Generalized superstatistics of nonequilibrium Markovian systems

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The paper is devoted to the *construction* of the superstatistical description for nonequilibrium Markovian systems. It is based on Kirchhoff's diagram technique and the assumption on the system under consideration to possess a wide variety of cycles with vanishing probability fluxes. The latter feature enables us to introduce equivalence classes called channels within which detailed balance holds individually. Then stationary probability as well as flux distributions are represented as some sums over the channels. The latter construction actually forms the superstatistical description, which, however, deals with a certain superposition of equilibrium subsystems rather then is a formal expansion of the nonequilibrium steady state distribution into terms of the Boltzmann type.

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SUPERSTATISTICS AND STATIONARY STATES OF NONEQUILIBRIUM MARKOVIAN SYSTEMS

During the last several years there has been considerably grown interest in the description of essentially nonequilibrium systems using quasiequilibrium notions, namely, the concept of "superstatistics" by Beck & Cohen [1]. Briefly, it assumes the stationary state $\mathcal{P}_i^{\rm st}$ of a nonequilibrium system can be written in the Boltzmann form with averaging over possible fluctuations in the inverse temperature β,

$$\mathcal{P}_i^{\rm st} = \int_0^\infty d\beta\, f(\beta) \frac{1}{Z(\beta)} \exp\{-\beta E_i\}\,, \qquad (1)$$

where $f(\beta)$ is the probability distribution of the inverse temperature, E_i is the effective energy of the system state i, and $Z(\beta)$ is the partition function for a fixed value of β. Representation (1) is actually a generalization of the so-called nonextensive statistics introduced by Tsallis [2] and in a integral form relates powerlike and Boltzmann distributions [3].

The main idea of superstatistics, however, seems to have a longer history [4], at least, expressions similar to Eq. (1) can be found in monograph by Lavenda [5] and this problem goes back to Szilard [6] and Mandelbrot [6] as well as the results of Hungarian school on information theory [8]. Moreover, in an effort to derive representation (1) starting from the general description of statistical systems ones have met fundamental problems and inconsistencies [9, 10, 11]. On the other hand, the superstatistical description is rather natural especially for systems exhibiting large-scale fluctuations in temperature gradients [12] or flow turbulence, where the spatiotemporal fluctuations in the energy dispersion is the standard fact going back to early work by Kolmogorov [13].

In spite of fundamental problems met in justifying expression (1) the number of papers dealing with the superstatistics has increased remarkably in the last years. In particular, it has been applied to Lagrangian [14, 15, 16, 17] and Eulerian turbulence [18, 19, 20], defect turbulence [21], atmospheric turbulence [22, 23], cosmic ray statistics [24], statistics of solar flares [25], hadronization of quark matter [26], small-world networks [27], multi-components self-gravitating systems and collisionless stellar systems [28], transitions between regular-chaotic dynamics [29, 30], particle ensembles with fractional reactions [31], analysis of time series [32], econophysics [33, 34, 35].

In some sense expression (1) can be interpreted in two fashions. The first way is to regard expression (1) as a rather formal expansion of the stationary distribution for a nonequilibrium system over terms having the Boltzmann form. In this case the main problem is the relationship between the weights $f(\beta)$ of this expansion and specific random processes governing the system dynamics, which is currently the main direction of researches carried out in this field (see, e.g., Ref. [36]). The other could be an attempt to represent a nonequilibrium system with nonzero stationary probability flux as a certain superposition of its subsystems being equilibrium, i.e. within which the detailed balance holds individually. Exactly this idea is the goal of our study. The purpose of the present paper is to implement this approach to describing the stationary properties of a nonequilibrium Markovian system. Naturally, the final integral over possible subsystems with local equilibrium has to be of a more general form than formula (1) and is reduced to it in special cases only, which is the reason of using the term “generalized superstatistics”.

There has been a great deal of studying nonequilibrium Markovian systems within the frameworks of the master equation, for a review see, e.g., Refs. [37, 38, 39]. During the last decades many nonequilibrium systems as well as systems of other nature where the equilibrium notion is irrelevant came into view of physical society. This, in particular, has reawaken the interest to the general steady state properties exhibited by Markovian systems without detailed balance and posed a question about their minimal mesoscopic description called dynamic equivalence classes [40, 41] (see also Ref. [42]). Within the latter approach it makes an attempt to find aggregated characteristics determining the steady state distribution as well as probability fluxes in a nonequilibrium system without detailed description of all the transition rates between the system states. In the present work we actually follow the spirit of this idea.

QUASIEQUILIBRIUM CHANNELS OF A NONEQUILIBRIUM MARKOVIAN SYSTEM

We consider a Markovian system with a finite number of states $\{i\}$. This number, however, may take any large value, so, there should be a feasibility to generalize the following constructions to systems with infinite number of states. The system evolution is described by the master equation

$$\frac{d\mathcal{P}_i}{dt} = \sum_{j\neq i} \langle i|j\rangle\, \mathcal{P}_j - \langle j|i\rangle\, \mathcal{P}_i \tag{2}$$

written for the distribution function $\mathcal{P}_i$, where $\langle i|j\rangle$ stands for the rate of system transitions to state i from state j. All the system states make up a graph $\mathbb{G}$ whose edges present possible transitions between the states. Without loss of generality we may adopt two assumptions about this graph, following, e.g., Ref. [37]. First, if there exists a transition from some state j to some state i, i.e. $\langle i|j\rangle > 0$, then the reverse transition is also possible, i.e. $\langle j|i\rangle > 0$. The opposite case is included within the limit $\langle j|i\rangle \to 0$. Second, the graph $\mathbb{G}$ is connected, which implies that for each pair of states, i.e. graph nodes (i,j), there exists at least one path $\mathbb{P}_{ij}$ on the graph $\mathbb{G}$ (sequence of joint edges) connecting them. Otherwise, the physical system behind the graph $\mathbb{G}$ can be decomposed into two or more independent subsystems analyzed individually.

