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A modified pseudospectral method for solving trajectory optimization problems with singular arc

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This paper presents a direct method based on Legendre–Radau pseudospectral method for efficient and accurate solution of a class of singular optimal control problems. In this scheme, based on *a priori* knowledge of control, the problem is transformed to a multidomain formulation, in which the switching points appear as unknown parameters. Then, by utilizing Legendre-Radau pseudospectral method, a nonlinear programming problem is derived which can be solved by the well-developed parameter optimization algorithms. The main advantages of the present method are its superior accuracy and ability to capture the switching times. Accuracy and performance of the proposed method are examined by means of some numerical experiments. Copyright © 2016 John Wiley & Sons, Ltd.

Keywords: pseudospectral method; singular optimal control problem; feedback rule; switching points; numerical solution; Legendre–Gauss–Radau

1. Introduction

A classical and challenging subject in optimal control field is singular optimal control problems. In these problems, the Pontryagin's maximum principle fails to directly determine the optimal control over the at least one interval. Singular optimal control problems arise in some well-known application areas, such as aerospace engineering. The sounding rocket problem proposed by Goddard [1], the analysis of wind shear during landing [2] and other optimal flight [3–5] all involve formulations with singular arcs.

Singular optimal control problems are, in general, not amenable to an analytical solution and must resort to numerical techniques. It seems that the classical direct and indirect numerical methods are suitable for solving singular optimal control problems. However, the accuracy of direct methods for the singular optimal control problems, especially in singular arcs, is not satisfactory; moreover, structure of the optimal control may not be detected adequately. On the other hand, for applying indirect methods, such as multiple shooting, *a priori* knowledge on the control structure is required.

Because of the mentioned difficulties, the simulation and numerical approximation of singular optimal control problems have received considerable attention. For instance, gradient technique [6], modified gradient technique [7], quasi Newton algorithm [8], quasi-linearization technique [9], indirect multiple shooting method [10, 11], direct shooting method [12], iterative dynamic programming method [13], and a lineup competition algorithm [14] can be referred to. In most of the mentioned papers, the control structure is assumed *a priori*. On the other hand, several authors have attempted to detect the structure of optimal control, for instance, the epsilon smoothing method [15], indirect shooting with epsilon smoothing [16], and other relevant methods [17–19]. As another family of methods for solving singular optimal control problems, we can refer to some two-phase methods, which are developed to reduce the drawbacks of the mentioned methods [20–23].

In this article, we consider the numerical solution of singular optimal control problems by a modified Legendre–Radau (LR) pseudospectral method [24]. To the best of the authors's knowledge, pseudospectral methods are employed widely for solving general

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optimal control problems but have not been improved or modified for the adaptive solution of singular optimal control problems. Similar to [10–12], only the problems were considered, in which control can be expressed as a function of the state variable. That is, the control has a feedback representation. Many practical singular problems belong to this family of singular optimal control problems. On the other hand, for developing a solver for general singular optimal control problems, we may need to solve a sequence of problems in the considered family. So an efficient and accurate method is essential for solving this family of singular problems.

In recent years, pseudospectral methods have been extensively used for the numerical solution of engineering problems [25–27] and optimal control problems [24, 28, 29]. In the pseudospectral method, the state and control variables are approximated by interpolating polynomials with specific collocation points such as Legendre–Gauss–Lobatto, Legendre–Gauss, and LR points [24]. Then, by collocating the state equations and path constraints and by utilizing differentiation matrix, the problem is transcribed to a nonlinear programming problem (NLP), which can be solved by a well-developed parameter optimization algorithm. The three most common types of pseudospectral method are Legendre–Gauss–Lobatto pseudospectral [30, 31], Legendre–Gauss pseudospectral [32], and LR pseudospectral [24, 33, 34] methods.

It is well known that pseudospectral methods, especially LR pseudospectral method, provide accurate approximations that converge exponentially for problems with smooth solution [30]. In contrast, in the case of the singular optimal control problems, due to nonsmoothness of control and state functions, using the pseudospectral methods can cause several issues, and high-order accuracy of the method is deteriorated. Furthermore, switching points in control cannot be captured by these methods, and adding more nodes, for overcoming these difficulties, could lead to inefficiencies and ill-conditioning of the resulted NLP. In this paper, to overcome all the mentioned numerical difficulties, a modified LR pseudospectral scheme is presented, which has two major differences from the traditional pseudospectral methods. First, instead of approximating the states by a polynomial in the whole computational domain, based on *a priori* knowledge of the structure of control, a piecewise smooth function and a piecewise continuous polynomial function are searched for state and control functions, respectively. Second, in the singular interval(s) by using feedback law, the control function is expressed by state functions.

The paper is organized as follows. In Section 2, formulation of singular optimal control problems and some necessary definitions are reviewed. Section 3 provides some background necessary for understanding the LR pseudospectral methods. In Section 4, a modified LR pseudospectral method is constructed and developed for solving the considered singular optimal control problems. The proposed methods are applied to two examples in Section 5. Finally, a conclusion is given in Section 6.

