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Finite-size effects in exponential random graphs

A. Gorsky
Institute for Information Transmission Problems RAS, Moscow, Russian Federation and Moscow Institute of Physics and Technology, Dolgoprudny 141700, Russian Federation

AND

O. Valba[†]
Department of Applied Mathematics, National Research University Higher School of Economics, 101000 Moscow, Russian Federation
[†]Corresponding author. Email: ovalba@hse.ru

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In this article, we show numerically the strong finite-size effects in exponential random graphs. Particularly, for the two-star model above the critical value of the chemical potential for triplets a ground state is a star-like graph with the finite set of hubs at network density $p < 0.5$ or as the single cluster at $p > 0.5$. We find that there exists the critical value of number of nodes $N^*(p)$ when the ground state undergoes clear-cut crossover. At $N > N^*(p)$, the network flows via a cluster evaporation to the state involving the small star in the Erdős–Rényi environment. The similar evaporation of the cluster takes place at $N > N^*(p)$ in the Strauss model. We suggest that the entropic trap mechanism is relevant for microscopic mechanism behind the crossover regime.

Keywords: random graphs; finite size effect; two-star model; phase transition.

1. Introduction

Exponential random graphs provide examples of the Ising type systems analysed via the conventional methods of the statistical physics and in general one can consider canonical, microcanonical and mixed ensembles. They can exhibit the critical behaviour and have a rich phase structure in the space of some control parameters.

Let us recall the known examples of a critical behaviour in exponential random graphs. The network ensemble supplemented with the chemical potential for the triplets (two-star model) has been considered in thermodynamic limit in mean-field approximation in dense regime [1] and in sparse regime under some assumptions [2]. The example of a phase transition in the canonical ensemble with the chemical potential for triangles has been found in [3]. The single complete graph above the critical value of positive chemical potential for triangles is formed, the phenomenon has been explained analytically in [4, 5]. At a negative chemical potential for triangles, the network becomes bipartite below some critical value of chemical potential [6–8]. The critical behaviour in microcanonical ensemble with fixed number of links and triangles has been considered in [9–11]. The mixed ensembles with fixed degrees and chemical potential for triangles has been investigated in [12] (see also [13] for the previous study). It was discovered

that above the critical value of a chemical potential the network gets defragmentated into the fixed number of weakly connected clusters. The spectrum of the defragmentated network has the unusual statistics and has the mobility edge [14].

The real networks are finite hence the role of the finite-size effects certainly is important for applications. Several aspects of the finite-size effects have been elaborated for the scale-free network where they induce the cut-off in a degree distribution [15–20]. The finite size effects influence the criticality of the epidemic processes [21, 22]. The crossover regime at some network size N^* has been found also in the synchronization problem in the scale-free networks [23]. Very recently the example of the strong finite-size effect in the peculiar phase transition in the microcanonical ensemble of exponential random graphs has been found in [24].

The interesting analysis of the finite-size effects in the Anderson model on the random regular graphs (RRG) has been performed in [25–27]. There was the long-standing controversy concerning the ergodicity of modes in the delocalized phase and the very existence of the delocalized non-ergodic phase in the Anderson model on RRG. The careful finite N analysis demonstrated that there is no additional phase transition between the ergodic and non-ergodic delocalized phases and instead there is the strong finite-size effect. At some value of N, the system undergoes the crossover behaviour from the near critical regime to the ergodic delocalized state. A kind of the renormalization group analysis has been developed in [27] which supports this picture.

The primary motivation for this study is the wish to have a clear-cut example of the crossover in N-dependence in simple exponential random graph without any approximation. We investigated numerically the two-star and the Strauss models, and it turned out that they undergo the crossover behaviour at some finite $N^*(p)$ above the critical value of the chemical potential. For the two-star model above the critical value of the chemical potential, the ground state is a star-like graph of the finite set of hubs at $p < 0.5$ or the single cluster at $p > 0.5$. It is found that there exists the critical value of nodes N^* when the ground state undergoes clear-cut crossover and at $N > N^*$ it flows to the small hub in the ER environment state via a cluster evaporation.

