

BOUNDEDNESS FOR FINITE SUBGROUPS OF LINEAR ALGEBRAIC GROUPS

CONSTANTIN SHRAMOV AND VADIM VOLOGODSKY

ABSTRACT. We show the boundedness of finite subgroups in any anisotropic reductive group over a perfect field that contains all roots of 1. Also, we provide explicit bounds for orders of finite subgroups of automorphism groups of Severi–Brauer varieties and quadrics over such fields.

1. INTRODUCTION

In this paper we study finite subgroups of linear algebraic groups. We say that a field \mathbb{K} *contains all roots of 1*, if, for every positive integer n , the polynomial $x^n - 1$ splits completely in $\mathbb{K}[x]$. An example of such a field is the field of rational functions on an irreducible variety defined over an algebraically closed field. Recall that a linear algebraic group is called *anisotropic* if it does not contain a subgroup isomorphic to the split one-dimensional torus \mathbb{G}_m . Our main result is the following.

Theorem 1.1. *Let \mathbb{K} be a perfect field that contains all roots of 1, and let G be an anisotropic reductive group over \mathbb{K} . Then there exists a constant $L = L(G)$ such that any finite subgroup of $G(\mathbb{K})$ has order at most L .*

Actually, we will prove a more precise Theorem 3.7 that gives some boundedness result for non-perfect fields and non-reductive linear algebraic groups as well, and also provides an explicit (multiplicative) bound for the orders of finite subgroups in terms of the rank of G , the number of its connected components, and the minimal dimension of a faithful representation of the maximal reductive quotient of the neutral component of $G_{\overline{\mathbb{K}}}$. Moreover, we will see in Corollary 3.8 that there is a bound that depends only on the rank and the number of connected components of G .

A particular case of Theorem 1.1 for the projective orthogonal groups was proved by T. Bandman and Yu. Zarhin in [BZ17, Corollary 4.11, Theorem 4.14]. They applied its rank 1 case to analyze fiberwise birational maps of varieties fibered into rational curves.

Let us say that a group G has *bounded finite subgroups*, if there exists a constant $L = L(G)$ such that, for any finite subgroup $\Gamma \subset G$, one has $|\Gamma| \leq L$. If this is not the case, we say that G has *unbounded finite subgroups*. Thus, Theorem 1.1

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claims that every anisotropic reductive group over a perfect field that contains all roots of 1 has bounded finite subgroups.

Recall that a *Severi–Brauer* variety is a variety X over a field \mathbb{K} such that its scalar extension to the algebraic closure of \mathbb{K} is isomorphic to a projective space. For instance, one-dimensional Severi–Brauer varieties are conics over \mathbb{K} . The automorphism group scheme of an $(n - 1)$ -dimensional Severi–Brauer variety is an inner form of the algebraic group PGL_n . One can apply Theorem 1.1 to study the automorphism groups of Severi–Brauer varieties. For a Severi–Brauer variety X associated to a central simple algebra A over a perfect field \mathbb{K} that contains all roots of 1, Theorem 1.1 implies that $\mathrm{Aut}(X)$ has bounded finite subgroups if and only if A is a division algebra; see Remark 4.6 for details. The following theorem, which we prove directly, amplifies this observation.

Theorem 1.2. *Let \mathbb{K} be a field that contains all roots of 1. Let X be a Severi–Brauer variety of dimension $n - 1$ over \mathbb{K} , and let A be the corresponding central simple algebra. Assume that the characteristic $\mathrm{char} \mathbb{K}$ of \mathbb{K} does not divide n . The following assertions hold.*

- (i) *The group $\mathrm{Aut}(X)$ has bounded finite subgroups if and only if A is a division algebra; in particular, if n is a prime number, then $\mathrm{Aut}(X)$ has bounded finite subgroups if and only if $X(\mathbb{K}) = \emptyset$, i.e., X is not isomorphic to \mathbb{P}^{n-1} .*
- (ii) *Suppose that A is a division algebra. Let $g \in \mathrm{Aut}(X)$ be an element of finite order, and $\Gamma \subset \mathrm{Aut}(X)$ be a finite subgroup. Then $g^n = 1$, and Γ is an abelian group whose order divides n^2 .*

In particular, if \mathbb{K} is a perfect field that contains all roots of 1, then Theorem 1.2 applies to all Severi–Brauer varieties over \mathbb{K} ; indeed, in this case $\mathrm{char} \mathbb{K}$ cannot divide the dimension of a central division algebra over \mathbb{K} , see Remark 4.7 below. In the case of an arbitrary field whose characteristic p divides the dimension of the corresponding division algebra, the structure of finite subgroups of the automorphism group is still rather simple, see Proposition 4.5; in particular, finite subgroups of the automorphism groups are always abelian in this case as well. However, for such Severi–Brauer varieties finite p -subgroups of the automorphism group can have arbitrarily large order, see Example 4.8.

Applying Theorem 1.1 to a projective orthogonal group, one can prove that the automorphism group of a smooth quadric Q over a field \mathbb{K} of characteristic different from 2 that contains all roots of 1 has bounded finite subgroups if and only if $Q(\mathbb{K}) = \emptyset$. We find explicit bounds for orders of finite automorphism groups of quadrics over appropriate fields, thus generalizing the results of [BZ17, §4] and making them more precise.

Theorem 1.3. *Let \mathbb{K} be a field that contains all roots of 1. Assume that $\mathrm{char} \mathbb{K} \neq 2$ or \mathbb{K} is perfect. Let $n \geq 3$ be an integer, and let $Q \subset \mathbb{P}^{n-1}$ be a smooth quadric hypersurface over \mathbb{K} . The following assertions hold.*

- (i) *The group $\mathrm{Aut}(Q)$ has bounded finite subgroups if and only if $Q(\mathbb{K}) = \emptyset$.*
- (ii) *If n is odd and $Q(\mathbb{K}) = \emptyset$, then every finite subgroup of $\mathrm{Aut}(Q)$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^m$, where $m \leq n - 1$.*
- (iii) *If n is even and $Q(\mathbb{K}) = \emptyset$, then every non-trivial element of finite order in the group $\mathrm{Aut}(Q)$ has order 2 or 4, and the order of every finite subgroup of $\mathrm{Aut}(Q)$ divides 8^{n-1} .*

According to Theorem 1.3(ii), if Q is a smooth odd-dimensional quadric without points over a field of characteristic different from 2 that contains all roots of 1, then every finite group faithfully acting on Q is abelian. If Q has even dimension, this is not always the case, see Example 5.5.

Note that Theorem 1.3 fails over a non-perfect field of characteristic 2, see Example 4.8. Note also that one cannot drop the assumption on the existence of roots of 1 in Theorems 1.2 and 1.3. Indeed, the conic over the field of real numbers defined by the equation $x^2 + y^2 + z^2 = 0$ has automorphisms of arbitrary finite order.

The plan of our paper is as follows. In §2 we prove Theorem 1.1 in the case when G is a torus. The proof is based on the Minkowski theorem on finite subgroups of $\mathrm{GL}_n(\mathbb{Z})$ and elementary Galois theory.

In §3 we study finite subgroups of linear algebraic groups and prove Theorem 3.7, which is a more precise version of Theorem 1.1. The idea of the proof is the following. According to a result of Borel and Tits, for every connected anisotropic reductive group G over a perfect field \mathbb{K} , every element $g \in G(\mathbb{K})$ is contained in $T(\mathbb{K})$, for some torus $T \subset G$. Using the results of §2 we bound the order of g . On the other hand, choosing a faithful representation of G we get an embedding $G(\mathbb{K}) \subset \mathrm{GL}_N(\mathbb{K})$ for some positive integer N . This, together with a Burnside type result due to [HP76] (see Theorem 3.6 below), proves that $G(\mathbb{K})$ has bounded finite subgroups.

In §4 we describe automorphism groups of Severi–Brauer varieties and prove Theorem 1.2. In §5 we prove Theorem 1.3.

Throughout the paper by $\bar{\mathbb{K}}$ we denote an algebraic closure of a field \mathbb{K} , and by \mathbb{K}^{sep} we denote a separable closure of \mathbb{K} (recall that $\mathbb{K}^{sep} = \bar{\mathbb{K}}$ provided that \mathbb{K} is perfect). Given a variety X defined over \mathbb{K} , for an arbitrary field extension $K \supset \mathbb{K}$ we denote by X_K the corresponding scalar extensions to K , and by $X(K)$ we denote the set of K -points of X . Abusing notation a bit, we write \mathbb{P}^n for a projective space over a field \mathbb{K} , and similarly write \mathbb{G}_m and \mathbb{G}_a for the multiplicative and additive groups, respectively.

2. TORI

In this section we study elements of finite order in algebraic tori.

The following result is a famous theorem of H. Minkowski, see [M1887, §1] or [Ser07, Theorem 1].

