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Structural feedback linearization based on nonlinearities rejection

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Abstract: In this paper, a structural feedback linearization technique is proposed. This is a quite simple and effective linear control scheme based on failure detection techniques. Our proposed linear control approach is intended to reject the nonlinearities, which are treated as failure signals affecting the systems dynamics. The proposed control methodology is illustrated via the attitude control of a quadrotor in hover flying.

Keywords: Nonlinear systems, feedback linearization, failure reconstruction.

NOTATION

$\underline{\chi}_{\mathbf{k}}^{\mathbf{i}}$ stands for the vector in $\mathbb{R}^k$, which the i-th entry is equal to 1 and the others are equal to 0. $\underline{\mathbf{0}}_{\mathbf{k}}$ stands for the zero vector in $\mathbb{R}^k$. $\mathrm{I}_{\mathbf{k}}$ stands for the identity matrix of size $k \times k$, or shortly I when k does not need to be explicited. $\mathbf{0}_{\mathbf{k}}$ stands for the zero matrix of size $k \times k$. $\mathbf{0}_{\mathbf{k},\ell}$ stands for the zero matrix of size $k \times \ell$. $\mathbf{N}_{\mathbf{k}}$ stands for the nilpotent matrix of size $k \times k$: $\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdots & 0 & 1 \\ 0 & \cdot & \cdot & \cdots & 0 \end{bmatrix}$. $\mathbf{C}_{\mathbf{k}}\left(\begin{bmatrix} a_1 & \cdots & a_k \end{bmatrix}\right) = \mathbf{N}_{\mathbf{k}} + \underline{\chi}_{\mathbf{k}}^{\mathbf{k}} \begin{bmatrix} a_1 & \cdots & a_k \end{bmatrix}$ stands for the companion matrix of size $k \times k$. $\mathbf{BD}\{\mathrm{X}_1, \ldots, \mathrm{X}_k\}$ stands for the block diagonal matrix, where $\mathrm{X}_1, \ldots, \mathrm{X}_k$ are on the diagonal and the other elements are zero.

1. INTRODUCTION

The basic idea of simplifying the form of a nonlinear system by choosing a different state representation has been known in the control community since the work of Isidori (1989) and Nijmeijer and Van der Schaft (1990). However, such a technique is similar to the choice of reference frames or coordinate systems in mechanics. Feedback linearization is equivalent to transforming original system models into equivalent models having a simpler form. In engineering, there are many applications for feedback linearization such as in helicopters, high-performance aircrafts, industrial robots manipulators, biomedical devices, vehicle control.

The central idea of feedback linearization is to algebraically transform nonlinear systems dynamics into (fully or partly) linear ones (see for example Slotine and Li (1991), Isidori (1989) and Khalil (1992)), so that linear control techniques can be applied.

Let us consider the nonlinear system:

$$\mathrm{d}\mathbf{x}/\mathrm{d}t = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}\ ;\ \ \mathbf{y} = \mathrm{C}_o\mathbf{x}, \tag{1.1}$$

where: $\mathbf{x} \in \mathbb{R}^n$ is the state variable, $\mathbf{y} = \begin{bmatrix} y_1 & \cdots & y_p \end{bmatrix}^T \in \mathbb{R}^p$ is the output variable, $\mathbf{u} = \begin{bmatrix} u_1 & \cdots & u_m \end{bmatrix}^T \in \mathbb{R}^m$ is the input variable, $\mathbf{f}(\mathbf{x})$ and $\mathbf{g}(\mathbf{x})$ are analytic vector fields on $\mathbb{R}^n$, and $\mathrm{C}_o \in \mathbb{R}^{p\times n}$ is an epic matrix, namely: $\ker \mathrm{C}_o^T = \{0\}$. We assume that $p = m$, and that the system has relative degree n at any point $\mathbf{x}_p \in \mathbb{R}^n$, the set of points where a relative degree can be defined is an open and dense set of the set U where the System (1.1) is defined (Isidori, 1989; Khalil, 1992; Slotine and Li, 1991).

Let us assume that:

Hypothesis 1. The origin $\mathbf{x}_o = \mathbf{0} \in \mathbb{R}^n$ is an equilibrium point of the autonomous system (1.1), namely: $\mathbf{f}(\mathbf{0}) = \mathbf{0}$.

Hypothesis 2. $\mathbf{f}$ and $\mathbf{g}$ are continuously differentiable.

Hypothesis 3. Given the defined matrices,

$$\mathrm{A}_o \doteq [\partial\mathbf{f}/\partial\mathbf{x}]_{\mathbf{x}=\mathbf{0}}\ , \quad \mathrm{B}_o \doteq [\mathbf{g}]_{\mathbf{x}=\mathbf{0}}\ , \tag{1.2}$$

(i) $\ker \mathrm{B}_o = \{0\}$, *(ii)* the pair $(\mathrm{A}_o, \mathrm{B}_o)$ is controllable, and *(iii)* the pair $(\mathrm{C}_o, \mathrm{A}_o)$ is observable.

Under these assumptions, it is well known that there exists a stabilizing feedback, $\mathbf{u}(t) = \mathrm{F}\mathbf{x}(t)$, $\mathrm{F} \in \mathbb{R}^{m\times n}$, such that: $\mathrm{A}_o + \mathrm{B}_o\mathrm{F}$ is a Hurwitz matrix, and $\mathbf{0}$ is an exponentially stable equilibrium point of the autonomous system $\mathrm{d}\mathbf{x}/\mathrm{d}t = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathrm{F}\mathbf{x}$ (see for example Vidyasagar (1993)). Thus, a common engineering practice is to control (1.1) with standard stabilizing linear control laws. Note

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that the stability neighborhood can be small for a control based on a Taylor approximation.

Let us briefly recall the input-output linearization technique for the case: $p = m = 1$.

Hypothesis 4. The relative degree and the dimension of the state $\mathbf{x}$ are equal to n.

Since the system has relative degree n, exactly equal to the dimension of state space, at some point $\mathbf{x} = \mathbf{x}_p$, there then exists a diffeomorphism T_{dif}, $\zeta(t) = \mathrm{T}_{dif}(\mathbf{x}(t))$, such that ($\zeta = [\,\zeta_1 \ \cdots \ \zeta_n\,]^T$):

$$\mathrm{d}\zeta/\mathrm{d}t = \mathbf{C_n}\left(\underline{\mathbf{0}}_{\mathbf{n}}^{\mathbf{T}}\right)\zeta + \underline{\chi}_{\mathbf{n}}^{\mathbf{n}}\big(\alpha(x) + \beta(x)\,u\big);\ y(t) = \left(\underline{\chi}_{\mathbf{n}}^{\mathbf{1}}\right)^T \zeta$$

Let us note that Hypothesis 4 implies that there exits a known positive constant $\bar{\beta}$ such that for all $\mathbf{x}$: $0 < \bar{\beta} < \beta(\mathbf{x})$. Thus, if we could have a complete knowledge of its parameters, then the ideal control law would be:

$$u(t) = \beta^{-1}(\mathbf{x}(t))\left(-\alpha(\mathbf{x}(t)) + [\,-a_1 \ \cdots \ -a_n\,]\,\zeta(t)\right).$$

With this ideal control law, the closed loop system is described by:

$$\mathrm{d}\zeta/\mathrm{d}t = \mathbf{C_n}\left(\underline{\mathbf{a}}^{\mathbf{T}}\right)\zeta(t)\ ;\ \ y(t) = \left(\underline{\chi}_{\mathbf{n}}^{\mathbf{1}}\right)^T \zeta(t),$$

where: $\underline{\mathbf{a}} = [\,-a_1 \ -a_2 \ -a_3 \ \cdots \ -a_{n-1} \ -a_n\,]^T$.

The feedback linearization problem is achieved by exact state transformation and feedback, rather than by linear approximations of the dynamics. This approach requires a complete knowledge of the model, with exact derivatives, and this is not always possible. There are still a number of shortcomings and limitations associated with the feedback linearization approach. In this paper, we propose a linearization methodology based on the internal structure of a linear part of the system, characterized by the pair $(\mathrm{A}_o,\, \mathrm{B}_o)$. For this, in Section 2, we first decompose the linear part of (1.1) in its Brunovsky canonical form $(\mathrm{A}_\mathcal{B},\, \mathrm{B}_\mathcal{B})$ (Brunovsky, 1970). And then, in Section 3, we propose an operator $\mathrm{X}(\mathrm{d}/\mathrm{d}t)$ depending on the structure of the Brunovsky matrix $\mathrm{A}_\mathcal{B}$, which aim is to concentrate the non-linear part of the system in the image of the Brunovsky input matrix $\mathrm{B}_\mathcal{B}$. In Section 4, we asymptotically reject the non-linear components by means of a failure reconstructor. In Section 5, we apply our methodology to a quadrotor in hover flying. In Section 6, we show simulation results. In Section 7, we conclude.

