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# A SEPARATION BASED OPTIMIZATION APPROACH TO DYNAMIC MAXIMAL COVERING LOCATION PROBLEMS WITH SWITCHED STRUCTURE

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Abstract. This paper extends a newly developed computational optimization approach to a specific class of Maximal Covering Location Problems (MCLPs) with a switched dynamic structure. Most of the results obtained for the conventional MCLP address the "static" case where an optimal decision is determined on a fixed time-period. In our contribution we consider a dynamic MCLP based optimal decision making and propose an effective computational method for the numerical treatment of the switched-type Dynamic Maximal Covering Location Problem (DMCLP). A generic geometrical structure of the constraints under consideration makes it possible to separate the originally given dynamic optimization problem and reduce it to a specific family of relative simple auxiliary problems. The generalized Separation Method (SM) for the DMCLP with a switched structure finally leads to a computational solution scheme. The resulting numerical algorithm also includes the classic Lagrange relaxation. We present a rigorous formal analysis of the DMCLP optimization methodology and also discuss computational aspects. The proposed SM based algorithm is finally applied to a practically oriented example, namely, to an optimal design of a (dynamic) mobile network configuration.

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1. **Introduction.** Applications of diverse methods from the modern Mathematical Optimization Theory and the corresponding numerical techniques are nowadays a usual and efficient approach to the development of engineering applications (see e.g., [7, 4, 3, 5, 2, 10, 11, 23, 27, 33, 34]). Optimal facility location methodology, amongst others, plays an important role in a success of Supply Chains and provides an important analytic tool for many real-world manufacturing and service problems [6, 8, 13, 15, 17, 20, 22, 25, 31, 36]. Let us recall that the conventional Maximal Covering Location Problem (MCLP) gives an optimal solution to cover a set of demands such that an objective is to be maximized. The basic MCLP was introduced by Church and ReVelle [13] and thereafter numerous practically important applications, theoretical and computational extensions to the classical MCLP have been developed. Let us mention here some known applications of the conventional MCLP for optimal location of industrial plants, landfills, hubs, cross-docks, networks, etc (see e.g., [8, 9, 12, 15, 16, 17, 19, 20, 22, 25, 30, 31, 36]). We also refer to [6] and references therein.

In our contribution, we propose a switched-type dynamic extension of the MCLP model with an incomplete information. We study a specific MCLP-type optimization problem with dynamic constraints. These constraints have a switched structure (depend on the switching time intervals) for some given switching times. The incomplete information of the Dynamic Maximal Covering Location Problem (DMCLP) under consideration is modelled by including the fuzzy-type eligibility matrices into the problem formulation. These two conceptual modifications of the generic MCLP involve more usability of the resulting DMCLP at the modelling stage and make it possible to incorporate the "resilience" or (and) "fuzzy" properties into the modelling approach. We give a self-closed and mathematically rigorous introduction to the new class of MCLP-type optimization problem, namely, to the DMCLPs and also develop a relative simple and implementable computational approach. In fact, the proposed methodology generalise the newly elaborated approach to the classic MCLP (see e.g. [6]).

Recall that a variety of computer oriented approaches have been proposed for an effective computational treatment of distinct classes of "static" MCLPs. Recently several heuristical methods are actively used in the practical numerical treatment of the MCLPs. We refer to [8, 16, 28, 30] for some effective heuristic and metaheuristic algorithms and for further references. Note that heuristics and metaheuristics have usually been employed in order to solve large size MCLPs (see e.g., [17, 19, 28, 30, 31]). However, the exact and various heuristic methods for the conventional MCLP are not sufficiently extended to a class of dynamic MCLP-type problems with the switched structure. The solution procedure we develop, namely, the generalized version of the Separation Method (SM) is based on an exact optimization procedure and constitutes an exact (non-heuristic) computational optimization method. Moreover, the analytic method we propose can also be easily incorporated (as an independent exact solution tool) into some existing optimization procedures.

The optimization approach we follow in our paper includes an equivalent transformation (called "separation") of the originally given DMCLP and further consideration of two auxiliary dynamic Knapsack-type problems (see e.g., [23] and references therein). The proposed dynamic version of the SM reduces the complexity of the original optimization problem in the presence of dynamic constraints. In this paper, we additionally use the celebrated Lagrange relaxation scheme for the purpose of a concrete computation [6, 18]. Let us also refer to [7, 2, 24, 29] for some advanced relaxation schemes of dynamic optimization.

It is necessary to stress that due to the extreme complexity of the general switched-type dynamic constraints and in particular to the complexity of the resulting dynamics the "generalization" of SM mentioned above constitutes a challenging theoretic and computational problem and cannot be considered as a simple "theory / facts transfer" from the conventional MCLP theory. Theoretic and numerical results obtained in this paper are next applied to a practically motivated example from the area of telecommunication engineering. The optimization problem we study in this example constitutes a (simplified) special case of the general Restricted Covering Problem (RCP) from the theory of mobile communication networks. Due to the dynamic nature of the communicative processes we try to maximize an average covering for a given system of the radio base stations. The requested optimal covering "design" and the resulting optimal management (decision making) can be formalized here as a specific switched DMCLP. In that case the natural information incompleteness of the used model can be adequately described by an eligibility matrix with a fuzzy structure.

The remainder of our paper is organized as follows: Section 2 contains a mathematically rigorous DMCLP problem formulation and some necessary theoretical concepts and facts. In Section 3 we extend the existing (static) SM to the switched-type dynamic MCLP. This extension constitutes a conceptually new solution approach and includes a necessary formal characterization of the resulting optimization problems obtained during the proposed separation procedure. Section 4 contains the concrete numerical schemes for the auxiliary optimization problems. Using the equivalence between the initially given and auxiliary problems (established in the previous sections), we finally develop a new implementable algorithm for a consistent numerical treatment of the initial DMCLP. The algorithm we consider also incorporates the Lagrange relaxation technique. Section 5 contains a concrete application of the proposed theoretic and numerical extensions of the classic MCLP techniques, namely, the dynamic version of the SM to a problem from the telecommunication engineering. We study a problem of the optimal covering of a cellular (mobile) communication network. This engineering examples shows the practical usability of the new solution approach we propose. It also illustrate the implementability and effectiveness of the resulting computational algorithm. Section 6 summarizes our contribution.