There are two ways to get the crossover regime. Following the first strategy, it is useful to choose the chemical potential for the corresponding motives well above the critical value and vary N. There is crossover regime at some N when the network ground state gets changed. Another way to capture the crossover regime involves the N dependence of the critical values for the chemical potentials. At $N < N^*$, critical chemical potential behaves as $1/N$ in agreement with the mean-field solutions to the two-star and Strauss models while at $N > N^*$ their N-dependence changes drastically. Let us emphasize that the ground state above N^* involves several components with the different mean degrees and cannot be captured via the mean-field analysis. That is why this crossover has not been seen in [1, 2].

We shall briefly discuss the possible microscopic description of the cluster evaporation which is responsible for the crossover regime. Due to non-homogeneity of the network, there are entropy gradients and therefore entropic forces. We argue that the entropic trap could be one of the mechanisms responsible for the cluster evaporation. The emerging clusters in the matrix models provide the toy model for the black holes formed from the D0 particles in the holographic framework [28, 29]. Our findings concerning the role of $1/N$ effects in the derivation of the ground state could be of some use for the microscopic analysis of the black hole formation and decay within the matrix model framework.

The note is organized as follows. In Section 2, we provide the model description, in Section 3 we describe numerical results, and in Section 4, we discuss the possible microscopic mechanism of crossover and comment on the gravity analogies.

2. Model description

We define an ensemble consisting of the set of all simple undirected ER graphs with N vertices and no self-edges (i.e. networks with either zero or one edge between each pair of distinct vertices). The partition function for the exponential networks reads as

$$Z = \sum_{G} e^{-H(G)}, \tag{2.1}$$

where the measure in the sum over all graphs involves the chemical potentials for some kind of subgraph. For the two-star model the Hamiltonian reads as

$$H = \theta L(G) - \mu s(G), \tag{2.2}$$

where $L(G)$ is the number of links and $s(G)$ is the number of triplets. For the Strauss model, we have

$$H = \theta L(G) - \nu T(G), \tag{2.3}$$

where $T(G)$ is the number of three-cycles.

In ER network, the links are independent with probability p, that is, $L = \dfrac{pN(N-1)}{2}$. The mean-field solution of two-star model yields the consistency equation for the control parameters

$$p = \frac{1}{2}\left[\tanh(\mu N p - \frac{1}{2}\theta) + 1\right]. \tag{2.4}$$

There is the critical line in the model $\theta = \mu N$ corresponding to a first-order phase transition which terminates at the second-order transition critical point $\mu N = 1$.

To proceed exponential random model with numeric, we use the dynamic algorithm which replaces the numerical evaluation of the combinatorial problem for (2.2) by running stochastic evolution of network (the discrete Langevin dynamics), starting from some initial configuration until the evolution converges. The initial state of the network is prepared by connecting any randomly taken pair of vertices with the probability p and the total number of links is fixed. Then, one randomly chooses arbitrary link, say, between vertices i and j (i,j) and switch it to some other pair (k,l) $(k \neq i,j;\ l \neq i,j)$. The acceptance probability this elementary permutation is defined by the standard Metropolis rule. According to the graph Hamiltonian, if under the elementary step the number of triplets is increased or remains unchanged, a move is accepted, and if the number of triplets is decreased by Δ, a move is accepted with the probability $e^{-\mu\Delta}$. The Metropolis algorithm runs repetitively for a large set of randomly chosen pairs of links, until it converges.

The Hamiltonian (2.2) can be written in the form:

$$H(G) = \theta \sum_{i<j} a_{i,j} - \mu \sum_{i<j<k} a_{ij}a_{jk}, \tag{2.5}$$

where $a_{ij} = a_{ji}$ is an element of the adjacency matrix having value 1 if an edge exists between vertices i and j and 0 otherwise. If we consider the complement graph $\tilde{G}$ ($b_{ij} = 1 - a_{ji}$), the Hamiltonian (2.5)