2. STRUCTURAL DECOMPOSITION

Let us assume that $p = m$ and that Hypothesis 1–3 are satisfied. Taking into account (1.2) in (1.1), we get:

$$\mathrm{d}\mathbf{x}/\mathrm{d}t = \mathrm{A}_o\mathbf{x} + \mathrm{B}_o\mathbf{u} + \mathbf{f}_o(\mathbf{x},\,\mathbf{u})\ ;\ \ \mathbf{y} = \mathrm{C}_o\mathbf{x}, \qquad (2.3)$$

where: $\mathbf{f}_o(\mathbf{x},\,\mathbf{u}) = \Delta\mathbf{f}(\mathbf{x}) + \Delta\mathbf{g}(\mathbf{x})\mathbf{u}$, $\Delta\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}) - \mathrm{A}_o\mathbf{x}$, and $\Delta\mathbf{g}(\mathbf{x}) = \mathbf{g}(\mathbf{x}) - \mathrm{B}_o$. Let us note that for a given bound, $\varepsilon_0 \in \mathbb{R}^+$, there exist a small neighborhood, $\delta_0 \in \mathbb{R}^+$, around the equilibrium point, $\mathbf{0}$, such that:

$$\|\Delta\mathbf{f}(\mathbf{x})\|_2 \le \varepsilon_0\|\mathbf{x}\|_2^2,\ \|\Delta\mathbf{g}(\mathbf{x})\|_2 \le \varepsilon_0\|\mathbf{x}\|_2,\ \forall\ \|\mathbf{x}\|_2 < \delta_0$$

Since the pair $(\mathrm{A}_o,\, \mathrm{B}_o)$ is controllable there exist a state feedback $\mathrm{F}_\mathcal{B} \in \mathbb{R}^{n\times m}$, and invertible matrices $\mathrm{T}_\mathcal{B} \in \mathbb{R}^{n\times n}$ and $\mathrm{G}_\mathcal{B} \in \mathbb{R}^{m\times m}$, such that (Brunovsky, 1970):

$$\mathrm{d}\xi/\mathrm{d}t = \mathrm{A}_\mathcal{B}\xi + \mathrm{B}_\mathcal{B}\bar{\mathbf{u}} + \mathrm{S}_o\mathbf{q}_o(\xi,\,\bar{\mathbf{u}})\ ;\ \ \mathbf{y} = \mathrm{C}_\mathcal{B}\xi, \qquad (2.4)$$

where:

$$\begin{array}{l} \mathbf{x} = \mathrm{T}_\mathcal{B}\xi,\ \ \mathbf{u} = \mathrm{F}_\mathcal{B}\mathbf{x} + \mathrm{G}_\mathcal{B}\bar{\mathbf{u}}, \\ \mathrm{A}_\mathcal{B} = \mathrm{T}_\mathcal{B}^{-1}\left(\mathrm{A}_o + \mathrm{B}_o\mathrm{F}_\mathcal{B}\right)\mathrm{T}_\mathcal{B} = \mathbf{BD}\,\{\mathrm{A}_1,\,\ldots,\,\mathrm{A}_m\}, \\ \mathrm{B}_\mathcal{B} = \mathrm{T}_\mathcal{B}^{-1}\mathrm{B}_o\mathrm{G}_\mathcal{B} = \mathbf{BD}\,\{\mathrm{B}_1,\,\ldots,\,\mathrm{B}_m\}, \\ \mathrm{C}_\mathcal{B} = \mathrm{C}_o\mathrm{T}_\mathcal{B} \doteq [\,\mathrm{C}_1\ \cdots\ \mathrm{C}_m\,], \\ \mathrm{S}_o\mathbf{q}_o = \mathrm{T}_\mathcal{B}^{-1}\mathbf{f}_o(\mathbf{x}(t),\,\mathbf{u}(t)),\ \ \mathrm{S}_o \doteq \left[\,\mathrm{S}_1^T\ \cdots\ \mathrm{S}_m^T\,\right]^T, \\ \mathrm{S}_i = [\,\mathbf{S}_{i,1}\ \cdots\ \mathbf{S}_{i,\ell}\,],\ i \in \{1,\,\ldots,\,m\}, \end{array} \qquad (2.5)$$

$$\mathrm{A}_i = \mathbf{C}_{\mathbf{n_i}}\left(\underline{\mathbf{0}}_{\mathbf{n_i}}^{\mathbf{T}}\right),\quad \mathrm{B}_i = \underline{\chi}_{\mathbf{n_i}}^{\mathbf{n_i}}. \qquad (2.6)$$

Thus, we have the following m *pseudo linear* state equations perturbed by the nonlinear perturbation signal $\mathbf{q}_o$:

$$\mathrm{d}\xi_i/\mathrm{d}t = \mathrm{A}_i\xi_i + \mathrm{B}_i\bar{u}_i + \mathrm{S}_i\mathbf{q}_o(\xi,\,\bar{\mathbf{u}}),\quad i \in \{1,\,\ldots,\,m\}, \quad (2.7)$$

where: $\bar{\mathbf{u}} \doteq [\,\bar{u}_1\ \cdots\ \bar{u}_m\,]^T \in \mathbb{R}^m$, $\xi \doteq \left[\,\xi_1^T\ \cdots\ \xi_m^T\,\right]^T \in \mathbb{R}^n$, $\xi_i \in \mathbb{R}^{n_i}$ $(i \in \{1,\,\ldots,\,m\})$, $\sum_{i=1}^m n_i = n$, $\mathbf{q}_o \in \mathbb{R}^\ell$.

3. STRUCTURAL EXACT FEEDBACK LINEARIZATION

Let us define the operator $\mathrm{X}(\mathrm{d}/\mathrm{d}t)$, as follows:

$$\mathrm{X}(\mathrm{d}/\mathrm{d}t) = -\mathrm{A}_\mathcal{B}^T\,\mathcal{C}_n(\mathrm{S}_o)\Psi_n(\mathrm{d}/\mathrm{d}t), \qquad (3.8)$$

$$\mathcal{C}_\kappa(\mathrm{S}_o) = \left[\,\mathrm{S}_o\ \ \mathrm{A}_\mathcal{B}^T\,\mathrm{S}_o\ \cdots\ (\mathrm{A}_\mathcal{B}^T)^{(\kappa-1)}\,\mathrm{S}_o\,\right], \qquad (3.9)$$

$$\Psi_\kappa(\mathrm{d}/\mathrm{d}t) = \left[\,\mathrm{I}\ \ \mathrm{I}\,\mathrm{d}/\mathrm{d}t\ \cdots\ \mathrm{I}\,\mathrm{d}^{\kappa-1}/\mathrm{d}t^{\kappa-1}\,\right]^T. \qquad (3.10)$$

Let us note that $\mathcal{C}_n(\mathrm{S}_o)$ is the controllability matrix of the pair $(\mathrm{A}_\mathcal{B}^T,\, \mathrm{S}_o)$. We do the following assumptions:

Hypothesis 5. The subspace $\mathrm{A}_\mathcal{B}^T\,\mathrm{Im}\,\mathrm{S}_o$ is contained in the unobservable subspace $\cap_{i=1}^n \ker \mathrm{C}_\mathcal{B}\,(\mathrm{A}_\mathcal{B}^T)^{i-1}$, namely:

$$\mathrm{C}_\mathcal{B}\,\mathrm{A}_\mathcal{B}^T\,\mathcal{C}_n(\mathrm{S}_o) = 0. \qquad (3.11)$$

Hypothesis 6. The state space description $\Sigma(\mathrm{A}_\mathcal{B}, \mathrm{B}_\mathcal{B}, \mathrm{C}_\mathcal{B})$ (2.4) has no finite invariant zeros at the origin, namely:

$$\mathrm{Im}\,\mathrm{B}_\mathcal{B} \cap \mathrm{A}_\mathcal{B}\ker \mathrm{C}_\mathcal{B} = \{0\}. \qquad (3.12)$$

As we will see later, Hypothesis 5 assures the rejectibility of the non-linear part q_o, and Hypothesis 6 assures the left invertivility of the static gain of $\Sigma_{ss}(\mathrm{A}_\mathcal{B},\, \mathrm{B}_\mathcal{B},\, \mathrm{C}_\mathcal{B})$.

We shall need the following two technical Lemmas:

Lemma 7. If (3.11) holds, then the operator $\mathrm{X}(\mathrm{d}/\mathrm{d}t)$, defined by (3.8), satisfies:

$$\begin{bmatrix} \mathrm{N}_\mathcal{B} & 0 \\ 0 & \mathrm{I}_m \end{bmatrix}\begin{bmatrix} (\mathrm{I}\mathrm{d}/\mathrm{d}t - \mathrm{A}_\mathcal{B}) & -\mathrm{S}_o \\ \mathrm{C}_\mathcal{B} & 0 \end{bmatrix}\begin{bmatrix} \mathrm{X}(\mathrm{d}/\mathrm{d}t) \\ \mathrm{I}_n \end{bmatrix} = 0, \quad (3.13)$$

where $\mathrm{N}_\mathcal{B}$ is the maximal annihilator of $\mathrm{Im}\,\mathrm{B}_\mathcal{B}$; namely: $\mathrm{N}_\mathcal{B}\,\mathrm{B}_\mathcal{B} = 0$, $\ker\begin{bmatrix}\mathrm{N}_\mathcal{B}\\ \mathrm{B}_\mathcal{B}^T\end{bmatrix} = \{0\}$, and $\mathrm{Im}\begin{bmatrix}\mathrm{N}_\mathcal{B}\\ \mathrm{B}_\mathcal{B}^T\end{bmatrix} = \mathbb{R}^n$.

