Received: 10 October 2017 | Revised: 24 November 2018 | Accepted: 26 November 2018
DOI: 10.1002/rnc.4446

RESEARCH ARTICLE

WILEY

Advances in attractive ellipsoid method for robust control design

V. Azhmyakov[1] | M. Mera[2,3] | R. Juárez[4]

[1]Departament of Basic Science, Universidad de Medellin, Medellin, Colombia
[2]ESIME-Instituto Politécnico Nacional, Mexico City, México
[3]UPIBI-Instituto Politécnico Nacional, Mexico City, Mexico
[4]Department of Accounting, Universidad Autonoma de Coahuila, Torreon, Mexico

Correspondence
V. Azhmyakov, Departament of Basic Science, Universidad de Medellin, Medellin, Colombia.
Email: vazhmyakov@udem.edu.co

Summary
Our contribution is devoted to a further theoretic development of the attractive ellipsoid method (AEM). We consider dynamic models given by nonlinear ordinary differential equations in the presence of bounded disturbances. The resulting robustness analysis of the closed-loop system incorporates the celebrated Clarke invariancy concept (an analytic extension of the celebrated Lyapunov methodology). We finally obtain a new general geometric characterization of the AEM-based approach to the robust systems design. Moreover, we also discuss the corresponding numerical aspects of the proposed theoretical extensions of the method. The theoretic results obtained in this contribution are finally illustrated by a practically oriented computational example.

KEYWORDS
attractive ellipsoid method, nonlinear systems, robust control

1 | INTRODUCTION

The problem of powerful computer-oriented algorithms for the robust control of nonlinear dynamic systems has attracted a lot of attention, thus both theoretical results and real-world applications were developed (see, eg, the work of Khalil[1]). The newly elaborated attractive ellipsoid method (AEM) (see other works[2-13]) constitutes a useful engineering oriented development of the theory of invariant sets and proposes a self-closed effective computational approach to the robust control design of systems with uncertainties. Recall that a set in the state space of a system is said to be positively invariant if any trajectory initiated in this set remains inside the set at all future time instants. The practically important problem of existence and constructive characterization of the invariant sets for general dynamical systems is in fact a sophisticated mathematical issue. For a specific class of systems, this complex problem is constructively solved in the framework of the AEM. A "small-size" ellipsoidal attractive (or invariant) set is evidently a geometrically motivated region that can also be defined as a "practical stability" region for the system under consideration. We refer to related works[3,6-13] for the necessary analytic background, some existing modifications of the AEM, and numerous applications of this methodology.

Our paper discusses a further theoretical and numerical extensions of the attractive ellipsoid (AE) techniques in the context of nonstationary dynamics. Recall that the conventional AEM is designed for a relatively restrictive class of stationary control systems. This stationarity assumption is a significant hypothesis and evidently has a strongly restrictive nature. Moreover, the generic version of the method is essentially limited to a subclass of models described by ordinary differential equations (ODEs) with the so-called "quasi-Lipschitz" right-hand sides.[12] The novel extension of the classic AEM, we propose in our contribution is free from the hard quasi-Lipschitz assumptions. It can be applied to a vide class of nonstationary control systems with (bounded) uncertainties.

A further implementation difficulty of the celebrated AE algorithm is characterized by a necessary linear matrix inequalities (LMIs) constrained optimization step. A concrete construction procedure of an AE and the corresponding robust feedback controller design include an auxiliary optimization problem, namely, sophisticated LMIs (or BMIs) constrained minimization (see other works[11,12,14]). This nonlinear high-dimensional constrained optimization problem constitutes a numerically sophisticated task and usually can not be effectively solved in the real time. Note that this LMI-based approach is a direct consequence of the Lyapunov-type analysis with a quadratic "storage" function. The alternative approach proposed in this paper uses an abstract concept of the flow-invariant sets associated with the ODEs-involved dynamic systems (see the works of Azhmyakov et al[15] and Clarke et al[16] for mathematical details). We firstly apply this concept to a linearized (nonstationary) dynamic system. We study a geometrical characterization of an AE for the linearized system and obtain some constructive estimations. Using a proved fundamental result about linearization (see Theorem 1 from the next section), we next derive the necessary ellipsoidal estimations of the attractive set for the originally given nonlinear control system. The resulting robust control algorithm does not include any optimization step. This algorithm generates a feedback-type robust control law and guarantees a practical system stability with respect to an a priory determined (small) ellipsoidal region. The geometrical approach we propose makes it possible designing the resulting nonstationary closed-loop system for an adequately chosen practical stability region.

The remainder of our paper is organized as follows. Section 2 contains a problem formulation, necessary description of the given nonlinearly-affine dynamic model, and introduces a linearized system. Furthermore, it includes a rigorous proof of the main analytic result that characterizes the "quality" of the obtained linear approximation. Section 3 is devoted to a geometrical description of an AE associated with the linearized system. We also discuss the resulting control design procedure for this linear dynamic model. The proposed robust control strategy is a formal consequence of the flow-invariancy condition. In Section 4, we come back to the originally given nonlinear nonstationary dynamic system and develop an AE-based robust control design. This feedback control strategy involves the previously obtained control design for the linearized system introduced in Section 3. We use the flow-invariancy property of the generated AE for linearized system and construct an AE and the robust control law for the original switched system. Section 5 discusses the numerical aspects of the proposed extension of the basic AEM. It also includes a practically oriented computational example. The effectiveness of the new approach proposed in this paper is illustrated by the obtained computational results. Section 6 summarizes our paper.

2 | PROBLEM FORMULATION AND PRELIMINARY RESULTS

Consider the following initial value problem for ODEs with a general control-affine structure:

$$\begin{aligned} \dot{x}(t) &= f\,(t, x(t)) + B(t)u(t) + \xi(t) \text{ a.e. on } \mathbb{R}_+, \\ x(0) &= x_0, \end{aligned} \tag{1}$$

where $x_0 \in \mathbb{R}^n$ is a fixed initial state. The given function $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be continuous on $\mathbb{R}_+$ and uniformly Lipschitz continuous on an open bounded set $\mathscr{R} \subseteq \mathbb{R}^n$. By

$$B(t) \in \mathbb{R}^{n\times m},\ t \in \mathbb{R}_+,$$

we denote here a control matrix. We next assume that $B(\cdot)$ is a continuous matrix-function. The uncertainties $\xi(\cdot)$ are assumed to be uniformly bounded

$$\sup_{t\in\mathbb{R}_+} \|\xi(t)\| \le M \in \mathbb{R}_+. \tag{2}$$

By $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$, we denote here the state and the control vector, respectively. Note that general affine control systems have become an important application focus of the modern control theory (see, eg, other works[1,15,17-20] and references therein). Let us firstly consider the basic system (1) over a control set $\mathscr{U}$ of essentially bounded measurable control inputs. Note that the possible unmodeled dynamics can be included into the uncertainty term $\xi(t)$ in (1).

In parallel with (1), we examine the corresponding linearized control system

$$\begin{aligned} \dot{y}(t) &= f_x\,(t, x^u(t))\, y(t) + B(t)v(t) + \xi(t) \text{ a.e. on } \mathbb{R}_+, \\ y(0) &= 0, \end{aligned} \tag{3}$$

where $u(\cdot) \in \mathscr{U}$ and $x^u(\cdot)$ is the absolutely continuous solution to the initial system (1) generated by an admissible $u(\cdot)$. The linearized system (3) is considered for a fixed admissible control function $u(\cdot)$ and the corresponding trajectory $x^u(\cdot)$. This preselected reference control strategy constitutes in fact so-called "tracking" control. We refer to the works of Khalil[1] and Rockafellar and Wets[21] for some basic facts related to the linearization techniques. Let us also note that the celebrated

Rademacher theorem guarantees the (almost everywhere) differentiability property of the function $f(t,\cdot)$ (see, eg, the work of Clarke et al[16]).

Let us continue by some additional technical assumptions. We next suppose that the pair $(A(t), B(t))$, where

$$A(t) := f_x(t, x^u(t))$$

is controllable for every $t \in \mathbb{R}_+$. We also assume that

$$\|f_x(t,x)\| \leq C, \tag{4}$$

for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$. In this paper, we also assume that a class of locally Lipschitz (feedback) control functions $w(\cdot,\cdot)$ such that

$$w(t, x^u(t)) = v(t)$$

in (3) constitutes an admissible set of inputs. This family of functions $w(\cdot,\cdot)$ is next denoted by $\mathscr{L}$. For each $u(\cdot) \in \mathscr{U}$ and $w(\cdot,\cdot) \in \mathscr{L}$, the initial value problem (3) has a unique solution denoted by $y^v(\cdot)$. We refer to the work of Hale and Lunel[22] for the necessary existence and uniqueness results. In this paper, we restrict our consideration to an important subclass of $\mathscr{L}$ and consider the "proportional" control design of the following type:

$$w(t, y(t)) = K(t)y(t), \quad K(t) \in \mathbb{R}^{m\times n}, \ t \in \mathbb{R}_+.$$

Here, $K(\cdot)$ is a gain matrix-function. This unknown gain matrix constitutes a free parameter of the control design under consideration and then linear closed-loop system can be written as

$$\begin{aligned} \dot{y}(t) &= \left(f_x(t, x^u(t)) + B(t)K(t)\right) y(t) + \xi(t), \text{ a.e. on } \mathbb{R}_+, \\ y(0) &= 0. \end{aligned} \tag{5}$$

Let us firstly describe the desired control design for the linearized system (5) in a qualitative manner. The trajectory $y^v(\cdot)$ of the closed-loop linearized system (5) with a concrete matrix-function $K(\cdot)$ needs to stay for $t \in \mathbb{R}_+$ in an ellipsoidal region (with the center at the origin), ie,

$$\mathscr{E} := \left\{ y \in \mathbb{R}^n \mid y^T P y \leq 1 \right\}.$$

Here, P is a positive defined symmetrical $n \times n$-dimensional matrix. Our main idea is to study the auxiliary linearized system (5) and propose a constructive geometrical characterization of the corresponding minimal-size AE $\mathcal{E}$. Various linearization methods associated with the dynamic models given by ODEs have been long time recognized as a powerful tool for stabilization of the conventional control systems (see, eg, related works[1,15,18-20,23-30] and the references therein).The geometric stability criterion for the linearized system (3) mentioned earlier is next used in the robust feedback control design procedure for the originally given nonlinear system (1).

