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MATHEMATICAL PROBLEMS OF NONLINEARITY

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On a Classification of Periodic Maps on the 2-Torus

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In this paper, following J. Nielsen, we introduce a complete characteristic of orientation-preserving periodic maps on the two-dimensional torus. All admissible complete characteristics were found and realized. In particular, each of the classes of orientation-preserving periodic homeomorphisms on the 2-torus that are nonhomotopic to the identity is realized by an algebraic automorphism. Moreover, it is shown that the number of such classes is finite. According to V. Z. Grines and A. Bezdenezhnykh, any gradient-like orientation-preserving diffeomorphism of an orientable surface is represented as a superposition of the time-1 map of a gradient-like flow and some periodic homeomorphism. Thus, the results of this work are directly related to the complete topological classification of gradient-like diffeomorphisms on surfaces.

Keywords: gradient-like flows and diffeomorphisms on surfaces, periodic homeomorphisms, torus

1. Introduction

According to the Nielsen – Thurston classification (see, for example, [1]), the set of homotopy classes of orientation-preserving homeomorphisms of orientable surfaces is split into four disjoint subsets. Each subset consists of homotopy classes of homeomorphisms of one of the following types: T_1) periodic homeomorphism; T_2) reducible nonperiodic homeomorphism of algebraically finite order; T_3) a reducible homeomorphism that is not a homeomorphism of algebraically finite

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order; T_4) pseudo-Anosov homeomorphism. It is known (see [2, p. 369, Theorem 13.1]) that the homotopic types of homeomorphisms of a torus are T_1, T_2, T_4 only.

Let S be a connected compact (possibly with boundary) orientable surface. Let us recall that homeomorphisms f, $f'\colon S \to S$ are called *topologically conjugate* if there is a homeomorphism $h\colon S \to S$ such that $f' = h \circ f \circ h^{-1}$.

A homeomorphism f is called periodic of period n if $f^n = id$ and $f^m \neq id$ for each natural $m < n$.

In class T_1 there is a classification of structurally stable gradient-like orientation-preserving diffeomorphisms of an orientable surface in [3] and [4]. It was proved that any such diffeomorphism is topologically conjugate to a diffeomorphism which is represented as a superposition of the time-1 map of a gradient-like flow and some periodic homeomorphism.

A homeomorphism $h\colon S \to S$ is called reducible by a system C of disjoint simple closed curves C_i, $i = \overline{1, \ldots, l}$, nonhomotopic to zero and pairwise nonhomotopic to each other if the system of curves C is invariant under h. A reducible nonperiodic homeomorphism $h\colon S \to S$ is called a homeomorphism of algebraically finite order if there exists an h-invariant neighborhood $\mathbb{C}$ of curves of the set C consisting of the union of two-dimensional annuli and such that for each connected component S_j, $j = \overline{1, \ldots, n}$ of the set $S \setminus \mathbb{C}$ there is a number $m_j \in \mathbb{N}$ such that $h^{m_j}|_{S_j} : S_j \to S_j$ is a periodic homeomorphism.

In class T_2 there is a classification of structurally stable Morse–Smale diffeomorphisms with orientable heteroclinic intersection in [5–8]. The paper [9] contains a review of the results obtained so far in the classification of Morse–Smale diffeomorphisms in general.

All representatives of class T_4 have chaotic dynamics. For the case of the 2-torus, structurally stable representatives of class T_4 are Anosov diffeomorphisms. Each Anosov 2-diffeomorphism is topologically conjugate to an algebraic automorphism (see [10, Theorem 5.2]). For the case of orientable surfaces of genus greater than one there is a classification of structurally stable orientation-preserving diffeomorphisms containing a one-dimensional spaciously situated perfect attractor in [11, Corollary 1.11 to Theorem 1.10].

In [12], J. Nielsen found necessary and sufficient conditions for topological conjugacy of periodic transformations of closed orientable surfaces. Describing all topological classes for periodic maps is a difficult and boundless task. However, this problem was completely solved in the case of the two-dimensional sphere by Kerkyarto in [14] and was partially solved in the case of the two-dimensional torus by Brauer in [15]. In addition, in [16] there is an algorithm to find the number of classes of topological conjugacy for periodic maps and there is a construction of the canonical representative for each homotopy class of the topological conjugacy. In the present paper we describe all classes of topological conjugacy for periodic maps on the two-dimensional torus by means of orientation-preserving homeomorphisms. Moreover, the work contains a realization of nonhomotopy of periodic maps to the identity on the two-dimensional torus by algebraic automorphisms. In addition, it is shown that the number of such classes is finite. Notice that there is an infinite set of topological conjugacy classes for periodic maps of the two-dimensional torus that are homotopic to the identity. The realization of such classes is presented in [17].

From the results of J. Nielsen [12, Sections 1–4] the following statements are true for any orientation-preserving periodic map f on a closed orientable surface S, whose period is n:

1. For each f we denote by $\overline{B}_f \subset S$ a set of *periodic points of the homeomorphism f with a period strictly less than n.* This set is either empty or consists of a finite number of orbits $\mathcal{O}_1, \ldots, \mathcal{O}_k$, $k \geqslant 1$. Denote by n_i the period of an orbit $\mathcal{O}_i$, $i \in \{1, \ldots, k\}$. Then n_i is a divisor of n. Set $\lambda_i = \frac{n}{n_i}$, then for any orbit $\mathcal{O}_i \subset \overline{B}_f$ there exists a unique num-

ber $\delta_i \in \{1, \ldots, \lambda_i - 1\}$ such that it is coprime to λ_i and there is some neighborhood $D_{\overline{x}_i}$ of a point $\overline{x}_i \in \mathcal{O}_i$ such that the restriction $f^{n_i}|_{D_{\overline{x}_i}}$ is topologically conjugate to the rotation by the angle $\frac{2\pi\delta_i}{\lambda_i}$ of the complex plane about the origin:

$$z \to e^{\frac{2\pi\delta_i}{\lambda_i}\mathbf{i}} z. \tag{1.1}$$

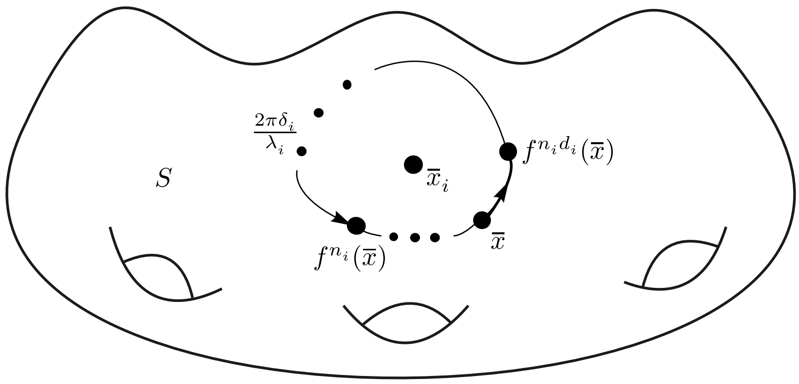

Fig. 1. The action of the homeomorphism f^{n_i} in some neighborhood of the point $\overline{x}_i$

Figure 1 illustrates the action of the homeomorphism f^{n_i} in the neighborhood of the point $\overline{x}_i$. This point has period n_i with respect to the homeomorphism f and is a fixed point with respect to the homeomorphism f^{n_i}. The action of the homeomorphism f^{n_i} is topologically conjugate to the rotation by the angle $\frac{2\pi\delta_i}{\lambda_i}$ about the point in the counterclockwise direction.

2. Denote by d_i the inverse for δ_i number in $\mathbb{Z}_{\lambda_i}$. Due to conjugacy with the map (1.1) there exists a curve which is homeomorpic to the circle and such that it is invariant under the homeomorphism f^{n_i} and bounds a disk containing a point $\overline{x}_i \in \overline{B}_f$. Then the number d_i has the following property: the open arc of the invariant curve joining the points $\overline{x}$ and $f^{n_i d_i}(\overline{x})$ (in the counterclockwise direction) does not contain points of the orbit of point $\overline{x}$. A pair of numbers (n_i, d_i) is called the *valency* of the orbit $\mathcal{O}_i$.

   Figure 1 illustrates the closed curve. This is invariant with respect to the homeomorphism f^{n_i} and contains each of points of the orbit of the point $\overline{x}$ under f^{n_i}. The open arc of the curve connecting points $\overline{x}$ and $f^{n_i d_i}(\overline{x})$ (in the counterclockwise direction) does not contain points of this orbit.

3. For each f denote by $G = \{id, f, \ldots, f^{n-1}\}$ the group which is isomorphic to $\mathbb{Z}_n = \{0, \ldots, n-1\}$. The orbit space $\Sigma = S/G$ is called the *modular surface*. Also, this is a closed surface, and the *natural projection* $\pi\colon S \to \Sigma$ is an n-fold covering map everywhere except at the points of the set $\overline{B}_f$. Put $x_i = \pi(\mathcal{O}_i)$.