Kirchhoff’s diagrams

The following part of the paper will deal only with the steady state properties of such Markovian systems described by the stationary solution $P_i = \mathcal{P}_i^{\rm st}$ of the master equation (2). This solution is represented using Kirchhoff’s diagram technique (see, e.g. Refs. [37, 43]). It applies to the notion of maximal trees. By definition, a maximal tree $\mathbb{T}$ for the given graph $\mathbb{G}$ is its subgraph without cycles that contains all the nodes. Figure 1 illustrates the division of a three state system into its maximal trees and Figure 2 depicts some maximal tree $\mathbb{T}$ for a six state system. Each node i specifies the realization $\mathbb{T}_i$ of a given tree $\mathbb{T}$ orienting its edges in such a way that for any node j they make up the path $\mathbb{P}_{ij}$ leading from this node j to the node i. Then ascribing the transition rate $\langle i|j\rangle$ to the directed edge (i,j) the stationary solution of equation (2) is written as

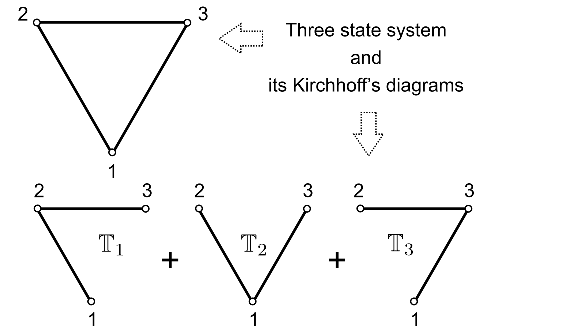

FIG. 1: A three state system and its division into maximal trees.

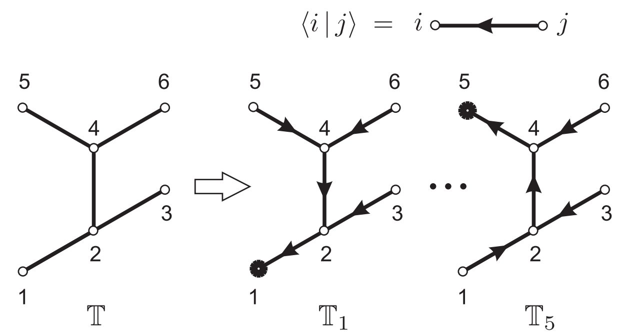

FIG. 2: Example of a maximal tree for a six state system and its realization for two states, 1 and 5. The correspondence between the graph edges and the transition rates $\langle i|j\rangle$ is shown above.

$$P_i = \frac{1}{Z}\sum_{\mathbb{T}} \prod_{(kl)\in\mathbb{T}_i} \langle k|l\rangle = \frac{1}{Z}\sum_{\mathbb{T}} \exp\left[-\mathcal{H}_i\left(\mathbb{T}\right)\right]\,, \tag{3}$$

where Z is the normalization constant (partition function) and the effective energy $\mathcal{H}_i\left(\mathbb{T}\right)$ of the state i within the tree $\mathbb{T}$ is defined as

$$\mathcal{H}_i\left(\mathbb{T}\right) = \sum_{(kl)\in\mathbb{T}_i} -\ln\left[\langle k|l\rangle\right]\,. \tag{4}$$

In particular, for a given tree $\mathbb{T}$ the difference $\mathcal{H}_i(\mathbb{T}) - \mathcal{H}_j(\mathbb{T})$ meets the relation

$$\mathcal{H}_i(\mathbb{T}) - \mathcal{H}_j(\mathbb{T}) = \ln\left[\prod_{(kl)\in\mathbb{P}_{ji}} \langle k|l\rangle\right] - \ln\left[\prod_{(kl)\in\mathbb{P}_{ij}} \langle k|l\rangle\right], \quad (5)$$

for example, for the trees shown in Fig. 2 we have

$$\mathcal{H}_5(\mathbb{T}) - \mathcal{H}_1(\mathbb{T}) = \ln\left[\langle 1|2\rangle\langle 2|4\rangle\langle 4|5\rangle\right] - \ln\left[\langle 5|4\rangle\langle 4|2\rangle\langle 2|1\rangle\right].$$

The steady state probability flux J_{ij} through the edge (ij) from the node j to the node i is, by definition, $J_{ij} := \langle i|j\rangle P_j - \langle j|i\rangle P_i$. So by virtue of (3) the equality

$$J_{ij} = \frac{1}{Z}\sum_{\mathbb{T}} \langle i|j\rangle e^{-\mathcal{H}_j\{\mathbb{T}\}} - \langle j|i\rangle e^{-\mathcal{H}_i\{\mathbb{T}\}} \equiv \frac{1}{Z}\sum_{\mathbb{T}} J_{\mathbb{C}\{(i,j),\mathbb{T}\}} \prod_{(kl)\in\mathbb{T}_{ij}} \langle k|l\rangle \quad (6)$$

holds. Here $\mathbb{C}\{(i,j),\mathbb{T}\}$ is the cycle created via connecting the node j to the node i and its forward tracing is given by the transition from j to i, the set $\mathbb{T}_{ij} = \mathbb{T}\setminus\mathbb{C}\{(i,j),\mathbb{T}\}$ is the collection of subtrees remaining after the edges of cycle $\mathbb{C}\{(i,j),\mathbb{T}\}$ having been removed from the tree $\mathbb{T}$, and

