

# Realization of Lévy flights as continuous processes

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On the basis of multivariate Langevin processes we present a realization of Lévy flights as a continuous process. For the simple case of a particle moving under the influence of friction and a velocity dependent stochastic force we explicitly derive the generalized Langevin equation and the corresponding generalized Fokker-Planck equation describing Lévy flights. Our procedure is similar to the treatment of the Kramers-Fokker Planck equation in the Smoluchowski limit. The proposed approach forms a feasible way of tackling Lévy flights in inhomogeneous media or systems with boundaries what is up to now a challenging problem.

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It has become evident that Brownian random walks may be a too simple description of diffusion processes in complex systems like the motion of tracer particles in turbulent flows [1], the diffusion of particles in random media [2], the motion of wandering albatrosses [3], human travel behavior and spreading of epidemics [4] or economic time series in finance [5]. A variable  $x$  corresponding to such a process can frequently exhibit the dynamics described by the notion of superfast diffusion, where the characteristic value  $\bar{x}$  of the variable  $x$  demonstrates scaling behavior  $[\bar{x}(t)]^2 \propto t^{2/\alpha}$  with  $\alpha < 2$  [6, 7].

Brownian motion is described on the basis of Langevin equations or, in a statistical sense, by the Fokker-Planck equation (cf. [8, 9]). A straightforward way to deal with anomalous diffusion is based on a generalization of the Langevin equations by replacing Gaussian white noise with Lévy noise [10]. Recently, there has been a great deal of research about superfast diffusion. It includes, in particular, a rather general analysis of the Langevin equation with Lévy noise (see, e.g., Ref. [11]) and the form of the corresponding Fokker-Planck equations [12, 13], description of anomalous diffusion with power law distributions of spatial and temporal steps [10, 14], Lévy flights in heterogeneous media [15, 16, 17] and in external fields [18, 19], first passage time analysis and escaping problem for Lévy flights [20, 21, 22, 23, 24, 25], as well as processing experimental data for detecting the Lévy type behavior [26]. Besides, it should be noted that the attempt to consider Lévy flights in bounded systems (see, e.g., Ref. [27, 28] and review [29] as well) has introduced the notion of Lévy walks being a non-Markovian process because of the necessity to bound the walker velocity.

The key point in constructing the mutually related pair of the stochastic Langevin equation and the non-local Fokker-Planck equation for superdiffusion is the Lévy-Gnedenko central limit theorem [12, 13, 30]. For the superdiffusion processes it specifies the possible step distributions  $P(\Delta x)$  which are universal and actually in-

dependent of the details in the system behavior at the microscopic level. In particular, for a symmetrical homogeneous 1D system superdiffusion can be regarded as a chain of steps  $\{\Delta x\}$  of duration  $\delta t$  whose distribution function  $P(\Delta x)$  exhibits the following asymptotic behavior for  $|\Delta x| \gg \bar{x}(\delta t)$

$$P(\Delta x) \sim \frac{[\bar{x}(\delta t)]^\alpha}{|\Delta x|^{\alpha+1}}. \quad (1)$$

In spite of the considerable success achieved in this field the theory of superdiffusion is far from being completed. For a given elementary step of any small duration it is impossible to single out some bounded domain that contains its initial  $x_i$  and terminal  $x_t$  points with the probability practically equal to unity because the second moment  $\langle (x_t - x_i)^2 \rangle$  diverges. This renders the description of Lévy flights in heterogeneous media or media with boundaries a challenging problem. Within the classical formulation the Lévy flight is not a spatially continuous processes. As a consequence it is not possible to attribute local characteristics to Lévy flights which might help to identify, e.g., the encounter time with boundaries. Bounding the particle velocity breaks the Lévy as well as the Markov properties.

The purpose of the present Letter is to formulate an approach to describing Lévy flights and Lévy walks using the *notion of continuous Markovian trajectories*. The key idea is to introduce the velocity as a second variable but staying with simple Gaussian noise. For a fixed time scale  $\delta t$  we can recover the standard behavior of Lévy flights. However, we have full locality in the sense that a trajectory can be determined with any desired resolution.

In the general form the model proposed for consideration is reduced to the class of coupled governing equations

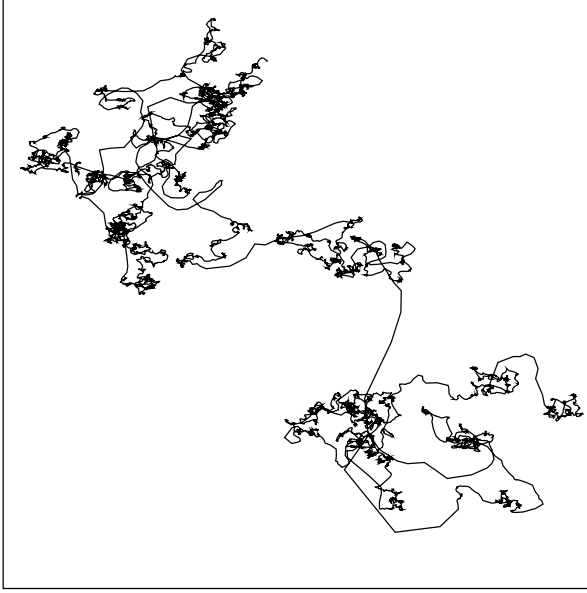


FIG. 1: Characteristic form of random walks described by 2D analogy of model (3). The used system parameters meet the Lévy exponent  $\alpha = 1.6$ .

for the state vectors  $\mathbf{x} = \{x_i\}$  and  $\mathbf{v} = \{v_i\}$

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}, \mathbf{v}), \quad (2a)$$

$$\frac{d\mathbf{v}}{dt} = \mathbf{H}(\mathbf{x}, \mathbf{v}) + \mathbf{G}(\mathbf{x}, \mathbf{v}) \cdot \boldsymbol{\xi}(t). \quad (2b)$$

Here the Langevin equation (2b) is written in the Itô form,  $\boldsymbol{\xi}(t) = \{\xi_i(t)\}$  are the collection of mutually independent Gaussian white noise components, the “forces”  $\mathbf{F}(\mathbf{x}, \mathbf{v})$ , and  $\mathbf{H}(\mathbf{x}, \mathbf{v})$  are given functions, and the matrix  $\mathbf{G}(\mathbf{x}, \mathbf{v})$  depending on the state variables specifies the intensity of Langevin “forces”. In some sense we reduce superdiffusion to a normal diffusion process expanding the phase space where a new variable, particle velocity, is governed by the Langevin equation with normal but multiplicative noise.

In this Letter we exemplify our procedure applying to the following 1D system with two variables, the coordinate of random walker  $x$  and its current velocity  $v$ ,

$$\frac{dx}{dt} = v, \quad (3a)$$

$$\frac{dv}{dt} = -\frac{(\alpha + 1)}{2\tau} v + \frac{1}{\sqrt{\tau}} g(v) * \xi(t). \quad (3b)$$

Here  $\tau$  is a certain time scale, the intensity of the Langevin random force is given by the function

$$g(v) = \sqrt{v_a^2 + v^2}, \quad (4)$$

with the parameter  $v_a$  measuring the intensity of the additive component of Langevin force,  $\xi(t)$  is white noise

such that  $\langle \xi(t)\xi(t') \rangle = \delta(t - t')$ , and the parameter  $\alpha \in (1, 2)$ . The Langevin equation (3b) is written in the Hänggi-Klimontovich form, which is indicated by the symbol  $*$ . The dynamics, resulting from a 2D version of these equations, is visualized in Fig. 1.

The corresponding forward Fokker-Planck equation for the distribution function  $\mathcal{P}(x - x_0, v, v_0, t)$  reads

$$\frac{\partial \mathcal{P}}{\partial t} = \frac{1}{2\tau} \frac{\partial}{\partial v} \left[ g^2(v) \frac{\partial \mathcal{P}}{\partial v} + (\alpha + 1)v\mathcal{P} \right] - \frac{\partial}{\partial x} [v\mathcal{P}], \quad (5)$$

where the values  $x_0$  and  $v_0$  specify the initial position of the walker.

