# Non-autonomous vector fields on $S^3$ : simple dynamics and wild separatrices embedding

V.Z. Grines<sup>1</sup>, L.M. Lerman<sup>1,2</sup>

<sup>1</sup>Higher School of Economics, Russia

e-mail: vgrines@yandex.ru

<sup>2</sup>Research and Educational Center "Mathematics for Future Technologies",

Lobachevsky National Research State University of Nizhny Novgorod,

e-mail: lermanl@mm.unn.ru

Abstract. We construct new substantive examples of non-autonomous vector fields on 3-dimensional sphere having a simple dynamics but non-trivial topology. The construction is based on two ideas: the theory of diffeomorpisms with wild separatrix embedding (Pixton, Bonatti-Grines, etc.) and the construction of a non-autonomous suspension over a diffeomorpism (Lerman-Vainshtein). As a result, we get periodic, almost periodic or even nonrecurrent vector fields which have a finite number of special integral curves possessing exponential dichotomy on  $\mathbb{R}$  such that among them there is one saddle integral curve (with an exponential dichotomy of the type (3,2)) having wildly embedded two-dimensional unstable separatrix and wildly embedded three-dimensional stable manifold. All other integral curves tend, as  $t \to \pm \infty$ , to these special integral curves. Also we construct another vector fields having  $k \geq 2$  special saddle integral curves with tamely embedded two-dimensional unstable separatrices forming mildly wild frames in the sense of Debrunner-Fox. In the case of periodic vector fields, corresponding specific integral curves are periodic with the period of the vector field, and they are almost periodic for the case of almost periodic vector fields.

## 1 Introduction

The aim of this paper is to present new substantive examples of non-autonomous periodic, almost periodic or even nonrecurrent vector fields on 3-dimensional sphere  $S^3$ . The main feature of constructed vector fields is that they have some saddle integral curve  $\gamma$  whose two-dimensional unstable and three-dimensional stable invariant manifolds wildly embed in the extended phase manifold  $S^3 \times \mathbb{R}$ . Thus, this provides new invariants of an uniform equivalence of non-autonomous vector fields (see definitions below). For the rest, from a dynamical viewpoint, the vector field has a quite simple structure of its foliation into integral curves (ICs, for briefness) in  $S^3 \times \mathbb{R}$ .

It will be seen from the construction that the method allows one to get non-autonomous uniformly dissipative vector fields in  $\mathbb{R}^3$  with a similar structure whose integral curves enter to some cylindrical domain of the form  $D^3 \times \mathbb{R}$ ,  $D^3 \subset \mathbb{R}^3$ , with its boundary manifold  $S^2 \times \mathbb{R}$  being uniformly transversal to integral curves.

To be more precise, let us recall the notion of a non-autonomous vector field on a smooth  $(C^{\infty})$  connected closed manifold M. Let  $\mathcal{V}^{r}(M)$  be a Banach space of  $C^{r}$ -smooth,  $r \geq 1$ , vector fields on M endowed with  $C^{r}$ -norm. By a  $C^{r}$ -smooth non-autonomous vector field on M (NVF, for brevity) it is understood an uniformly continuous bounded map  $v : \mathbb{R} \to \mathcal{V}^{r}(M)$ . We endow the set of non-autonomous vector fields with the supremum norm of the related maps. As a particular case, one may think on a periodic NVF, if the map v is periodic: there exist a positive  $T \in \mathbb{R}$  such that  $v(t + T) \equiv v(t)$  for all  $t \in \mathbb{R}$ . If the map v is almost periodic [17, 9], they say on an almost periodic non-autonomous vector field.

We recall a solution to the vector field v be a  $C^1$ -differentiable map  $x : I \to M$ , I is an interval of  $\mathbb{R}$ , such that for any  $t \in I$  tangent vector  $x'(t) = Dx_t(1) \in TM$  coincides with the vector  $v_t(x(t))$ . Here we identify in the standard way the tangent space  $T_t\mathbb{R}$  at the point  $t \in \mathbb{R}$  with  $\mathbb{R}$  itself by shifts in  $\mathbb{R}$ .

By the standard existence and uniqueness theorem, there is a unique solution through any initial point  $(x_0, \tau) \in M \times \mathbb{R}$ . On a closed manifold M any solution of v is extended on the whole  $\mathbb{R}$ . The graph of the map x, that is the set  $\cup_t (x(t), t) \subset M \times \mathbb{R}$ , is the integral curve of the solution x. Thus, every non-autonomous vector field v generates a foliation  $\mathcal{L}_v$ of manifold  $M \times \mathbb{R}$  into ICs of v. An example of such foliation for the case M = I is plotted on Fig. 1.

Following [4] we call two non-autonomous vector fields  $v_1, v_2$  on M to be uniformly equivalent, if there is an equimorphism (see subsection 5 for definition)  $\Phi : M \times \mathbb{R} \to M \times \mathbb{R}$ that transforms  $\mathcal{L}_{v_1}$  to  $\mathcal{L}_{v_2}$  preserving orientation in  $\mathbb{R}$ . Here we consider manifold  $M \times \mathbb{R}$ with the uniform structure of the direct product of the unique uniform structure on M given by the topology of the manifold M (it is a compact manifold) and the standard uniform structure on  $\mathbb{R}$  invariant w.r.t. shifts on the Abelian group (see subsection 5 for definitions of uniform structure and equimorphism).

NVFs, we construct below, fall into the class of gradient-like non-autonomous NVFs singled out in [4, 5]. They satisfy several restrictions on the structure of their foliation  $\mathcal{L}_v$ , one of them is the claim of an exponential dichotomy for any of their solutions on both semi-axes  $\mathbb{R}_+$  and  $\mathbb{R}_-$  (types of dichotomy may distinct) [13]. This assumption allows one to get invariant stable manifolds and also unstable manifolds. Thus, the whole extended phase manifold  $M \times \mathbb{R}$  is partitioned into smooth stable manifolds, another partition is generated by unstable manifolds. One more assumption is the finiteness of both partitions (though they can be completely different). We shall not deep into details of these restrictions, since in the examples we construct these restrictions are given explicitly.

The NVF on  $S^3$ , we shall construct, has four special ICs possessing an exponential dichotomy on the whole  $\mathbb{R}$ . One such IC  $\gamma_{\alpha}$  is exponentially unstable on  $\mathbb{R}$ , there is also the only IC  $\gamma_{\sigma}$  of a saddle type, it possesses an exponential dichotomy on  $\mathbb{R}$  of the type (3,2), that is, such IC has 3-dimensional stable manifold and 2-dimensional unstable manifold on  $\mathbb{R}$ . Finally, this NVF has two exponentially stable on  $\mathbb{R}$  ICs  $\gamma_{\omega_1}$ ,  $\gamma_{\omega_2}$  whose stable manifolds are 4-dimensional ones (the dichotomy of the type (1,4)).