2. Problem statement and some preliminaries

Consider the following optimal control problems, in which the single control function is appeared linearly in dynamic system and the cost functional is of Mayer type

$$\min \mathcal{J}(\mathbf{x}, u, t_f) = g(\mathbf{x}(t_0), \mathbf{x}(t_f), t_f), \tag{1}$$

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}(t), u(t), t) = \mathbf{f}_1(\mathbf{x}(t), t) + \mathbf{f}_2(\mathbf{x}(t), t)u(t), \tag{2}$$

$$\boldsymbol{\psi}(\mathbf{x}(t_0), \mathbf{x}(t_f), t_f) = 0, \tag{3}$$

$$u \in \mathcal{U} := \left\{u \mid u(\cdot) \in [u^{\min}, u^{\max}] \text{ is piecewise constant function}\right\}. \tag{4}$$

Here, the state variable $\mathbf{x}(t) = [x_1(t), \cdots, x_p(t)]^T \in \mathbb{R}^p$ is a continuous vector function in $[t_0, t_f]$, where t_f may be free or fixed. The functions $g, \mathbf{f}_1, \mathbf{f}_2$, and $\boldsymbol{\psi}$ are sufficiently continuously differentiable in all arguments and defined by the following mappings:

$$\begin{aligned} &g : \mathbb{R}^{2p+1} \to \mathbb{R}, \\ &\mathbf{f}_1,\ \mathbf{f}_2 : \mathbb{R}^{p+1} \to \mathbb{R}^p, \\ &\boldsymbol{\psi} : \mathbb{R}^{2p+1} \to \mathbb{R}^r, \quad 0 \le r \le 2p. \end{aligned}$$

The Hamiltonian function for the aforementioned problem is defined by

$$\mathscr{H}(\mathbf{x}, u, \boldsymbol{\lambda}, t) := \boldsymbol{\lambda}^T \mathbf{f}_1(\mathbf{x}, t) + \boldsymbol{\lambda}^T \mathbf{f}_2(\mathbf{x}, t)u, \tag{5}$$

where $\boldsymbol{\lambda}(t) \in \mathbb{R}^p$ is the so-called adjoin or co-state vector function.

According to the Pontryagin's minimum principle [35], the solution of the problem (1)–(4) requires minimization of the Hamiltonian function (5) with respect to $u \in \mathcal{U}$ along the entire trajectories, which satisfy (2) and (3) and the following conditions:

$$\dot{\boldsymbol{\lambda}}^*(t) = -\mathscr{H}_x(\mathbf{x}^*(t), u^*(t), \boldsymbol{\lambda}^*, t), \tag{6}$$

$$\boldsymbol{\lambda}^*(t_0) = -\mathbf{l}_{x_0}(\mathbf{x}^*(t_0), u^*(t_f), t_f^*, \boldsymbol{\rho}), \tag{7}$$

$$\boldsymbol{\lambda}^*(t_f) = \mathbf{l}_{x_f}(\mathbf{x}^*(t_0), u^*(t_f), t_f^*, \boldsymbol{\rho}), \tag{8}$$

$$\mathscr{H}(t_f) + \mathbf{l}_{t_f}(\mathbf{x}^*(t_0), u^*(t_f), t_f^*, \boldsymbol{\rho}) = 0, \text{ if } t_f \text{ is free,} \tag{9}$$

where

$$\mathbf{l}(\mathbf{x}_0, \mathbf{x}_f, t_f, \boldsymbol{\rho}) := g(\mathbf{x}_0, \mathbf{x}_f, t_f) + \boldsymbol{\rho}^T \boldsymbol{\psi}(\mathbf{x}_0, \mathbf{x}_f, t_f). \tag{10}$$

In the considered problem, u appears linearly in the dynamic equations. So the Hamiltonian is linear in the control u, too. The factor u in the Hamiltonian is called the switching function and denoted by

$$\sigma(\mathbf{x}, \boldsymbol{\lambda}, t) := \boldsymbol{\lambda}^T \mathbf{f}_2(\mathbf{x}, t). \tag{11}$$

As a result of the Pontryagin's minimum principle, if in the time interval $[t_1, t_2] \in [t_0, t_f]$, the switching function $\sigma(t)$ be positive (negative), then $u(t)$ takes the smallest (largest) admissible control value $u^{\min}$ ($u^{\max}$). So if the switching function in the time interval $[t_1, t_2] \in [t_0, t_f]$ has isolated finite zeros, then the optimal control $u^*(t)$ fulfills

$$u^*(t) \in \{u^{\min}, u^{\max}\}, \qquad \forall t \in [t_1, t_2]. \tag{12}$$

In this case, the optimal control is called bang-bang in the interval $[t_1, t_2]$. If, however, there is a time interval $[t_1, t_2] \in [t_0, t_f]$ in which the switching function $\sigma(t)$ vanishes, then the Pontryagin's minimum principle provides no information about how to select $u^*(t)$. In this case, it is said that the problem is *singular* and the interval $[t_1, t_2]$ is called a *singular interval*. The control over a singular interval is referred to a *singular arc*. An optimal control problem, whose optimal control involves a singular arc is called *singular optimal control problem*.

