Available online at www.sciencedirect.com
ScienceDirect
IFAC PapersOnLine 51-25 (2018) 18–23

Robust Optimal Control of Linear-Type Dynamic Systems with Random Delays

Vadim Azhmyakov[1], **Erik I. Verriest**[2], **Luz A. Guzman Trujillo**[1,3], **Sebastien Lahaye**[3], **Nicolas Delanoue**[3]

1 Department of Basic Sciences, Universidad de Medellin, Medellin, Colombia (e-mail: vazhmyakov@udem.edu.co; jpfernandez@udem.edu.co)
2 School of Electrical and Computer Engineering, Georgia Institute of Technology, Atlanta, GA-30332 USA (e-mail: erik.verriest@ece.gatech.edu)
3 School for Engineers in Sciences and Technology, University of Angers, Angers, France (luzadriguz@gmail.com)

Abstract: Our contribution deals with a class of Optimal Control Problems (OCPs) of dynamic systems with randomly varying time delays. We study the minimax-type OCPs associated with a family of delayed differential equations. The presented minimax dynamic optimization has a natural interpretation as a robustness (in optimization) with respect to the possible delays in control system under consideration. A specific structure of a delayed model makes it possible to reduce the originally given sophisticated OCP to an equivalent convex program in an Euclidean space. This analytic transformation implies a possibility to derive the necessary and sufficient optimality conditions for the original OCP. Moreover, it also allows consideration of the wide range of effective numerical procedures for the constructive treatment of the obtained convex-like OCP. The concrete computational methodology we follow in this paper involves a gradient projected algorithm. We give a rigorous formal analysis of the proposed solution approach and establish the necessary numerical consistence properties of the resulting robust optimization algorithm.

1. INTRODUCTION

Theoretical approaches to robust optimization of sophisticated dynamic models and the resulting numerical schemes constitute a challenging and important part of the modern systems optimization methodology (5; 6; 7; 8; 15; 20; 21; 28; 31; 33; 35; 42). The minimax based optimal control (we discuss in this paper) is one possible modelling framework for a robust optimal systems design (17). Let also refer to (16) for some modern robust minimax OCPs and further analytic results. The minimax optimal control techniques can usually be interpreted as an engineering optimization under the "worst case" scenarios. Evidently, the possible "un-friendly" scenarios are extremely important for a resulting safety of a dynamic system under consideration and also for the robust software oriented development of several types of modern engineering systems.

Our paper deals with a particular class of control systems, namely, with the delayed models described by differential equations. We study systems with random delays and the related OCPs. Let us note that differential equations with stochastically varying time delays constitute an adequate and widely used modelling approach to the genetic regulatory networks (see (23)). In general control systems determined by delay differential equations describe a wide range of real-world problems in engineering, economy, social and bioscience (see e.g., (1; 2; 9; 11; 12; 24; 25; 26; 29; 36; 38; 39; 40; 41)). Various classes of OCPs associated with sophisticated dynamic systems have been comprehensively studied due to their natural engineering applications (see (8; 11; 32) and References therein). Let us mention here some notable applications from the mobile robot technology, automotive and aerospace control, electrical engineering and telecommunications. We generally refer to (15) for various examples and real-world applications of the optimized control systems. On the other side analytic and computational techniques for OCPs are not sufficiently advanced to optimal control processes governed by delayed differential equations with random delays. Let us mention some particular results for OCPs involved specific classes of delayed dynamic systems (12; 29). The aim of our contribution is to develop a self-contained and relatively simple numerical approach to the minimax type OCPs associated with a class of linear systems with stochastic delays. We use the convex analysis approach for this purpose and propose a specific gradient-projected method for a constructive numerical treatment of the OCP involved closed-loop delay systems. We follow here the feedback control methodology, give a rigorous convexity proof for the resulting OCP and establish the numerical consistence (numerical stability) of the proposed numerical algorithm.

The remainder of our paper is organized as follows: Section 2 contains a formal problem statement and some necessary theoretic concepts. In this section we give a practical motivation and interpretation of the main optimization problem in the framework of the minimax type systems

2405-8963 © 2018, IFAC (International Federation of Automatic Control) Hosting by Elsevier Ltd. All rights reserved.
Peer review under responsibility of International Federation of Automatic Control.
10.1016/j.ifacol.2018.11.075

robustness. In Section 3 we prove our main convexity result, namely, Theorem 1 and give a constructive characterization of the obtained auxiliary problems. Section 4 deals with the gradient based solution scheme applied to the closed-loop realisation of the delay system. We use our main theoretic results and establish the numerical consistency of the proposed algorithm (Theorem 3). The finally obtained computational algorithm guarantees an effective numerical treatment of the initially given sophisticated OCP. Section 5 summarizes our paper.