The sense of the term "wild embedding" we use below for some embedded submanifold is the following. Take any section  $t = t_0$  in  $S^3 \times \mathbb{R}$ . Then the intersection of the wildly embedded 2-dimensional unstable manifold with the section is a 1-dimensional ray in  $S^3 \times \{t_0\}$  wildly

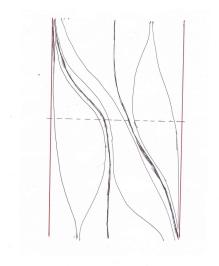


Figure 1: A foliation into IC on the segment, M = I.

embedded in the topological sense [1, 18, 6] (see below). The closure of this ray is a point being the trace of the exponentially stable IC  $\gamma_{\omega_2}$ . All ICs in this unstable manifold tend to  $\gamma_{\sigma}$  as  $t \to -\infty$  and to  $\gamma_{\omega_2}$  as  $t \to \infty$ . Also, the trace of 3-dimensional stable manifold of  $\gamma_{\sigma}$ on  $M_{t_0}$  is an embedded  $\mathbb{R}^2$  and its closure in  $M_{t_0}$  is an embedded sphere being wild at one point [].

The construction of such non-autonomous vector fields exploits two ideas. One belongs to Pixton [18] and further was developed by Bonatti, Grines, Pochinka and others [6, 7, 8]. In [18] a simple 3-dimensional Morse-Smale diffeomorphism on  $S^3$  was constructed that has, as its non-wandering set, only four hyperbolic fixed points: one source  $\alpha$ , one saddle  $\sigma$  of (2,1) type and two sinks  $\omega_1, \omega_2$ . Moreover, the closure of the stable manifold  $w^s_{\sigma}$  of  $\sigma$  is homeomorphic to the sphere smoothly embedded in each point except the point  $\alpha$  where it is wildly embedded (see, on the figure 2). The point  $\sigma$  divides unstable manifold  $w^u_{\sigma}$  into two separatrices  $l_1^u, l_2^u$ . The closure of one of these separatrices  $(l_2^u \text{ on the figure 2})$  is a simple arc wildly embedded in the point  $\omega_2$  but the closure of another one  $(l_1^u)$  is tamely embedded arc in each point. The separatrix  $l_2^u$  tends to the sink  $\omega_2$  in such a way that the fundamental domain near this sink after identifying points on the boundary of this domain contains the image of the separtrix which makes up a nontrivial knot in the related factor space (being the manifold  $S^2 \times S^1$ ). This will be explained in more details in section 2. The non-triviality of this knot implies the wild embedding of the separatrix for the diffeomorphism, moreover, if knots for two different diffeomorphisms are non-homeomorphic in  $S^2 \times S^1$ , then they are not topologically conjugated. This fact leads to the existence of infinite number of topologically non-conjugated Morse-Smale diffeomorphisms of considered types [6].

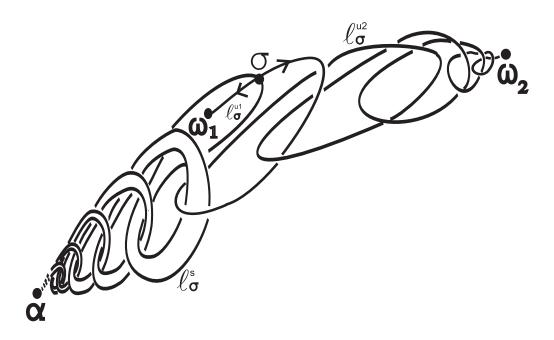


Figure 2: Phase portrait of diffeomorphism on  $S^3$  with wildly embedded separatices.

The second idea is borrowed from [11]. It uses the construction of a so-called nonautonomous suspension over a diffeomorphism introduced in [10] and developed further in [11]. Recall this construction. Suppose  $f: M \to M$  be some diffeomorphism of a smooth  $(C^{\infty})$  closed manifold M. To avoid a discussion on the class of smoothness for the suspension, we assume f to be  $C^{\infty}$ -smooth. Its (usual) suspension is a smooth closed manifold  $M_f$ of dimension  $\dim M + 1$  with a flow defined as follows. Let us identify in the cylinder  $M \times I$ , I = [0, 1], points (x, 1) and (f(x), 0). It is more convenient to consider the manifold  $M \times \mathbb{R}$  with an action F of the group Z by the rule: for  $m \in \mathbb{Z}$  the related  $F^m$  acts as  $F^m(x,s) = (f^m(x), s - m)$ . This action is free and discrete (any orbit of the action has not accumulation points). Thus, the factor-manifold  $M_f = (M \times \mathbb{R})/F$  is a smooth manifold being a smooth bundle over the circle  $S^1$ ,  $p: M_f \to S^1$ , with the leaf M. A vector field on  $M_f$  is generated by the constant vector field V = (0, 1) on  $M \times \mathbb{R}$  (its orbits are straightlines  $(x, t), t \in \mathbb{R}$ ). After factorizing one gets a smooth vector field  $v_f$  on  $M_f$  with the global cross-section, as such one can choose any  $M_{\theta} = p^{-1}(\theta), \ \theta \in S^1$ . The Poincaré map defined on this cross-section is conjugated to f. This construction allows one to get vector fields with the dynamics similar to that as for iterations of the mapping f, see [22].

To proceed let us consider the covering manifold  $M_f$  for  $M_f$  generated by the standard covering  $\mathbb{R} \to S^1$ ,  $t \to \exp[2\pi i t]$  that gives a commutative diagram

$$\begin{array}{ccc} \tilde{M}_f & \stackrel{\text{exp}}{\longrightarrow} & M_f \\ & & \downarrow^{\tilde{p}} & & \downarrow^{p} \\ & & \mathbb{R} & \stackrel{\text{exp}}{\longrightarrow} & S^1 \end{array}$$

Topologically,  $\tilde{M}_f$  is homeomorphic to  $M \times \mathbb{R}$ , since  $\mathbb{R}$  is contractible. The manifold  $M_f$  is compact, hence it has the unique uniform structure compatible with the topology [16]. The uniform structure in  $\tilde{M}_f$  is defined by lifting the uniform structure in  $M_f$  by means of the

map  $\widetilde{\exp}$ . This is more easily understood, if we endow  $M_f$  with a smooth Riemannian metrics and lift this metrics to  $\tilde{M}_f$  by the covering map  $\widetilde{\exp}$ . Since  $\widetilde{\exp}$  is a local diffeomorphism, we get a Riemannian metrics on  $\tilde{M}_f$  such that  $\widetilde{\exp}$  is the local isometry. The foliation in  $M_f$  into orbits of the vector field  $v_f$  is lifted as some foliation  $\mathcal{L}_{v_f}$  into infinite curves in  $\tilde{M}_f$ . This foliation is homeomorphic to the foliation of  $M \times \mathbb{R}$  into straight-lines  $(x, t), t \in \mathbb{R}$ , since  $\tilde{M}_f$  is homeomorphic to  $M \times \mathbb{R}$ , but generically the foliation in  $\tilde{M}_f$  is not equimorphic to the foliation into straight-lines. Moreover, even the manifold  $\tilde{M}_f$  itself with its uniform structure lifted from  $M_f$  is not always equimorphic to  $M \times \mathbb{R}$ . For instance, it is the case for  $M = T^2 = \mathbb{R}^2/\mathbb{Z}^2$  with f being an Anosov diffeomorphism (see details in [11]). Next proposition which is Corollary 4.1 from [11] will be useful for us.