We now explain (qualitatively) the main advances in AEM we developed in this paper. The nonstationary character of the dynamic systems (1) and (5) makes it impossible a direct application of the conventional LMI-based robust control design that involves the AEM techniques (see other works[3,6,7,9,12,31,32]). Recall that the classic AEM was developed under the strict assumptions of stationarity for the given dynamic system. Moreover, it also involves the restrictive "quasi-Lipschitz" condition for the right-hand side of the initial differential equation (see the work of Poznyak et al[12]). However, many modern engineering control systems involve a sophisticated nonstationary systems modeling framework. These nonstationary dynamic models with time-depending parameters are adequately modeled by time-depending systems of the type (1) closed by a nonstationary feedback (for example, by the law $w(t,x)$ given earlier).

We next need an exact analytic result that establishes the quality of the linear approximation (5) for system (1). We use here the notation $\mathbb{L}_r^\infty$ for a standard Lebesgue space of measurable essentially bounded r-dimensional vector functions defined on a time interval $I \subset \mathbb{R}_+$.

Theorem 1. *Assume that the initial system (1) given on a time interval I satisfies all the aforementioned technical assumptions. Then, there exists a function* $o : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $s^{-1}o(s) \to 0$ *as* $s \downarrow 0$ *and*

$$\left\| x^{u+v}(\cdot) - \left(x^u(\cdot) + y^v(\cdot)\right)\right\|_{\mathbb{L}_n^\infty} \leq o\left(\|v(\cdot)\|_{\mathbb{L}_m^\infty}\right),$$

for all $u(\cdot) \in \mathcal{U}$ *and* $v(\cdot) \in \mathbb{L}_m^\infty$.

Proof. Assume $u(\cdot) \in \mathbb{L}^\infty_m$. For a function $w(\cdot,\cdot) \in \mathcal{L}$, we have $v(\cdot) \in \mathbb{L}^\infty_m$. From the well-known comparison theorem (see the work of Khalil[1]) with the following selected comparison functions:

$$z(t) := x^u(t) + y^v(t),$$
$$\psi(t,x) = f(t,x) + B(t)u + v(t),$$

where $t \in \mathbb{R}_+$, we obtain

$$\left\|x^{u+v}(\cdot) - \left(x^u(\cdot) + y^v(\cdot)\right)\right\|_{\mathbb{L}^\infty_n} \leq e^C \int_I \left\|\dot{x}^u(t) + \dot{y}^v(t) - f\left(t, x^u(t) + y^v(t)\right) - B(t)\left(u(t) + v(t)\right)\right\| dt$$
$$= e^C \int_I \left\|\left\langle \left(f_x\left(t, x^u(t)\right), B(t)\right), \left(y^v(t), v(t)\right)\right\rangle - \left[f\left(t, x^u(t) + y^v(t)\right) + B(t)v(t) - f(t, x^u(t)\right]\right\| dt. \tag{6}$$

Here, C is a constant in (4). From the component-wise variant of the mean value theorem (see the work of Aliprantis and Border[33]), we next deduce

$$f_i(t, x^u(t)) + y^v(t)) + B(t)\left(u(t) + v(t)\right) - \left(f_i\left(t, x^u(t)\right) + B(t)u(t)\right)$$
$$= \langle (f_i)_x\left(t, x^u(t) + v_i(t)\right), B(t))\left(y^v(t), v(t)\right)\rangle,$$

for $i = 1, \ldots, n$ and a suitable bounded function $v(\cdot)$. Assumption (4) and the Lipschitz continuity of $f(t,\cdot)$ on a bounded set $\mathcal{R}$ imply the existence of a (continuous) function $o_1 : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$s^{-1} o_1(s) \to 0,$$

as $s \downarrow 0$ and

$$\|\langle\left(f_x\left(t, x^u(t)\right), B(t)\right), \left(y^v(t), v(t)\right) - \left[f\left(t, x^u(t) + y^v(t)\right) + B(t)v(t) - f\left(x^u(t)\right)\right] \leq o_1\left(\|v(\cdot)\|_{\mathbb{L}^\infty_m}\right),$$

for all $t \in I$. From (6), we finally deduce the expected estimation

$$\left\|x^{u+v}(\cdot) - \left(x^u(\cdot) + y^v(\cdot)\right)\right\|_{\mathbb{L}^\infty_n} \leq o\left(\|v(\cdot)\|_{\mathbb{L}^\infty_m}\right),$$

with $o(s) := e^C o_1(s)$. The proof is finished. □

Theorem 1 will next be used in a concrete robust control design procedure for the original nonlinear system (1) (see Section 4). Let us discuss the celebrated Clarke flow-invariance concept from the work of Clarke et al.[16]

Definition 1. A smooth manifold S in an Euclidean space is called a flow-invariant in the sense of a given dynamic system

$$\dot{z}(t) = \phi\left(z(t)\right), \ t \in \mathbb{R}_+,$$
$$z(0) = 0 \tag{7}$$

if $z(t) \in S$ for all $t \geq T \in \mathbb{R}_+$.

The next abstract theorem gives a general criterion of a flow-invariant manifold.

Theorem 2. *A smooth manifold S is flow-invariant for system (7) if and only if $\phi(x)$ belongs to the tangent space T_S of S for all x from the given Euclidian space.*

The formal proof of this basic theorem is based on an extended Lyapunov-type technique. We refer to the work of Clarke et al[16] for the additional mathematical details. Note that Theorem 2 has a very natural geometrical interpretation.

Let us also recall the general invariance concept, ie, a set $\mathcal{D}$ in the state space of a dynamic system is called to be (positively) invariant if an admissible trajectory initiated in this set remains inside the set at all future time instants. Let us denote by $\Omega(z(0))$ a (positive) limit set of system (5) (the set of all positive limit points, see, eg, related works[1,27,30,34]). We now give a related Lyapunov set stability concept (see the works of Blanchini and Miani[24] and Michel et al[30]).

Definition 2. A compact invariant set $\mathcal{D} \subset \mathbb{R}^n$ of the closed-loop dynamic system (5) is called asymptotically Lyapunov stable if $\Omega(z(0)) \subset \mathcal{D}$ and

- for all $\varepsilon > 0$, there exists $\delta_1 > 0$ such that the initial condition $\text{dist}[z(0), \mathcal{D}] \leq \delta_1$ implies

  $$\text{dist}\left[z(t), \mathcal{D}\right] \leq \varepsilon,$$

  for all $t \in \mathbb{R}_+$ (the Lyapunov stability of the set);

- there exists $\delta_2 > 0$ such that $\text{dist}[z(0), \mathcal{D}] \leq \delta_1$ implies
$$\lim_{t\to\infty} \text{dist}\,[z(t), \mathcal{D}]$$
(the attraction property of the set).

Here,

$$\text{dist}[z; \mathcal{D}] := \min_{\tilde{z}\in\mathcal{D}} ||z - \tilde{z}||_{R^n}$$

is the usual Euclidean distance between a point $z \in \mathbb{R}^n$ and $\mathcal{D}$. As mentioned earlier, the existence and a constructive characterization of an invariant set for a general dynamical system (7) constitute a very sophisticated theoretic questions. Our aim is to specify an invariant set for the closed-loop linear system (5) in the form of an ellipsoid $\mathcal{E}$ introduced in Section 2. Applying the classic concept of an asymptotically stable set, namely, Definition 1.2, we now introduce the main concept of an AE.

Definition 3. An ellipsoid $\mathcal{E}$ is called an AE for system (5) if it is an asymptotically stable invariant set of this system.

It is evident that an AE $\mathcal{E}$ for system (5) is determined by the matrices P and $K(\cdot)$. The chosen gain matrix-function $K(\cdot)$ determines in fact the resulting dynamic behavior of the trajectory $y^v(t)$ such that the basic inequality

$$\left(y^v(t)\right)^T Py(t) \leq 1,\ \ t \in R_+$$

is satisfied. From the point of view of a practical engineering robust control design, the last condition can of course be considered in a (suitable) approximative sense. The resulting $\mathcal{E}$-restricted dynamic behavior of the linear system (5) closed by the nonstationary linear feedback $w(\cdot,\cdot)$ can finally be interpreted as a practical stability of this dynamic system.

3 | APPLICATION OF THE AES METHOD TO THE LINEARIZED SYSTEM

The previously discussed general facts and concepts are used in this section for a constructive geometric interpretation of the basic AEM in the context of the linearized system (5). We first introduce an auxiliary dynamic variable $\theta(\cdot)$, $\theta(0) = 0$ and determine the smooth manifold in the (extended) Euclidian state space $\mathbb{R}^{n+1}$

$$S_P := \left\{z \in \mathbb{R}^{n+1} \mid y^T(t)Py(t) - 1 + \theta(t) = 0\right\}.$$

Here, $z := (y, \theta)^T$, and $y(\cdot)$ corresponds to system (5). Introduce the following additional notation:

$$h(z) := z^T Pz - 1 + \theta.$$

The necessary and sufficient condition for the flow-invariance of S_P can now be determined using Theorem 2, ie,

$$\begin{aligned} \langle \nabla h(z), \phi(z)\rangle &= 0, \\ \theta(t) &\geq 0,\ \forall t \in \mathbb{R}_+. \end{aligned} \tag{8}$$

By $\nabla h(\cdot)$, we denote the gradient of the function $h(\cdot)$ introduced earlier. The vector field $\phi(\cdot)$ corresponds to the right-hand side of the following system of equations:

$$\begin{aligned} &\dot{y}(t) = \left(f_x\,(t, x^u(t)) + B(t)K(t)\right) y(t) + \xi(t), \\ &\dot{\theta}(t) = -2y^T(t)P\dot{y}(t), \\ &y(0) = 0,\ \ \theta(0) = 0. \end{aligned} \tag{9}$$

We now use the dynamics of system (5) and rewrite the second differential equation from (9) in the equivalent form

$$\dot{\theta}(t) = -2y^T(t)P\left[\left(f_x\,(t, x^u(t)) + B(t)K(t)\right) y(t) + \xi(t)\right]. \tag{10}$$

Note that (9) constitutes a self-closed initial-value problem for a generic ODE. The closed-loop linear system (5) is extended to (9) by an additional (scalar) equation with respect to θ. Assuming we have a trajectory $y^v(\cdot)$ of (5), the resulting function $\theta(\cdot)$ is next clearly determined.