4. Denote by D_i $(i = \overline{1, k})$ an open disk on the modular surface Σ such that it contains the only point $x_i \in B_f$. Let c_i denote the boundary of D_i and let $\pi^{-1}(D_i)$ denote the full preimage of D_i on the surface S. Put $\dot{S} = S \setminus \bigcup\limits_{i=1}^{k} \pi^{-1}(D_i)$, $\dot{\Sigma} = \Sigma \setminus \bigcup\limits_{i=1}^{k} D_i$. Choose a point x on $\dot{\Sigma}$

and draw a closed path $c(t)$ ($t \in [0, 1]$) through x on $\dot{\Sigma}$ such that $x = c(0) = c(1)$. Then, by the monodromy theorem (see, for example, [18, Statement 10.27]), for a point $\overline{x} \in S$ such that $\pi(\overline{x}) = x$ there is a unique path $\overline{c}(t)$ on $\dot{S}$ which is a lift of $c(t)$ and such that $\overline{c}(0) = \overline{x}$. Then there is $m \in \mathbb{Z}_n$ such that $\overline{c}(1) = f^m(\overline{x})$ (see Fig. 2).

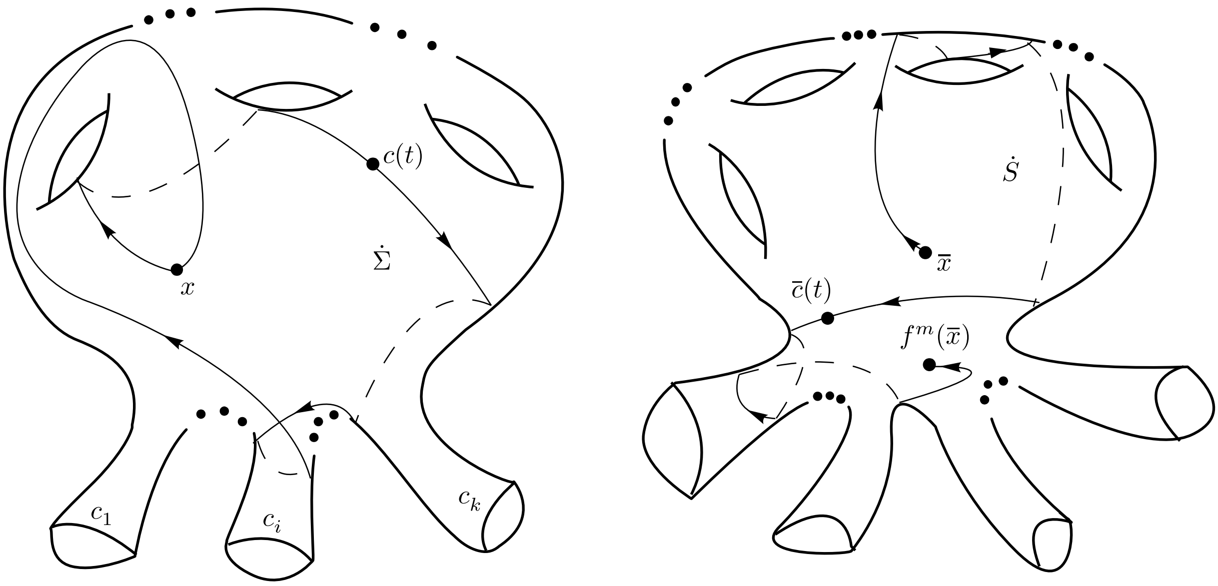

Fig. 2. Path $c(t)$ on surface $\dot{\Sigma}$ and its lift $\overline{c}(t)$ on surface $\dot{S}$

Denote by η the map from the fundamental group of the surface $\pi_1(\dot{\Sigma})$ to the group $\mathbb{Z}_n$ by the rule $\eta([c]) = m$, where $[c]$ is the class of loops homotopic to c. Then $\eta\colon \pi_1(\dot{\Sigma}) \to \mathbb{Z}_n$ is a group homomorphism.

Let p be the genus of the surface S and g be the genus of the modular surface Σ. It follows from the Riemann–Hurwitz formula (see, for example, [25]) that

$$2p + \sum_{i=1}^{k} n_i - 2 = n(2g + k - 2). \tag{1.2}$$

For each periodic transformation f of a closed orientable surface S we define the collection of the numbers

$$\kappa = (n,\, p,\, n_1,\, \ldots,\, n_k,\, d_1,\, \ldots,\, d_k),$$

which we call the *complete characteristic* of the periodic transformation f.

Let us choose a point O on $\dot{\Sigma}$ and draw the canonical system of loops

$$\alpha_1,\, \beta_1,\, \ldots,\, \alpha_g,\, \beta_g,\, \gamma_1,\, \ldots,\, \gamma_k$$

on $\dot{\Sigma}$ starting and finishing at the point O such that it defines the family of generators of the fundamental group $\pi_1(\dot{\Sigma})$ of the surface $\dot{\Sigma}$ (see Fig. 3). The only defining relation of the fundamental group $\pi_1(\dot{\Sigma})$ is the equality

$$K_1 \cdot \ldots \cdot K_g \cdot \gamma_1 \cdot \ldots \cdot \gamma_k = 1, \tag{1.3}$$

where $K_j = \alpha_j \beta_j \alpha_j^{-1} \beta_j^{-1}$.

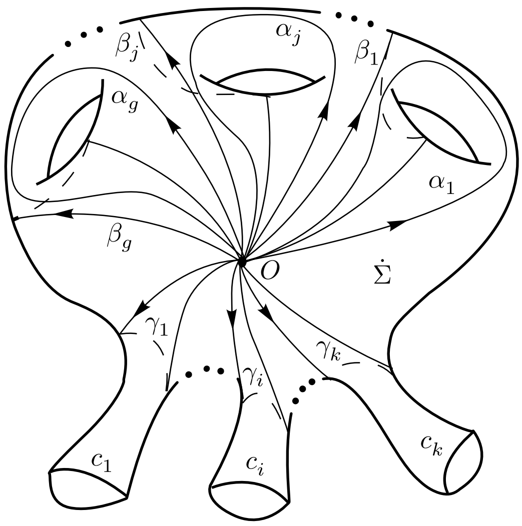

Fig. 3. The family of generators of the fundamental group $\pi_1(\dot{\Sigma})$

Let us choose a point $\overline{O}$ on the surface $\dot{S}$ such that $\pi(\overline{O}) = O \in \dot{\Sigma}$. For any $m \in \mathbb{Z}_n$ through the point $\overline{O} \in \dot{S}$ it is possible to draw a path $\overline{l}(t)$ $(t \in [0, 1])$ on the surface $\dot{S}$ such that $\overline{l}(0) = \overline{O}$, $\overline{l}(1) = f^m(\overline{O}) \in \dot{S}$. Its projection $l(t) = \pi(\overline{l})$ is a closed path on the surface $\dot{\Sigma}$ such that $l(0) = l(1) = O$. Therefore, the map $\eta\colon \pi_1(\dot{\Sigma}) \to \mathbb{Z}_n$ defined in 4. is an epimorphism. Therefore, the following equality holds:

$$gcd(\eta([\alpha_1]), \eta[\beta_1], \ldots, \eta[\alpha_g], \eta[\beta_g], \eta[\gamma_1], \ldots, \eta[\gamma_k], n) = 1, \tag{1.4}$$

where $gcd(a_1, \ldots, a_m)$ is the greatest common divisor of integers $a_1, \ldots, a_m$.

By definition of the valence of the ith orbit $\eta[\gamma_i] = n_i d_i$ $(i = \overline{1, k})$, and $\eta[K_j] = \eta[\alpha_j\beta_j\alpha_j^{-1}\beta_j^{-1}] = \eta[\alpha_j] + \eta[\beta_j] + \eta[\alpha_j^{-1}] + \eta[\beta_j^{-1}] \equiv 0 \pmod{n}$ $(j = \overline{1, q})$ since $\eta[c] + \eta[c^{-1}] = \eta[cc^{-1}] = \eta[1] = 0 \pmod{n}$ for any path $c \in \dot{\Sigma}$. From this, taking Eq. (1.3) into account, we get

$$\sum_{i=1}^{k} d_i n_i \equiv 0 \pmod{n}. \tag{1.5}$$

For the case where the modular surface is a sphere $(g = 0)$ it follows from Eq. (1.4) that

$$gcd(n_1 d_1, \ldots, n_k d_k, n) = 1. \tag{1.6}$$

Statement 1 ([12, p. 56]).