$$J_{\mathbb{C}\{(i,j),\mathbb{T}\}} = \prod_{(kl)\in\mathbb{C}^+\{(i,j),\mathbb{T}\}} \langle k|l\rangle - \prod_{(kl)\in\mathbb{C}^-\{(i,j),\mathbb{T}\}} \langle k|l\rangle \quad (7)$$

is the partial probability flux through the edge (i,j) related to the cycle $\mathbb{C}\{(i,j),\mathbb{T}\}$, where the superscripts $+$ and $-$ label the forward and backward directions of the cycle tracing. We point our that the quantity $J_{\mathbb{C}\{(i,j),\mathbb{T}\}}$ as well as the total probability flux J_{ij} is antisymmetric with respect to the index interchange, whereas the set $\mathbb{T}_{ij}$ and, as a consequence, its contribution to the flux J_{ij} remain the same under this transformation. The items in sum (6) are depicted by the diagram in Fig. 3. The existence of nonzero probability flux under the steady state conditions is actually the manifestation of the system being nonequilibrium and detailed balance not holding.

Three state system. Illustrating example

To illustrate Kirchhoff's technique described above we consider the three state system shown in Fig. 1 and imitating particle hopping in an random medium. Without some driving field $\mathcal{E}$ the system is assumed to be equilibrium one with the detailed balance, which is reduced to the equality

$$\langle 1|2\rangle\langle 2|3\rangle\langle 3|1\rangle = \langle 1|3\rangle\langle 3|2\rangle\langle 2|1\rangle,$$

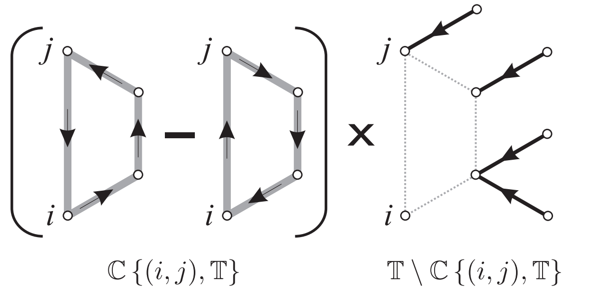

FIG. 3: Diagram visualizing an item in sum (6). Here $\mathbb{C}\{(i,j),\mathbb{T}\}$ is the cycle created by connecting the nodes j, i of the minimal tree $\mathbb{T}$, where the forward direction of cycle tracing is given by transition from the node j to the node i.

implying the absence of the probability flux along the cycle 1-2-3-1. The particle hopping between potential traps $\{u_j\}$ is described by the transition rates

$$\langle i|j\rangle = \omega e^{-u_j},$$

where ω is some kinetic coefficient. The field $\mathcal{E}$ breaks the detailed balance disturbing the transition rates as follows

$$\begin{aligned} \langle 1|2\rangle_{\mathcal{E}} &= \langle 1|2\rangle e^{\mathcal{E}}, & \langle 2|1\rangle_{\mathcal{E}} &= \langle 2|1\rangle e^{-\mathcal{E}}, \\ \langle 2|3\rangle_{\mathcal{E}} &= \langle 2|3\rangle e^{\mathcal{E}}, & \langle 3|2\rangle_{\mathcal{E}} &= \langle 3|2\rangle e^{-\mathcal{E}}, \\ \langle 3|1\rangle_{\mathcal{E}} &= \langle 3|1\rangle e^{\mathcal{E}}, & \langle 1|3\rangle_{\mathcal{E}} &= \langle 1|3\rangle e^{-\mathcal{E}}. \end{aligned}$$

Then using the general expression (3) for the stationary distribution and applying to Fig. 1 showing the corresponding collection of maximal trees we get the expression for the partition function

$$Z(\mathcal{E}) = \omega^2 e^{-u_1-u_2-u_3}\left(e^{u_1}+e^{u_2}+e^{u_3}\right)\left[1+2\cosh(2\mathcal{E})\right].$$

Then by virtue of (6) the probability flux along any edge of this system is

$$J = \frac{2\omega\sinh\mathcal{E}}{(e^{u_1}+e^{u_3}+e^{u_3})}.$$

Below the effective energy $\mathcal{H}_i(\mathbb{T})$ for the state i within a given tree $\mathbb{T}$ has been introduced by expression (4). For the three state system under consideration applying to Fig. 1 we can write, for example,

$$\begin{aligned} \mathcal{H}_2(\mathbb{T}_1) - \mathcal{H}_1(\mathbb{T}_1) &= 2\mathcal{E} - (u_2-u_1), \\ \mathcal{H}_3(\mathbb{T}_1) - \mathcal{H}_1(\mathbb{T}_1) &= 4\mathcal{E} - (u_3-u_1), \\ \mathcal{H}_2(\mathbb{T}_2) - \mathcal{H}_1(\mathbb{T}_2) &= 2\mathcal{E} - (u_2-u_1), \\ \mathcal{H}_3(\mathbb{T}_2) - \mathcal{H}_1(\mathbb{T}_2) &= -2\mathcal{E} - (u_3-u_1). \end{aligned}$$

Whence it follows that the effective energies $\mathcal{H}_i(\mathbb{T}_1)$, $\mathcal{H}_i(\mathbb{T}_2)$ of the trees $\mathbb{T}_1$, $\mathbb{T}_2$ cannot be represented in terms of one energy H_i^{ss} multiplied by some individual cofactors β_1, β_2, i.e. there is no function H_i^{ss} such that $\mathcal{H}_i(\mathbb{T}_1)\mapsto\beta_1 H_i^{ss}$ and $\mathcal{H}_i(\mathbb{T}_2)\mapsto\beta_2 H_i^{ss}$. Indeed, otherwise, the relations $\beta_1=\beta_2$ and $\beta_1\neq\beta_2$ have to hold simultaneously.

