Proceedings of the 45th ISCIE International Symposium
on Stochastic Systems Theory and Its Applications
Okinawa, Nov. 1-2, 2013

Towards Multi-Dimensional Nonlinear Langevin Equation

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Abstract

The work is devoted to possible generalizations of the Langevin equation based on the notion of the intermediate point determining the contribution of nonlinear random forces and different channels of noise action.

1 Introduction

Within the rigor typical for physical constructions there can be singled out several basic approaches to describing stochastic Markov processes which are reducible to one another, at least, in principle. In the present section we will focus out attention on two of them. The first one is the Fokker-Planck equation

$$\partial_t P = \sum_{i=N}^{N} \partial_i \left\{ \sum_{j=N}^{N} \partial_j [D_{ij}(\mathbf{x},t)P] - V_i(\mathbf{x},t)P \right\}, \quad (1)$$

written here in the "forward" form to be specific. In Exp. (1) the symbols ∂_t, ∂_i stand for the derivative operators with respect to the time t and the spatial coordinates of the point $\mathbf{x} = \{x_i\}_{i=1}^{i=N} \in \mathbb{R}^N$ and $P(\mathbf{x},t)$ is the probability density function. The latter is usually subjected to the condition

$$P(\mathbf{x},t)|_{t=t_0} = \delta(\mathbf{x}-\mathbf{x}_0) \quad (2)$$

at the initial moment of time t_0, where $\delta(\mathbf{x}-\mathbf{x}_0)$ is the Dirac δ-function; in this case the probability density $P(\mathbf{x},t) \equiv G(\mathbf{x},t|\mathbf{x}_0,t_0)$ is also called the Green function. In terms of Brownian particles whose random motion implements a Markovian process under consideration the Green function can be interpreted as the probability density of finding such a particle at the point $\mathbf{x}$ at time t provided at the initial moment of time t_0 it was located at the point $\mathbf{x}_0$. The kinetic coefficients entering the Fokker-Planck equation (1), i.e., the matrix $\mathbf{D} = \|D_{ij}\|$ of diffusion coefficients and the velocity drift $\mathbf{V} = \{V_i\}$ in the "phase" space $\mathbb{R}^N$ are introduced via the expressions

$$D_{ij}(\mathbf{x},t) = \lim_{\tau\to 0} \frac{1}{2\tau} \left\langle (x'_i - x_i)(x'_j - x_j) \right\rangle_{x':(t+\tau|\mathbf{x},t)} \quad (3)$$

$$V_i(\mathbf{x},t) = \lim_{\tau\to 0} \frac{1}{\tau} \left\langle (x'_i - x_i) \right\rangle_{x':(t+\tau|\mathbf{x},t)}\,, \quad (4)$$

where we have used the symbol

$$\langle \ldots \rangle_{x':(t'|\mathbf{x},t)} = \int\limits_{\mathbb{R}^N} d\mathbf{x}'(\ldots) G(\mathbf{x}',t'|\mathbf{x},t)\,.$$

The technique of the Wiener path integrals (see, e.g., [1, 2]) enables us to write the solution of the Fokker-Planck equation (1) subject to the initial condition (2) as a statistical integral over all the possible random trajectories of the Brownian particle $\{\mathbf{x}(t)\}$. These trajectories originate at the point $\mathbf{x}_0$ at time t_0, terminate at the point $\mathbf{x}$ at time t, and can be made of any sequence $\{\mathbf{x}(t')\}_{t'=t_0}^{t'=t}$ of the points in the space $\mathbb{R}^N$. Using the notions of the nonstandard analysis (see, e.g., [3]) we can deal with individual trajectories of Brownian particle motion ascribing to them some probability weights. It enables us to regard the kinetic coefficients $\mathbf{D}$ and $\mathbf{V}$ given by Exps. (3) and (4) as some probabilistic characteristics of the particle elementary movements.