There is a periodic homeomorphism whose complete characteristic is $\kappa = (n, p, n_1, \ldots, n_k, d_1, \ldots, d_k)$ *and the genus of the modular surface is* g *if and only if the following conditions are satisfied*:

1) $\sum\limits_{i=1}^{k} d_i n_i \equiv 0 \pmod{n}$;

2) $2p + \sum\limits_{i=1}^{k} n_i - 2 = n(2g + k - 2)$;

3) *if* $g = 0$: $gcd(n_1 d_1, \ldots, n_k d_k, n) = 1$.

Statement 2 ([12, p. 53]). *Two periodic maps f and f' on the orientable surface S are topologically conjugate by an orientation-preserving homeomorphism if and only if the complete characteristic of f coincides with the complete characteristic of f'.*

If $\overline{B}_f = \varnothing$ the periodic transformation f is completely described by the set of the numbers (n, p). In this case the natural projection $\pi\colon S \to \Sigma$ is an n-fold covering map of the modular surface Σ of genus g by the surface S of genus p. This implies the following statement.

Statement 3. *Two periodic transformations f, f' of the surface S such that $\overline{B}_f = \varnothing$, $\overline{B}_{f'} = \varnothing$ are topologically conjugate by an orientation-preserving homeomorphism if and only if f and f' have the same periods.*

An orientation-preserving homeomorphism $f\colon \mathbb{T}^2 \to \mathbb{T}^2$ induces a group automorphism f_* of $\pi_1(\mathbb{T}^2) = \mathbb{Z} \oplus \mathbb{Z}$ given by matrix $A_f \in SL(2, \mathbb{Z})$ (see, for example, in [22, Section D]). Homeomorphism f is isotopic to the identity if and only if $A_f = I$. Recall that a matrix A is called periodic if there exists $m \in \mathbb{N}$ such that $A^m = I$. Since the action of the induced automorphism f_*^n is defined by the matrix A_f^n, $A_f^n = E$ for homeomorphism f of period n. Therefore, matrix A_f is periodic for any periodic homeomorphism f. However, the period of matrix A_f may not coincide with period n of homeomorphism f in the general case, but does not exceed n.

The main results of this work are the following theorems.

Theorem 1. *There is an orientation-preserving periodic homeomorphism $f\colon \mathbb{T}^2 \to \mathbb{T}^2$ such that $\overline{B}_f \neq \varnothing$ if and only if the complete characteristic of f coincides exactly with one of seven complete characteristics*:

1) κ_1: $n = 2$, $n_1 = n_2 = n_3 = n_4 = 1$, $d_1 = d_2 = d_3 = d_4 = 1$;

2) κ_2: $n = 3$, $n_1 = n_2 = n_3 = 1$, $d_1 = d_2 = d_3 = 1$;

3) κ_3: $n = 3$, $n_1 = n_2 = n_3 = 1$, $d_1 = d_2 = d_3 = 2$;

4) κ_4: $n = 6$, $n_1 = 3$, $n_2 = 2$, $n_3 = 1$, $d_1 = d_2 = d_3 = 1$;

5) κ_5: $n = 6$, $n_1 = 3$, $n_2 = 2$, $n_3 = 1$, $d_1 = 1$, $d_2 = 2$, $d_3 = 5$;

6) κ_6: $n = 4$, $n_1 = 2$, $n_2 = n_3 = 1$, $d_1 = d_2 = d_3 = 1$;

7) κ_7: $n = 4$, $n_1 = 2$, $n_2 = n_3 = 1$, $d_1 = 1$, $d_2 = d_3 = 3$.

Theorem 2. *Let $f\colon \mathbb{T}^2 \to \mathbb{T}^2$ be an orientation-preserving periodic homeomorphism of period $n \in \mathbb{N}$. Then the following conditions are equivalent*:

1) *f is homotopic to the identity*;

2) $\overline{B}_f = \varnothing$;

3) $g = 1$;

4) *f is topologically conjugate to the shift on the torus* $\Psi_n\left(e^{i2x\pi}, e^{i2y\pi}\right) = \left(e^{i2\pi(x+1/n)}, e^{i2y\pi}\right)$.

Corollary 1. *Periodic homeomorphisms of the two-dimensional torus satisfying complete characteristics κ_j $(j = \overline{1, 7})$ and only they are orientation-preserving periodic homeomorphisms of the two-dimensional torus that are nonhomotopic to the identity.*

Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{Z})$, that is, let A be a second-order integer square matrix with $\det A = 1$. Then A induces an *orientation-preserving algebraic automorphism of the two-dimensional torus* $f_A\colon \mathbb{T}^2 \to \mathbb{T}^2$ given by the formula

$$f_A\colon \begin{cases} \overline{x} = ax + by \pmod 1, \\ \overline{y} = cx + dy \pmod 1. \end{cases}$$

Theorem 3. *Any orientation-preserving periodic homeomorphism $f\colon \mathbb{T}^2 \to \mathbb{T}^2$ nonhomotopic to the identity is conjugate by an orientation-preserving homeomorphism with exactly one map f_{A_j} induced by the matrix A_j $(j = \overline{1, 7})$:*

$$A_1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}; \quad A_2 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}; \quad A_3 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}; \quad A_4 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix};$$

$$A_5 = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}; \quad A_6 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}; \quad A_7 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The complete characteristic of f_{A_j} coincides with a complete characteristic κ_j.

1.1. Structure of the paper

- Section 2 contains lemmas necessary for the proof of Theorem 1.
- Section 3 is the proof of Theorem 1.
- Section 4 is the proof of Theorem 2.
- Section 4 contains a classification of algebraic automorphisms of the two-dimensional torus and the proof of Theorem 3.

2. Auxiliary inequalities for finding complete characteristics of periodic transformations

In this section we prove some useful consequences from Eq. (1.2):

$$2p + \sum_{i=1}^{k} n_i - 2 = n(2g + k - 2).$$

Lemma 1. *For any orientation-preserving periodic homeomorphism $f\colon S \to S$ such that $\overline{B}_f = \varnothing$ the following equality holds:*

$$p = n(g - 1) + 1. \tag{2.1}$$

Proof. Due to $\overline{B}_f = \varnothing$, the number $k = 0$. Using Eq. (1.2) $2p + \sum\limits_{i=1}^{k} n_i - 2 = n(2g + k - 2)$, we get $2p - 2 = n(2g - 2)$. Therefore, $p = n(g - 1) + 1$. $\square$

Lemma 2. *For any orientation-preserving periodic homeomorphism $f\colon S \to S$ such that $\overline{B}_f \neq \varnothing$ the following equality holds*:

$$p > n(g-1) + 1. \tag{2.2}$$

Proof. By definition n_i is a divisor of n not exceeding n for all $i = \overline{1,\,k}$, therefore, $n_i \leqslant \frac{n}{2}$ and the following relation holds:

$$0 < k \leqslant \sum_{i=1}^{k} n_i \leqslant \frac{nk}{2} < nk. \tag{2.3}$$

Let us consider all possible cases:

1. $p = 0$. Then Eq. (1.2) is equivalent to the equality $\sum\limits_{i=1}^{k} n_i - 2 = n(2g+k-2)$. Transforming this, we get $\sum\limits_{i=1}^{k} n_i - 2 = n(2g-2)+nk$. It follows from Eq. (2.3) that $nk-2 > n(2g-2)+nk$. Hence, $n(2g-2) < -2$ and $n(1-g) > 1$. Then $1 - g > 0$. Considering that $g \in \mathbb{N}$, we get $g = 0$.

2. $p = 1$. Then Eq. (1.2) is equivalent to the equality $\sum\limits_{i=1}^{k} n_i = n(2g+k-2)$. Using Eq. (2.3), we obtain $n(2g+k-2) < nk$. Therefore, $g - 1 < 0$ and $g < 1$. Considering $g \in \mathbb{N}$, we get $g = 0$.

3. $p > 1$. Using Eqs. (1.2) and (2.3), we find that $n(2g+k-2) < 2p+nk-2$. Hence, $n(g-1) < p-1$ and $p > n(g-1)+1$.

Thus, in all cases the following inequality holds: $p > n(g-1)+1$. □

Lemma 3. *For any orientation-preserving periodic homeomorphism $f\colon \mathbb{T}^2 \to \mathbb{T}^2$ such that $\overline{B}_f \neq \varnothing$ the following equality holds*:

$$2 < \frac{2n}{n-1} \leqslant k \leqslant 4. \tag{2.4}$$

Proof. In the case of the two-dimensional torus $p = 1$. Then it follows from Lemma 2 that the genus of the modular surface is equal to 0. Therefore, Eq. (1.2) is equivalent to the equality

$$nk = \sum_{i=1}^{k} n_i + 2n. \tag{2.5}$$

Using Eq. (2.3), we see that $2n + k \leqslant nk \leqslant 2n + \frac{nk}{2}$. From the left side of this inequality we find that $k \geqslant \frac{2n}{n-1} > \frac{2n-2}{n-1} = 2$. From the right side we obtain $\frac{nk}{2} \leqslant 2n$. Therefore, $n(4-k) \geqslant 0$. Considering that $k,\, n \in \mathbb{N}$, we have $k \leqslant 4$. □

3. Complete characteristics of periodic transformations of the two-dimensional torus

It follows from Statement 3 that, if the set $\overline{B}_f = \varnothing$ for the homeomorphism f, then the complete characteristic of f is determined by the genus p of the surface and the period n of the transformation. Thus, complete characteristics of periodic homeomorphisms on the two-dimensional torus corresponding to the empty set $\overline{B}_f$ have the form $(n, 1)$, where n is the period of f.