The distribution of the walker velocities  $v_w$  is determined by the partial distribution function

$$P_v(v, v_0, t) := \langle \delta(v - v_w) \rangle \quad (6)$$

and, by virtue of (5), the stationary velocity distribution  $P_v^{\text{st}}(v)$  meets the equality

$$g^2(v) \frac{\partial P_v^{\text{st}}}{\partial v} + (\alpha + 1)vP_v^{\text{st}} = 0,$$

whence we immediately get the expression

$$P_v^{\text{st}}(v) = \frac{\Gamma(\frac{\alpha+1}{2})}{\sqrt{\pi} \Gamma(\frac{\alpha}{2})} \frac{v_a^\alpha}{[g(v)]^{\alpha+1}} \quad (7)$$

where  $\Gamma(\dots)$  is the Gamma-function. In addition, using the Fokker-Planck equation for function (6) following directly from (5) we find the expressions

$$\langle v(t) \rangle = v_0 \exp \left[ -\frac{(\alpha - 1)}{2} \frac{t}{\tau} \right], \quad (8a)$$

$$\langle v^2(t) \rangle = v_0^2 \exp \left[ (2 - \alpha) \frac{t}{\tau} \right] \quad \text{for } v_0 \gtrsim v_a \quad (8b)$$

characterizing actually the relaxation of the initial velocity distribution to its steady state form.

The found exponential decay of the first velocity moment demonstrates the fact that the Lévy walker “remembers” its velocity practically on time scales not exceeding the value  $\tau$ . The exponential divergence of the second moment (8b) indicates that the system relaxes to the stationary distribution (7) on time scales  $t \gg \tau$ . So, in some sense, the spatial steps of duration about  $\tau$  are mutually independent. In other words, the value  $\tau$  separates the time scales into two groups. On scales less than  $\tau$  the particle motion is strongly correlated and has to be considered using both the phase variables  $x$  and  $v$ . Thus, on a time scale  $\delta t \gg \tau$  the particle displacements are mutually independent and the succeeding steps of the Lévy walker form a Markovian chain, with the particle velocity playing the role of Lévy noise. This scenario is exemplified in Fig. 2 for some realization of  $v(t)$  following from equation (3b). Lévy flight events, i.e. the long-distance jumps of a Lévy walker, are due to large spikes

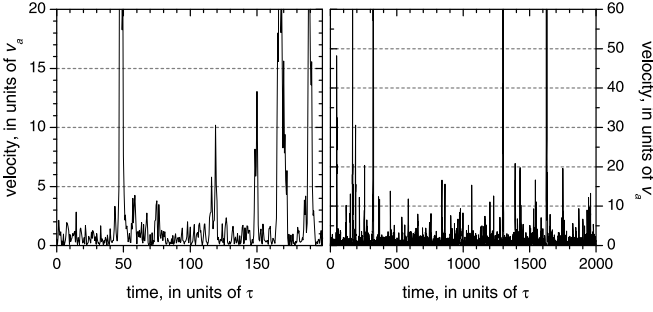


FIG. 2: Characteristic form of the time pattern  $v(t)$  exhibited by the stochastic system (3). The individual windows depict the patterns on various scales. In simulation the parameter  $\alpha = 1.6$  was used.

of the time pattern  $v(t)$  whose duration is about several  $\tau$ . More precisely, the long-distance displacement  $\Delta x$  of a walker during a certain time interval  $\delta t$  is mainly caused by the velocity spike of maximal amplitude  $\vartheta$  attained during the given interval, i.e.  $\Delta x \sim \vartheta \tau$ . For  $\delta t \gg \tau$  the quantity  $\{\vartheta\}$  is statistically uncorrelated during succeeding time intervals.

Now we proceed in a two steps. First, we use this simple physical picture to show via a combination of analytical and numerical evidence that the distribution function  $P_x(\Delta x, v_0, t) := \langle \delta(x - x_0 - \Delta x) \rangle$  indeed is of form (1) for  $t \gg \tau$ . Second, we strictly show that the corresponding generating function fulfills

$$G_x(\varkappa, t) := \left\langle e^{i(\varkappa \Delta x)/(v_a \tau)} \right\rangle \simeq \exp \left[ -\frac{\Gamma\left(\frac{2-\alpha}{2}\right)}{\Gamma(\alpha)\Gamma\left(\frac{\alpha}{2}\right)} \frac{t}{\tau} \varkappa^\alpha \right]. \quad (9)$$

The latter expression is the standard generating function of Lévy flights with exponent  $\alpha$  and matches the distribution (1).

If the spikes in Fig. 2 had the same shape and  $\delta t \gg \tau$  the normalized walker displacement  $\Delta x/\vartheta$  would be a constant of the order of  $\tau$  (in the limit of large  $\vartheta$  where  $\Delta x$  is largely determined by a single peak). Then  $P_x(\Delta x)$  would directly follow from the distribution of maximum velocities. To proceed we, first, make use of the relation between the extremum statistics of Markovian processes and the first passage time distribution [31]. Namely, the probability function  $\Phi(\vartheta, v, t)$  of the random variable  $\vartheta$  and the probability  $F(\vartheta, v, t)$  of passing the boundary  $v = \pm \vartheta$  for the first time at moment  $t$  are related as

$$\Phi(\vartheta, v, t) = -\frac{\partial}{\partial \vartheta} \int_0^t dt' F(\vartheta, v, t'). \quad (10)$$

Here  $v$  is the initial position of the Lévy walker. Then analyzing the Laplace transform of the first passage time

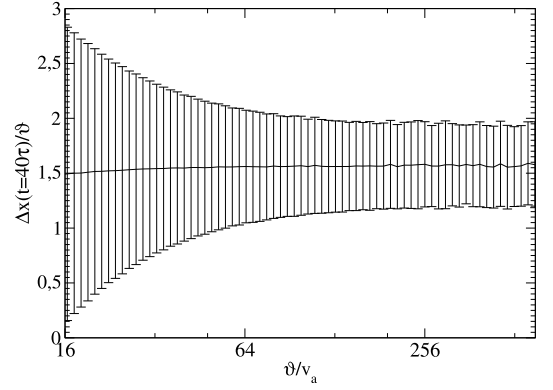


FIG. 3: The ratio  $\Delta x/\vartheta$  for individual steps vs the values of the random variable  $\vartheta$ . In simulation  $\alpha = 1.6$  was used.

distribution  $F_L(\vartheta, v, s)$  we will get the conclusion that the distribution function  $\Phi(\vartheta, v, t)$  of the velocity extrema  $\vartheta$  is of the form (see the supplementary materials)

$$\Phi(\vartheta, v, t) = \frac{1}{\vartheta(t)} \phi \left[ \frac{\vartheta}{\vartheta(t)} \right] \quad (11)$$

for  $v \ll \bar{\vartheta}(t)$  and  $\vartheta \gtrsim \bar{\vartheta}(t)$ . Here the quantity  $\bar{\vartheta}(t) = v_a(t/\tau)^{1/\alpha}$  is the velocity scale characterizing variations of the random value  $\vartheta$  and the function  $\phi(\zeta)$  possesses the asymptotics

$$\phi(\zeta) = \frac{\alpha^2 \Gamma\left(\frac{\alpha+1}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{\alpha}{2}\right)} \frac{1}{\zeta^{\alpha+1}}. \quad (12)$$

Via numerical simulation we have determined the distribution of  $\Delta x/\vartheta$  for given velocity extremum  $\vartheta$ . The first and second moment of this distribution is shown in Fig. 3. As expected the average value of  $\Delta x/\vartheta$  indeed approaches a constant  $c_\tau$  (for  $\alpha = 1.6$  the value  $c_\tau \approx 1.6\tau$ ). However, the finite variance shows that the velocity spikes have some distribution in their shape. Thus a priori the distributions  $\phi\left[\frac{\vartheta}{\vartheta(t)}\right]$  and  $P(\Delta x)$  are not identical when replacing  $\Delta x$  by  $c_\tau \vartheta$ . However, since the distribution of  $\Delta x/\vartheta$  for fixed  $\vartheta$  does *not* depend on  $\vartheta$  (for large  $\vartheta$ ) one can directly write

$$P_x(\Delta x) \propto \int d\epsilon d\vartheta q(\epsilon) \vartheta^{-(1+\alpha)} \delta(\epsilon \vartheta + c_\tau \vartheta - \Delta x) \propto \int d\epsilon q(\epsilon) [(\epsilon + c_\tau)/\Delta x]^{(1+\alpha)} \propto \Delta x^{-(1+\alpha)}, \quad (13)$$

where  $q(\epsilon)$  is the distribution of the random variable  $\epsilon := \Delta x/\vartheta - c_\tau$ . Thus, despite the variance in peak shapes the algebraic distribution of  $\vartheta$  directly translates into an identical distribution for  $\Delta x$ .