In summary, minimization of the Hamiltonian function leads to the following control law [35, 36]:

$$u^*(t) = \begin{cases} u^{\min}, & \text{if} \quad \sigma(t) > 0, \\ u^{\max}, & \text{if} \quad \sigma(t) < 0, \\ u^{\sin}, & \text{if} \quad \sigma(t) = 0. \end{cases} \tag{13}$$

Accordingly, in general, singular optimal control contains both bang-bang and singular sub-arcs. Each point that is a transition between one bang-bang arc and another bang-bang or singular arc is called *switching point*.

2.1. *Order of the singular optimal control problems*

Note that $\frac{d}{dt}\sigma(\mathbf{x}, \boldsymbol{\lambda}, t)$ is explicitly a function of $\mathbf{x}$, $\boldsymbol{\lambda}$, $\dot{\mathbf{x}}$, $\dot{\boldsymbol{\lambda}}$, and t. By substituting $\dot{\mathbf{x}}$ and $\dot{\boldsymbol{\lambda}}$ from (2) and (6), $\frac{d}{dt}\sigma(\mathbf{x}, \boldsymbol{\lambda}, t)$ can be expressed as a function of $\mathbf{x}$, $\boldsymbol{\lambda}$, and t. It is easy to show that the control u does not appear in $\frac{d}{dt}\sigma$ [37]. By repeating this manner, $\frac{d^j}{dt^j}\sigma(\mathbf{x}, \boldsymbol{\lambda}, t)$ can be expressed as a function of $\mathbf{x}$, $\boldsymbol{\lambda}$, t, and maybe u. Furthermore, if u appears in $\frac{d^j}{dt^j}\sigma$, then it appears linearly [37]. It is possible that the control u does not appear in $\frac{d^j}{dt^j}\sigma$ for any j. However, if w be the first integer number that u appears in $\frac{d^w}{dt^w}\sigma$, then w is always even [38]. In the former case, the order of singular optimal control problem is defined to be infinite, and in the latter case, the integer number $\kappa = \frac{w}{2}$ is called order of singular problem.

Definition 2.1 (Order of singular problem [39])
The integer number κ is called the order of singular problem when 2κ is the lowest order derivative of switching function σ such that u appears explicitly. In other words,

$$\frac{d^{2\kappa}}{dt^{2\kappa}}\sigma(\mathbf{x}, \boldsymbol{\lambda}, t) \equiv e(\mathbf{x}, \boldsymbol{\lambda}, t) + d(\mathbf{x}, \boldsymbol{\lambda}, t)u, \qquad d \neq 0. \tag{14}$$

If u never appears explicitly in the differentiation process, then the optimal control problem is called an infinite-order singular problem.

Let the problem (1)–(4) be a singular problem of order κ and $[t_1, t_2]$ be the singular interval. So the control u appears explicitly in the 2κth derivative of the switching function σ with respect to t, as Eq. (14). Therefore, by noting that $\sigma = 0$ for $t \in [t_1, t_2]$, we conclude

$$\frac{d^{2\kappa}}{dt^{2\kappa}}\sigma(\mathbf{x}, \boldsymbol{\lambda}, t) = 0 = e(\mathbf{x}, \boldsymbol{\lambda}, t) + d(\mathbf{x}, \boldsymbol{\lambda}, t)u, \qquad d \neq 0. \tag{15}$$

Now, by solving Eq. (15) for u, we obtain

$$u = u(\mathbf{x}, \boldsymbol{\lambda}, t) = -\frac{d(\mathbf{x}, \boldsymbol{\lambda}, t)}{e(\mathbf{x}, \boldsymbol{\lambda}, t)}, \quad t \in [t_1, t_2].$$

In other words, if the singularity order of problem be finite, then by successive differentiation of the switching function, the control function u can be expressed as a function of $\mathbf{x}$, $\boldsymbol{\lambda}$, and t.

In some cases, the right-hand side of Eq. (15) does not depend on $\boldsymbol{\lambda}$, or $\boldsymbol{\lambda}$ can be eliminated because of

$$\frac{d^j}{dt^j}\sigma(\mathbf{x}, \boldsymbol{\lambda}, t) \equiv 0, \qquad j = 0, \ldots, 2\kappa - 1. \tag{16}$$

In these cases, in the singular interval $[t_1, t_2]$, the control u can be obtained in the following feedback form:

$$u(t) = \hat{u}(t; \mathbf{x}(t)), \qquad \forall t \in [t_1, t_2]. \tag{17}$$

2.2. Assumption on the problem

Similar to [10–12], we consider a family of optimal control problems, which is stated with the following assumptions.

