{i=1}^k a_i}. \quad (4.12)$$

for some  $c > 0$  and numbers  $a_i$ .

(ii) Let  $\lambda \in [\frac{1}{2}, 1]$  and suppose the  $V_i$  are such that

$$\prod_{i=1}^k e^{-V_i^*(x_i)} \leq \rho \left( \sum_{i,j=1, i < j}^k \langle x_i, x_j \rangle \right), \text{ for every } x_i, x_j \in \mathbb{R}^n \text{ satisfying } \langle x_i, x_j \rangle \geq 0.$$

Then the equality characterizations in inequality (3.6) respectively (3.7) are as in (i) with  $V_i^*$  instead of  $V_i$ .

*Proof.* (i) It is clear that if (4.11) and (4.12) hold, then there is equality in (3.4) and the assumption (4.10) holds. On the other hand, by Theorem 3.2, equality holds in the first

inequality of the proof of Proposition 3.3, if and only if there exist  $a_i \in \mathbb{R}$  and positive definite matrices  $A_i$  such that for every  $x \in \mathbb{R}^n$ , for all  $1 \leq i \leq k$ ,

$$V_i(x) = \langle A_i x, x \rangle + a_i. \quad (4.13)$$

By Theorem 4.2, equality holds in the second inequality of the proof of Proposition 3.3, if and only if there exist constants  $c_i$ ,  $1 \leq i \leq k$ , such that  $\prod_{i=1}^k c_i = 1$  and such that for all  $1 \leq i \leq k$ ,

$$e^{-V_i(x)} = c_i \rho^{\frac{1}{k}} \left( \frac{k(k-1)}{2} |x|^2 \right), \quad (4.14)$$

almost everywhere, and the function  $\rho$  satisfies the inequality

$$\prod_{i=1}^k \rho^{\frac{1}{k}} \left( \frac{k(k-1)}{2} |x_i|^2 \right) \leq \rho \left( \sum_{i,j=1, i < j}^k \langle x_i, x_j \rangle \right).$$

It follows from (4.13) and (4.14) that for almost all  $x$ , for all  $i$

$$e^{-\langle A_i x, x \rangle} e^{-a_i} = c_i \rho^{\frac{1}{k}} \left( \frac{k(k-1)}{2} |x|^2 \right).$$

In particular, for  $x = 0$ , we get that for all  $i$ ,  $\rho^{\frac{1}{k}}(0) = \frac{e^{-a_i}}{c_i}$  and thus for all  $i$

$$e^{-\langle A_i x, x \rangle} = \rho^{-\frac{1}{k}}(0) \rho^{\frac{1}{k}} \left( \frac{k(k-1)}{2} |x|^2 \right).$$

This clearly means that  $A_i = \frac{c}{2} Id$  for some  $c > 0$  and  $\rho(t) = C e^{-\frac{c}{k-1}t}$  and we easily complete the proof.

(ii) The proof of (ii) is done in the same way. □

## 5 The Blaschke–Santaló inequality and the affine isoperimetric inequality for many sets

The classical Blaschke–Santaló inequality for symmetric sets can be stated in the following way,

$$\int_{S^{n-1}} f^n dx \int_{S^{n-1}} g^n dy \leq n^2 (\text{vol}_n(B_2^n))^2 = (\text{vol}_{n-1}(S^{n-1}))^2,$$

where  $f, g$  are positive symmetric functions on  $S^{n-1}$  satisfying

$$f(x)g(y) \leq \frac{1}{\langle x, y \rangle_+},$$

and where for  $a \in \mathbb{R}$ ,  $a_+ = \max\{a, 0\}$ . Note that if  $x$  and  $y$  are orthogonal, then the right hand side of the inequality is infinite. This happens only for set of measure zero. The latter inequality is satisfied, in particular, if

$$f(x) = r_K(x), \quad g(y) = \frac{1}{h_K(y)} = r_{K^\circ}(y),$$

where  $r_K(x) = \max\{\lambda \geq 0 : \lambda x \in K\}$  is the radial function of the convex body  $K$ ,  $h_K(y) = \sup\{\langle x, y \rangle : x \in K\}$  is the support function of  $K$  and where for a 0-symmetric convex body  $K$  with non-empty interior,

$$K^\circ = \{y \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \forall x \in K\}$$

is the polar body of  $K$ . We can then write the above as follows,

$$\text{vol}_n(K_1) \text{vol}_n(K_2) \leq (\text{vol}_n(B_2^n))^2,$$

provided

$$\langle x, y \rangle \leq 1, \forall x \in K_1, \forall y \in K_2.$$

We now prove a Blaschke–Santaló inequality for multiple sets. We recall that a subset  $K$  in  $\mathbb{R}^n$  is unconditional if its characteristic function  $\mathbb{1}_K$  is unconditional.

**Theorem 5.1.** *Let  $K_i$ ,  $1 \leq i \leq k$ , be unconditional convex bodies in  $\mathbb{R}^n$  such that*

$$\prod_{i=1}^k e^{-\frac{1}{2}\|x_i\|_{K_i}^2} \leq \rho \left( \sum_{i,j=1, i < j}^k \langle x_i, x_j \rangle \right) \text{ for every } x_i, x_j \in \mathbb{R}_+^n,$$

where  $\rho$  is a positive non-increasing function on  $[0, \infty)$  such that  $\int_{\mathbb{R}} \rho^{\frac{1}{k}}(t^2) dt < \infty$ . Then

$$\prod_{i=1}^k \text{vol}_n(K_i) \leq \left( \frac{\text{vol}_n(B_2^n)}{(2\pi)^{\frac{n}{2}}} \right)^k \left( \int_{\mathbb{R}^n} \rho^{\frac{1}{k}} \left( \frac{k(k-1)}{2} |x|^2 \right) dx \right)^k.$$

For  $k > 2$ , equality holds if and only if  $K_i = r B_2^n$  and  $\rho(t) = e^{-\frac{t}{(k-1)r^2}}$  for some  $r > 0$ .

In particular, if  $\rho(t) = e^{-\frac{t}{k-1}}$ , then, if  $\sum_{i=1, i < j}^k \langle x_i, x_j \rangle \leq \frac{k-1}{2} \sum_{i=1}^k \|x_i\|_{K_i}^2$ , we have that

$$\prod_{i=1}^k \text{vol}_n(K_i) \leq (\text{vol}_n(B_2^n))^k$$

and for  $k > 2$  equality holds if and only if  $K_i = B_2^n$  for all  $1 \leq i \leq k$ .

*Proof.* As for a convex body with 0 in its interior  $\text{vol}_n(K) = \frac{\text{vol}_n(B_2^n)}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{1}{2}\|x\|_K^2} dx$ , we get from Theorem 3.1 that

$$\prod_{i=1}^k \text{vol}_n(K_i) = \left( \frac{\text{vol}_n(B_2^n)}{(2\pi)^{\frac{n}{2}}} \right)^k \prod_{i=1}^k \int_{\mathbb{R}^n} e^{-\frac{1}{2}\|x_i\|_{K_i}^2} dx \leq \left( \frac{\text{vol}_n(B_2^n)}{(2\pi)^{\frac{n}{2}}} \right)^k \left( \int_{\mathbb{R}^n} \rho^{\frac{1}{k}} \left( \frac{k(k-1)}{2} |x|^2 \right) dx \right)^k$$

provided that

$$\prod_{i=1}^k e^{-\frac{1}{2}\|x_i\|_{K_i}^2} \leq \sum_{i,j=1, i < j}^k \rho(\langle x_i, x_j \rangle).$$

The equality characterizations follow from Theorem 4.2 and Corollary 4.4. Indeed, by Theorem 4.2, equality holds for  $k > 2$  if and only if there exist constants  $c_i$ ,  $1 \leq i \leq k$ , such that  $\prod_{i=1}^k c_i = 1$ , and such that