Theorem 2.1. *For any positive integer n , the group $\mathrm{GL}_n(\mathbb{Z})$ has bounded finite subgroups.*

Theorem 2.1 tells us that the maximal order $\Upsilon_A(n)$ of a finite subgroup in $\mathrm{GL}_n(\mathbb{Z})$ and the least common multiple $\Upsilon_M(n)$ of the orders of such subgroups are well defined constants depending only on n . For small n , one can compute the values of $\Upsilon_A(n)$ and $\Upsilon_M(n)$. For instance, we have

$$\Upsilon_A(1) = \Upsilon_M(1) = 2, \quad \Upsilon_A(2) = 12, \quad \Upsilon_M(2) = 24, \quad \Upsilon_A(3) = \Upsilon_M(3) = 48,$$

see e.g. [Ser07, §1.1] and [Tah71, §1]. In particular, neither of the bounds given by $\Upsilon_A(2)$ and $\Upsilon_M(2)$ is strictly stronger than the other one.

The following is the main technical result of this section.

Lemma 2.2. *Let n and d be positive integers. Let \mathbb{K} be a field such that the characteristic of \mathbb{K} does not divide d , and \mathbb{K} contains a primitive d -th root of 1. Let T*

be an anisotropic n -dimensional algebraic torus over \mathbb{K} such that $T(\mathbb{K})$ contains a point of order d . Then $d \leq \Upsilon_A(n)$ and d divides $\Upsilon_M(n)$.

Proof. Let $\check{T} = \text{Hom}(\mathbb{G}_m, T_{\mathbb{K}^{sep}})$ be the lattice of cocharacters of T . Recall (see [Bor91, §8.12]) that the functor $T \mapsto \check{T}$ induces an equivalence between the category of algebraic tori over \mathbb{K} and the category of free abelian groups of finite rank equipped with an action of the Galois group $\text{Gal}(\mathbb{K}^{sep}/\mathbb{K})$ such that the image of the homomorphism $\text{Gal}(\mathbb{K}^{sep}/\mathbb{K}) \rightarrow \text{Aut}(\check{T})$ is finite. Denote this image by Θ .

The group of d -torsion elements of $T(\mathbb{K}^{sep})$ is isomorphic, as a Galois module, to $\check{T} \otimes \mu_d$, where μ_d is the group of d -th roots of unity in \mathbb{K}^{sep} . Since \mathbb{K} contains a primitive d -th root of 1, the Galois module μ_d is the trivial module $\mathbb{Z}/d\mathbb{Z}$, so that the Galois module $\check{T} \otimes \mu_d$ is isomorphic to $\check{T}/d\check{T}$. Hence, a point $x \in T(\mathbb{K})$ of order d can be viewed as a $\text{Gal}(\mathbb{K}^{sep}/\mathbb{K})$ -invariant element $\bar{v} \in \check{T}/d\check{T}$ of order d (so that $m\bar{v} \neq 0$ for $m < d$). Let $v \in \check{T}$ be any preimage of \bar{v} under the projection $\check{T} \rightarrow \check{T}/d\check{T}$, and let

$$w = \sum_{\theta \in \Theta} \theta(v).$$

Since \bar{v} is $\text{Gal}(\mathbb{K}^{sep}/\mathbb{K})$ -invariant, the image of w in $\check{T}/d\check{T}$ is equal to $|\Theta|\bar{v}$. On the other hand, it is clear that w is a $\text{Gal}(\mathbb{K}^{sep}/\mathbb{K})$ -invariant element of \check{T} . By the above mentioned equivalence of categories, w gives rise to a non-trivial homomorphism $\mathbb{G}_m \rightarrow T$, provided that w itself is non-zero. Since T is anisotropic, we conclude that $w = 0$. Therefore, $|\Theta|$ divides d , and the required assertion follows. \square

Remark 2.3. J.-L. Colliot-Thélène pointed out to us that the proof of Lemma 2.2 can be reformulated in the following way. The short exact sequence of Θ -modules

$$0 \longrightarrow \check{T} \xrightarrow{d} \check{T} \longrightarrow \check{T}/d\check{T} \longrightarrow 0$$

gives rise to the long exact sequence of cohomology groups

$$(2.1) \quad \dots \rightarrow H^0(\Theta, \check{T}) \rightarrow H^0(\Theta, \check{T}/d\check{T}) \rightarrow H^1(\Theta, \check{T}) \rightarrow \dots$$

Since T is anisotropic, we have that $H^0(\Theta, \check{T}) = 0$. Thus, the second map in (2.1) is injective. On the other hand, the group $H^1(\Theta, \check{T})$ is annihilated by $|\Theta|$ (see for instance [CF67, Proposition IV.6.3]). It follows that the group $H^0(\Theta, \check{T}/d\check{T})$ of d -torsion points of $T(\mathbb{K})$ is also annihilated by $|\Theta|$.

Lemma 2.2 implies the following result.

Corollary 2.4. *Let \mathbb{K} be a field that contains all roots of 1, and let T be an anisotropic n -dimensional torus over \mathbb{K} . Let $g \in T(\mathbb{K})$ be an element of finite order d , and let $\Gamma \subset T(\mathbb{K})$ be a finite subgroup. Then d is not divisible by $\text{char } \mathbb{K}$, one has $d \leq \Upsilon_A(n)$, and d divides $\Upsilon_M(n)$. Moreover, one has $|\Gamma| \leq \Upsilon_A(n)^n$, and $|\Gamma|$ divides $\Upsilon_M(n)^n$.*

Proof. Note that if $\text{char } \mathbb{K} = p$ is positive, then $T(\mathbb{K})$ does not contain elements of order p , because $T_{\mathbb{K}^{sep}} \cong \mathbb{G}_m^n$, and there are no such elements in $(\mathbb{K}^{sep})^*$. Thus, if there is an element of finite order d in $T(\mathbb{K})$, then d is not divisible by p , so that $d \leq \Upsilon_A(n)$ and d divides $\Upsilon_M(n)$ by Lemma 2.2. It remains to notice that every finite subgroup Γ of $T(\mathbb{K})$ is an abelian group generated by at most n elements. Therefore, $|\Gamma| \leq \Upsilon_A(n)^n$ and $|\Gamma|$ divides $\Upsilon_M(n)^n$. \square

Example 2.5. Let \mathbb{K} be a field that contains all roots of 1, and let T be a one-dimensional torus over \mathbb{K} that is different from \mathbb{G}_m . Since $\Upsilon_A(2) = \Upsilon_M(2) = 2$, we conclude from Corollary 2.4 that every non-trivial finite subgroup of $T(\mathbb{K})$ has order 2. If $\text{char } \mathbb{K} = 2$, then $T(\mathbb{K})$ does not contain non-trivial finite subgroups at all, because there are no elements of order 2 in $(\mathbb{K}^{sep})^*$.

In certain cases the bound provided by Corollary 2.4 can be improved.

Example 2.6. Let \mathbb{K} be a field that contains all roots of 1, and let $\mathbb{K} \subset \mathbb{L}$ be a Galois extension of degree n . Consider the torus $T = R_{\mathbb{L}/\mathbb{K}}\mathbb{G}_m/\mathbb{G}_m$ over \mathbb{K} , where $R_{\mathbb{L}/\mathbb{K}}$ denotes the Weil restriction of scalars, and the embedding $\mathbb{G}_m \hookrightarrow R_{\mathbb{L}/\mathbb{K}}\mathbb{G}_m$ comes by adjunction from the identity morphism $\mathbb{G}_m \rightarrow \mathbb{G}_m$. Let Γ be a finite subgroup of $T(\mathbb{K}) \cong \mathbb{L}^*/\mathbb{K}^*$. We claim that $|\Gamma|$ divides n . Indeed, since \mathbb{K} contains all roots of 1, there is a well-defined homomorphism

$$\zeta: (\mathbb{L}^*/\mathbb{K}^*)_{tors} \rightarrow \text{Hom}(\text{Gal}(\mathbb{L}/\mathbb{K}), \mu_\infty(\mathbb{K}))$$

which sends a torsion element $\alpha \in \mathbb{L}^*/\mathbb{K}^*$ to the map

$$\gamma \mapsto \frac{\gamma(\tilde{\alpha})}{\tilde{\alpha}},$$

where $\tilde{\alpha}$ is an arbitrary preimage of α in \mathbb{L}^* . It is easy to see that ζ is injective. Since Γ is a subgroup of $(\mathbb{L}^*/\mathbb{K}^*)_{tors}$, this implies that $|\Gamma|$ divides n .

3. LINEAR ALGEBRAIC GROUPS

In this section we study finite subgroups of linear algebraic groups and prove Theorem 1.1.

Recall that a linear algebraic group G over a field \mathbb{K} is a smooth closed subgroup scheme of GL_N over \mathbb{K} . In particular, the group $G(\mathbb{K})$ of its \mathbb{K} -points has a faithful finite-dimensional representation in a \mathbb{K} -vector space. We refer the reader to [Bor91] and [Spr98] for the basics of the theory of linear algebraic groups.