2. PROBLEM FORMULATION AND ROBUSTNESS WITH RESPECT TO DELAYS

In this paper we study the following initial-value problem for differential equations with delays

$$\begin{aligned} &\dot{x}(t) = A_1 x(t) + A_2 x(t-\tau) + Bv(t) \quad \forall t \in [0, t_f], \\ &x(t) = \phi(t) \quad \forall t \in [-\tau_{\max}, 0], \end{aligned} \tag{1}$$

where $t_f \in \mathbb{R}_+$ and $A_1, A_2 \in \mathbb{R}^{n\times n}$ are given systems matrices and τ is a random delay. More precisely we assume that the delay τ constitutes a random value with a given probability distribution $\mathcal{F}(\tau),\ \tau \in [\tau_{\min}, \tau_{\max}]$, where $\tau_{\min} \geq 0,\ \tau_{\max} > 0$. In this paper we assume that the delay τ may only take M possible discrete values from the set $D := \{\tau_1, \tau_2, ..., \tau_M\}$. The corresponding probability can be calculated as follows

$$P[\tau = \tau_i] = w_i,\ i = 1, ..., M,\ \sum_{i=1}^{M} w_i = 1.$$

Without loss of generality we assume

$$\tau_{\min} = \tau_1 < \tau_2 < ... < \tau_M = \tau_{\max}.$$

The corresponding probability density function $\rho(\cdot)$ has the generic expression

$$\rho(\tau) = \sum_{i=1}^{M} w_i \delta(\tau - \tau_i),\ \int_{\tau_{\min}}^{\tau_{\max}} \rho(\tau) d\tau = 1$$

Here $\delta(\cdot)$ denotes the Dirac delta. As mentioned in Section 1 delayed dynamic systems of the type (1) with the specific discrete-valued stochastic delays constitute adequate modelling framework for some protein production processes (23). One uses the dynamic model of the type (1) for control of a genetic regulatory network that correspond to the automated protein production process.

Let $\mathbb{C}(-\tau_{\max}, 0)$ be the Banach space of all continuous functions $y(\cdot)$ from the interval $[-\tau_{\max}, 0]$ into $\mathbb{R}^n$ equipped with the usual norm

$$||y(\cdot)||_{\mathbb{C}(-\tau_{\max},0)} := \max_{s\in[-\tau_{\max},0]} ||y(s)||_{\mathbb{R}^n}.$$

Assume that $\phi(\cdot) \in \mathbb{C}(-\tau_{\max}, 0)$. We call a function $x(\cdot)$ a solution (or trajectory) of (1), if $x : [\tau_{\max}, t_f) \to \mathbb{R}^n$ is absolutely continuous and satisfies (1) for all time instants $t \in [-\tau_{\max}, t_f),\ \tau \in D$.

An admissible control input $v(\cdot)$ in (1) is assumed to be of a proportional feedback type. Taking into consideration the usual delays in the "state-controller channel" of a dynamic system with the feedback-type control, we next consider the delayed feedback control function

$$v(t) = u(x(t-\tau))$$

such that

$$u(x(t-\tau)) = Kx(t-\tau).$$

Here $K \in \mathbb{R}^{m\times n}$ is a bounded control gain matrix with the properties

$$||K||_{\mathbb{R}^{m\times n}} \leq k_{\max} \in \mathbb{R}_+.$$

By $||\cdot||_{\mathbb{R}^{m\times n}}$ we denote here the usual operator norm (induced norm) on the Euclidean space $\mathbb{R}^{m\times n}$). Note that the above boundeness condition expresses the natural restrictions of the technical resources of the usual feedback-type controllers. The resulting closed-loop version of system (1) has the following simple form

$$\begin{aligned} &\dot{x}(t) = A_1 x(t) + (A_2 + BK)x(t-\tau) \\ &\forall t \in [0, t_f], \\ &x(t) = \phi(t) \quad \forall t \in [-\tau_{\max}, 0], \end{aligned} \tag{2}$$

A solution of (2) associated with an admissible control gain K is next denoted by $x^K(\cdot)$.