The other approach is based on the Langevin equation assuming the system dynamics to be governed by (i) deterministic forces causing the regular drift $\mathbf{V}(\mathbf{x},t)$ and (ii) random forces giving rise to the Brownian motion of the system (see, e.g., [4]). In terms of infinitesimals (infinitely small nonstandard reals) the Langevin equation can be written as

$$dx_i = V_i(\mathbf{x},t)dt + \sum_{\alpha=1}^{M} G_{i\alpha}(\mathbf{x},t)\cdot dw_\alpha(t)\,, \quad (5)$$

where $\{dw_\alpha(t)\}_{\alpha=1}^{\alpha=M}$ are the infinitesimal increments of a certain M-dimensional Wiener process $\{\mathbf{w}(t)\}$ and the matrix $\mathbf{G}(\mathbf{x},t) = \|G_{i\alpha}(\mathbf{x},t)\|$ specifies the intensity of its components. The symbol "$\cdot$" of multiplication denotes that the stochastic differential equation (5) is written in the Itô form. The components of this Wiener process are assumed to be mutually independent and meet the condition

$$\begin{aligned} \langle dw_\alpha(t) dw_\beta(t)\rangle &= \delta_{\alpha\beta}\, dt \\ \langle dw_\alpha(t) dw_\beta(t')\rangle &= 0 \qquad \text{for } t \neq t'. \end{aligned} \quad (6)$$

The relationship between the two approaches is determined by the equality

$$2\mathbf{D} = \mathbf{G}\mathbf{G}^T\,. \quad (7)$$

When, for example, the dimension of the Wiener process exceeds the system dimension, $M > N$, there are its several implementations, in particular, of different dimensions that lead to the same Fokker-Planck equation and, thus, have to be regarded as equivalent. It means that the approach based on the Langevin equation is wider than the former one, it includes some additional component, the Wiener process $\{\mathbf{w}(t)\}$ endowing the system $\mathbf{x}$ with stochastic dynamics. Indeed, in describing the corresponding physical objects only the statistical properties of the trajectories $\{\mathbf{x}(t)\}$ matter. A particular implementation of the Wiener process $\{dw_\alpha(t)\}_{\alpha,t}$ may be treated as their internal probabilistic feature and chosen for the sake of convenience. Keeping the aforesaid in mind let us introduce the notion of equivalence of stochastic processes.

Assertion 1 (Physical equivalence of stochastic processes)**.** *Two stochastic processes $\{\mathbf{x}(t), \mathbf{w}_1(t)\}$ and $\{\mathbf{x}(t), \mathbf{w}_2(t)\}$ described by the corresponding Langevin equations are equivalent if both of them obey the same Fokker-Planck equation.*

The Langevin equation is a natural gate to generalizations including different types of stochastic process, in particular, the Itô, Fisk-Stratonovich, and Hänngi-Klimontovich interpretations. For example, dealing with an one-dimensional single-noise stochastic process $x(t)$ the following implicit Langevin equation of the θ-type

$$dx = V(x,t)dt + G(x + \theta\, dx, t)\, dw\,, \tag{8}$$

represents these interpretations. Namely, the particular cases of $\theta = 0$, $\theta = 1/2$, and $\theta = 1$ match the Itô, Fisk-Stratonovich, and Hänngi-Klimontovich interpretations, respectively. To demonstrate this directly let use a technique described in [5]. Namely, by virtue of Exp. (6) equation (8) can be written as

$$dx = G(x,t)\, dw + o\big(\sqrt{dt}\big) \tag{9}$$

within the accuracy of order of $\sqrt{dt}$, i.e., in the linear approximation with respect to dw. Then expanding the last term in Eq. (8) in the Taylor series and truncating all the terms of order larger than dt we get

$$dx = V(x,t)dt + G(x,t)\, dw + \partial_x G(x,t)\, \theta\, dx\, dw\,. \tag{10}$$