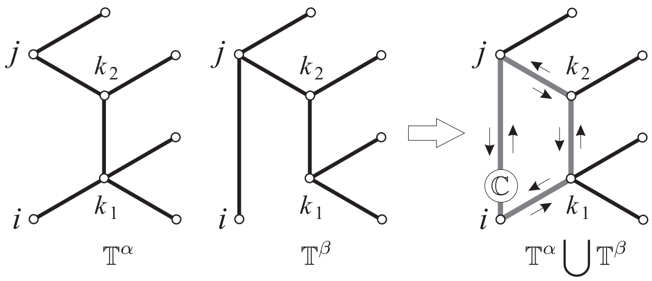

FIG. 4: Illustration of the cycle formation by the superposition of maximal trees.

Equivalence classes of maximal trees. Channels

At the next step a special case of Markovian systems is analyzed, where there are collections of many maximal trees within which the detailed balance holds individually. Namely, let us consider some two maximal trees $\mathbb{T}^\alpha$, $\mathbb{T}^\beta$, and the union of their edges $\mathbb{T}^\alpha \bigcup \mathbb{T}^\beta$ referred below as to the tree superposition. The two trees are said to be equivalent, $\mathbb{T}^\alpha \sim \mathbb{T}^\beta$, if their superposition does not contain any cycle $\mathbb{C}$ of edges with nonzero probability flux $J_\mathbb{C} \neq 0$, which is illustrated in Fig. 4. Applying, for example, to the cycle $\mathbb{C}$ shown in Fig. 4 the condition of zero probability flux reads

$$\langle i|j\rangle \langle j|k_2\rangle \langle k_2|k_1\rangle \langle k_1|i\rangle = \langle i|k_1\rangle \langle k_1|k_2\rangle \langle k_2|j\rangle \langle j|i\rangle$$

thus

$$\frac{\langle j|k_2\rangle \langle k_2|k_1\rangle \langle k_1|i\rangle}{\langle i|k_1\rangle \langle k_1|k_2\rangle \langle k_2|j\rangle} = \frac{\langle j|i\rangle}{\langle i|j\rangle}$$

whence, by virtue of (5), we get the identity

$$\mathcal{H}_i(\mathbb{T}^\alpha) - \mathcal{H}_j(\mathbb{T}^\alpha) = \mathcal{H}_i\left(\mathbb{T}^\beta\right) - \mathcal{H}_j\left(\mathbb{T}^\beta\right) .$$

This example demonstrates the fact that for equivalent trees, e.g., $\mathbb{T}^\alpha$and $\mathbb{T}^\beta$ their effective energies differ in constant value only, with latter statement being actually an equipollent definition of the tree equivalence,

$$\mathbb{T}^\alpha \sim \mathbb{T}^\beta \Longleftrightarrow \mathcal{H}_i(\mathbb{T}^\alpha) - \mathcal{H}_i\left(\mathbb{T}^\beta\right) = \text{const}. \tag{8}$$

Thereby the introduced relationship between the maximal trees really form an equivalence relation because the condition $\mathbb{T}^\alpha \sim \mathbb{T}^\beta$ and $\mathbb{T}^\alpha \sim \mathbb{T}^\gamma$ gives rise to $\mathbb{T}^\beta \sim \mathbb{T}^\gamma$. Therefore the collection of all the maximal trees $\{\mathbb{T}\}$ of the state graph $\mathbb{G}$ can be divided in the classes $\{\mathbb{K}\}$ of equivalent trees that will be called channels. The superposition of all the maximal trees belonging to one channel $\mathbb{K}$ i.e. its implementation as a subgraph of graph $\mathbb{G}$ will be also referred to as just the channel $\mathbb{K}$.

The fact that the notion of channels does have some meaning is justified in Fig. 5 exhibiting a case where there are channels contaning more then one maximal tree and not coinciding with the system as a whole. Namely, this figure depicts a four state system for which there are two elementary (three state) cycles with zero probability flux (cycles 1-3-4-1 and 2-3-4-2) and two ones with nonzero flux (cycles 1-3-2-1 and 1-2-4-1). These cycles, indeed, can have such properties, which requires some comments. The matter is that the fluxes $J_{\text{1-3-2-1}}$ and $J_{\text{1-2-4-1}}$ are not independent because the compound cycle 1-3-2-4-1 can be obtained unifying either the cycles 1-3-2-1 and 1-2-4-1 or the cycles 1-3-4-1 and 2-3-4-2. The unification of cycles with zero probability flux inevitably gives rise to the compound cycle with zero flux too. So the fluxes $J_{\text{1-3-2-1}}$ and $J_{\text{1-2-4-1}}$ should also meet the condition of zero probability flux for the cycle 1-3-2-4-1. The detailed analysis of the relationship between the probability fluxes of elementary and compound cycles is beyond the scope of the present paper. Here we note only that the adopted flux pattern can be implemented for the given graph when all its edges are symmetrical with respect to the transition rate, $\langle i|j\rangle = \langle j|i\rangle$, except for the edge $(1,2)$, where $\langle 1|2\rangle \neq \langle 2|1\rangle$. The induced partition of the maximal tree collection into the channels and their implementation in graph form are shown in Fig. 5.