**Proposition 1** If  $\tilde{M}_f$  is equimorphic to  $M \times \mathbb{R}$ , then there is such  $n \in \mathbb{Z}$  that  $f^n$  is homotopic to  $id_M$ .

Let a diffeomorphism  $f : M \to M$  on a smooth  $(C^{\infty})$  closed manifold M be given. An important question here is if there exists a non-autonomous vector field v on M such that its foliation  $\mathcal{L}_v$  into ICs in  $M \times \mathbb{R}$  (with its uniform structure of the direct product) is equimorphic to the foliation  $\mathcal{L}_{v_f}$  into infinite curves generated by vector field  $v_f$  in  $\tilde{M}_f$ ? It is evident the first condition this to be true is an equimorphness of uniform spaces  $M \times \mathbb{R}$ and  $\tilde{M}_f$ . This gives a meaning to the definition introduced in [11]

**Definition 1** A diffeomorphism  $f : M \to M$  is reproduced by a non-autonomous vector field v on M (or, equivalently, v reproduces the structure of f), if foliations  $\mathcal{L}_v$  in  $M \times \mathbb{R}$  and  $\mathcal{L}_{v_f}$  in  $\tilde{M}_f$  are equimorphic. In particular, this implies the uniform spaces  $M \times \mathbb{R}$  and  $\tilde{M}_f$  be equimorphic.

**Remark 1** It follows from Proposition 2.5 (item a) in [11] that diffeomorphisms f and  $f^n$  are reproduced simultaneously for any  $n \in \mathbb{Z}$ .

**Remark 2** As is known, for a given diffeomorphism  $f : M \to M$  it is possible that its suspension  $M_f$  is not diffeomorphic to the direct product  $M \times S^1$  but for some its iteration  $f^n$  the related suspension  $M_{f^n}$  is diffeomorphic to  $M \times S^1$ . In fact, the manifold  $M_{f^n}$  is the k-fold covering of  $M_f$ . As a simplest example of such situation, one can take  $M = S^1$  with a coordinate  $\varphi \pmod{2\pi}$  and a diffeomorphism  $f(\varphi) = -\varphi(\mod 2\pi)$ . Then  $S_f^1$  is the Klein bottle but  $f^2(\varphi) = \varphi$ , hence  $S_{f^2}^1 = S^1 \times S^1 = \mathbb{T}^2$  (that is, a 2-dimensional torus being a 2-fold covering of the Klein bottle). Thus, according to remark 1 the manifold  $\tilde{S}_f^1$  is equimorphic to  $S^1 \times \mathbb{R}$ .

Denote  $\pi_M : M \times \mathbb{R} \to M$  the standard projection onto the first co-factor. For any non-autonomous vector field v on the manifold M the map  $\Phi_0^t : M_0 \to M_t$  from the section  $M_0 = M \times \{0\}$  to the manifold  $M_t = M \times \{t\}$ , generated by solutions of v with initial points on  $M_0, \pi_M(M_t) = M$ , is diffeotopic to the identity map  $id_M$  for all  $t \in \mathbb{R}$ . In particular, if vis a periodic vector field on manifold M, then its Poincaré map over the period is diffeotopic to identity map  $id_M$ .

Our first result is theorem 1 below which gives sufficient condition for diffeomorphism f to be reproduced by a flow generated by a non-autonomous periodic vector field v.

First we formulate an obvious lemma.

**Lemma 1** If f is diffeotopic to  $id_M$ , then there is a diffeotopy  $F_t : M \to M, t \in [0, 1]$ , joining  $id_M$  and f such that diffeomorphisms  $F_t$  depend smoothly on t and for some  $\varepsilon > 0$  small enough one gets  $F_t \equiv id_M$ , as  $t \in [0, \varepsilon]$ , and  $f_t \equiv f$  as  $t \in [1 - \varepsilon, 1]$ .

**Theorem 1** Suppose for some  $n \in \mathbb{N}$  diffeomorphism  $f^n : M \to M$  be diffeotopic to the identity map  $id_M$ . Then

- 1.  $M_f$  is fiber-wisely<sup>1</sup> diffeomorphic to  $M \times S^1$ ;
- 2. there is a periodic vector field v on M such that v reproduces the structure of f.

**Proof.** To ease the exposition we assume that f itself is diffeotopic to  $id_M$ . At the first step we shall construct a 1-periodic vector field  $v_t$  on M such that the vector field on  $M \times S^1$ given as  $(v_t, 1)$  is diffeomorphic to the vector field of the suspension of f on  $M_f$ . To this end, we need to endow the manifold  $M_f$  with the structure of the direct product in the explicit form. This means that one needs to define two foliations of  $M_f$ . One of them is given by leaves of the bundle over  $S^1$ , these leaves are diffeomorphic to M. The second foliation into closed curves is defined as follows. Suppose  $F_t: M \to M, t \in [0,1]$ , be a diffeotopy  $F_t: M \to M$  joining  $id_M$  and f, that is,  $F_t$  are diffeomorphisms,  $F_0 = id_M$  and  $F_1 = f$ . We assume, by Lemma 1, that  $F_t = id_M$  for  $t \in [0, \varepsilon]$  and  $F_t = f$  for  $t \in [1 - \varepsilon, 1]$ . If  $p: M_f \to S^1$  is the bundle map, then for any point  $x \in p^{-1}(0)$  we define the curve through point  $(x,0) \in M \times [0,1]$  given as  $(F_t^{-1}(x),t)$  for  $t \in [0,1]$ , and then we apply the factor map using the identification (x,t) = (f(x), t-1). The curve in  $M \times \mathbb{R}$  with the initial point (x,0) has the extreme point  $(F_1^{-1}(x),1) = (f^{-1}(x),1)$  at t = 1. After identifying this point becomes  $(f \circ f^{-1}(x), 0) = (x, 0)$ . Thus, all curves constructed in the manifold  $M_f$  are closed and we get the homeomorphism  $h: M_f \to M \times S^1$ . The map h is defined as follows. Take any point  $a \in M_f$  and denote  $l_a$  a closed curve through the point a of the second foliation constructed. Define a map  $p_1: M_f \to M_0, M_0 = p^{-1}(0)$ , by the rule  $p_1(a) = l_a \cap M_0$ . We get a homeomorphism  $h: a \to (p_1(a), p(a))$ .

Generically, h is a homeomorphism, since the dependence of  $F_t$  on t can be only continuous. But we need a diffeomorphism between bundles  $M_f$  and  $M \times S^1$  in order orbits of the suspended flow would be transformed to smooth curves in  $M \times S^1$ . Lemma 1 above guarantees that if  $f: M \to M$  is diffeotopic to  $id_M$ , then a diffeotopy  $F_t$  exists joining  $id_M$ and f such that curves constructed above give a smooth foliation, that is, curves are smooth and their dependence of the points is smooth, so the map  $p_2$  is smooth. This proves the first item of the theorem.