## 2. **Problem formulation and preliminaries.**

We study a specific case of the integer programming problem with some dynamic variables and parameters

$$
\begin{aligned}
&\text{maximize} \quad J(z(\cdot)) := \frac{1}{(t_f - t_0)} \int_{t_0}^{t_f} \sum_{j=1}^{n} w_j(t) z_j(t)\,dt \\
&\text{subject to} \\
&\begin{cases}
\sum_{i=1}^{l} y_i(t) = k^s, \;\; t \in (t_{s-1}, t_s], \\
z_j(t) \leq \sum_{i=1}^{l} a_{ij}^s y_i(t), \;\; t \in (t_{s-1}, t_s], \\
z(t) \in \mathbb{B}^n, \;\; y(t) \in \mathbb{B}^l, \;\; t \in [t_0, t_f], \\
s = 1, ..., S \in \mathbb{N}, \\
j = 1, ..., n \in \mathbb{N}, \;\; i = 1, ..., l \in \mathbb{N}.
\end{cases}
\end{aligned}
\tag{1}
$$

Here $w_j(t) \in \mathbb{R}_+,\ j = 1, ..., n$ are the given dynamic weights of a demand node $j$ for $t \in [t_0, t_f]$ and $z_j(t)$ is a binary "state" variable which is equal to 1 if a $j$-demand node is covered by at least one facility at the time $t$, otherwise $z_j(t) = 0$. By $y_i(t)$ we denote a binary "decision" variable which is equal to 1 if a $i$-facility is opened at the time $t$, otherwise $y_i(t) = 0$. We next assume that the complete operational time interval $[t_0, t_f]$ for the given model is divided into $S$ adjoint intervals

$$(t_{s-1}, t_s],\ s = 1, ..., S$$

by switching times:

$$t_0 < t_1 ... t_{S-1} < t_S = t_f.$$

Additionally $k^s \in \mathbb{N}$ for $s = 1, ..., S$ in (1) describes a total number of facilities to be located on every time-interval $(t_{s-1}, t_s]$. Evidently, $k^s \leq l$ for every $s = 1, ..., S$. Coefficients

$$a^s_{ij},\ s = 1, ..., S,$$

where

$$1 \geq a^s_{ij} \geq 0,\ \sum_{i=1}^{l} a^s_{ij} \geq 1,$$

are constant on every interval $(t_{s-1}, t_s]$ and constitute components of the "eligibility matrix" associated with the optimization problem (1)

$$A^s := (a^s_{ij})^{i=1,...,l}_{j=1,...,n}.$$

This matrix describes a "resilient" (or fuzzy) covering of the demand nodes indexed by $j = 1, ..., n$ for every specific time-interval

$$(t_{s-1}, t_s],\ s = 1, ..., S.$$

Let us note that the index $i = 1, ..., l$ in (1) is related to the given facilities. The admissible sets $\mathbb{B}^n$ (sometimes called "state space") and $\mathbb{B}^l$ ("decision space") in (1) are defined as follows:

$$\mathbb{B}^n := \{0, 1\}^n,\ \ \mathbb{B}^l := \{0, 1\}^l.$$

We use here the natural notation

$$z := (z_1, ..., z_n)^T,\ \ y := (y_1, ..., y_l)^T.$$

Motivated from practical applications we next assume that all dynamic components in the basic problem (1) have a structure of piecewise-constant functions determined on the full time interval $[t_0, t_f]$:

$$\begin{aligned} &w_j(t) = c^s_j > 0\ \ \forall t \in (t_{s-1}, t_s], \\ &z_j(t),\ y_i(t) = 0 \text{ or } 1\ \ \forall t \in (t_{s-1}, t_s],\ s = 1, ..., S \end{aligned}$$

where $j = 1, ..., n$ and $i = 1, ..., l$. In this specific case we evidently have

$$J(z(\cdot)) := \sum_{s=1}^{S} \frac{(t_s - t_{s-1})}{(t_f - t_0)} \sum_{j=1}^{n} (w_j(t) z_j(t))_{|t \in (t_{s-1}, t_s]} =$$

$$\sum_{s=1}^{S} \frac{(t_s - t_{s-1})}{(t_f - t_0)} \langle w(t), z(t) \rangle_{|t \in (t_{s-1}, t_s]}.$$

Here $w := (w_1, ..., w_n)^T$. By $\langle \cdot, \cdot \rangle$ we denote here the scalar product in the corresponding Euclidean space. As we can see the objective functional $J(\cdot)$ in (1) has a linear structure and can be interpreted as optimal average costs over the complete operating time-interval. Note that the implicit dependence of $z(\cdot)$ on the decision function $y(\cdot)$ is given by the inequalities constraints

$$z(t) \leq (A^s)^T y(t),$$
$$t \in (t_{s-1}, t_s], \ s = 1, ..., S$$

in (1). A pair $(z(\cdot), y(\cdot))$ of piecewise-constant functions that satisfies all the constraints in (1) is next called an admissible pair of this problem.

The basic optimization framework (1) provides a useful modelling approach to the variety of real-world applications (see e.g., [8, 9, 12, 15, 16, 17, 19, 20, 22, 25, 30, 31, 36]). Following [17] we next call the main optimization problem (1) a Dynamic Maximal Covering Location Problem (DMCLP). Evidently, the given DMCLP has a specific switched dynamic structure. Let us also refer to [6, 16] for a detailed discussion on the conventional (static) MCLPs and some possible generalizations. The main DMCLP (1) is formulated here using the general (non-binary) values of the elements $a^s_{ij}$ of matrix $A^s$. This fact is motivated by a possible incomplete "eligibility" information in the practical optimal design (see e.g., [8, 9] and references therein). The resulting abstract framework can also be interpreted as a "resilience" modelling approach. In that case an admissible value of a parameter $a^s_{ij}$ has a fuzzy nature (see e.g., [6, 30]). The fuzzy DMCLP (1) provides an adequate formal model for many applied engineering problems, for example, for the optimal mobile networking design and for the optimization of Resilient Supply Chain Management Systems (RSCMSs). We refer to [6, 31] for the necessary technical details and some interesting example.