The aforementioned analytic observations can now be summarized in the form of a theorem.

Theorem 3. *The ellipsoidal set $\mathcal{E}$ is an invariant set for the closed-loop system (5) if and only if the variable $\theta(t)$ determined by the initial-value problem (9)-(10) is nonnegative for all $t \in \mathbb{R}_+$.*

Proof. The scalar product in (8) $\langle \nabla h(z), \phi(z)\rangle|_{z=z(t)}$ can be easily calculated, ie,

$$\langle (2y^T(t)P, 1), (\dot{y}(t), -2y^T(t)P\dot{y}(t))\rangle$$
$$= 2y^T(t)P\dot{y}(t) - 2y^T(t)P\dot{y}(t)) = 0.$$

Evidently, the first condition from (8) is true. Therefore, (8) is reduced to the second condition, namely, to the nonnegativity of the variable $\theta(t)$ determined by system (9). The proof is completed. □

The formal solution of the initial-value problem on a time interval I for the artificial variable $\theta(\cdot)$ can be written as

$$\theta(t) = -\int_I \left[2y^T(t)P\left[f_x(t, x^u(t)) + B(t)K(t)\right] y(t) + y^T(t)P\xi(t)\right] dt.$$

Therefore, the nonnegativity condition for $\theta(t)$ required by Theorem 2 implies the next inequality, ie,

$$\int_I \left[y^T(t)P\left[f_x(t, x^u(t)) + B(t)K(t))\right] y(t) + y^T(t)P\xi(t)\right] dt \le 0. \quad (11)$$

Taking into account the robust control applications, we are naturally interested to construct an AE $\mathcal{E}$ of a minimal "size" (volume). This natural requirement and the corresponding selection of the matrix P need to be adequately formalized. Let us recall the well-established formalization procedure in the form of a specific LMI-constrained minimization problem (see, eg, other works[24,25,30]). In the case of a stationary dynamic system, one can use a specific optimization problem (see related works[5,13,14,35]) for a minimal-size AE design and for the calculation of the corresponding gain matrix-function $K(\cdot)$

$$\begin{aligned} &\text{minimize } \operatorname{tr}(P^{-1}) \\ &\text{subject to } P^T = P,\ P > 0,\ \{K(t)\} \in \mathcal{K}_t, \end{aligned} \quad (12)$$

where $t \in \mathbb{R}_+$ and $\mathcal{K}_t \subset \mathbb{R}^{m\times n}$ is the set of admissible matrices that ensure invariance of AE $\mathcal{E}$ for system (5). A constructive description of the set $\mathcal{K}_t$ constitutes a sophisticated mathematical problem. This problem can be constructively solved in the case of stationary dynamic systems and, moreover, under some additional hard conditions. The same observation is also true for the originally given nonlinear system (1). Control systems discussed earlier, namely, dynamic models (1) and (5) have a nonstationary nature. Therefore, in that case, the nature of the resulting systems make it impossible a direct application of the celebrated LMI-based approach. Roughly speaking, one cannot give a constructive description of the set $\mathcal{K}$ of admissible gain matrices in the form of a unique LMI. The restrictions set $\mathcal{K}_t$ in (12) evidently has a dynamic structure and contains time-depending admissible matrices. A possible solution of the optimization problem (12) with the time-depending restriction $\mathcal{K}_t$ evidently involves a very massive calculation associated with every time instant.

The critical analysis of the conventional AEM implies the necessity to generalize the usual AEM and to develop a conceptually new approach to the parameter selection of an AE (the ellipsoid matrix P). Then, same observation is also true with respect to the resulting feedback control design, namely, to the gain matrix $K(t)$. In this paper, we follow a "guaranteed" approach and select (a priory) a suitable matrix $\hat{P}$ that involves a relative "small" ellipsoidal invariant set $\mathcal{E}$ for the closed-loop linear system (5).

Theorem 4. *For a given (admissible) ellipsoid matrix $\hat{P}$ with $\hat{P} = \hat{P}^T > 0$ associated with the linearized system (5), the suitable gain matrix $K(t) \in \mathcal{K}_t,\ t \in \mathbb{R}_+$ can be found as solutions of the following LMI:*

$$\hat{P}\left[f_x(t, x^u(t)) + B(t)K(t)\right] + \sqrt{M}||\hat{P}|| \times E \le 0, \quad (13)$$

where E is a $n \times n$-dimensional unit matrix.

Proof. The first summand in (13) evidently coincides with the matrix of the first summand in the integrand in (11). Let us now estimate the expression $y^T(t)P\xi(t)$ from (11). We get

$$\left(y^T(t)P\xi(t)\right)^2 = y^T(t)P\xi(t)\xi^T(t)Py \le ||\hat{P}||^2 y^T(t)\xi(t)\xi^T(t)y(t).$$

For the Frobenius norm $||\xi(t)\xi^T(t)||_{Fr}$ of the matrix $\xi(t)\xi^T(t)$, we obtain (see assumptions of Section 2)

$$\left\|\xi(t)\xi^T(t)\right\|_{Fr} = \|\xi(t)\|^2 \le \sup_{t\in\mathbb{R}_+} \|\xi(t)\|^2 \le M.$$

Therefore,

$$\left(y^T(t)P\xi(t)\right)^2 \le y^T(t)\left(M||\hat{P}||^2 E\right) y(t)$$

and the matrix condition (13) implies the integral inequality (11). Using (11) and Theorem 3, we deduce that the obtained LMI (13) determines the admissible gain matrices $K(t) \in \mathcal{K}_t$. By definition of the set $\mathcal{K}_t$, the aforementioned inclusion guarantees invariance of AE $\mathcal{E}$ for system (5). The proof is completed. □

Finally, note that the LMI-type matrix inequality (13) provides a theoretic fundament for an effective and implementable computational procedure. Note that Theorem 4 expresses only a sufficient condition for the inclusion $K(t) \in \mathcal{K}_t,\ t \in \mathbb{R}_+$, and, thereby, for the invariance of the set $\mathcal{E}$. Solutions to the obtained LMI (13) depend on the selected reference (tracking) trajectory $x^u(\cdot)$ for the given system (1).

4 | FROM LINEARIZED TO THE ORIGINAL NONLINEAR SYSTEM

Results for linear systems obtained in Section 3 can now be applied to the robust control design of the originally given nonlinearly affine control system (1). Using the feedback input $w(\cdot,\cdot)$ implemented in the linearized system (5), we consider the following simple control strategy associated with the initial dynamics (1):

$$\begin{aligned} u(t) &:= u^{\text{ref}}(t) + w\left(t, y^{\nu}(t)\right), \\ w\left(t, y^{\nu}(t)\right) &= K(t)y^{\nu}(t), \end{aligned} \tag{14}$$

where $y^{\nu}(\cdot)$ is a solution to the linearized system (5) and $u^{\text{ref}}(\cdot)$ is a selected reference (tracking) control that corresponds to a reference trajectory $x^{\text{ref}}(\cdot)$ of (1). We assume that $u^{\text{ref}}(\cdot)$ is an essentially bounded measurable function from $\mathcal{U}$ (see Section 2). This assumption evidently guarantee the admissibility $u(\cdot) \in \mathcal{U}$. An absolutely continuous solution of the initial system (1) generated by $u^{\text{ref}}(\cdot)$ is denoted by $x^{\text{ref}}(\cdot)$. We next choose a "combined" control input $u(\cdot)$ in (14) such that the closed-loop variant of system (1)

$$\begin{aligned} \dot{x}(t) &= f\left(t, x(t)\right) + B(t)u^{\text{ref}}(t) + B(t)K(t)y^{\nu}(t) + \xi(t), \\ x(0) &= x_0, \end{aligned} \tag{15}$$

possesses the required robustness property. This basically means that (15) admits an AE $\mathcal{E}_0$, $0 \in \mathcal{E}_0$. One can also use here the meaningful concept of a "practical stability" as an alternative terminology. Note that the feedback-type control design (14) depends on the state vector of system (5). Our aim is to establish the practical stability (in the AE framework) of the original nonlinear system (1) using the robust control design developed for the linearized model (5). For a given admissible matrix $\hat{P}$, we next give a constructive estimation of the AE $\mathcal{E}_0$ associated with the nonlinearly affine switched system (1). At this point, we use the abstract results from Section 2, namely, the basic Theorem 1.

Theorem 5. *Consider a system (1) that satisfies all the basic assumptions from Section 2 and the corresponding closed-loop realization (15). Assume that matrices $\hat{P}$ and $K(t)$ are determined by Theorem 4. Let the reference trajectory $x^{\text{ref}}(\cdot)$ be uniformly bounded $||x^{\text{ref}}(t)|| \leq \chi,\ t \in \mathbb{R}_+$. Then, the AE $\mathcal{E}_0$ associated with the closed-loop version (15) of system (1) admits the following estimation:*

$$\mathcal{E}_0 := (||K(\cdot)|| + 1)\ \mathcal{E} + \chi. \tag{16}$$

Proof. The fundamental Theorem 1 implies the following simple estimation:

$$\left\| x^u(\cdot) - \left(x^{\text{ref}}(\cdot) + y(\cdot)\right) \right\|_{\mathbb{L}_n^\infty} \leq o\left(\|K(\cdot)y(\cdot)\|_{\mathbb{L}_m^\infty} \right).$$

The last inequality leads to the next result

$$\left\| x^u(\cdot) - x^{\text{ref}}(\cdot) \right\|_{\mathbb{L}_m^\infty} \leq o\left(\|K(\cdot)y^{\nu}(\cdot)\|_{\mathbb{L}_m^\infty} \right) + \|y(\cdot)\|_{\mathbb{L}_n^\infty} \leq (\|K(\cdot)\| + 1)\, \|y(\cdot)\|_{\mathbb{L}_n^\infty}.$$

Finally, we deduce

$$\|x^u(\cdot)\| \leq (||K(\cdot)|| + 1)\, ||y(\cdot)||_{\mathbb{L}_n^\infty} + \left\| x^{\text{ref}}(\cdot) \right\|_{\mathbb{L}_m^\infty} \leq (||K(\cdot)|| + 1)\, ||y(\cdot)||_{\mathbb{L}_n^\infty} + \chi, \tag{17}$$

where $t \in \mathbb{R}_+$.