1.  $e^{-\frac{1}{2}\|x\|_{K_i}^2} = c_i \rho^{\frac{1}{k}} \left( \frac{k(k-1)}{2} |x|^2 \right)$  and

2. The function  $\rho$  satisfies

$$\prod_{i=1}^k \rho^{\frac{1}{k}} \left( \frac{k(k-1)}{2} |x_i|^2 \right) \leq \rho \left( \sum_{i,j=1, i < j}^k \langle x_i, x_j \rangle \right).$$

From the first identity we get for  $x = 0$  that  $c_i = \frac{1}{\rho^{\frac{1}{k}}(0)}$  for all  $i$ . As  $\prod_{i=1}^k c_i = 1$ , this implies that  $\rho(0) = 1$  and hence  $c_i = 1$  for all  $i$ . In particular, this implies that almost everywhere on  $\mathbb{R}^n$ , for all  $i, j$ ,  $\|x\|_{K_i} = \|x\|_{K_j} = \|x\|_K$  and thus  $K_i = K$  for all  $i$ . From the relation  $e^{-\frac{1}{2}\|x\|_{K_i}^2} = \rho^{\frac{1}{k}} \left( \frac{k(k-1)}{2} |x|^2 \right)$  we get that  $K_i = K = rB_2^n$ , hence  $e^{-\frac{t}{2r^2}} = \rho^{\frac{1}{k}} \left( \frac{k(k-1)}{2} t \right)$ , equivalently  $e^{-\frac{s}{r^2(k-1)}} = \rho(s)$ . The proof is complete.  $\square$

**Remark 5.2.** Note that for  $k = 2$  the above equality characterization clearly fails: the equality  $\text{vol}_n(K)\text{vol}_n(K^\circ) = (\text{vol}_n(B_2^n))^2$  holds if and only if  $K$  is an ellipsoid. This follows obviously from the linear invariance of the Blaschke–Santaló functional for two sets.

The Blaschke–Santaló inequality for convex bodies is closely related to affine isoperimetric inequalities which involve the  $L_p$ -affine surface area. For a convex body  $K$  with centroid at 0, and for  $-\infty \leq p \leq \infty$ ,  $p \neq -n$  it is defined as (see, e.g., [32, 38]),

$$as_p(K) = \int_{\partial K} \frac{\kappa_K(x)^{\frac{p}{n+p}}}{\langle x, N_K(x) \rangle^{\frac{n(p-1)}{n+p}}} d\mu_K(x), \quad (5.1)$$

where  $\mu_K$  the Hausdorff measure on  $\partial K$ , the boundary of  $K$ ,  $N_K(x)$  is the outer unit normal at  $x \in \partial K$  and  $\kappa_K(x)$  is the generalized Gauss curvature at  $x \in \partial K$ . Note that  $as_0(K) = n \text{vol}_n(K)$ , and if  $K$  is  $C_+^2$ , then  $as_{\pm\infty}(K) = n \text{vol}_n(K^\circ)$ .

The  $L_p$ -affine isoperimetric inequalities state that for  $0 \leq p \leq \infty$ ,

$$\frac{as_p(K)}{as_p(B_2^n)} \leq \left( \frac{\text{vol}_n(K)}{\text{vol}_n(B_2^n)} \right)^{\frac{n-p}{n+p}} \quad (5.2)$$

and for  $-n < p \leq 0$ ,

$$\frac{as_p(K)}{as_p(B_2^n)} \geq \left( \frac{\text{vol}_n(K)}{\text{vol}_n(B_2^n)} \right)^{\frac{n-p}{n+p}}.$$

Equality holds trivially if  $p = 0$ . In both cases equality holds for  $p \neq 0$  if and only if  $K$  is an ellipsoid. If  $-\infty \leq p < -n$  and  $K$  is  $C_+^2$ , then

$$c^{\frac{np}{n+p}} \left( \frac{\text{vol}_n(K)}{\text{vol}_n(B_2^n)} \right)^{\frac{n-p}{n+p}} \leq \frac{as_p(K)}{as_p(B_2^n)}, \quad (5.3)$$

with a constant  $c > 0$  not depending on the dimension. These inequalities were proved by Lutwak [32] for  $p > 1$  and for all other  $p$  by Werner and Ye [40]. The case  $p = 1$  is the classical case.

Theorem 5.1 leads to a multi-set “affine” isoperimetric inequality.

**Proposition 5.3.** *Let  $K_i$ ,  $1 \leq i \leq k$ , be unconditional convex bodies in  $\mathbb{R}^n$  such that*

$$\prod_{i=1}^k e^{-\frac{1}{2}\|x_i\|_{K_i}^2} \leq \rho \left( \sum_{i=1, i < j}^k \langle x_i, x_j \rangle \right) \text{ for every } x_i, x_j \in \mathbb{R}_+^n,$$

where  $\rho$  is a positive non-increasing function on  $[0, \infty)$  such that  $\int_{\mathbb{R}} \rho^{\frac{1}{k}}(t^2) dt < \infty$ . Then we have for  $0 \leq p < n$

$$\prod_{i=1}^k \frac{as_p(K_i)}{as_p(B_2^n)} \leq \left( \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \rho^{\frac{1}{k}} \left( \frac{k(k-1)}{2} |x|^2 \right) dx \right)^{k \frac{n-p}{n+p}}.$$

For  $k > 2$ , equality holds if and only if

1.  $K_i = r B_2^n$  for all  $i$ , where  $r > 0$  is a constant,
2.  $\rho(t) = e^{-\frac{t}{(k-1)r^2}}$ .

In particular, if  $\rho(t) = e^{-\frac{t}{k-1}}$  and if  $\sum_{i=1, i < j}^k \langle x_i, x_j \rangle \leq \frac{k-1}{2} \sum_{i=1}^k \|x_i\|_{K_i}^2$ , then we have that

$$\prod_{i=1}^k as_p(K_i) \leq (as_p(B_2^n))^k \tag{5.4}$$

and equality holds if and only if  $K_i = B_2^n$  for all  $1 \leq i \leq k$ .

If  $p = n$ , then

$$\prod_{i=1}^k as_p(K_i) \leq (as_p(B_2^n))^k \tag{5.5}$$

and equality holds if and only if  $K_i$  is an ellipsoid for all  $1 \leq i \leq k$ .

*Proof.* Let  $0 \leq p \leq n$ . By the affine isoperimetric inequality and Theorem 5.1 we get

$$\begin{aligned} \prod_{i=1}^k as_p(K_i) &\leq (as_p(B_2^n))^k \prod_{i=1}^k \left( \frac{\text{vol}_n(K_i)}{\text{vol}_n(B_2^n)} \right)^{\frac{n-p}{n+p}} \\ &\leq (as_p(B_2^n))^k \left( \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \rho^{\frac{1}{k}} \left( \frac{k(k-1)}{2} |x|^2 \right) dx \right)^{k \frac{n-p}{n+p}}. \end{aligned} \tag{5.6}$$

The first inequality shows that for  $p = n$ ,

$$\prod_{i=1}^k as_n(K_i) \leq (as_n(B_2^n))^k.$$

If  $\rho(t) = e^{-\frac{t}{k-1}}$ , then we have for all  $0 \leq p \leq n$ ,

$$\prod_{i=1}^k as_p(K_i) \leq (as_p(B_2^n))^k \prod_{i=1}^k \left( \frac{\text{vol}_n(K_i)}{\text{vol}_n(B_2^n)} \right)^{\frac{n-p}{n+p}} \leq (as_p(B_2^n))^k \tag{5.7}$$

The equality characterizations follow from Theorem 5.1 and the equality characterizations of the above affine isoperimetric inequalities.