We have performed a stick derivation of formula (9) in the following way (for details see the supplementary materials available online). The appropriate Fokker-Planck

equation should be written for the full generating function  $\mathcal{G}(\varkappa, k, t)$  for system (3) which depends on two wave numbers,  $\varkappa$  as before and  $k$  related to the velocity variations. Then the corresponding eigenvalue problem can be analyzed assuming the wave number  $\varkappa$  to be a small parameter. It turns out that the perturbation caused by the  $\varkappa$ -term is singular which affects essentially the minimal eigenvalue, making it dependent on  $\varkappa$  as  $\Lambda_{\min} \propto \varkappa^\alpha$ . In this way expression (9) is obtained. Furthermore the specific value of  $c_\tau$  equal to

$$c_\tau = \left[ \frac{2 \sin\left(\frac{\pi\alpha}{2}\right) \Gamma\left(\frac{2-\alpha}{2}\right)}{\sqrt{\pi}\alpha\Gamma\left(\frac{\alpha+1}{2}\right)} \right]^{\frac{1}{\alpha}} \quad (14)$$

follows directly from the comparison of the asymptotics of  $P_x(\Delta x)$  determined by (9) and the asymptotics (12) of the velocity extremum distribution. In particular, for  $\alpha = 1.6$  we have  $\Delta x \approx 1.6\vartheta\tau$  in agreement with the simulation data.

The developed model (3) actually gives us the implementation of Lévy flights at the “microscopic” level admitting the notion of continuous trajectories. Indeed, fixing any small duration  $\delta t$  of the Lévy walker steps we can choose the time scale  $\tau$  of model (3) such that  $\delta t \gg \tau$  and, as a result, receive the Lévy statistics for the corresponding spatial steps. Moreover, the found expression (9) demonstrates the equivalence of all the systems in asymptotic behavior for which the parameters  $v_a$  and  $\tau$  are related by the expression  $v_a^\alpha \tau^{\alpha-1} = \sigma$ . In some sense, all the details of the microscopic implementation of Lévy flights are aggregated in two constants: the exponent  $\alpha$  and the superdiffusion coefficient  $\sigma$ . In particular, the characteristic scale of the walker displacement during time  $t$  is  $\bar{x}(t) \sim (\sigma t)^{1/\alpha}$ .

Our approach has several immediate consequences. First of all, it yields an easily implementable procedure for the numerical simulation of Lévy processes based on the simulation of the Langevin equations (2). Second, it seems to be possible to attack the yet unsolved problem of the formulation of accurate boundary conditions for the generalized Fokker-Planck equations describing Lévy processes in finite domains and heterogeneous media. The crucial point of our treatment is the existence of quantities varying on three widely separated time scales  $\delta t \ll \tau \ll \delta t$ . On time scales  $\delta t$  the Langevin equation is updated. In the well-defined limit of small  $\delta t$  the trajectory can be constructed with arbitrary precision. Furthermore,  $\tau$  is connected with the relaxation time of the variable  $v$  and sets the overall time scale of the model. Finally, for  $\delta t$  the variation of the position  $x$  is fully Markovian and the systems behaves according to the standard Lévy flight scenario. A similar approach is the treatment of the Kramers-Fokker-Planck equation describing diffusion of particles, which is obtained from eq. (5) for the case of purely additive noise  $g = \text{const}$ . The so-called Smoluchowski limit  $\tau \rightarrow 0$  leads to Einstein’s diffusion

equation. For equilibrium systems the fluctuation dissipation theorem relates linear damping and purely additive noise. The emergence of Lévy flights, however, is related with the presence of multiplicative noise, and, in turn, with nonequilibrium situations.

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- [1] T.H. Solomon, E.R. Weeks, and H. L. Swinney, Phys. Rev. Lett. **71**, 3975 (1993).
  - [2] J.-P. Bouchaud, A. Georges, Phys. Rep. **195**, 127 (1990).
  - [3] G.M. Viswanathan, V. Afanasyev, S. Buldyrev, E. J. Murphy, P. A. Prince, H.E. Stanley, Nature **381**, 413 (1996).
  - [4] D. Brockmann, L. Hufnagel, T. Geisel, Nature **439**, 462 (2006).
  - [5] R.N. Mategna, H.E. Stanley, Nature **376**, 46 (1995).
  - [6] J. Klafter, M. F. Shlesinger, and G. Zumofen, Phys. Today **49**(2), 33 (1996).
  - [7] *Lévy Flights and Related Topics in Physics*, edited by M. Shlesinger, G. M. Zaslavsky, and U. Frisch (Springer Verlag, Berlin, 1994); *Anomalous Diffusion*, edited by A. Pekalski, K. Sznajd-Weron, Lecture notes in Physics Vol. 519, (Springer Verlag, Berlin, 1999).
  - [8] H. Risken, *The Fokker-Planck Equation: Methods of Solution and Applications* 2-nd ed. (Springer-Verlag, Berlin, 1996).
  - [9] C. W. Gardiner, *Handbook of Stochastic Methods for Physics, Chemistry, and the Natural Sciences*, 3-rd ed. (Springer-Verlag, Berlin, 2004).
  - [10] H. C. Fogedby, Phys. Rev. E **50**, 1657 (1994).
  - [11] A. Weron, K. Burnecki, S. Mercik, and K. Weron, Phys. Rev. E **71**, 016113 (2005).
  - [12] V. V. Yanovsky, A. V. Chechkin, D. Schertzer, and A. V. Tur, Physica A **282**, 13 (2000).
  - [13] D. Schertzer, M. Larchevêque, J. Duan, V. V. Yanovsky, and S. Lovejoy, J. Math. Phys. **42**, 200 (2001).
  - [14] I. M. Sokolov, Phys. Rev. E **63**, 011104 (2000).
  - [15] H. C. Fogedby, Phys. Rev. Lett. **73**, 2517 (1994).
  - [16] J. Honkonen, Phys. Rev. E **53**, 327 (1996).
  - [17] D. Brockmann and T. Geisel, Phys. Rev. Lett. **90**, 170601 (2003).
  - [18] D. Brockmann, I. M. Sokolov, Chem. Phys. **284** 409 (2002).
  - [19] S. Jespersen, R. Metzler, and H. C. Fogedby, *Lévy Flights in External Force Fields: Langevin and Fractional Fokker-Planck Equations, and their Solutions* arXiv.org:cond-mat/9810176 (1998).
  - [20] S. V. Buldyrev, S. Havlin, A. Ya. Kazakov, M. G. E. da Luz, E. P. Raposo, H. E. Stanley, and G. M. Viswanathan, Phys. Rev. E **64**, 041108 (2001).
  - [21] A. V. Chechkin, V. Yu. Gonchar, J. Klafter, and R. Metzler, Europhys. Lett., **72**, 348 (2005).
  - [22] P. Imkeller and I. Pavlyukevich, J. Phys. A **39**, L237 (2006).
  - [23] B. Dybiec, E. Gudowska-Nowak, and P. Hänggi, Phys.