A connected semi-simple algebraic group G is said to be *simply connected* if every central isogeny $\tilde{G} \rightarrow G$, where \tilde{G} is a connected semi-simple group, is necessarily an isomorphism. Recall that every connected semi-simple group G admits a *universal cover* which is a pair consisting of a connected semi-simple simply connected group \tilde{G} and a central isogeny $\tilde{G} \rightarrow G$. The group scheme theoretic kernel of the latter isogeny is called the *algebraic fundamental group* of G and is denoted by $\pi_1(G)$. This is a finite group scheme whose order $|\pi_1(G)|$ equals the order of the topological fundamental group of the connected semi-simple group over \mathbb{C} constructed from the the root datum of $G_{\mathbb{K}}$.

Let H be a quasi-simple algebraic group over an algebraically closed field (that is, H has no proper infinite normal closed subgroups). One defines the set $\mathcal{T}(H)$ of *torsion primes of H* to be the empty set if H has type A_n or C_n . If H has type B_n , D_n , or G_2 , we set $\mathcal{T}(H) = \{2\}$; if H has type F_4 , E_6 , or E_7 , we set $\mathcal{T}(H) = \{2, 3\}$; if H has type E_8 , we set $\mathcal{T}(H) = \{2, 3, 5\}$. Given any connected semi-simple algebraic group G over a field \mathbb{K} we say that a prime p is a *torsion prime of G* if p is a torsion prime for some quasi-simple direct factor of $\tilde{G}_{\mathbb{K}}$, where \tilde{G} is the universal cover of G .

Similarly to the case of an algebraically closed field, many properties of linear algebraic groups are determined by their maximal tori. Note that in general a

linear algebraic group G over a non-algebraically closed field \mathbb{K} may contain non-isomorphic maximal tori, but their dimension still equals the dimension of maximal tori in $G_{\bar{\mathbb{K}}}$, see [Spr98, Theorem 13.3.6(i)] and [Spr98, Remark 13.3.7]; this dimension is called the *rank* of G .

Recall that an element $g \in G(\mathbb{K})$ is called semi-simple if its image in $\mathrm{GL}_N(\mathbb{K})$ is diagonalizable over an algebraic closure $\bar{\mathbb{K}}$ of \mathbb{K} . The notion of a semi-simple element is intrinsic, that is, it does not depend on the choice of N and an embedding $G \hookrightarrow \mathrm{GL}_N(\mathbb{K})$; see [Spr98, §2.4]. The main tool that will allow us to apply the results of §2 is the following theorem.

Theorem 3.1 (see [Spr98, Corollary 13.3.8(i)]). *Let G be a connected linear algebraic group over a field \mathbb{K} , and let $g \in G(\mathbb{K})$ be a semi-simple element. Then there exists a torus $T \subset G$ such that g is contained in $T(\mathbb{K})$.*

Corollary 3.2. *Let G be a connected linear algebraic group over a field \mathbb{K} , and let $g \in G(\mathbb{K})$ be a finite order element whose order is not divisible by the characteristic of \mathbb{K} . Then there exists a torus $T \subset G$ such that g is contained in $T(\mathbb{K})$.*

For anisotropic reductive groups over perfect fields and for simply connected semi-simple anisotropic groups over arbitrary fields whose characteristic is large enough, one has a stronger result.

Theorem 3.3 (see [BT71, Corollary 3.8] and [Tit86, Corollary 2.6]). *Let G be a connected anisotropic reductive group over \mathbb{K} . Assume, in addition, that either \mathbb{K} is perfect, or G is semi-simple, simply connected, and $\mathrm{char} \mathbb{K} = p > 0$ is not a torsion prime for G . Then, for every element $g \in G(\mathbb{K})$, there exists a torus $T \subset G$ such that g is contained in $T(\mathbb{K})$.*

Corollary 3.4. *Under the assumptions of Theorem 3.3 the order of every finite order element of $G(\mathbb{K})$ is not divisible by the characteristic of \mathbb{K} .*

Proof. By Theorem 3.3 it suffices to prove the assertion in the case when G is a torus, in which case it is given by Corollary 2.4. \square

Note that over fields of positive characteristic non-reductive linear algebraic groups may have unbounded finite subgroups. For instance, the p -torsion subgroup of \mathbb{G}_a over an infinite field of characteristic p is an infinite-dimensional vector space over the field \mathbf{F}_p of p elements. However, this example is in a certain sense the only source of unboundedness for unipotent groups.

Lemma 3.5. *Let G be a unipotent group over a field \mathbb{K} . If $\mathrm{char} \mathbb{K} = 0$, then the group $G(\mathbb{K})$ does not contain elements of finite order greater than 1. If $\mathrm{char} \mathbb{K} = p > 0$, then $G(\mathbb{K})$ is a p -primary torsion group.*

Proof. Without loss of generality we may assume that \mathbb{K} is algebraically closed. Then G is isomorphic to a closed subgroup of the group U_n of unipotent upper triangular matrices (see e.g. [Spr98, Proposition 2.4.12]). Thus, we may assume that $G = U_n$. The lemma follows since U_n can be obtained as a consecutive extension of groups isomorphic to \mathbb{G}_a . \square

We will need the following auxiliary fact about orders of finite groups with given exponents proved in [HP76].

Theorem 3.6. *Let n and d be positive integers, and let \mathbb{K} be a field. Let $\Gamma \subset \mathrm{GL}_n(\mathbb{K})$ be a finite subgroup. If $\mathrm{char} \mathbb{K} > 0$, denote by $|\Gamma|'$ the largest factor of $|\Gamma|$ which is not divisible by $\mathrm{char} \mathbb{K}$; otherwise put $|\Gamma|' = |\Gamma|$. Suppose that for every $g \in \Gamma$ such that the order of g is not divisible by the characteristic of \mathbb{K} , one has $g^d = 1$. Then $|\Gamma|'$ divides d^n .*

Proof. It is proved in [HP76, Theorem 1] that under the above assumptions one has $|\Gamma|' \leq d^n$. Applying this to the q -Sylow subgroups Γ_q for all primes $q \neq \mathrm{char} \mathbb{K}$, we see that $|\Gamma_q| \leq d_q^n$, where d_q is the largest power of q dividing d . This immediately implies that $|\Gamma_q|$ divides d_q^n , and hence $|\Gamma|' = \prod_q |\Gamma_q|$ divides the number $\prod_q d_q$, which in turn divides d . \square

Now we state and prove a more precise version of Theorem 1.1.

Theorem 3.7. *Let r and n be positive integers. Let \mathbb{K} be a field that contains all roots of 1, and let G be an anisotropic linear algebraic group over \mathbb{K} such that the number of \mathbb{K} -points of the group of connected components of G is at most r and the rank of G is at most n . Denote by $N(G)$ the minimal dimension of a faithful representation of the maximal reductive quotient of the neutral component of $G_{\bar{\mathbb{K}}}$. Let Γ be a finite subgroup of $G(\mathbb{K})$. The following assertions hold.*

- (i) *If G is reductive and \mathbb{K} is perfect, then $|\Gamma|$ divides $r\Upsilon_M(n)^{N(G)}$.*
- (ii) *Suppose that G is an arbitrary linear algebraic group. If $\mathrm{char} \mathbb{K} > 0$, denote by $|\Gamma|'$ the largest factor of $|\Gamma|$ which is not divisible by $\mathrm{char} \mathbb{K}$; otherwise put $|\Gamma|' = |\Gamma|$. Then $|\Gamma|'$ divides $r\Upsilon_M(n)^{N(G)}$.*
- (iii) *Assume that G is connected, semi-simple, and $\mathrm{char} \mathbb{K} = p > 0$ is not a torsion prime for G . Write $|\pi_1(G)| = lp^m$, for some non-negative integers m and l such that l is not divisible by p . Then Γ is a semi-direct product $\Gamma = \Gamma_1 \rtimes \Gamma_2$ of its normal subgroup Γ_1 whose order is not divisible by p , and an abelian p -group Γ_2 of exponent less than or equal to p^m . Moreover, $|\Gamma_1|$ divides $r\Upsilon_M(n)^{N(G)}$.*

Proof. Clearly, we may assume that G is connected, so that in particular $r = 1$. Also, we assume that the rank of G equals n . Note that assertion (i) follows from assertion (ii) and Corollary 3.4.