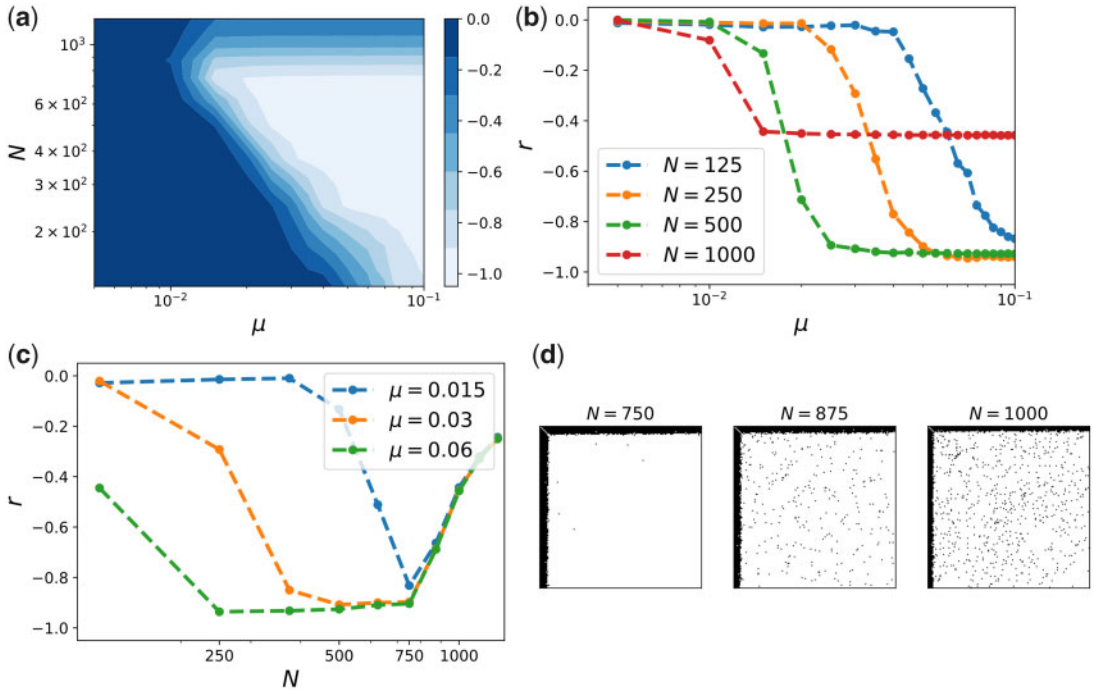

FIG. 1. Phase portrait of the two-star model in (μ, N)-plane for random graphs of $p = 0.1$: (a) density plot of the degree assortativity coefficient $r(\mu, N)$; (b) the assortativity in dependence on the chemical potential μ for different network sizes; (c) the assortativity in dependence on the network size N for different value of μ; (d) typical adjacency matrices for different network size and $\mu = 0.1$. The results are performed over averaging of 100 realization.

keeps the same form:

$$H(\tilde{G}) \sim \alpha \sum_{i<j} b_{i,j} - \mu \sum_{i<j<k} b_{ij} b_{jk}, \tag{2.6}$$

where α is linear combination of θ, μ and the number of the vertices N.

In the two-star model [1], there is the phase transition at some critical μ_{cr} and at $\mu > \mu_{cr}$ a kind of dense component gets emerged which depends on the mean vertex degree pN. Hence similar phase transition with the complete graph formation occurs for the complement graph $\tilde{G}$. The high-density network state described in [1] is a complete graph, for the fixed number of vertices the complement graph of this state has a star-like configuration (see Fig. 1(d)). We compare analytically the ground state energies of two optimal configurations and found the following picture. If the link probability $p < 0.5$, the optimal configuration is a star-like graph, if $p > 0.5$, the ground state is a complete graph, $p_1 = 0.5$ is a natural critical point (see Figs 1(d) and 2(d)). We note that in the grand canonical ensemble, when the number of nodes is supplemented by the chemical potential, the critical density separating two different topologies remains the same. It happens because the number of triplets depends as N^3 and contributes mainly to the ground state energy.