The DMCLP under consideration has a structure of an integer programming problem (see e.g., [10, 13, 27] for mathematical details). Let us note that (1) possesses an optimal solution (an optimal pair) $(z^{opt}(\cdot), y^{opt}(\cdot))$

$$(z^{opt}(t), y^{opt}(t)) \in \mathbb{B}^n \bigotimes \mathbb{B}^l,$$

where

$$z^{opt} := (z^{opt}_1, ..., z^{opt}_n)^T, \ y^{opt} := (y^{opt}_1, ..., y^{opt}_l)^T.$$

The main optimization problem (1) has an evident switched structure related to the given time-intervals $(t_{s-1}, t_s], \ s = 1, ..., S$. For the objective functional $J(\cdot)$ in (1) we obtain the natural decomposition

$$J(z(\cdot)) = \sum_{s=1}^{S} J_s(z(\cdot)), \tag{2}$$

where

$$J_s(z(\cdot), y(\cdot)) := \frac{(t_s - t_{s-1})}{(t_f - t_0)} \langle w(t), z(t) \rangle_{|t \in (t_{s-1}, t_s]}.$$

The following theorem uses the additivity property (2) of the objective functional and is in fact a compilation of the celebrated Bellman Optimality Principle from Optimal Control (see e.g., [2, 27])).

**Theorem 2.1.** *The restriction* $(z^{opt}(\cdot), y^{opt}(\cdot))_s$ *of an optimal solution* $(z^{opt}(\cdot), y^{opt}(\cdot))$ *on the time interval* $(t_{s-1}, t_s], \ s = 1, ..., S$ *is an optimal solution to the following particular* MCLP

$$\begin{aligned} &\text{maximize} \;\; J_s(z(\cdot)) \\ &\text{subject to} \\ &\begin{cases} \sum_{i=1}^{l} y_i(t) = k^s, \;\; t \in (t_{s-1}, t_s], \\ z_j(t) \le \sum_{i=1}^{l} a_{ij}^s y_i(t), \;\; t \in (t_{s-1}, t_s], \\ z(t) \in \mathbb{B}^n, \;\; y(t) \in \mathbb{B}^l, \;\; t \in [t_{s-1}, t_s], \\ s \in \mathbb{N}, \;\; s \le S, \\ j = 1, ..., n \in \mathbb{N}, \;\; i = 1, ..., l \in \mathbb{N}. \end{cases} \end{aligned} \tag{3}$$

The presented results means that the total optimal solution $(z^{opt}(\cdot), y^{opt}(\cdot))$ can be found sequentially. This optimal solution constitutes a formal union of optimal solutions for the particular MCLPs of the type (3) determined on intervals

$$(t_{s-1}, t_s], \;\; s = 1, ..., S.$$

This fact is an immediate consequence of the "independence" of the particular problems of the type (3): an optimal solution of problem (3) for $s+1$ does not depend on the previous solution to MCLP (3) with the index $s$.

We refer to [2, 35] for some general theoretic and computational results related to switched dynamic optimization problems. The aim of our contribution is to propose an effective approach for the numerical treatment of the sophisticated DMCLP (1). We generalize here the newly elaborated ("static") separation method for this purpose (see [6]) and next combine it with the Lagrange relaxation scheme (see e.g., [23]).

3. **Theoretical foundations of the general separation method.** Let us now introduce a sequence $Pr^S$ of the following auxiliary problems (indicated by $s = 1, ..., S$):

$$\begin{aligned} &\text{maximize} \;\; \sum_{j=1}^{n} \mu_j^s \sum_{i=1}^{l} a_{ij}^s y_i(t), \;\; t \in (t_{s-1}, t_s], \\ &\text{subject to} \\ &\left\{ \begin{array}{l} \sum_{i=1}^{l} y_i(t) = k^s, \;\; t \in (t_{s-1}, t_s], \\ y(t) \in \mathbb{B}^l, \;\; t \in (t_{s-1}, t_s], \\ \mu_j^s \in [0,1], \;\; j = 1, ..., n, \end{array} \right. \end{aligned} \tag{4}$$

Assume $\hat{y}^s(\cdot)$, where $\hat{y}^s(t) \in \mathbb{B}^l$, $s = 1, ..., S$ are optimal solutions to the given problems (4). The components of $\hat{y}^s(t)$ are denoted by $\hat{y}_i^s(t)$, $i = 1, ..., l$. In parallel to (4) consider the single auxiliary problem

$$\begin{aligned} &\text{maximize} \;\; J(z(\cdot)) \\ &\text{subject to} \\ &\begin{cases} z_j(t) \le \sum_{i=1}^{l} a_{ij}^s \hat{y}_i^s(t), \;\; t \in (t_{s-1}, t_s], \\ z(t) \in \mathbb{B}^n, \;\; t \in [t_0, t_f] \\ s = 1, ..., S, \;\; j = 1, ..., n. \end{cases} \end{aligned} \tag{5}$$

Note that the existence of optimal solutions to the auxiliary problems (4) and (5) is a direct consequence of the general results from [10, 23, 27]. Let us also underline here that every problem from $Pr^S$ can be solved independently from problem (5). Therefore, we next define a total optimal vector $\hat{y}(t) \in \mathbb{B}^l$ for $Pr^S$ determined by the natural composition

$$\hat{y}(t) := \hat{y}^s(t) \ \forall t \in [t_0, t_f], \ s = 1, ..., S. \tag{6}$$

By $\hat{z}(\cdot)$, where $\hat{z}(t) \in \mathbb{B}^n$ and $\hat{z}(\cdot) := (\hat{z}_1(\cdot), ..., \hat{z}_n(\cdot))^T$, we next denote an optimal solution to the auxiliary problem (5). It is easy to see that problem (5) coincides with the originally given DMCLP (1) for the specific case

$$y(\cdot) = \hat{y}(\cdot).$$

In the general case we evidently have $\hat{y}(\cdot) \neq y^{opt}(\cdot)$ and hence

$$(z^{opt}(\cdot), y^{opt}(\cdot)) \neq (\hat{z}(\cdot), \hat{y}(\cdot)).$$

We next call the pair $(\hat{z}(\cdot), \hat{y}(\cdot))$ an optimal solution of the family of auxiliary problems (4)-(5).