Let $g \in G(\mathbb{K})$ be an element of finite order not divisible by $\mathrm{char} \mathbb{K}$. Then, by Corollary 3.2, the element g is contained in some subtorus of G . Thus, it follows from Corollary 2.4 that

$$g^{\Upsilon_M(n)} = 1.$$

Let $R_u(G_{\bar{\mathbb{K}}})$ be the unipotent radical of $G_{\bar{\mathbb{K}}}$. (Note that unless \mathbb{K} is perfect the group $R_u(G_{\bar{\mathbb{K}}})$ need not be defined over \mathbb{K} .) Then $G_{\bar{\mathbb{K}}}/R_u(G_{\bar{\mathbb{K}}})$ is a reductive group over $\bar{\mathbb{K}}$. By assumption, the group $(G_{\bar{\mathbb{K}}}/R_u(G_{\bar{\mathbb{K}}}))(\bar{\mathbb{K}})$ admits a faithful representation in an $N(G)$ -dimensional vector space over $\bar{\mathbb{K}}$:

$$(G_{\bar{\mathbb{K}}}/R_u(G_{\bar{\mathbb{K}}}))(\bar{\mathbb{K}}) \hookrightarrow \mathrm{GL}_{N(G)}(\bar{\mathbb{K}}).$$

Composing this embedding with the projection $G(\mathbb{K}) \rightarrow (G_{\bar{\mathbb{K}}}/R_u(G_{\bar{\mathbb{K}}}))(\bar{\mathbb{K}})$ we construct a homomorphism

$$\phi: G(\mathbb{K}) \rightarrow \mathrm{GL}_{N(G)}(\bar{\mathbb{K}}),$$

whose kernel is contained in $R_u(G_{\bar{\mathbb{K}}})(\bar{\mathbb{K}})$. By Lemma 3.5, every element of finite order in $R_u(G_{\bar{\mathbb{K}}})(\bar{\mathbb{K}})$ has order divisible by $\mathrm{char} \mathbb{K}$. This means that the image $\phi(\Gamma)$ of a finite subgroup $\Gamma \subset G(\mathbb{K})$ in $\mathrm{GL}_{N(G)}(\bar{\mathbb{K}})$ has order divisible by the largest

factor $|\Gamma'|$ of $|\Gamma|$ not divisible by $\text{char } \mathbb{K}$; in particular, if $\text{char } \mathbb{K} = 0$, then Γ projects isomorphically to $(G_{\overline{\mathbb{K}}}/R_u(G_{\overline{\mathbb{K}}}))(\overline{\mathbb{K}})$. Theorem 3.6 applied to $\phi(\Gamma)$ implies that $|\Gamma'|$ divides $\Upsilon_M(n)^{N(G)}$. This proves assertion (ii).

For the proof of assertion (iii), observe that since the group scheme $\pi_1(G)$ is commutative, we have that $\pi_1(G) \cong Z \times Z'$, where Z is a group scheme of order p^m and Z' is a group scheme whose order is not divisible by p . The central extensions

$$\begin{aligned} Z' &\rightarrow \tilde{G} \rightarrow \tilde{G}/Z', \\ Z &\rightarrow \tilde{G}/Z' \rightarrow G \end{aligned}$$

give rise to the exact sequences of groups

$$\begin{aligned} Z'(\mathbb{K}) &\longrightarrow \tilde{G}(\mathbb{K}) \longrightarrow (\tilde{G}/Z')(\mathbb{K}) \longrightarrow H_{f_l}^1(\text{spec } \mathbb{K}, Z'), \\ Z(\mathbb{K}) &\longrightarrow (\tilde{G}/Z')(\mathbb{K}) \longrightarrow G(\mathbb{K}) \xrightarrow{N} H_{f_l}^1(\text{spec } \mathbb{K}, Z), \end{aligned}$$

where the groups on the right stand for cohomology of Z' and Z regarded as sheaves for the fppf topology on $\text{spec } \mathbb{K}$, and $H_{f_l}^1$ denotes the first cohomology group for the fppf topology (see, for example, [Mil80, § III.4]). Set $\Gamma_1 = \Gamma \cap \ker N$. By Corollary 3.4 the group $\tilde{G}(\mathbb{K})$ has no elements of order p . Since the multiplication by p is invertible in Z' , the same is true for $H_{f_l}^1(\text{spec } \mathbb{K}, Z')$. Hence Γ_1 has no elements of order p . Thus, by assertion (ii) the order of Γ_1 divides $\Upsilon_M(n)^{N(G)}$. On the other hand, by construction Γ_1 is a normal subgroup of Γ , and Γ/Γ_1 is a subgroup of $H_{f_l}^1(\text{spec } \mathbb{K}, Z)$. The latter is an abelian group annihilated by p^m . Hence, the same is true for Γ/Γ_1 . Finally, since Γ_1 has no elements of order p , a p -Sylow subgroup of Γ projects isomorphically to Γ/Γ_1 . Thus, the group Γ is isomorphic to a semi-direct product of Γ_1 and Γ/Γ_1 . \square

Recall that for every positive integer n there exists a finite collection of reductive group schemes $\mathcal{G}_1, \dots, \mathcal{G}_{r(n)}$ over $\text{spec } \mathbb{Z}$, such that every connected reductive group of rank at most n over an algebraically closed field \mathbb{K} can be obtained from one of \mathcal{G}_i 's via the base change along the morphism $\text{spec } \mathbb{K} \rightarrow \text{spec } \mathbb{Z}$ (see [SGA3, Theorem XXV.1.1]). By [SGA3, Proposition VI_B.13.2], every such \mathcal{G}_i admits an embedding into a group GL_{N_i} over $\text{spec } \mathbb{Z}$ for some positive integer N_i . Thus, there exists a number $N(n)$ with the following property: for every algebraically closed field \mathbb{K} and every connected reductive group G of rank at most n over \mathbb{K} , the group $G(\mathbb{K})$ has a faithful representation of dimension at most $N(n)$.

Corollary 3.8. *In the notation of assertions (i), (ii), and (iii) of Theorem 3.7, the orders $|\Gamma|$, $|\Gamma'|$, and $|\Gamma_1|$, respectively, divide the number $r\Upsilon_M(n)^{N(n)}$.*

Proof. Let us use the notation of the proof of Theorem 3.7. The group $G_{\overline{\mathbb{K}}}/R_u(G_{\overline{\mathbb{K}}})$ has the same rank as $G_{\overline{\mathbb{K}}}$ (which is equal to the rank of G). Indeed, the rank of a unipotent algebraic group is zero. Hence, the rank of $G_{\overline{\mathbb{K}}}/R_u(G_{\overline{\mathbb{K}}})$ is greater than or equal to the rank of $G_{\overline{\mathbb{K}}}$. On the other hand, by [Bor91, Theorem 10.6(4)], every extension of a torus by a connected unipotent group over $\overline{\mathbb{K}}$ admits a section, which means that the rank of $G_{\overline{\mathbb{K}}}$ is greater than or equal to the rank of the quotient $G_{\overline{\mathbb{K}}}/R_u(G_{\overline{\mathbb{K}}})$. On the other hand, $G_{\overline{\mathbb{K}}}/R_u(G_{\overline{\mathbb{K}}})$ is a connected reductive group. Therefore, it admits a faithful representation in an $N(n)$ -dimensional vector space over $\overline{\mathbb{K}}$. \square

4. SEVERI–BRAUER VARIETIES

In this section we describe automorphism groups of Severi–Brauer varieties and prove Theorem 1.2, as well as some more special results for Severi–Brauer varieties over non-perfect fields. We refer the reader to [Art82] for the definition and basic facts concerning Severi–Brauer varieties.

Let A be a central simple algebra of dimension n^2 over an arbitrary field \mathbb{K} , and let \mathcal{A}^* be the corresponding inner form of the algebraic group GL_n . By definition, for every scheme S over \mathbb{K} , the group $\mathcal{A}^*(S)$ is the group of invertible elements in the algebra $A \otimes_{\mathbb{K}} \mathcal{O}(S)$. Also denote by $\mathcal{A}^*/\mathbb{G}_m$ the quotient of \mathcal{A}^* by its center. The latter is an inner form of PGL_n . Let X be the Severi–Brauer variety corresponding to A . Recall that X represents the functor that takes a scheme S over \mathbb{K} to the set of right ideals I in the sheaf of algebras $A \otimes_{\mathbb{K}} \mathcal{O}_S$ which are locally free of rank n as \mathcal{O}_S -modules and such that the quotient $(A \otimes_{\mathbb{K}} \mathcal{O}_S)/I$ is also locally free. The action of the group $\mathcal{A}^*(S)$ on $A \otimes_{\mathbb{K}} \mathcal{O}_S$ by conjugation induces a homomorphism

$$(4.1) \quad \mathcal{A}^*/\mathbb{G}_m \rightarrow \underline{\mathrm{Aut}}(X),$$

where the target is the group scheme of automorphisms of X .

The following fact is well known to experts (cf. Theorem E on page 266 of [Châ44], or [Art82, §1.6.1]), but for the reader’s convenience we provide a proof.