Assumption 2.2
It is supposed that, 'chattering phenomenon' [40] does not occur; that is, we consider problems with a finite number of singular arcs and switching points.

Assumption 2.3
We assume that the singularity order of problem be finite and, in the singular arcs, the control function can be obtained in the feedback form (17).

Assumption 2.4
Assume that the structure of the optimal control problem is known; that is, the number of the switching points is known, and in each arc, it is known that $u(t)$ is either singular or nonsingular. Moreover, in nonsingular arcs, it is known that $u(t)$ takes its maximum value $u^{\max}$ or minimum value $u^{\min}$. Therefore, the unknowns are the positions of the switching points, singular control arcs, and state functions.

3. Background of Legendre–Gauss–Radau pseudospectral method

In the pseudospectral method [41], the unknown solution is expanded as global polynomial interpolants based on some suitable points. Also, the derivatives are approximated by discrete derivative operator (the differentiation matrix). So the concepts of interpolation and differentiation matrices are useful for understanding the pseudospectral method.

3.1. Approximation by polynomial interpolation

A function g defined on $[-1, 1]$ may be approximated by Lagrange polynomials as

$$g(\tau) \simeq \sum_{i=0}^{n} g(\xi_i)\ell_i(\tau), \tag{18}$$

where ξ_i, $i = 0, \cdots, n$ are distinct points in $[-1, 1]$, which are called collocation points, and $\ell_i(\tau)$, $i = 0, \ldots, n$ are the Lagrange polynomials corresponding to the considered collocation points, which are expressed as

$$\ell_i(\tau) = \prod_{j=0, j\neq i}^{n} \frac{\tau - \xi_j}{\xi_i - \xi_j}, \quad i = 0, \cdots, n,$$

with the Kronecker property

$$\ell_i(\xi_j) = \delta_{ij} = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j. \end{cases} \tag{19}$$

It is a well-established fact that a proper choice of collocation points is crucial in terms of accuracy and computational stability of the approximation (18). As a typically good choice of such collocation points, we can refer to the well-known Gauss, Gauss–Lobatto, and Gauss–Radau points [42], which lie on $[-1, 1]$ and are clustered near the endpoints.

In Legendre–Gauss–Radau (LGR) pseudospectral method for optimal control problems [24], the first n nodes are LGR nodes, and the last node is selected as $\xi_n = +1$. It is noted that LGR nodes are the roots of $P_{n-1}(\tau) + P_n(\tau)$, where $P_n(\tau)$ is the well-known Legendre polynomial of degree n. No explicit formula of the LGR nodes is known. However, these points can be determined by accurate and stable numerical methods [43].

To develop matrix-oriented methods, we express Eq. (18) in the following matrix from

$$g(\tau) \simeq [\boldsymbol{\phi}(\tau)]^T \mathbf{g},$$

where $\mathbf{g} = [g(\xi_0), \cdots, g(\xi_n)]^T$ and $\boldsymbol{\phi}(\tau)$ is a $(n+1)$-dimensional vector function as

$$\boldsymbol{\phi}(\tau) = [\ell_0(\tau), \cdots, \ell_n(\tau)]^T.$$

3.2. Differentiation matrix

In pseudospectral methods, it is crucial to express the derivative $\dot{g}(\tau)$ in terms of $g(\tau)$ at the collocation points ξ_i, which can be carried out using the so-called differentiation matrices.

Let g be a function with a sufficient degree of smoothness and approximated as (18). The first derivative of g can be approximated by

$$\dot{g}(\tau) \simeq \sum_{i=0}^{n} g(\xi_i)\dot{\ell}_i(\tau).$$

By noting that $\dot{\ell}_i(\tau)$ is a polynomial of degree n, we can write

$$\dot{\ell}_i(\tau) = \sum_{j=0}^{n} \dot{\ell}_i(\xi_j)\ell_j(\tau).$$

Using the aforementioned two equations, we obtain

$$\dot{g}(\tau) \simeq \sum_{i=0}^{n}\sum_{j=0}^{n} \dot{\ell}_i(\xi_j)g(\xi_i)\ell_j(\tau),$$

so the value of $\dot{g}(\tau)$ in $\tau = \xi_j$ can be approximated as

$$\dot{g}(\xi_j) \simeq \sum_{i=0}^{n} d_{ij}g(\xi_i),$$

where

$$d_{ij} = \dot{\ell}_i(\xi_j), \;\; i,j = 0,\cdots,n, \tag{20}$$

is the (i,j)th component of a matrix $\mathbf{D}$, which is called differentiation matrix [41]. According to (20), the entries of differentiation matrix $\mathbf{D}$ are computed by taking the analytical derivative of $\ell_i(\tau)$ and evaluating it at collocation points ξ_j for $i,j = 0,\cdots,n$. However, more computationally practical methods for deriving these entries, in an accurate and stable manner, can be found in [44–46].