Every problem (for a fixed index $s$) from the family $Pr^S$ of problems (4) was obtained by a standard linear scalarization of the following multiobjective optimization problem (sometimes also called "vector optimization"):

$$\begin{aligned} &\text{maximize } \ \{\sum_{i=1}^{l} a^s_{i1} y_i(t), ..., \sum_{i=1}^{l} a^s_{in} y_i(t)\} \\ &\text{subject to} \\ &\begin{cases} \sum_{i=1}^{l} y_i(t) = k^s, \\ y(t) \in \mathbb{B}^l, \ t \in (t_{s-1}, t_s], \end{cases} \end{aligned} \tag{7}$$

where $s = 1, ..., S$. Let us recall that a suitable scalarizing of a multi-objective optimization problem is an adequate theoretic and computational approach to an initially given vector optimization problem. We refer to [14, 21] for the corresponding technical details and algorithms. From the general results of the vector optimization it follows that there exist vectors $\mu^s, \ s = 1, ..., S$ of multipliers $\mu^s_j, \ j = 1, ..., n$ in (4) such that an optimal solution to the scalarized optimization problem (4) is a Pareto optimal solution for the originally given multi-objective optimization problem. In this case problems (4) and (2.2a) are called equivalent (see e.g., [10] for the necessary mathematical foundations).

We now assume that the multipliers $\mu^s_j, \ j = 1, ..., n$ in (4) are chosen such that problems (4) and (7) are equivalent for every $s = 1, ..., S$. In this case we call (4) an adequate scalarizing of the auxiliary problem (7). We now observe that every problem (4) from the family $Pr^S$ and problem (5) have a structure of the celebrated Knapsack problem (see [23] and references therein). Let us recall that many powerful computational algorithms are recently proposed for an effective numerical treatment of the generic Knapsack problem. We refer to [16] for a comprehensive overview of the theory and modern numerical schemes for this celebrated optimization problem.

The importance of the family $Pr^S$ and of the auxiliary optimization problem (5) introduced above can be recognized from the following theoretic result.

**Theorem 3.1.** *Assume* (4) *is an adequate scalarizing of problem* (7). *Then an optimal solution* $(\hat{z}(\cdot), \hat{y}(\cdot))$ *of the family of auxiliary problems* $\{Pr^S, (5)\}$ *is also an optimal solution to the originally given* DMCLP (1) *and*

$$J(z^{opt}(\cdot)) = J(\hat{z}). \tag{8}$$

*If additionally all problems* (1), (4) *and* (5) *have unique optimal solutions, then*

$$(z^{opt}(\cdot), y^{opt}(\cdot)) = (\hat{z}(\cdot), \hat{y}(\cdot)). \tag{9}$$

*Proof.* Since

$$\hat{y}^s(\cdot), \; s = 1, ..., S$$

and $\hat{z}(\cdot)$ are admissible solutions for (4) and (5), we have

$$\sum_{i=1}^{l} \hat{y}_i^s(t) = k^s,$$

$$\hat{z}_j(t) \leq \sum_{i=1}^{l} a_{ij}^s \hat{y}_i(t)$$

for every $t \in (t_{s-1}, t_s]$ and $s = 1, ..., S$. Therefore, $(\hat{z}, \hat{y})$ is also an admissible pair for the originally given DMCLP (1). By the property of an optimal pair

$$(z^{opt}(\cdot), y^{opt}(\cdot))$$

we next obtain

$$J(\hat{z}(\cdot)) \leq J(z^{opt}(\cdot)). \tag{10}$$

Let us introduce the set of all optimal solutions (solution set) associated with the original problem (1)

$$\mathcal{F} := \mathcal{F}_z \bigotimes \mathcal{F}_y$$

Evidently,

$$z^{opt}(\cdot) \in \mathcal{F}_z,$$

$$y^{opt}(\cdot) \in \mathcal{F}_y.$$

The particular solution sets of problems (4) and (5) denote by

$$\mathcal{F}^s_{(2.2)}, \; s = 1, ...S$$

and $\mathcal{F}_{(2.3)}$, respectively. From (10) it follows that

$$\mathcal{F}_{(2.3)} \bigotimes \{ \bigcup_{s=1,\dots,S} \mathcal{F}^s_{(2.2)} \} \subset \mathcal{F}. \tag{11}$$

Observe that the constraints for the decision variable $y(\cdot)$ in (1) and in all problems (4) from $Pr^S$ are the same. This fact implies

$$\mathcal{F}_y \equiv \{ \bigcup_{s=1,\dots,S} \mathcal{F}^s_{(2.2)} \}. \tag{12}$$

Since (4) is an adequate scalarization of the multi-objective maximization problem (7) and taking into consideration the inequality constraints for the state variable $z(\cdot)$ in (1), we next deduce

$$z_j^{opt}(t) \leq \max_{\begin{cases} \sum_{i=1}^{l} y_i(t) = k^s, \\ y(t) \in \mathbb{B}^l \end{cases}} \sum_{i=1}^{l} a_{ij}^s y_i(t) \equiv \sum_{i=1}^{l} a_{ij}^s \hat{y}_i^s(t). \tag{13}$$

for every

$$t \in (t_{s-1}, t_s], \; s = 1, ..., S.$$

Here

$$z_j^{opt}(\cdot), \; j = 1, ..., n$$

is the $j$-component of $z^{opt}(\cdot)$. The obtained condition (13) implies that $z^{opt}(\cdot)$ is an admissible solution of the auxiliary problem (5). Since $\hat{z}$ is an optimal solution of (5), we get

$$J(\hat{z}(\cdot)) \geq J(z^{opt}(\cdot)) \tag{14}$$

for the admissible solution $z^{opt}(\cdot)$ in (5). The obtained inequalities (10) and (15) for the objective functional $J(\cdot)$ imply the expected result (8).