Lemma 4.1. *Homomorphism (4.1) is an isomorphism. Moreover, it induces an isomorphism $\mathrm{Aut}(X) \cong A^*/\mathbb{K}^*$.*

Proof. For the first assertion, it suffices to prove that (4.1) is an isomorphism after the base change to \mathbb{K}^{sep} . But $A \otimes_{\mathbb{K}} \mathbb{K}^{sep}$ is the matrix algebra and $X_{\mathbb{K}^{sep}} \cong \mathbb{P}_{\mathbb{K}^{sep}}^{n-1}$. Thus, the base change (4.1) boils down to the natural homomorphism $\mathrm{PGL}_n \rightarrow \underline{\mathrm{Aut}}(\mathbb{P}_{\mathbb{K}^{sep}}^{n-1})$ which is known to be an isomorphism.

For the second assertion, consider the exact sequence of groups with $\mathrm{Gal}(\mathbb{K}^{sep}/\mathbb{K})$ -action

$$1 \rightarrow (\mathbb{K}^{sep})^* \rightarrow \mathcal{A}^*(\mathbb{K}^{sep}) \rightarrow \underline{\mathrm{Aut}}(X)(\mathbb{K}^{sep}) \rightarrow 1$$

and the corresponding exact sequence of Galois cohomology groups

$$1 \rightarrow \mathbb{K}^* \rightarrow A^* \rightarrow \mathrm{Aut}(X) \rightarrow H^1(\mathrm{Gal}(\mathbb{K}^{sep}/\mathbb{K}), (\mathbb{K}^{sep})^*).$$

The latter cohomology group vanishes by Hilbert’s Theorem 90, and the assertion of the lemma follows. \square

Recall the reduced norm homomorphism:

$$\mathrm{Norm}: A^* \rightarrow \mathbb{K}^*.$$

One has $\mathrm{Norm}(cx) = c^n \mathrm{Norm}(x)$, for every $c \in \mathbb{K}^*$ and $x \in A^*$. Hence, Norm induces a homomorphism

$$(4.2) \quad A^*/\mathbb{K}^* \rightarrow \mathbb{K}^*/(\mathbb{K}^*)^n,$$

where $(\mathbb{K}^*)^n \subset \mathbb{K}^*$ is the subgroup of n -th powers.

Lemma 4.2. *Let n be a positive integer, let \mathbb{K} be a field that contains all roots of 1, and let A be a central division algebra of dimension n^2 over \mathbb{K} . Then, for every finite subgroup $\Gamma \subset A^*/\mathbb{K}^*$, the restriction of the homomorphism (4.2) to Γ is injective:*

$$\Gamma \hookrightarrow \mathbb{K}^*/(\mathbb{K}^*)^n.$$

In particular, Γ is abelian and, for every element $g \in A^*/\mathbb{K}^*$ of finite order, the order of g divides n .

Proof. Denote by A_1^* the kernel of the reduced norm homomorphism $\text{Norm}: A^* \rightarrow \mathbb{K}^*$. We have an exact sequence of groups

$$1 \rightarrow \mu_n(\mathbb{K}) \rightarrow A_1^* \rightarrow A^*/\mathbb{K}^* \rightarrow \mathbb{K}^*/(\mathbb{K}^*)^n,$$

where $\mu_n(\mathbb{K}) \subset \mathbb{K}^*$ is the subgroup of n -th roots of unity. In particular, every element $g \in A^*/\mathbb{K}^*$ of finite order whose image in $\mathbb{K}^*/(\mathbb{K}^*)^n$ is 1 lifts to an element $\tilde{g} \in A_1^* \subset A^*$, which also has finite order. Thus it suffices to prove that every element $x \in A^*$ of finite order belongs to $\mathbb{K}^* \subset A^*$. Assume that $x^d = 1$. Using that \mathbb{K} contains all roots of 1, we get

$$0 = x^d - 1 = \prod_{\epsilon \in \mu_d(\mathbb{K})} (x - \epsilon).$$

Since A has no zero divisors, we conclude that $x \in \mu_d(\mathbb{K})$ as desired. \square

We need an auxiliary result about bilinear forms on finite abelian groups.

Lemma 4.3. *Let Γ be a finite abelian group, and let $B: \Gamma \otimes_{\mathbb{Z}} \Gamma \rightarrow \mathbb{Q}/\mathbb{Z}$ be a homomorphism such that $B(g, g) = 0$ for every $g \in \Gamma$. Then there exists a subgroup $\Lambda \subset \Gamma$ such that the restriction of B to $\Lambda \otimes_{\mathbb{Z}} \Lambda$ is zero, and $|\Gamma|$ divides $|\Lambda|^2$.*

Proof. It is enough to prove the assertion in the case when Γ is an ℓ -group for some prime number ℓ . We will do this by induction on the order of Γ .

Choose an element g of maximal possible order ℓ^r in Γ , and let $\langle g \rangle \cong \mathbb{Z}/\ell^r\mathbb{Z}$ be the cyclic group generated by g . Set

$$\Gamma' = \{g' \in \Gamma \mid B(g, g') = 0\}.$$

Then Γ' is a subgroup of Γ , and there is an injective homomorphism

$$\Gamma/\Gamma' \hookrightarrow \text{Hom}(\langle g \rangle, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}/\ell^r\mathbb{Z}.$$

Thus

$$|\Gamma| \leq |\langle g \rangle| \cdot |\Gamma'| = \ell^r |\Gamma'|.$$

Note that Γ' contains the cyclic group $\langle g \rangle$. Since g has maximal possible order in Γ , we have $\Gamma' \cong \langle g \rangle \times \Gamma''$ for some subgroup $\Gamma'' \subset \Gamma'$. By induction, the group Γ'' contains a subgroup Λ'' such that the restriction of B to $\Lambda'' \otimes_{\mathbb{Z}} \Lambda''$ is zero, and $|\Gamma''| \leq |\Lambda''|^2$. Set $\Lambda = \langle g \rangle \times \Lambda''$. Then the restriction of B to $\Lambda \otimes_{\mathbb{Z}} \Lambda$ is zero, and

$$|\Lambda|^2 = \ell^{2r} |\Lambda''|^2 \geq \ell^{2r} |\Gamma''| = \ell^r |\Gamma'| \geq |\Gamma|.$$

Since all these numbers are powers of ℓ , the assertion of the lemma follows. \square

Corollary 4.4. *Let n be a positive integer, let \mathbb{K} be a field of characteristic $p \geq 0$ that contains all roots of 1, and let A be a central division algebra of dimension n^2 over \mathbb{K} . Let $\Gamma \subset A^*/\mathbb{K}^*$ be a finite subgroup. Then $\Gamma \cong \Gamma_1 \times \Gamma_2$, where Γ_2 is a p -group, while the order of Γ_1 is not divisible by p and divides n^2 .*

Proof. According to Lemma 4.2, the group Γ is abelian. Thus, if $p > 0$, then

$$\Gamma \cong \Gamma_1 \times \Gamma_2$$

for its p -Sylow subgroup Γ_2 and a subgroup Γ_1 of order not divisible by p . If $p = 0$, we set $\Gamma_1 = \Gamma$ and let Γ_2 be a trivial group. Denote by $\mu_{\infty}(\mathbb{K})$ the subgroup of roots of 1 in \mathbb{K} .

Let the (infinite) group $\tilde{\Gamma} \subset A^*$ be the preimage of Γ under the projection

$$A^* \rightarrow A^*/\mathbb{K}^*.$$

Consider the commutator pairing

$$B: \Gamma \otimes_{\mathbb{Z}} \Gamma \rightarrow \mu_{\infty}(\mathbb{K}), \quad (g, h) \mapsto \tilde{g}\tilde{h}\tilde{g}^{-1}\tilde{h}^{-1},$$

where \tilde{g} and \tilde{h} are arbitrary preimages of g and h in $\tilde{\Gamma}$. Since $\tilde{\Gamma}$ is a central extension of the abelian group Γ , the map B is a well-defined homomorphism. Choosing an embedding $\mu_{\infty}(\mathbb{K}) \hookrightarrow \mathbb{Q}/\mathbb{Z}$, we may consider B as a \mathbb{Z} -bilinear pairing with values in \mathbb{Q}/\mathbb{Z} .

Apply Lemma 4.3 to the restriction of the pairing B to $\Gamma_1 \otimes_{\mathbb{Z}} \Gamma_1$. We infer that there exists a subgroup $\Lambda \subset \Gamma_1$ such that B is trivial on $\Lambda \otimes_{\mathbb{Z}} \Lambda$ and $|\Gamma_1|$ divides $|\Lambda|^2$. In particular, the preimage $\tilde{\Lambda}$ of Λ in $\tilde{\Gamma}$ is abelian. Let \mathbb{L} be the subalgebra of A generated by $\tilde{\Lambda}$. Then \mathbb{L} is commutative, and thus it is a field. Furthermore, \mathbb{L} is contained in the composite of field extensions such that each of them is obtained by adjoining to \mathbb{K} some element $a \in A$ with $a^k \in \mathbb{K}$ for some positive integer k not divisible by p . Since \mathbb{K} contains all roots of 1, an extension of this form is a splitting field of the polynomial $x^k - a$; thus, it is a Galois extension of \mathbb{K} . Therefore, \mathbb{L} is a Galois extension of \mathbb{K} as well. On the other hand, the number $|\text{Gal}(\mathbb{L}/\mathbb{K})| = [\mathbb{L} : \mathbb{K}]$ divides n , see for instance [Bou58, §10.3].