Proof of Lemma 7 The maximal annihilator $\mathrm{N}_\mathcal{B}$ is:

$$\mathrm{N}_\mathcal{B} = \mathbf{BD}\,\{\mathrm{N}_1,\,\cdots,\,\mathrm{N}_m\}, \qquad (3.14)$$

where: $\mathrm{N}_i = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \in \mathbb{R}^{(n_i-1)\times n_i}$. Then:[1]

$$\begin{aligned}&\mathrm{N}_{\mathcal{B}}(\mathrm{Id}/\mathrm{d}t - \mathrm{A}_{\mathcal{B}})\mathrm{X}(\mathrm{d}/\mathrm{d}t) = \\ &= \mathrm{N}_{\mathcal{B}}(\mathrm{A}_{\mathcal{B}}\,\mathrm{A}_{\mathcal{B}}^T - \mathrm{A}_{\mathcal{B}}^T\mathrm{d}/\mathrm{d}t)\mathcal{C}_n(\mathrm{S}_o)\Psi_n(\mathrm{d}/\mathrm{d}t) \equiv \mathrm{N}_{\mathcal{B}}\,\mathrm{S}_o\end{aligned} \tag{3.15}$$

And from (3.11), we get (3.13). □

Lemma 8. The operator $\mathrm{X}(\mathrm{d}/\mathrm{d}t)$, defined by (3.8), satisfies:

$$\begin{aligned}&\mathrm{B}_{\mathcal{B}}^T\left(-(\mathrm{Id}/\mathrm{d}t - \mathrm{A}_{\mathcal{B}})\mathrm{X}(\mathrm{d}/\mathrm{d}t) + \mathrm{S}_o\right) \\ &\qquad = \mathrm{B}_{\mathcal{B}}^T\,\mathcal{C}_{(n+1)}(\mathrm{S}_o)\,\Psi_{(n+1)}(\mathrm{d}/\mathrm{d}t).\end{aligned} \tag{3.16}$$

Furthermore:

$$\mathrm{B}_{\mathcal{B}}^T\,\mathcal{C}_{(n+1)}(\mathrm{S}_o)\,\Psi_{(n+1)}(\mathrm{d}/\mathrm{d}t) = \Sigma_{\mathrm{S}}(\mathrm{d}/\mathrm{d}t), \tag{3.17}$$

where:

$$\begin{aligned}\Sigma_{\mathrm{S}}(\mathrm{d}/\mathrm{d}t) &= \left[\,\sigma_{\mathrm{S}_1}(\mathrm{d}/\mathrm{d}t)\,\cdots\,\sigma_{\mathrm{S}_m}(\mathrm{d}/\mathrm{d}t)\,\right]^T, \\ \sigma_{\mathrm{S}_i}^T(\mathrm{d}/\mathrm{d}t) &= \left[\,\sigma_{\mathrm{S}_{i,1}}(\mathrm{d}/\mathrm{d}t)\,\cdots\,\sigma_{\mathrm{S}_{i\ell}}(\mathrm{d}/\mathrm{d}t)\,\right], \\ \sigma_{\mathrm{S}_{i,j}}(\mathrm{d}/\mathrm{d}t) &= \det\begin{bmatrix}(\mathrm{I}_{n_i}\mathrm{d}/\mathrm{d}t - \mathrm{A}_i) & \mathbf{S}_{i,j} \\ -\mathbf{c}_i^T & 0\end{bmatrix},\end{aligned} \tag{3.18}$$

$i \in \{1,\ldots,m\}$, $j \in \{1,\ldots,\ell\}$, and the $\mathbf{c}_i$ are vectors in $\ker \mathrm{N}_i\mathrm{A}_i$ such that: $\mathbf{c}_i^T\mathbf{c}_i = 1$.

Proof of Lemma 8 Equation (3.16) directly follows from (3.8) and the fact: $\mathrm{B}_{\mathcal{B}}^T\,\mathrm{A}_{\mathcal{B}}\,\mathrm{A}_{\mathcal{B}}^T = 0$. Equation (3.17) directly follows from (3.18), (2.5) and (2.6). □

In the next Theorem, we introduce a change of variable which aim is to map the nonlinear perturbation signal, $\mathbf{q}_o$, into the image of $\mathrm{B}_{\mathcal{B}}$.

Theorem 9. Let ζ be the state variable defined as follows:

$$\zeta = \xi - \mathrm{X}(\mathrm{d}/\mathrm{d}t)\mathbf{q}_o(\xi,\,\bar{\mathbf{u}}). \tag{3.19}$$

where $\mathrm{X}(\mathrm{d}/\mathrm{d}t)$ is the operator defined by (3.8). Then the state representation (2.4) takes the following form:

$$\mathrm{d}\zeta/\mathrm{d}t = \mathrm{A}_{\mathcal{B}}\zeta + \mathrm{B}_{\mathcal{B}}\left(\bar{\mathbf{u}} + \mathbf{q}_*(\xi,\,\bar{\mathbf{u}})\right)\,;\ \mathbf{y}(t) = \mathrm{C}_{\mathcal{B}}\zeta(t), \tag{3.20}$$

where: $\mathbf{q}_*(\xi,\,\bar{\mathbf{u}}) = \Sigma_{\mathrm{S}}(\mathrm{d}/\mathrm{d}t)\mathbf{q}_o(\xi,\,\bar{\mathbf{u}})$.

Proof of Theorem 9 Substituting (3.19) into (2.4), we get from Lemmas 7 and 8: $\mathrm{N}_{\mathcal{B}}\big(\mathrm{Id}/\mathrm{d}t - \mathrm{A}_{\mathcal{B}}\big)\zeta = 0$, $\mathrm{C}_{\mathcal{B}}\zeta = \mathbf{y}$, and:[2]

$$\begin{aligned}&\mathrm{B}_{\mathcal{B}}^T\,(\mathrm{Id}/\mathrm{d}t - \mathrm{A}_{\mathcal{B}})\,\zeta = \\ &\mathrm{B}_{\mathcal{B}}^T\,(\mathrm{Id}/\mathrm{d}t - \mathrm{A}_{\mathcal{B}})\,(\xi - \mathrm{X}(\mathrm{d}/\mathrm{d}t)\mathbf{q}_o(\xi,\,\bar{\mathbf{u}})) = \\ &\mathrm{B}_{\mathcal{B}}^T\,(\mathrm{Id}/\mathrm{d}t - \mathrm{A}_{\mathcal{B}})\,\xi + \big(\Sigma_{\mathrm{S}}\,(\mathrm{d}/\mathrm{d}t) - \mathrm{B}_{\mathcal{B}}^T\mathrm{S}_o\big)\,\mathbf{q}_o(\xi,\,\bar{\mathbf{u}}) = \\ &\qquad = \bar{\mathbf{u}} + \Sigma_{\mathrm{S}}\,(\mathrm{d}/\mathrm{d}t)\,\mathbf{q}_o(\xi,\,\bar{\mathbf{u}}).\end{aligned}$$ □

Thus, with the help of the change of variable (3.19), we have carried the state representation (2.4) to (3.20), which has the particularity that the nonlinearities $\mathbf{q}_*(\xi,\,\bar{\mathbf{u}})$ are contained in $\operatorname{Im}\mathrm{B}_{\mathcal{B}}$. Then, with the ideal state feedback:

$$\bar{\mathbf{u}}_* = \mathrm{F}\zeta - \mathbf{q}_*(\xi,\,\bar{\mathbf{u}}), \tag{3.21}$$

we get an exact feedback linearization, where $\mathrm{F} \in \mathbb{R}^{n\times m}$ is any stabilizing feedback of the pair $(\mathrm{A}_{\mathcal{B}},\,\mathrm{B}_{\mathcal{B}})$ namely:[3]

$$\sigma\left\{(\mathrm{A}_{\mathcal{B}} + \mathrm{B}_{\mathcal{B}}\mathrm{F})\right\} \subset \mathbb{C}^-. \tag{3.22}$$

4. STRUCTURAL ASYMPTOTIC FEEDBACK LINEARIZATION

Remember that the exact feedback linearization (3.21) requires a complete knowledge of the model, and particularly the time derivatives of the nonlinear variable $\mathbf{q}_o(\xi,\,\bar{\mathbf{u}})$. But this approach has the advantage, over the previous classical feedback linearization, that the unknown nonlinearities $\mathbf{q}_*(\xi,\,\bar{\mathbf{u}})$ belong to $\operatorname{Im}\mathrm{B}_{\mathcal{B}}$. So, they can be asymptotically attenuated with the help of disturbance rejectors based on standard state observers. For example, Bonilla *et al* (2016) have proposed the following disturbance rejector based on the Beard-Jones filter (*cf.* Isermann (1984); Massoumnia (1986); Saberi *et al* (2000)):

$$\begin{aligned}\mathrm{d}\mathbf{w}/\mathrm{d}t &= (\mathrm{A}_{\mathcal{B}} + \mathrm{K}\mathrm{C}_{\mathcal{B}})\,\mathbf{w} - \mathrm{K}\mathbf{y} + \mathrm{B}_{\mathcal{B}}\bar{\mathbf{u}}, \\ \bar{\mathbf{r}} &= \mathrm{G}^{\ell}\left(\mathrm{C}_{\mathcal{B}}\,\mathbf{w} - \mathbf{y}\right), \quad \bar{\mathbf{u}} = \mathrm{F}\zeta + \bar{\mathbf{r}},\end{aligned} \tag{4.23}$$

where $\mathrm{K} \in \mathbb{R}^{\hat{n}\times p}$ is an output injection to be computed, such that:

$$\sigma\left\{(\mathrm{A}_{\mathcal{B}} + \mathrm{K}\mathrm{C}_{\mathcal{B}})\right\} \subset \mathbb{C}^-. \tag{4.24}$$