We are now in a position to formulate the main OCP problem associated with the given delayed dynamics (1):

$$\begin{aligned} &\min_{K} \max_{\tau\in D} \{J(K)\} \\ &\text{subjet to (2)}, \\ &||K||_{\mathbb{R}^{m\times n}} \leq k_{\max}. \end{aligned} \tag{3}$$

where $J(K) := \psi(x^K(t_f))$ for a convex, continuously differentiable function $\psi(\cdot)$. As we can see the main OCP under consideration is a so called "minimax" problem with a Mayer-type objective functional. Since the terminal state vector $x^K(t_f)$ depends on the concrete realizations of the stochastic delays $\tau \in D$, the maximization procedure

$$\max_{\tau\in D} \psi(x^K(t_f)) \tag{4}$$

in (3) is well defined. We next search in (3) for a minimum (with respect to the admissible control gains) of this realised maximum value $\max_{\tau\in D}\{J(K)\ \}$.

Problem (3) evidently expresses a "worst-case" realization (with respect to the possible delays from D) of the minimal value $\min\{J(K)\}$ of the objective functional. Since D is a finite (discrete) set, there exists a (nonempty) $\Gamma \subseteq D$ such that maximum in (3) will be realized for a constant $\tau^{opt} \in \Gamma$. Here Γ is in fact a "solution set" for the maximization procedure in the main OCP (3).

3. CONVEXITY PROPERTY OF THE ROBUST OPTIMAL CONTROL OF DELAY SYSTEMS

In this section we establish the useful convexity properties of the main optimization problem (3). For an "optimal" (in the sense of (4)) delay $\tau^{opt} \in \Gamma \subseteq D$ we next consider the natural approximations of the autonomous delayed system (2). Let $N\Delta = \tau^{opt}$. Then a suitable N-order forward differences based approximations of the closed-loop delay system (2) are given by the following system

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ x(t-\Delta) \\ \vdots \\ x(t-N\Delta) \end{pmatrix} = \begin{pmatrix} A_1 & 0 & \cdots & \chi \\ \frac{1}{\Delta}I & -\frac{1}{\Delta}I & & \\ & \ddots & \ddots & \\ & 0 & \frac{1}{\Delta}I & -\frac{1}{\Delta}I \end{pmatrix}\begin{pmatrix} x(t) \\ x(t-\Delta) \\ \vdots \\ x(t-N\Delta) \end{pmatrix}. \tag{5}$$

where $\chi := (A_2+BK)$. Introduce the following augmented state

$$X_N(t) := ((x^K)^T(t),\ (x^K)^T(t-\Delta),\ \ldots\ (x^K)^T(t-N\Delta))^T$$

of dimension $n(N+1)$. Here $x^K(\cdot)$ is a solution to (2) for a fixed admissible matrix K. The initial condition for the approximating system (5) can be written as follows

$$X_N(0) = (\phi^T(0),\ \phi^T(-\Delta),\ \ldots\ \phi^T(-\tau^{opt}))^T.$$

We finally get the following convex characterization of the main OCP (3).

Theorem 1. Assume that all the technical conditions from Section 2 are satisfied. Then the (combined) objective functional $J : \mathbb{R}^{m\times n} \to \mathbb{R}$ in (3) is convex and the OCP (3) constitutes a convex minimization problem in $\mathbb{R}^{m\times n}$.

Proof: The rewritten system (5) has a linear structure and it can be represented as follows:

$$\frac{d}{dt}X_N(t) = \mathcal{A}_N X_N(t)$$

where $\mathcal{A}_N$ denotes the full system matrix in (5). We now introduce the uniformly continuous $\mathbb{C}_0$-semigroup $\{\mathcal{T}_N(t)\}_{t\geq 0}$ associated with the given generator $\mathcal{A}_N$. Then

$$X_N(t) = \mathcal{T}_N(t)X_N(0).$$

The continuity property in the strong operator topology of the map $\mathcal{A}_N,\ \forall N \in \mathbb{N}$ implies the following relation

$$\lim_{N\to\infty} X_N(t) = ((x^K)^T(t),\ (x^K)^T(t),\ \ldots\ (x^K)^T(t-\tau^{opt}))^T,$$

where

$$x^K(t) = \exp(A_1 t)\phi(0) + \int_0^t \exp(A_1(t-s))(A_2+BK)x^K(s-\tau^{opt})ds. \tag{6}$$

for all $t \geq 0$. From (6) and taking into consideration the fundamental Implicit Function Theorem (see e.g., (3)) we deduce the affine structure of the mapping

$$x^K : \mathbb{R}^{m\times n} \to \mathbb{R}^n,\quad K \to x^K(t_f)$$

where $x^K(t_f)$ constructively determined by (6).