Moreover, inequality (15) also implies the following inclusion

$$\mathcal{F}_z \subset \mathcal{F}_{(2.3)}. \tag{15}$$

From (11), (12) and (15) we immediately deduce the crucial consequence

$$\mathcal{F}_{(2.3)} \bigotimes \{ \bigcup_{s=1,\dots,S} \mathcal{F}^s_{(2.2)} \} \equiv \mathcal{F}. \tag{16}$$

If the solutions sets $\mathcal{F}, \mathcal{F}_{(2.3)}$ and $\mathcal{F}^s_{(2.2)},\ s = 1, \dots, S$ are one-point-sets, we deduce the expected result (9) as a direct consequence of the obtained equivalence (16). The proof is completed. □

As we can see Theorem 3.1 separates equivalently the originally given sophisticated DMCLP (1) into two relatively simple optimization problems. It provides the theoretical foundation of the Separation Method we propose and generates effective numerical approaches to the dynamic MCLPs of the type (1). In fact, Theorem 3.1 reduce a dynamic optimization problem (1) with a switched structure to a family of specific Knapsack problems. This complexity reduction is a direct consequence of the geometrical structure of constraints in the originally given DMCLP (1).

Let us also note that an adequate scalarizing of problem (7) is a most problematic point of the method we proposed. On the other side, the same question related to an adequate scalarizing can also be stated in the framework of a general multi-objective optimization (vector optimization) problem. An adequate scalarizing in the case of the general multi-objective problem (7) constitutes in fact an open question of the mathematical optimization theory. This problem is solved for some specific cases and partial problem formulations (see e.g., [14, 21, 32] for the corresponding technical details and algorithms. It is also necessary to underline that the initially given multiobjective optimization problem (7) can also be solved directly (without any scalarizing (4)) by an appropriate heuristic / metaheuristic method (see e.g., [26, 32] and references therein).

4. **The separation based computational approach to DMCLP.** We now consider a particular problem from the family $Pr^S$, namely, problem (4) and introduce the following notation

$$\begin{aligned} \hat{I}^s &:= \{1 \leq i \leq l \mid \mathcal{S}_{\mathcal{A}^s_i} \in \max_{k_s} \{\mathcal{S}_{\mathcal{A}^s_1}, \dots, \mathcal{S}_{\mathcal{A}^s_l}\}\}, \\ \mathcal{S}_{\mathcal{A}^s_i} &:= \sum_{j=1}^{n} \mu^s_j a_{ij}, \\ \mathcal{A}^s_i &:= (a^s_{i1}, \dots, a^s_{in})^T,\ s = 1, \dots, S. \end{aligned} \tag{17}$$

Here $\mathcal{A}^s_i$ is a vector of $i$-row of the eligibility matrix $A^s$ and operator $\max_{k_s}\{\cdot\}$ determines an array of $k_s$-largest numbers from the given array. Since every problem (4) has a trivial combinatorial structure it can be easily solved by the naturally combinatorial algorithm.

**Algorithm 1.**

0. *Input: the eligibility matrices* $A^s$*, numbers* $k_s$ *multipliers* $\mu_j^s$*,* $j = 1, ..., n$ *for each* $s = 1, ..., S$*;*
1. *Calculate* $\hat{I}^s$*,* $s = 1, ..., S$ *from (17);*
2. *For each* $s = 1, ..., S$ *determine the solution* $\hat{y}^s$ *of every auxiliary problem (4) as*
$$\begin{aligned} &\hat{y}_i^s = 1 \text{ if } i \in \hat{I}^s; \\ &\hat{y}_i^s = 0 \text{ if } i \in \{1, ..., l\} \setminus \hat{I}^s; \end{aligned} \tag{18}$$
*STOP.*
3. *Output: the optimal (for the family of problems (4)) solution vector* $\hat{y}^s$*,* $s = 1, ..., S$*.*

The following result establishes the consistency of the proposed Algorithm 1 for the auxiliary problem (4).

**Theorem 4.1.** *Algorithm 1 and the corresponding optimal choice* (18) *determines an optimal solution of problem* (4) *for every* $s = 1, ..., S$*.*

*Proof.* The finite selection algorithm (18) assigns the maximal (from admissible) value $\hat{y}_i^s = 1$ for all vectors $\mathcal{A}_i^s$ such that the weighted sum of components $\mathcal{S}_{\mathcal{A}_i^s}$ of this vector belongs to the $k^s$-dimensional array of largest sums of components of all vectors
$$\mathcal{A}_\rangle^\int, \; i = 1, ..., l.$$
Hence the resulting sum
$$\sum_{j=1}^{n} \mu_j^s \sum_{i=1}^{l} a_{ij}^s \hat{y}_i^s(t)$$
for $t \in (t_{s-1}, t_s]$ is maximal. The proof is completed. □

Let us also note that for the given eligibility matrix $A^s$ with the specific positive elements $a_{ij}^s$ (as determined in Section 2) the sum of components $\mathcal{S}_{\mathcal{A}_i^s}$ constitutes a specific norm of the row-vector $\mathcal{A}_i^s$. The total complexity of the proposed combinatorial Algorithm 1 for a fixed index $s$ can be calculated as follows (see e.g., [23] for details)
$$O(l \times \log k^s) + O(k^s).$$
We now turn back to the second auxiliary problem (5) and define
$$c^s := \sum_{j=1}^{n} \sum_{i=1}^{l} a_{ij}^s \hat{y}_i^s,$$
$$c := \max_{s=1,\dots,S} \{c^s\}$$
Then the inequality constraints in (5) imply the generic Knapsack-type constraint with uniform weights for every index $s = 1, ..., S$
$$\sum_{j=1}^{n} z_j \leq c^s.$$
Let us present a fundamental solvability result for the second auxiliary optimization problem, namely, the "switched-type" Knapsack problem (5).

**Theorem 4.2.** *The* Knapsack *problem* (5) *can be solved in maximum $O(nc)$ time and $O(n+c)$ space.*

A formal proof of Theorem 4.2 can be found in [23].