Indeed, by the affine isoperimetric inequality, equality holds in the first inequality of (5.6) if and only if  $K_i = T_i B_2^n$ , where  $T_i$  is a linear invertible map. By Theorem 5.1, equality holds in the second inequality of (5.6) if and only if  $K_i = r B_2^n$  for all  $i$ , where  $r > 0$  is a constant, and  $\rho(t) = e^{-\frac{t}{(k-1)r^2}}$ .  $\square$

**Remark 5.4.** (i) For  $p = n$ , the inequality is just the affine isoperimetric inequality (5.2). As  $as_0(K) = n \text{vol}_n(K)$ , the inequalities of the theorem for  $p = 0$  are just the inequalities of Theorem 5.1.

(ii) The corresponding inequalities for  $-\infty \leq p < -n$  also hold, using (5.3).

A further multiple set version of the Blaschke–Santaló inequality is given in the next proposition.

**Proposition 5.5.** Let  $K_i$ ,  $1 \leq i \leq k$ , be unconditional convex bodies in  $\mathbb{R}^n$  with non-empty interior and radial functions  $r_i = r_{K_i}$ . Assume that for all  $x_i = ((x_i)_1, \dots, (x_i)_n) \in S^{n-1}$ ,

$$\prod_{i=1}^k r_i(x_i) \leq \frac{1}{\left(\sum_{j=1}^n (|(x_1)_j|^{\frac{1}{k}} \cdots |(x_k)_j|^{\frac{1}{k}})^2\right)^{\frac{k}{2}}}. \quad (5.8)$$

Then

$$\prod_{i=1}^k \text{vol}_n(K_i) \leq (\text{vol}_n(B_2^n))^k.$$

*Proof.* Let  $m \in \mathbb{R}$ ,  $1 \leq m < n$  and put  $x_i = e^{t_i}$ . Set  $w = \frac{1}{k} \sum_{i=1}^k t_i$ . Then

$$\prod_{i=1}^k r_i^m(e^{t_i}) \mathbf{1}_{\{|e^{t_i}| \leq 1\}} e^{\sum_{i,j} (t_i)_j} \leq \frac{\mathbf{1}_{\{|e^w| \leq 1\}} e^{\sum_{i,j} (t_i)_j}}{\left(\sum_{j=1}^n e^{2w_j}\right)^{\frac{km}{2}}} = \frac{\mathbf{1}_{\{|e^w| \leq 1\}} e^{k \sum_{j=1}^n (w)_j}}{\left(\sum_{j=1}^n e^{2w_j}\right)^{\frac{km}{2}}}. \quad (5.9)$$

We now apply again the change of variables  $x_i = e^{t_i}$ ,  $1 \leq i \leq k$ , the Prékopa–Leindler inequality and (5.9),

$$\begin{aligned} \left(\prod_{i=1}^k \int_{B_2^n \cap \mathbb{R}_+^n} r_i^m dx_i\right)^{\frac{1}{k}} &= \left(\prod_{i=1}^k \int_{\mathbb{R}^n} r_i^m(e^{t_i}) \mathbf{1}_{\{|e^{t_i}| \leq 1\}} e^{\sum_j (t_i)_j} dt_i\right)^{\frac{1}{k}} \\ &\leq \int_{\mathbb{R}^n} \sup_{w = \frac{1}{k} \sum_{i=1}^k t_i} \left[\prod_{i=1}^k \left(r_i^{\frac{m}{k}}(e^{t_i}) \mathbf{1}_{\{|e^{t_i}| \leq 1\}} e^{\frac{1}{k} \sum_{i,j} (t_i)_j}\right)\right] dw \\ &\leq \int_{\mathbb{R}^n} \frac{\mathbf{1}_{\{|e^w| \leq 1\}} e^{\sum_{j=1}^n (w)_j}}{\left(\sum_{j=1}^n e^{2w_j}\right)^{\frac{m}{2}}} dw = \int_{B_2^n \cap \mathbb{R}_+^n} \frac{dx}{|x|^m}. \end{aligned}$$

Hence by symmetry

$$\left(\prod_{i=1}^k \int_{B_2^n} r_i^m dx_i\right)^{\frac{1}{k}} \leq \int_{B_2^n} \frac{dx}{|x|^m}.$$

Next we observe that every radial function  $r_i$  satisfies

$$r_i(x_i) = r_i\left(\frac{x_i}{|x_i|}\right) \frac{1}{|x_i|}.$$

For every  $1 \leq m < n$ ,  $m \in \mathbb{R}$ , we introduce the finite probability measure  $d\mu_m = \frac{\mathbf{1}_{B_2^n}(u)}{\int_{B_2^n} \frac{du}{|u|^m}} \frac{du}{|u|^m}$ . The inequality above can then be rewritten as follows,

$$\prod_{i=1}^k \int_{B_2^n} r_i^m\left(\frac{x_i}{|x_i|}\right) d\mu_m \leq 1.$$

Since  $\mu_m$  is rotational invariant, the above inequality can be rewritten as

$$\prod_{i=1}^k \int_{S^{n-1}} r_i^m(\theta) d\sigma(\theta) \leq \sigma(S^{n-1})^k, \quad (5.10)$$

where  $\sigma$  is the  $(n-1)$ -dimensional Hausdorff measure. Passing to the limit  $m \rightarrow n$  and applying the Fatou's Lemma one gets that (5.10) holds for  $m = n$ . On the other hand, for  $m = n$  one has for all  $i$

$$\int_{S^{n-1}} r_i^n(\theta) d\sigma(\theta) = \sigma(S^{n-1}) \frac{\text{vol}_n(K_i)}{\text{vol}_n(B_2^n)}. \quad (5.11)$$

From this we derive the desired estimate.  $\square$

## 6 Prékopa–Leindler and displacement convexity inequalities: refinement of the transportational argument

In this section we recall the transportational arguments of F. Barthe [7] in his proof of the reverse Brascamp–Lieb inequality. We show that the use of barycenters gives certain refinements of the Prékopa–Leindler inequality.

**In this section we do not assume that the functions  $f_i$  are even.**

Let  $f_i$ ,  $1 \leq i \leq k$ , be nonnegative integrable functions and  $\lambda_i \in [0, 1]$  be numbers such that  $\sum_{i=1}^k \lambda_i = 1$ , and let  $d\mu = p(x)dx$  be a probability measure. For every  $i$ ,  $\nabla\Phi_i$  is the optimal transportation mapping that pushes forward  $\mu$  onto  $\mu_i = f_i dx$ .

In what follows we apply the change of variables formula for the optimal transportation mapping. In that form it was established by R. McCann (see [39], Theorem 4.8),

$$p(x) = \frac{f_i(\nabla\Phi_i)}{\int f_i dx_i} \det D_a^2\Phi_i(x),$$

where  $D_a^2\Phi_i$  is the absolutely continuous part of the distributional Hessian  $D^2\Phi_i$  of  $\Phi_i$ . In particular, it is a nonnegative matrix-valued measure. This formula holds almost everywhere with respect to Lebesgue measure. We will also apply below the following results

- The arithmetic-geometric mean inequality

$$\prod_{i=1}^k (\det A_i)^{\lambda_i} \leq \det \left( \sum_{i=1}^k \lambda_i A_i \right),$$

where the  $A_i$  are symmetric nonnegative matrices,  $\lambda_i \geq 0$ ,  $\sum_{i=1}^k \lambda_i = 1$ .