At the next step Exp. (9) is substituted into the last term on the right-hand side of Eq. (10) yielding

$$dx = V(x,t)dt + G(x,t)dw + \theta G(x,t)\partial_x G(x,t)(dw)^2. \tag{11}$$

Finally we note that the replacement

$$(dw)^2 \to \big\langle (dw)^2 \big\rangle = dt \tag{12}$$

does not change the form of the corresponding Fokker-Planck equation, which results from Exps. (3) and (4) specifying its kinetic coefficients. Therefore the given stochastic process is equivalent to the stochastic process governed by the following Langevin equation of the Itô type

$$dx = \left[V(x,t) + \frac{\theta}{2}\partial_x G^2(x,t)\right] dt + G(x,t)dw\,. \tag{13}$$

Whence it stems, in particular, that the given stochastic process of the θ-type and the stochastic processes of the Itô, Fisk-Stratonovich, and Hänngi-Klimontovich types are equivalent provided their regular drifts $V^0(x,t)$, $V^{1/2}(x,t)$, and $V^1(x,t)$, respectively, are related to one another via the expressions

$$\begin{aligned} V^0(x,t) &= V(x,t) + \frac{\theta}{2}\partial_x G^2(x,t)\,, \\ V^{1/2}(x,t) &= V(x,t) + \frac{(2\theta-1)}{4}\,\partial_x G^2(x,t)\,, \\ V^1(x,t) &= V(x,t) + \frac{(\theta-1)}{2}\,\partial_x G^2(x,t)\,. \end{aligned} \tag{14}$$

Besides, the Fokker-Planck equation matching the Langevin equation (8) can be rewritten also in the form

$$\partial_t P = \partial_x \left\{ \frac{1}{2} G^{2\theta}(x,t)\partial_x[G^{2(1-\theta)}(x,t)P] - V(x,t)P \right\}. \tag{15}$$

To complete the description of a stochastic process its kinetic coefficients should be specified appealing to the physical properties of the system at hand. As far as the Langevin equation is concerned, they are the noise intensity matrix $\mathbf{G}$, the drift rate $\mathbf{V}$, and maybe the collection of parameters $\{\theta\}$. it is worthwhile to underline the following.

Assertion 2 (Characteristics of the Langevin approach)**.** *The specification of the parameters $\{\theta\}$ is a question of physical interpretation of a given system and its properties rather than a mathematical problem.*

We can justify this statement appealing, for example, to the following argumentation. Let us assume that the microscopic description of a given physical system is finally reduced to a certain parabolic differential equation written for the probability density function P. It has to be of a form similar to Eq. (15). Then analyzing its structure we can find the corresponding kinetic coefficients and the collection of parameters $\{\theta\}$, in the given case, they are $G(x,t)$, $V(x,t)$, and θ. A discussion of this aspect from another point of view can be fond also in [6].

2 Characteristic examples

In order to simplify understanding the basic points of the further mathematical constructions let us discuss some characteristic systems exemplifying them.

(**1**) *Electron diffusion in semiconductors with nonuniform distribution of impurities and temperature gradient*

As follows from the standard gas-kinetic theory the diffusion of electrons in p-type semiconductors is described by the equation

$$\partial_t n + \sum_{i=1}^{3} \partial_i J_i = 0\,, \tag{16}$$

where $n(\mathbf{x}, t)$ is the electron density and the diffusion flux $\mathbf{J} = \{J_i\}_{i=1}^{i=3}$ is specified by Fick's law with the thermal diffusion effect

$$J_i = -D\partial_i n - \frac{D}{T} n \partial_i T\,. \tag{17}$$

Here $T(\mathbf{x}, t)$ is the local temperature of the semiconductor and the electron diffusivity D is proportional to T and inversely proportional to the concentration N of impurities scattering electrons, i.e.,

$$D = c\,\frac{T}{N}\,, \tag{18}$$

with c being a certain fixed characteristics of the material. In its turn the impurity distribution $N(\mathbf{x})$ can be also nonuniform in space.