Finally, recall from Example 2.6 that $|\Lambda|$ divides $|\text{Gal}(\mathbb{L}/\mathbb{K})|$. Therefore, $|\Gamma_1|$ divides n^2 . \square

The following is a more precise version of Theorem 1.2(ii).

Proposition 4.5. *Let \mathbb{K} be a field of characteristic $p \geq 0$ that contains all roots of 1. Let A be a central division algebra over \mathbb{K} of dimension n^2 , and let X be the corresponding Severi–Brauer variety. Write $n = n'p^m$ for some non-negative integers m and n' such that n' is not divisible by p . Then every finite subgroup $\Gamma \subset \text{Aut}(X)$ is a direct product $\Gamma = \Gamma_1 \times \Gamma_2$ of an abelian group Γ_1 whose exponent divides n' and whose order divides n'^2 , and an abelian p -group Γ_2 whose exponent divides p^m .*

Proof. By Lemma 4.1, one has $\Gamma \subset A^*/\mathbb{K}^*$. Therefore, the assertion about the product structure and about the order of Γ_1 follows from Corollary 4.4. The assertion about the exponents of Γ_1 and Γ_2 follows from Lemma 4.2. \square

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Assertion (ii) is a particular case of Proposition 4.5. Also, Proposition 4.5 tells us that if A is a central division algebra, then the group $\text{Aut}(X)$ has bounded finite subgroups.

Now suppose that A is not a division algebra. Then

$$A \cong D \otimes_{\mathbb{K}} \text{Mat}_m(\mathbb{K})$$

for some $2 \leq m \leq n$, where $\text{Mat}_m(\mathbb{K})$ denotes the algebra of $m \times m$ -matrices, and D is a central division algebra over \mathbb{K} , see for instance [GS06, Theorem 2.1.3]. Thus A contains a subalgebra isomorphic to $\text{Mat}_m(\mathbb{K})$. Since the field \mathbb{K} contains roots of 1 of arbitrarily large degree, we see from Lemma 4.1 that the group $\text{Aut}(X)$ contains elements of arbitrarily large finite order. This completes the proof of assertion (i). \square

Remark 4.6. Let A be a central simple algebra over a field \mathbb{K} . Denote by \mathcal{A}^* the algebraic group whose S -points are invertible elements in the algebra $A \otimes_{\mathbb{K}} \mathcal{O}_S$. We have a natural embedding $\mathbb{G}_m \hookrightarrow \mathcal{A}^*$ induced by the homomorphism $\mathcal{O}_S^* \hookrightarrow (A \otimes_{\mathbb{K}} \mathcal{O}_S)^*$. The quotient group scheme $\mathcal{A}^*/\mathbb{G}_m$ is anisotropic if and only if A is a division algebra (see, for instance, [Bor91, §23.1]). In particular, if \mathbb{K} is perfect, then Theorem 1.2(i) follows from Theorem 1.1 applied to the reductive group $\mathcal{A}^*/\mathbb{G}_m$.

Similarly, under the assumptions of Proposition 4.5, the fact that every finite subgroup of $\mathcal{A}^*/\mathbb{K}^*$ is a *semi*-direct product of its abelian p -Sylow subgroup and a normal subgroup of bounded order is a special case of Theorem 3.7(iii) applied to the reductive group $\mathcal{A}^*/\mathbb{G}_m$. Indeed, the algebraic fundamental group of $\mathcal{A}^*/\mathbb{G}_m$ is isomorphic to the group scheme $\mu_n = \ker(\mathbb{G}_m \xrightarrow{n} \mathbb{G}_m)$, which has order n . Also, since $\mathcal{A}^*/\mathbb{G}_m$ has type A_{n-1} , the set of torsion primes for $\mathcal{A}^*/\mathbb{G}_m$ is empty.

Remark 4.7. Let \mathbb{K} be a perfect field of positive characteristic p , and let A be a central division algebra of dimension n^2 over \mathbb{K} . Then p does not divide n . Indeed, the Frobenius morphism $\text{Fr}: \bar{\mathbb{K}}^* \rightarrow \bar{\mathbb{K}}^*$ is an isomorphism and, hence, the Brauer group

$$\text{Br}(\mathbb{K}) \cong H^2(\text{Gal}(\bar{\mathbb{K}}/\mathbb{K}), \bar{\mathbb{K}}^*)$$

of \mathbb{K} has no p -torsion elements and it is p -divisible. Therefore, our claim follows from the fact that, over any field, the dimension of a central division algebra and the order of its class in the Brauer group have the same prime factors (see for instance [Lie08, Lemma 2.1.1.3]).

The restriction on the characteristic of \mathbb{K} in Theorem 1.2 is essential for validity of the statement.

Example 4.8. Let F be a field of characteristic $p > 0$, and let $\mathbb{K} = F(x, y)$ be the field of rational functions in two variables, so that \mathbb{K} is a non-perfect field of characteristic p . Let A be an algebra over \mathbb{K} with generators u and v and relations

$$v^p = x, \quad u^p = y, \quad vu - uv = 1.$$

Then A is a central division algebra of dimension p^2 over \mathbb{K} . This is a special case of the Azumaya property of the ring of differential operators in characteristic p (see [BMR08, Theorem 2.2.3]), but can be also checked directly. Indeed, it suffices to check that $A_{\bar{\mathbb{K}}} = A \otimes_{\mathbb{K}} \bar{\mathbb{K}}$ is the algebra of $p \times p$ -matrices. To see this, take the elements

$$v' = v - x^{\frac{1}{p}}, \quad u' = u - y^{\frac{1}{p}}$$

in $A_{\bar{\mathbb{K}}}$ with $x^{\frac{1}{p}}, y^{\frac{1}{p}} \in \bar{\mathbb{K}}$. Then $v'^p = u'^p = 0$ and $v'u' - u'v' = 1$. Define an action of $A_{\bar{\mathbb{K}}}$ on the $\bar{\mathbb{K}}$ -vector space $V = \bar{\mathbb{K}}[z]/(z^p)$ letting u' act as the multiplication by z and v' as $\frac{d}{dz}$. It is easy to verify that V is an irreducible representation of $A_{\bar{\mathbb{K}}}$. Hence, by the Jacobson density theorem the homomorphism $A_{\bar{\mathbb{K}}} \rightarrow \text{End}_{\bar{\mathbb{K}}}(V)$ is surjective. Since the dimension of $A_{\bar{\mathbb{K}}}$ is at most p^2 for obvious reasons, the latter homomorphism is actually an isomorphism.

Now, since the intersection of $F(v)^*$ with \mathbb{K}^* is $F(x)^* = F(v^p)^*$, we see that the group $\mathcal{A}^*/\mathbb{K}^*$ contains $F(v)^*/F(v^p)^*$ as a subgroup. The latter group is p -torsion, because the p -th power of any rational function in v is a rational function in v^p . In other words, it can be regarded as a vector space over the field \mathbf{F}_p of p elements. At the same time, $F(v)$ is a vector space of dimension p over $F(v^p)$, so that $F(v)^*/F(v^p)^*$ can be thought of as a projective space of dimension $p - 1 > 0$ over an infinite field $F(v^p)$. Thus, the set $F(v)^*/F(v^p)^*$ is infinite, which implies

that $F(v)^*/F(v^p)^*$ is infinite-dimensional as a vector space over \mathbf{F}_p . In particular, for $p = 2$ this construction provides an example of a conic C over a non-perfect field of characteristic 2 such that C is acted on by elementary 2-groups of arbitrarily large order.

However, for central division algebras whose dimension is divisible by the characteristic of \mathbb{K} one can prove the following result.

Proposition 4.9. *Let A be a central division algebra of dimension n^2 over a field \mathbb{K} of finite characteristic p . If there exists an element $v \in A$ that is inseparable over \mathbb{K} (i.e., the field extension $\mathbb{K}(v) \supset \mathbb{K}$ is inseparable), then the group A^*/\mathbb{K}^* has unbounded finite subgroups. On the other hand, if \mathbb{K} contains all roots of 1, and every element $v \in A$ is separable over \mathbb{K} , then the group A^*/\mathbb{K}^* has bounded finite subgroups.*

Proof. If $v \in A$ is not separable, then there exists a subfield

$$\mathbb{K} \subset \mathbb{L} \subset \mathbb{K}(v)$$

which is a purely inseparable extension of \mathbb{K} of degree p . Then every non-trivial element of the group $\mathbb{L}^*/\mathbb{K}^*$ has order p . Since $|\mathbb{L}^*/\mathbb{K}^*| = \infty$, we conclude that the group $\mathbb{L}^*/\mathbb{K}^* \subset A^*/\mathbb{K}^*$ has unbounded finite subgroups.