- The inequality between the distributional Hessian and its absolutely continuous part

$$0 \leq D_a^2\Phi_i \leq D^2\Phi_i.$$

First, we get by the arithmetic-geometric mean inequality

$$\begin{aligned} p(x) &= \prod_{i=1}^k \left( \frac{f_i(\nabla\Phi_i)(x)}{\int f_i dx_i} \det D_a^2\Phi_i(x) \right)^{\lambda_i} \leq \prod_{i=1}^k \left( \frac{f_i(\nabla\Phi_i)(x)}{\int f_i dx_i} \right)^{\lambda_i} \det \left( \sum_{i=1}^k \lambda_i D_a^2\Phi_i(x) \right) \\ &\leq \sup_{\{y_i: \sum_i \lambda_i y_i = \sum_i \lambda_i \nabla\Phi_i(x)\}} \prod_{i=1}^k \left( \frac{f_i(y_i)}{\int f_i dx_i} \right)^{\lambda_i} \det \left( \sum_{i=1}^k \lambda_i D_a^2\Phi_i(x) \right). \end{aligned} \quad (6.1)$$

In the proof of Barthe, one fixes an *arbitrary* measure  $\mu$  and integrates inequality (6.1). By the change of variables  $y = \sum_i \lambda_i \nabla \Phi_i(x)$ , we get the Prékopa–Leindler inequality

$$\prod_{i=1}^k \left( \int f_i dx_i \right)^{\lambda_i} \leq \int \sup_{\{y_i: \sum_i \lambda_i y_i = y\}} \prod_{i=1}^k f_i^{\lambda_i}(y_i) dy.$$

If instead of an arbitrary measure  $\mu$ , we apply this result to the barycenter of the  $\mu'_i$ s, we obtain the following pointwise refinement of the Prékopa–Leindler inequality.

**Theorem 6.1. (Pointwise Prékopa–Leindler inequality)** *Let  $\mu$  be the barycenter of the  $\mu_i$  with weights  $\lambda_i$ . Then it has a density  $p$  satisfying*

$$\prod_{i=1}^k \left( \int f_i dx_i \right)^{\lambda_i} p(x) \leq \sup_{x = \sum_{i=1}^k \lambda_i y_i} \prod_{i=1}^k f_i^{\lambda_i}(y_i), \text{ for } p - \text{a.e.} \quad (6.2)$$

*Proof.* By the arithmetic-geometric mean inequality one has

$$\prod_{i=1}^k \left( \det D_a^2 \Phi_i(x) \right)^{\lambda_i} \leq \det \left( \sum_{i=1}^k \lambda_i D_a^2 \Phi_i(x) \right).$$

Since  $\sum_{i=1}^k \lambda_i \Phi_i(x) = \frac{|x|^2}{2}$  for  $p(x) dx$ -almost all  $x$ , (see Theorem 2.4, 3), one gets

$$\sum_{i=1}^k \lambda_i D_a^2 \Phi_i(x) \leq D^2 \left( \sum_{i=1}^k \lambda_i \Phi_i(x) \right) = I$$

$p(x) dx$ -a.e. Using this inequality and inequality (6.1) one gets the result.  $\square$

**Remark 6.2.** *Following the proof, one can easily get the equality characterization for the Prékopa–Leindler inequality. Indeed, we have equality in the arithmetic-geometric mean inequality if and only if  $D_a^2 \Phi_i(x)$  are all equal for almost all  $x$ . Next, from the relation  $\sum_{i=1}^k \lambda_i \Phi_i(x) = \frac{|x|^2}{2}$  one can easily get that every  $\Phi_i$  has the form  $\Phi_i(x) = \frac{|x|^2}{2} + \langle x, a_i \rangle + b_i$ . This easily implies that the  $f_i$  differ by shifts. The rest of the proof is standard.*

Let us rewrite (6.1) in terms of the standard Gaussian reference measure  $d\gamma = \frac{e^{-\frac{|x|^2}{2}}}{(2\pi)^{\frac{n}{2}}} dx$ .

**Corollary 6.3.** *Let  $f_i dx_i = \rho_i \cdot d\gamma$  be probability measures and let  $d\mu = \rho \cdot d\gamma$  be their barycenter. Then  $\mu$ -a.e.*

$$\rho(x) e^{\frac{1}{2} \sum_{i=1}^k \lambda_i |\nabla \Phi_i(x) - x|^2} \leq \prod_{i=1}^k \rho_i^{\lambda_i} (\nabla \Phi_i). \quad (6.3)$$

*Proof.* Applying the first inequality of (6.1) to  $f_i = \rho_i \frac{e^{-\frac{|x|^2}{2}}}{(2\pi)^{\frac{n}{2}}}$  and  $p = \rho \frac{e^{-\frac{|x|^2}{2}}}{(2\pi)^{\frac{n}{2}}}$ , we get

$$\rho(x) e^{-\frac{|x|^2}{2}} \leq \prod_{i=1}^k \rho_i^{\lambda_i} (\nabla \Phi_i) e^{-\lambda_i \frac{|\nabla \Phi_i|^2}{2}}.$$



Also using Theorem 2.4, 3., we finally observe that

$$\begin{aligned} \sum_{i=1}^k \lambda_i \left( \frac{|\nabla \Phi_i(x)|^2}{2} - \frac{|x|^2}{2} \right) &= \sum_{i=1}^k \lambda_i \left( \frac{|\nabla \Phi_i(x)|^2}{2} - \frac{|x|^2}{2} - \langle \nabla \Phi_i(x) - x, x \rangle \right) \\ &= \frac{1}{2} \sum_{i=1}^k \lambda_i |\nabla \Phi_i(x) - x|^2. \end{aligned}$$

□

Integrating pointwise inequality (6.2) we get the Prékopa–Leindler inequality. Taking logarithm of (6.3) and integrating we get the displacement convexity property of the Gaussian entropy,

$$\text{Ent}_\gamma(\mu) + \frac{1}{2} \sum_{i=1}^k \lambda_i W_2^2(\mu, \mu_i) \leq \sum_{i=1}^k \lambda_i \text{Ent}_\gamma(\mu_i). \quad (6.4)$$

This result was proved in [1].

Mimicking the arguments that were used in the proof of (6.1) leads to the following result.

**Theorem 6.4.** *Let  $f_i$ ,  $1 \leq i \leq k$ , be integrable functions satisfying*

$$\prod_{i=1}^k f_i^{\lambda_i}(x_i) \leq g \left( \sum_{i=1}^k \lambda_i x_i \right), \quad (6.5)$$

where  $\lambda_i \in [0, 1]$ ,  $\sum_{i=1}^k \lambda_i = 1$  and  $g$  is a nonnegative function. Then for  $pdx$ -almost all  $x$ ,

$$\prod_{i=1}^k \left( \int f_i dx_i \right)^{\lambda_i} \rho(x) \leq g(x), \quad (6.6)$$

where  $\rho(x)dx$  is the barycenter of the measures  $\frac{f_i}{\int f_i dx_i} dx_i$  with weights  $\lambda_i$ .

*Proof.* Applying inequality (6.1) and the relation  $\sum_{i=1}^k \lambda_i \nabla \Phi_i(x) = x$  one immediately gets

$$\prod_{i=1}^k \left( \int f_i(x_i) dx_i \right)^{\lambda_i} \rho(x) \leq \sup_{\{y_i: \sum_i \lambda_i y_i = x\}} \prod_{i=1}^k f_i^{\lambda_i}(y_i) \det \left( \sum_{i=1}^k \lambda_i D^2 \Phi_i(x) \right) \leq g(x).$$

□

**Remark 6.5.** *Assuming (6.5) and integrating (6.6) one gets the inequality*

$$\prod_{i=1}^k \left( \int f_i dx_i \right)^{\lambda_i} \leq \int g(x) dx, \quad (6.7)$$

which can be considered as a weak form of the Blaschke–Santaló functional inequality, because it is equivalent to (7.16) (see the explanations in Remark 7.4), which is a weaker version of the displacement convexity property (6.4). Inequality (6.7) follows, of course, directly from the Prékopa–Leindler inequality.