In particular, in one-dimensional case this diffusion equation can be represented as

$$\partial_t n = \partial_x \left\{ \frac{c}{N} \partial_x \left(Tn\right) \right\}. \tag{19}$$

The comparison of the latter equation and Eq. (15) enables us to regards such diffusion of electrons as random walks $\{x(t)\}$ of certain Brownian particles with the diffusion coefficient $D(x, t) = cT(x, t)/N(x)$, zero velocity drift $V(x, t) = 0$, and a certain value of the parameter $\theta(x, t)$ depending on the spatial coordinate x and time t.

This type diffusion can be also represented as an one-dimensional stochastic process $x(t)$ affected by a single random force via two channels, namely, the scattering of electron momentum by impurities (channel 1) and the energy dissipation due to the interaction with phonons (channel 2). The corresponding Langevin equation reads

$$dx = V(x, t)dt + G_1(x + dx, t)G_2(x, t)\, dw\,. \tag{20}$$

Using the technique introduced in the previous Section Eq. (20) is reduced to the following equivalent Langevin equation of the Itô type

$$\begin{aligned} dx = \left[V(x, t) + \frac{1}{2} G_2^2(x, t) \partial_x G_1^2(x, t) \right] dt \\ + G_1(x, t)G_2(x, t)\, dw\,. \end{aligned} \tag{21}$$

The corresponding Fokker-Planck equation can be written in the form

$$\partial_t P = \partial_x \left\{ \frac{1}{2} G_1^2(x, t) \partial_x \left[G_2^2(x, t) P \right] \right\} \tag{22}$$

which coincides with the diffusion equation (19) in form.

(**2**) *Particles diffusion in strong magnetic field*

A fully ionized plasma, confined by a magnetic field $\mathbf{B}$, diffuses across the filed due to particle collisions. In this case a transverse gradient of the density n or temperature T gives rise, first, to the electron (e) and ion (i) flows in the third direction

$$n\mathbf{u}_L^{e,i} = \pm \frac{c}{e} \cdot \frac{\mathbf{B} \times \nabla(nkT)}{B^2} \tag{23}$$

where e is the electron charge, c is the speed of light, and k is the Boltzmann constant. It should be noted that Eq. (23) follows from the general laws of motion in magnetic field rather than caused by the particle collisions. In this case it actually does not contribute to the diffusion process. The particle collisions induce the drift of electrons and ions with the same velocity in the direction of the gradients

$$\mathbf{u}_d^{e,i} = -\frac{\eta c^2}{B^2} \nabla(nkT) + \frac{3\eta n c^2}{4B^2} \nabla(kT) + c\frac{\mathbf{B} \times \mathbf{E}}{B^2}, \tag{24}$$

where η is some kinetic coefficient and E is a possible electric field caused by the spatial separation of electrons and ions. For details see, e.g., [10]. In more complex topology of magnetic fields, for example, in the heliosphere both the components affect the plasma transport phenomena, which is described, in particular, with the anomalous diffusion tensor and the drift being highly inhomogeneous in space (see, e.g., [11]). Nevertheless, even such an anomalous process can be mimics by the Langevin equation [12]. Figure 1 illustrates one of anomalous properties of particle diffusion in the normal direction to a strong magnetic field $\mathbf{B}$. Without collisions and an electric field a charged particle moves along the Larmor circle with a fixed position of the center. So the diffusion process has to be described as random walks of the Larmor circle center rather than the particle motion. As the result, it is possible to relate the center random walks with particle collisions some where inside the circle. So it is naturally to expect that the point of scattering may be outside the line connecting the initial and terminal points of the elementary displacement of the circle center. This scattering point can be treated as the intermediate point $\boldsymbol{\chi}$ at which the kinetic coefficients have to be taken. As a result the relation between the point $\boldsymbol{\chi}$, and the points $\mathbf{x}$, $\mathbf{x} + \mathbf{dx}$ of elementary step of the effective Brownian particle is of tensor form

$$(\chi_i - x_i) = \sum_{j=1}^{N} \theta_{ij} dx_j\,,$$

which is certain generalization of the constructions presented in the previous example.