- Rev. E **73**, 046104 (2006).
- [24] B. Dybiec, E. Gudowska-Nowak, P. Hänggi, Phys. Rev. E **75**, 021109 (2007).
- [25] A. V. Chechkin, O. Yu. Sliusarenko, R. Metzler, J. Klafter, Phys. Rev. E **75**, 041101 (2007).
- [26] S. Siegert, R. Friedrich, Phys. Rev. E **64**, 041107 (2001).
- [27] G. Zumofen and J. Klafter, Phys. Rev. E **51**, 2805 (1995).
- [28] P. M. Drysdale and P. A. Robinson, Phys. Rev. E **58**, 5382 (1998).
- [29] R. Metzler and J. Klafter, Phys. Rep. **339**, 1 (2000).
- [30] P. Lévy, Théorie de l'Addition des Variables Aléatoires (Gauthier-Villars, Paris, 1954); B.V. Gnedenko and A.N. Kolmogorov, Limit Distributions for Sums of Random Variables, (Addison-Wesley, Reading, MA. 1954).
- [31] K. Lindenberg, K. E. Shuler, J. Freeman and T. J. Lie, J. Stat. Phys., **12**, 217 (1975).

## Supplementary material to paper “Realization of Lévy flights as continuous processes”

### STOCHASTIC SYSTEM AND ITS GOVERNING EQUATION

We consider continuous 1D random walks governed by the following stochastic differential equations of the Hänggi-Klimontovich type [1, 2, 3]

$$\frac{dx}{dt} = v, \quad (15)$$

$$\frac{dv}{dt} = -\frac{(\alpha+1)}{2\tau}v + \frac{1}{\sqrt{\tau}}g(v) * \xi(t), \quad (16)$$

where  $x$  is the position of a walker,  $v$  is its current velocity,  $\xi(t)$  is white noise such that  $\langle \xi(t)\xi(t') \rangle = \delta(t-t')$ , and the function

$$g(v) = \sqrt{v_a^2 + v^2} \quad (17)$$

specifies the intensity of random Langevin forces. The dimensionless coefficient  $\alpha$ , the time scale  $\tau$ , and the characteristic velocity  $v_a$  quantifying the additive component of the Langevin forces are the system parameters. The Lévy flights arise when the coefficient  $\alpha$  belongs to the interval

$$1 < \alpha < 2 \quad (18)$$

which, thereby, is assumed to hold beforehand.

For the given system the distribution function  $\mathcal{P}(x - x_0, v, v_0, t)$  obeys the following Fokker-Planck equation written in the kinetic form

$$\frac{\partial \mathcal{P}}{\partial t} = \frac{1}{2\tau} \frac{\partial}{\partial v} \left[ g^2(v) \frac{\partial \mathcal{P}}{\partial v} + (\alpha+1)v\mathcal{P} \right] - \frac{\partial}{\partial x} [v\mathcal{P}] \quad (19)$$

subject to the initial condition

$$\mathcal{P}(x - x_0, v, v_0, 0) = \delta(x - x_0)\delta(v - v_0), \quad (20)$$

where,  $x_0$  and  $v_0$  are the initial position and velocity of the walker and, an addition, the system translation invariance with respect to the variable  $x$  is taken into account.

### VELOCITY DISTRIBUTION

#### General relations

It is the statistical properties of the walker velocity  $v$  that give rise to Lévy flights. So the present section is devoted to them individually. The velocity distribution is given by the partial distribution function

$$P_v(v, v_0, t) = \int_{\mathbb{R}} dx \mathcal{P}(x - x_0, v, v_0, t) \quad (21)$$

which, by virtue of (19), obeys the reduced forward Fokker-Planck equation

$$2\tau \frac{\partial P_v}{\partial t} = \frac{\partial}{\partial v} \left[ g^2(v) \frac{\partial P_v}{\partial v} + (\alpha+1)vP_v \right] \quad (22)$$

written in the kinetic form whose right-hand side acts on the variable  $v$ . Simultaneously, the function  $P_v(v, v_0, t)$  meets the backward Fokker-Planck equation

$$2\tau \frac{\partial P_v}{\partial t} = g^2(v_0) \frac{\partial^2 P_v}{\partial v_0^2} - (\alpha-1)v_0 \frac{\partial P_v}{\partial v_0} \quad (23)$$

written in the Itô form and acting on the variable  $v_0$  (see, e.g., [4]). The two equations are supplemented with the initial condition

$$P_v(v, v_0, 0) = \delta(v - v_0). \quad (24)$$

In particular, as stems from (22), the stationary velocity distribution  $P_v^{\text{st}}(v)$  is the solution of the equation

$$g^2(v) \frac{\partial P_v^{\text{st}}}{\partial v} + (\alpha+1)vP_v^{\text{st}} = 0, \quad (25)$$

which together with the normalization condition

$$\int_{\mathbb{R}} dv P_v^{\text{st}}(v) = 1 \quad (26)$$

gives us the expression

$$P_v^{\text{st}}(\mathbf{v}) = \frac{\Gamma\left(\frac{\alpha+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\alpha}{2}\right)} \frac{v_a^\alpha}{[g(v)]^{\alpha+1}}, \quad (27)$$

where  $\Gamma(\dots)$  is the Gamma function. For the exponent  $\alpha$  belonging to interval (18) the first moment of the velocity  $v$  converges, whereas the second one diverges what actually was the reason for specifying the region of  $\alpha$  under consideration.

### First passage time problem and extremum distribution

In order to establish some kinematic relationship between the Lévy type behavior exhibited by the given random walks on time scales  $t \gg \tau$  and properties of the velocity distribution we will make use of the first passage time statistics. The probability  $F(v_0, \vartheta, t)$  for the walker with initial velocity  $v_0$  such that  $|v_0| < \vartheta$  to gain the velocity  $v = \pm\vartheta$  for the first time at the moment  $t$  is directly described by the backward Fokker-Planck equation (23). In particular its Laplace transform

$$F_L(v_0, \vartheta, s) = \int_0^\infty dt e^{-st} F(v_0, \vartheta, t)$$

obeys the equation (see, e.g., [4])

$$2\tau s F_L = g^2(v_0) \frac{\partial^2 F_L}{\partial v_0^2} - (\alpha - 1)v_0 \frac{\partial F_L}{\partial v_0} \quad (28)$$

subject to the boundary condition

$$F_L(v_0, \vartheta, s)|_{v_0=\pm\vartheta} = 1. \quad (29)$$

The introduced first passage time probability is necessary to analyze the extremum statistics. Namely we need the probability  $\Phi(v_0, \vartheta, t)$  for the velocity pattern  $v(t)$  originating from the point  $v_0 \in (-\vartheta, +\vartheta)$  to the get the extremum equal to  $\pm\vartheta$  somewhere during the time interval  $t$  is related to the probability  $F(v_0, \vartheta, t)$  by the expression [5]

$$\Phi(v_0, \vartheta, t) = -\frac{\partial}{\partial \vartheta} \int_0^t dt' F(v_0, \vartheta, t') \quad (30)$$

or for the Laplace transforms

$$\Phi_L(v_0, \vartheta, s) = -\frac{1}{s} \frac{\partial}{\partial \vartheta} F_L(v_0, \vartheta, s). \quad (31)$$