And G^{ℓ} is a left inverse of the static gain, $-\mathrm{C}_{\mathcal{B}}\mathrm{A}_{\mathcal{B}_{\mathrm{K}}}^{-1}\mathrm{B}_{\mathcal{B}}$, of the remainder generator:[4]

$$\mathrm{d}\mathbf{e}/\mathrm{d}t = \mathrm{A}_{\mathcal{B}_{\mathrm{K}}}\,\mathbf{e} - \mathrm{B}_{\mathcal{B}}\,\mathbf{q}_*(\xi,\,\bar{\mathbf{u}})\,;\ \mathbf{r} = \mathrm{C}_{\mathcal{B}}\,\mathbf{e}, \tag{4.25}$$

where: $\mathrm{A}_{\mathcal{B}_{\mathrm{K}}} \doteq (\mathrm{A}_{\mathcal{B}} + \mathrm{K}\mathrm{C}_{\mathcal{B}})$, $\mathbf{e}(t) = \mathbf{w}(t) - \zeta(t)$ and $\mathbf{r} = \mathrm{C}_{\mathcal{B}}\,\mathbf{w} - \mathbf{y}$. Let us note that:[5] $\bar{\mathbf{r}}(\mathrm{s}) = -\,\mathrm{G}^{\ell}\,\mathsf{F}(\mathrm{s})\,\mathbf{q}_*(\mathrm{s})$, where: $\mathsf{F}(\mathrm{s}) \doteq \mathrm{C}_{\mathcal{B}}\,(\mathrm{sI} - \mathrm{A}_{\mathcal{B}_{\mathrm{K}}})^{-1}\,\mathrm{B}_{\mathcal{B}}$. Then, under the assumption that $\mathbf{q}_*(\xi,\,\bar{\mathbf{u}})$ is a bounded limited band frequency signal[6], we only need to synthesize a Hurwitz low-pass filter $\mathsf{F}(\mathrm{s})$, with a corner frequency ω_c, to achieve an asymptotic feedback linearization, indeed:

$$\|\bar{\mathbf{u}}_*(\omega) - \bar{\mathbf{u}}(\omega)\| \le \left\|\mathrm{G}^{\ell}\,\mathsf{F}(\jmath\omega) - \mathrm{I}\right\|\,\|\mathbf{q}_*(\omega)\|\,. \tag{4.26}$$

In (Bonilla *et al*, 2016) is shown (see their Lemma 1 and Theorem 1):

Under the Hurwitz stability assumptions (4.24) and (3.22), there exist $k_3,\,k_4,\,\alpha_2 \in \mathbb{R}^+$ such that the closed loop system behaves as:[7]

$$\left|\mathbf{y}(t) - \mathrm{C}_{\mathcal{B}}\int_0^t \exp\left(\mathrm{A}_{\mathcal{B}_{\mathrm{F}}}(t-\tau)\right)\mathrm{B}_{\mathcal{B}}\,\mathbf{u}_r(\tau)\,\mathrm{d}\tau\right| \le \left(k_3/\omega_c\right)\|\mathrm{d}\mathbf{q}_*(t)/\mathrm{d}t\|_\infty + k_4\,\mathrm{e}^{-\alpha_2 t},$$

where: $\mathrm{A}_{\mathcal{B}_{\mathrm{F}}} \doteq (\mathrm{A}_{\mathcal{B}} + \mathrm{B}_{\mathcal{B}}\,\mathrm{F})$. Moreover, the steady state response of y for high corner frequencies is:

$$\mathbf{y}_{ssr} = \lim_{\substack{t\to\infty \\ \omega_c\to\infty}} \mathbf{y}(t) = \mathrm{C}_{\mathcal{B}}\int_0^\infty \exp\left(\mathrm{A}_{\mathcal{B}_{\mathrm{F}}}(t-\tau)\right)\mathrm{B}_{\mathcal{B}}\,\mathbf{u}_r(\tau)\,\mathrm{d}\tau$$

5. QUADROTOR

Let us consider a Quadrotor where the total mass is M, the moment of inertia about axis ox, oy and oz are:[8] $\mathcal{I}_{xx}$,

[1] Let us note that (recall (2.6), (3.14) and (3.8)-(3.10)): *(i)* $\mathrm{N}_{\mathcal{B}}\mathrm{A}_{\mathcal{B}}\mathrm{A}_{\mathcal{B}}^T = \mathrm{N}_{\mathcal{B}}$, *(ii)* $(A_{\mathcal{B}}^T)^n = 0$ and *(iii)* $\mathcal{C}_n\Psi_n(\mathrm{d}/\mathrm{d}t) = \left(\mathrm{I} + (\mathrm{A}_{\mathcal{B}}^T)\mathrm{d}/\mathrm{d}t + \cdots + (\mathrm{A}_{\mathcal{B}}^T)^{(n-1)}\mathrm{d}^{n-1}/\mathrm{d}t^{n-1}\right)\mathrm{S}_o$.

[2] Recall (3.19) (3.16), (3.17) and (2.4), and that: $\mathrm{B}_{\mathcal{B}}^T\mathrm{B}_{\mathcal{B}} = \mathrm{I}$.

[3] Given a square matrix A, $\sigma\{\mathrm{A}\}$ stands for its spectrum.

[4] Recall (3.12), and note that: $(\mathrm{A}_{\mathcal{B}} + \mathrm{K}\mathrm{C}_{\mathcal{B}})\ker\mathrm{C}_{\mathcal{B}} = \mathrm{A}_{\mathcal{B}}\ker\mathrm{C}_{\mathcal{B}}$.

[5] In order to clarify ideas, let us assume for a while the existence of the Laplace transform of $\mathbf{q}_*(\xi,\,\bar{\mathbf{u}})$.

[6] Since F is a stabilizing feedback of the pair $(\mathrm{A}_{\mathcal{B}},\,\mathrm{B}_{\mathcal{B}})$, this assumption is satisfied in a local neighborhood around the equilibrium point $(\xi,\,\bar{\mathbf{u}}) = (\mathbf{0},\,\mathbf{0})$.

[7] $\mathbf{u}_r$ is some reference signal added to the control law (4.23).

[8] Since the Quadrotor is mechanically symmetric its products inertia are zero.

$\mathcal{I}_{yy}$ and $\mathcal{I}_{zz}$, and the distance of each rotor with respect to the centre of gravity of the quadrotor is: L (see Fig. 1).

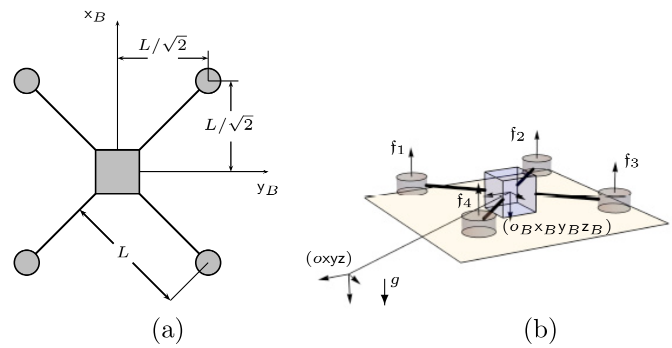

Fig. 1. Schematic diagram of the quadrotor. (a) Upper view. (b) Perspective view.

The motion is referred to a fixed orthogonal axis set (airframe) ($o\mathsf{xyz}$), where $o\mathsf{z}$ points vertically down along the gravity vector $[\,0\ 0\ g\,]^T$, and the origin o is located at the desired height $\bar{\mathsf{z}}$, above the ground level. ϕ, θ and ψ are the Euler angles, roll, pitch and yaw, measured respectively over the axis $o_B\mathsf{x}_B$, $o_B\mathsf{y}_B$ and $o_B\mathsf{z}_B$; where $(o_B\mathsf{x}_B\mathsf{y}_B\mathsf{z}_B)$ is the body axis system, with its origin o_B fixed at the centre of gravity of the quadrotor. We denote: $\eta = [\,\phi\ \theta\ \psi\,]^T$. Usually, the Quadrotor is represented by the following behavioral equations (see for example García *et al* (2013) and Cook (2013)):

Translational dynamics The translational behavior is represented by the non-linear state description:

$$\ddot{\mathsf{x}} = -\theta\, g + q_{\mathsf{x}}(\Delta u_{\mathsf{z}};\eta), \quad \ddot{\mathsf{y}} = \phi\, g + q_{\mathsf{y}}(\Delta u_{\mathsf{z}};\eta), \quad \ddot{\mathsf{z}} = \Delta u_{\mathsf{z}}/M + q_{\mathsf{z}}(\Delta u_{\mathsf{z}};\eta), \tag{5.27}$$

where: $\ddot{\mathsf{x}} \doteq \mathrm{d}^2\mathsf{x}/\mathrm{d}t^2$, $\ddot{\mathsf{y}} \doteq \mathrm{d}^2\mathsf{y}/\mathrm{d}t^2$, $\ddot{\mathsf{z}} \doteq \mathrm{d}^2\mathsf{z}/\mathrm{d}t^2$, and q_{x}, q_{y} and q_{z} are the variables describing the non-linear part of the translational model (see the Appendix), and Δu_{z} is the incremental control action: $\Delta u_{\mathsf{z}} = u_{\mathsf{z}} + Mg$, and u_{z} is defined hereafter in the thrusters model.

Attitude dynamics The attitude behavior is represented by the non-linear state description:

$$\ddot{\phi} = u_{\mathsf{y}}/\mathcal{I}_{xx} + q_\phi(\eta), \quad \ddot{\theta} = u_{\mathsf{x}}/\mathcal{I}_{yy} + q_\theta(\eta), \quad \ddot{\psi} = u_\psi/\mathcal{I}_{zz} + q_\psi(\eta), \tag{5.28}$$

where: $\ddot{\phi} \doteq \mathrm{d}^2\phi/\mathrm{d}t^2$, $\ddot{\theta} \doteq \mathrm{d}^2\theta/\mathrm{d}t^2$, $\ddot{\psi} \doteq \mathrm{d}^2\psi/\mathrm{d}t^2$, and q_ϕ, q_θ and q_ψ are the variables describing the non-linear part of the attitude model (see the Appendix), and u_{y}, u_{x} and u_ψ are the incremental control actions: $u_{\mathsf{y}} = \tau_\phi$, $u_{\mathsf{x}} = \tau_\theta$ and $u_\psi = \tau_\psi$, defined hereafter.