In particular, assuming that the functions  $V_i$  satisfy

$$\sum_{i=1}^k \lambda_i V_i(x_i) \geq \frac{1}{2} \left| \sum_{i=1}^k \lambda_i x_i \right|^2,$$

one gets

$$\left( \prod_{i=1}^k \int e^{-V_i(x_i)} dx_i \right)^{\lambda_i} \rho(x) \leq e^{-\frac{|x|^2}{2}}.$$

Rewriting this inequality with respect to the Gaussian reference measure  $\gamma$ , one gets the following equivalent formulation.

**Corollary 6.6.** *Assume that the measurable functions  $F_i$  satisfy*

$$\sum_{i=1}^k \lambda_i F_i(x_i) \leq \frac{1}{2} \left[ \sum_{i=1}^k \lambda_i |x_i|^2 - \left| \sum_{j=1}^k \lambda_j x_j \right|^2 \right].$$

Then

$$\left( \prod_{i=1}^k \int e^{F_i} d\gamma \right)^{\lambda_i} p(x) \leq 1,$$

where  $p \cdot \gamma$  is the barycenter of  $\frac{e^{F_i}}{\int e^{F_i} d\gamma} \cdot \gamma$ .

## 7 Talagrand-type estimates for the barycenter functional

In this section we show that a weak form of the Blaschke–Santaló inequality is related to the displacement convexity property of the Gaussian entropy. The conjectured strong form of the Blaschke–Santaló inequality is equivalent to a certain strong entropy- $W_2$ -bound, a particular case of this bound for two functions was proved by M. Fathi in [17].

Let us briefly recall the main transportation Gaussian inequalities.

1. Every probability measure  $f \cdot \gamma$  (not necessary centered) satisfies the Talagrand transportation inequality

$$\frac{1}{2} W^2(f \cdot \gamma, \gamma) \leq \text{Ent}_\gamma f := \int f \log f d\gamma.$$

2. In the case when one of the measures  $f \cdot \gamma, g \cdot \gamma$  is centered, a stronger inequality holds (see Remark 7.2 and the comments after it)

$$\frac{1}{2} W_2^2(f \cdot \gamma, g \cdot \gamma) \leq \int f \log f d\gamma + \int g \log g d\gamma. \quad (7.1)$$

3. Displacement convexity of the Gaussian entropy for arbitrary measures  $\mu_i, 1 \leq i \leq k$ , which states that

$$\text{Ent}_\gamma(\mu) + \frac{1}{2} \sum_{i=1}^k \lambda_i W_2^2(\mu, \mu_i) \leq \sum_{i=1}^k \lambda_i \text{Ent}_\gamma(\mu_i), \quad (7.2)$$

where  $\mu$  is the barycenter of the  $\mu_i$  with weights  $\lambda_i$ .

We have seen above that (7.2) follows from Theorem 6.1 (pointwise Prékopa–Leindler inequality). We show below that the following weaker version of (7.2)

$$\frac{1}{2} \sum_{i=1}^k \lambda_i W_2^2(\mu, \mu_i) \leq \sum_{i=1}^k \lambda_i \text{Ent}_\gamma(\mu_i) \quad (7.3)$$

is equivalent to some form of the Prékopa–Leindler inequality (see Remark 7.4).

In this section we establish the equivalence (and verify it in the unconditional case in Theorem 7.1) between the conjectured Blaschke–Santaló inequality and the inequality

$$\frac{1}{2k} \sum_{i=1}^k W_2^2(\mu_i, \mu) \leq \frac{k-1}{k^2} \sum_{i=1}^k \text{Ent}_\gamma(\mu_i),$$

for symmetric measures, which is stronger than (7.3) for the choice of weights  $\lambda_i = \frac{1}{k}$  and generalizes (7.1) for  $k > 2$ .

In what follows,  $\pi$  denotes the solution to the multimarginal Kantorovich problem with marginals  $\mu_i$ . Note that

$$\sum_{i=1}^k \left| x_i - \frac{1}{k} \sum_{j=1}^k x_j \right|^2 = \frac{1}{k} \sum_{i,j=1, i < j}^k |x_i - x_j|^2.$$

Hence one gets by Theorem 2.4

$$\mathcal{F}(\mu) = \frac{1}{2k^2} \int \sum_{i,j=1, i < j}^k |x_i - x_j|^2 d\pi = \frac{1}{2k} \sum_{i=1}^k W_2^2(\mu_i, \mu).$$

**Theorem 7.1.** *Assume that for  $1 \leq i \leq k$ ,  $\mu_i = \rho_i \cdot \gamma$  are probability measures and the  $\rho_i$  are unconditional and let  $\mu$  be the barycenter of the  $\mu_i$  with weights  $\lambda_i = \frac{1}{k}$ . Then*

$$\mathcal{F}(\mu) \leq \frac{k-1}{k^2} \sum_{i=1}^k \int \rho_i \log \rho_i d\gamma = \frac{k-1}{k^2} \sum_{i=1}^k \text{Ent}_\gamma(\mu_i). \quad (7.4)$$

*Proof.* Using standard approximation arguments and lower semicontinuity of the functional  $\mathcal{F}$  one can reduce the general case to the case of compactly supported densities  $\rho_i$ . By the Kantorovich duality (see e.g., [39]),

$$\begin{aligned} \mathcal{F}(\mu) &= \frac{1}{2k^2} \int \sum_{i,j=1, i < j}^k |x_i - x_j|^2 d\pi = \frac{k-1}{k^2} \frac{1}{2(k-1)} \int \sum_{i,j=1, i < j}^k |x_i - x_j|^2 d\pi \\ &= \frac{k-1}{k^2} \int \sum_{i=1}^k f_i(x_i) d\pi = \frac{k-1}{k^2} \sum_{i=1}^k \int f_i(x_i) d\mu_i, \end{aligned}$$

for some measurable functions  $f_i$  satisfying

$$\sum_{i=1}^k f_i(x_i) \leq \frac{1}{2(k-1)} \sum_{i,j=1, i < j}^k |x_i - x_j|^2, \quad (7.5)$$

with equality  $\pi$ -a.e.

Note that we can assume that the functions  $f_i$  are unconditional. Indeed, if not, replace  $f_i$  for all  $i$  by

$$g_i(x_i) = \frac{1}{2^n} \sum_{\varepsilon} f_i(\varepsilon x_i),$$

where  $\varepsilon x_i = (\varepsilon_1 x_i^1, \varepsilon_2 x_i^2, \dots, \varepsilon_n x_i^n)$  and  $\varepsilon_l = \pm 1$ ,  $1 \leq l \leq n$ . Then the functions  $g_i$  are unconditional. They also satisfy the dual problem as the measures  $\mu_i$  are unconditional and as the cost function does not change under  $x_i \rightarrow \varepsilon x_i$ .

Inequality (7.5) is equivalent to

$$\sum_{i=1}^k (f_i(x_i) - \frac{1}{2}|x_i|^2) \leq -\frac{1}{k-1} \sum_{i,j=1, i < j} \langle x_i, x_j \rangle.$$

We will apply Theorem 3.1 to the functions  $f_i(x_i) - \frac{1}{2}|x_i|^2$ . To this end we need to show that  $e^{f_i(x_i) - \frac{1}{2}|x_i|^2}$  are integrable functions. Moreover, let us show that  $f_i(x_i) - \frac{1}{2}|x_i|^2 \in L^\infty(\mu_i)$  for every  $i$ .