We now need to determine an adequate implementable numerical procedure for problem (5). This auxiliary optimization problem, which is $\mathcal{NP}$-hard, has been comprehensively studied in the last few decades and several exact algorithms for its solution can be found in the literature (see [6, 23] and the references therein). Computational algorithms for Knapsack problems are mostly based on two basic approaches: the celebrated branch-and-bound methods and dynamic programming techniques. Moreover, one can combine these two basic numerical approaches. These two main solution procedures can also be used in combination with a relaxation scheme applied to an initial model.

"Relaxing a problem" has various meanings in applied mathematics, depending on the areas where it is defined, depending also on what one relaxes (a functional, the underlying space, etc.). We refer to [7, 5, 2, 18, 23, 24, 29] for various implementable relaxation techniques. In this contribution we apply the celebrated Lagrange relaxation scheme to the second auxiliary problem, namely, to the Knapsack-type problem (5). Let us introduce the Lagrange function for (5)

$$\mathcal{L}(z(\cdot),\lambda(\cdot)) := \sum_{s=1}^{S} \frac{(t_s - t_{s-1})}{(t_f - t_0)} \langle w(t), z(t)\rangle_{|t\in(t_{s-1},t_s]} - \sum_{j=1}^{n} \lambda_j(t)\big(z_j - \sum_{i=1}^{l} a_{ij}^s \hat{y}_i^s\big)$$

where $\lambda_j(\cdot),\ j=1,...,n$ are piecewise constant on the time intervals $(t_{s-1},t_s],\ s=1,...,S$ functions. Let us denote

$$\lambda(t) := (\lambda_1(t),...,\lambda_n(t))^T,\ t\in[t_0,t_f].$$

The following problem is called Lagrange relaxation of the auxiliary optimization problem (5)

$$\begin{aligned} &\text{maximize} \ \ \mathcal{L}(z(\cdot),\lambda(\cdot)) \\ &\text{subject to} \ \ z(t)\in\mathbb{B}^n \ t\in[t_0,t_f]. \end{aligned} \tag{19}$$

Let us refer to [2, 18, 23] for the foundations of the Lagrange relaxation in optimization. Note that the relaxed problem (19) does not contain the unpleasant inequality constraints which are included in the objective (Lagrange) function $\mathcal{L}(\cdot,\cdot)$ as a penalty term. All feasible solutions to (5) are also feasible solutions for problem (19). Moreover, the objective value calculated for the feasible solutions to (5) is not larger than the objective value obtained using a solution of (19) (see [1, 23] for the necessary proofs). This fact implies that the optimal solution value to the relaxed problem (19) provides an upper bound to the original problem (5) for any vector of non-negative Lagrange multipliers

$$\lambda(\cdot),\ \lambda_j(t)\geq 0,\ t\in[t_0,t_f].$$

Applying the branch-and-bound algorithm to the relaxed problem we are interested in achieving the tightest upper bound for the objective functional in (19). Hence, we would like to choose a vector of non-negative Lagrange multipliers (piecewise constant functions)

$$\hat{\lambda}^{\mathcal{L}}(t) := (\hat{\lambda}_1^{\mathcal{L}}(t),...,\hat{\lambda}_n^{\mathcal{L}}(t))^T,\ \ \hat{\lambda}_j^{\mathcal{L}}\geq 0$$

such that $L(\cdot,\cdot)$ in (19) is minimized. This consideration strongly motivates the celebrated concept of a Lagrangian dual problem (see e.g., [23] for details)

$$\begin{aligned} &\text{minimize } \ \mathcal{L}(z(\cdot),\lambda(\cdot)) \\ &\text{subject to } \ \lambda(t) \geq 0 \ t \in [t_0, t_f] \end{aligned} \tag{20}$$

By $(\hat{z}^{\mathcal{L}}(\cdot), \hat{\lambda}^{\mathcal{L}}(\cdot))$ we next denote the optimal solution pair to the primal-dual system (19)-(20). It is well-known that the Lagrangian dual problem (20) yields the least upper bound for (5) available from all possible Lagrangian relaxations. Note that the dual problem (20) is in fact a linear (integer) programming problem [6, 10]. The following basic result is a direct consequence of the basic properties of the above primal-dual system (19)-(20) and of our main theoretic result, namely, of Theorem 3.1.

**Theorem 4.3.** *Let $(\hat{z}^{\mathcal{L}}(\cdot), \hat{\lambda}^{\mathcal{L}}(\cdot))$ be an optimal solution of the primal-dual system* (19)-(20) *associated with the auxiliary problem* (5). *Assume that all conditions of* Theorem 3.1 *are satisfied. Then*

$$J(z^{opt}(\cdot)) = J(\hat{z}(\cdot)) \leq J(\hat{z}^{\mathcal{L}}(\cdot)). \tag{21}$$

*where $(\hat{z}(\cdot), \hat{y}(\cdot))$ is an optimal solution of the family of auxiliary problems* (4)-(5). *Moreover, inequality* (21) *constitutes a tightest upper bound available from all possible* Lagrangian *relaxations* (19) *of the optimization problem* (5).

Theorem 4.3 can be proved by a direct calculation using the corresponding result from [23].

The proposed SM belongs to the family of exact numerical optimization methods, it is a non-heuristic method. Recall that the modern development in computational optimization is mostly focused on the heuristic and metaheuristic approaches. On the contrary, the recent development of new exact numerical optimization methods is a slowly process. Since an effective optimization procedure from the family of exact computational approaches has evident advantages in comparison to the heuristics and metaheuristics, the presented development of the SM method is highly motivated.

5. **Numerical aspects.** In this section we apply the generalized SM developed in Section 3 and Section 4 and study a specific real-world example from the area of telecommunication. Our aim is to optimize a cellular mobile communication network, namely, to find an optimal solution to a specific restricted covering problem (see [9, 15]). Note that the mobile communication network constitutes a (dynamic) switching system. Our objective leads to the definition of a transmitter location problem as a locating problem that does not require the coverage of all demand nodes. The coverage model of a mobile communication network includes a limited budget as a constraint on the number of facilities to be located. The optimization model we implement has a fuzzy nature and includes a fixed number

$$k^s, \ s = 1, ..., S \in \mathbb{N}$$

of the totally $l \in \mathbb{N}$ base stations such that a specific average covering (considered on a fixed time interval) is maximized.