Thrusters model The control actions, u_{z}, τ_ϕ, τ_θ and τ_ψ, are related with the thrusters of the four rotors, $\mathfrak{f}_1$, $\mathfrak{f}_2$, $\mathfrak{f}_3$ and $\mathfrak{f}_4$, by means of the following invertible matrix:

$$\begin{bmatrix} u_{\mathsf{z}} \\ u_{\mathsf{y}} \\ u_{\mathsf{x}} \\ u_\psi \end{bmatrix} = \begin{bmatrix} -1 & -1 & -1 & -1 \\ k_\ell & k_\ell & -k_\ell & -k_\ell \\ -k_\ell & k_\ell & k_\ell & -k_\ell \\ -\gamma & \gamma & -\gamma & \gamma \end{bmatrix} \begin{bmatrix} \mathfrak{f}_1 \\ \mathfrak{f}_2 \\ \mathfrak{f}_3 \\ \mathfrak{f}_4 \end{bmatrix}$$

where: $k_\ell = L/\sqrt{2}$ and $\gamma = k_\tau/k_f$ [9] (the determinant of this matrix is: $-16\,\gamma\, k_\ell^2$).

[9] The thrusters and momentums generated by each rotor are: $\mathfrak{f}_i = k_f\omega_i^2$ and $\tau_i = k_\tau\omega_i^2$, where ω_i is the angular velocity of each rotor. Thus: $\tau_i = (k_\tau/k_f)\mathfrak{f}_i$, which implies (Powers *et al*, 2014): $\gamma = k_\tau/k_f$.

5.1 State Representations

From (5.27) and (5.28), we get the following state representations (*cf.* (2.3) and (2.4)):

State representation of x-dynamics :

$$\mathrm{d}\xi_{\mathsf{x}}/\mathrm{d}t = \mathrm{A}_{\mathcal{B}\mathsf{x}}\xi_{\mathsf{x}} + \mathrm{B}_{\mathcal{B}\mathsf{x}}u_{\mathsf{x}} + \mathrm{S}_{o\mathsf{x}}\mathbf{q}_{o\mathsf{x}}\,;\ \mathbf{y}_{\mathsf{x}} = \mathrm{C}_{\mathcal{B}\mathsf{x}}\xi_{\mathsf{x}}, \tag{5.29}$$

$$\mathrm{A}_{\mathcal{B}\mathsf{x}} = \mathbf{C_4}\left(\underline{\mathbf{0}}_{\mathbf{4}}^{\mathbf{T}}\right),\ \mathrm{B}_{\mathcal{B}\mathsf{x}} = \underline{\chi}_{\mathbf{4}}^{\mathbf{4}},\ \mathrm{C}_{\mathcal{B}\mathsf{x}}^T = (-g/\mathcal{I}_{yy})\,\underline{\chi}_{\mathbf{4}}^{\mathbf{1}},\quad \mathrm{S}_{o\mathsf{x}} = \left[\,(-\mathcal{I}_{yy}/g)\,\underline{\chi}_{\mathbf{4}}^{\mathbf{2}} \quad (\mathcal{I}_{yy})\,\underline{\chi}_{\mathbf{4}}^{\mathbf{4}}\,\right], \tag{5.30}$$

where: $\xi_{\mathsf{x}} = \mathrm{T}_{\mathcal{B}\mathsf{x}}^{-1}\mathbf{x}_{\mathsf{x}}$, $\mathbf{x}_{\mathsf{x}} = [\,\mathsf{x}\ \mathrm{d}\mathsf{x}/\mathrm{d}t\ \theta\ \mathrm{d}\theta/\mathrm{d}t\,]^T$, $\mathrm{T}_{\mathcal{B}\mathsf{x}} = \mathbf{BD}\{-g/\mathcal{I}_{yy},\, -g/\mathcal{I}_{yy},\, 1/\mathcal{I}_{yy},\, 1/\mathcal{I}_{yy}\}$, $\mathbf{q}_{o\mathsf{x}} = [\,q_{\mathsf{x}}\ q_\theta\,]^T$, $\mathrm{A}_{\mathcal{B}\mathsf{x}} = \mathrm{T}_{\mathcal{B}\mathsf{x}}^{-1}\,\mathrm{A}_{o\mathsf{x}}\,\mathrm{T}_{\mathcal{B}\mathsf{x}}$, $\mathrm{B}_{\mathcal{B}\mathsf{x}} = \mathrm{T}_{\mathcal{B}\mathsf{x}}^{-1}\,\mathrm{B}_{o\mathsf{x}}$, $\mathrm{C}_{\mathcal{B}\mathsf{x}} = \mathrm{C}_{o\mathsf{x}}\,\mathrm{T}_{\mathcal{B}\mathsf{x}}$, $\mathrm{S}_{o\mathsf{x}}\mathbf{q}_{o\mathsf{x}} = \mathrm{T}_{\mathcal{B}\mathsf{x}}^{-1}\mathbf{f}_{o\mathsf{x}}$ and $\mathbf{f}_{o\mathsf{x}} = (q_{\mathsf{x}})\underline{\chi}_{\mathbf{2}}^{\mathbf{4}} + (q_\theta)\underline{\chi}_{\mathbf{4}}^{\mathbf{4}}$.

Let us note that (3.11) is satisfied: $\mathrm{C}_{\mathcal{B}\mathsf{x}}\,\mathrm{A}_{\mathcal{B}\mathsf{x}}^T = 0$.

State representation of y-dynamics :

$$\mathrm{d}\xi_{\mathsf{y}}/\mathrm{d}t = \mathrm{A}_{\mathcal{B}\mathsf{y}}\xi_{\mathsf{y}} + \mathrm{B}_{\mathcal{B}\mathsf{y}}u_{\mathsf{y}} + \mathrm{S}_{o\mathsf{y}}\mathbf{q}_{o\mathsf{y}}\,;\ \mathbf{y}_{\mathsf{y}} = \mathrm{C}_{\mathcal{B}\mathsf{y}}\xi_{\mathsf{y}}, \tag{5.31}$$

$$\mathrm{A}_{\mathcal{B}\mathsf{y}} = \mathbf{C_4}\left(\underline{\mathbf{0}}_{\mathbf{4}}^{\mathbf{T}}\right),\ \mathrm{B}_{\mathcal{B}\mathsf{y}} = \underline{\chi}_{\mathbf{4}}^{\mathbf{4}},\ \mathrm{C}_{\mathcal{B}\mathsf{x}}^T = (g/\mathcal{I}_{xx})\,\underline{\chi}_{\mathbf{4}}^{\mathbf{1}},\quad \mathrm{S}_{o\mathsf{y}} = \left[\,(\mathcal{I}_{xx}/g)\,\underline{\chi}_{\mathbf{4}}^{\mathbf{2}} \quad (\mathcal{I}_{xx})\,\underline{\chi}_{\mathbf{4}}^{\mathbf{4}}\,\right], \tag{5.32}$$

where: $\xi_{\mathsf{y}} = \mathrm{T}_{\mathcal{B}\mathsf{y}}^{-1}\mathbf{x}_{\mathsf{y}}$, $\mathbf{x}_{\mathsf{y}} = [\,\mathsf{y}\ \mathrm{d}\mathsf{y}/\mathrm{d}t\ \phi\ \mathrm{d}\phi/\mathrm{d}t\,]^T$, $\mathrm{T}_{\mathcal{B}\mathsf{y}} = \mathbf{BD}\{g/\mathcal{I}_{xx},\, g/\mathcal{I}_{xx},\, 1/\mathcal{I}_{xx},\, 1/\mathcal{I}_{xx}\}$, $\mathbf{q}_{o\mathsf{y}} = [\,q_{\mathsf{y}}\ q_\phi\,]^T$, $\mathrm{A}_{\mathcal{B}\mathsf{y}} = \mathrm{T}_{\mathcal{B}\mathsf{y}}^{-1}\,\mathrm{A}_{o\mathsf{y}}\,\mathrm{T}_{\mathcal{B}\mathsf{y}}$, $\mathrm{B}_{\mathcal{B}\mathsf{y}} = \mathrm{T}_{\mathcal{B}\mathsf{y}}^{-1}\,\mathrm{B}_{o\mathsf{y}}$, $\mathrm{C}_{\mathcal{B}\mathsf{y}} = \mathrm{C}_{o\mathsf{y}}\,\mathrm{T}_{\mathcal{B}\mathsf{y}}$, $\mathrm{S}_{o\mathsf{y}}\mathbf{q}_{o\mathsf{y}} = \mathrm{T}_{\mathcal{B}\mathsf{y}}^{-1}\mathbf{f}_{o\mathsf{y}}$ and $\mathbf{f}_{o\mathsf{y}} = (q_{\mathsf{y}})\underline{\chi}_{\mathbf{2}}^{\mathbf{4}} + (q_\phi)\underline{\chi}_{\mathbf{4}}^{\mathbf{4}}$.

Let us note that (3.11) is satisfied: $\mathrm{C}_{\mathcal{B}\mathsf{y}}\,\mathrm{A}_{\mathcal{B}\mathsf{y}}^T = 0$.

State representation of ψ-dynamics :

$$\mathrm{d}\mathbf{x}_\psi/\mathrm{d}t = \mathrm{A}_{o\psi}\mathbf{x}_\psi + \mathrm{B}_{o\psi}u_\psi + \mathbf{f}_{o\psi}\,;\quad \psi = \mathrm{C}_{o\psi}\mathbf{x}_\psi$$

$$\mathrm{A}_{o\psi} = \mathbf{C_2}\left(\underline{\mathbf{0}}_{\mathbf{2}}^{\mathbf{T}}\right),\ \mathrm{B}_{o\psi} = (\mathcal{I}_{zz}^{-1})\underline{\chi}_{\mathbf{2}}^{\mathbf{2}},\ \mathbf{f}_{o\psi} = (q_\psi)\underline{\chi}_{\mathbf{2}}^{\mathbf{2}},\ \mathrm{C}_{o\psi}^T = \underline{\chi}_{\mathbf{2}}^{\mathbf{1}}, \tag{5.33}$$

where: $\mathbf{x}_\psi = [\,\psi\ \mathrm{d}\psi/\mathrm{d}t\,]^T$.