Since $x^{\text{ref}}(\cdot)$ is assumed to be bounded and $\mathcal{E}$ is an invariant ellipsoid for (5), the inequality (17) implies the required estimation (16). From (17), we also deduce the invariance property of the ellipsoidal set $\mathcal{E}_0$ for the closed-loop system (15). The proof is completed. □

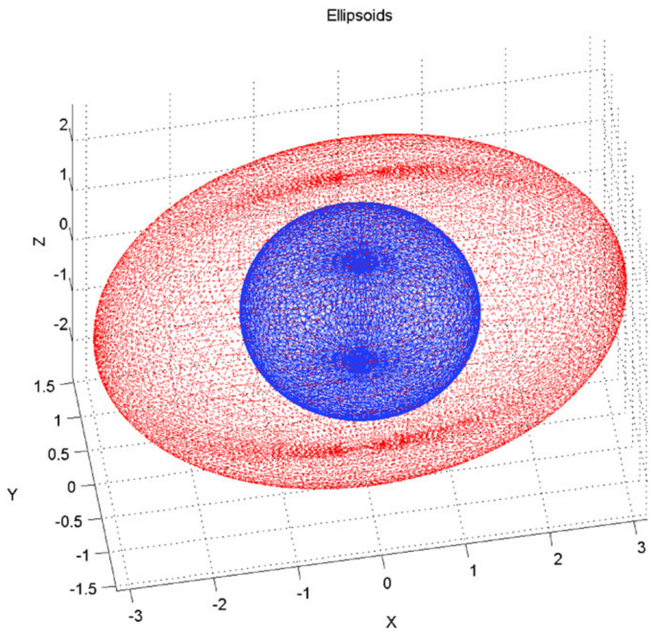

FIGURE 1 The geometrical relationship between $\mathcal{E}$ and $\mathcal{E}_0$ [Colour figure can be viewed at wileyonlinelibrary.com]

Theorem 5 makes it possible to characterize an invariant ellipsoidal set $\mathcal{E}_0$ associated with the sophisticated nonlinearly affine system (1). We use the minimal-size invariant ellipsoid $\mathcal{E}$ for this purpose. Evidently, the robust control design $w(t, y)$ for the linearized system (5) has a simple feedback-type structure. Theorem 5 opens up the possibility to use this simplified linear control design for robust control of the sophisticated nonlinear system (1). Note that the size of the ellipsoid $\mathcal{E}_0$ depends on the size of $\mathcal{E}$ and, in fact, by an adequate choice of the matrix $\hat{P}$. Moreover, we evidently have $\mathcal{E} \subset \mathcal{E}_0$. Geometrically, this situation is illustrated on Figure 1.

The blue ellipsoid corresponds to $\mathcal{E}$ and the red ellipsoid illustrates $\mathcal{E}_0$. Recall that a linearization procedure with respect to a reference pair $(u^{\text{ref}}(\cdot), x^{\text{ref}}(\cdot))$ is adequate only on a finite (small) time interval. One has to update the corresponding reference trajectory $x^{\text{ref}}(\cdot)$ after the control design on the selected small time interval is completed.

Let us finally make some observations related to an adequate choice of the (symmetric and positive definite) matrix $\hat{P}$ for $\mathcal{E}$. It is easy to see that the basic integral inequality (11) contains two unknown "variables", namely, matrices $\hat{P}$ and $K(\cdot)$. Therefore, condition (11) can be denoted as "underdetermined". Taking into consideration the result of Theorem 4 (a sufficient condition for the requested robust control design $K(t) \in \mathcal{K}_t,\ t \in \mathbb{R}_+$), we can choose the necessary matrix $\hat{P}$ such that

$$||\hat{P}|| = \max_{i=1,\dots,n} \{\lambda_i\} \leq \tau,$$

where $\lambda_i, i = 1, \dots, n$ are eigenvalues of $\hat{P}$ and τ is a prescribed (small) positive number. Using (13), we get the following implementable rule (LMI) associated with a reference trajectory:

$$B(t)K(t) \leq -f_x\left(t, x^{\text{ref}}(t)\right) - \tau\sqrt{M}\hat{P}^{-1}, \tag{18}$$

where $\hat{P}^{-1}$ denotes the inverse matrix of $\hat{P}$. Finally, note that a "small" AE is determined by a correspondingly chosen "small" matrix $\hat{P}$. This fact implies big components of the inverse $\hat{P}^{-1}$ on the right-hand side of (18). Consequently, the norm of the gain matrix $K(t)$ also achieves big values. Therefore, an adequate selection of the ellipsoid $\mathcal{E}$ and the associated matrix $\hat{P}$ constitutes a crucial "a priori" aspect of the proposed methodology.

5 | COMPUTATIONAL ASPECTS AND NUMERICAL RESULTS

In this section, we apply the theoretical results obtained earlier and propose an implementable computational algorithm for the robust control design of the type (14) for system (1). We consider here one academic and one practically oriented numerical examples. Let us note that the combined feedback control design (14) incorporates a selected matrix $\hat{P}$ as well as the (precomputed) trajectory $y^v(\cdot)$ of the linearized system (5). This fact eliminates a main technical difficulties of the conventional AEM, namely, a hard high-dimensional LMI constrained minimization problem (see the work of Poznyak et al[13]). Secondly, the conventional AE methodology was developed under the so-called quasi-Lipschitz conditions. In the case of an affine nonstationary system (1), these restrictive conditions can be written as follows:

$$\|f(t,x)+B(t)u-\mathscr{A}x\|_{Q_1}^2 \le \delta_1+\delta_2\|x\|_{Q_2}^2, \tag{19}$$

where

$$\delta_1, \delta_2 > 0$$

and $||\cdot||_{Q_1}$, $||\cdot||_{Q_2}$ are weighted (by some positive matrices Q_1 and Q_2) vector norms. Evidently, condition (19) has no sense for the basic nonstationary system (1). The right-hand side of inequality (19) does not depend on the dynamic variable t. Note that this absence of the "dynamics" in (19) makes it impossible to apply the conventional AEM even in the case of systems with a simple nonstationary structure (see Example 1).

5.1 | Numerical examples

Let us firstly apply the proposed robust control methodology to an academic example.

Example 1. Consider a simple two-dimensional control system of the type (1)

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)+u_1(t)+\xi_1(t)\\ \dot{x}_2(t) &= \alpha(t)\sin(x_2(t))+u_2(t)+\xi_2(t)\\ x_1(0) &= x_2(0)=2.\end{aligned} \tag{20}$$

Assume that

$$\sup_{t\in\mathbb{R}_+} ||\xi(t)|| \le 1$$

and $||u(t)|| \le 2$, where

$$\xi := (\xi_1, \xi_2)^T,$$
$$u := (u_1, u_2)^T.$$

Moreover, assume that $\alpha(\cdot)$ is a continuously differentiable function such that

$$0 < \alpha(t) \le 1,$$

for all $t \in \mathbb{R}_+$. Evidently, we have here $n = m = 2$,

$$f(t,x) := (x_2, \alpha(t)\sin(x_2))^T$$

and $B := \mathrm{diag}(1,1)$. Consider the following (admissible) reference control:

$$u(t) = (0, \cos(t) - \alpha(t)\sin(\sin(t)))^T$$

and the corresponding reference trajectory

$$x^u(t) = (-\cos(t), \sin(t))^T.$$

The linearized system (3) can now be written with the specific system matrix

$$A(t) \equiv f_x(t, x^u(t)) = \begin{pmatrix} 0 & 1 \\ 0 & \alpha(t)\cos(\sin(t)) \end{pmatrix}.$$

Note that $||A(\cdot)||$ is bounded (see Section 2). We consider here the diagonal gain matrix

$$K(t) = \mathrm{diag}(k_1(t), k_2(t))^T$$

for the control

$$v(t) = w(t, y(t)) = K(t)y(t)$$

associated with the corresponding linearized system (3). The closed-loop system (5) for (20) has the following form:

$$\begin{pmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \end{pmatrix} = \begin{pmatrix} k_1(t) & 1 \\ 0 & k_2(t)+\alpha(t)\cos(\sin(t)) \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} + \xi(t) \tag{21}$$

$$y_1(0) = y_2(0) = 0.$$

We now select a (diagonal) matrix

$$\hat{P} := \text{diag}(p_1, p_2),$$
$$p_2 > p_1 > 0$$

associated with (21). Application of the basic Theorem 4 makes it now possible to define the corresponding robust gain matrix $K(\cdot)$. One can use the main condition (13) from Theorem 4 as well as the simplified condition (18) for this purpose. Note that

$$||\hat{P}|| = p_2.$$

The main condition (13) implies the following matrix inequality:

$$\begin{pmatrix} p_1 k_1(t) + p_2 & p_1 \\ 0 & p_2 + p_2\left[k_2(t) + \alpha(t)\cos(\sin(t)) + 1\right] \end{pmatrix} \leq 0. \tag{22}$$

Condition (22) constitutes a "negative semidefiniteness" requirement for the resulting matrix. In that simple case, one can use the formal definition of a negative semidefinite matrix and derive the constructive expressions for elements $k_1(\cdot), k_2(\cdot)$ of the gain matrix $K(\cdot)$. Consider a nonzero vector $z \in \mathbb{R}^2$. The negative semidefiniteness concept in combination with a suitable negative majorant implies that

$$(p_1 k_1(t) + p_2) z_1^2 + p_1 z_1 z_2 + \left(p_2 + p_2\left[k_2(t) + \alpha(t)\cos(\sin(t)) + 1\right]\right) z_2^2 \leq -(q_1 z_1 - q_2 z_2)^2 < 0,$$

where z_1, z_2 are components of the vector $z \in \mathbb{R}^2$ and q_1, q_2 are some positive constants. We now put

$$q_1 = q_2 = \sqrt{p_1/2}$$

and obtain the particular conditions for $k_1(\cdot)$ and $k_2(\cdot)$, ie,

$$(p_1 k_1(t) + p_2) \leq -p_1/2,$$
$$\left(p_2 + p_2\left[k_2(t) + \alpha(t)\cos(\sin(t)) + 1\right]\right) \leq -p_1/2.$$