Let  $R > 0$  be a number such that  $\text{supp}(\mu_i) \subset B_R$ . Then it follows from Theorem 6.1 that  $\text{supp}(\mu) \subset B_R$ . Hence the optimal transportation mapping  $\nabla \Phi_i^*$  of  $\mu$  onto  $\mu_i$  satisfies the estimate  $|\nabla \Phi_i^*| \leq R$ . By Theorem 2.4  $\Phi_i^*(x_i) = \frac{1}{2k}|x_i|^2 + kv_i(x_i) + C_i$ , where  $v_i$  and  $f_i$  are related as follows

$$\frac{k^2}{k-1}v_i(x_i) = \frac{1}{2}|x_i|^2 - f_i(x_i).$$

To show that  $\frac{1}{2}|x_i|^2 - f_i(x_i) \in L^\infty(\mu_i)$  it is sufficient to show that  $\nabla v_i$  is bounded on the support of  $\mu_i$ . Indeed,  $|\nabla v_i| = \frac{1}{k}|\nabla \Phi_i^*(x_i) - \frac{1}{k}x_i| \leq \frac{k+1}{k^2}R$ .

It now follows from Theorem 3.1 that

$$\prod_{i=1}^k \int e^{f_i(x_i) - \frac{1}{2}|x_i|^2} dx_i \leq (2\pi)^{k\frac{n}{2}},$$

or, equivalently,

$$\prod_{i=1}^k \int e^{f_i(x_i)} d\gamma \leq 1. \tag{7.6}$$

The claim follows from the estimates

$$\begin{aligned} \int \sum_{i=1}^k f_i(x_i) d\mu_i &\leq \int \sum_{i=1}^k (f_i - \log \int e^{f_i} d\gamma) \rho_i d\gamma \leq \sum_{i=1}^k \int (\rho_i \log \rho_i - \rho_i + e^{\int (f_i - \log \int e^{f_i} d\gamma)}) d\gamma \\ &= \sum_{i=1}^k \int \rho_i \log \rho_i d\gamma. \end{aligned}$$

Here the first inequality follows from (7.6) and in the second inequality we apply the inequality  $xy \leq e^x + y \log y - y$ , which is valid for  $x \in \mathbb{R}, y \geq 0$ .  $\square$

**Remark 7.2.** *This result is a generalization in the unconditional setting of a result of M. Fathi [17] for two functions:*

Let  $\rho_0, \rho_1$  be two Gaussian unconditional probability densities and  $\rho_{1/2}$  be the corresponding barycenter. Then inequality 7.4 implies

$$\begin{aligned} \frac{1}{2}W_2^2(\rho_0 \cdot \gamma, \rho_1 \cdot \gamma) &= 2W_2^2(\rho_0 \cdot \gamma, \rho_{1/2} \cdot \gamma) = W_2^2(\rho_0 \cdot \gamma, \rho_{1/2} \cdot \gamma) + W_2^2(\rho_1 \cdot \gamma, \rho_{1/2} \cdot \gamma) \\ &\leq \int \rho_0 \log \rho_0 d\gamma + \int \rho_1 \log \rho_1 d\gamma. \end{aligned} \quad (7.7)$$

This is a particular case of Fathi's inequality.

Fathi has shown that in the class of symmetric functions inequality (7.7) is equivalent to a Blaschke–Santaló inequality involving two exponential functions. Already earlier, in [3], it was noted that the Blaschke–Santaló inequality can be re-written in terms of a property  $\tau$  introduced by Maurey [33] which is dual to the transportation inequality. We follow the approach in [17] to show that the inequality of Theorem 7.1 is also equivalent to a functional Blaschke–Santaló for multiple exponential functions.

Indeed, letting  $\rho(t) = e^{-\frac{t}{k-1}}$  in Theorem 3.1, we get the following multifunctional Blaschke–Santaló inequality:

Let  $f_i: \mathbb{R}^n \rightarrow \mathbb{R}_+$ ,  $1 \leq i \leq k$ , be measurable unconditional functions with  $\int e^{f_i}$  integrable such that

$$\sum_{i=1}^k f_i(x_i) \leq -\frac{1}{k-1} \sum_{i,j=1, i < j}^k \langle x_i, x_j \rangle. \quad (7.8)$$

Then

$$\prod_{i=1}^k \int_{\mathbb{R}^n} e^{f_i} dx \leq (2\pi)^{k\frac{n}{2}}. \quad (7.9)$$

**Proposition 7.3.** *Inequality (7.4) is equivalent to the functional Blaschke–Santaló inequality (7.9).*

*Proof.* One implication is just Theorem 7.1.

For the other implication, we first rewrite inequality (7.4). Thus, let  $\mu$  be the barycenter of the  $\mu_i = \rho_i \cdot \gamma$  with coefficients  $\frac{1}{k}$  and unconditional  $\rho_i$ . We recall that for a probability measure  $\nu$

$$\text{Ent}_\gamma(\nu) = \text{Ent}_{dx}(\nu) + \frac{n}{2} \log(2\pi) + \frac{1}{2} \int |x|^2 d\nu$$

and use this and the definition of the Kantorovich distance to get that (7.4) is equivalent to

$$-\frac{2}{k} \inf_P \sum_{i,j=1, i < j}^k \int \langle x_i, x_j \rangle dP \leq \frac{2(k-1)}{k} \sum_{i=1}^k \text{Ent}_{dx}(\mu_i) + (k-1) \log(2\pi)^n. \quad (7.10)$$

Let now the  $f_i$  be unconditional and such that they satisfy (7.8). We apply (7.10) to  $\mu_i = \rho_i \gamma = \frac{e^{f_i}}{\int e^{f_i}} \gamma$ . We also use that for a probability measure  $\nu$

$$\text{Ent}_{dx}(\nu) = \sup_f \int f d\nu - \log \int e^f dx \quad (7.11)$$

and get

$$\begin{aligned} &-\frac{2}{k} \inf_P \sum_{i,j=1, i < j}^k \int \langle x_i, x_j \rangle dP \\ &\leq \frac{2(k-1)}{k} \sum_{i=1}^k \left( \int f_i d\mu_i - \log \int e^{f_i} dx \right) + (k-1) \log(2\pi)^n. \end{aligned} \quad (7.12)$$

By the Kantorovich duality, the left hand side of this inequality equals

$$\begin{aligned}
& -\frac{2}{k} \inf_P \sum_{i,j=1, i<j}^k \int \langle x_i, x_j \rangle dP \\
&= \frac{2(k-1)}{k} \sup_{\sum_{i=1}^k h_i(x_i) \leq -\frac{1}{k-1} \sum_{i,j=1, i<j}^k \langle x_i, x_j \rangle} \sum_{i=1}^k \int h_i d\mu_i \\
&\geq \frac{2(k-1)}{k} \sum_{i=1}^k \int f_i d\mu_i.
\end{aligned} \tag{7.13}$$

Putting this into (7.12) and removing terms that appear on both sides gives the inequality (7.9).  $\square$

**Remark 7.4.** *Mimicking the proofs of Theorem 7.1 and Proposition 7.3 one can show that the inequality*

$$\left( \prod_{i=1}^k \int e^{F_i} d\gamma \right)^{\lambda_i} \leq 1, \tag{7.14}$$

where the functions  $F_i$  satisfy

$$\sum_{i=1}^k \lambda_i F_i(x_i) \leq \frac{1}{2} \left[ \sum_{i=1}^k \lambda_i |x_i|^2 - \left| \sum_{j=1}^k \lambda_j x_j \right|^2 \right], \tag{7.15}$$

is equivalent to the inequality

$$\sum_{i=1}^k \lambda_i \text{Ent}_\gamma(\mu_i) \geq \frac{1}{2} \sum_{i=1}^n \lambda_i W_2^2(\mu_i, \mu), \tag{7.16}$$

where  $\mu$  is the barycenter of the  $\mu_i$ 's with weights  $\lambda_i$ .