The objective function $\psi : \mathbb{R}^n \to \mathbb{R}$ was assumed to be convex and continuously differentiable. Recall that a superposition of a convex functional $\mathbb{R}^n \to \mathbb{R}$ and an affine function $\mathcal{H} \to R^n$, where $\mathcal{H}$ is a real Hilbert space, is convex (see e.g., (5; 11)). We have in our case $\mathcal{H} \equiv \mathbb{R}^{m\times n}$ and can conclude that $J(\cdot)$ is convex.

The inequality constraint in (3) evidently determines a closed (convex) ball in the Euclidean space $\mathbb{R}^{m\times n}$. The proof is completed. □

Let us note that Theorem 1 gives a useful "convex characterization" of the sophisticated (constrained and "delayed") OCP (3). Moreover, Theorem 1 implies the existence of an optimal solution K^{opt} to (3). We refer to (11; 31; 32) for the general existence results for convex minimization problems in Hilbert spaces.

Finally note that the generic quadratic objective function

$$\psi(x(t_f)) = x^T(t_f)x(t_f) \tag{7}$$

evidently satisfies conditions of Theorem 1. Therefore this results generalizes the Linear Quadratic (LQ) optimal control of the delay systems under consideration (11; 37; 38).

4. NUMERICALLY STABLE APPROACHES TO THE ROBUST OPTIMAL CONTROL OF DELAY SYSTEMS

In this section we propose a conceptual solution algorithm for an effective treatment of the main OCP (3) associated with the delay system (1). We finally prove the numerical consistence (numerical stability) of the resulting approach.

4.1 On the Reduced Gradient of the Objective Functional

Taking into consideration the practical usability of the first-order algorithms for convex programming we next consider a concrete calculation scheme for the gradient $\nabla J(\cdot)$ of the objective functional $J(\cdot)$ in the main problem (3). The combined structure of $J(\cdot)$ implies the notation "reduced gradient" (see e.g., (6; 7; 8; 9; 10; 11; 33; 35; 40)) Since $\psi(x^K(t_f))$ is assumed to be continuously differentiable, we deduce

$$\nabla J(K) = \nabla\psi(x^K(t_f))(\nabla x^K(t_f))^T, \tag{8}$$

where $\nabla\psi(x)$ denotes the derivative of $\psi(\cdot)$ at a vector $x \in \mathbb{R}^n$ and $\nabla x^K(t)$ is a derivative of the state mapping $x^K : \mathbb{R}^{m\times n} \to \mathbb{R}^n$ introduced in Section 3. As established in the Proof of Theorem 1 this mapping has an affine structure and therefore is differentiable.

The main challenge with respect to the gradient formulae (8) is the calculation of the derivative $\nabla x^K(t_f)$. With the aim of this practical question we now consider a general framework and introduce the real Banach and Hilbert spaces $X,\ Y$. Let $P : X \times Y \to X$ be a known "state" mapping. Here $P(\cdot,\cdot)$ is assumed to be continuously differentiable. Let us note that all derivatives considered in our contribution are Fréchet derivatives (denoted by ∇). Assume that the abstract state equation $P(\xi,\nu) = 0$ can be solved with respect to ξ, i.e.,

$$\xi = \omega(\nu),$$

where the mapping $\omega : Y \to X$ is a differentiable mapping. We call the variable ξ state variable and ν is a parameter.

We use the standard notation $P_\xi(\cdot,\cdot), P_\nu(\cdot,\cdot)$ for the partial derivatives of the mapping $P(\cdot,\cdot)$. Moreover, we introduce the corresponding adjoint operators

$$P^*_\xi(\cdot,\cdot), P^*_\nu(\cdot,\cdot)$$

to the derivatives (linear operators) given above. We next prove the following simple technical result.

Lemma 1. Assume that $P(\cdot,\cdot)$ is continuously differentiable and the parametric state equation is solvable. In addition, assume that there exists the continuous inverse operator $(P^*_\xi)^{-1}(\cdot,\cdot)$ to $P_\xi(\cdot,\cdot)$. Then the reduced gradient $\nabla\omega(\cdot)$ of the mapping $\omega(\cdot)$ can be calculated as follows

$$\nabla\omega(\nu) = -(P_\xi)^{-1}(\xi,\nu)P_\nu(\xi,\nu). \qquad (9)$$

Proof: Differentiating the state equation we obtain

$$P_\xi(\xi,\nu)\nabla\omega(\nu) + P_\nu(\xi,\nu) = 0.$$

The existence of $(P^*_\xi)^{-1}$ implies the final formula (9). The proof is completed. □