Now we construct a periodic vector field v on M such that its foliation  $\mathcal{L}_v$  into ICs is uniformly diffeomorphic to the foliation  $\mathcal{L}_{v_f}$  into infinite curves in  $\tilde{M}_f$ . Diffeomorphism  $h: M_f \to M \times S^1$ , defined above, allows us to identify  $M_f$  with  $M \times S^1$ . Thus, the suspended vector field is given as (V(x,t), n(x,t)) with V, n being 1-periodic in t and n > 0. Its flow therefore has a cross-section, for instance, as such one can take the section t = 0. The Poincaré map  $g: M_0 \to M_0$  on this cross-section is evidently conjugated to f. Hence, we can consider instead of diffeomorphism f on M the diffeomorphism g. Since f, g are conjugated, their non-autonomous suspensions are equimorphic along with the related foliations.

<sup>&</sup>lt;sup>1</sup>The term "fiber-wisely" means the existence of a diffeomorphism  $\Psi: M_f \to M \times S^1$  acting as  $(x, s) \to (\psi(x, s), s)$ .

Compactness of M and  $S^1$  implies that n is strictly positive. We can define a periodic vector field on M as v(x,t) = V(x,t)/n(x,t). Integral curves in  $M \times \mathbb{R}$  of this periodic vector field coincide with orbits the vector field (V(x,t), n(x,t)) since they are obtained by the change of time being uniformly bounded from above and below. So, the item 2 has also been proved.

## 2 Diffeomorphisms with wildly embedded separatricies

This section contains some definitions and results which are contained in the book [8], we present them here for of the reader convenience.

### 2.1 Wild embedding

**Definition 2** A topological embedding  $\lambda : X \to Y$  of an m-dimensional manifold X into a n-dimensional manifold Y ( $m \leq n$ ) is said to be locally flat at the point  $\lambda(x) \in Y$ , if there is a chart  $(U, \psi), \lambda(x) \in U, \psi : U \to \mathbb{R}^n$ , in the manifold Y such that  $\psi(\lambda(X) \cap U) = D^m \subset \mathbb{R}^m$ , here  $\mathbb{R}^m \subset \mathbb{R}^n$  is the set of points for which the last n - m coordinates equal to zero or  $\psi(\lambda(X) \cap U) = \mathbb{R}^m_+ (\mathbb{R}^m_+ \subset \mathbb{R}^m)$  is the set of points with non-negative last coordinate).

Now, an embedding  $\lambda$  is said to be *tame* and the manifold X is said to be *tamely embedded*, if  $\lambda$  is locally flat at every point  $\lambda(x) \in Y$ . Otherwise, the embedding  $\lambda$  is said to be *wildl* and the manifold X is said to be *wildly embedded*. If the embedding  $\lambda$  is not locally flat at the point  $\lambda(x)$ , this point is said to be a *point of wildness*.

It is worth remarking that the definition of a tamely embedded manifold coincides with the definition of a topological submanifold.

Every topological embedding into the space  $\mathbb{R}^2$  (respectively,  $S^2$ ) is tame. In the space  $\mathbb{R}^3$  (respectively,  $S^3$ ) there are wild arcs and wild 2-spheres. As an example of a wild arc, we recall the construction by Artin and Fox [1]. The related arc is smooth everywhere except for its boundary point.

Consider a linear contraction  $\phi : \mathbb{R}^3 \to \mathbb{R}^3$  defined in spherical coordinates  $(\rho, \varphi, \theta)$  as  $\phi(\rho, \varphi, \theta) = (\frac{1}{2}\rho, \varphi, \theta)$ , and denote  $L \subset \mathbb{R}^3$  a spherical layer defined by inequalities  $\frac{1}{2} \leq \rho \leq 1$ . Its boundary spheres are  $V_{\frac{1}{2}} = \{(\rho, \varphi, \theta) | \rho = \frac{1}{2}\}$  and  $V_1 = \{(\rho, \varphi, \theta) | \rho = 1\}$ . Let  $a, b, c \subset L$  be pairwise disjoint simple arcs with their respective boundary points  $\alpha_1, \alpha_2; \beta_1, \beta_2; \gamma_1, \gamma_2$  (see Figure 3 (a)) such that

- 1.  $\alpha_1, \alpha_2, \gamma_1 \subset V_1; \beta_1, \beta_2, \gamma_2 \subset V_{\frac{1}{2}};$
- 2.  $\phi(\alpha_1) = \gamma_2, \ \phi(\alpha_2) = \beta_1, \ \phi(\gamma_1) = \beta_2.$

Let us choose arcs a, b, c in such a way that the arc  $\ell_O \subset \mathbb{R}^3$  defined as  $\ell_O = \bigcup_{k \in \mathbb{Z}} \phi^k (a \cup b \cup c) \cup O$  (see Figure 3 (b)), is smooth at every point except O. Artin and Fox proved that  $\ell_O$  is wildly embedded into  $\mathbb{R}^3$  and the point O is point of wildness. This fact also follows from the criterion below proved in [14].

**Proposition 2** Let  $\ell$  be a compact arc in  $\mathbb{R}^3$  which is smooth everywhere except its boundary point O. Then  $\ell$  is locally flat at O, iff for every  $\varepsilon$ -ball  $B_{\varepsilon}(O)$  centered at O there is a subset  $U \subset B_{\varepsilon}(O)$  diffeomorphic to the closed 3-ball such that O is an interior point of U, and the intersection  $\partial U \cap \ell$  is the only point.

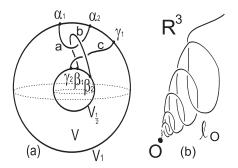


Figure 3: Constructions of wild curves in  $\mathbb{R}^3$ 

Now we consider a standard sphere  $S^3 = \{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 | x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$ . The point  $N(0, 0, 0, 1) \in S^3$  (respectively,  $S(0, 0, 0, -1) \in S^3$ ) will be called the north (respectively, the south) pole.

For each point  $x \in S^3 \setminus \{N\}$  there is the unique straight line in  $\mathbb{R}^4$  containing N and x. This line cuts the plane  $(x_1, x_2, x_3, 0)$  at exactly one point  $\vartheta(x)$ ,  $\vartheta(x_1, x_2, x_3, x_4) = \frac{x_1}{1-x_4}, \frac{x_2}{1-x_4}, \frac{x_3}{1-x_4}$ . The stereographic projection of the point x is defined as the point  $\vartheta(x)$ . The stereographic projection is a diffeomorphism of  $S^3 \setminus \{N\}$  to  $\mathbb{R}^3$  (see Figure 4, where it is shown the stereographic projection from  $S^2 \setminus \{N\}$  to  $\mathbb{R}^2$ ).

Let  $\ell = \vartheta_+^{-1}(\ell_O) \cup S$  (see Figure 5 (a)), then the arc  $\ell_N(\ell_S)$  in Figure 5 (b) is a sub-arc of the arc  $\ell$  from the point  $\vartheta_+^{-1}(\alpha_1)$  to the point N (from the point  $\vartheta_+^{-1}(\alpha_1)$  to the point S). The arc  $\ell_N(\ell_S)$  is wildly embedded into  $\mathbb{S}^3$ .

Now we can inflate the arcs on Figure 5 (a), (b) to get closed 3-balls whose boundaries are 2-spheres wildly embedded to  $\mathbb{S}^3$  and whose wild points are the poles.