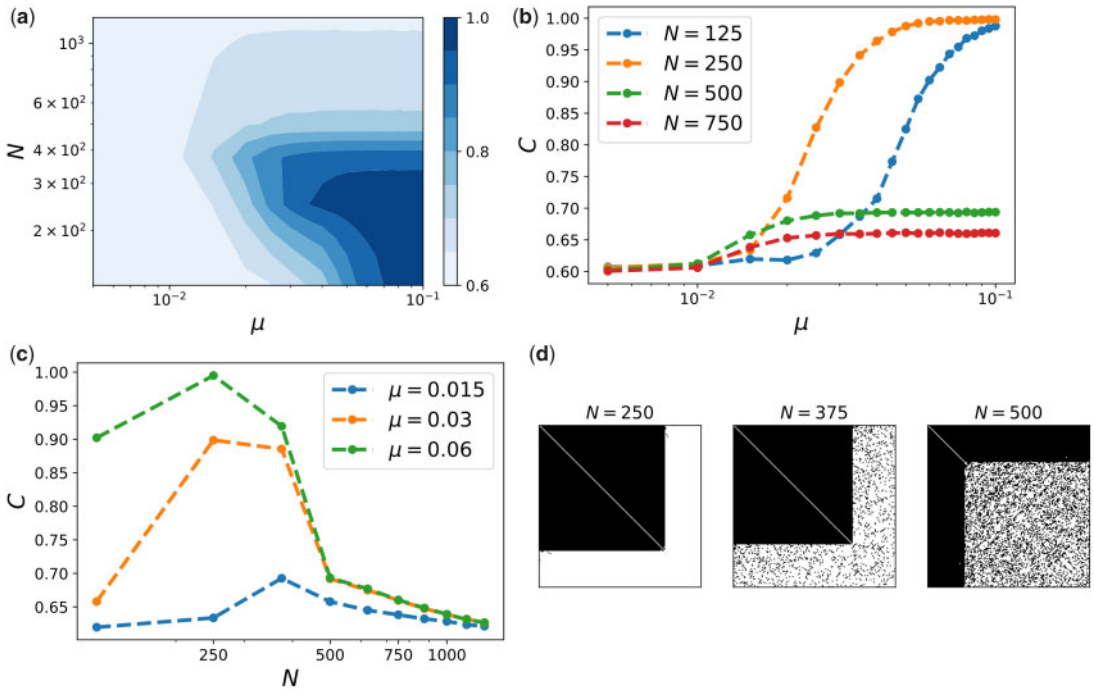

FIG. 2. Phase portrait of the two-star model in (μ, N)-plane for random graphs of $p = 0.6$: (a) density plot of the clustering coefficient $C(\mu, N)$; (b) the clustering coefficient in dependence on the chemical potential μ for different network sizes; (c) the clustering coefficient in dependence on the network size N for different value of μ; (d) typical adjacency matrices for different network size and $\mu = 0.1$. The results are performed over averaging of 100 realization.

3. Numerical results

3.1 *Crossover behaviour at finite N*

We have revisited numerically the transition to the thermodynamical limit in the two-star and the Strauss model. Contrary to the general expectations it turned out that the transition is not smooth and there is the crossover value N^*, where the ground state of network gets changed. The typical phase portrait of the model in the parameter (μ, N) plane is shown in Fig. 1(a). For $p < 0.5$, it is convenient to calculate the degree assortativity coefficient which is defined as the Pearson correlation coefficient of degree between pairs of linked nodes:

$$r = \frac{L\sum_{i=1}^{L} j_i k_i - \left[\sum_{i=1}^{L} j_i\right]^2}{L\sum_{i=1}^{L} j_i^2 - \left[\sum_{i=1}^{L} j_i\right]^2}, \tag{3.1}$$

where L is the number of edges, j_i, k_i are the degrees of the nodes, connected by the link i. If the nodes with large degrees are connected with nodes with a small number of links, the assortativity is close to -1, random graphs have the zero assortativity. The ground state for $N < N^*$ gets identified with the dissortative star-like state while at $N > N^*$ we observe the smaller star-like graph with few hubs in the

ER environment. The critical size is $N^* \approx 750$ for the network density $p = 0.1$. Let us emphasize that it differs from the mean-field solution when the ER network without any defects was predicted. In the mean-field solution, it is assumed that the mean degree of the nodes is homogeneous and corresponds to the dense cluster or ER phases. Our numerical analysis demonstrates that there is the phase at $N > N^*$ when the ground state involves at least two components with different degrees: few linked hubs and the ER vapour around.

For $p > 0.5$ we observe the same crossover behaviour, but now from the clustered network state at $N < N^*$ to the smaller star-like graph with few hubs in the ER environment $N > N_{cr}$. The respective phase portrait is shown in Fig. 2(b).

If we fix the value of μ a bit above critical point it is possible to estimate the $N^*(p)$ dependence, substituting in (2.4) the critical line equation $\theta = \mu N$:

$$N^* = \frac{\ln p - \ln(1-p)}{\mu(2p-1)}. \tag{3.2}$$

The observed crossover behaviour is associated with a fixed number of vertices and it seems to disappear in the grand canonical ensemble.