From the aforementioned inequalities, we immediately deduce

$$k_1(t) \leq -1/2 - p_2/p_1,$$
$$k_2(t) \leq -p_1/(2p_2) - 1 - \alpha(t)\cos(\sin(t))$$

and can finally put

$$\begin{aligned} k_1 &= -1/2 - p_2/p_1 - \varepsilon, \\ k_2(t) &= -p_1/(2p_2) - 1 - \alpha(t), \end{aligned} \tag{23}$$

where $\varepsilon > 0$. With the aims of a concrete calculation, we have selected here the following parameters:

$$\alpha(t) = 0.9 \times |\sin(t)| + e^{-6},$$
$$p_1 = 0.7927,\ p_2 = 0.8188,\ \varepsilon = 0.81297.$$

Following (14), we now are ready to define the resulting robust control design for the originally given nonlinear system (20)

$$u(t) = \begin{pmatrix} 0 \\ \cos(t) - \alpha(t)\sin(\sin(t)) \end{pmatrix} + K(t)y^{\nu}(t) = \begin{pmatrix} k_1 y_1^{\nu}(t) \\ \cos(t) - \alpha(t)\sin(\sin(t)) + k_2(t)y_1^{\nu}(t) \end{pmatrix}, \tag{24}$$

where k_1 and $k_2(\cdot)$ are calculated in (23) and $y^{\nu}(\cdot)$ is a solution to the closed-loop system (21) with the gain matrix $K(\cdot)$ determined by (23).

We now apply Theorem 5 and get the expected estimation (16) for the ellipsoid $\mathcal{E}_0$ of the originally given nonlinear system (20). Recall that the ellipsoid $\mathcal{E}$ of the linearized system (21) is determined by the aforementioned matrix $\hat{P}$. Since for the aforementioned reference trajectory $x^u(\cdot)$ of (20) we have

$$\|x^u(t)\| \leq \chi = 1,\ t \in \mathbb{R}_+,$$

we obtain the following (conservative) estimation of the ellipsoid $\mathcal{E}_0$ for the original system (20):

$$\mathcal{E}_0 = \left(\sqrt{k_1^2 + k_2^2} + 1\right)\mathcal{E} + 1,$$

where k_1 and $k_2(\cdot)$ are defined in (23) and $\mathcal{E}$ is an ellipsoid determined by matrix $\hat{P} = \text{diag}(p_1, p_2)$.

The corresponding simulation results are shown on Figure 2 and Figure 3.

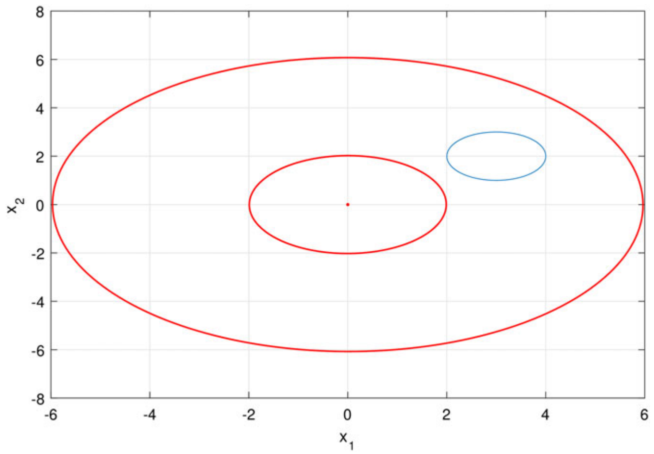

FIGURE 2 Phase plane dynamics and attractive ellipsoids $\mathcal{E}$ and $\mathcal{E}_0$ [Colour figure can be viewed at wileyonlinelibrary.com]

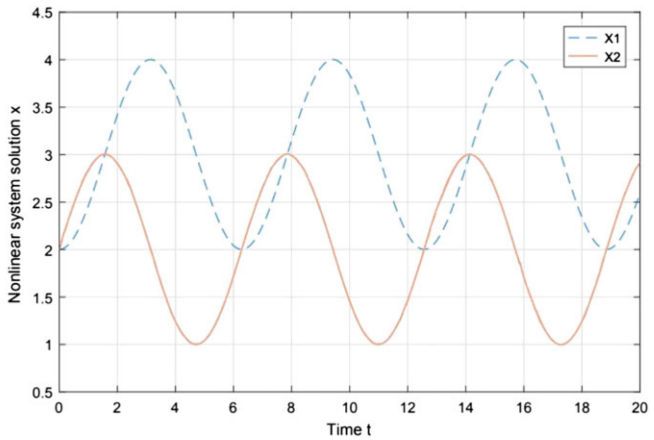

FIGURE 3 Dynamics of the closed-loop nonlinear system [Colour figure can be viewed at wileyonlinelibrary.com]

The (external) ellipsoid $\mathcal{E}_0$ for the originally given nonlinear system (20) contains the auxiliary (internal) ellipsoid $\mathcal{E}$ associated with the linearized system (21) (see Figure 2). The phase trajectory of the closed-loop nonlinear system is indicated in blue (Figure 2). The dynamic evolution of the closed-loop version of the original system (20) closed by the proposed robust feedback control (24) is illustrated on Figure 3. Let us finally note that the existence of a periodic solution (as depicted in Figure 2) of the closed-loop version of system (20) can be easily proved.

We next apply the proposed robust control design approach (14) to an engineering motivated example.

Example 2. Consider the mathematical model of a separately excited DC motor used in the electric wire manufacturing (see the works of Leonhard[36] and Poznyak et al[12])

$$\begin{aligned} J(t)\frac{dx_1(t)}{dt} &= cx_2(t)x_3(t) - Bx_1(t) - \xi(t), \\ L_r(t)\frac{dx_2(t)}{dt} &= U_r(t) - R_r x_2(t) - cx_1(t)x_3(t), \\ \frac{dx_3(t)}{dt} &= U_s(t) - R_s x_3(t). \end{aligned} \tag{25}$$

Here, x_1 is the angular velocity of the shaft. By x_2, we denote the current of the rotor circuit, R_r and R_s stand for the corresponding resistances. The rotor and stator voltages are expressed by U_r and U_s. The rotor inductance is assumed to be time-depending and is denoted by $L_r(t)$ and x_3 describes the stator flux. The mechanical parameters $J(\cdot)$ and B

in (20) express the time-depending moment of inertia of the rotor and the viscous friction coefficient, respectively. Finally, c represents a constant parameter that depends on the spatial architecture of the drive. We also assume that

$$\sup_{t\in\mathbb{R}_+} ||\xi(t)|| \le 0.1.$$

The control variables in (25) are the aforementioned voltages U_r and U_s. Note that the increment of the moment of inertia associated with the rotor in the electric wire (coil) production can be modeled by a suitable linear function, ie, $J(t) = g_1 t + J_0$, where $g_1 > 0$ and $J_0 > 0$ is the proper moment of inertia of the rotor. The temperature gradient of the running motor causes the corresponding change (increment) of the rotor inductivity $L_r(t)$. This augmentation can be modeled (for a restricted time-range) as follows: $L_r(t) = g_2 t + L_r^0$. In this example, we select the following model parameters:

$$J_0 = 0.02 \ (\mathrm{kg} \times \mathrm{m}^2),$$
$$g_1 = 10^{-3} \ (\mathrm{kg} \times \mathrm{m}^2/\,\mathrm{sec}),$$
$$B = 0.23 \ (\mathrm{N} \times \mathrm{m} \times \mathrm{sec}).$$

Moreover, we also put $L_r^0 = 10^{-3}$ (H), $g_2 = 10^{-3}(H/\mathrm{sec})$, $c = 1$. The set of admissible initial conditions in (20) was chosen as follows:

$$x_1^0 = 0,\ x_2^0 = 1,\ x_3^0 = 1.$$

The originally given nonlinearly-affine control system (25) implies the corresponding linearized model of the type (3)

$$\begin{aligned}
\dot{y}_1 &= -\frac{B}{J(t)}y_1 + \frac{cx_3^{\mathrm{ref}}(t)}{J(t)}y_2 + \frac{cx_2^{\mathrm{ref}}(t)}{J(t)}y_3 - \hat{\xi}(t),\\
\dot{y}_2 &= -\frac{cx_3^{\mathrm{ref}}(t)}{L_r(t)}y_1 - \frac{R}{L_r(t)}y_2 - \frac{cx_1^{\mathrm{ref}}(t)}{L_r(t)}y_3 + \frac{1}{L_r(t)}v_1(t),\\
\dot{y}_3 &= -R_s y_3 + v_2(t),
\end{aligned} \tag{26}$$

where $y := (y_1, y_2, y_3)^T$ is a state vector of the linearized model and $\hat{\xi}(t) = \frac{1}{J(t)}\xi(t)$. The linear state space model (26) is characterized by the following matrices:

$$A(t) = \begin{bmatrix} -\frac{B}{J(t)} & \frac{cx_3^{\mathrm{ref}}}{J(t)} & \frac{cx_2^{\mathrm{ref}}}{J(t)} \\ -\frac{cx_3^{\mathrm{ref}}}{L_r(t)} & -\frac{R_r}{L_r(t)} & -\frac{cx_1^{\mathrm{ref}}}{L_r(t)} \\ 0 & 0 & -R_s \end{bmatrix}, \quad B(t) = \begin{bmatrix} 0 & 0 \\ \frac{1}{L_r(t)} & 0 \\ 0 & 1 \end{bmatrix}.$$

Let us consider the following reference control (taking for simplicity $R_s = 1, R_r = 1$):

$$U_s(t) = R_s x_3(t) = 1,$$
$$U_r(t) = R_r x_2(t) + cx_1(t)x_3(t) = 2 - e^{-t}.$$

The corresponding reference trajectory can be easily evaluated

$$x_1 = 1 - e^{-t},\ x_2 = 1,\ x_3 = 1.$$

The obtained simulation results are now shown on Figure 4 and Figure 5. The ellipsoid $\mathcal{E}_0$ for the originally given nonlinear system (25) is presented in red (Figure 4).