In this section we find all complete characteristics of an orientation-preserving periodic homeomorphism of the two-dimensional torus corresponding to the nonempty set $\overline{B}_f$ and prove that there are only 7 such complete characteristics. Using transformations of a modular surface, we realize all such classes by the periodic homeomorphism.

Proof of Theorem 1. **Let us prove the necessity.** Let f be an orientation-preserving periodic homeomorphism of the two-dimensional torus such that $\overline{B}_f \neq \varnothing$.

Let us find all admissible complete characteristics of the transformation f. Due to Lemma 3, we have that the number k of orbits of the set $\overline{B}_f$ should only be equal to 3 or 4. Consider these cases separately.

Case 1: $k = 4$. Substituting 4 for k in Eq. (2.5), we get $2n = \sum\limits_{i=1}^{4} n_i$. Due to $n_i \leqslant \frac{n}{2}$ Eq. (2.5) holds only if $n_i = \frac{n}{2}$ for all $i = 1, \ldots, 4$. Substituting $\frac{n}{2}$ for n_i in Eq. (1.6), we get $gcd\left(\frac{n}{2}, \frac{n}{2}, \frac{n}{2}, \frac{n}{2}, n\right) = 1$. Therefore, $n = 2$. Since $n_i < n$, we have $n_i = 1$ for each $i = \overline{1, 4}$. Hence, $\lambda_i = \frac{2}{1} = 2$. Since $d_i < \lambda_i$, we have $d_i = 1$ for each $i = \overline{1, 4}$. We obtain the complete characteristic of $(2, 1, 4, 1, 1, 1, 1, 1, 1, 1, 1)$, which we denote by κ_1.

Case 2: $k = 3$. Substituting 3 for k in Eq. (2.5), we get $n = \sum\limits_{i=1}^{3} n_i$. From this it is obvious that at least one n_i is not less than $\frac{n}{3}$. For definiteness put $n_1 \geqslant \frac{n}{3}$. Consider 2 admissible cases: a) $n_1 = \frac{n}{3}$ and b) $n_1 = \frac{n}{2}$.

a) $n_1 = \frac{n}{3}$. Then $n_2 + n_3 = \frac{2n}{3}$. Therefore, at least one of n_2, n_3 is not less than $\frac{n}{3}$. For definiteness put $n_2 \geqslant \frac{n}{3}$. Then a_1) $n_2 = \frac{n}{3}$ and $n_3 = \frac{n}{3}$, or a_2) $n_2 = \frac{n}{2}$ and $n_3 = \frac{n}{6}$.

a_1) Substituting $\frac{n}{3}$ for n_i in Eq. (1.6), we get $gcd\left(\frac{nd_1}{3}, \frac{nd_2}{3}, \frac{nd_3}{3}, n\right) = 1$. Hence, $n = 3$. Then, substituting the data $n_1 = n_2 = n_3 = 1$ for n_i in Eq. (1.5), we get $\sum\limits_{i=1}^{3} d_i \equiv 0 \pmod{n}$. Due to $d_i \in \{1, 2\}$ there are 2 admissible number collections: $d_1 = d_2 = d_3 = 1$ (the complete characteristic $(3, 1, 3, 1, 1, 1, 1, 1, 1)$, which we denote by κ_2), or $d_1 = d_2 = d_3 = 2$ (the complete characteristic $(3, 1, 3, 1, 1, 1, 2, 2, 2)$, which we denote by κ_3).

a_2) Taking Eq. (1.6) into account, we get $n = 6$. Using Eq. (1.5) and the fact that $d_i \in \{1, \ldots, 5\}$, we obtain 2 admissible number collections: $d_1 = d_2 = d_3 = 1$ (the complete characteristic $(6, 1, 3, 3, 2, 1, 1, 1, 1)$, which we denote by κ_4) and $d_1 = 1$, $d_2 = 2$, $d_3 = 5$ (the complete characteristic $(3, 1, 3, 1, 1, 1, 1, 2, 5)$, which we denote by κ_5).

b) $n_1 = \frac{n}{2}$. Then $n_2 + n_3 = \frac{n}{2}$, whence $\frac{n}{2} > n_2 \geqslant \frac{n}{4}$. Therefore, b_1) $n_2 = \frac{n}{3}$ and $n_3 = \frac{n}{6}$, or b_2) $n_2 = \frac{n}{4}$ and $n_3 = \frac{n}{4}$.

b_1) The complete characteristics for this number collection is found in (a_2).

b_2) For these number collections, taking Eq. (1.6) into account, we have $n = 4$. Using Eq. (1.5) and the fact that $d_i \in \{1, \ldots, 5\}$, we obtain 2 admissible number collections: $d_1 = d_2 = d_3 = 1$ (the complete characteristic $(4, 1, 3, 2, 1, 1, 1, 1, 1)$, which we denote by κ_6)

and $d_1 = 1$, $d_2 = d_3 = 3$ (the complete characteristic (4, 1, 3, 2, 1, 1, 1, 3, 3), which we denote by κ_7).

Let us prove the sufficiency. Let us show that for each complete characteristic κ_i ($i = \overline{1, 7}$) there is an orientation-preserving periodic homeomorphism of the two-dimensional torus. Using Eq. (1.2), we find that for such complete characteristics the genus of the modular surface is equal to 0. Therefore, a modular surface is the sphere.

Let us construct the homeomorphism f_1 satisfying the complete characteristic κ_1 thus found. We mark on the sphere (modular surface) 4 points: x_1, x_2, x_3, x_4, each of which is the projection of orbits with a period less than the period of the homeomorphism f_3. Construct an arc with the beginning at the point x_1 and the end at the point x_4 on the sphere such that it contains points x_2 and x_3 and the point x_2 is located between the points x_1 and x_3 (see Fig. 4a). Cutting the sphere along the constructed arc, we obtain a disk with the boundary $x_1x_2x_3x_4x_3x_2x_1$ (see Fig. 4b). Gluing two such disks along the boundary $x_3x_4x_3$, we get a square (see Fig. 4c). Having identified the sides of the square as shown in Fig. 4c), we obtain the two-dimensional torus (see Fig. 4d).

Define the homeomorphism f_1 by the following rule. Rotating disk 1 in Fig. 4c by the angle π in the counterclockwise direction, we map it into disk 2. Similarly, we map disk 2 into disk 1 by the angle π in the counterclockwise direction. The described map is an orientation-preserving periodic homeomorphism of the two-dimensional torus satisfying the complete characteristic κ_1.

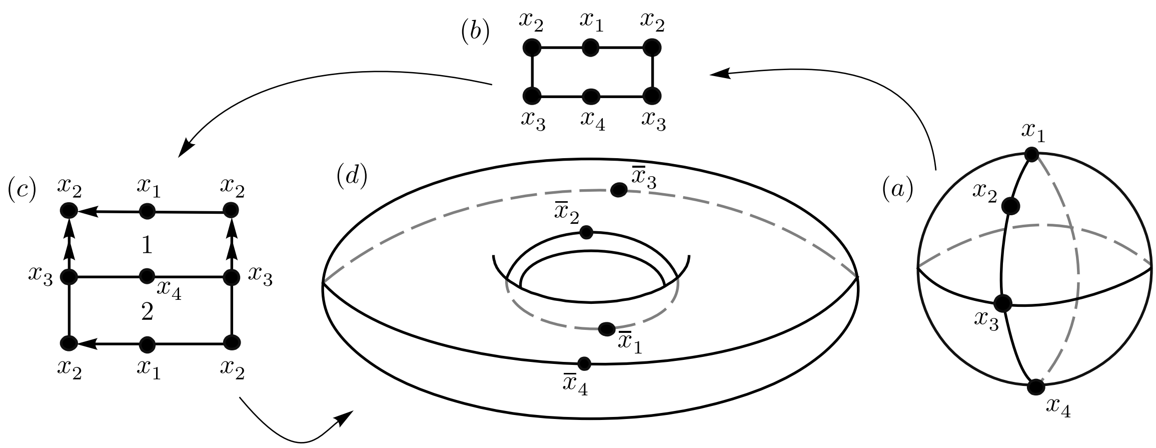

Fig. 4. The construction of the homeomorphism with the complete characteristic κ_1

Let us construct homeomorphisms f_2 and f_3 satisfying the complete characteristics κ_2 and κ_3, respectively. We mark 3 points on the sphere: x_1, x_2, x_3, each of which is the projection of orbits with a period less than the period of homeomorphisms f_2 and f_3. Construct an arc with the beginning at the point x_1 and the end at the point x_3 on the sphere such that it contains the point x_2 (see Fig. 5a). Cutting the sphere along the constructed arc, we obtain a disk with the boundary $x_1x_2x_3x_2x_1$ (see Fig. 5b). Gluing in pairs three such disks along the boundary x_1x_2, we obtain a hexagon (see Fig. 5c). Having identified the sides of the hexagon as shown in Fig. 5c), we obtain the two-dimensional torus (see Fig. 5d).