To examine the characteristic properties of the first passage time statistics let us consider two limit cases,  $s \rightarrow 0$  and  $\vartheta \rightarrow \infty$ . Their analysis starts at the first step with the same procedure. Namely, we assume the function  $F_L(v_0, \vartheta, s)$  to be approximately constant,  $F_L(v_0, \vartheta, s) \simeq F_0(\vartheta, s)$  inside some neighborhood  $\mathbb{Q}_0$  of the origin  $v_0 = 0$ . For  $s \rightarrow 0$  it is the domain  $(-\vartheta, \vartheta)$  itself and  $F_0(\vartheta, s) = 1$  by virtue of (29). For  $\vartheta \rightarrow \infty$  the thickness of this neighborhood is much larger than  $v_a$  as it will be seen below. Under such conditions equation (28) can be integrated directly inside the domain  $\mathbb{Q}_0$  with respect to the formal variable  $f(v_0) = \partial F_L / \partial v_0$  using the standard parameter-variation method. In this way taking into account that  $f(0) = 0$  due to the system symmetry we obtain the expression

$$\frac{\partial F_L(v_0, \vartheta, s)}{\partial v_0} \simeq \frac{2\tau s}{v_a} F_0(\vartheta, s) \left( \frac{v_0^2}{v_a^2} + 1 \right)^{\frac{\alpha-1}{2}} \int_0^{v_0/v_a} \frac{d\xi}{(\xi^2 + 1)^{\frac{\alpha+1}{2}}} \quad (32)$$

and for  $|v_0| \gg v_a$

$$\frac{\partial F_L(v_0, \vartheta, s)}{\partial v_0} \simeq \sqrt{\pi} \tau s F_0(\vartheta, s) \frac{\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{\alpha+1}{2}\right)} \frac{|v_0|^{\alpha-1}}{v_a^\alpha}. \quad (33)$$

Expression (33) demonstrates us that, first, the implementation of the limit case of small values of  $s$  (formally,  $s \rightarrow 0$ ) is the validity of the inequality

$$F_0(\vartheta, s) \gg v_0 \frac{\partial F_L(v_0, \vartheta, s)}{\partial v_0} \Rightarrow \tau s \frac{|v_0|^\alpha}{v_a^\alpha} \ll 1,$$

which can be rewritten as

$$\bar{\vartheta}_L(s) := \left( \frac{1}{\tau s} \right)^{\frac{1}{\alpha}} v_a \gg |v_0| \quad (34)$$

or converting to the time dependence

$$\bar{\vartheta}(t) := \left(\frac{t}{\tau}\right)^{\frac{1}{\alpha}} v_a \gg |v_0|. \quad (35)$$

So the characteristic velocity scale characterizing the first passage time probability and aggregating its time dependence is  $\bar{\vartheta}(t)$ . As a consequence, the limit of small values of  $s$  is actually defined by the inequality  $|v_0| \ll \bar{\vartheta}(t)$ . Correspondingly, the limit of large values of  $\vartheta$  is implemented by the inequality  $\vartheta \gg \bar{\vartheta}(t)$  or  $\vartheta \gg \bar{\vartheta}_L(s)$ .

Second, for  $\vartheta \gg \bar{\vartheta}_L(s)$  there is a region, namely,  $v_a \ll |v_0| \ll \bar{\vartheta}(t)$  wherein the assumption  $F_L(v_0, \vartheta, s) \simeq F_0(\vartheta, s)$  holds whereas the derivative  $\partial F_L / \partial v_0$  scales with  $v_0$  as  $\partial F_L / \partial v_0 \propto |v_0|^{\alpha-1}$ . This asymptotic behavior can be obtained also by analyzing the solution of equation (28) for  $|v_0| \gg v_a$  where  $g^2(v_0) \simeq v_0^2$ . In this case equation (28) admits two solutions of the form

$$F_L(v_0, \vartheta, s) \propto v_0^{g_{1,2}}$$

with

$$g_1 \simeq \alpha \quad \text{and} \quad g_2 \simeq -\frac{2\tau s}{\alpha}. \quad (36)$$

The second solution is relevant to the function  $F_L(v_0, \vartheta, s)$  only within the crossover from  $F_L(v_0, \vartheta, s) \propto v_0^\alpha$  to  $F_L(v_0, \vartheta, s) \approx F_0(\vartheta, s)$  and even in this region, i.e.  $|v_0| \lesssim \bar{\vartheta}(t)$  the derivative  $\partial F_L / \partial v_0$  is determined by its asymptotics  $F_L(v_0, \vartheta, s) \propto v_0^\alpha$ . For larger values of  $v_0$ , i.e.,  $|v_0| \gg \bar{\vartheta}(t)$  the first passage time distribution is given by the expression

$$F_L(v_0, \vartheta, s) \simeq \left(\frac{|v_0|}{\vartheta}\right)^\alpha \quad (37)$$

taking into account the boundary condition (29). So we can write

$$\frac{\partial F_L(v_0, \vartheta, s)}{\partial v_0} \simeq \alpha \frac{|v_0|^{\alpha-1}}{\vartheta^\alpha} \quad (38)$$

also for  $|v_0| \lesssim \bar{\vartheta}(t)$ . Expressions (33) and (38) describe the same asymptotic behavior of the function  $F_L(v_0, \vartheta, s)$ . Thereby we can “glue” them together, obtaining the expression for

$$F_0(\vartheta, s) = \frac{\alpha \Gamma\left(\frac{\alpha+1}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{\alpha}{2}\right)} \frac{1}{\tau s} \frac{v_a^\alpha}{\vartheta^\alpha}. \quad (39)$$

It should be noted that this procedure is the kernel of the singular perturbation technique which will be also used below. Expression (39) immediately gives us the desired formula for the extremum distribution  $\Phi_L(v_0, \vartheta, s)$ . Namely, by virtue of (31), for  $|v_0| \lesssim \bar{\vartheta}_L(s)$  we have

$$\Phi_L(v_0, \vartheta, s) = \frac{\alpha^2 \Gamma\left(\frac{\alpha+1}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{\alpha}{2}\right)} \frac{1}{\tau s^2} \frac{v_a^\alpha}{\vartheta^{\alpha+1}} \quad (40)$$

and restoring the time dependence of the extremum distribution from its Laplace transform the asymptotic behavior for  $\vartheta \gg \bar{\vartheta}(t)$  we get

$$\Phi(v_0, \vartheta, t) = \frac{\alpha^2 \Gamma\left(\frac{\alpha+1}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{\alpha}{2}\right)} \frac{t}{\tau} \frac{v_a^\alpha}{\vartheta^{\alpha+1}}. \quad (41)$$

Finalizing the present subsection we draw the conclusion that for  $|v_0| \ll \bar{\vartheta}(t)$  the extremum distribution is described by a certain function

$$\Phi(v_0, \vartheta, t) = \frac{1}{\bar{\vartheta}(t)} \Phi_0\left(\frac{\vartheta}{\bar{\vartheta}(t)}\right) \quad (42)$$

with the asymptotics

$$\Phi_0(\xi) = \frac{\alpha^2 \Gamma\left(\frac{\alpha+1}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{\alpha}{2}\right)} \frac{1}{\xi^{\alpha+1}}. \quad (43)$$

Here the velocity scale  $\bar{\vartheta}(t)$  is given by expression (35). We remind that distribution (41) describes the amplitude of the velocity extrema, so, as the velocity extrema themselves are concerned their distribution is characterized by the function

$$\tilde{\Phi}(v_0, \vartheta, t) = \frac{1}{2} \Phi(v_0, |\vartheta|, t) \quad (44)$$

because of the symmetry in the velocity fluctuations.