The auxiliary ellipsoid $\mathcal{E}$ associated with the linearized system (26) is indicated in blue. We also depict components of the trajectory of (25) closed by the control of the type (14) (see Figure 5).

The numerical results obtained in Examples 1 and 2 illustrate the effectiveness and implementability of the robust control methodology we developed.

5.2 | On a robust control approach to a class of general linear systems

As one can conclude from Section 4, the robust control design for the originally given uncertain dynamic system (1) involves a (more simple) robust control strategy for the linearized system (5) of the type $w(t, y^v(t)) = K(t)y^v(t)$. At the same

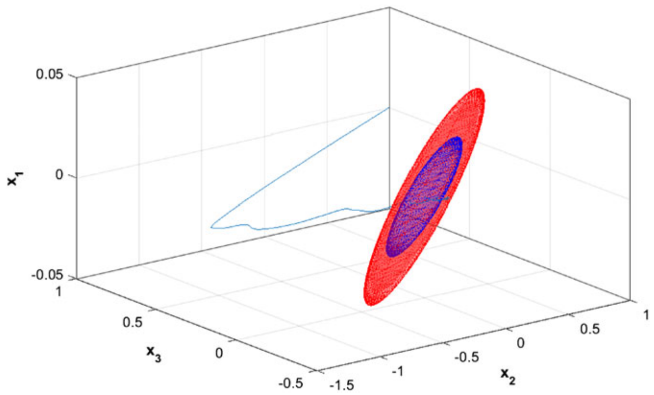

FIGURE 4 Phase plane dynamics and attractive ellipsoids $\mathcal{E}$ and $\mathcal{E}_0$ [Colour figure can be viewed at wileyonlinelibrary.com]

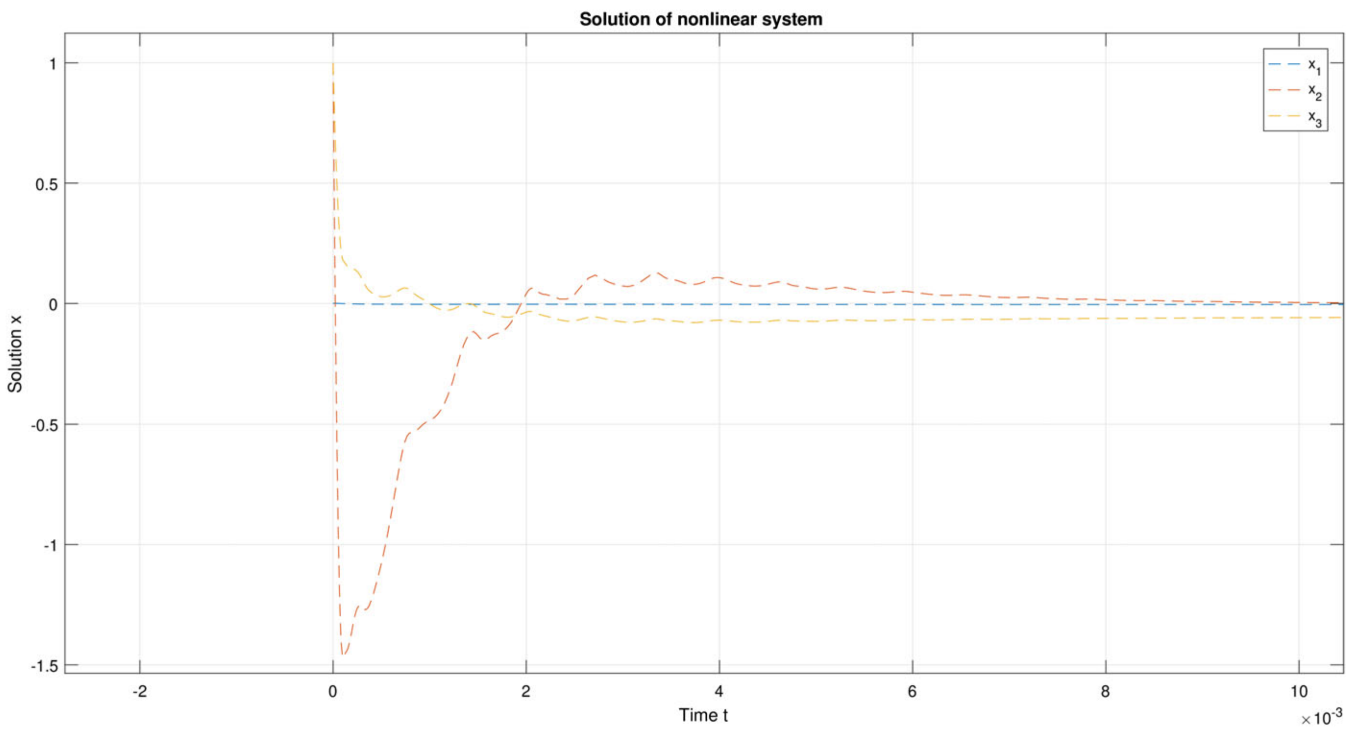

FIGURE 5 Dynamics of the closed-loop nonlinear system [Colour figure can be viewed at wileyonlinelibrary.com]

time, the main theoretical result we obtained, namely, Theorem 5, does not depend on a specific design method for the gain matrix $K(\cdot)$. In this section, we propose an effective method for the evaluation of this necessary "robust" gain matrix $K(\cdot)$ for (5). Recall that the aforementioned gain matrix is next used in the control design procedure for the originally given nonlinear system.

Consider a general case of a linear system (5) with

$$A(t) \in co\{A_1, A_2, \dots, A_N\}, \; B(t) \in co\{B_1, B_2, \dots, B_N\}. \tag{27}$$

Evidently,

$$A(t) = \sum_{i=1}^{N} \lambda_i(t) A_i, \quad B(t) = \sum_{i=1}^{N} \lambda_i(t) B_i,$$

where $\lambda_i(t) > 0,\ t \in \mathbb{R}_+$ and $\sum_{i=1}^{N} \lambda_i(t) = 1$. Taking into consideration the aforementioned framework, it is possible to propose a gain matrix $K(\cdot)$ (generated by a combination of some convex functions) such that the solution of the resulting closed-loop system converges to a minimal size set generated by the convex hull of some ellipsoids. We refer to the work of Hu and Lin[37] for technical details. We now extend the main result from the aforementioned work[37] to the time-varying case determined earlier.

Theorem 6. *Assume there exist a positive scalar α, some positive definite symmetric matrices $P_i \in \mathbb{R}^{n\times n}$, matrices $Y_i \in \mathbb{R}^{m\times n}$, and positive semidefinite continuous functions $\gamma_i(t)$, for $j = 1, \dots, N$ with*

$$\sum_{j=1}^{N}\gamma_j(t) = 1, \quad \bar{\gamma}_D := \sup\left\{|\dot{\gamma}_j(t)| \,:\, t \geq 0,\ j = 1, \dots, N\right\}.$$

Consider (5), where the system and control matrices are determined by (27) such that

$$\sum_{i=1}^{N}\lambda_i(t)\left(\sum_{j=1}^{N}\gamma_j(t)\left(A_iP_j^{-1} + P_j^{-1}A_i^T + B_iY_j + Y_j^TB_i^T + \alpha P_j^{-1}\right) + \overline{M}\right) < 0, \tag{28}$$

and

$$\overline{M} = \bar{\gamma}_D\sum_{j=1}^{N}P_j^{-1} + \alpha^{-1}MI_{n\times n}.$$

Then, the following set

$$\mathcal{E}\left(Q(t)\right) := \left\{x(t) \in \mathbb{R}^n \,:\, x^T(t)Q(t)x(t) \leq 1,\ \forall t \geq 0,\ \gamma \in \Gamma\right\},$$

$Q(t) := \left(\sum_{i=j}^{N}\gamma_j(t)P_j^{-1}\right)^{-1}$ *is an invariant set for (5) with the control input $u(t) = K(t)x(t)$, where*

$$K(t) := \left(\sum_{j=1}^{N}\gamma_j(t)Y_i\right)\left(\sum_{j=1}^{N}\gamma_j(t)P_j^{-1}\right)^{-1}.$$

Proof. Introduce $V(x) = x^T(t)Q(t)x(t)$ as the Lyapunov candidate. Calculating the corresponding time derivative, we obtain

$$\dot{V}(x) = 2x^T(t)Q(t)\left(A(t) + B(t)K(t)\right)x(t) + 2x^T(t)Q(t)\xi(t) + 2x^T(t)\dot{Q}(t)x(t). \tag{29}$$

We now add and subtract the terms $\frac{\alpha}{M}\xi^T(t)\xi(t)$ and $\alpha x^T(t)Q(t)x(t)$ and introduce the extended vector $\eta(t) = [\,x^T(t),\ \xi^T(t)\,]^T$. Equation (29) can now be expressed as follows:

$$\dot{V}(x) = \eta^T(t)\Omega(t)\eta(t) + \frac{\alpha}{M}\xi^T(t)\xi(t) - \alpha x^T(t)Q(t)x(t),$$

with

$$\Omega(t) = \begin{bmatrix} \begin{matrix} Q(t)A(t) + A^T(t)Q(t) + Q(t)B(t)K(t)+ \\ K^T(t)B^T(t)Q(t) + \alpha Q(t) + \dot{Q}(t) \end{matrix} & Q(t) \\ Q(t) & -\frac{\alpha}{M}I_{n\times n} \end{bmatrix},$$

and

$$P(t) = Q^{-1}(t),\ Y(t) = K(t)Q^{-1}(t).$$

Note that

$$\dot{Q}(t) = -Q(t)\dot{P}(t)Q(t),$$

and pre and post multiplying $\Omega(t)$ by $\mathrm{diag}\left[\,P(t),\ I_{n\times n}\,\right]$ involves the following expression:

$$\overline{\Omega}(t) = \begin{bmatrix} \begin{matrix} A(t)P(t) + P(t)A^T(t) + B(t)Y(t)+ \\ Y^T(t)B^T(t) + \alpha P(t) - \dot{P}(t) \end{matrix} & I_{n\times n} \\ I_{n\times n} & -\frac{\alpha}{M}I_{n\times n} \end{bmatrix}.$$

We now evaluate an upper bound for $-\dot{P}(t)$, ie,

$$-\dot{P}(t) = -\sum_{j=1}^{N}\dot{\gamma}(t)P_j^{-1} \leq \bar{\gamma}_D\sum_{j=1}^{N}P_j^{-1}.$$

Using this estimation, we next obtain the upper bound for the derivative of $V(x)$, ie,

$$\dot{V}(x) \leq x^T(t)\Omega_0(t)x(t) + \alpha\left(1 - x^T(t)Q(t)x(t)\right), \tag{30}$$

$$\Omega_0(t) = \begin{bmatrix} \begin{array}{c} A(t)P(t) + P(t)A^T(t) + B(t)Y(t)+ \\ Y^T(t)B^T(t) + \alpha P(t) + \bar{\gamma}_D \sum_{j=1}^{N} P_j^{-1} \end{array} & I_{n\times n} \\ I_{n\times n} & -\frac{\alpha}{M} I_{n\times n} \end{bmatrix}.$$

Applying the celebrated Schur complement, we obtain inequality (28).