Channel superstatistics of the steady state distribution

In order to describe the properties of a maximal tree $\mathbb{T}$ within its channel $\mathbb{K}$ containing $N_\mathbb{K}$ trees, first, the effective energy $\mathcal{H}_i\{\mathbb{K}\}$ averaged over the given channel

$$H_\mathbb{K}(i) = \frac{1}{N_\mathbb{K}} \sum_{\mathbb{T}'\in\mathbb{K}} \mathcal{H}_i(\mathbb{T}') \tag{9}$$

is introduced. The quantity $H_i(\mathbb{K})$ will be called the effective energy of the state i within the channel $\mathbb{K}$. Then, using the constructed function the effective energy $\mathcal{H}_i(\mathbb{T})$ of the state i within the tree $\mathbb{T}$ in rewritten as

$$\mathcal{H}_i(\mathbb{T}) = H_\mathbb{K}(i) + U(\mathbb{T}|\mathbb{K}) , \tag{10}$$

where the value

$$U(\mathbb{T}|\mathbb{K}) := \frac{1}{N_\mathbb{K}} \sum_{\mathbb{T}'\in\mathbb{K}} \left[\mathcal{H}_i(\mathbb{T}') - \mathcal{H}_i(\mathbb{T})\right] \tag{11}$$

is regarded as the effective energy of the tree $\mathbb{T}$ within the channel $\mathbb{K}$ because due to property (8) it does not depend on the state i and, thus, characterizes the tree $\mathbb{T}$ as a whole. Naturally, the mean value of $U(\mathbb{T}|\mathbb{K})$ is equal to zero,

$$V_\mathbb{K} = \frac{1}{N_\mathbb{K}} \sum_{\mathbb{T}\in\mathbb{K}} U(\mathbb{T}|\mathbb{K}) = 0 . \tag{12}$$

The introduced quantities permit us to rewrite expression (3) for the steady state distribution of the given

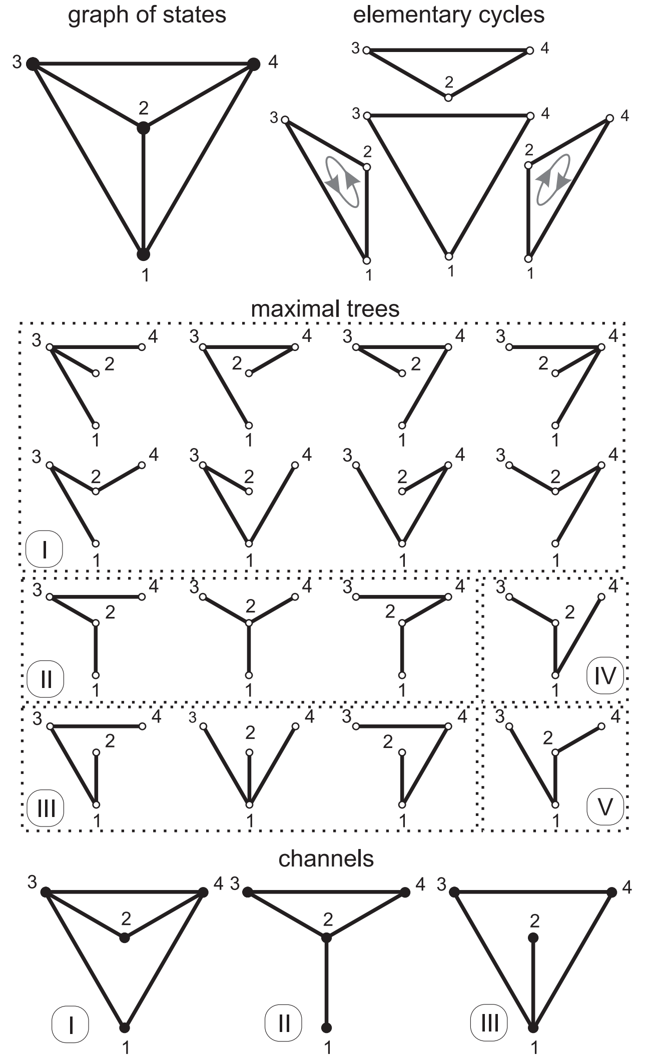

FIG. 5: An example of four state system and its division into channels I–V. In addition the elementary cycles with zero and nonzero probability flux are shown.

Markovian system as a sum running over the channels

$$P_i = \frac{1}{Z}\sum_{\mathbb{K}} N_{\mathbb{K}} \mathrm{e}^{-F_{\mathbb{K}}} \exp\left\{-H_{\mathbb{K}}(i)\right\}$$
$$= \frac{1}{Z}\sum_{\mathbb{K}} w(\mathbb{K}) \exp\left\{-H_{\mathbb{K}}(i)\right\} . \qquad (13)$$

Here the quantity $F_{\mathbb{K}}$ has appeared in formula (13) via the sum over all the maximal trees $\{\mathbb{T}\}_{\mathbb{K}}$ composing the channel $\mathbb{K}$

$$\exp\left(-F_{\mathbb{K}}\right) := \frac{1}{N_{\mathbb{K}}}\sum_{\mathbb{T}\in\mathbb{K}} \exp\left\{-U\left(\mathbb{T}|\mathbb{K}\right)\right\} \qquad (14)$$

and specifies the statistical weight of channel $\mathbb{K}$

$$w(\mathbb{K}) = N_{\mathbb{K}} e^{-F_{\mathbb{K}}} .$$

In order to write the desired expression for $F_{\mathbb{K}}$ we make use of the expansion

$$e^{-U} = 1 - U + \varphi(U)\,, \text{ where } \varphi(U) > 0 \text{ for } U \neq 0\,.$$