For the closed-loop delay system (2) we now define the concrete spaces

$$X = \mathbb{W}^{1,\infty}_n(0,t_f)\times\mathbb{C}(-\tau_{\max},0),\; Y = \mathbb{R}^{m\times n}.$$

and introduce the corresponding state operator

$$P : \mathbb{W}^{1,\infty}_n(0,t_f)\times\mathbb{C}(-\tau_{\max},0)\times\mathbb{R}^{m\times n}\to \mathbb{W}^{1,\infty}_n(0,t_f)\times\mathbb{C}(-\tau_{\max},0).$$

The constructive definition of the state mapping $P(\cdot,\cdot)$ can be given as follows

$$P(x(\cdot),K)\Big|_t := \big[x(t) - \exp(A_1 t)\phi(0) - \int_0^t \exp(A_1(t-s))(A_2+BK)x(s-\tau^{opt})ds\big].$$

Evidently, the resulting operator equation $P(x(\cdot),K) = 0$ is consistent with the abstract state equation considered in Lemma 1. Recall that a solution $x(\cdot)$ of the above equation is an element of the space $\mathbb{W}^{1,\infty}_n(0,t_f)\times\mathbb{C}(-\tau_{\max},0)$.

In view of Lemma 1 and the basic formulae (9) we next have to give explicit expressions for the partial derivative $P_{x(\cdot)}(x(\cdot),K)$ and $P_K(x(\cdot),K)$ of the concrete state operator $P(x(\cdot),K)$.

Theorem 2. Assume that the conditions of Lemma 1 are satisfied for the concrete operator $P(\cdot,\cdot)$ associated with the closed-loop delay system (2). Moreover, assume that there exist an inverse

$$\big(I - \int_0^{t_f}\exp(A_1(t-s))(A_2+BK)ds\big)^{-1}$$

to the matrix $\big(I-(A_2+BK)\int_0^{t_f}\exp(A_1(t-s))ds\big)$. Then the reduced gradient $\nabla x^K(t_f)$ of the mapping $x^K(\cdot)$ at t_f can be calculated as follows

$$\nabla x^K(t_f) = \big(I - \int_0^{t_f}\exp(A_1(t-s))(A_2+BK)ds\big)^{-1}\times BI_{m,n}\big[\int_{-\tau^{opt}}^0\phi(s)ds + \int_0^{t_f-\tau^{opt}}x(s)ds\big] \qquad (10)$$

Moreover, the the gradient $\nabla J(\cdot)$ of the objective functional $J(\cdot)$ in the main problem (3) has the following expression

$$\nabla J(K) = \nabla\psi(x^K(t_f))(\nabla x^K(t_f))^T. \qquad (11)$$

Proof: Since the state operator $P(\cdot,\cdot)$ associated with the closed-loop delay system (2) is Fréchet differentiable, there exist the Gâteaux derivatives. Using the corresponding differentiability concept we easy deduce

$$P_K(x(\cdot),K)\Big|_{t_f} = BI_{m,n}\int_0^{t_f}x(s-\tau^{opt})ds = BI_{m,n}\big[\int_{-\tau^{opt}}^0\phi(s)ds + \int_0^{t_f-\tau^{opt}}x(s)ds\big], \qquad (12)$$

where $I_{m,n}$ is an identity matrix in $\mathbb{R}^{m\times n}$.

In the same manner we obtain

$$P_{x(\cdot)}(x(\cdot),K)\Big|_{t_f} = \big[I - \int_0^{t_f}\exp(A_1(t_f-s))(A_2+BK)ds\big] \qquad (13)$$

for the derivative $P_{x(\cdot)}(x(\cdot),K)$ calculated for $t=t_f$. Here I denotes an identity matrix in $\mathbb{R}^{n\times n}$.

Taking into consideration the invertibility of the matrix

$$\big(I - \int_0^{t_f}\exp(A_1(t-s))(A_2+BK)ds\big)$$

we next apply Lemma 1 to the concrete operator $P(\cdot,\cdot)$ associated with the closed-loop delay system (2) and finally obtain (10). The gradient formula (11) is a direct consequence of (10) and the formulae (8) of the derivative of a combined function. The proof in finished. □

Let us finally refer to (11; 14; 33; 35; 40; 42) for some results related to the advanced computational techniques for functional gradients and for the corresponding control theoretical applications.

4.2 A Gradient Based Solution Algorithm for Robust Optimal Control

We now use the convex structure of the main optimization problem (3) (established in Theorem 1), the explicit expression of the gradient for the corresponding objective functional (see Theorem 2) and develop a gradient based numerical scheme for (3). We also prove the numerical consistence of the resulting algorithm. To make a step forward we next discuss some related numerical aspects and a complete conceptual solution scheme for the minimax OCP (3).