From (28) and taking into consideration conditions $\Omega_0 < 0$ and

$$\dot{V}(x) \leq \alpha\left(1 - x^T(t)Q(t)x(t)\right),$$

we obtain $\dot{V}(x) < 0$ for $x^T(t)Q(t)x(t) > 1$. The proof is completed. □

Let us finally illustrate the robust design of the gain matrix $K(\cdot)$ proposed in Theorem 6 and consider a simple example.

Example 3. Consider (5) with the system and control matrices $A(t)$, $B(t)$ in (27) determined by

$$A_1 = \begin{bmatrix} -4 & 4 \\ -5 & 0 \end{bmatrix}, \; A_2 = \begin{bmatrix} -6 & 6 \\ -6 & 4 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.1 \\ -0.5 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0.5 \\ -0.8 \end{bmatrix}.$$

The generic functions $\lambda(\cdot)$ is selected as follows:

$$\lambda_1(t) = 0.5 + 0.5\sin(10t), \;\; \lambda_2(t) = 0.5 - 0.5\sin(10t).$$

We next apply Theorem 6 and obtain the following matrices:

$$P_1^{-1} = \begin{bmatrix} 0.4841 & 0.2386 \\ 0.2386 & 1.4052 \end{bmatrix}, \; P_2^{-1} = \begin{bmatrix} 0.7504 & 0.0380 \\ 0.0380 & 1.2247 \end{bmatrix}$$

$$Y_1 = \begin{bmatrix} 4.2658 & -1.3673 \end{bmatrix}, \; Y_2 = \begin{bmatrix} 7.5237 & 6.4755 \end{bmatrix}.$$

Figure 6 shows the dynamic system solutions for the initial condition $x(0) = [\,4,\ -4\,]^T$ as well as the ellipsoids represented by the matrices P_1^{-1} (red) and P_2^{-1} (in yellow). The corresponding time-dependent evolution of the trajectory is presented on Figure 7.

The presented Example 3 also shows that the control design methodology we propose easily interacts with the classic Lyapunov-based techniques.

Let us now make some general observation related to the computational aspects of the proposed robust control design. With the exception of some easy cases, one always has to consider a suitable time discretization (a time grid) for a

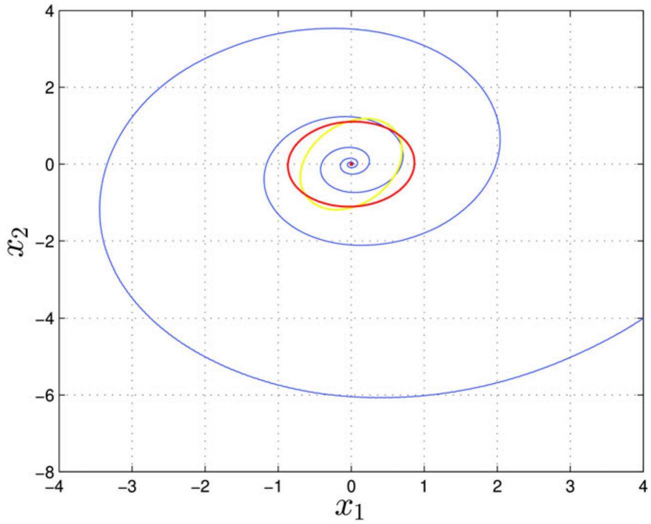

FIGURE 6 Phase plane dynamics and attractive ellipsoids for P_1^{-1} and P_2^{-1} [Colour figure can be viewed at wileyonlinelibrary.com]

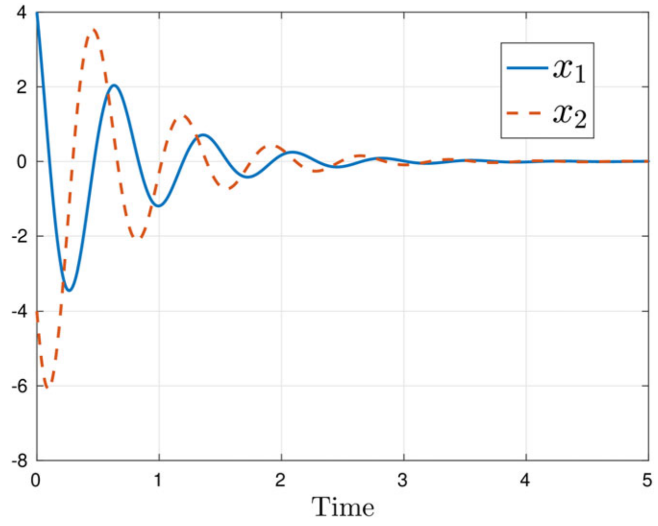

FIGURE 7 Dynamics of the closed-loop nonlinear system [Colour figure can be viewed at wileyonlinelibrary.com]

consistent numerical treatment of the nonstationary continuous time dynamics. Similar to many well-established robust control design methods for the nonstationary systems, the proposed control strategy (based on the main LMI (13)) implies a relative big computational effort. However, from the computational point of view, the control approach proposed in this paper has the following decisive advantages.

- The dimension of the main LMI (13) is equal to $(n \times n)$, where n is the dimension of the initially given control system (1). Recall that, even in the case of a stationary system, many LMI-based robust control design strategies use LMIs of an increased dimension $(Mn) \times (Mn)$, where $M \in \mathbb{N}$ (see, eg, the work of Poznyak et al[13] and the references therein). This fact makes it very difficult a practical application of the conventional LMI-based methods (with the high-dimensional LMIs) to the nonstationary control systems studied in our paper.
- The method proposed in this paper is an a priori control design approach. One can proceed with all the necessary calculations before the functioning of the designed system.
- Under some weak assumptions (see Section 4), the basic LMI (13) can be reduced to the simplified LMI (18).

Finally, note that the implementation of the extended AE-based computational techniques proposed in this paper was carried out, using the standard MATLAB packages.

6 | CONCLUDING REMARKS

Our paper generalizes the conventional AEM-based robust control design for a wide class of control-affine systems. We have studied here an important class of nonstationary dynamic models with an affine structure and proposed a constructive approach to the AEM-involved robust stability methodology for nonstationary dynamic systems. The proposed practical stabilization method includes two main steps. We firstly study an associated piecewise-linear (linearized) control system, apply the conventional AEM, and design the corresponding ellipsoidal invariant set. The generic feedback type control strategy obtained for the aforementioned linearized system is next directly used in the control design procedure for the original switched system. The two-steps approach we developed makes it possible to obtain an a priori constructive estimation of the ellipsoidal invariant set for the originally given nonlinear dynamic systems using the LMI-based control strategy for the linearized systems.

Let us note that various approaches to the robust control design of the nonstationary dynamic systems have been established (see other works[13,24,38] for a comprehensive overview). We shortly discuss here some of these classic robust control ideas in comparison with the advanced AEM we developed.

It is common knowledge that the classic and extended Lyapunov stability approaches constitute a generic theoretical basis for the robust control of various dynamic systems.[1,27,38,39] However, the conceptual limitations of the methods involving the Lyapunov functions are also well known and does not exist a general method for the Lyapunov function design for a nonstationary closed-loop realization of system (1). Let us also refer to the celebrated (relatively sophisticated)

Lyapunov-Krasovskii functional method in that connection.[40] In contrast to the general Lyapunov-based approach, the extended AEM we propose involves, in fact, some relative simple LMIs, namely, inequalities (13) and (18). A solution of these LMIs guarantees robustness for a trajectory of the corresponding closed-loop system. Robust control design using H-infinity theory constitutes one of the most traditional robust control approaches (see, eg, the works of Li et al[41] and Kwakernaak[42]). On the other hand, the H-infinity methodology is developed under the restrictive assumption of the decreasing (over time) uncertainties. Evidently, this is not the case in many real-world engineering systems. The advanced AEM robust control scheme for system (1) does not involve any similar restrictive uncertainty characterization.

The robust control design based on the LQ type optimal control was proposed in the work of Otsuki and Yoshida.[43] However, it is evident that the LQ optimal control techniques for robust control are restricted to very specific (linear) classes of dynamic systems and do not include general control-affine systems (1). We also refer to the work of Lin[44] for optimal control techniques in robust control. Recently, many LMI-based techniques are proposed for a robust control design of the nonstationary systems. Let us refer to the works of Boyd et al[25] and Daher and Stoustrup[45] for a comprehensive overview on this subject. However, as mentioned in Section 5, applications of the LMI-based control design techniques to the nonstationary systems involve numerical solutions of the high-dimensional LMI constrained optimization problems at every step of a time grid. Therefore, the control approach presented in this paper (the n-dimensional LMIs (13) and (18)) constitutes a favorable and numerically tractable solution procedure.

Recall that the control design approaches based on the traditional sliding mode control technologies and on the high-order extensions are nowadays a mature methodology for the constructive synthesis of several types of robust controllers for the control-affine systems (1) (see the work of Shtessel et al[46]). However, the sliding mode control strategy can formally be implemented assuming a hard system dimensionality restriction (see, eg, the work of Shtessel et al[46]). Moreover, the control techniques based on the sliding mode methodology usually imply a high frequency system chattering. This fact makes it impossible any application of the conventional and high-order sliding mode control strategies to a broad class of dynamic models, for example, to some sensitive mechanical systems. Generalization of the AEM approach proposed in this paper is free from this negative ancillary effect and, moreover, it can be applied to a vide class of control-affine dynamic systems (without any dimensionality restriction).