Whence, by virtue of (12),

$$\exp\left(-F_{\mathbb{K}}\right) = 1 + \frac{1}{N_{\mathbb{K}}}\sum_{\mathbb{T}\in\mathbb{K}} \varphi\left[U\left(\mathbb{T}|\mathbb{K}\right)\right] \overset{\text{def}}{=} \exp\left\{\theta_{\mathbb{K}} S_{\mathbb{K}}\right\}\,, \qquad (15)$$

where $S_K := \ln N_{\mathbb{K}}$ has the meaning of channel entropy. Due to the zero value of the mean channel energy $V_{\mathbb{K}} = 0$ the quantity $F_{\mathbb{K}}$ can be rewritten as $F_{\mathbb{K}} = V_{\mathbb{K}} - \theta_{\mathbb{K}} S_{\mathbb{K}}$ and keeping in mind the standard notions of statistical physics it will be called the free energy of channel $\mathbb{K}$. In particular, the quantity

$$\theta_{\mathbb{K}} = \frac{1}{\ln N_{\mathbb{K}}} \ln\left\{1 + \frac{1}{N_{\mathbb{K}}}\sum_{\mathbb{T}\in\mathbb{K}} \varphi\left[U\left(\mathbb{T}|\mathbb{K}\right)\right]\right\} \qquad (16)$$

is an order parameter of the channel structure; when all the trees of a channel $\mathbb{K}$ have the same energy $U\left(\mathbb{T}|\mathbb{K}\right) = U\left(\mathbb{K}\right)$ the value $\theta = 0$ and the wider the distribution of the tree energies, the large the value $\theta_{\mathbb{K}}$.

Finalizing this section we rewrite formula (13) as

$$P_i = \frac{1}{Z}\sum_{\mathbb{K}} N_{\mathbb{K}} \exp\left\{\theta_{\mathbb{K}} S_{\mathbb{K}} - H_{\mathbb{K}}(i)\right\}\,. \qquad (17)$$

Expression (17) is the desired superstatistics representation of the steady state distribution for a nonequilibrium Markovian system. It should be noted that the obtained effective energy $H_{\mathbb{K}}(i)$ of the system states $\{i\}$ within the channel $\mathbb{K}$ can depend on many parameters of this channel and is reduced to some fixed function $H(i)$ multiplied by an inverse channel "temperature", $H_{\mathbb{K}}(i) = \beta_{\mathbb{K}} H(i)$, in special cases only. This could take place if, for example, the distribution of the tree energies within one channel and the distribution of the system states are caused by the same mechanism. In this case it might be expected that the channel temperature will be specified by the order parameter $\theta_{\mathbb{K}}$, i.e. $1/\beta_{\mathbb{K}} = \theta_{\mathbb{K}}$.

Figure 6 visualizes an example of systems where differen channels, at least some ones, can be ascribed with the effective energies for the system states that differ only by some constant prefactors. It is a stair case system with the transition rates shown in Fig. 6, where the lower two diagrams depict equivalent maximal trees. Rigorously speaking, these trees become equivalent only on scales $\delta x \gg a$ or, what is actually the same, in the limit $a \to 0$. Indeed, at the first approximation the energy of a state i located near the point x along the system (Fig. 6) can be written as

$$H_{\mathbb{K}}(x) = 2\left(q_u\mathcal{E}_u + q_l\mathcal{E}_l\right)\frac{x}{a} + \text{const}\,,$$

where $q_{u,l}$ is the relative portion of edges on the upper and lower branches belonging to the trees of a given channel, $q_u + q_l = 1$. In some sense the value q_u (or q_l, what is

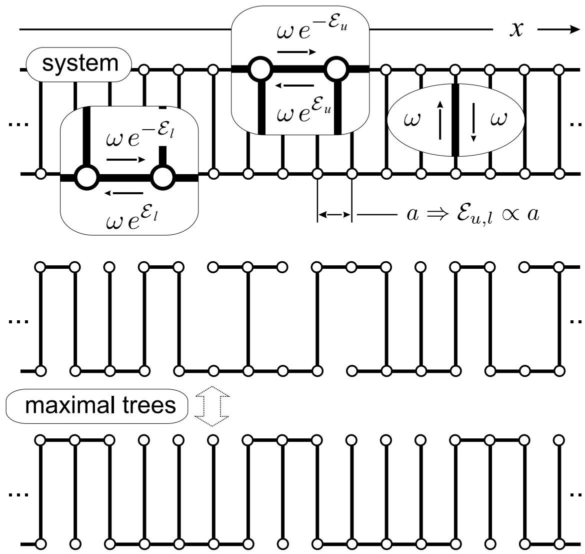

FIG. 6: Example of a system where at least some channels can be characterized by the effective energies of the states differing only by constant prefactors.

the same) is the main characteristics of such a channel. It is assumed to be distributed rather uniformly along the system because exactly the uniform distribution matches the maximal number of the channel realizations in the maximal trees. Naturally, there are channels in this system where the portion q_u and, thus, also q_l can depend on the coordinate x and, thereby, their effective energies are not reduced to the presented form. However, their contribution to the steady state distribution could be rather small.