### 2.2 Diffeomorphisms of Pixton type on $S^3$

Let V be a smooth closed orientable 3-manifold whose fundamental group admits a nontrivial homomorphism  $\eta_V : \pi_1(V) \to \mathbb{Z}$ . Denote by  $(V, \eta_V)$  the manifold V equipped with the

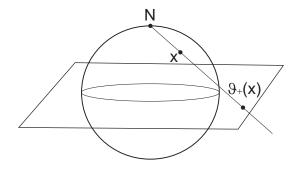


Figure 4: The stereographic projection

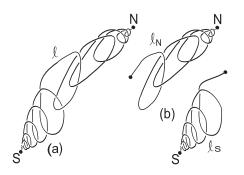


Figure 5: Constructions of wild curves in  $\mathbb{S}^3$ 

homomorphism  $\eta_{v}$ .

**Definition 3** Manifolds  $(V, \eta_V)$  and  $(V', \eta_{V'})$  are said to be equivalent, if there is a homeomorphism  $\varphi: V \to V'$  such that  $\eta_{V'}\varphi_* = \eta_V$ .

**Definition 4** Two smooth submanifolds  $a \subset V$  and  $a' \subset V'$  are said to be equivalent, if there is a homeomorphism  $\varphi: V \to V'$  such that  $\eta_{V'}\varphi_* = \eta_V$  and  $\varphi(a) = a'$ .

**Definition 5** A smooth submanifold  $a \subset V$  is said to be  $\eta_V$ -essential, if  $\eta_V(i_{a*}(\pi_1(a))) \neq 0$ , where  $i_a : a \to V$  is the inclusion map.

Let us illustrate these definitions for the manifold  $S^2 \times S^1$ . We represent the manifold  $S^2 \times S^1$  as the orbit space of the homothety  $a^s \mapsto a^s(x) = 0.5x$   $(x = (x_1, x_2, x_3))$ ,  $(\mathbb{R}^3 \setminus \{O\})/a^s$ . It is easy to check that the natural projection  $p : \mathbb{R}^3 \setminus O \to S^2 \times S^1$  is the covering map, it induces the epimorphism  $\eta_{S^2 \times S^1} : \pi_1(S^2 \times S^1) \to \mathbb{Z}$ .<sup>2</sup>

Denote  $\gamma_0 = p(Ox_1^+), \lambda_0 = p(Ox_2x_3)$ , where  $Ox_1^+$  is positive semi-axis and  $Ox_2x_3$  is coordinate plane  $x_1 = 0$ . On Fig.6 it is shown the spherical layer bounded by spheres of radii 1 and 0.5. If we identify points which lie on the boundary of the spherical layer and belong to the same ray through O, we get the manifold  $S^2 \times S^1$ . Moreover, if we identify the extreme points of the segment with the same numbers (1), we get the knot  $\hat{\gamma}_0$  and if we identify extreme points lying on the same ray that belongs to circles with the same numbers (2) and bounding 2-annulus, we get the torus  $\hat{\lambda}_0$  ( $\hat{\gamma}_0$  and  $\hat{\lambda}_0$  are embedded to  $S^2 \times S^1$ ).

It is easy to check that  $\hat{\gamma}_0$  (respectively,  $\hat{\lambda}_0$ ) is a  $\eta_{\mathbb{S}^2 \times \mathbb{S}^1}$ -essential knot (respectively, torus) in the manifold  $(S^2 \times S^1, \eta_{\mathbb{S}^2 \times \mathbb{S}^1})$ .

**Definition 6** A knot (torus)  $\hat{\gamma}(\hat{\lambda})$  in the manifold  $(S^2 \times S^1, \eta_{\mathbb{S}^2 \times \mathbb{S}^1})$  is said to be trivial if it is equivalent to the knot (torus)  $\hat{\gamma}_0(\hat{\lambda}_0)$ .

**Proposition 3** Every  $\eta_{S^2 \times S^1}^s$ -essential torus  $\hat{\lambda} \subset (S^2 \times S^1, \eta_{S^2 \times S^1}^s)$  bounds a solid torus in  $S^2 \times S^1$ .

<sup>&</sup>lt;sup>2</sup>Take the homotopy class  $[c] \in \pi_1(S^2 \times S^1)$  of a loop  $c: \mathbb{R}/\mathbb{Z} \to S^2 \times S^1)$ . Then  $c: [0,1] \to S^2 \times S^1$  lifts to a curve  $c: [0,1] \to \mathbb{R}^3 \setminus \{O\}$  joining a point x with a point  $(a^s)^n(x)$  for some  $n \in \mathbb{Z}$ , where n is independent of the lift. So, we define  $\eta^s_{S^2 \times S^1}([c]) = n$ .

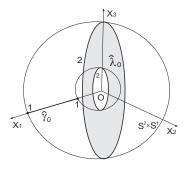


Figure 6: Construction of an essential knot and torus embedded in  $S^2 \times S^1$ 

**Proposition 4** A knot  $\hat{\gamma}$  (torus  $\hat{\lambda}$ ) in the manifold  $(S^2 \times S^1, \eta^s_{S^2 \times S^1})$  is trivial if and only if there is a tubular neighborhood  $N(\hat{\gamma})$   $(N(\hat{\lambda}))$  of it in the manifold  $S^2 \times S^1$  such that the manifold  $(S^2 \times S^1) \setminus N(\hat{\gamma})$   $((S^2 \times S^1) \setminus N(\hat{\lambda}))$  is homeomorphic to the solid torus (a pair of the solid tori).

Denote by  $\mathcal{P}$  the class of the Morse-Smale diffeomorphisms whose non-wandering set consists of the source  $\alpha_f$ , the saddle  $\sigma_f$  and the sinks  $\omega_f^1$ ,  $\omega_f^2$ . The phase portrait of a diffeomorphism of the class  $\mathcal{P}$  is shown in Figure 7. Pixton has constructed example from class  $\mathcal{P}$  mentioned above, so we call the class  $\mathcal{P}$  the *Pixton class*. We omit below the index f in the notations of fixed points.

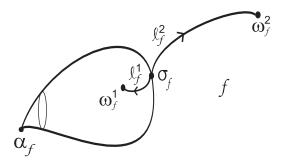


Figure 7: The phase portrait of a diffeomorphism of the class  $\mathcal{P}$ 

A surprising fact is the existence of a countable set of non-conjugated diffeomorphisms in the class  $\mathcal{P}$ . To understand this, we describe below knot topological invariant suggested in [6]. Moreover, this invariant explains existence in the class  $\mathcal{P}$  of diffeomorphisms for which a saddle fixed point possesses wildly embedded one-dimensional and two-dimensional separatrices.