4. Presented method for solving singular optimal control problems

Based on Assumption 2.2, it is assumed that the problem (1)–(4) has a solution with $s \geq 1$ switching point(s) denoted by $t_i, i = 1,\ldots,s$ such that

$$t_0 < t_1 < \cdots < t_s < t_f. \tag{21}$$

So if we set $t_{s+1} = t_f$, then the time interval $[t_0, t_f]$ is divided into the following $s+1$ subintervals:

$$[t_0, t_1], [t_1, t_2], \ldots, [t_s, t_{s+1}], \tag{22}$$

where $\cup_{k=0}^{s}[t_k, t_{k+1}] = [t_0, t_f]$ and $\cap_{k=0}^{s}[t_k, t_{k+1}) = \emptyset$.

For $k = 0,\ldots,s$, let the restriction of $\mathbf{x}(t)$ and $u(t)$ in the kth subinterval $[t_k, t_{k+1}]$ be denoted by $\mathbf{x}^{[k]}(t) = \left[x_1^{[k]}(t),\cdots,x_p^{[k]}(t)\right]^T$ and $u^{[k]}(t)$, respectively. So the control and state functions can be expressed as

$$u(t) = \begin{cases} u^{[0]}(t), \;\; t\in[t_0,t_1), \\ u^{[1]}(t), \;\; t\in[t_1,t_2), \\ \quad\vdots \\ u^{[s]}(t), \;\; t\in[t_s,t_{s+1}], \end{cases}, \quad \mathbf{x}(t) = \begin{cases} \mathbf{x}^{[0]}(t), \;\; t\in[t_0,t_1), \\ \mathbf{x}^{[1]}(t), \;\; t\in[t_1,t_2), \\ \quad\vdots \\ \mathbf{x}^{[s]}(t), \;\; t\in[t_s,t_{s+1}]. \end{cases}$$

Then, the problem (1)–(4) is reformulated as the following multidomain minimization problem:

$$\min \quad \mathcal{J} = g(\mathbf{x}^{[0]}(t_0), \mathbf{x}^{[s]}(t_f), t_f), \tag{23}$$

$$\dot{\mathbf{x}}^{[k]}(t) = \mathbf{f}(\mathbf{x}^{[k]}(t), u^{[k]}(t), t), t\in[t_k, t_{k+1}), k = 0,\ldots,s, \tag{24}$$

$$u^{\min} \leq u^{[k]}(t) \leq u^{\max}, t\in[t_k, t_{k+1}), \; k = 0,\ldots,s, \tag{25}$$

$$\boldsymbol{\psi}(\mathbf{x}^{[0]}(t_0), \mathbf{x}^{[s]}(t_f), t_f) = 0, \tag{26}$$

$$\mathbf{x}^{[k]}(t_{k+1}) = \mathbf{x}^{[k+1]}(t_{k+1}), \quad k = 0,\ldots,s-1. \tag{27}$$

Note that Eq. (27) is considered to guarantee the continuity of state functions in the switching points.

Based on Assumption 2.4, if $[t_k, t_{k+1}]$ be a nonsingular interval, then the control arc $u^{[k]}(t)$ is known and is equal to $u^{\min}$ or $u^{\max}$. Otherwise, if the interval $[t_k, t_{k+1}]$ be singular, then the control arc $u^{[k]}(t)$ is unknown but can be expressed by $\mathbf{x}^{[k]}(t)$ and t, as mentioned in (17). In other words, the control arcs $u^{[k]}(t)$, $k = 0, \ldots, s$, can be obtained based on $u^{\min}$, $u^{\max}$, or $\mathbf{x}^{[\mathbf{k}]}(t)$. Accordingly, in the kth interval $[t_k, t_{k+1}]$, we can denote the control function u, by $\hat{u}^{[k]}(t; \mathbf{x}^{[k]})$, where the function $\hat{u}^{[k]}(\cdot;\cdot)$ is a known function. By this consideration, the problem (23)–(27) can be written as

$$\min \quad \mathcal{J} = g(\mathbf{x}^{[0]}(t_0), \mathbf{x}^{[s]}(t_f), t_f), \tag{28}$$

$$\dot{\mathbf{x}}^{[k]}(t) = \mathbf{f}(\mathbf{x}^{[k]}(t), \hat{u}^{[k]}(t; \mathbf{x}^{[k]}), t \in [t_k, t_{k+1}), k = 0, \ldots, s, \tag{29}$$

$$u^{\min} \leq \hat{u}^{[k]}(t; \mathbf{x}^{[k]}) \leq u^{\max}, \ \text{ if } [t_k, t_{k+1}) \text{ be a sigular arc}, \tag{30}$$

$$\boldsymbol{\psi}(\mathbf{x}^{[0]}(t_0), \mathbf{x}^{[s]}(t_f), t_f) = 0, \tag{31}$$

$$\mathbf{x}^{[k]}(t_{k+1}) = \mathbf{x}^{[k+1]}(t_{k+1}), \quad k = 0, \ldots, s-1. \tag{32}$$

It is noted that, in the problem just shown, functions $\hat{u}^{[k]}(\cdot;\cdot)$, $k = 0, \ldots, s$, are known functions. So the unknowns are functions $\mathbf{x}^{[k]}(t)$, $k = 0, \cdots, s$, and maybe the final time t_f.