Channel superstatistics of the steady state probability flux

The steady state probability flux $\{J_{ij}\}$ through the graph edges can be also reduced to a sum over the channels. In order to do this we make use of formula (6). Let us consider an edge (i,j) and two trees $\mathbb{T}^\alpha$, $\mathbb{T}^\beta \in \mathbb{K}$ belonging to one channel $\mathbb{K}$. The given edge (i,j) is assumed beforehand not to belong to the channel $\mathbb{K}$ because otherwise all its maximal trees do not contribute to the flux J_{ij}. The edge (i,j) with the edges of the trees $\mathbb{T}^\alpha$ and $\mathbb{T}^\beta$ form two cycles $\mathbb{C}^\alpha = (i,j)\bigcup\mathbb{P}^\alpha_{i,j}$ and $\mathbb{C}^\beta = (i,j)\bigcup\mathbb{P}^\beta_{i,j}$, where $\mathbb{P}^\alpha_{i,j}$, $\mathbb{P}^\beta_{i,j}$ are paths on the trees $\mathbb{T}^\alpha$, $\mathbb{T}^\beta$, respectively, connecting the nodes i and j (Fig. 7). The paths $\mathbb{P}^\alpha_{i,j}$, $\mathbb{P}^\beta_{i,j}$ can coincide with each other partly or even completely. Since the two trees belong to one channel the probability flux along the composite cycle $\mathbb{C}^{\alpha\beta} = \mathbb{C}^\alpha\bigcup\mathbb{C}^\beta = \mathbb{P}^\alpha_{i,j}\bigcup\mathbb{P}^\beta_{i,j}$ is equal to zero. The latter condition is implemented by the equality

$$\prod_{(kl)\in\mathbb{P}^{\alpha+}}\langle k|l\rangle\prod_{(kl)\in\mathbb{P}^{\beta-}}\langle k|l\rangle=\prod_{(kl)\in\mathbb{P}^{\alpha-}}\langle k|l\rangle\prod_{(kl)\in\mathbb{P}^{\beta+}}\langle k|l\rangle\,, \tag{18}$$

in particular, for the case shown in Fig. 7

$$\langle j|\alpha_1\rangle\langle\alpha_1|\alpha_2\rangle\langle\alpha_2|i\rangle\langle i|\beta_2\rangle\langle\beta_2|\beta_1\rangle\langle\beta_1|j\rangle = \langle j|\beta_1\rangle\langle\beta_1|\beta_2\rangle\langle\beta_2|i\rangle\langle i|\alpha_2\rangle\langle\alpha_2|\alpha_1\rangle\langle\alpha_1|j\rangle\,.$$

Let us introduce the intensities R^α_{ij}, R^β_{ij}, and the asymmetry $\mathcal{A}_{ij}(\mathbb{K})$ of the transitions along the paths $\mathbb{P}^\alpha_{i,j}$, $\mathbb{P}^\beta_{i,j}$ via the expressions

$$R^{\alpha,\beta}_{ij} = \left[\prod_{(kl)\in\mathbb{P}^{\alpha,\beta+}}\langle k|l\rangle\prod_{(kl)\in\mathbb{P}^{\alpha,\beta-}}\langle k|l\rangle\right]^{1/2}, \tag{19}$$

$$\begin{aligned} e^{\mathcal{A}_{ij}(\mathbb{K})} &= \left[\prod_{(kl)\in\mathbb{P}^{\alpha+}}\langle k|l\rangle\right]^{1/2}\cdot\left[\prod_{(kl)\in\mathbb{P}^{\alpha-}}\langle k|l\rangle\right]^{-1/2} \\ &= \left[\prod_{(kl)\in\mathbb{P}^{\beta+}}\langle k|l\rangle\right]^{1/2}\cdot\left[\prod_{(kl)\in\mathbb{P}^{\beta-}}\langle k|l\rangle\right]^{-1/2} \end{aligned} \tag{20}$$

with the value $\mathcal{A}_{ij}(\mathbb{K})$ being the same for both the paths $\mathbb{P}^\alpha_{i,j}$, $\mathbb{P}^\beta_{i,j}$ due to (18). In other words, the quantity $\mathcal{A}_{ij}(\mathbb{K})$ is the characteristics of the edge (i,j) with respect to the channel $\mathbb{K}$ rather than to its trees individually. It should be pointed out that the quantities R^α_{ij} are symmetrical whereas $\mathcal{A}_{ij}(\mathbb{K})$ is asymmetrical with respect to the index interchange, i.e., $R^\gamma_{ij} = R^\gamma_{ji}$ and $\mathcal{A}_{ij}(\mathbb{K}) = -\mathcal{A}_{ji}(\mathbb{K})$. In addition, let the quantities P^α_{ij} and P^β_{ij} stand for the contributions of the subtree collections $\mathbb{T}^\alpha\setminus\mathbb{C}^\alpha$ and $\mathbb{T}^\beta\setminus\mathbb{C}^\beta$ (Fig. 3) to the tree weights $\exp\{-\mathcal{H}_i(\mathbb{T}^\alpha)\}$ and $\exp\{-\mathcal{H}_i(\mathbb{T}^\beta)\}$, respectively. As noted above the quantities P^γ_{ij} are symmetrical within the index interchange. Applying to definition (4) of the effective energy $\mathcal{H}_i(\mathbb{T}^\gamma)$ we write $(\gamma = \alpha,\beta)$

$$\begin{aligned} \exp\{-\mathcal{H}_i(\mathbb{T}^\gamma)\} &= R^\gamma_{ij}P^\gamma_{ij}\exp\{-\mathcal{A}_{ij}(\mathbb{K})\}\,, \\ \exp\{-\mathcal{H}_j(\mathbb{T}^\gamma)\} &= R^\gamma_{ij}P^\gamma_{ij}\exp\{\mathcal{A}_{ij}(\mathbb{K})\}\,, \end{aligned}$$

whence taking also into account relationship (10) between the effective energies of the system state i within the tree $\mathbb{T}^\gamma$ and the corresponding channel $\mathbb{K}$ we get

$$\begin{aligned} R^\gamma_{ij}P^\gamma_{ij} &= \exp\left\{-\frac{1}{2}\left[\mathcal{H}_i(\mathbb{T}^\gamma)+\mathcal{H}_j(\mathbb{T}^\gamma)\right]\right\} \\ &= \exp\left\{-\frac{1}{2}\left[H_{\mathbb{K}}(i)+H_{\mathbb{K}}(j)\right]-U(\mathbb{T}^\gamma|\mathbb{K})\right\}\,. \end{aligned}$$