It should be also noted that the asymptotics  $\Phi(v_0, \vartheta, t) \propto \vartheta^{-(\alpha+1)}$  for  $\vartheta \gg \bar{\vartheta}(t)$  could be obtained immediately from equation (28). In fact, formally assuming  $\vartheta \rightarrow \infty$  and taking into account the boundary condition (29) we can represent the solution of equation (28) in form (37) for  $v_0 \lesssim \vartheta$  because, first,  $g^2(v_0) = v_0^2$  in this case and, second, the function  $F_L(v_0, \vartheta, s)$  must be decreasing with  $|v_0|$ . It is the only one place where the variable  $\vartheta$  enters the function  $F_L(v_0, \vartheta, s)$ , thus, for  $\vartheta \gg \bar{\vartheta}(t)$

$$F_L(v_0, \vartheta, s) \propto \frac{1}{\vartheta^\alpha}$$

and relationship (31) directly gives rise to

$$\Phi(v_0, \vartheta, t) \propto \frac{1}{\vartheta^{\alpha+1}}.$$

## GENERATING FUNCTION

### General relations

To analyze the given stochastic process the generating function

$$\mathcal{G}(k, \varkappa, t) = \left\langle \exp \left\{ \frac{i}{v_a \tau} [\tau v k + (x - x_0) \varkappa] \right\} \right\rangle \quad (45)$$

is introduced. As follows from the Fokker-Planck equation (19) it obeys the governing equation

$$2\tau \frac{\partial \mathcal{G}}{\partial t} = \frac{\partial}{\partial k} \left( k^2 \frac{\partial \mathcal{G}}{\partial k} \right) + [2\varkappa - (\alpha + 1)k] \frac{\partial \mathcal{G}}{\partial k} - k^2 \mathcal{G} \quad (46)$$

subject to the initial condition

$$\mathcal{G}(k, \varkappa, 0) = \exp \left\{ \frac{i}{v_a} v_0 k \right\}. \quad (47)$$

At the origin  $k = 0$  and  $\varkappa = 0$  function (45) meets also the identity

$$\mathcal{G}(0, 0, t) = 1 \quad (48)$$

which follows directly from the meaning of probability. In deriving equation (46) the following relationships between the operators acting in the spaces  $\{x, v\}$  and  $\{\varkappa, k\}$

$$\frac{\partial}{\partial x} \leftrightarrow -\frac{i}{v_a \tau} \varkappa, \quad \frac{\partial}{\partial v} \leftrightarrow -\frac{i}{v_c} k, \quad v = -i v_a \frac{\partial}{\partial k}$$

as well as the commutation rule

$$\frac{\partial}{\partial k} k - k \frac{\partial}{\partial k} = 1$$

have been used.

The argument  $\varkappa$  enters equation (46) as a parameter; the given equation does not contain any differential operator acting upon the function  $\mathcal{G}(k, \varkappa, t)$  via the argument  $\varkappa$ . This property enables us to pose a question about the spectrum of equation (46), where the variable  $\varkappa$  plays the role of a parameter. The desired eigenfunctions and their eigenvalues

$$\{\Psi_\Lambda(k|\varkappa)\}, \quad \{\Lambda(\varkappa)\} \quad (49)$$

obey the equation

$$-2\Lambda\Psi_\Lambda = \frac{d}{dk} \left( k^2 \frac{d\Psi_\Lambda}{dk} \right) + [2\varkappa - (\alpha + 1)k] \frac{d\Psi_\Lambda}{dk} - k^2 \Psi_\Lambda. \quad (50)$$

In deriving equation (50) the time dependence  $\exp(-\Lambda t/\tau)$  corresponding to eigenfunctions (49) has been assumed.

In these terms the solution of equation (46) is reduced to the series

$$\mathcal{G}(k, \varkappa, t) = \sum_{\Lambda} f_{\Lambda}(\varkappa, |v_0) \Psi_{\Lambda}(k|\varkappa) \exp \left\{ -\Lambda(\varkappa) \frac{t}{\tau} \right\} \quad (51)$$

whose the coefficients  $\{f(\varkappa, |v_0)\}$  meet the equality

$$\sum_{\Lambda} f_{\Lambda}(\varkappa, |v_0) \Psi_{\Lambda}(k|\varkappa) = \exp \left\{ \frac{i}{v_a} v_0 k \right\} \quad (52)$$

stemming from the initial condition (47). In agreement with the results to be obtained, the spectrum of the Fokker-Planck equation (46) is bounded from below by a nondegenerate minimal eigenvalue  $\Lambda_{\min}(\varkappa) \geq 0$  whereas the other eigenvalues are separated from it by a final gap of order unity.

So, as time goes on and the inequality  $t \gg \tau$  holds, the term corresponding to the minimal eigenvalue will be dominant and sum (51) is reduced to

$$\mathcal{G}(k, \varkappa, t) = f_{\min}(\varkappa, |v_0) \Psi_{\min}(k|\varkappa) \exp \left\{ -\Lambda_{\min}(\varkappa) \frac{t}{\tau} \right\} \quad (53)$$

on large time scales. Here  $\Psi_{\min}(k|\varkappa)$  is the eigenfunction of the eigenvalue  $\Lambda_{\min}(\varkappa)$ .

Whence several consequences follow. First, the identity (48) holds at any time moment, thereby

$$\Lambda_{\min}(0) = 0. \quad (54)$$

Second, in the limit case  $t \gg \tau$  the system has to “forget” the value  $v_0$  of initial velocity, so the coefficient  $f_{\min}(\varkappa)$  does not depend on the argument  $v_0$  and, therefore, can be aggregated into the function  $\Psi_{\min}(k|\varkappa)$ . In this way the initial condition expansion (52) reads

$$\Psi_{\min}(k|\varkappa) + \sum_{\Lambda > \Lambda_{\min}} f_{\Lambda}(\varkappa, |v_0) \Psi_{\Lambda}(k|\varkappa) = \exp \left\{ \frac{i}{v_a} v_0 k \right\} \quad (55)$$

for any  $v_0$ .

The terms in sum (55) with  $\Lambda > \Lambda_{\min}$  determine the dependence of the generating function  $\mathcal{G}(k, \varkappa, t)$  on the initial velocity  $v_0$ , so, the corresponding coefficients  $f_{\Lambda}(\varkappa, |v_0)$  must depend on  $v_0$ . Finding the first derivative of both the sides of this equality with respect to  $v_0$  we have

$$\sum_{\Lambda > \Lambda_{\min}} \frac{\partial}{\partial v_0} f_{\Lambda}(\varkappa, |v_0) \Psi_{\Lambda}(k|\varkappa) = \frac{i}{v_a} k \exp \left\{ \frac{i}{v_a} v_0 k \right\}.$$

Whence it follows that, third, the eigenfunctions  $\Psi_{\Lambda}(k|\varkappa)$  for  $\Lambda > \Lambda_{\min}$  must exhibit the asymptotic behavior  $\Psi_{\Lambda}(k|\varkappa) \rightarrow 0$  as  $k \rightarrow 0$  because of their linear independence. Fourth, setting  $k = 0$  in expression (55) we get the conclusion that the eigenfunction  $\Psi_{\min}(k|\varkappa)$  has to meet the normalization condition

$$\Psi_{\min}(0|\varkappa) = 1 \quad \text{at} \quad k = 0. \quad (56)$$

Summarizing the aforementioned we see that on large time scales  $t \gg \tau$  the desired asymptotic behavior of the given system is described by the generating function

$$\mathcal{G}(k, \varkappa, t) = \Psi_{\min}(k|\varkappa) \exp \left\{ -\Lambda_{\min}(\varkappa) \frac{t}{\tau} \right\}, \quad (57)$$

and by virtue of (56)

$$\mathcal{G}(0, \varkappa, t) = \exp \left\{ -\Lambda_{\min}(\varkappa) \frac{t}{\tau} \right\}. \quad (58)$$

In what follows the calculation of the eigenvalue  $\Lambda_{\min}(\varkappa)$  will be the main goal.



The given random walks should exhibit the Lévy flight behavior on large spatial and temporal scales, i.e.  $x \gg v_a \tau$  and  $t \gg \tau$ . It allows us to confine our analysis to the limit of small values of  $\varkappa$ , i.e. assume that  $|\varkappa| \ll 1$ , where also the eigenvalue  $\Lambda_{\min}(\varkappa) \ll 1$ . In this case the spectrum of equation (46) may be studied using perturbation technique, where the term

$$\hat{V}_\varkappa \Psi = 2\varkappa \frac{d\Psi}{dk} \quad (59)$$

plays the role of perturbation.