Define the homeomorphism f_2 by the following rule. Rotating disk 1 in Fig. 5c by the angle $\frac{2\pi}{3}$ in the counterclockwise direction, we map it into disk 2. Similarly, we map disk 2 into disk 3, and disk 3 into disk 1. Figure 5e$_1$ illustrates the action of the map in the neighborhood of a fixed point. The described map is an orientation-preserving periodic homeomorphism of the two-dimensional torus satisfying the complete characteristic κ_2.

Define the homeomorphism f_3 by the following rule. Rotating disk 1 in Fig. 5c by the angle $\frac{4\pi}{3}$ in the counterclockwise direction, we map it into disk 3. Similarly, we map disk 3 into disk 2, and disk 2 into disk 1. Figure 5e$_1$ illustrates the action of the map in the neighborhood of a fixed point. The described map is an orientation-preserving periodic homeomorphism of the two-dimensional torus satisfying the complete characteristic κ_3.

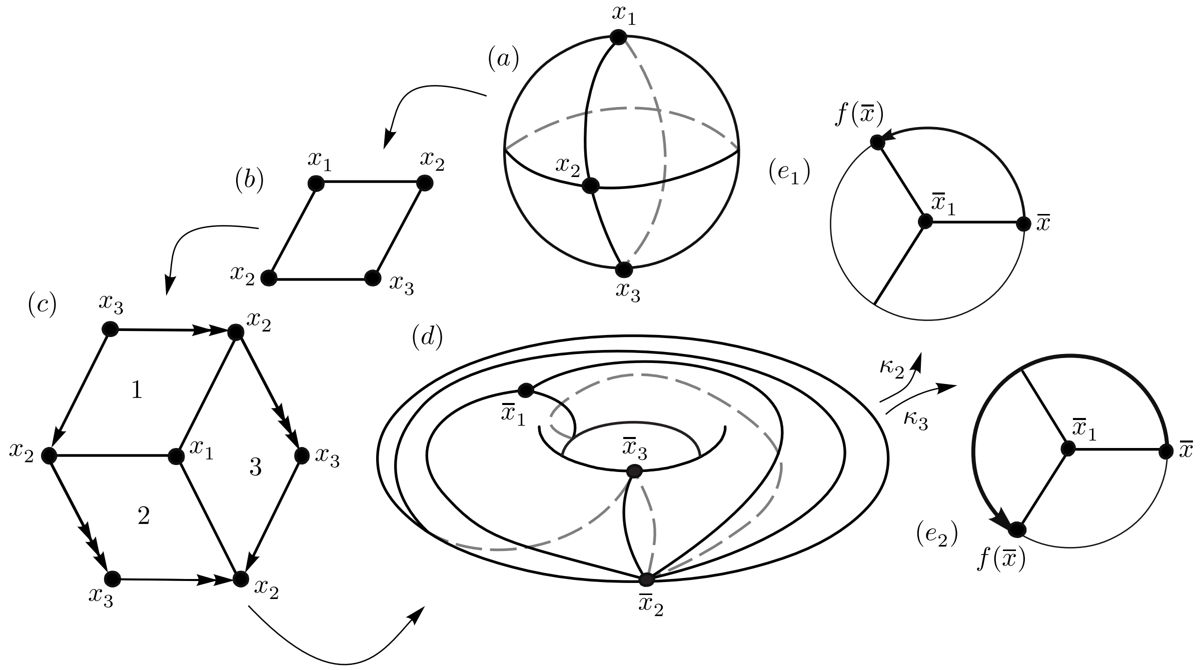

Fig. 5. The construction of homeomorphisms with complete characteristics κ_2 and κ_3

Let us construct homeomorphisms f_4 and f_5 satisfying the complete characteristics κ_4 and κ_5, respectively. We mark 3 points on the sphere: x_1, x_2, x_3, each of which is the projection of orbits with a period less than the period of homeomorphisms f_4 and f_5. Construct an arc with the beginning at the point x_2 and the end at the point x_3 on the sphere such that it contains the point x_2 (see Fig. 6a). Cutting the sphere along the constructed arc, we obtain a disk with the boundary $x_3x_1x_2x_1x_3$ (see Fig. 6b). Gluing in pairs six such disks along the boundary x_1x_3, we get a polygon (see Fig. 6c). Having identified the sides of the polygon as shown in Fig. 6c, we obtain the two-dimensional torus.

Define the homeomorphism f_4 by the following rule. Rotating disk 1 in Fig. 6c by the angle $\frac{\pi}{3}$ in the counterclockwise direction, we map disk 1 into disk 2. Similarly, we map disk 2 into disk 3, disk 3 into disk 4, disk 4 into disk 5, disk 5 into disk 6, and disk 6 into disk 1. Figure 6d$_1$ illustrates the location of orbits with a period less than the period of the homeomorphism f_4. The described map is an orientation-preserving periodic homeomorphism of the two-dimensional torus satisfying the complete characteristic κ_4.

Define the homeomorphism f_5 by the following rule. Rotating disk 1 in Fig. 6c by the angle $\frac{5\pi}{3}$ in the counterclockwise direction, we map disk 1 into disk 6. Similarly, we map disk 6 into disk 5, disk 5 into disk 4, disk 4 into disk 3, disk 3 into disk 2, and disk 2 into disk 1. Figure 6d$_2$ illustrates the location of orbits with a period less than the period of the homeomorphism f_5.

The described map is an orientation-preserving periodic homeomorphism of the two-dimensional torus satisfying the complete characteristic κ_5.

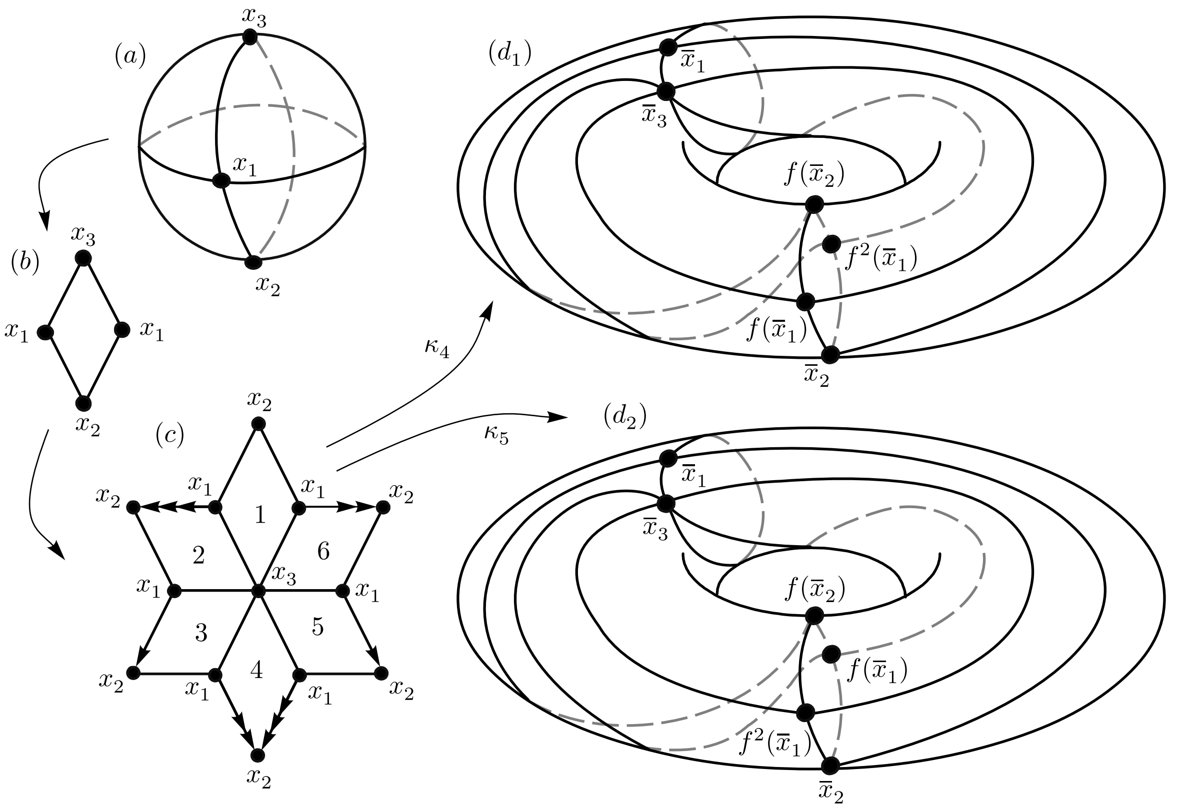

Fig. 6. The construction of homeomorphisms with complete characteristics κ_4 and κ_5

Let us construct homeomorphisms f_6 and f_7 satisfying the complete characteristics κ_6 and κ_7, respectively. We mark 3 points on the sphere: x_1, x_2, x_3, each of which is the projection of orbits with a period less than the period of homeomorphisms f_6 and f_7. Construct an arc with the beginning at the point x_2 and the end at the point x_3 on the sphere such that it contains the point x_3 (see Fig. 7a). Cutting the sphere along the constructed arc, we obtain a disk with the boundary $x_3x_1x_2x_1x_3$ (see Fig. 7b). Gluing in pairs four such disks along the boundary x_1x_3, we get a square (see Fig. 7c). Having identified the sides of the square as shown in Fig. 7c, we obtain the two-dimensional torus (see Fig. 7d).