In these terms the partial probability flux (see (6) and (7)) for the two trees $\mathbb{T}^\alpha$, $\mathbb{T}^\beta$ individually becomes ($\gamma =$

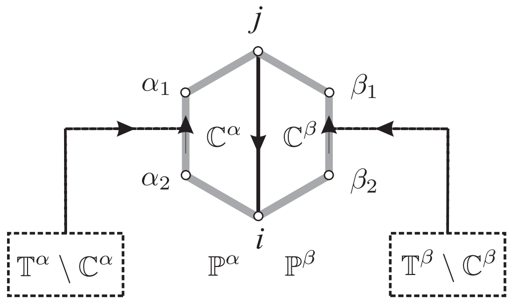

FIG. 7: Two cycles $\mathbb{C}^\alpha$ and $\mathbb{C}^\beta$ created via the connection of the nodes i and j on the maximal trees $\mathbb{T}^\alpha$ and $\mathbb{T}^\beta$ belonging to one channel $\mathbb{K}$. Arrows specify the forward direction of the cycle tracing.

$\alpha, \beta)$

$$\begin{aligned} J_{ij}\left(\mathbb{T}^\gamma\right) &= J_{\mathbb{C}\{(i,j),\mathbb{T}^\gamma\}} \prod_{(kl)\in\mathbb{T}^\gamma_{ij}} \langle k|l\rangle \\ &= \left[\langle i|j\rangle \exp\left\{\mathcal{A}_{ij}\left(\mathbb{K}\right)\right\} - \langle j|i\rangle \exp\left\{-\mathcal{A}_{ij}\left(\mathbb{K}\right)\right\}\right] R^\gamma_{ij} P^\gamma_{ij} \\ &= \mathcal{E}_{ij}\left(\mathbb{K}\right) \exp\left\{-\frac{1}{2}\left[H_\mathbb{K}(i) + H_\mathbb{K}(j)\right] - U\left(\mathbb{T}^\gamma|\mathbb{K}\right)\right\}, \end{aligned} \tag{21}$$

where the quantity $\mathcal{E}_{ij}\left(\mathbb{K}\right)$ introduced by the expression

$$\mathcal{E}_{ij}\left(\mathbb{K}\right) := \left[\langle i|j\rangle \exp\left\{\mathcal{A}_{ij}\left(\mathbb{K}\right)\right\} - \langle j|i\rangle \exp\left\{-\mathcal{A}_{ij}\left(\mathbb{K}\right)\right\}\right] \tag{22}$$

is ascribed directly to the edge (i,j) with respect to the channel $\mathbb{K}$. It characterizes the contribution of the channel $\mathbb{K}$ to the stationary probability flux through the edge (i,j), namely, by virtue of (6), (14), and (21)

$$J_{ij} = \frac{1}{Z}\sum_{\mathbb{T}} \mathcal{E}_{ij}\left(\mathbb{K}\right) N_\mathbb{K} e^{-F_\mathbb{K}} \exp\left\{-\frac{1}{2}\left[H_\mathbb{K}(i) + H_\mathbb{K}(j)\right]\right\}. \tag{23}$$

Expression (23) is the desired superstatistics representation of the stationary flux of probability. In some sense the nonequilibrium properties of the Markovian system under consideration are behind the driven forces $\{\mathcal{E}_{ij}\left(\mathbb{K}\right)\}$ induced by channel $\mathbb{K}$. In particular, if the edge (i,j) belongs to the channel $\mathbb{K}$ then $\mathcal{E}_{ij}\left(\mathbb{K}\right) = 0$.

CONCLUSION

In conclusion we have derived a general form of the stationary probability distribution for Markovian systems. This has allowed us to obtain a novel interpretation of probability distributions used in the framework of superstatistics, or more rigorously, generalized superstatistics. This notion has been successful in reproducing the probability distribution of many systems ranging from turbulence to economics. However, the justification of the form of the probability distribution as a weighted sum over equilibrium distributions up to now is based on purely phenomenological arguments. No fundamental derivation from the basic principles has been given.

In the present paper we have been able to derive an analogy to the superstatistics approximation with respect to the probability distribution as well as the stationary probability fluxes. This means that superstatistical representations can also be formulated for fluxes, which has not been discussed up to now. Our treatment is based on Kirchhoff's diagram technique applied to the master equation for Markovian processes. This techniques has been known for many years. We have included an assumption on the properties of the system graphs. As a main point we have focused on systems possessing a wide collection of cycles with vanishing fluxes. In this case it is possible to construct equivalence classes of Kirchhoff's maximal trees called channels, for which detailed balance holds individually. The nonequilibrium properties initially attributed to cycles can be assigned to individual edges, i.e. to individual transitions between a pair of states within one channel.

It is worthwhile to underline the fact that in the present paper we actually have formulated an original approach to describing steady states of nonequilibrium systems. In its spirit it is rather similar to the widely used notion of superstatistics but, nevertheless, differs from at the basics. The key point of the developed approach is the representation of a nonequilibrium system as a superposition of its equilibrium subsystems (channels) with local detailed balance rather then a formal expansion of the steady state distribution into the sum over terms of the Boltzmann type. Our considerations shed light on the network structures of complex systems and may help to understand systems in flux equilibrium like turbulent flows, traffic flows and economic systems. Beside, the introduced notion of channels enables one to pose a question as to whether it is possible to describe the dynamics of Markovian systems as transient processes in the channels individually and their interaction with one another.

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