Denote by  $\ell_1$ ,  $\ell_2$  the unstable 1-dimensional separatrices of the point  $\sigma$ . It follows from Smale [12] that the closure  $cl(\ell_i)$  (i = 1, 2) is homeomorphic to a simple compact arc which consists of the separatrix itself and two its extreme points:  $\sigma$  and the sink (see Proposition 2.3 in [8]). Moreover, the closures of the separatrices  $\ell_1$  and  $\ell_2$  contain different sinks (see Corollary 2.2 in [8]). To be definite, let  $\omega_i$  belongs to  $cl(\ell_i)$  (see Figure 7). For i = 1, 2 denote  $V_i = W^s(\omega_i) \setminus \{\omega_i\}$ . Denote by  $\hat{V}_i = V_i/f$  the corresponding orbit space and let  $p_i : V_i \to \hat{V}_i$ be the natural projection that is the covering map inducing the epimorphism  $\eta_i : \pi_1(\hat{V}_i) \to$   $\mathbb{Z}$ . As for the sink  $\omega_i$  is concerned, the restriction  $f|_{V_i}$  is topologically conjugated to the diffeomorphism  $a : \mathbb{R}^3 \setminus \{O\} \to \mathbb{R}^3 \setminus \{O\}$ , then the manifold  $(\hat{V}_i, \eta_i)$  is equivalent to the manifold  $(S^2 \times S^1, \eta^s_{S^2 \times S^1})$  and the set  $\hat{\ell}_i = p_i(\ell_i)$  is the  $\eta_i$ -essential knot in the manifold  $\hat{V}_i$  such that  $\eta_i(i_{\hat{\ell}_i*}(\pi_1(\hat{\ell}_i))) = \mathbb{Z}$  (see Theorem 2.3 in [8]).

It was proved in [6] (Theorem 1) that at least one of the knots  $\hat{\ell}_1$ ,  $\hat{\ell}_2$  is trivial (see also [8], Proposition 4.3). To be definite we assume below the knot  $\hat{\ell}_1$  be trivial.

Next result was proved in [6] (Theorem 3) (see also [8], Theorem 4.3).

**Proposition 5** Diffeomorphisms  $f, f' \in \mathcal{P}$  are topologically conjugated if and only if the knots  $\hat{\ell}_2(f)$  and  $\hat{\ell}_2(f')$  are equivalent.

Therefore the equivalence class of the knot  $\hat{\ell}_2(f)$  is a complete topological invariant for diffeomorphisms from the Pixton class. Moreover, the following realization theorem holds (see [6] Theorem 2 and [8] Theorem 4.4).

**Proposition 6** For every knot  $\hat{\ell} \subset (S^2 \times S^1, \eta^s_{S^2 \times S^1})$  such that  $\eta^s_{S^2 \times S^1}(i_{\hat{\ell}*}(\pi_1(\hat{\ell}))) = \mathbb{Z}$  there is a diffeomorphism  $f : S^3 \to S^3$  from the class  $\mathcal{P}$  such that the knots  $\hat{\ell}$  and  $\hat{\ell}^2(f)$  are equivalent.

Masur constructed an example of an essential and nontrivial knot embedded to  $S^2 \times S^1[15]$ . According to Proposition 6, there exists a diffeomorphism f of the Pixton class such that exactly one unstable one-dimensional separatix and stable two-dimensional separatix of the saddle point  $\sigma$  are wildly embedded.

On Fig. 8 it is shown the Masur's knot  $\hat{l}_{\sigma}^{u}$  which appears in factor-space  $\hat{W}^{s}(\omega^{2})$  and essential torus  $\hat{l}_{\sigma}^{s}$  embedded to  $\hat{W}^{u}(\alpha)$  which is tubular neighborhood of the a Masur knot.

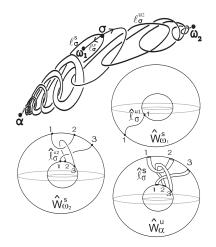


Figure 8: The phase portrait of a diffeomorphism of the class  $\mathcal{P}$  and projection of saddle separatrices in factor spaces

### 2.3 Diffeomorphisms with wildly embedded frames

**Definition 7** For  $k \in \mathbb{N}$  a k-frame  $F_k$  in  $\mathbb{R}^n$  at the point p is an union of k simple curves  $A_1, \ldots, A_k, F_k = \bigcup_{i=1}^k A_i$ , with a single common point p such that p is the boundary point of each  $A_k, k \geq 1$  and  $A_i \cap A_j = p, i \neq j$ .

#### Definition 8

- The k-frame  $F_k = \bigcup_{i=1}^k A_i$  is said to be standard if each arc  $A_i$  lies in the plain  $Ox_1x_2$ and it is defined  $\varphi = \frac{2\pi(i-1)}{k}$  where  $\rho$ ,  $\varphi$  are the polar coordinates in the plain  $Ox_1x_2$ .
- A k-frame  $F_k = \bigcup_{i=1}^k A_i$  is said to be tame, if there is a homeomorphism  $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ such that  $\varphi(F_k)$  is standard. Otherwise, the frame  $F_k$  is said to be wild.
- A k-frame  $F_k = \bigcup_{i=1}^{k} A_i$  is said to be mildly wild, if the frame  $F_k \setminus (A_i \setminus p)$  is tame for every  $i \in \{1, \ldots, k\}$ .

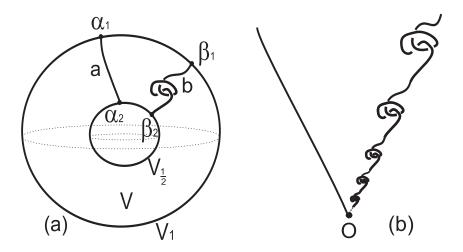


Figure 9: A construction of a wild 2-frame in  $\mathbb{R}^3$ 

One can easily construct a wild k-frame, if one assumes the arc  $A_1$  be the wild arc  $\tilde{\ell}$  of Artin-Fox's example [1]. But the fact that each arc  $A_i$  is tame does not mean that the frame  $F_k$  is tame. Figure 9 (b) shows an example of the wild 2-frame. Similarly to the Artin-Fox's example this frame is constructed using the arcs a, b shown in Figure 10 (a). The boundary points  $\alpha_1, \alpha_2, \beta_1, \beta_2$  of the respective arcs  $\alpha, \beta$  are glued by  $\phi(\alpha_1) = \alpha_2, \phi(\beta_1) = \beta_2$  and  $A_1 = \bigcup_{k \in \mathbb{Z}} \phi^k(a) \cup O, A_2 = \bigcup_{k \in \mathbb{Z}} \phi^k(b) \cup O, F_2 = A_1 \cup A_2$ . From the Statement 2 it follows that both  $A_1, A_2$  are tame. Debrunner and Fox [2] presented the construction of a mildly wild k-frame for every k > 1. Figure 10 shows this construction for k = 6. This 6-frame is constructed using the arcs  $a_1, \ldots, a_6$  shown in Figure 10 (a). The boundary points  $\alpha_1^i, \alpha_2^i$ , of the arc  $a_i, i \in \{1, \ldots, 6\}$  are glued by  $\phi(\alpha_1^i) = \alpha_2^i$  and  $A_i = \bigcup_{k \in \mathbb{Z}} \phi^k(a_i) \cup O, F_6 = \bigcup_{i=1}^6 A_i$ .