(**3**) *Stochastic self-acceleration*

Usually the concept of stochastic self-acceleration is used to describe anomalous transport phenomena

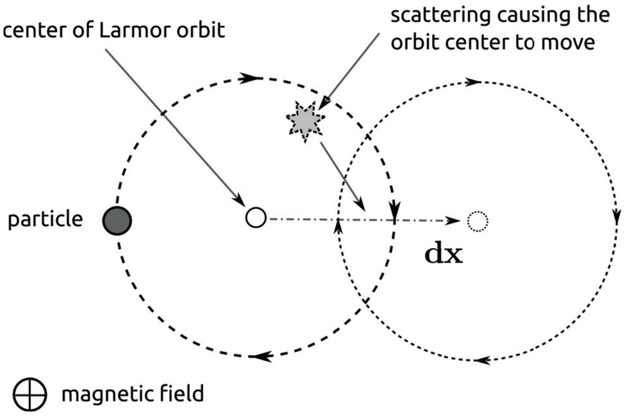

Fig. 1: Illustration of the mechanism causing particle diffusion across a strong magnetic field.

observed in highly non-equilibrium media like atmospheric and cosmic plasma or liquids with developed turbulence. In a simple case, this process is represented as one-dimensional Brownian motion $\{x(t)\}$ of a particle whose acceleration dx^2/dt^2 is governed by independent additive and multiplicative random forces:

$$dv = -kv + g_a dw_a + g_m \cdot (v + \theta dv)\, dw_m \,, \tag{25}$$

where $v = dx/dt$ is the particle velocity and k, $g_{a,m}$ are some constants. An equivalent single-noise noise process is described by the Langevin equation

$$dv = -kv + g(v + \theta dv)\, dw \,, \tag{26}$$

where the function $g(v)$ is specified by the expression

$$g(v) = \sqrt{g_a^2 + g_m^2 v^2} \,. \tag{27}$$

Both of these stochastic differential equations match the same Fokker-Planck equation

$$\partial_t P = \partial_v \left\{ \frac{1}{2} g^{2\theta}(v) \partial_v \left[g^{2(1-\theta)}(v) P \right] + kvP \right\} . \tag{28}$$

In particular, this process generates Lévy flights in the physical space $\mathbb{R}_x$ (see, e.g., [7, 8, 9]).

In these examples we have met the notions of anisotropic θ_{ij} and $\theta(x,t)$ depending on both the random variable x and the time t as well as the notions of different channels of noise action. It is a certain background of the developed model.

3 Generalized Langevin equation

The aforesaid prompts us to pose for consideration the following generalization of the Langevin equation. A given system is characterized by the regular drift $V_i(\mathbf{x}, t)$, the noise intensity matrix $G^{(k)}_{i\alpha}(\mathbf{x}, t)$, K various channels of noise action (indexed by k's), and the following construction of an *intermediate point* $\boldsymbol{\chi}^{(k,\alpha)}(\mathbf{x}, d\mathbf{x}, t) \in \mathbb{R}^N$ at which the components of the matrix $\mathbf{G}$ should be taken:

$$\chi_i^{(k,\alpha)} = x_i + \sum_{j=1}^{N} \theta_{ij}^{(k,\alpha)}(\mathbf{x}, t)\, dx_j \,. \tag{29}$$

Then the generalized implicit Langevin equation reads

$$dx_i = V_i(\mathbf{x}, t) dt + \sum_{\alpha=1}^{M} \prod_{k=1}^{K} G^{(k)}_{i\alpha} \left[\boldsymbol{\chi}^{(k,\alpha)}(\mathbf{x}, d\mathbf{x}, t), t \right] dw_\alpha . \tag{30}$$

It should be noted that the given Langevin equation (30) is not strictly of the tensor form. However, in constructing the mathematical description of a given stochastic process we have to appeal to its physical properties. Thereby, the choice of the coordinate frame is considered to be chosen for certain physical reasons. Figure 2 illustrates the given description.