Let $\mathcal{K}$ be the set of admissible gain matrices K in (3) that satisfy the inequality $||K||_{\mathbb{R}^{m\times n}} \le k_{\max}$. The iterative gradient-projected scheme applied to the main OCP (3) can now be explicitly written

$$K^{(l+1)} = \gamma_l\mathcal{P}_{\mathcal{K}}\Big[K^{(l)} - \alpha_l\nabla J(K^{(l)})\Big] + (1-\gamma_l)K^{(l)}, \quad (14)$$

where $l\in\mathbb{N}$. By $\mathcal{P}_{\mathcal{K}}$ we denote here the projection operator on the convex set $\mathcal{K}\subset\mathbb{R}^{m\times n}$ (of admissible gains). Moreover, $\{\alpha_l\}$, $\{\gamma_l\}$ are sequences of some suitable step sizes associated with the gradient-projected method. Recall that several choices are possible for the step sizes α_l and γ_l in (9). It is well known that the mostly used realization of (14) is the celebrated Armijo rule (Armijo line search) (4; 9; 42). The simplest step size selection, namely, the constant step sizes strategy was analysed in (22). In that case we have

$$\gamma_l = 1, \quad \alpha_l = \alpha > 0 \ \forall l \in \mathbb{N}.$$

Under some general assumptions the gradient iterations (14) generate a minimizing sequence $\{K^l\}, \ l \in \mathbb{N}$ for the optimization problem (3) (see (31)). That means

$$\lim_{l\to\infty} J(K^{(l)}) = J(K^{opt}), \ ||K||_{\mathbb{R}^{m\times n}} \leq k_{\max}.$$

The existence of an optimal solution K^{opt} to (3) was established in Section 3.

Many useful and mathematically exact convergence theorems for iterations (9) can be found in (22; 31). A comprehensive discussion of the weakly and strongly convergent realizations of the basic gradient method can be found in (13; 14). We also refer to (7; 20; 21; 42) for some specific convergence results obtained for the gradient-based schemes applied to hybrid and switched OCPs. We now formulate a convergence result associated with the proposed gradient-projected scheme (14).

Theorem 3. Consider the main OCP (3) and assume that all the conditions from Section 2 are satisfied. Let

$$\{K^{(l)}\} \subset \mathcal{K}, \ l \in \mathbb{N}$$

be a sequence generated by the gradient method (14) with $\gamma = 1$. Assume that

$$\int_0^{t_f} \exp\left(A_1(t-s)\right)(A_2 + BK^{(l)})ds \neq I \qquad (15)$$

for all $l \in \mathbb{N}$. Then there exists a constant step sizes $\alpha > 0$ such that for an admissible initial point $K^{(0)} \in \mathcal{K}$ we obtain

$$\lim_{l\to\infty} J(K^{(l)}) = J(K^{opt}).$$

Moreover, the sequence $\{K^{(l)}\}$ converges in norm $||\cdot||_{\mathbb{R}^{m\times n}}$ to an optimal solution of (3) (to an optimal gain matrix) $K^{opt} \in \mathcal{K}$.

Proof: Since OCP (3) is a convex minimization problem in the Euclidean space $\mathbb{R}^{m\times n}$ (see Theorem 1), it follows that $\{K^{(l)}\}$ is a minimizing sequence for (3) (see (14; 31)).

From (15) it follows the invertibility of the matrix

$$\big(I - \int_0^{t_f} \exp\left(A_1(t-s)\right)(A_2 + BK^{(l)})ds\big)$$

for all $K^{(l)}$ generated by methor (14). We now consider the explicit representation (10)-(11) of the derivative $\nabla J(\cdot)$ in Theorem 2 and deduce the Lipschitz continuity of the derivative $\nabla J(\cdot)$ on the set $\mathrm{co}\{K^{(l)}\}$. Let L be the corresponding Lipschitz constant. From the main result of (22) we now deduce the weak convergence of the minimizing sequence $\{K^{(l)}(\cdot)\}$ to K^{opt} where $\alpha \in (0, 2/L)$. The weak convergence in the Euclidean space $\mathbb{R}^{m\times n}$ coincides with a strong convergence. The proof is completed. □

As we can see Theorem 3 establishes a strong convergence of the minimizing sequence $\{K^{(l)}\}, \ l \in \mathbb{N}$ generated by the gradient-projected algorithm (14). Roughly speaking the proposed solution method (14) is numerically stable in the strong sense (in the sense of the norm convergence). Moreover, the Proof of Theorem 3 determines the concrete choice of the parameter α.