Conversely, suppose that every element of A is separable over \mathbb{K} . Denote by n' the largest factor of n which is not divisible by $\text{char } \mathbb{K}$. Then for every element $v \in A^*$ whose image in A^*/\mathbb{K}^* has finite order, one has $v^{n'} \in \mathbb{K}^*$. Indeed, otherwise $v^{n'}$ would be inseparable over \mathbb{K} , because

$$\left(v^{n'}\right)^{\frac{n}{n'}} = v^n \in \mathbb{K}^*$$

by Lemma 4.2. Hence, the assertion follows from Proposition 4.5. \square

Remark 4.10. If $p = 2$ or $p = 3$, then every central division algebra A of dimension p^2 over a field \mathbb{K} of characteristic p contains an element $v \in A$ which is inseparable over \mathbb{K} . For $p = 3$ this result was proved in [Wed21]. For $p = 2$ the assertion is easy. Indeed, choose a subfield $\mathbb{L} \subset A$ of degree 2 over \mathbb{K} . If the extension $\mathbb{K} \subset \mathbb{L}$ is inseparable, then we are done. Suppose that it is separable. Then $\mathbb{K} \subset \mathbb{L}$ is a Galois extension. Let σ be the generator of its Galois group. By the Skolem–Noether theorem (see e.g. [Bou58, §10.1, Théorème 1]) the action of σ on \mathbb{L} extends to an inner automorphism of A given by an element $v \in A^*$, that is,

$$vuv^{-1} = \sigma(u)$$

for every $u \in \mathbb{L}$. Since the degree of \mathbb{L} over \mathbb{K} is 2, the automorphism σ^2 is trivial. Hence v^2 commutes with \mathbb{L} . On the other hand, the centralizer of \mathbb{L} in A is \mathbb{L} itself, and so we have $v^2 \in \mathbb{L}$. Observe that

$$\sigma(v^2) = vv^2v^{-1} = v^2.$$

Since \mathbb{L} is a Galois extension of \mathbb{K} , this means that the element v^2 is, in fact, contained in \mathbb{K} . Thus the field extension $\mathbb{K} \subset \mathbb{K}(v)$ is not separable as desired. In particular, using Proposition 4.9 we see that, for $p = 2$ or $p = 3$ and a central division algebra A of dimension p^2 over a field \mathbb{K} of characteristic p , the group A^*/\mathbb{K}^* has unbounded finite subgroups.

5. QUADRICS

In this section we study automorphism groups of quadrics and prove Theorem 1.3.

Let V be a finite-dimensional vector space over a field \mathbb{K} , and let q be a non-degenerate quadratic form on V . Recall that a quadratic form q is said to be non-degenerate if the associated symmetric bilinear form

$$B_q: V \times V \rightarrow \mathbb{K}, \quad B_q(v, w) = q(v + w) - q(v) - q(w),$$

is non-degenerate. If $\text{char } \mathbb{K} = 2$ the form B_q is also alternating. Hence, in this case the dimension of V must be even.

Denote by $O(V, q)$ the orthogonal (linear algebraic) group corresponding to q . The following result is well-known (see, for instance, [Bor91, §§22.4, 22.6]).

Lemma 5.1. *The group $O(V, q)$ is reductive. It is anisotropic if and only if q does not represent 0.*

We will also need the following structural result on quadratic forms over a perfect field of characteristic 2 due to Arf ([Arf41]).

Lemma 5.2. *Let V be a finite-dimensional vector space over a perfect field \mathbb{K} of characteristic 2, and let q be a non-degenerate quadratic form on V (so that in particular $\dim V = 2k$ is even). Then, for some coordinates x_1, \dots, x_{2k} on V , the quadratic form q is given by*

$$(5.1) \quad q_a(x_1, \dots, x_{2k}) = x_1^2 + x_1x_2 + ax_2^2 + x_3x_4 + \dots + x_{2k-1}x_{2k},$$

where a is an element of \mathbb{K} . Moreover, two quadratic forms $q_a(x_1, \dots, x_{2k})$ and

$$q_{a'}(x_1, \dots, x_{2k})$$

are equivalent if and only if a and a' have the same image in the cokernel of Artin-Schreier homomorphism

$$\mathbb{K} \rightarrow \mathbb{K}, \quad c \mapsto c^2 - c,$$

which is the Arf invariant of the quadratic form. In particular, if $\dim V > 2$ then every non-degenerate quadratic form on V represents 0.

Lemma 5.3 (cf. [GA13, Lemma 2.1]). *Let \mathbb{K} be a field of characteristic $p > 2$. Let V be a vector space over \mathbb{K} , and let q be a non-degenerate quadratic form on V . Assume that q does not represent 0. Then the group of \mathbb{K} -points of $O(V, q)$ has no elements of order p .*

Proof. Assuming the contrary, let $g \in O(V, q)(\mathbb{K})$ be an element of order p . Viewing g as a linear endomorphism of V , we have

$$g^p - 1 = (g - 1)^p = 0.$$

Applying the Jordan Normal Form theorem to $g - 1$, we can find linearly independent vectors $v_1, v_2 \in V$ such that $g(v_1) = v_1$ and $g(v_2) = v_1 + v_2$. Thus, we have

$$B_q(v_1, v_2) = B_q(g(v_1), g(v_2)) = B_q(v_1, v_1) + B_q(v_1, v_2).$$

Hence, we obtain $2q(v_1) = B_q(v_1, v_1) = 0$, that is, q represents 0. \square

Lemma 5.4 (cf. [BZ17, Corollary 4.4]). *Let \mathbb{K} be a field that contains all roots of 1. Assume that $\text{char } \mathbb{K} \neq 2$ or \mathbb{K} is perfect. Suppose that q does not represent 0. Then every finite subgroup of $O(V, q)(\mathbb{K})$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^m$, where $m \leq \dim V$.*

Proof. By Lemma 5.2, if \mathbb{K} is perfect and $\text{char } \mathbb{K} = 2$, then we must have $\dim V = 2$. In this case, by Lemma 5.1, the group $O(V, q)$ is anisotropic and, thus, isomorphic to a semi-direct product of an anisotropic torus T of rank 1 and the finite group $\mathbb{Z}/2\mathbb{Z}$. According to Example 2.5, the group $T(\mathbb{K})$ does not contain non-trivial finite subgroups, and hence every non-trivial finite subgroup of $O(V, q)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

Now assume that $\text{char } \mathbb{K} \neq 2$. Let $g \in O(V, q)(\mathbb{K})$ be an element of finite order. By Lemma 5.3 the order of g is not divisible by the characteristic of \mathbb{K} . Since \mathbb{K} contains all roots of unity, it follows that every such element $g \in O(V, q)(\mathbb{K})$ viewed as a linear endomorphism of V is diagonalizable in an appropriate basis for V . Moreover, since q does not represent 0, the diagonal entries of the matrix of g in this basis must be equal to ± 1 . Hence $g^2 = 1$. It follows that every finite subgroup $\Gamma \subset O(V, q)(\mathbb{K})$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^m$ for some non-negative integer m . In particular, Γ is abelian, and thus it is conjugate to a subgroup of the group of diagonal matrices in $GL(V)$. Hence, one has $m \leq \dim V$. \square

Now we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. Let V be an n -dimensional vector space such that \mathbb{P}^{n-1} is identified with the projectivization $\mathbb{P}(V)$, and let q be a quadratic form corresponding to the quadric Q .

First, assume that \mathbb{K} is a perfect field of characteristic 2. If n is even, then the quadratic form is non-degenerate; indeed, otherwise its kernel T would be at least two-dimensional, so that the singular locus of Q , which is $Q \cap \mathbb{P}(T)$, would be non-empty. Thus, by Lemma 5.2 one has $Q(\mathbb{K}) \neq \emptyset$. Moreover, writing q in the form (5.1), we see that $\text{Aut}(Q)$ contains a subgroup isomorphic to \mathbb{K}^* . Hence, $\text{Aut}(Q)$ has unbounded finite subgroups. If n is odd, the symmetric bilinear form $B_q: V \times V \rightarrow \mathbb{K}$ associated to q has a one-dimensional kernel. In this case q can be written as

$$q(x_1, \dots, x_n) = x_1^2 + r(x_2, \dots, x_n)$$

for some coordinates x_1, \dots, x_n on V and some non-degenerate quadratic form r in $n - 1$ variables. Applying Lemma 5.2 to r , we see that $Q(\mathbb{K}) \neq \emptyset$ and $\text{Aut}(Q)$ has unbounded finite subgroups. In fact, in this case $\text{Aut}(Q)$ is isomorphic to the group of linear transformations of the quotient \bar{V} of V by T which preserve the induced bilinear form \bar{B}_q on \bar{V} , i.e., to the symplectic group $\text{Sp}(\bar{V}, \bar{B}_q)(\mathbb{K})$; see [Bor91, §22.6]. We see that in the case when $\text{char } \mathbb{K} = 2$, assertion (i) holds, while the assumptions of assertions (ii), (iii), and (iv) do not hold.