**Example 1.** Consider a (simplified) mathematical abstraction of the cellular mobile communication network from [9, 15]. Optimization of the communication flows in this network can be formalized as a generic DMCLP (1). Let us assume that a specific region covered by a communication network is divided into $n = 15$ subregions. A concrete demand node is located into the "center" of a subregion and is associated with an amount of demands for a fixed number of call requests per time unit. We next assume to have $l = 5$ totally feasible locations for the base stations. As mentioned above for every interval $(t_{s-1}, t_s]$ we have

$$k^s, \ s = 1, ..., S \in \mathbb{N}$$

"active" base stations (the stations in service). In our example we put $t_0 = 0$ and $t_f = 2$ and the (unique) switching time $t_1 = 1$. Moreover, let $S = 2, \ k^1 = 2, \ k^2 = 4$.

Let us now specify the further variables and parameters from the general DMCLP framework (1): the weights vectors associated with the corresponding time intervals are selected as follow

$$w_1 = (27, 42, 11, 29, 43, 16, 14, 19, 45, 30, 0, 0, 16, 42, 0)^T,$$
$$w_2 = (27, 26, 0, 48, 0, 35, 49, 10, 49, 48, 16, 14, 41, 28, 0)^T.$$

The eligibility matrices $A^s \in \mathbb{R}^{5\times 15}, \ s = 1, 2$ associated with the corresponding time intervals are selected as follows (given by rows $\mathcal{A}_i$ of $A^s$ for $i = 1, ..., 5$)

$$\mathcal{A}_1 = (0.0, 0.959, 0.832, 0.965, 0.0, 0.0, 0.9380, 1.0, 0.0, 0.0, 0.816,$$
$$0.528, 0.378, 0.0, 0.998),$$
$$\mathcal{A}_2 = (0.5069, 0.846, 0.388, 0.624, 0.970, 0.0, 0.0, 0.0, 0.653, 0.779, 0.701,$$
$$0.511, 0.876, 0.566, 0.963),$$
$$\mathcal{A}_3 = (0.340, 0.222, 0.211, 0.745, 0.0, 0.0, 0.0, 0.0, 0.0, 0.542, 0.0,$$
$$0.0, 0.730, 0.832, 0.0),$$
$$\mathcal{A}_4 = (0.0, 0.478, 0.593, 0.0, 0.0, 0.939, 0.0, 0.0, 0.736, 0.0, 0.0,$$
$$0.0, 0.794, 0.406, 0.0),$$
$$\mathcal{A}_5 = (0.0, 0.437, 0.0, 0.0, 0.959, 0.596, 0.619, 0.0, 0.0, 0.819, 0.0,$$
$$0.0, 0.611, 0.0, 0.806),$$
$$\text{for } s = 1$$

and

$$\mathcal{A}_1 = (0.531, 0.283, 0.0, 1.0, 0.975, 0.0, 0.383, 1.0, 0.479, 0.829, 0.0,$$
$$0.0, 0.707, 0.688, 0.0),$$
$$\mathcal{A}_2 = (0.558, 0.512, 1.0, 0.0, 0.691, 0.949, 0.929, 0.0, 0.650, 0.979, 0.438,$$
$$0.0, 0.517, 0.0, 0.0),$$
$$\mathcal{A}_3 = (0.0, 0.563, 0.0, 0.0, 0.530, 0.532, 0.0, 0.0, 0.0, 0.843, 0.0,$$
$$0.0, 0.0, 0.605, 1.0),$$
$$\mathcal{A}_4 = (0.0, 0.375, 0.0, 0.0, 0.241, 0.0, 0.0, 0.0, 0.0, 0.0, 0.360,$$
$$0.739, 0.928, 0.0, 0.0),$$
$$\mathcal{A}_5 = (0.559, 0.0, 0.0, 0.0, 0.615, 0.606, 0.969, 0.0, 0.0, 0.0, 0.0,$$
$$0.432, 0.470, 0.686, 0.0),$$
$$\text{for } s = 2.$$

Both the eligibility matrices have a fuzzy structure and express the technical reliability of the communication network under consideration.

Recall that the decision variable from (1), namely, $y_i(t),\ i = 1, ...5$ represents here the "activating" of the $i$ - base station at the time instant $t \in [0, 2]$. Additionally the specific state variable

$$z_j(t),\ j = 1, ..., 15$$

from the general model (1) describes here the binary state of the $j$ - demand node. Recall that it is equal to 1 if the $j$ - demand node is covered by at least one base station (facility). Otherwise

$$z_j(t) = 0.$$

Application of the numerical Algorithm 1 leads to the computational results for the optimal variables $z^{opt}(t)$ and $y^{opt}(t)$, where

$$t \in [0, 1] \text{ for } s = 1$$

and

$$t \in [1, 2] \text{ for } s = 2.$$

We have obtained the following optimal pairs $(z^{opt}(t),\ y^{opt}(t))$:

$$\begin{aligned} z^{opt}(t) &= (0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 1, 1, 0)^T, \\ y^{opt}(t) &= (0, 1, 1, 0, 0)^T, \end{aligned} \tag{22}$$

for $s = 1,\ t \in [0, 1]$ and

$$\begin{aligned} z^{opt}(t) &= (1, 1, 0, 1, 0, 1, 1, 1, 1, 1, 0, 1, 1, 1, 0)^T, \\ y^{opt}(t) &= (1, 1, 0, 1, 1)^T, \end{aligned} \tag{23}$$

where

$$s = 2,\ t \in [1, 2].$$

The optimal value

$$\max\{J(z(\cdot))\}$$

of the objective functional $J(\cdot)$ in (1) is as follow:

$$\max\{J(z(\cdot))\} = 534.$$

The dynamic behavior of the optimal decision variable $y^{opt}(t)$ is also presented in Fig. 1.

Let us now give a practical interpretation of the obtained computational (optimal) results (22)-(23). From (22) we deduce that the active base stations are the stations no.2 and no.3. Moreover, the following dynamic demand nodes no.2, no.4, and no.10 are covered. The optimal pair in (23) indicates that the active base stations are the stations no.1, no.2, no.4, and no.5. In that case the demand nodes excluding the following nodes: { no.3, no.5, no.11, no.15 } are covered.