The complete solution procedure for the minimax OCP (3) evidently includes determination of the "worst case" delay $\tau^{opt} \in \Gamma \subseteq \mathcal{D}$. This step in fact implies a finite search and need to be combined with the proposed gradient-projected scheme (14).

5. CONCLUDING REMARK

In this contribution, we have studied optimal control processes governed by dynamic systems with random delays. The minimax-type OCPs we consider can be interpreted in the framework of a robust optimization with respect to the possible delay realisations. The paper is mainly focused on an analytic development of a constructive computational solution procedure and on the corresponding numerical analysis of the resulting algorithm. The convex structure of the main OCPs associated with the delay systems makes it possible to apply some basic techniques of the conventional convex programming. We propose the (first order) gradient projected algorithm for the concrete numerical treatment of the OCP under consideration.

The theoretical solution methodology we proposed needs a comprehensively numerical examination, namely, simulations of several robust (minimax-type) OCPs for the differential equations with random delays.

REFERENCES

[1] A. Ahmed, E.I. Verriest, Estimator design for a subsonic rocket car (soft landing) based on state-dependent delay measurement, in: *Proceedings of the 52th IEEE Conference on Decision and Control*, Florence, Italy, 2013, pp. 5698 – 5703.

[2] W.G. Aiello, H.I. Freedman, J. Wu, Analysis of a model representing stage-structured population growth with state-dependent time delay, *SIAM Journal on Applied Mathematics*, vol. 52, 1992, pp. 855 - 869.

[3] C.D. Aliprantis, K.C. Border, *Infinite Dimensional Analysis*, Springer, Berlin, 2006.

[4] L. Armijo, Minimization of functions having Lipschitz continuous first partial derivatives, *Pacific Journal of Mathematics*, vol. 16, 1966, pp. 13.

[5] V. Azhmyakov, J. Raisch, Convex control systems and convex optimal control problems with constraints, *IEEE Transactions on Automatic Control*, vol. 53, 2008, pp. 993 – 998.

[6] V. Azhmyakov, M. Basin, J. Raisch, A Proximal point based approach to optimal control of affine switched systems, *Discrete Event Dynamic Systems*, vol. 22, no. 1 (2012), pp 61 - 81.

[7] V. Azhmyakov, M. Basin, C. Reincke-Collon, Optimal LQ-type switched control design for a class of linear systems with piecewise constant inputs, in: *Proceedings of the 19th IFAC World Congress*, Cape Town, South Africa, 2014, pp. 6976 – 6981.

[8] V. Azhmyakov, J.Cabrera Martinez, A. Poznyak, Optimal fixed-levels control for nonlinear systems with quadratic cost-functionals, *Optimal Control Applications and Methods*, vol. 37, no. 5, 2016, pp. 1035 – 1055.

[9] V. Azhmyakov, A. Ahmed, E.I Verriest, On the optimal control of systems evolving with state suprema, in: *Proceedings of the 55th IEEE Conference on Decision and Control*, Las Vegas, USA, 2016, pp. 3617 – 3623.

[10] V. Azhmyakov, R. Juarez, A first-order numerical approach to switched-mode systems optimization, *Nonlinear Analysis: Hybrid Systems*, vol. 25, no. 1, 2017, pp. 126 – 137.

[11] V. Azhmyakov, *A Relaxation Based Approach to Optimal Control of Hybrid and Switched Systems*, Elsevier, Massachusetts, USA, to appear in 2018.

[12] M. Basin, Optimal control for linear systems with multiple time delays in control input, *IEEE Transactions on Automatic Control*, vol. 51, 2006, pp. 91 –97.

[13] J.Y. Bello Cruz, C.W. de Oliveira On weak and strong convergence of the projected gradient method for convex optimization in Hilbert spaces, *arXiv*, pp. 1 – 18, 2014.

[14] D. Bertsekas, *Nonlinear Programming*, Athena Scientific, Belmont, USA, 1995.

[15] J. Betts, *Practical Methods for Optimal Control Problems Using Nonlinear Programming*, SIAM, Philadelphia, USA, 2001.

[16] V. Boltyanski, H. Martini and V. Soltan, *Geometric Methods and Optimization Problems*, Kluver Academic Publishers, Dordrecht, 1999.

[17] A.E. Bryson, Y-C. Ho, *Applied Optimal Control: Optimization, Estimation and Control*, CRC Press, New York, 1975.