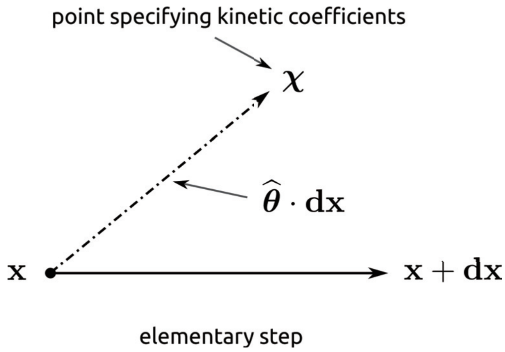

Fig. 2: Illustration of the relationship between the intermediate point $\boldsymbol{\chi}^{(k,\alpha)}$, where the corresponding coefficients have to be taken, and the elementary step $\mathbf{dx}$ for a given channel k and noise component α.

Let us consider the basic features of the developed model in more details.

3.1 Equivalence relation for different types of stochastic process

Using the technique discussed in Introduction we may rewrite Eq. (30) within the accuracy $\sqrt{dt}$ as follows

$$dx_i = \sum_{\alpha=1}^{M} \prod_{k=1}^{K} G^{(k)}_{i\alpha}(\mathbf{x}, t)\, dw_\alpha + o\big(\sqrt{dt}\big) \,. \tag{31}$$

Then dealing with the terms on the right-hand side of Eq. (30) containing the points $\{\boldsymbol{\chi}^{k,\alpha}\}$, we have to expand them near the point $\mathbf{x}$ into the Taylor series and cut off all the components whose order is higher than dt. In this way adopting the replacement rule

$$dw_\alpha(t) dw_\beta(t) \to \langle dw_\alpha(t) dw_\beta(t) \rangle \propto dt \tag{32}$$

and using properties (6) we construct the equivalent stochastic process of the Itô type

$$dx_i = V_i^{\text{Itô}}(\mathbf{x},t)dt + \sum_{\alpha=1}^{M}\prod_{k=1}^{K} G_{i\alpha}^{(k)}(\mathbf{x},t)\,dw_\alpha\,, \qquad (33)$$

where the drift velocity is specified via the expression

$$\begin{aligned} V_i^{\text{Itô}}(\mathbf{x},t) &= V_i(\mathbf{x},t) \\ &+ \sum_{j=1}^{N}\sum_{\alpha=1}^{M}\sum_{k=1}^{K}\left[\prod_{\substack{k'=1\\k'\neq k}}^{K} G_{i\alpha}^{(k')}(\mathbf{x},t)\cdot G_{j\alpha}^{(k')}(\mathbf{x},t)\right] \\ &\times \frac{\partial G_{i\alpha}(\mathbf{x},t)}{\partial x_j}\cdot\theta_{jl}^{(k,\alpha)}(\mathbf{x},t)\cdot G_{l\alpha}(\mathbf{x},t)\,. \qquad (34)\end{aligned}$$

Actually Exp. (34) is the main result enabling us construct equivalent stochastic processes of different types. Indeed, if after transformation (34) several systems lead to the same Langevin equation (33) then they are equivalent in the sense given by Proposition 1.