From now on we assume that $\text{char } \mathbb{K} \neq 2$. The group $\text{Aut}(Q)$ is isomorphic to the group of \mathbb{K} -points of the group scheme quotient $G = O(V, q)/\mu_2$, where $\mu_2 \subset O(V, q)$ is the central subgroup of order 2. (More geometrically, $\text{Aut}(Q)$ can be identified with the group $\text{PO}(V, q)$ of automorphisms of the projective space $\mathbb{P}(V)$ that preserve q up to a scalar multiple.) By Lemma 5.1, the group G is anisotropic if and only if $Q(\mathbb{K}) = \emptyset$. The connected component of identity $G^\circ \subset G$ is a connected semi-simple algebraic group. Moreover, the algebraic fundamental group of G° has order 2 if n is odd and order 4 if n is even. Also, note that no prime other than 2 is a torsion prime for G° . Thus, assertion (i) of the proposition follows from Theorem 1.1 applied to G° . (It also follows that the group $\text{Aut}(Q)$ has no elements of order $\text{char } \mathbb{K}$; cf. Lemma 5.1.) The reader will see that the argument

we give below for the remaining assertions of the proposition also furnishes a direct proof of (i).

If n is odd, then the embedding $\mu_2 \hookrightarrow \mathrm{O}(V, q)$ splits, so that $\mathrm{Aut}(Q)$ is isomorphic to the subgroup $\mathrm{SO}(V, q)(\mathbb{K}) \subset \mathrm{O}(V, q)(\mathbb{K})$ of the orthogonal group that consists of matrices whose determinant is equal to 1. Thus, assertion (ii) follows from Lemma 5.4.

Suppose that n is even. We know from Lemma 5.4 that every non-trivial element of $\mathrm{O}(V, q)(\mathbb{K})$ of finite order has order 2. Hence, using the exact sequence

$$0 \rightarrow \mu_2 \rightarrow \mathrm{O}(V, q)(\mathbb{K}) \rightarrow \mathrm{Aut}(Q) \rightarrow \mathbb{K}^*/(\mathbb{K}^*)^2,$$

we see that every non-trivial element of $\mathrm{Aut}(Q)$ of finite order has order 2 or 4. Let $\Gamma \subset \mathrm{Aut}(Q)$ be a finite subgroup. Consider the embedding

$$\mathrm{Aut}(Q) \cong G(\mathbb{K}) \hookrightarrow G(\bar{\mathbb{K}}) \cong \mathrm{O}(V, q)(\bar{\mathbb{K}})/\{\pm 1\},$$

and let $\tilde{\Gamma}$ be the preimage of Γ in $\mathrm{O}(V, q)(\bar{\mathbb{K}})$. The order of every element of $\tilde{\Gamma}$ divides 8, and the same is true for the subgroup $\hat{\Gamma} \subset \mathrm{GL}(V)(\bar{\mathbb{K}})$ generated by $\tilde{\Gamma}$ and scalar matrices whose orders divide 8. Thus, we conclude from Theorem 3.6 that $|\hat{\Gamma}|$ divides 8^n . On the other hand, we have $|\hat{\Gamma}| = 8|\Gamma|$. This proves assertion (iii). \square

The next example shows that there exist even-dimensional quadrics satisfying the assumptions of Theorem 1.3 and having a faithful action of non-abelian finite groups.

Example 5.5. Choose an integer $k \geq 3$. Let \mathbb{k} be an algebraically closed field of characteristic zero, and let a_1, \dots, a_k be independent transcendental variables. Set $\mathbb{K} = \mathbb{k}(a_1, \dots, a_k)$. Consider the Pfister quadratic form

$$q_k = \sum_I a_I x_I^2$$

in the variables x_I , $I \subset \{1, \dots, k\}$, where $a_I = \prod_{i \in I} a_i$. Let Q be the quadric defined by the equation $q_k = 0$ in the projective space \mathbb{P}^{2^k-1} with homogeneous coordinates x_I .

We claim that $Q(\mathbb{K}) = \emptyset$. Indeed, if P is a \mathbb{K} -point on Q , its homogeneous coordinates can be written as polynomials p_I in a_1, \dots, a_k , such that at least one of them is a non-zero polynomial. Let M denote the maximal degree of the polynomials p_I in the variable a_k . Write

$$p_I = c_{I,M} a_k^M + c_{I,M-1} a_k^{M-1} + \dots + c_{I,0},$$

where $c_{I,j}$ are polynomials in a_1, \dots, a_{k-1} ; at least one of $c_{I,M}$ is a non-zero polynomial. The fact that the quadratic form q_k vanishes at P implies that

$$a_k^{2M} \sum_{k \notin I} a_I c_{I,M}^2 = 0, \quad a_k^{2M+1} \sum_{k \notin I} a_I c_{I \cup \{k\}, M}^2 = a_k^{2M} \sum_{k \in I} a_I c_{I,M}^2 = 0.$$

Thus, at least one of the 2^{k-1} -tuples

$$(c_{I,M}), \quad k \notin I, \quad (c_{I \cup \{k\}, M}), \quad k \in I,$$

provides a point of the projective space $\mathbb{P}^{2^{k-1}-1}$ over the field $\mathbb{k}(a_1, \dots, a_{k-1})$ where the quadratic form

$$q_{k-1} = \sum_I a_I x_I^2, \quad I \subset \{1, \dots, k-1\},$$

vanishes. Proceeding by induction, we arrive to the conclusion that the quadratic form

$$q_1(x_0, x_1) = x_0^2 + a_{\{1\}}x_{\{1\}}^2$$

represents zero over the field $\mathbb{k}(a_1)$, which is absurd.

Now consider the elements $\sigma, \tau \in \mathrm{PGL}_{2^k}(\mathbb{K})$ acting as

$$\sigma(x_{\{1\}}) = -x_{\{1\}}, \quad \sigma(x_{\{2\}}) = -x_{\{2\}}, \quad \sigma(x_I) = x_I \text{ for } I \neq \{1\}, \{2\},$$

and

$$\tau(x_I) = a_{\bar{I}}x_{\bar{I}},$$

where $\bar{I} = \{1, \dots, k\} \setminus I$. Then both σ and τ preserve the quadric Q , and $\sigma^2 = \tau^2 = 1$. The commutator $\iota = \sigma \circ \tau \circ \sigma \circ \tau$ acts as

$$\begin{aligned} \iota(x_I) &= -x_I \text{ for } I = \{1\}, \{2\}, \overline{\{1\}}, \overline{\{2\}}, \\ \iota(x_I) &= x_I \text{ for } I \neq \{1\}, \{2\}, \overline{\{1\}}, \overline{\{2\}}. \end{aligned}$$

In particular, ι is a non-trivial element of $\mathrm{PGL}_{2^k}(\mathbb{K})$, i.e. σ and τ do not commute with each other. On the other hand, it is clear that σ and τ generate a finite subgroup in $\mathrm{Aut}(Q)$.

Note that the algebraic group $\mathrm{Aut}(Q)$ has two connected components (both over \mathbb{K} and over $\bar{\mathbb{K}}$), corresponding to the connected components of the orthogonal group $\mathrm{O}(q_k)$. The element $\sigma \in \mathrm{Aut}(Q)$ clearly lifts to an element in the neutral component $\mathrm{SO}(q_k)$ of $\mathrm{O}(q_k)$, while τ lifts to an element of $\mathrm{SO}(q_k)_{\bar{\mathbb{K}}}$. Thus, both σ and τ are contained in the neutral component of $\mathrm{Aut}(Q)$. This gives an example of a finite non-abelian subgroup of a connected anisotropic reductive group over a field containing all roots of unity, cf. Lemma 4.2.

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STEKLOV MATHEMATICAL INSTITUTE OF RAS, 8 GUBKINA STREET, MOSCOW 119991, RUSSIA;
and NATIONAL RESEARCH UNIVERSITY HIGHER SCHOOL OF ECONOMICS, LABORATORY OF ALGEBRAIC GEOMETRY, NRU HSE, 6 USACHEVA STR., MOSCOW 117312, RUSSIA

Email address: costya.shramov@gmail.com

NATIONAL RESEARCH UNIVERSITY HIGHER SCHOOL OF ECONOMICS, LABORATORY OF MIRROR SYMMETRY, NRU HSE, 6 USACHEVA STR., MOSCOW 117312, RUSSIA

Email address: vologod@gmail.com