Now, to utilize Radau pseudospectral method, the time domain $[t_{k-1}, t_k], k = 0, \ldots, s$, is mapped to $[-1, 1]$ via the following affine transformations:

$$t = \frac{t_{k+1} - t_k}{2}\tau + \frac{t_{k+1} + t_k}{2}, \qquad k = 0, \ldots, s. \tag{33}$$

So by using the mapping (33) and by noting that $\frac{dt}{d\tau} = \frac{t_{k+1} - t_k}{2}$, the optimal control problem given in (23)–(27) is converted to the following minimization problem in the time domain $[-1, 1]$:

$$\min \quad \mathcal{J} := g(\mathbf{x}^{[0]}(-1), \mathbf{x}^{[s]}(1), t_f), \tag{34}$$

$$\begin{aligned} \dot{\mathbf{x}}^{[k]}(\tau) = \frac{t_{k+1} - t_k}{2} \mathbf{f}(\mathbf{x}^{[k]}(\tau), \hat{u}^{[k]}(\tau; \mathbf{x}^{[k]}), \tau), \\ \tau \in [-1, 1], k = 0, \ldots, s, \end{aligned} \tag{35}$$

$$u^{\min} \leq \hat{u}^{[k]}(\tau; \mathbf{x}^{[k]}) \leq u^{\max}, \ \text{ if } [t_k, t_{k+1}) \text{ be a singular arc}, \tag{36}$$

$$\boldsymbol{\psi}(\mathbf{x}^{[0]}(-1), \mathbf{x}^{[s]}(1), t_f) = 0, \tag{37}$$

$$\mathbf{x}^{[k]}(1) = \mathbf{x}^{[k+1]}(-1), \quad k = 0, \ldots, s-1. \tag{38}$$

It is noted that, by applying this transformation, the symbols of variables will change and new symbols should be used for them. For simplicity, however, we will retain the symbols already used.
Now, considering (18) for $l = 1, \ldots, p$, the lth component of state $\mathbf{x}^{[k]}(t)$ is approximated by Lagrange polynomials as

$$x_l^{[k]}(\tau) \simeq \sum_{i=0}^{n} x_l^{[k]}(\xi_i)\ell_i(\tau), \qquad k = 0, \ldots, s. \tag{39}$$

So the vector function $\mathbf{x}^{[k]}(t)$ is approximated as

$$\mathbf{x}^{[k]}(\tau) \simeq \sum_{i=0}^{n} \boldsymbol{\alpha}_i^k \ell_i(\tau), \qquad k = 0, \ldots, s, \tag{40}$$

where for $k = 0, \ldots, s$, $i = 0, \ldots, n$, the coefficients $\boldsymbol{\alpha}_i^k$ are unknown p-vector and

$$\boldsymbol{\alpha}_i^k = \mathbf{x}^{[k]}(\xi_i) = \left[x_1^{[k]}(\xi_i), \cdots, x_p^{[k]}(\xi_i)\right]^T.$$

Using (40), we have

$$\dot{\mathbf{x}}^{[k]}(\tau) \simeq \sum_{i=0}^{n} \boldsymbol{\alpha}_i^k \dot{\ell}_i(\tau), \qquad k = 0, \dots, s. \tag{41}$$

By substituting approximations (40) and (41) in the dynamics equations (35) and then by collocating it in LGR points ξ_j, $j = 0, \dots, n-1$, we acquire

$$\sum_{i=0}^{n} \boldsymbol{\alpha}_i^k \dot{\ell}_i(\xi_j) - \frac{t_{k+1} - t_k}{2} \mathbf{f}\left(\sum_{i=0}^{n} \boldsymbol{\alpha}_i^k \ell_i(\xi_j), \hat{u}^{[k]} \left(\xi_j; \sum_{i=0}^{n} \boldsymbol{\alpha}_i^k \ell_i(\xi_j) \right), \xi_j \right) = 0, \qquad k = 0, \dots, s,\ j = 0, \dots, n-1.$$

It is worthwhile to note that, although $\xi_n = +1$ is used beside the LGR points to approximate $\mathbf{x}^{[k]}(\tau)$, but this point is not used for collocation.