3.2 Post-point and kinetic types of stochastic processes

Following the notion of the K-type stochastic processes introduced by Klimontovich [13] also referred to as the processes of the Hänngi-Klimontovich type their characteristic property is the following form of the diffusion flux

$$J_i = \sum_{j=N}^{N} D_{ij}(\mathbf{x},t)\partial_j P - V_i^K(\mathbf{x},t)P\,. \qquad (35)$$

In this case if the regular drift is absent, then the stationary distribution $P^{\text{st}}(\mathbf{x})$ of the corresponding Brownian particles is uniform in the space $\mathbb{R}^N$. For isotropic media, in particular, for one-dimensional systems such processes can be described by the parameter $\theta = 1$ (see, e.g., [1, 2]). However, for anisotropic media this relation does not hold. To demonstrate this fact let us consider the simplest situation when there is only one channel, $K = 1$, and the quantity $\theta_{ij} = \delta_{ij}$. Under such conditions the relationship (34) between the given stochastic process and the equivalent Itô process reads

$$\begin{aligned} V_i^{\text{Itô}}(\mathbf{x},t) &= V_i^{\theta=1}(\mathbf{x},t) \\ &+ \sum_{j=1}^{N}\sum_{\alpha=1}^{M}\frac{\partial G_{i\alpha}(\mathbf{x},t)}{\partial x_j}\cdot G_{j\alpha}(\mathbf{x},t)\,. \qquad (36)\end{aligned}$$

The substitution of Exp. (36) into the Fokker-Planck equation (1) leads us to the following expression for the diffusion flux

$$J_i = \sum_{j=1}^{N} D_{ij}(\mathbf{x},t)\partial_j P - V_i^{\theta=1}(\mathbf{x},t)P + \frac{1}{2}\sum_{j=1}^{N}\sum_{\alpha=1}^{M}$$
$$\left[G_{i\alpha}(\mathbf{x},t)\frac{\partial G_{j\alpha}(\mathbf{x},t)}{\partial x_j} - \frac{\partial G_{i\alpha}(\mathbf{x},t)}{\partial x_j}G_{j\alpha}(\mathbf{x},t)\right]P\,. \qquad (37)$$

In deriving the latter we have used relationship (7) between the diffusions tensor D_{ij} and the noise intensity matrix $G_{i\alpha}$, i.e.,

$$D_{ij}(\mathbf{x},t) = \frac{1}{2}\sum_{\alpha=1}^{M} G_{i\alpha}(\mathbf{x},t)\cdot G_{j\alpha}(\mathbf{x},t)\,. \qquad (38)$$

In the general case for anisotropic media the last term in Exp. (37) does not equal to zero, so, a ($\theta = 1$)-stochastic process is not of the K-type. It seems that the K-type stochastic processes are characterized by a non-diagonal and, maybe, non-constant tensor $\theta_{ij}(\mathbf{x},t)$.

3.3 Change-of-variables and the Stratonovich type processes

Let us remind, first, (see [1, 2] for details) that the Stratonovich type processes match the quantity $\theta_{ij} = \frac{1}{2}\delta_{ij}$. So, by virtue of expression (34) the relationship between a given general type stochastic process and its equivalent one of the Stratonovich type is written as

$$\begin{aligned} V_i^{\text{Str}}(\mathbf{x},t) &= V_i(\mathbf{x},t) \\ &+ \sum_{j=1}^{N}\sum_{\alpha=1}^{M}\sum_{k=1}^{K}\left[\prod_{\substack{k'=1\\k'\neq k}}^{K} G_{i\alpha}^{(k')}(\mathbf{x},t)\cdot G_{j\alpha}^{(k')}(\mathbf{x},t)\right] \\ &\times \frac{\partial G_{i\alpha}(\mathbf{x},t)}{\partial x_j}\cdot\left[\theta_{jl}^{(k,\alpha)}(\mathbf{x},t) - \frac{1}{2}\delta_{jl}\right]\cdot G_{l\alpha}(\mathbf{x},t)\,. \qquad (39)\end{aligned}$$