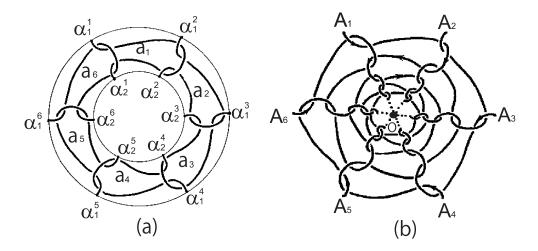


Figure 10: Debrunner-Fox's example

Let f be a Morse-Smale diffeomorphism and suppose that there is a sink  $\omega \in NW(f)$ and the set  $L_{\omega}$  of all unstable one-dimensional different separatrices  $\ell_1, \ldots, \ell_k$  of saddles  $\sigma_1, \ldots, \sigma_r, k, r \in \mathbb{N}, k \leq r \ (\sigma_i \text{ may coincide with } \sigma_j)$  such that for any j closure of separatrix  $\ell_j$  consists of exactly two points:  $\omega$  and saddle point  $\sigma$  for which  $\ell_j$  is separatrix. Since  $W^s_{\omega}$ is homeomorphic to  $\mathbb{R}^3$  and since the set  $L_{\omega} \cup \omega$  is the union of the simple arcs with the unique common point  $\omega$  belonging to each arc, analogously to a frame of arcs in  $\mathbb{R}^3$  we call  $L_{\omega} \cup \omega$  the frame of 1-dimensional unstable separatrices.

**Definition 9** A frame of separatrices  $L_{\omega} \cup \omega$  is tame if there is a homeomorphism  $\psi_{\omega}$ :  $W^s_{\omega} \to \mathbb{R}^3$  such that  $\psi_{\omega}(L_{\omega} \cup \omega)$  is the standard frame of arcs in  $\mathbb{R}^3$ . Otherwise the frame of separatrices is called wild.

In [19] by the method, similar to that described in subsection 2.2, a Morse-Smale diffeomorphism was constructed having a mildly wild frame of one-dimensional separatrices (see fig. 11).

## 3 Periodic vector field on $S^3$ with wildly embedded separatrix set

Now we present a periodic vector field on  $S^3$  with a wild embedding of a 2-dimensional unstable separatrix manifold and 3-dimensional stable separatrix manifold for the saddle IC with exponential dichotomy on  $\mathbb{R}$  of the type (3, 2). Also we present another periodic vector field on  $S^3$  that has a mildly wild frame of 2-dimensional separatrix manifolds.

We start with some diffeomorphism f of the Pixton class on  $S^3$  that has one hyperbolic source  $\alpha$ , one saddle  $\sigma$  of the type (2, 1) (2-dimensional stable manifold and 1-dimensional unstable one) and two hyperbolic sinks  $\omega_1, \omega_2$ . Stable 2-dimensional manifold of  $\sigma$  contains in its closure the point  $\alpha$ , that is, all orbits of f with initial points on  $W^s(\sigma)$ , except  $\sigma$  itself, have the only  $\alpha$ -limit point  $\alpha$  and the  $\omega$ -limit point  $\sigma$ . The closure of  $W^s(\sigma)$  is a topologically embedded sphere  $\Sigma$  in  $S^3$ , being the boundary of two open 3-balls  $D_1$ ,  $D_2$  in  $S^3$ . The fixed

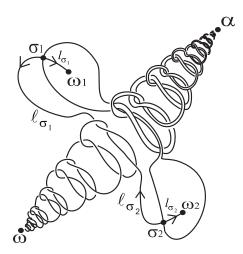


Figure 11: Phase portrait of a Morse-Smale diffeomorphism on  $\mathbb{S}^3$  with the mildly wild frame of separatrices

point  $\omega_1$  (sink) lies inside of the ball  $D_1$ , another sink  $\omega_2$  lies inside another ball  $D_2$ . We suppose that a 1-dimensional separatrix of  $\sigma$  which enters to  $D_2$  is wildly embedded. This implies stable manifold  $W^s(\sigma)$  be also wildly embedded (see figure 2).

Now consider the suspension over f. Since f is diffeotopic to  $id_{S^3}$ , the manifold  $M_f$  is topologically a direct product  $S^3 \times S^1$ , moreover, this direct product structure can be chosen by means of a some diffeomorphism (see above). We fix this product structure and consider henceforth the suspension as the standard  $S^3 \times S^1$ . Thus, the suspension flow in  $S^3 \times S^1$ has one totally unstable periodic orbit, one saddle periodic orbit of the type (3, 2) and two totally stable periodic orbits, all of them are hyperbolic periodic orbits. The projection of any of these periodic orbits onto the base  $S^1$  is 1-1 correspondence.

Now recall that the suspension flow is Morse-Smale one. All its periodic orbits are hyperbolic and any other orbit tends to some of four periodic orbits as  $t \to \pm \infty$ . This implies, by the construction, all four periodic ICs of the non-autonomous vector field on  $S^3$ , to possess an exponential dichotomy on  $\mathbb{R}$ . Types of an exponential dichotomy are different: two stable periodic orbits give rise to two completely stable periodic ICs, their type of an exponential dichotomy is (4, 1) (four is the dimension of their stable manifolds), the saddle periodic orbit gives rise to the saddle periodic IC with the dichotomy on  $\mathbb{R}$  of the type (3, 2), and the completely unstable periodic orbit gives rise to the IC with the dichotomy of the type (1, 4). All other ICs tend to these four ICs and hence they possess an exponential dichotomy on  $\mathbb{R}_-$  and  $\mathbb{R}_+$  separately depending on which of four ICs they approach to.

Now remind that diffeomorphism f has a smooth curve being the unstable separatrix  $W^u(\sigma)$  for the saddle point  $\sigma$ . For the suspended flow in  $S^3 \times S^1$  we get the two-dimensional smooth unstable submanifold  $W^u(\gamma_{\sigma})$  of the saddle periodic orbit  $\gamma_{\sigma}$ . The manifold  $W^u(\gamma_{\sigma})$  is a direct product of  $W^u(\sigma) \times S^1$ , this follows from the suspension construction. If one of two unstable separatrices of  $\sigma$  is wildly embedded into  $S^3$  (see above), then one of connected component of intersection  $S^3_{\tau} \cap W^u(\gamma_{\sigma}), \tau \in S^1$ , (denote it as  $\Sigma_{\tau}$ ) is a wildly embedded curve in  $S^3_{\tau}$ . Suppose, to be definite, that stable sink  $\omega_2$  be the  $\omega$ -limit set for all orbits of the wildly embedded separatix of  $\sigma$ . We will say that the related component of  $W^u(\sigma) \times S^1 \setminus \gamma_{\sigma}$  is wildly embedded in  $S^3 \times S^1$ . The characterization of this wild embedding is the following.

Choose any smooth 3-disk D being transversal to a point on the periodic orbit  $\gamma_{\omega_2}$ . Then the wildly embedded component intersects this disk along a smooth ray with the extreme point  $D \cap \gamma_{\omega_2}$ .

**Lemma 2** If flows  $f^t, f'^t \in \tilde{\mathcal{P}}$  are topologically equivalent and  $f^t$  possesses a wildly embedded connected component of the set  $W^u(\gamma_S) \setminus \gamma_S$ , then the same holds true for the flow  $f'^t$ .