3.2 *Comments on the spectral behaviour*

A spectral density is a good indicator of the ground state of a network. There are some general observations concerning the evolution of a spectral density as a function of chemical potential. First, the spectral density in the continuum part of the spectrum gets modified. Secondly, the isolated eigenvalues appear at the clusterized phase since the number of the isolated eigenvalues corresponds to the number of clusters. The creation of the cluster has been identified as the eigenvalue instanton in [12].

Can we recognize the crossover behaviour in the spectral evolution? The answer turns out to be positive and is pictured at Fig. 3. We have investigated the dependence of the spectral density on the chemical potential below and above N^*. At $N < N^*$, the spectral density becomes more narrow at the phase transition point. On the other hand at $N > N^*$, the behaviour is more rich. First, as result of evaporation the isolated eigenvalues get appeared symmetrically with respect to the central zone. Since all hubs are identical in the star-like configuration, the eigenvalues at fixed network are degenerate. However, since the number of links is different in each ensemble they form two zones which are analogue of the non-perturbative zone found in [12]. Increasing N we increase the number of evaporated nodes which results into the interaction between the star nodes and lifting the degeneration of the eigenvalues.

4. Discussion

4.1 *Localization and entropic trap*

Let us discuss the possible microscopic mechanisms responsible for the crossover behaviour. It is natural to assume that mechanism of the cluster evaporation is of the entropic nature. Indeed, due to non-homogeneity of network the entropic forces are present and could 'keep the nodes' around some center of disorder if the entropic force is strong enough. Hence, we could look for the examples of the localization of trajectories around some center of disorder at the random graph. We could mention two relevant examples.

First, strong finite-size effects and crossover behaviour at some N have been found in the Anderson model at the regular random graph [25–27]. The crossover size N_{cr} has been interpreted as the existence

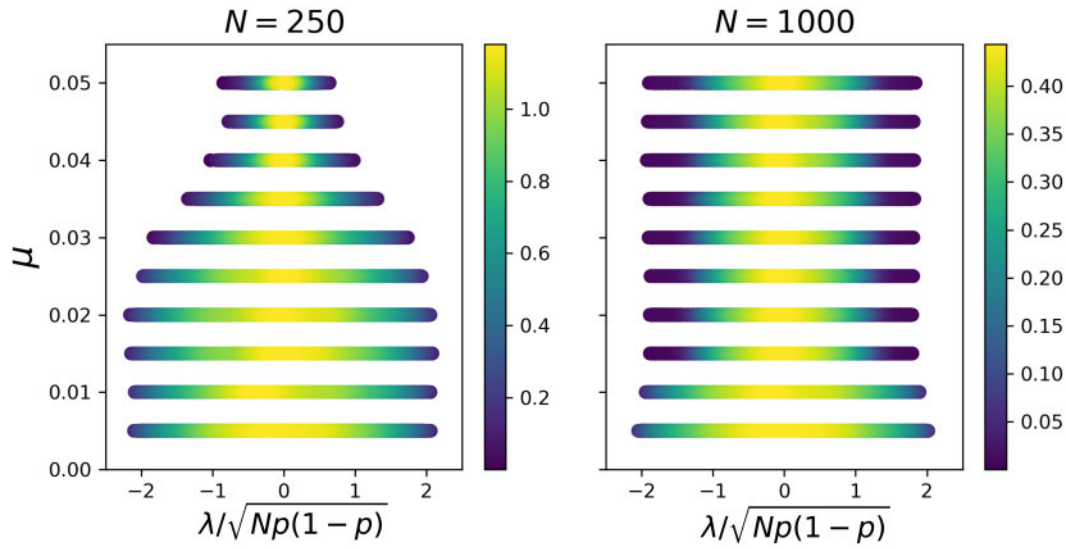

FIG. 3. Spectral behaviour of the two-star network below and above the crossover. The results are obtained after averaging over 100 realization ($p = 0.1$).

of hidden intrinsic localization scale in the model. It was observed the if $N < N_{cr}$ the system behaves almost as critical even near the critical line while at $N > N_{cr}$ the RG flow pushes the system in the different direction to the ergodic state. This is similar to what we have found: at small N we have the critical star-like state or cluster even apart from the critical line while at $N > N_{cr}$ they start to evaporate down to ergodic ER network with small inhomogeneity.