[18] O. Diekmann, S.A. van Gils, S.M. Verduyn Lunel, H.O. Walther, *Delay Equations: Functional-, Complex-, and Nonlinear Analysis*, Springer, New York, 1995.

[19] P. Colanery, R.H. Middleton, Z. Chen, D. Caporale, F. Blanchini, Convexity of the cost functional in an optimal control problem for a class of positive switched systems, *Automatica*, vol. 50, 2014, pp. 1227 – 1234.

[20] M. Egerstedt, Y. Wardi, H. Axelsson, Transition-time optimization for switched-mode dynamical systems, *IEEE Transactions on Automatic Control*, vol. 51, 2006, pp. 110 – 115.

[21] M. Egerstedt, C. Martin, *Control Theoretic Splines: Optimal Control, Statistics, and Path Planning*, Princeton University Press, Princeton, USA, 2009.

[22] A.A. Goldstein, Convex programming in Hilbert space, *Bulletin of the American Mathematical Society*, vol. 70, 1964, pp. 709 - 710.

[23] M. M. Gomez, M. Sadeghpour, M. R. Bennett, G. Orosz, R. M. Murray, Stability of systems with stochastic delays and applications to genetic regulatory networks, *SIAM Journal of Applied Dynamic Systems*, vol. 15, no. 4, 2016, pp 1844 – 1873.

[24] K.P. Hadeler, *Delay Equations in Biology*, Springer, New York, 1979.

[25] J.K. Hale, S.M.V. Lunel, *Introduction to Functional Differential Equations*, Springer-Verlag, New York, 1993.

[26] F. Hartung, M. Pituk (eds.), *Recent Advances in Delay Differential and Difference Equations*, Springer, Basel, 2014.

[27] J.B. Hiriart-Urruty and C. Lemarchal, *Convex Analysis and Minimization Algorithms*, vol. 305 and 306, Springer, Berlin, Germany, 1996.

[28] A.D. Ioffe and V.M. Tichomirov, *Theory of Extremal Problems*, North Holland, Amsterdam, 1979.

[29] M. Malek-Zavarei, M. Jamshidi, *Time-Delay Systems: Analysis, Optimization and Applications*, North Holland, Amsterdam, 1987.

[30] D. Otrocol, I.A. Rus, Functional-differential equations with maxima via weakly Picard operators theory, *Bull. Math. Soc. Sci. Math.*, vol. 51, 208, pp. 253 – 261.

[31] E. Polak, *Optimization*, Springer-Verlag, New York, USA, 1997.

[32] A. Poznyak, A. Polyakov, V. Azhmyakov, *Attractive Ellipsoids in Robust Control*, Birkhäuser, Basel, Switzerland, 2014.

[33] R. Pytlak, *Numerical Methods for Optimal Control Problems with State Constraints*, Springer, Berlin, Germany, 1999.

[34] T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, 1970.

[35] K.L. Teo, C.J. Goh, K.H. Wong, *A Unifed Computational Approach to Optimal Control Problems*, Wiley, New York, 1991.

[36] M. Thitsa, S. Williams, E. Verriest, Formal power series method for nonlinear time delay systems with analytic initial data, in: *Proceedings of the 54rd IEEE Conference on Decision and Control*, Kita-KuOsaka, Japan, 2015, pp 6478 – 6483.

[37] E.I. Verriest, Pseudo-continuous multi-dimensional multi-mode systems, *Discrete Event Dynamic Systems*, 22, 2012, 27 – 59.

[38] E.I. Verriest, G. Dirr, U. Helmke, O. Mitesser, Explicitly solvable bilinear optimal control problems with applications in ecology, in: *Proceedings of the 22nd International Symposium on Mathematical Theory of Networks and Systems*, Minneapolis, MN, 2016.

[39] E.I. Verriest, G. Dirr, U. Helmke, Delayed resource allocation optimization with applications in population dynamics, *IFAC-PapersOnline*, vol. 49, no. 10, 2016, pp. 1 – 6.

[40] E. I. Verriest, Azhmyakov, Advances in optimal control of differential systems with state suprema, in: Proceedings of the 56th IEEE Conference on Decision and Control, Melbourne, Australia, 2017, pp. 739 – 744.

[41] H.O. Walther, Linearized stability for semiflows generated by a class of neutral equations, with applications to state-dependent delays, *Journal of Dynamics and Differential Equations*, vol. 22, 2010, pp. 439 - 462.

[42] Y. Wardi Optimal control of switched-mode dynamical systems, in: *Proceedings of the 11th International Workshop on Discrete Event Systems*, Guadalajara, Mexico, 2012, pp. 4 – 8.