Define the homeomorphism f_6 by the following rule. Rotating disk 1 in Fig. 7c by the angle $\frac{\pi}{4}$ in the counterclockwise direction, we map disk 1 into disk 2. Similarly, we map disk 2 into disk 3, disk 3 into disk 4, and disk 4 into disk 1. Figure 7e$_1$ illustrates the action of the map in the neighborhood of a fixed point. The described map is an orientation-preserving periodic homeomorphism of the two-dimensional torus satisfying the complete characteristic κ_6.

Define the homeomorphism f_7 by the following rule. Rotating disk 1 in Fig. 7c by the angle $\frac{3\pi}{4}$ in the counterclockwise direction, we map disk 1 into disk 4. Similarly, we map disk 4 into disk 3, disk 3 into disk 2, and disk 2 into disk 1. Figure 7e$_2$ illustrates the action of the map in the neighborhood of a fixed point. The described map is an orientation-preserving periodic homeomorphism of the two-dimensional torus satisfying the complete characteristic κ_7.

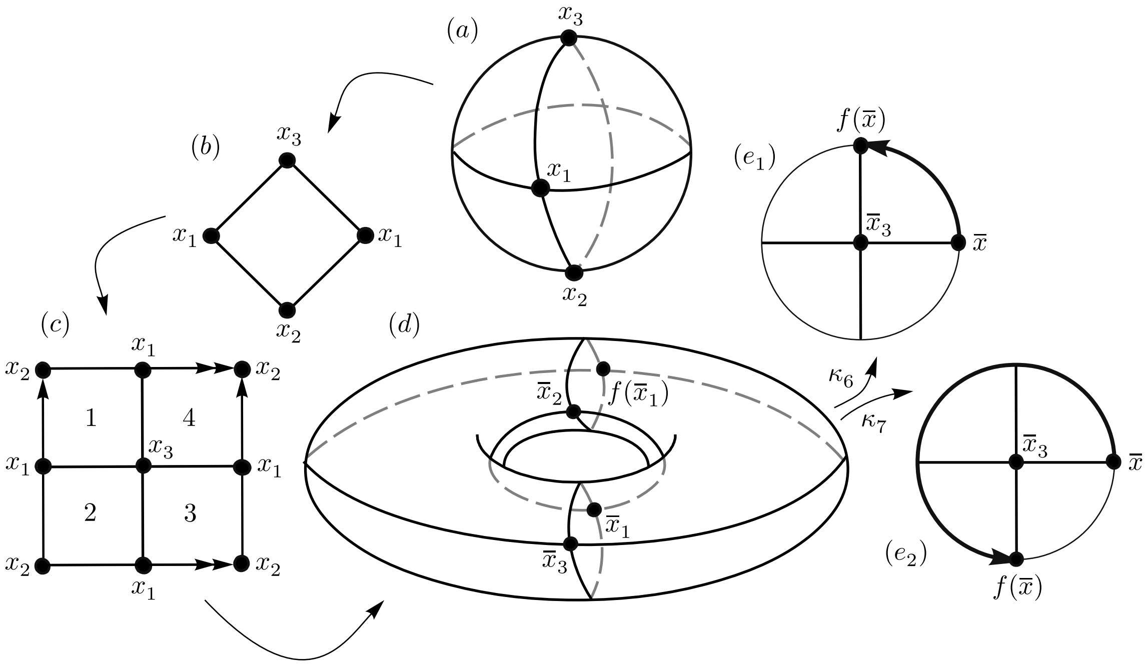

Fig. 7. The construction of homeomorphisms with complete characteristics κ_6 and κ_7

Remark 1. By construction the homeomorphism f_1 is mutually inverse to itself and each pair of homeomorphisms $\{f_2, f_3\}$, $\{f_4, f_5\}$ and $\{f_6, f_7\}$ is mutually inverse.

4. Classification of periodic homeomorphisms of the two-dimensional torus that are homotopic to the identity

In this section we prove Theorem 2.

Let the vector field $\xi(x, y)$ be defined and continuous at each of the points of the plane $\mathbb{R}^2$ except perhaps at some points. Denote by $M_0(x_0, y_0)$ the singular point of the vector field $\xi(x, y)$. Choose $r > 0$ such that the disk $d_r = \left\{(x, y) \in \mathbb{R}^2 \colon \sqrt{(x-x_0)^2 + (y-y_0)^2} \leqslant r\right\}$ does not contain singular points of the vector field different from the point $M_0(x_0, y_0)$. The number of rotations of the vector field $\xi|_{\partial d_r}$ in the counterclockwise direction in the neighborhood of the point M_0 is called the *index of the singular point* M_0 and is denoted by $I(M_0)$ (for detailed definitions, see [19, Section V, 10]).

Consider λ and δ such that $\lambda \in \mathbb{N}$, $\delta \in \{1, \ldots, \lambda-1\}$ and $(\delta, \lambda) = 1$. Then the index $I(M_0)$ of the fixed point M_0 of the plane rotation by the angle $\frac{2\pi\delta}{\lambda}$ in the counterclockwise direction is equal to δ if $\delta \leqslant \frac{\lambda}{2}$ and is equal to $-\delta$ if $\delta > \frac{\lambda}{2}$.

Let $z \in \mathbb{T}^2$ be an isolated fixed point of a continuous map $g \colon \mathbb{T}^2 \to \mathbb{T}^2$. Let $\varphi \colon U \to \mathbb{R}^2$ be a chart at z; put $\varphi(z) = w$. The vector field $f(x, y) = \varphi g \varphi^{-1}(x, y) - (x, y)$ is defined on a neighborhood of w in $\mathbb{R}^2$ and has w for an isolated zero. Define $Lef_z(g) = I(w)$. This is independent of (φ, U) (see [20, p. 139, Ex. 10]). Put $Lef(g) = \sum\limits_z Lef_z(g)$, where z is a fixed point of g.

For the surface S of genus p the map $f\colon S \to S$ homotopic to the identity, with a finite number of fixed points, we have *the Lefschetz – Hopf formula* [21]:

$$Lef(g) = 2 - 2p. \tag{4.1}$$

Proof of Theorem 2. Let $f\colon \mathbb{T}^2 \to \mathbb{T}^2$ be an orientation-preserving periodic homeomorphism of period $n \in \mathbb{N}$.

Let us prove that the following implications are true: $1 \to 2$, $2 \to 3$, $3 \to 4$, $4 \to 1$.

$1 \to 2$. Assume the converse. Let the homeomorphism f be homotopic to the identity and $\overline{B}_f \neq \varnothing$. In this case, taking Theorem 1 into account, we have 7 admissible complete characteristics κ_j $(j = \overline{1,\, 7})$ for the homeomorphism f. On the one hand, the homeomorphism f is homotopic to the identity and by Eq. (4.1) we get $Lef(f) = 0$. On the other hand, by direct calculation we get a nonzero value of $Lef(f)$ (see Table 4.2) for all admissible complete characteristics κ_j $(j = \overline{1,\, 7})$. This contradiction proves the implication.

κ	κ_1	κ_2	κ_3	κ_4	κ_5	κ_6	κ_7
$Lef(f)$	4	3	−6	1	−5	2	−6

(4.2)

$2 \to 3$. If $\overline{B}_f = \varnothing$, then it follows from Lemma 1 that $p - 1 = n(g - 1)$. Substituting 1 for p in this equality, we obtain $n(g - 1) = 0$. Therefore, $g = 1$.