Now, by using Eq. (20) and Kronecker property (19), the aforementioned equation is reduced to the following algebraic equations:

$$\sum_{i=0}^{n} d_{ij} \boldsymbol{\alpha}_i^k - \frac{t_{k+1} - t_k}{2} \mathbf{f}\left(\boldsymbol{\alpha}_j^k, \hat{u}^{[k]} \left(\xi_j; \boldsymbol{\alpha}_j^k \right), \xi_j \right) = 0, \qquad k = 0, \dots, s,\ j = 0, \dots, n-1. \tag{42}$$

By noting that, for $k = 0, \dots, s$, $\mathbf{x}^{[k]}(-1) = \boldsymbol{\alpha}_0^{[k]}$ and $\mathbf{x}^{[k]}(1) = \boldsymbol{\alpha}_n^{[k]}$, finally, the optimal control problem (34)–(38) is converted to the following finite-dimensional NLP corresponding to the Radau pseudospectral method:

$$\min \quad \mathcal{J} := g\left(\boldsymbol{\alpha}_1^0, \boldsymbol{\alpha}_{n+1}^s, t_f \right), \tag{43}$$

$$\sum_{i=0}^{n} d_{ij} \boldsymbol{\alpha}_i^k - \frac{t_{k+1} - t_k}{2} \mathbf{f}\left(\boldsymbol{\alpha}_j^k, \hat{u}^{[k]} \left(\xi_j; \boldsymbol{\alpha}_j^k \right), \xi_j \right) = 0, \qquad k = 0, \dots, s,\ \ j = 0, \dots, n-1, \tag{44}$$

$$u^{\min} \le \hat{u}^{[k]} \left(\xi_j; \boldsymbol{\alpha}_j^k \right) \le u^{\max}, \tag{45}$$

$$\boldsymbol{\varphi}\left(\boldsymbol{\alpha}_0^0, \boldsymbol{\alpha}_n^s, t_f \right) = 0, \tag{46}$$

$$\boldsymbol{\alpha}_n^k = \boldsymbol{\alpha}_0^{k+1}, k = 0, \dots, s-1. \tag{47}$$

Here, decision variables of the aforementioned NLP problem are $\boldsymbol{\alpha}_i^k$, t_k, and maybe t_f, where $i = 0, \dots, n,\ k = 0, \dots, s$.

5. Illustrative examples

In this section, we implemented the proposed method in Section 3 with MATLAB on a personal computer for two examples. The final NLP (43)–(47) is solved by MATLAB function `fminco,`, and we set this solver to use sequential quadratic programming algorithm. For adjusting the accuracy of the obtained solution, the termination tolerance on the objective function value, tolerance on the constraint violation, and termination tolerance on decision variables are set to `TolFun=` $1e-12$, `TolCon=` $1e-14$, and `TolX=` $1e-10$, respectively.

5.1. Example 1 (Van der Pol oscillator problem)

Consider the following Van der Pol oscillator problem with fixed final time $t_f = 4$:

$$\begin{aligned} \min \quad & x_3(t_f), \\ & \dot{x}_1 = x_2, \\ & \dot{x}_2 = -x_1 + x_2\left(1 - x_1^2\right) + u, \\ & \dot{x}_3 = \tfrac{1}{2}\left(x_1^2 + x_2^2\right), \\ & x_1(0) = 0, \quad x_2(0) = 1, \quad x_3(0) = 0, \\ & -1 \le u(t) \le 1, \quad \forall t \in [0, t_f]. \end{aligned}$$

According to [12], it is found that this problem is a singular optimal control problem and the singular optimal control is composed of three arcs as

$$u(t) = \begin{cases} -1, & \text{if } 0 \le t \le t_1, \\ +1, & \text{if } t_1 \le t \le t_2, \\ u^{\text{sing}}(\mathbf{x}(t)), & \text{if } t_2 \le t \le t_f = 4. \end{cases}$$

Table I. van der Pol oscillator problem: the obtained values of switching times and performance index for various values n.

n	t_1	t_2	$\mathcal{J}$
4	**1.36**563582151	**2.4**5036140595	**0.75**9602956999
6	**1.3667**3389486	**2.460**79265972	**0.7576**28002959
8	**1.36674001**110	**2.46087**414442	**0.757617**964172
10	**1.36674004**533	**2.46087**450899	**0.757617928**785
15	**1.36674003**055	**2.46087448**102	**0.757617928659**
20	**1.36674003**414	**2.46087448**419	**0.757617928659**

The Hamiltonian of the optimal control problem is

$$\mathscr{H}(\mathbf{x}, u, \boldsymbol{\lambda}, t) = \lambda_1 x_2 + \lambda_2 \left(-x_1 + x_2 \left(1 - x_1^2\right) + u\right) + \tfrac{1}{2}\lambda_3 \left(x_1^2 + x_2^2\right).$$