Second, the conversion from the original variables to new ones: $\mathbf{x} \mapsto \mathbf{z} = \Phi(\mathbf{x})$, can be implemented in the stochastic differential equation (30) using the standard change-of-variable rules. Indeed in this case we can write

$$\mathbf{x} = \boldsymbol{\chi} - \frac{1}{2}\mathbf{dx} \quad \text{and} \quad \mathbf{x} + \mathbf{dx} = \boldsymbol{\chi} + \frac{1}{2}\mathbf{dx}\,, \qquad (40)$$

thus,

$$\begin{aligned} dz_i &= \Phi_i(\mathbf{x}+\mathbf{dx}) - \Phi_i(\mathbf{x}) \\ &= \sum_{j=1}^{N}\left.\frac{\partial\Phi_i(\mathbf{x})}{\partial x_j}\right|_{\mathbf{x}=\boldsymbol{\chi}} dx_j + O\left[(dt)^{3/2}\right]\,. \qquad (41)\end{aligned}$$

As a result, in the substitution of (30) into (41) no other terms similar to $dx_i dx_j$ have to be taken into account. Let summarize the aforesaid.

Assertion 3. *The change-of-variables $\mathbf{z} = \Phi(\mathbf{x})$ in the Langevin equation (30) obeys the standard rules for ordinary differential equations provided a stochastic process at hand is of the Stratonovich type.*

Dealing with a general type stochastic process the particular details of the change-of-variables can be figured out using the following diagram

$$\begin{array}{ccc} \text{general type:} & \mathbf{x}(t) \xrightarrow{\ \Phi\ } \mathbf{z}(t) & \\ & \downarrow \qquad\quad \uparrow & (42) \\ \text{Stratonovich type:} & \mathbf{x}(t) \xrightarrow{\ \Phi\ } \mathbf{z}(t) & \end{array}$$

3.4 One open problem

Finalizing the present work we would like to pose the following problem for discussion. Dealing with a particular physical system the diffusion coefficients D_{ij} and the regular drift velocity V_i can be evaluated experimentally. In contrast, the noise components $\{dw_\alpha\}$ are hidden in some sense, they can be evaluated using only random variations in the phase variables $\mathbf{x}(t)$. However, the terms entering, for example, transformation (34) to the effective Itô stochastic process cannot be reduced directly to some operations with the diffusion tensor D_{ij} related to the noise intensity matrix $G_{i\alpha}$ via formula (38).

So, let us consider a general type stochastic process whose dynamics is determined by the regular drift velocity $\{V_i(\mathbf{x}, t)\}$, the noise intensity $\{G^{(k)}_{i\alpha}(\mathbf{x}, t)\}$ known for various channels, and the tensor $\theta^{(k,\alpha)}(\mathbf{x}, t)$. The question is whether there is an equivalent stochastic process such that its description is specified completely by the drift velocity and the diffusion coefficients and, maybe, some differential operations with them.

4 Conclusion

The present work discusses the relationship between two approaches to describing nonlinear Markovian processes. One of them is the Fokker-Planck equation, the other is the Langevin equation. The characteristic feature of the processes at hand is the dependence of the diffusion tensor and, correspondingly, the intensity of noise entering the Langevin equation on the random variable, which explains the use of the term "nonlinear Markovian process."

There are many particular forms of the Langevin equation leading to the same Fokker-Planck equation, which enables us regards them as different equivalent representations of one Markovian process. The point pursued in the work is that the choice of a particular form of the Langevin equation is a question of physical interpretation of a given system rather than a mathematical problem.

Appealing to several physical systems exemplifying various stochastic processes, the work proposes a certain generalization of the Langevin equation. It considers several channels of noise actions characterized individually by (i) the noise intensity and (ii) the intermediate point of general position at which this noise intensity should be calculated. Based on this approach the relationship between different interpretations of stochastic processes is analyzed as well as the generalization of the change-of-variables rule is developed.

Acknowledgments

This work was supported in part by the JSPS Grants-in-Aid for Scientific Research Program, Grant 24540410-0001.

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