$3 \to 4$. Substituting 1 for g and 1 for p in Eq. (1.2), we get $\sum\limits_{i=1}^{k} n_i = nk$. Using Eq. (2.3), we have $\sum\limits_{i=1}^{k} n_i \leqslant \frac{nk}{2}$. Therefore, $k = 0$ and $\overline{B}_f = \varnothing$. It follows from Statement 3 that the periodic homeomorphism f of period n is topologically conjugate to the diffeomorphism $\Psi_n\left(e^{i2x\pi},\, e^{i2y\pi}\right) = \left(e^{i2\pi(x+1/n)},\, e^{i2y\pi}\right)$, which has period n.

$4 \to 1$. By construction, the diffeomorphism $\Psi_n\left(e^{i2x\pi},\, e^{i2y\pi}\right) = \left(e^{i2\pi(x+1/n)},\, e^{i2y\pi}\right)$ acts identically on the fundamental group and, therefore, it is isotopic to the identity map (see, for example, [22]).

Remark 2. Implication $2 \to 3$ is known to specialists since it is a consequence of some homotopy facts about covering maps and surface topology, but we have found a proof of this fact using the technique implication from Lemma 1.

The following statement is a corollary to Theorem 1 and Theorem 2.

Corollary 1. Periodic homeomorphisms of the two-dimensional torus satisfying the complete characteristics κ_j $(j = \overline{1,\, 7})$ and only they are orientation-preserving periodic homeomorphisms of the two-dimensional torus that are nonhomotopic to the identity.

In each topological conjugacy class of orientation-preserving periodic homeomorphisms of the two-dimensional torus that are homotopic to the identity there is the representative $f_\alpha\colon \mathbb{T}^2 \to \mathbb{T}^2$, which is a translation of the two-dimensional torus by the vector $\alpha = (\alpha_1,\, \alpha_2)$ defined by the formula $f_\alpha(\overline{x},\, \overline{y}) = ((x + \alpha_1) \pmod 1;\, (y + \alpha_2) \pmod 1)$, where $\alpha_1,\, \alpha_2 \in \mathbb{Q}$. Necessary and sufficient conditions for the conjugacy of two translations on the 2-torus can be found in [17].

5. Classification of periodic algebraic automorphisms of the two-dimensional torus

Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL(2, \mathbb{Z})$, that is, let A be a second-order integer square matrix with $\det A = \pm 1$. Then A induces the map $f_A \colon \mathbb{T}^2 \to \mathbb{T}^2$ given by the formula

$$f_A \colon \begin{cases} \overline{x} = ax + by \pmod 1, \\ \overline{y} = cx + dy \pmod 1, \end{cases}$$

which is an *algebraic automorphism of the two-dimensional torus.*

If the eigenvalues of $A \in Gl(2, \mathbb{Z})$ are not equal in modulus to unity, then the algebraic automorphism of the two-dimensional torus induced by the matrix A is called a *hyperbolic* algebraic automorphism of the two-dimensional torus. Otherwise, the automorphism f_A is called a *nonhyperbolic* algebraic automorphism of the two-dimensional torus.

Two algebraic automorphisms of the 2-torus f and g are called *conjugate* if there exists an automorphism h such that $gh = hf$.

The set $K_f = \{hfh^{-1} \mid h \text{ is an algebraic automorphism of the 2-torus}\}$ is called the *conjugacy class* of the automorphism f.

We denote by $\mathbb{Z}^{2\times 2}$ the set of integer matrices of order 2. The matrix $B \in \mathbb{Z}^{2\times 2}$ is called *similar over* $\mathbb{Z}$ to the matrix $A \in \mathbb{Z}^{2\times 2}$ if there exists a matrix $S \in Gl(2, \mathbb{Z})$ such that $B = S^{-1}AS$. If $A \in \mathbb{Z}^{2\times 2}$, then the set $K_A = \{S^{-1}AS \mid S \in Gl(2, \mathbb{Z})\}$ is called the *similarity class of the matrix A*.

Thus, the problem of finding the conjugacy classes of nonhyperbolic algebraic automorphisms of the two-dimensional torus is reduced to the problem of finding the similarity classes of second-order integer unimodular matrices, whose eigenvalues are equal in modulus to unity. The problem of finding similarity classes for second-order integer unimodular matrices whose eigenvalues are roots of unity was solved in [23] in the form of the following statement:

Statement 4 ([23, Lemma 3]). *Let $A \in Gl(2, \mathbb{Z})$ and suppose that both eigenvalues of A are roots of unity. Then A is similar over $\mathbb{Z}$ to exactly one of the following matrices*:

$$\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & m \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix},$$
$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad m \in \{0, 1, 2, \ldots\}.$$

Lemma 4. *The eigenvalues of second-order integer unimodular matrices are equal to the root of unity if and only if they are equal in modulus to unity.*

Proof. Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in Gl(2, \mathbb{Z})$. Then $\Delta = ad - bc$ is a determinant of the matrix A ($\Delta = \pm 1$) and $f(\lambda) = \lambda^2 - \lambda(a + d) + ad - bc$ is a characteristic polynomial of the matrix A. Denote by $\lambda_{1,2}$ the eigenvalues of the matrix A.

Suppose that there exists $n \in \mathbb{N}$ such that $\lambda_i^n = 1$ ($i \in \{1, 2\}$). Then $|\lambda_i| = 1$.

Let us prove the statement in the opposite direction. Let $|\lambda_i| = 1$ ($i \in \{1, 2\}$).

In [24], Kronecker proved that, if each of the roots of an integer polynomial with the leading coefficient 1 is equal in modulus to unity, then these roots are roots of unity. Since $a, b, c, d \in \mathbb{Z}$ and $|\lambda_1| \cdot |\lambda_2| = |ad - bc| = 1$, we have that all eigenvalues of the matrix A are roots of unity. $\square$

The next corollaries follow from Statement 4 and Lemma 4.

Corollary 2. *Each conjugacy class of nonhyperbolic algebraic automorphisms of the two-dimensional torus is given by exactly one of the following matrices*:

$$M_1(m) = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}, \quad M_2(m) = \begin{pmatrix} -1 & m \\ 0 & -1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad M_4 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix},$$

$$M_5 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad M_6 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \quad M_7 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad m \in \{0, 1, 2, \ldots\}.$$

Recall that a nonidentity matrix A is called *periodic* if there exists a number $n \in \mathbb{N}$ such that $A^n = I$. The smallest of such n is called the *period* of the matrix A.

By direct calculation, we have that for $m \neq 0$ matrices $M_1(m)$ and $M_2(m)$ are not periodic. For $m = 0$, the matrix $M_1(m)$ is identical and the matrix $M_2(m)$ is a matrix of period 2. Matrices M_3 and M_4 are also matrices of period 2. The periods of matrices M_5, M_6 and M_7 are equal to 4, 3 and 6, respectively.

Corollary 3. *There are* 6 *classes of periodic algebraic automorphisms of the two-dimensional torus, each of which is given by exactly one of the following matrices*:

$$M_2(0) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad M_4 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix},$$

$$M_5 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad M_6 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \quad M_7 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

Put $A_1 = M_2(0)$, $A_2 = M_6^{-1}$, $A_3 = M_6$, $A_4 = M_7$, $A_5 = M_7^{-1}$, $A_6 = M_5^{-1}$, $A_7 = M_5$.

Remark 3. Consider the modular group $PSL(2, \mathbb{Z}) := SL(2, \mathbb{Z})/\{\pm I\}$ which acts on the upper-half of the complex plane by fractional linear transformations: $z \to \frac{az+b}{cz+d}$ $(a, b, c, d \in \mathbb{Z}, ad - bc = 1, z \in \mathbb{C}$ and $\mathrm{Im}(z) > 0)$ (see, for example, [13]). It is well known that the standard generators of this group are two matrices $S = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ so that $PSL(2, \mathbb{Z}) = \langle S, T \mid S^2 = I, (ST)^3 = I \rangle$.

Consider the projection $p\colon SL(2, \mathbb{Z}) \to PSL(2, \mathbb{Z})$. Let $A'_j = p(A_j)$ $(j = \overline{1, 7})$ be the image of the matrix A_j by p. Then we have $A'_1 = I$, $A'_6 = A'_7 = S$, $A'_3 = A'_4 = (ST)$, and $A'_2 = A'_5 = (ST)^2$.

Next we consider a subgroup $H' = \langle S \mid S^2 = I \rangle \times \langle (ST) \mid (ST)^3 = I \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \cong \mathbb{Z}_6$ of $PSL(2, \mathbb{Z})$.

Thus, there is a connection between algebraic automorphisms, induced matrices $A_1, \ldots, A_7$ and actions of homeomorphisms on the upper-half of the complex plane. The images of these matrices under p generate a cyclic subgroup H' of the group of all such homeomorphisms. It is noteworthy that automorphism f_{A_3} of period 3 and automorphism f_{A_4} of period 6 correspond to the same homeomorphism $z \to -\frac{1}{z+1}$, automorphism f_{A_2} of period 3 and automorphism f_{A_5} of period 6 correspond to homeomorphism: $z \to -1 - \frac{1}{z}$, automorphism f_{A_6} of period 4 and automorphism f_{A_7} of period 4 correspond to homeomorphism: $z \to -\frac{1}{z}$ and automorphism f_{A_1} of period 2 correspond to the identical map: $z \to z$ $(z \in \mathbb{C}$ and $\mathrm{Im}(z) > 0)$.