Letting  $V_i = -F_i + \frac{|x_i|^2}{2}$  and  $\lambda_i = \frac{1}{k}$ ,  $1 \leq i \leq k$ , we note that inequality (7.14) has the following equivalent ‘‘Euclidean’’ formulation:

$$\prod_{i=1}^k \int e^{-V_i} dx_i \leq (2\pi)^{k \frac{n}{2}},$$

provided  $\sum_{i=1}^k V_i(x_i) \geq \frac{1}{2k} |x_1 + \dots + x_k|^2$ . This inequality is a direct consequence of the Prékopa–Leindler inequality and here we do not assume that the  $V_i$  are even.

**Remark 7.5.** *See also the notes to the first version of the article which contained another proof of (7.14) based on a symmetrization procedure.*

Inequality (7.16) is, in fact, a weaker version of the displacement convexity property (6.4). It follows, for instance, from inequality (6.3).

What happens, if in the derivation of the Talagrand type bounds instead of (7.14) one applies the stronger pointwise inequality  $\left( \prod_{i=1}^k \int e^{F_i} d\gamma \right)^{\lambda_i} p(x) \leq 1$ , (see Corollary 6.6)? The answer is given in the next theorem.

**Theorem 7.6.** Let  $\mu_i = \rho_i \cdot \gamma$  be probability measures and  $f_i(x_i)$  be the solution to the dual multimarginal problem with marginals  $\mu_i$  and the cost function  $\frac{1}{2k} \sum_{i,j=1, i < j}^k |x_i - x_j|^2$ . Let  $\mu = p(x) \cdot \gamma$  be the barycenter of probability measures  $\frac{e^{f_i}}{\int e^{f_i} d\gamma} \cdot \gamma$  with weights  $\frac{1}{k}$ . Then

$$p(x) \leq e^{\frac{1}{k} \sum_{i=1}^k (\int \rho_i \log \rho_i d\gamma - \frac{1}{2} W_2^2(\mu, \mu_i))}. \quad (7.17)$$

*Proof.* Let  $\pi$  be the solution to the corresponding primary problem. By the Kantorovich duality

$$\frac{1}{2k} \sum_{i=1}^k W_2^2(\mu_i, \mu) = \frac{1}{2k^2} \int \sum_{i,j=1, i < j}^k |x_i - x_j|^2 d\pi = \frac{1}{k} \sum_{i=1}^k \int f_i d\mu_i.$$

Then

$$\frac{1}{2k} \sum_{i=1}^k W_2^2(\mu_i, \mu) = \frac{1}{k} \sum_{i=1}^k \int f_i d\mu_i = \log \left( \prod_{i=1}^k \int e^{f_i} d\gamma \right)^{\frac{1}{k}} + \frac{1}{k} \sum_{i=1}^k \int (f_i - \log \int e^{f_i} d\gamma) \rho_i d\gamma.$$

Using Corollary 6.6 one has

$$\log \left( \prod_{i=1}^k \int e^{f_i} d\gamma \right)^{\frac{1}{k}} \leq -\log p(x).$$

Then we apply Young inequality and get that  $\int (f_i - \log \int e^{f_i} d\gamma) \rho_i d\gamma \leq \int \rho_i \log \rho_i d\gamma$ . Finally one obtains  $\log p(x) \leq \frac{1}{k} \sum_{i=1}^k (\int \rho_i \log \rho_i d\gamma - \frac{1}{2} W_2^2(\mu, \mu_i))$ .  $\square$

Taking logarithm of both sides of (7.17) and integrating with respect to  $\nu = p \cdot \gamma$  we obtain, in particular, the following estimate

$$\text{Ent}_\gamma(\nu) + \frac{1}{2k} \sum_{i=1}^k W_2^2(\mu, \mu_i) \leq \frac{1}{k} \sum_{i=1}^k \text{Ent}_\gamma(\mu_i),$$

which is reminiscent to (7.2), but it is not completely clear how they can be compared.

## 8 Monotonicity of the Blaschke-Santaló functional

In this section we prove a remarkable monotonicity property of the Blaschke-Santaló functional which appears naturally with respect to the barycenter problem.

### 8.1 The case of two functions

We start with the case of two functions,  $k = 2$ . We first recall for  $\lambda \in \mathbb{R}$ , the definition of the  $\lambda$ -affine surface area of a convex function  $V$  introduced in [12] and already given in (3.2).

$$as_\lambda(V) = \int_{\Omega_V} e^{(2\lambda-1)V(x) - \lambda \langle x, \nabla V(x) \rangle} (\det D^2 V(x))^\lambda dx,$$

where  $D^2 V$  is the Hessian of  $V$ .

We consider now two functionals on convex functions  $V$ , the Blaschke-Santaló functional

$$\mathcal{BS}(V) = \int e^{-V} dx \int e^{-V^*} dx$$

and the  $\frac{1}{2}$ -affine surface area functional,

$$\mathcal{J}(V) = as_{\frac{1}{2}}(V) = \int e^{-\frac{1}{2}\langle x, \nabla V(x) \rangle} \sqrt{\det D^2 V} dx.$$

To avoid technicalities, we assume that  $V$  is  $C^2$  and strictly convex.

**Proposition 8.1.** *Let  $V$  be a strictly convex  $C^2$ -function such that  $e^{-V}, e^{-V^*}$  are integrable functions. Let  $\nabla\Psi$  be the optimal transportation of  $\frac{e^{-V} dx}{\int e^{-V} dx}$  onto  $\frac{e^{-V^*} dx}{\int e^{-V^*} dx}$ . Then*

$$\mathcal{BS}(V) \leq \mathcal{J}^2(\Psi) \leq \mathcal{BS}(\Psi). \quad (8.1)$$

Equivalently

$$\int e^{-V} dx \int e^{-V^*} dx \leq \left( \int e^{-\frac{1}{2}\langle x, \nabla\Psi \rangle} \sqrt{\det D^2\Psi} dx \right)^2 \leq \int e^{-\Psi} dx \int e^{-\Psi^*} dx.$$

*Proof.* The second inequality is just Theorem 3.2 (iii). To prove the first inequality, we apply the change of variables formula

$$\frac{e^{-V}}{\int e^{-V} dx} = \frac{e^{-V^*(\nabla\Psi)}}{\int e^{-V^*} dx} \det D^2\Psi.$$

Note that regularity of  $V, V^*$  imply that  $\Psi$  is sufficiently regular, hence  $D^2\Psi$  is absolutely continuous (see, for instance, [39]). Then

$$\int e^{-\frac{1}{2}\langle x, \nabla\Psi \rangle} \sqrt{\det D^2\Psi} dx = \sqrt{\frac{\int e^{-V^*} dx}{\int e^{-V} dx}} \int e^{\frac{V^*(\nabla\Psi) - V(x) - \langle x, \nabla\Psi \rangle}{2}} dx.$$

The result follows from the inequality  $V^*(\nabla\Psi) + V(x) \geq \langle x, \nabla\Psi \rangle$ .  $\square$

Let us outline (without rigorous justifications) the idea of alternative proof of the Blaschke–Santaló inequality. It can be easily seen from the proof that equality in  $V^*(\nabla\Psi) + V(x) \geq \langle x, \nabla\Psi \rangle$  (and hence in (8.1)) is attained if and only if  $V = \Psi + a$  for some constant  $a$ . Thus, within a certain appropriate class of functions, e.g., symmetric, the maximum of the Blaschke–Santaló functional must satisfy that the measure  $\frac{e^{-\Psi^*}}{\int e^{-\Psi^*} dx}$  is the push-forward measure of  $\frac{e^{-\Psi}}{\int e^{-\Psi} dx}$  under the mapping  $\nabla\Psi$ . This means that  $\Psi$  solves the following Monge–Ampère equation

$$\frac{e^{-\Psi}}{\int e^{-\Psi} dx} = \frac{e^{-\Psi^*(\nabla\Psi)}}{\int e^{-\Psi^*} dx} \det D^2\Psi. \quad (8.2)$$

It was shown in [13] that this equation admits the following family of solutions, provided  $\frac{e^{-\Psi}}{\int e^{-\Psi} dx}$  has logarithmic derivatives,

$$\Psi = \frac{\langle Ax, x \rangle}{2} + c,$$

where  $A$  is a positive definite matrix and  $c$  is a constant. These are exactly the maximizers of the Blaschke–Santaló functional.