Another interesting possibility for the crossover mechanism can be traced from the heavy root problem [30]. In that paper, the infinite tree-like network with equal node degrees d and single node with large degree d_0 has been considered. It was demonstrated that depending on a degree of the heavy root the random walks are localized around the heavy root or escape to infinity. The result has been derived analytically and confirmed numerically.

4.2 *Gravity analogy*

It is well-known that large N matrix model provides the effective summation over two-dimensional geometries (see [31] for review). The partition function of the matrix model is presented in the following form

$$Z(t_k) = \int DM exp(\sum_k t_k TrM^k), \tag{4.1}$$

where t_k are the chemical potentials for single trace observables and matrix M belongs to some matrix ensemble. The rich critical behaviour of the partition function corresponds to the criticality of the pure topological two-dimensional gravity if only t_2, t_3 are non-vanishing or topological gravity coupled to matter in a general case with multiple chemical potentials. The analogy with the exponential graph

partition function is clear. In that case, t_k counts the number of k-cycles in the network and for instance the critical behaviour in the Strauss model corresponds to the critical behaviour in the pure topological two-dimensional gravity. The Laplacian matrix of the random network $L = D - A$, where D is diagonal matrix of the mode degrees $D = \mathrm{diag}(d_1, \ldots d_N)$ and A is a adjacency matrix, is the discretization of the two-dimensional Laplace operator on the Riemann surface.

The size of the matrix plays the role of the inverse coupling constant in the quantum gravity $g_s = 1/N$ hence large N limit corresponds to the weak coupling regime and the finite-size effects correspond to the 'perturbative' corrections. There are also the non-perturbative effects with e^{-N} dependence which are identified in matrix models as eigenvalue instantons or as the creation of the baby universes in the gravity language [32]. These non-perturbative effects have been identified in the constrained ER networks and RRG in [12] as the cluster formation. Our findings in this study suggest that account of the finite-size effects $1/N$ could lead to the crossover-like regime in the matrix models. Note that in the two-star model, we take into account both singlet and the non-singlet sectors in matrix model language since the number of two-stars cannot be presented as the single trace of some power of adjacency matrix.

Another viewpoint concerning the gravity interpretation of our results is suggested by the holographic approach for large N boundary theories. The holographic setup provides the way to describe the dual black holes (BH) in terms of large number of D_0 branes in the boundary theory [28, 29]. We can consider our exponential random network as the toy boundary large N theory. The single BH in the holographic dual looks as the full block in the large N matrix where the diagonal matrix elements represent the positions of the D0 branes and the off-diagonal elements correspond to the strings connecting D0 branes. The several blocks at the diagonal represent the configuration of the several black holes while the emission of the single D_0 brane from the black hole looks as one separated element in the matrix on the diagonal outside the cluster [28, 29].

Could we learn some lessons from our findings for the toy matrix BH? First, the analogy suggests the microscopic mechanism for the toy BH formation via phase transition. The next point concerns a evaporation of the toy BH which we could relate with the evaporation of the cluster. Since the coupling constant behaves as $1/N$ increasing the network size, we decrease the perturbative gravity effects and finally at some critical N^* the BH starts to evaporate since microscopic degrees of freedom can escape. Certainly this picture is quite qualitative but it could shed some light on the entropic mechanism behind the BH formation and evaporation.

The evaporation of the star-like configuration in the two-star model has similar interpretation with additional subtlety. Due to 'open-string' non-singlet sector in the two-star model, there is a kind of hair in the ground state which presumably corresponds to the black hole hair. Indeed apart from the full graph block which represents the horizon there are finite number of nodes with the smaller degrees which are placed at the different radial coordinate. Since they are connected with the full graph core, we have a kind of BH hair. Upon the evaporation the star-like graph gets transforms into the single hub but the hair remains intact.

5. Conclusion

In this article, we have considered numerically the finite-size effects in exponential random graphs. There are no approximations and assumptions implied in the previous studies. We have found critical network sizes $N^*(p)$ when the crossover takes place. The key process at the crossover region is a evaporation of the star-like configuration or the single cluster which represents the ground state at $N < N^*$. The final stage of the cluster evaporation in the two-star model is the single complete star.

We have briefly discussed the possible microscopic explanation of the crossover regime but additional analysis is certainly required. It is also highly desirable to elaborate the analogy with a BH evaporation and localization at heavy root in the random network environment further.

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