### Zero-th approximation. Spectral properties of the velocity distribution

The zero-th approximation of (46) in perturbation (59) matches the case  $\varkappa = 0$ , where the generating function (45) actually describes the velocity distribution (21). Setting  $\varkappa = 0$  reduces the eigenvalue equation (50) to the following

$$-2\lambda\Phi_\lambda = \frac{d}{dk} \left( k^2 \frac{d\Phi_\lambda}{dk} \right) - (\alpha + 1)k \frac{d\Phi_\lambda}{dk} - k^2\Phi_\lambda, \quad (60)$$

where

$$\Phi_\lambda(k) = \Psi_\Lambda(k|0) \quad \text{and} \quad \lambda = \Lambda(0). \quad (61)$$

Having in mind different goals we consider the conversion of equation (60) under the replacement

$$\Phi_\lambda(k) = |k|^{\beta_i} \phi_{\lambda,i}(k) \quad (62)$$

for two values of the exponent  $\beta_i$ .

First, for  $\beta_1 = (\alpha + 1)/2$  equation (60) is converted into

$$2\lambda\phi_{\lambda,1} = -\frac{d}{dk} \left( k^2 \frac{d\phi_{\lambda,1}}{dk} \right) + \left[ k^2 + \frac{1}{4}(\alpha^2 - 1) \right] \phi_{\lambda,1}. \quad (63)$$

The operator on the right-hand side of equation (63) is Hermitian within the standard definition of scalar product. So all the eigenvalues  $\lambda$  are real numbers and the corresponding eigenfunctions form a basis. It should be noted that the given conclusion coincides with the well known property of the Fokker-Planck equations with the detailed balance [6]. In addition the eigenfunctions  $\phi_{\lambda,1}(k)$  can be chosen so that the identity

$$\int_{\mathbb{R}} dk \phi_{\lambda,1}^*(k) \phi_{\lambda',1}(k) = \delta_{\lambda\lambda'} \quad (64)$$

holds for all of them except for the eigenfunction  $\phi_{\min}(k)$  corresponding to the minimal eigenvalue  $\lambda_{\min} = \Lambda_{\min}(0) = 0$  by virtue of (54). We note that the latter eigenfunction describes the stationary velocity distribution (27) and its normalization is determined by condition (56). Treating the eigenfunction  $\Phi_{\min}(k)$  individually releases the remainders from the necessity to take a

nonzero value at the origin  $k = 0$  and, thereby, enables the eigenfunction problem (63) to be considered within  $L^2$ -space.

Second, for  $\beta_2 = \alpha/2$  equation (60) is reduced to the modified Bessel differential equation

$$k^2 \frac{d^2 \phi_{\lambda,2}}{dk^2} + k \frac{d\phi_{\lambda,2}}{dk} - \left[ k^2 + \frac{1}{4}\alpha^2 - 2\lambda \right] \phi_{\lambda,2} = 0. \quad (65)$$

Since the desired eigenfunctions should decrease as  $k \rightarrow \infty$  the solution of equation (65) is given by the modified Bessel function of the second kind

$$\phi_{\lambda,2}(k) \propto K_\nu(|k|) \quad (66)$$

with the order  $\nu = \sqrt{\frac{1}{4}\alpha^2 - 2\lambda}$  because

$$K_\nu(|k|) \sim \sqrt{\frac{\pi}{2|k|}} e^{-|k|} \quad \text{as} \quad k \rightarrow \infty$$

for any value of the parameter  $\nu$  [7].

Whence it follows that there are no eigenfunctions with  $\lambda < \frac{1}{8}\alpha^2$  and  $\lambda \neq 0$ . Indeed, when  $\lambda < 0$  the function

$$\Phi(k) := |k|^{\frac{1}{2}\alpha} K_\nu(|k|) \propto |k|^{-(\nu - \frac{1}{2}\alpha)} \quad \text{for} \quad |k| \ll 1$$

diverges as  $k \rightarrow 0$ . In the region  $0 < \lambda < \frac{1}{8}\alpha^2$  the corresponding eigenfunctions

$$\phi_{\lambda,1}(k) = \phi_{\lambda,3}(k) |k|^{\beta_2 - \beta_1} \propto |k|^{-\frac{1}{2}} K_\nu(|k|)$$

would give rise to a strong divergency in the normalization condition (64). When  $\lambda > \frac{1}{8}\alpha^2$  the solution of equation (65) is described by the modified Bessel functions of pure imaginary order which exhibit strongly oscillatory behavior as  $\varkappa \rightarrow 0$  and describe the continuous spectrum of the Fokker-Planck equation (46) for  $\varkappa = 0$ . Due to result (66) the eigenfunction  $\Phi_{\min}(k)$  corresponding to the eigenvalue  $\lambda = 0$  and meeting the normalization condition (56) is of the form

$$\begin{aligned} \Phi_{\min}(k) &= \frac{2^{\frac{2-\alpha}{2}}}{\Gamma(\frac{\alpha}{2})} k^{\frac{\alpha}{2}} K_{\frac{\alpha}{2}}(|k|) \\ &= 1 - \left( \frac{|k|}{2} \right)^\alpha \frac{\Gamma(\frac{2-\alpha}{2})}{\Gamma(\frac{3+\alpha}{2})} + O(k^2). \end{aligned} \quad (67)$$

In deriving expression (67) the following expansion of the function  $K_\nu(k)$  has been used

$$K_\nu(|k|) = \frac{\Gamma(\nu)}{2^{1-\nu} |k|^\nu} \left[ 1 - \left( \frac{|k|}{2} \right)^{2\nu} \frac{\Gamma(1-\nu)}{\Gamma(1+\nu)} + O(k^2) \right] \quad (68)$$

which is justified for the order  $0 < \nu < 1$  (see, e.g., Ref. [7]). The latter inequality holds due to the adopted assumption (18) about the possible values of the parameter  $\alpha$ .

Expression (67) finalizes the analysis of the zero-th approximation. Summarizing the aforementioned we draw the conclusion that at  $\varkappa = 0$  the spectrum of the Fokker-Planck equation (46) for the generating function (45) does contain zero eigenvalue  $\Lambda_{\min}(0) = 0$  corresponding to eigenfunction (67) which is separated from higher eigenvalues by a gap equal to  $\alpha^2/8$  (in units of  $\tau$ ). We note that the given statement is in agreement with the conclusion about the spectrum properties for a similar stochastic process with multiplicative noise [8, 9, 10].

**The eigenvalue  $\Lambda_{\min}(\varkappa)$  for  $|\varkappa| \ll 1$ . Singular perturbation technique**

When  $\varkappa \neq 0$  the perturbation term (59) mixes the eigenfunctions of zero-th approximation and, as a result, the eigenfunctions  $\Phi_\lambda(k)$  with  $\lambda > 0$  contribute also to the eigenfunction  $\Psi_{\min}(k|\varkappa)$ . However, because their eigenvalues are about unity or larger,  $\lambda \gtrsim 1$ , the perturbation can be significant only in the domain  $|k| \lesssim |\varkappa|$ . Outside this domain the perturbation is not essential and the eigenfunction  $\Psi_{\min}(k|\varkappa)$  practically coincides with its zero-th approximation  $\Phi_{\min}(k)$ . So in the case when  $|\varkappa| \ll 1$  there should be an interval  $|\varkappa| \ll |k| \ll 1$  where, on one hand, the eigenfunction  $\Psi_{\min}(k|\varkappa)$  can be already approximated by  $\Phi_{\min}(k)$  and, on the other hand, the expansion (67) still holds, in particular,  $\Psi_{\min}(k|\varkappa) \approx 1$  in this region. Leaping ahead, we note that  $\Lambda_{\min} \sim |\varkappa|^\alpha$  so inside the subinterval  $|\varkappa| \ll |k| \ll |\varkappa|^{\frac{\alpha}{2}}$  the last term on the right-hand