Thus, we have proved the assertion

**Theorem 2** There exists a smooth 1-periodic vector field v on  $S^3$  such that v is gradientlike one with only four 1-periodic ICs possessing exponential dichotomies on  $\mathbb{R}$ : completely unstable (of the type (1,4)), saddle one of the type (3,2), two completely stable ones and its saddle periodic IC has wildly embedded two-dimensional and three-dimensional separatrix sets.

In the same way, starting with a diffeomorphism of  $S^3$  having a mildly wild frame of separatrices described above, we get a 1-periodic gradient-like vector field on  $S^3$  such that it has one completely unstable IC, one completely stable IC and  $n \ge 2$  saddle periodic ICs with an exponential dichotomy of the type (3,2) whose n two-dimensional unstable separatrices form a mildly wild frame along with their n three-dimensional stable separatrices which also form a mildly wild frame similar to that plotted in Fig.11.

We formulate some assertion concerning non-autonomous vector fields we have constructed.

**Theorem 3** Any sufficiently small uniform perturbation of such vector field v gives a nonautonomous vector field v' that is uniformly equivalent to the initial one, i.e. there exists an equimorphism h of the extended phase manifold  $S^3 \times \mathbb{R}$  which transforms the foliation  $\mathcal{L}_v$  to that of  $\mathcal{L}_{v'}$ .

Due to the very simple structure of  $\mathcal{L}_v$  the proof is almost evident, nevertheless, because of some technicalities, it will be performed elsewhere.

## 4 Perturbations

In this section we perturb periodic vector fields constructed in the preceding section in such a way that its uniform structure stays the same, but, in dependence on the perturbation chosen, the perturbed non-autonomous vector field would be almost periodic or even be nonrecurrent in time.

By the theorem 3, v is structurally stable w.r.t. small uniform perturbations of the form  $v + \varepsilon v_1$  given by a bounded uniformly continuous map  $v_1 : \mathbb{R} \to V^r(S^3)$  into the Banach space  $V^r(S^3)$ . In particular, such a perturbation can be chosen being an almost periodic map. In this case, since any of four periodic ICs possess an exponential dichotomy on  $\mathbb{R}$  (of different types) the perturbed almost periodic vector field will have in a small enough uniform neighborhood of each periodic IC an almost periodic IC with the same type of exponential dichotomy. Moreover, the perturbed vector field will be also gradient-like one and any of its IC will possess an exponential dichotomy on  $\mathbb{R}_{\pm}$  and will tend to some of four almost periodic ICs.

The periodic vector field, we constructed above, has a very simple structure of its foliation into ICs. Namely, it has a unique totally unstable IC  $\gamma_{\alpha}$  possessing an exponential dichotomy on  $\mathbb{R}$  of the type (4,1), one saddle IC  $\gamma_{\sigma}$  possessing an exponential dichotomy on  $\mathbb{R}$  of the type (3,2) and two totally stable ICs  $\gamma_{\omega_1}$  and  $\gamma_{\omega_2}$  possessing both an exponential dichotomy on  $\mathbb{R}$ of the type (4,1). All other ICs tend to one of these specific ICs as  $t \to \pm \infty$ . One important thing exists. Let us choose some sufficiently thin uniform neighborhoods  $U_i$ , j = 1 - 4, of all specific ICs. One can choose these neighborhoods in such a way that the passage time within  $M \times \mathbb{R} \subset \bigcup_j U_j$  would be uniformly bounded from above and below. This allow us to prove that the non-autonomous vector field is structurally stable with respect to a small enough uniformly bounded perturbations. This means that there is an equimorphism  $\Phi: M \times \mathbb{R} \to M \times \mathbb{R}$  such that it transforms a foliation into ICs of the periodic vector filed into the foliation of the perturbed vector field. In particular,  $\Phi$  preserves the properties of a wild embedding or mildly wild separatrix frame. If  $v_1$  is nonrecurrent but uniformly continuous bounded map, then the perturbed NVF will have the same uniform structure but its four specific ICs will lie in thin uniform neighborhoods of those for the constructed periodic NVF. The same holds true for the periodic NVF with the mildly wild k-frame of separatrices.

### 5 Addendum: Elements of the uniform topology

For the reader convenience, we present here some notions from the uniform topology. Recall some basic definitions of the theory of uniform spaces (see details in [16]). A set X is called an *uniform space*, if on  $X \times X$  is defined a collection  $\mathcal{U}$  of its subsets satisfying the following conditions (if so,  $\mathcal{U}$  is called *the uniformity*)

- 1. each element of  $\mathcal{U}$  contains diagonal  $\Delta = \bigcup_{x \in X} \{(x, x)\};$
- 2. if  $U \in \mathcal{U}$ , then  $U^{-1} \in \mathcal{U}$ , where  $U^{-1}$  is the set of all pairs (y, x) for which  $(x, y) \in U$ ;
- 3. for any  $U \in \mathcal{U}$  some  $V \in \mathcal{U}$  exists such that  $V \circ V \subset \mathcal{U}$ , here  $V \circ V$  denotes the composition:  $(x, z) \in V \circ V$  if there is  $y \in X$  such that  $(x, y) \in V$  and  $(y, z) \in V$ ;
- 4. if  $U, V \in \mathcal{U}$ , then  $U \cap V \in \mathcal{U}$ ;
- 5. if  $U \in \mathcal{U}$  and  $U \subset V \subset X \times X$ , then  $V \in \mathcal{U}$ .

If X is a metric space with metrics d, then 1) corresponds to the property d(x, x) = 0, 2) corresponds to the symmetry of d: d(x, y) = d(y, x). The property 3) is of the type of the triangle inequality: for any ball of the radius r a ball of the radius r/2 should exist. The third and fifth conditions are similar to the axioms of neighborhoods near a point for the topology that is defined by the uniformity.

The uniformity on a given set X can be defined by many ways providing different uniform spaces. This was used above where on the set  $M \times \mathbb{R}$  different uniform structures were defined. When  $(X, \mathcal{U}), (Y, \mathcal{V})$  are two uniform spaces, then a notion of a uniformly continuous map  $h: X \to Y$  is defined. Namely, a map  $h: X \to Y$  is uniformly continuous w.r.t.  $\mathcal{U}, \mathcal{V}$ , if for any  $V \in \mathcal{V}$  the set  $\{(x, y) | (h(x), h(y)) \in V\}$  belongs to  $\mathcal{U}$ . When  $h: X \to Y$  is one-to-one and both  $h, h^{-1}$  are uniformly continuous, then h is called to be an equimorphism. In this case uniform spaces  $(X, \mathcal{U}), (Y, \mathcal{V})$  are called uniformly equivalent or equimorphic ones.

An uniformity on the set X making it the uniform space  $(X, \mathcal{U})$  generates the definite topology on X making it a topological space. This space can possess various topological properties. Conversely, each regular<sup>3</sup> topology  $\mathcal{T}$  on X is an uniform topology which corresponds to some uniformity, but such uniformity is, in general, not unique. But if the topological space is compact and regular one, then there is an unique uniformity generating the topology  $\mathcal{T}$ .

## 6 Acknowledgement

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## 7 Conflict of Interest

The authors declare they have not conflict of interest.

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<sup>&</sup>lt;sup>3</sup>A topological space is regular, iff for every its point x and any its neighborhood U there is a closed neighborhood V of x such that  $V \subset U$ .

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