The presented extension of the classic AEM in combination with the generic linearization technique constitutes a novel analytic foundation for novel computational approaches in the robust control design of control-affine systems. We are convinced that the proposed "overestimations" of AEs associated with the closed-loop system can be applied to various alternative typos of "nonstandard" control processes. We expect that the proposed approach can guarantee an adequate robust control design for implicit systems (see the work of Azhmyakov et al[5]), to dynamic models with piecewise constant control inputs (see the works of Azhmyakov et al[15,35]), to systems evolving with state suprema (see, eg, the work of Azhmyakov et al[17]), and to the various types of hybrid and switched control systems.[23,47-50]

ORCID

V. Azhmyakov https://orcid.org/0000-0003-3634-6786

REFERENCES

1. Khalil HK. *Nonlinear Systems*. Upper Saddle River, NJ: Prentice Hall; 2002.
2. Alazki J, Poznyak A. Robust output stabilization for a class of nonlinear uncertain stochastic systems under multiplicative and additive noises: the attractive ellipsoid method. *J Ind Manag Optim*. 2016;12(1):169-186.
3. Azhmyakov V. On the geometric aspects of the invariant ellipsoid method: application to the robust control design. In: Proceedings of the 50th IEEE Conference on Decision and Control and European Control Conference; 2011; Orlando, FL.
4. Azhmyakov V, Poznyak A, Gonzalez O. On the robust control design for a class of nonlinearly affine control systems: the attractive ellipsoid approach. *J Ind Manag Optim*. 2013;9(3):579-593.
5. Azhmyakov V, Poznyak A, Juárez R. On the practical stability of control processes governed by implicit differential equations: the invariant ellipsoid based approach. *J Franklin Inst*. 2013;350(8):2229-2243.
6. González O, Poznyak A, Azhmyakov V. On the robust control design for a class of nonlinear affine control systems: the invariant ellipsoid approach. In: Proceedings of the 6th International Conference on Electrical Engineering, Computing Science and Automatic Control (CCE); 2009; Toluca, Mexico.
7. Mera M, Castaños F, Poznyak A. Quantised and sampled output feedback for nonlinear systems. *Int J Control*. 2014;87(12):2475-2487.
8. Mera M, Polyakov A, Perruquetti W, Zheng G. Finite-time attractive ellipsoid method using implicit Lyapunov functions. In: Proceedings of the 54th Conference on Decision and Control (CDC); 2015; Osaka, Japan.

9. Polyakov A, Poznyak A. Lyapunov function design for finite-time convergence analysis: "twisting" controller for second-order sliding mode realization. *Automatica*. 2009;45(2):444-448.
10. Polyakov A. Minimization of disturbances effects in time delay predictor-based sliding mode control systems. *J Franklin Inst*. 2012;349(4):1380-1396.
11. Poznyak AS. *Advanced Mathematical Tools for Automatic Control Engineers: Deterministic Techniques*. Vol 1. Amsterdam, The Netherlands: Elsevier; 2008.
12. Poznyak A, Azhmyakov V, Mera M. Practical output feedback stabilisation for a class of continuous-time dynamic systems under sample-dada outputs. *Int J Control*. 2011;84(8):1408-1416.
13. Poznyak A, Polyakov A, Azhmyakov V. *Attractive Ellipsoids in Robust Control*. Basel, Switzerland: Springer International Publishing Switzerland; 2014.
14. Poznyak T, Chairez I, Perez C, Poznyak A. Switching robust control for ozone generators using the attractive ellipsoid method. *ISA Trans*. 2014;53(6):1796-1806.
15. Azhmyakov V, Cabrera Martinez J, Poznyak A, Serrezuela RR. Optimization of a class of nonlinear switched systems with fixed-levels control inputs. In: Proceedings of the 2015 American Control Conference (ACC); 2015; Chicago, IL.
16. Clarke FH, Ledyaev YS, Stern RJ, Wolenski PR. *Nonsmooth Analysis and Control Theory*. New York, NY: Springer-Verlag New york; 1998.
17. Azhmyakov V, Ahmed A, Verriest EI. On the optimal control of systems evolving with state suprema. In: Proceedings of the 55th IEEE Conference on Decision and Control (CDC); 2016; Las Vegas, NV.
18. Bonilla M, Malabre M, Azhmyakov V. Decoupling of internal variable structure for a class of switched systems. In: Proceedings of the 2015 European Control Conference (ECC); 2015; Linz, Austria.
19. Bonilla M, Malabre M, Azhmyakov V. An implicit systems characterization of a class of impulsive linear switched control processes. Part 1: modeling. *Nonlinear Anal Hybrid Syst*. 2015;15:157-170.
20. Zubov VI. *Mathematical Methods for the Study of Automatic Control Systems*. New York, NY: Pergamon Press; 1962.
21. Rockafellar RT, Wets RJ-B. *Variational Analysis*. Berlin, Germany: Springer-Verlag Berlin Heidelberg; 1998.
22. Hale JK, Lunel SMV. *Introduction to Functional Differential Equations*. New York, NY: Springer Science+Business Media New York; 1993.
23. Azhmyakov V, Juarez R. On the projected gradient methods for switched-mode systems optimization. *IFAC-Pap*. 2015;48(27):181-186.
24. Blanchini F, Miani S. *Set-Theoretic Methods in Control*. Basel, Switzerland: Birkhäuser; 2008.
25. Boyd S, Ghaoui LE, Feron E, Balakrishnan V. *Linear Matrix Inequalities in System and Control Theory*. Philadelphia, PA: Society for Industrial and Applied Mathematics; 1994.
26. Dahleh MA, Pearson JB. Optimal rejection of persistent disturbances, robust stability, and mixed sensitivity minimization. *IEEE Trans Autom Control*. 1988;33(8):722-731.
27. Haddad W, Chellaboina V. *Nonlinear Dynamical Systems and Control: a Lyapunov-Based Approach*. Princeton, NJ: Princeton University Press; 2008.
28. Kurzhanski AB, Varaiya P. Ellipsoidal techniques for reachability under state constraints. *SIAM J Control Optim*. 2006;45(4):1369-1394.
29. Kurzhanski AB, Veliov VM. *Modeling Techniques for Uncertain Systems*. Basel, Switzerland: Birkhäuser; 1994.
30. Michel AN, Hou L, Liu D. *Stability of Dynamical Systems: Continuous, Discontinuous, and Discrete Systems*. New York, NY: Birkhäuser; 2007.
31. Polyak BT, Nazin SA, Durieu C, Walter E. Ellipsoidal parameter or state estimation under model uncertainty. *Automatica*. 2004;40(7):1171-1179.
32. Polyak BT, Topunov MV. Suppression of bounded exogenous disturbances: output control. *Autom Remote Control*. 2008;69(5):801-818.
33. Aliprantis CD, Border KC. *Infinite Dimensional Analysis: A Hitchhiker's Guide*. Berlin, Germany: Springer-Verlag Berlin Heidelberg; 2006.
34. Glover J, Schweppe F. Control of linear dynamic systems with set constrained disturbances. *IEEE Trans Autom Control*. 1971;16(5):411-423.
35. Azhmyakov V, Basin M, Reincke-Collon C. Optimal LQ-type switched control design for a class of linear systems with piecewise constant inputs. In: Proceedings of the 19th IFAC World Congress; 2014; Cape Town, South Africa.
36. Leonhard W. *Control of Electrical Drives*. Berlin, Germany: Springer-Verlag Berlin Heidelberg; 1996.
37. Hu T, Lin Z. Composite quadratic Lyapunov functions for constrained control systems. *IEEE Trans Autom Control*. 2003;48(3):440-450.
38. Qu Z. *Robust Control of Nonlinear Uncertain Systems*. New York, NY: John Wiley & Sons; 1998.
39. Freeman RA, Kokotović P. Robust control Lyapunov functions. In: *Robust Nonlinear Control Design: State-Space and Lyapunov Techniques*. Boston, MA: Birkhäuser; 2008:33-63.
40. Kharitonov VL, Zhabko AP. Lyapunov–Krasovskii approach to the robust stability analysis of time-delay systems. *Automatica*. 2003;39(1):15-20.
41. Li XP, Chang BC, Banda S, Yeh H. Robust control systems design using H-infinity optimization theory. *J Guid Control Dyn*. 1992;15(4):944-952.
42. Kwakernaak H. Robust control and *H*-infinity optimization—tutorial paper. *Automatica*. 1993;29(2):255-273.
43. Otsuki M, Yoshida K. Nonstationary robust control for time-varying system. In: Proceedings of the 2004 American Control Conference; 2004; Boston, MA.
44. Lin F. *Robust Control Design: An Optimal Control Approach*. Chichester, UK: John Wiley & Sons; 2007.
45. Daher Adegas F, Stoustrup J. Robust structured control design via LMI optimization. *IFAC Proc Vol*. 2011;44(1):7933-7938.

46. Shtessel Y, Edwards C, Fridman L, Levant A. *Sliding Mode Control and Observation*. New York, NY: Birkhäuser; 2013.
47. Liberzon D. *Switching in Systems and Control*. Boston, MA: Birkhäuser; 2003.
48. Lygeros J. *Lecture Notes on Hybrid Systems*. Cambridge, UK: Cambridge University Press; 2003.
49. Shaikh MS, Caines PE. On the hybrid optimal control problem: theory and algorithms. *IEEE Trans Autom Control*. 2007;52(9):1587-1603.
50. Wardi Y, Egerstedt M, Twu P. A controlled-precision algorithm for mode-switching optimization. In: Proceedings of the 51st IEEE Conference on Decision and Control (CDC); 2012; Maui, HI.

How to cite this article: Azhmyakov V, Mera M, Juárez R. Advances in attractive ellipsoid method for robust control design. *Int J Robust Nonlinear Control*. 2018;1–19. https://doi.org/10.1002/rnc.4446