Finally, note that the implementation code of the computational Algorithm 1 was carried out by using the Python optimization packages and an author-written program.

In the above illustrative Example 1 we have considered some given eligibility matrices $A^s$. In the telecommunication engineering this technical parameter has a specific structure and is usually a function of the required cover range of the existing base stations.

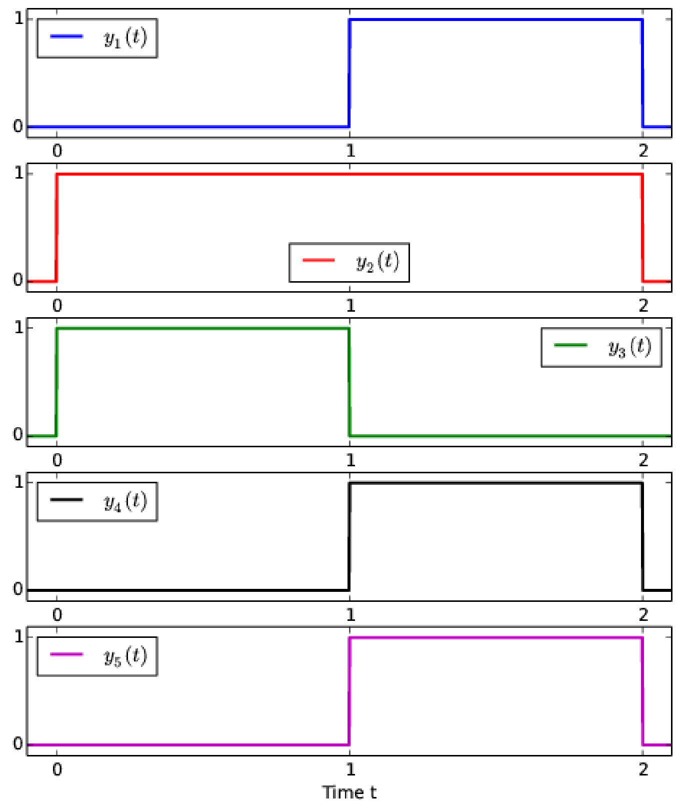


FIGURE 1. Optimal dynamics of the switched decision variables $y^{opt}(t)$

6. **Conclusion.** This paper proposes advanced theoretic and computational approaches to a new class of the Dynamic Maximal Covering Location Problems with a switched structure. In our contribution, we have theoretically extended the classic static Maximal Covering Location Problems and also implemented some additional novel modelling approaches. First of all we have proposed a switched type dynamical generalization of the conventional MCLP problem statement. This new approach makes it possible to incorporate a wide range of real-world application and engineering effects into the proposed framework. Evidently, switched dynamic processes constitute in general a more adequate modelling approach to the technical processes and systems. Moreover, we have considered the DMCLP optimization with the fuzzy-type eligibility matrices. This fact also involves more flexibility of the resulting optimization model that finally includes some necessary "resilience" or (and) "fuzzy" interpretations. In this paper, we have followed a mathematically rigorous considerations and developed a consistent numerical solution approach.

Let us mention two kinds of the applied mathematical works: the first type includes applications of the existing methods to a new problem and the second one constitutes a development of new methods. Our paper comes under the second type of works and is dedicated to a fully new problem. We consider a new MCLP related problem, namely, the dynamic generalization of the classic MCLP (called DMCLP). Since the problem under consideration as well as the proposed solution method are very new, we cannot implement here any comparative analysis of the possible numerical approaches to the DMCLP. The DMCLP firstly appears in this our paper and the necessary comparative study constitutes a future work.

The obtained computational algorithm was next applied to an engineering motivated problem of optimal covering in a cellular communication network. Application of the proposed algorithm makes it possible to optimize a specific RCP type problem (see Section 5) and maximize an average covering for a given configuration of the base stations. The SM and the corresponding numerical scheme we developed reduce the originally given sophisticated optimization problem to two Knapsack-type auxiliary problems. The first one is in fact a linear scalarization of a specific multiobjective program. The second Knapsack-type model in the framework of SM constitutes a classic problem. We have studied the proposed dynamic generalization of the SM in the context of the conventional Lagrange relaxation scheme and in combination with an additional combinatorial algorithm. The obtained theoretic results finally lead to an implementable computational algorithm for the switched type DMCLPs with a fuzzy-type eligibility matrix.

Let us summarize briefly the main novelty aspects of this paper. We propose a new exact numerical optimization method for a dynamic version of the MCLP problem (called DMCLP). Moreover, the given problem formulation (1) is also new and it constitutes a generalization of the conventional MCLP to a case of dynamic (time-dependent) constraints. The SM method we proposed for the DMCLP reduces an initially given sophisticated dynamic optimization problem (1) to two auxiliary problems, namely, to the well-known Knapsack problems (4) and to (5). The equivalence of the initially given problem (1) and the "decomposed" problems (4)-(5) is proven in Theorem 3.1. This theorem is in fact a new theoretical decomposition result. Additionally, a suitable version of the celebrated Bellman Optimality Principle for problem (1) was established (see Theorem 2.1). The proposed decomposition of problem (1) makes it possible to apply the known effective numerical procedures developed for the Knapsack type problems to the given DMCLP. We discuss the possible algorithmic part of the solution procedure (Section 4) and give a rigorous convergence proof of the proposed numerical procedure, namely, of Algorithm 1. The strong convergence property of the proposed algorithm is the essence of Theorem 4.1.

The rigorously proven theoretical results related to the DMCLP make it possible to apply consistently some known numerical tools to the initially given sophisticated dynamic optimization problem (1). The analytic and numeric methodologies we propose in our paper can be applied to various further generalizations of the generic MCLP. Moreover, the developed SM for a class of DMCLP under consideration can be combined with the celebrated branch-and-bound method and with the dynamic programming approach. It can also be included (as an auxiliary computational tool) into the modern heuristic solution approaches. Let us note that our paper mainly discussed the theoretic foundations of the newly elaborated SM for DMCLPs. The novel solution methodology we proposed needs further comprehensively numerical examinations and computer based simulations of several switched-type DMCLPs.

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