State representation of z-dynamics :

$$\mathrm{d}\mathbf{x}_{\mathsf{z}}/\mathrm{d}t = \mathrm{A}_{o\mathsf{z}}\mathbf{x}_{\mathsf{z}} + \mathrm{B}_{o\mathsf{z}}\Delta u_{\mathsf{z}} + \mathbf{f}_{o\mathsf{z}}\,;\quad \mathsf{z} = \mathrm{C}_{o\mathsf{z}}\mathbf{x}_{\mathsf{z}}$$

$$\mathrm{A}_{o\mathsf{z}} = \mathbf{C_2}\left(\underline{\mathbf{0}}_{\mathbf{2}}^{\mathbf{T}}\right),\ \mathrm{B}_{o\mathsf{z}} = (M^{-1})\underline{\chi}_{\mathbf{2}}^{\mathbf{2}},\ \mathbf{f}_{o\mathsf{z}} = (q_{\mathsf{z}})\underline{\chi}_{\mathbf{2}}^{\mathbf{2}},\ \mathrm{C}_{o\mathsf{z}}^T = \underline{\chi}_{\mathbf{2}}^{\mathbf{1}}, \tag{5.34}$$

where: $\mathbf{x}_{\mathsf{z}} = [\,\mathsf{z}\ \mathrm{d}\mathsf{z}/\mathrm{d}t\,]^T$.

The state representations (5.33) and (5.34) are already in the form (3.20) of Theorem 9.

5.2 Structural Decomposition

In order to get the state representation (3.20) of Theorem 9, we need the operator (*cf.* (3.8)-(3.10)):

$$\mathrm{X}_{\mathsf{x}}(\mathrm{d}/\mathrm{d}t) = -\mathrm{A}_{\mathcal{B}\mathsf{x}}^T\,\mathcal{C}_4(\mathrm{S}_{o\mathsf{x}})\Psi_4(\mathrm{d}/\mathrm{d}t) = \frac{\mathcal{I}_{yy}}{g}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ \mathrm{d}/\mathrm{d}t & 0 \end{bmatrix}$$

Thus:

$$\mathrm{C}_{\mathcal{B}\mathsf{x}}\mathrm{X}_{\mathsf{x}}(\mathrm{d}/\mathrm{d}t) \equiv 0,$$

$$\mathrm{S}_{o\mathsf{x}} - (\mathrm{I}_4\mathrm{d}/\mathrm{d}t - \mathrm{A}_{\mathcal{B}\mathsf{x}})\,\mathrm{X}_{\mathsf{x}}(\mathrm{d}/\mathrm{d}t) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -\left((\mathcal{I}_{yy}/g)\mathrm{d}^2/\mathrm{d}t^2\right) & \mathcal{I}_{yy} \end{bmatrix},$$

namely:

$$q_{*,\mathsf{x}} = -(\mathcal{I}_{yy}/g)\,\mathrm{d}^2q_{\mathsf{x}}/\mathrm{d}t^2 + \mathcal{I}_{yy}\,q_\theta. \tag{5.35}$$

In the same way:

$$q_{*,\mathsf{y}} = (\mathcal{I}_{xx}/g)\,\mathrm{d}^2q_{\mathsf{y}}/\mathrm{d}t^2 + \mathcal{I}_{xx}\,q_\phi. \tag{5.36}$$

6. NUMERICAL SIMULATIONS

We have considered the following numerical values, taken from a laboratory prototype:

$$M = 0.60\,\mathrm{kg},\ \mathcal{I}_{xx} = 0.00,32\,\mathrm{kg\,m^2},\ \mathcal{I}_{yy} = 0.00,32\,\mathrm{kg\,m^2},\ \mathcal{I}_{zz} = 0.00,58\,\mathrm{kg\,m^2},\ L = 0.17\,\mathrm{m},\ g = 9.81\,\mathrm{m\,s^{-2}}.$$

The control laws are the following (*cf.* (4.23)):

x-*dynamics* :

$$\begin{aligned} \mathrm{d}\mathbf{w}_{\mathsf{x}}/\mathrm{d}t &= \mathrm{A}_{\overline{K}_{\mathsf{x}}}\,\mathbf{w}_{\mathsf{x}} - \overline{\mathrm{K}}_{\mathsf{x}}\mathsf{x} + \overline{\mathrm{B}}_{\mathcal{B}\mathsf{x}}u_{\mathsf{x}}, \\ u_{\mathsf{x}} &= \mathrm{F}_{\mathsf{x}}\zeta_{\mathsf{x}} + \bar{r}_{\mathsf{x}},\quad \bar{r}_{\mathsf{x}} = \overline{\mathrm{G}}^{\ell}_{\mathsf{x}}\left(\overline{\mathrm{C}}_{\mathcal{B}\mathsf{x}}\,\mathbf{w}_{\mathsf{x}} - \mathsf{x}\right), \end{aligned} \tag{6.37}$$

where: $k_{\mathsf{x}} = -g\,\mathcal{I}_{yy}^{-1}$, $\overline{\mathrm{B}}_{\mathcal{B}\mathsf{x}} = k_{\mathsf{x}}\mathrm{B}_{\mathcal{B}\mathsf{x}}$, $\overline{\mathrm{C}}_{\mathcal{B}\mathsf{x}} = k_{\mathsf{x}}^{-1}\mathrm{C}_{\mathcal{B}\mathsf{x}}$, $\mathrm{A}_{\overline{K}_{\mathsf{x}}} = \left(\mathrm{A}_{\mathcal{B}\mathsf{x}} + \overline{\mathrm{K}}_{\mathsf{x}}\overline{\mathrm{C}}_{\mathcal{B}\mathsf{x}}\right)$, $\overline{\mathrm{G}}^{\ell}_{\mathsf{x}} = -\overline{\mathrm{C}}_{\mathcal{B}\mathsf{x}}\mathrm{A}^{-1}_{\overline{K}\mathsf{x}}\overline{\mathrm{B}}_{\mathcal{B}\mathsf{x}}$, $\bar{\zeta}_{\mathsf{x}} = \left[\,\mathsf{x}\ \ \mathrm{d}\mathsf{x}/\mathrm{d}t\ \ \mathrm{d}^2\mathsf{x}/\mathrm{d}t^2\ \ \mathrm{d}^3\mathsf{x}/\mathrm{d}t^3\,\right]^T$, $\zeta_{\mathsf{x}} = k_{\mathsf{x}}^{-1}\bar{\zeta}_{\mathsf{x}}$.

y-*dynamics* :

$$\begin{aligned} \mathrm{d}\mathbf{w}_{\mathsf{y}}/\mathrm{d}t &= \mathrm{A}_{\overline{K}_{\mathsf{y}}}\,\mathbf{w}_{\mathsf{y}} - \overline{\mathrm{K}}_{\mathsf{y}}\mathsf{y} + \overline{\mathrm{B}}_{\mathcal{B}\mathsf{y}}u_{\mathsf{y}}, \\ u_{\mathsf{y}} &= \mathrm{F}_{\mathsf{y}}\zeta_{\mathsf{y}} + \bar{r}_{\mathsf{y}},\quad \bar{r}_{\mathsf{y}} = \overline{\mathrm{G}}^{\ell}_{\mathsf{y}}\left(\overline{\mathrm{C}}_{\mathcal{B}\mathsf{y}}\,\mathbf{w}_{\mathsf{y}} - \mathsf{y}\right), \end{aligned} \tag{6.38}$$

where: $k_{\mathsf{y}} = g\,\mathcal{I}_{xx}^{-1}$, $\overline{\mathrm{B}}_{\mathcal{B}\mathsf{y}} = k_{\mathsf{y}}\mathrm{B}_{\mathcal{B}\mathsf{y}}$, $\overline{\mathrm{C}}_{\mathcal{B}\mathsf{y}} = k_{\mathsf{y}}^{-1}\mathrm{C}_{\mathcal{B}\mathsf{y}}$, $\mathrm{A}_{\overline{K}_{\mathsf{y}}} = \left(\mathrm{A}_{\mathcal{B}\mathsf{y}} + \overline{\mathrm{K}}_{\mathsf{y}}\overline{\mathrm{C}}_{\mathcal{B}\mathsf{y}}\right)$, $\overline{\mathrm{G}}^{\ell}_{\mathsf{y}} = -\overline{\mathrm{C}}_{\mathcal{B}\mathsf{y}}\mathrm{A}^{-1}_{\overline{K}\mathsf{y}}\overline{\mathrm{B}}_{\mathcal{B}\mathsf{y}}$, $\bar{\zeta}_{\mathsf{y}} = \left[\,\mathsf{y}\ \ \mathrm{d}\mathsf{y}/\mathrm{d}t\ \ \mathrm{d}^2\mathsf{y}/\mathrm{d}t^2\ \ \mathrm{d}^3\mathsf{y}/\mathrm{d}t^3\,\right]^T$, $\zeta_{\mathsf{y}} = k_{\mathsf{y}}^{-1}\bar{\zeta}_{\mathsf{y}}$.