side of equation (50) is also ignorable in comparison with its left-hand side. Under these conditions the eigenvalue equation (50) is reduced to the following

$$2\Lambda_{\min} = \frac{d}{dk} \left( k^2 \frac{d\psi}{dk} \right) + [2\varkappa - (\alpha + 1)k] \frac{d\psi}{dk} \quad (69)$$

for the function  $\psi(k|\varkappa) = 1 - \Psi_{\min}(k|\varkappa)$ . In the given case the singular perturbation technique is implemented within the replacement  $k = \zeta \varkappa$  converting equation (69) into one of the form

$$2\Lambda_{\min} = \frac{d}{d\zeta} \left( \zeta^2 \frac{d\psi}{d\zeta} \right) + [2 - (\alpha + 1)\zeta] \frac{d\psi}{d\zeta} \quad (70)$$

subject to the effective “boundary” conditions by virtue of (67)

$$\psi(\zeta|\varkappa) \sim |\zeta|^\alpha \left( \frac{|\varkappa|}{2} \right)^\alpha \frac{\Gamma(\frac{2-\alpha}{2})}{\Gamma(\frac{3+\alpha}{2})} \quad \text{as } \zeta \rightarrow \pm\infty. \quad (71)$$

In some sense the condition (71) “glues” the asymptotic behavior of the eigenfunction  $\Psi_{\min}(k|\varkappa)$  resulting from its properties for sufficiently large values of  $k$  together with one stemming from small values of  $k$ , in this case, specified by the solution of equation (71). Exactly such a procedure is the essence of the singular perturbation technique.

Equation (70) with respect to the variable  $d\psi/d\zeta$  can be solved directly using the standard parameter-variation method. In this way we get for  $\zeta < 0$

$$\frac{d\psi}{d\zeta} = |\zeta|^{\alpha-1} \left[ \exp\left(\frac{2}{\zeta}\right) C_{-\infty} + 2^{1-\alpha} \Lambda_{\min} \int_{2/\zeta}^0 \xi^{\alpha-1} \exp\left(\frac{2}{\zeta} - \xi\right) d\xi \right] \quad (72)$$

and for  $\zeta > 0$

$$\frac{d\psi}{d\zeta} = |\zeta|^{\alpha-1} \exp\left(\frac{2}{\zeta}\right) \left[ C_{+\infty} - 2^{1-\alpha} \Lambda_{\min} \int_0^{2/\zeta} \xi^{\alpha-1} \exp(-\xi) d\xi \right], \quad (73)$$

where the constants  $C_{\pm\infty}$  specify the asymptotic behavior of the derivative

$$\frac{d\psi}{d\zeta} \sim |\zeta|^{\alpha-1} C_{\pm\infty} \quad \text{as } \zeta \rightarrow \pm\infty$$

and according to condition (71)

$$C_{+\infty} = -C_{-\infty} = \alpha \left( \frac{|\varkappa|}{2} \right)^\alpha \frac{\Gamma(\frac{2-\alpha}{2})}{\Gamma(\frac{3+\alpha}{2})}. \quad (74)$$

Expression (73) diverges as  $\zeta \rightarrow 0$  unless the equality

$$\frac{\Gamma(1-\nu)}{\Gamma(\nu)} - \Lambda_{\min} \int_0^\infty \exp(-\xi) \xi^{\alpha-1} d\xi = 0$$

holds, whence we find the desired expression for the

eigenvalue  $\Lambda_{\min}$

$$\Lambda_{\min} = \frac{\Gamma\left(\frac{2-\alpha}{2}\right)}{\Gamma(\alpha)\Gamma\left(\frac{\alpha}{2}\right)} |\varkappa|^\alpha. \quad (75)$$

Expression (75) finalizes the analysis of the generating function (45). In particular, together with expression (58) it gives the desired formula for the generating function

$$\begin{aligned} \mathcal{G}(0, \varkappa, t) &= \left\langle \exp \left\{ i \frac{(x - x_0) \varkappa}{v_a \tau} \right\} \right\rangle \\ &= \exp \left\{ - \frac{\Gamma\left(\frac{2-\alpha}{2}\right)}{\Gamma(\alpha)\Gamma\left(\frac{\alpha}{2}\right)} |\varkappa|^\alpha \frac{t}{\tau} \right\} \end{aligned} \quad (76)$$

demonstrating the fact that the given random walks exhibit Lévy flight statistics on time scales  $t \gg \tau$ . Expression (76) in turn gives us the asymptotics of the  $x$ -distribution function

$$P_x(x - x_0, v_0, t) = \int_{\mathbb{R}} dv \mathcal{P}(x - x_0, v, v_0, t)$$

for  $|x - x_0| \gg \bar{x}(t)$  in the form

$$P_x(x - x_0, t) = \frac{\sin\left(\frac{\pi\alpha}{2}\right) \alpha \Gamma\left(\frac{2-\alpha}{2}\right)}{\pi \Gamma\left(\frac{\alpha}{2}\right)} \frac{\bar{x}^\alpha(t)}{|x - x_0|^{\alpha+1}} \quad (77)$$

where the length

$$\bar{x}(t) = (\sigma t)^{\frac{1}{\alpha}} \quad (78)$$

with  $\sigma = v_a^\alpha \tau^{\alpha-1}$  specifies the characteristic scales of the walker displacement during the time interval  $t$ .

### THE LÉVY FLIGHT BEHAVIOR AND THE EXTREMUM STATISTICS OF THE WALKER VELOCITIES

Comparing expressions (44) and (77) describing the asymptotic behavior of the given random walks with re-

spect to the walker displacement  $x - x_0$  and its velocity extrema  $\vartheta$  we get the relationship between their characteristic scales

$$\bar{x}(t) = \bar{\vartheta}(t) \tau \quad (79)$$

and the asymptotic equivalence within the replacement  $(x - x_0) = \vartheta T$ , where

$$T = \left[ \frac{2 \sin\left(\frac{\pi\alpha}{2}\right) \Gamma\left(\frac{2-\alpha}{2}\right)}{\sqrt{\pi} \alpha \Gamma\left(\frac{\alpha+1}{2}\right)} \right]^{\frac{1}{\alpha}} \tau. \quad (80)$$

The obtained expressions allow us to consider the long distance displacements of the walker within the time interval  $t$  to be implemented during one spike of duration  $\tau$  in the pattern  $v(t)$  that has the maximal amplitude. In particular, for  $\alpha = 1.6$  the ratio  $T/\tau \simeq 1.6$

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- [1] P. Hänggi, *Helv. Phys. Acta* **51**, 183 (1978).
  - [2] P. Hänggi, *Helv. Phys. Acta* **53**, 491 (1980).
  - [3] Yu. L. Klimontovich, *Physics-Uspekhi* **37**, 737 (1994).
  - [4] C. W. Gardiner, *Handbook of Stochastic Methods for Physics, Chemistry, and the Natural Sciences*, 3-rd ed. (Springer-Verlag, Berlin, 2004).
  - [5] K. Lindenberg, K. E. Shuler, J. Freeman, and T. J. Lie, *J. Stat. Phys.* **12**, 217 (1975).
  - [6] H. Risken, *The Fokker Planck Equation: Methods of Solution and Applications*, 2nd ed. (Springer-Verlag, Berlin, 1989).
  - [7] *Handbook of Mathematical Functions: with Formulas, Graphs, and Mathematical Tables*, M. Abramowitz and I. A. Stegun (editors), (Dover Publications, Inc., New York, 1972) 10-th edt5v.
  - [8] A. Schenzle and H. Brand, *Phys. Let. A* **69**, 313 (1979).
  - [9] A. Schenzle and H. Brand, *Phys. Rev. A* **20**, 1628 (1979).
  - [10] A. Schenzle, H. Brand, and Schröder, *Phys. Rev. A* **25**, 2324 (1982).