Proof of Theorem 3. It follows from Statement 2 that two periodic surface transformations are conjugate by an orientation-preserving homeomorphism if and only if their complete characteristics coincide. It follows from Corollary 1 that any orientation-preserving periodic homeomorphism of the two-dimensional torus that is nonhomotopic to the identity satisfies exactly one of the complete characteristics κ_j $(j = \overline{1, 7})$. Let us show that the map f_{A_j} has the complete characteristic $\varkappa_j$ and, therefore, any orientation-preserving periodic homeomorphism of the two-dimensional torus that is nonhomotopic to the identity is conjugate by an orientation-preserving homeomorphism with exactly one automorphism f_{A_j}.

Let us find the complete characteristic of the periodic transformation f_{A_5} induced by the matrix $A_5 = \left(\begin{smallmatrix} 1 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. The complete characteristics of automorphisms given by the remaining matrices are found in a similar way.

The period of A_5 is 6. Consequently, it induces a periodic homeomorphism of the two-dimensional torus f_{A_5} of period 6.

Let us find the set $\overline{B}_{f_{A_5}}$ with a period less than the period of the homeomorphism f_{A_5}. As mentioned before, the periods of such points are divisors of six, that is, they are fixed or have period 2 or 3.

We find fixed points from the system

$$\begin{cases} x = x + y \pmod 1, \\ y = -x \pmod 1, \end{cases}$$

decomposed into a countable set of systems

$$\begin{cases} x + k = x + y, & k \in \mathbb{Z}, \\ y + m = -x, & m \in \mathbb{Z}. \end{cases}$$

These systems are equivalent to

$$\begin{cases} x = -m - k, \\ y = k, \end{cases} \quad k \in \mathbb{Z}, \quad m \in \mathbb{Z}.$$

Hence, by direct calculation we see that the map f_{A_5} has a unique fixed point $p(0, 0)$, where $p\colon \mathbb{R}^2 \to \mathbb{T}^2$ is the natural projection.

Finding fixed points of the map $f^2_{A_5}$, induced by the matrix $A_5^2 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix}\right)$, we get points of period 2 of the map f_{A_5}.

By analogy with finding fixed points of the map f_{A_5}, we consider a countable set of systems

$$\begin{cases} x + k = y, & k \in \mathbb{Z}, \\ y + m = -x - y, & m \in \mathbb{Z}. \end{cases}$$

These systems are equivalent to

$$\begin{cases} x = \dfrac{1}{3}(-2m - k), \\ y = \dfrac{1}{3}(k - m), \end{cases} \quad k \in \mathbb{Z}, \quad m \in \mathbb{Z},$$

whence, by direct calculation we make sure that the map $f^2_{A_5}$ has three fixed points $p(0, 0)$, $p\left(\frac{1}{3}, \frac{1}{3}\right)$, $p\left(\frac{2}{3}, \frac{2}{3}\right)$. Therefore, the map f_{A_5} has a unique orbit of period two $\mathcal{O}_2 = \left\{p\left(\frac{1}{3}, \frac{1}{3}\right), p\left(\frac{2}{3}, \frac{2}{3}\right)\right\}$.

Finding fixed points of the map $f^3_{A_5}$ induced by the matrix $A_5^3 = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, we get points of period 3 of the map f_{A_5}.

By analogy with finding fixed points of the map f_{A_5}, we consider a countable set of systems

$$\begin{cases} x+k=-x, & k\in\mathbb{Z}, \\ y+m=-y, & m\in\mathbb{Z}. \end{cases}$$

These systems are equivalent to

$$\begin{cases} x=-\dfrac{1}{2}k, \\ y=-\dfrac{1}{2}m, \end{cases} \quad k\in\mathbb{Z}, \quad m\in\mathbb{Z},$$

whence, by direct calculation we have that the map $f^3_{A_5}$ has 4 fixed points: $p(0,\,0)$, $p\left(\frac{1}{2},\,\frac{1}{2}\right)$, $p\left(\frac{1}{2},\,0\right)$, $p\left(0,\,\frac{1}{2}\right)$. Therefore, the map f_{A_5} has a unique orbit $\mathcal{O}_3=\left\{p\left(\frac{1}{2},\,\frac{1}{2}\right),\,p\left(\frac{1}{2},\,0\right),\,p\left(0,\,\frac{1}{2}\right)\right\}$ of period 3.

Thus, the set $\overline{B}_{f_{A_5}}$ of the map f_{A_5} consists of three orbits: $\mathcal{O}_1=\{p(0,\,0)\}$, $\mathcal{O}_2 = \left\{p\left(\frac{1}{3},\,\frac{1}{3}\right),\,p\left(\frac{2}{3},\,\frac{2}{3}\right)\right\}$, $\mathcal{O}_3=\left\{p\left(\frac{1}{2},\,\frac{1}{2}\right),p\left(\frac{1}{2},\,0\right),\,p\left(0,\,\frac{1}{2}\right)\right\}$.

Consider the map $\widehat{A}_1$: $\begin{cases}\overline{x}=x+y,\\ \overline{y}=-y.\end{cases}$ This map is covering with respect to f_{A_5}. The point $O_1(0,\,0)$ is fixed under $\widehat{A}_1$.

In some neighborhood of the point O_1 we fix the point $Q(1,\,0)$. Under the action of $\widehat{A}_1$ the orbit of the point O_1 is the set

$$\mathcal{O}_Q=\{(1,\,0),\,(0,\,1),\,(-1,\,1),\,(-1,\,0),\,(0,\,-1),\,(1,\,-1)\}.$$

Connect each of the points of the set $\mathcal{O}_Q$ by means of a segment to the point O_1. Consider a circle of center O_1 such that it intersects each of the constructed segments at some point (see Fig. 8).

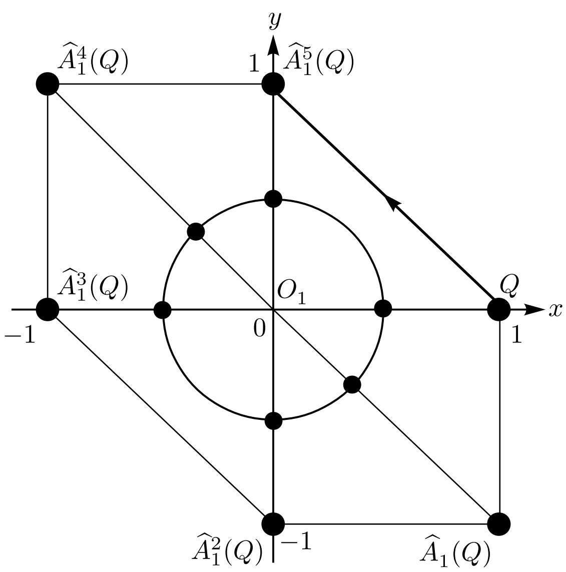

Fig. 8. The action of $\widehat{A}_1$ in some neighborhood of the fixed point

Connect points of the set $\mathcal{O}_Q$ in pairs by means of the segment following the counterclockwise order on the circle. We obtain a closed curve (see Fig. 8). It is an invariant curve of the map $\widehat{A}_1$. The open arc $(Q, \widehat{A}_1^5(Q))$ does not contain points of the orbit of the point Q in the counterclockwise direction. Therefore, $d_3 = 5$.

Consider the map $\widehat{A}_2\colon \begin{cases} \overline{x} = y, \\ \overline{y} = -x - y + 1. \end{cases}$ This map is covering with respect to $f_{A_5}^2$. The point $O_2\left(\frac{1}{3}, \frac{1}{3}\right)$ is fixed under $\widehat{A}_2$.

Analogously to finding d_3, we construct a closed curve invariant under the map $f_{A_5}^2$ and find the parameter $d_2 = 2$.

Consider the map $\widehat{A}_3\colon \begin{cases} \overline{x} = -x + 1, \\ \overline{y} = -y + 1. \end{cases}$ This map is covering with respect to $f_{A_5}^3$. The point $O_3\left(\frac{1}{2}, \frac{1}{2}\right)$ is fixed under $\widehat{A}_3$.

The map $\widehat{A}_3$ defines the rotation by the angle π about the point $O_3\left(\frac{1}{2}, \frac{1}{2}\right)$ of the plane. From this we find $d_1 = 1$.

Thus, the complete characteristic of the map f_{A_5} given by the matrix A_5 is the following collection of the numbers: $n = 6$, $n_1 = 3$, $n_2 = 2$, $n_3 = 1$, $d_1 = 1$, $d_2 = 2$, $d_3 = 5$ (the complete characteristic κ_5).

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