ψ-*dynamics* :

$$\begin{aligned} \mathrm{d}\mathbf{w}_{\psi}/\mathrm{d}t &= \mathrm{A}_{\overline{K}_{\psi}}\,\mathbf{w}_{\psi} - \overline{\mathrm{K}}_{\psi}\psi + \overline{\mathrm{B}}_{\mathcal{B}\psi}u_{\psi}, \\ u_{\psi} &= \mathrm{F}_{\psi}\zeta_{\psi} + \bar{r}_{\psi},\quad \bar{r}_{\psi} = \overline{\mathrm{G}}^{\ell}_{\psi}\left(\overline{\mathrm{C}}_{\mathcal{B}\psi}\,\mathbf{w}_{\psi} - \psi\right), \end{aligned} \tag{6.39}$$

where: $\mathrm{A}_{\mathcal{B}\psi} = \mathrm{A}_{o\psi}$, $\overline{\mathrm{B}}_{\mathcal{B}\psi} = \mathrm{B}_{o\psi}$, $\overline{\mathrm{C}}_{\mathcal{B}\psi} = \mathrm{C}_{o\psi}$, $\mathrm{A}_{\overline{K}_{\psi}} = \left(\mathrm{A}_{\mathcal{B}\psi} + \overline{\mathrm{K}}_{\psi}\overline{\mathrm{C}}_{\mathcal{B}\psi}\right)$, $\overline{\mathrm{G}}^{\ell}_{\psi} = -\overline{\mathrm{C}}_{\mathcal{B}\psi}\mathrm{A}^{-1}_{\overline{K}\psi}\overline{\mathrm{B}}_{\mathcal{B}\psi}$, $\bar{\zeta}_{\psi} = \mathbf{x}_{\psi} = \left[\,\psi\ \ \mathrm{d}\psi/\mathrm{d}t\,\right]^T$, $\zeta_{\psi} = k_{\psi}^{-1}\bar{\zeta}_{\psi}$, $k_{\psi} = \mathcal{I}_{zz}^{-1}$.

z-*dynamics* From (5.34), we have that: $\bar{r}_{\mathsf{z}} = -Mq_{\mathsf{z}}$. And from (A.1(c)), it follows that: $\bar{r}_{\mathsf{z}} = M\,g\,\bar{q}_{\mathsf{z}}/(1+\bar{q}_{\mathsf{z}})$, then:

$$\Delta u_{\mathsf{z}} = \mathrm{F}_{\mathsf{z}}\zeta_{\mathsf{z}} + \bar{r}_{\mathsf{z}},\quad \bar{r}_{\mathsf{z}} = k_{\mathsf{z}}^{-1}\,g\,\bar{q}_{\mathsf{z}}/(1+\bar{q}_{\mathsf{z}}), \tag{6.40}$$

where: $\bar{\zeta}_{\mathsf{z}} = \mathbf{x}_{\mathsf{z}} = \left[\,\mathsf{z}\ \ \mathrm{d}\mathsf{z}/\mathrm{d}t\,\right]^T$, $\zeta_{\mathsf{z}} = k_{\mathsf{z}}^{-1}\bar{\zeta}_{\mathsf{z}}$, $k_{\mathsf{z}} = M^{-1}$.

The state feedbacks, F_{x}, F_{y}, F_{ψ} and F_{z}, were computed to minimize the criterions:

$$J_i = \int_0^{\infty}\left(\zeta_i^T\zeta_i + (1/5)u_i^2\right)\mathrm{d}t,\quad i\in\{\mathsf{x},\,\mathsf{y},\,\psi,\,\mathsf{z}\},$$

obtaining:

$$\begin{aligned} \mathrm{F}_{\mathsf{x}} = \mathrm{F}_{\mathsf{y}} &= \left[\,-0.4472\ \ -1.5814\ \ -2.5725\ \ -2.3119\,\right], \\ \mathrm{F}_{\psi} = \mathrm{F}_{\mathsf{z}} &= \left[\,-0.4472\ \ -1.0461\,\right], \end{aligned}$$

with the following dynamics (spectrums):[10]

$$\begin{aligned} \Lambda_{\mathrm{A}_{F_i}}(\mathrm{s}) &= \{-0.4162\pm 0.7387\,\imath,\ -0.7398\pm 0.2736\,\imath\}, \\ \Lambda_{\mathrm{A}_{F_j}}(\mathrm{s}) &= \{-0.5231\pm 0.4167\,\imath\},\quad i\in\{\mathsf{x},\,\mathsf{y}\},\ j\in\{\psi,\,\mathsf{z}\}. \end{aligned}$$

[10] $\mathrm{A}_{F_k} = (\mathrm{A}_{\mathcal{B}k} + \mathrm{B}_{\mathcal{B}k}\mathrm{F}_k)$, $k = i, j$.

The output injections, $\overline{\mathrm{K}}_{\mathsf{x}}$, $\overline{\mathrm{K}}_{\mathsf{y}}$, $\overline{\mathrm{K}}_{\psi}$ and $\overline{\mathrm{K}}_{\mathsf{z}}$, were computed for having a second and a forth order approximations of the Gaussian filter ($\kappa = 2,\,4$): $H(\mathrm{s})H(-\mathrm{s}) = 1\Big/\sum_{i=0}^{\kappa}\frac{(-1)^i}{i\,!}\mathrm{s}^{2i}$ (see for example Blinchikoff and Zverev (1976)), with the spectral radii, ρ_K, of the $\left(\mathrm{A}_{\mathcal{B}i} + \overline{\mathrm{K}}_i\overline{\mathrm{C}}_{\mathcal{B}i}\right)$ 100 times greater than the ones, ρ_F, of the $(\mathrm{A}_{\mathcal{B}i} + \mathrm{B}_{\mathcal{B}i}\mathrm{F}_i)$, $i\in\{\mathsf{x},\,\mathsf{y},\,\psi\}$, obtaining:

$$\begin{aligned} \overline{\mathrm{K}}_i &= \left[\,-430.1\ \ -78,122\ \ -6.962\times 10^6\ \ -2.532\times 10^8\,\right], \\ & i\in\{\mathsf{x},\,\mathsf{y}\},\quad \overline{\mathrm{K}}_{\psi} = \left[\,-146.9\ \ -6,325\,\right]. \end{aligned}$$

with the following spectra:

$$\begin{aligned} \Lambda_{\mathrm{A}_{\overline{K}_i}}(\mathrm{s}) &= \{-100.1\pm 89.9\,\imath,\ -114.9\pm 27.81\,\imath\}, \\ \Lambda_{\mathrm{A}_{\overline{K}_j}}(\mathrm{s}) &= \{-73.47\pm 30.43\,\imath\},\quad i\in\{\mathsf{x},\,\mathsf{y}\},\ j\in\{\psi,\,\mathsf{z}\}. \end{aligned}$$

Let us show some simulation results obtained in a MATLAB® platform. We have considered that the air-frame ($o\mathsf{xyz}$) is located at height $\bar{\mathsf{z}} = 0.3\,[\mathrm{m}]$, above the ground level, and with the initial conditions: $\mathsf{x}(0) = \mathsf{y}(0) = \mathsf{z}(0) = 0$ [m], $\mathrm{d}\mathsf{x}(0)/\mathrm{d}t = \mathrm{d}\mathsf{y}(0)/\mathrm{d}t = \mathrm{d}\mathsf{z}(0)/\mathrm{d}t = 0$ [$\mathrm{m\,s^{-1}}$], $\phi(0) = \theta(0) = \psi(0) = \alpha_0$, $\alpha_0 = \pi/6$ [rad], $\mathrm{d}\phi(0)/\mathrm{d}t = \mathrm{d}\theta(0)/\mathrm{d}t = \mathrm{d}\psi(0)/\mathrm{d}t = 0$ [$\mathrm{rad\,s^{-1}}$].

The initial conditions of the Beard-Jones filters (6.37(a)), (6.38(a)) and (6.39(a)) were set up as (see the Appendix):

$$\begin{aligned} \mathbf{w}_{\mathsf{x}}(0) &= \left[\,0\ \ 0\ \ -g\tan\alpha_0(\cos\alpha_0 + \tan\alpha_0)\ \ 0\,\right]^T, \\ \mathbf{w}_{\mathsf{y}}(0) &= \left[\,0\ \ 0\ \ g\tan\alpha_0(1-\sin\alpha_0)\ \ 0\,\right]^T, \mathbf{w}_{\psi}(0) = \left[\,\alpha_0\ \ 0\,\right]^T. \end{aligned}$$

In Fig. A.1, we show the simulation results.

7. CONCLUSION

In this paper, we have proposed a structural feedback linearization based on failure detection techniques. We have considered a non-linear system, represented by a non-linear state representation (1.1), for which there exists a structural differential operator $\mathrm{X}(\mathrm{d}/\mathrm{d}t)$, (3.8)-(3.8), which transforms (1.1) into the state description (3.20), where the non-linear components $\mathbf{q}_*$ are contained in the image of its constant input matrix $\mathrm{B}_{\mathcal{B}}$. The operator $\mathrm{X}(\mathrm{d}/\mathrm{d}t)$ depends on the way the constant matrix, S_o, of the non-linear components, $\mathbf{q}_o$, is related with the internal structure of the linear part, defined by the pair $(\mathrm{A}_o,\,\mathrm{B}_o)$, *cf.* (1.2).

Once arrived to (3.20), we only need to directly cancel $\mathbf{q}_*$, if it is of course known. And if this is not the case, it can be simply asymptotically rejected with a failure reconstructor, as for example, the one proposed in (Bonilla *et al*, 2016).

One advantage of this control schema is that it enlarges the linearity neighborhood around the equilibrium point $\mathbf{0}$. Also, the possible uncertainty's model is absorbed by $\mathbf{q}_*$.

We have used a quadrotor example to illustrate this performant alternative technique. In the simulation results, we have obtained a good control performance until a perturbation of $\pi/4$ rad in the Euler angles (ϕ, θ, ψ); here is reported the case $\pi/3$ rad.

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Appendix A. VARIABLES DEFINITIONS

In this Appendix, we use the abbreviated notations: $(c_\phi,\ c_\theta,\ c_\psi)$ for $(\cos\phi,\ \cos\theta,\ \cos\psi)$, $(s_\phi,\ s_\theta,\ s_\psi)$ for $(\sin\phi,\ \sin\theta,\ \sin\psi)$ and $(\dot\phi,\ \dot\theta,\ \dot\psi)$ for $(\mathrm{d}\phi/\mathrm{d}t,\ \mathrm{d}\theta/\mathrm{d}t,\ \mathrm{d}\psi/\mathrm{d}t)$. We also omit the explicit dependence, $(\Delta u_{\mathsf{z}};\ \eta)$, from the variables definitions, q_{x}, q_{y} and q_{z}.