Thus, this observation suggests the following (so far heuristic) approach to the Blaschke–Santaló inequality. Let  $\Psi_0 = V$ , and consider iterations  $\Psi_l$ ,  $l \in \mathbb{N}$ , where  $\Psi_{l+1}$  is the optimal transportation potential pushing forward  $\frac{e^{-\Psi_l} dx}{\int e^{-\Psi_l} dx}$  onto  $\frac{e^{-\Psi_l^*} dx}{\int e^{-\Psi_l^*} dx}$ . By Proposition 8.1, one gets an increasing sequence  $\mathcal{BS}(\Psi_l)$ ,  $l \in \mathbb{N}$ . From this, one can try to extract convergence of  $\Psi_l$  to a potential  $\Psi$ , which gives a maximum to the Blaschke–Santaló functional. Then prove that  $\Psi$  solves (8.2), and by uniqueness deduce that  $\Psi$  is quadratic.



## 8.2 The multimarginal case

Next we generalize the previous result to the multimarginal case,  $k > 2$ .

**Theorem 8.2.** *Assume that  $V_i(x_i)$ ,  $1 \leq i \leq k$ , are measurable functions such that  $e^{-V_i}$  are integrable, satisfying*

$$\sum_{i=1}^k \lambda_i V_i(x_i) \geq C \sum_{i,j=1,i < j}^k \lambda_i \lambda_j \langle x_i, x_j \rangle$$

for some  $C > 0$  and  $\lambda_i \in (0, 1)$  with  $\sum_{i=1}^k \lambda_i = 1$ .

Let the tuple of functions  $\lambda_i U_i(x_i)$  be the solution to the dual multimarginal maximization problem with marginals  $\frac{e^{-V_i dx_i}}{\int e^{-V_i dx_i}}$  and the cost function  $C \sum_{i,j=1,i < j}^k \lambda_i \lambda_j \langle x_i, x_j \rangle$ . Then

$$\prod_{i=1}^k \left( \int e^{-V_i} dx_i \right)^{\lambda_i} \leq \prod_{i=1}^k \left( \int e^{-U_i} dx_i \right)^{\lambda_i}.$$

*Proof.* Let  $\rho dx$  be the barycenter of  $d\mu_i = \frac{e^{-V_i dx_i}}{\int e^{-V_i dx_i}}$  with weights  $\lambda_i$  and  $\nabla \Phi_i$  be the optimal transportation mapping pushing forward  $\rho dx$  onto  $d\mu_i$ . Recall that for  $\rho dx$ -almost all  $y$  one has (see Theorem 2.4),

$$\sum_{i=1}^k \lambda_i U_i(\nabla \Phi_i(y)) = C \sum_{i,j=1,i < j}^k \lambda_i \lambda_j \langle \nabla \Phi_i(y), \nabla \Phi_j(y) \rangle.$$

Apply the change of variables formula

$$\rho(y) = \frac{e^{-V_i(\nabla \Phi_i(y))}}{\int e^{-V_i} dx_i} \det D_a^2 \Phi_i(y).$$

One has

$$\begin{aligned} \prod_{i=1}^k \left( \int e^{-V_i} dx_i \right)^{\lambda_i} \rho(y) &= e^{-\sum_{i=1}^k \lambda_i V_i(\nabla \Phi_i(y))} \prod_{i=1}^k (\det D_a^2 \Phi_i(y))^{\lambda_i} \\ &\leq e^{-C \sum_{i,j=1,i < j}^k \lambda_i \lambda_j \langle \nabla \Phi_i(y), \nabla \Phi_j(y) \rangle} \prod_{i=1}^k (\det D_a^2 \Phi_i(y))^{\lambda_i} \\ &= \prod_{i=1}^k e^{-\lambda_i U_i(\nabla \Phi_i(y))} (\det D_a^2 \Phi_i(y))^{\lambda_i}. \end{aligned}$$

Integrating both sides and using Hölder's inequality, we get

$$\begin{aligned} \prod_{i=1}^k \left( \int e^{-V_i} dx_i \right)^{\lambda_i} &\leq \int \prod_{i=1}^k e^{-\lambda_i U_i(\nabla \Phi_i)} (\det D_a^2 \Phi_i(y))^{\lambda_i} dy \\ &\leq \prod_{i=1}^k \left( \int e^{-U_i(\nabla \Phi_i)} (\det D_a^2 \Phi_i(y)) dy \right)^{\lambda_i} = \prod_{i=1}^k \left( \int_{\nabla \Phi_i(\mathbb{R}^n)} e^{-U_i} dx_i \right)^{\lambda_i} \\ &\leq \prod_{i=1}^k \left( \int e^{-U_i} dx_i \right)^{\lambda_i}. \end{aligned}$$

Here we use the change of variables and the fact that the image of  $\det D_a^2 \Phi_i(y) dy$  under  $\nabla \Phi_i$  is the Lebesgue measure on  $\nabla \Phi_i(\mathbb{R}^n)$ . This follows, for instance from the aforementioned result of McCann ([39], Theorem 4.8).

□

Let us informally analyze the equality case. Clearly, in this case one has for almost all  $y$ ,

$$\sum_{i=1} \lambda_i V_i(\nabla \Phi_i(y)) = \sum_{i=1} \lambda_i U_i(\nabla \Phi_i(y)).$$

Integrating over  $\rho dy$  we get that  $(\lambda_i V_i)$  is a dual Kantorovich solution as well. Hence, by uniqueness of the dual solution

$$V_i = U_i + C_i, \quad \sum_{i=1}^k C_i = 0.$$

In addition, one has for all  $i$  that

$$\frac{e^{-U_i(\nabla \Phi_i)}}{\int e^{-U_i} dx_i} \det D^2 \Phi_i = \rho,$$

or, equivalently,

$$\frac{e^{-U_i}}{\int e^{-U_i} dx_i} = \rho(\nabla \Phi_i^*) \det D^2 \Phi_i^*.$$

In particular, since (see Theorem 2.4)

$$\Phi_i^*(x_i) = \lambda_i \frac{|x_i|^2}{2} + \frac{U_i(x_i)}{C} + C_i,$$

every function  $U_i$  must satisfy

$$\frac{e^{-U_i}}{\int e^{-U_i} dx_i} = \rho \left( \frac{\nabla U_i(x_i)}{C} + \lambda_i x_i \right) \det \left( \frac{D^2 U_i}{C} + \lambda_i I \right). \quad (8.3)$$

Thus, a maximizer of the Blaschke–Santaló inequality, if it exists, must satisfy the system of equations (8.3), where every  $U_i$  is convex.

**Remark 8.3.** Equation (8.3) is an equation of the Kähler–Einstein type. We do not know whether (8.3) admits a unique solution. The well posedness of the classical Kähler–Einstein equation

$$\frac{e^{-\Phi}}{\int e^{-\Phi} dx} = \rho(\nabla \Phi) \det D^2 \Phi$$

was proved under broad assumptions in [14].

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