The variables q_{x}, q_{y} and q_{z}, describing the non-linear part are:

$$\begin{gathered} q_{\mathsf{x}} = \theta\, g + (\Delta u_{\mathsf{z}}/M - g)\,\bar q_{\mathsf{x}}, \quad q_{\mathsf{y}} = -\phi g + (\Delta u_{\mathsf{z}}/M - g)\,\bar q_{\mathsf{y}}, \\ q_{\mathsf{z}} = (\Delta u_{\mathsf{z}}/M - g)\,\bar q_{\mathsf{z}}, \end{gathered} \tag{A.1}$$

$$\bar q_{\mathsf{x}} = c_\phi s_\theta c_\psi + s_\phi s_\psi,\ \bar q_{\mathsf{y}} = c_\phi s_\theta s_\psi - s_\phi c_\psi,\ \bar q_{\mathsf{z}} = c_\phi c_\theta - 1. \tag{A.2}$$

Note that: $q_{\mathsf{x}}(\Delta u_{\mathsf{z}};\ \mathbf{0}) = 0$, $q_{\mathsf{y}}(\Delta u_{\mathsf{z}};\ \mathbf{0}) = 0$ and $q_{\mathsf{z}}(\Delta u_{\mathsf{z}};\ \mathbf{0}) = 0$.

The variables q_ϕ, q_θ and q_ψ, describing the non-linear part of the attitude model, are defined as follows ($\mathbf{q}_\eta = \begin{bmatrix} q_\phi & q_\theta & q_\psi \end{bmatrix}^T$):

$$\mathbf{q}_\eta = \left(\mathrm{J}^{-1}(\eta) - \mathrm{J}^{-1}(0)\right)\tau - \mathrm{J}^{-1}(\eta)\,\mathrm{C}(\eta,\,\dot\eta)\,\dot\eta \tag{A.3}$$

The elements of the symmetric inertial matrix J are:

$$\begin{gathered} J_{11} = \mathcal{I}_{xx},\ J_{22} = \mathcal{I}_{yy} c_\phi^2 + \mathcal{I}_{zz} s_\phi^2,\ J_{23} = (\mathcal{I}_{yy} - \mathcal{I}_{zz})\, c_\phi s_\phi c_\theta, \\ J_{12} = 0,\ J_{13} = -\mathcal{I}_{xx} s_\theta,\ J_{33} = \mathcal{I}_{xx} s_\theta^2 + (\mathcal{I}_{yy} - \mathcal{I}_{zz})\, c_\phi s_\phi c_\theta \end{gathered}$$

The elements of the Coriolis matrix $C(\eta, \dot\eta)$ are ($\Delta_{\mathsf{zy}} = (\mathcal{I}_{zz} - \mathcal{I}_{yy})$):

$$\begin{aligned} & c_{11} = 0,\ c_{12} = -\mathcal{I}_{xx} c_\theta \dot\psi - \Delta_{\mathsf{zy}}\left(c_\phi s_\phi \dot\theta + \left(s_\phi^2 - c_\phi^2\right) c_\theta \dot\psi\right), \\ & c_{13} = \Delta_{\mathsf{zy}} c_\phi s_\phi c_\theta^2 \dot\psi,\ c_{21} = \mathcal{I}_{xx} c_\theta \dot\psi + \Delta_{\mathsf{zy}}\left(c_\phi s_\phi \dot\theta + \left(s_\phi^2 - c_\phi^2\right) c_\theta \dot\psi\right), \\ & c_{22} = \Delta_{\mathsf{zy}} c_\phi s_\phi \dot\phi,\ c_{23} = \left(\mathcal{I}_{zz} c_\phi^2 + \mathcal{I}_{yy} s_\phi^2 - \mathcal{I}_{xx}\right) c_\theta s_\theta \dot\psi, \\ & c_{31} = -\mathcal{I}_{xx} c_\theta \dot\theta - \Delta_{\mathsf{zy}} c_\phi s_\phi c_\theta^2 \dot\psi,\ c_{32} = \mathcal{I}_{xx} c_\theta s_\theta \dot\psi + \\ & \qquad \mathcal{I}_{yy}\left(\left(c_\phi^2 - s_\phi^2\right) c_\theta\,\dot\phi - c_\phi s_\phi s_\theta \dot\theta - s_\phi^2 c_\theta s_\theta \dot\psi\right) - \\ & \qquad \mathcal{I}_{zz}\left(\left(c_\phi^2 - s_\phi^2\right) c_\theta \dot\phi - c_\phi s_\phi s_\theta\,\dot\theta + c_\phi^2 c_\theta s_\theta \dot\psi\right), \\ & c_{33} = \mathcal{I}_{xx} c_\theta s_\theta \dot\theta + \mathcal{I}_{yy}\left(c_\phi s_\phi c_\theta \dot\phi - s_\phi^2 s_\theta \dot\theta\right) c_\theta - \\ & \qquad \mathcal{I}_{zz}\left(s_\phi c_\theta^2 \dot\phi + c_\phi c_\theta s_\theta \dot\theta\right) c_\phi \end{aligned}$$

Note that: $\mathrm{C}(\mathbf{0},\,\dot\eta)\,\dot\eta = \begin{bmatrix} -(\mathcal{I}_{xx} + \mathcal{I}_{yy} - \mathcal{I}_{zz})\,\dot\theta\,\dot\psi \\ (\mathcal{I}_{xx} - \mathcal{I}_{yy} + \mathcal{I}_{zz})\,\dot\phi\,\dot\psi \\ -(\mathcal{I}_{xx} - \mathcal{I}_{yy} + \mathcal{I}_{zz})\,\dot\phi\,\dot\theta \end{bmatrix}$.

From (5.27(a)) and (A.1(a)), we get: $\mathrm{d}^2\mathsf{x}(0)/\mathrm{d}t^2 = -g\theta(0) + q_{\mathsf{x}}(0) = (\Delta u_{\mathsf{z}}(0)/M - g)\ \bar q_{\mathsf{x}}(\eta(0))$. From (6.40) and (A.2(c)), we get ($\zeta_{\mathsf{z}}(0) = 0$): $\Delta u_{\mathsf{z}}(0)/M - g = -g/(c_{\phi(0)} c_{\theta(0)})$. Then: $\mathrm{d}^2\mathsf{x}(0)/\mathrm{d}t^2 = -g\left(s_{\theta(0)} c_{\theta(0)}^{-1} c_{\psi(0)} - s_{\phi(0)} c_{\phi(0)}^{-1} c_{\theta(0)}^{-1} s_{\psi(0)}\right)$ (see (A.2(a))). In a similar way: $\mathrm{d}^2\mathsf{y}(0)/\mathrm{d}t^2 = -g\left(s_{\theta(0)} c_{\theta(0)}^{-1} s_{\psi(0)} - s_{\phi(0)} c_{\phi(0)}^{-1} c_{\theta(0)}^{-1} c_{\psi(0)}\right)$. Since: $\dot\phi(0) = \dot\theta(0) = \dot\psi(0) = 0$, then: $\mathrm{d}^3\mathsf{x}(0)/\mathrm{d}t^3 = \mathrm{d}^3\mathsf{y}(0)/\mathrm{d}t^3 = 0$.

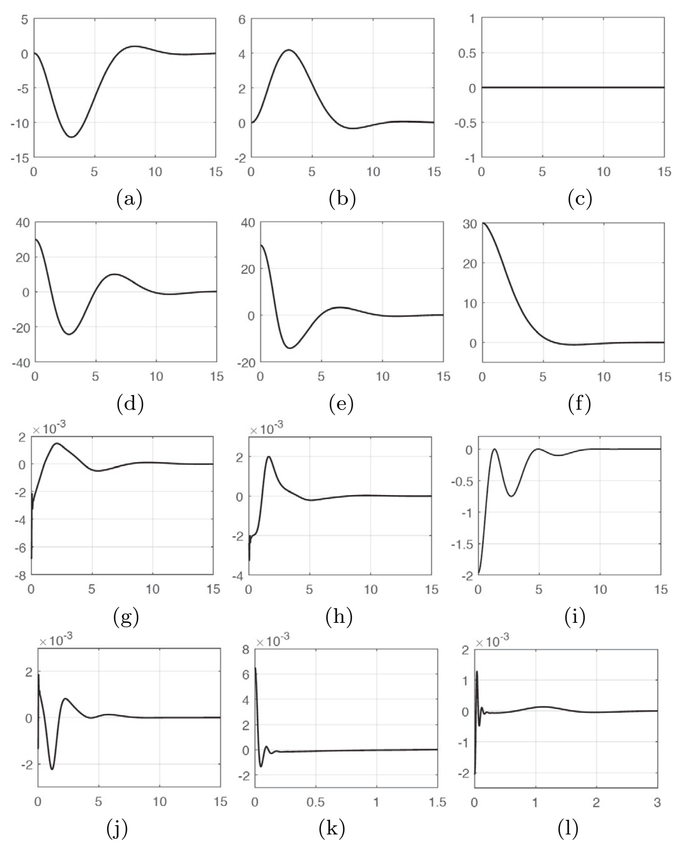

Fig. A.1. Center of mass position: (a) x [m], (b) y [m], (c) z [m]. Quadrotor attitude: (d) θ [°], (e) ϕ [°], (f) ψ [°]. Control signals: (g) u_{x}, (h) u_{y}, (i) Δu_{z}, (j) u_ψ. $\mathbf{q}_*$ estimation: (k) $\tilde q_{\mathsf{x}} = q_{*,\mathsf{x}} - \bar r_{\mathsf{x}}$, (l) $\tilde q_{\mathsf{y}} = q_{*,\mathsf{y}} - \bar r_{\mathsf{y}}$.