Applying Pontryagins maximum principle leads to the following adjoint equations:

$$\dot{\lambda}_1 = -x_1 + \lambda_2(1 + 2x_1x_2), \tag{48}$$

$$\dot{\lambda}_2 = -x_2 - \lambda_1 - \lambda_2 \left(1 - x_1^2\right), \tag{49}$$

$$\dot{\lambda}_3 = 0. \tag{50}$$

The factor u in the Hamiltonian is $\lambda_2(t)$, so the switching function is given by $\sigma(\mathbf{x}, \boldsymbol{\lambda}, t) = \lambda_2(t)$. Therefore, $\frac{d}{dt}\sigma(\mathbf{x}, \boldsymbol{\lambda}, t) = \dot{\lambda}_2(t)$, and by using (49), we have

$$\frac{d}{dt}\sigma(\mathbf{x}, \boldsymbol{\lambda}, t) = \dot{\lambda}_2(t) = -\lambda_1(t) - x_2(t)\lambda_3(t).$$

It can be seen that u does not appear in $\frac{d}{dt}\sigma$, so we continue with $\frac{d^2}{dt^2}\sigma$. Similarly, using Eqs (48) and (50) leads to

$$\frac{d^2}{dt^2}\sigma(\mathbf{x}, \boldsymbol{\lambda}, t) = 2x_1(t) - x_2(t)\left(1 - x_1^2(t)\right) - u(t).$$

The control u appears in the second derivative of σ; therefore, the order of the problem is $\kappa = 1$. Moreover, by extracting u from $\frac{d^2}{dt^2}\sigma = 0$, the control function on the singular interval $[t_2, t_f]$ is obtained as

$$u = u^{\text{sing}}(\mathbf{x}) = 2x_1 - x_2\left(1 - x_1^2\right).$$

By applying the technique described in the preceding section, the state and control functions can be obtained. Moreover, the switching times are obtained, too. The computational results for t_1, t_2, and performance index are reported in Table I. It is seen that, by the present method, the switching times are provided accurately. Also, in Figure 1, the state and control functions obtained by the aforementioned method with $n = 20$ are plotted.

5.2. Example 2 (Goddard problem)

In this example, we consider Goddard problem, a benchmark for singular optimal control problems, which was introduced by Bryson and Ho [47]. The Goddard problem is to maximize the final altitude of a vertically ascending rocket under the influence of atmospheric drag and the gravitational field. The final time is free, and the state variables are altitude h, speed v, and mass m of the rocket during the flight, that is, $\mathbf{x} = (h, v, m)$. The control u is the trust curve of the rocket.

$$\begin{aligned}
\max \quad & h(t_f), \\
& \dot{h} = v, \\
& \dot{v} = \frac{1}{m}(cu - D(v,h)) - g(h), \\
& \dot{m} = -u, \\
& h(0) = 0, \quad v(0) = 0, \quad m(0) = m_0, \quad m(t_f) = m_f, \\
& 0 \le u(t) \le u^{\max}, \quad \forall t \in [0, t_f],
\end{aligned}$$

where the drag function $D(v, h)$ and gravity function $g(h)$ are defined as

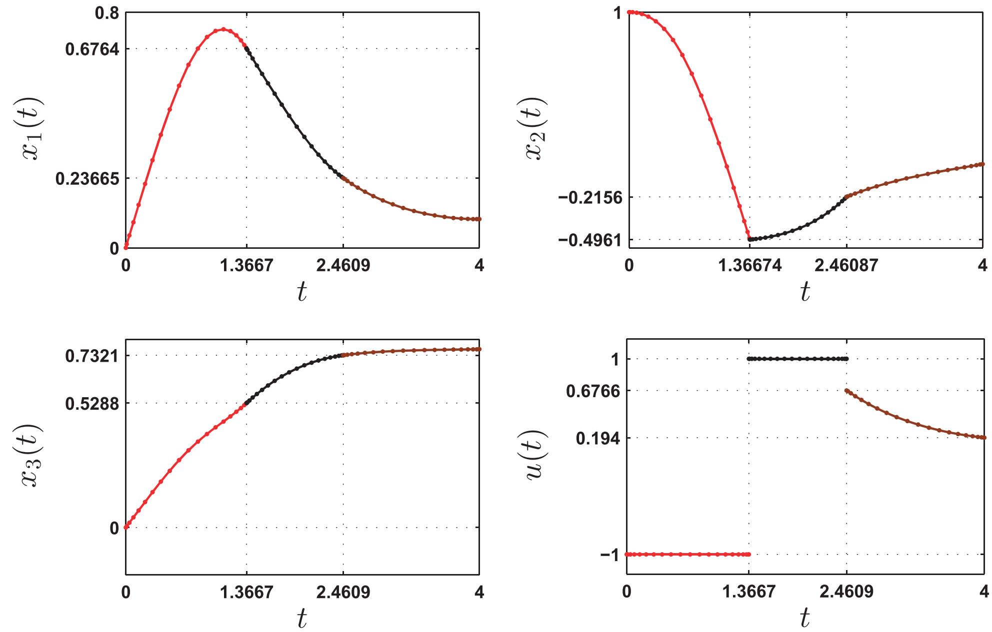

Figure 1. van der Pol oscillator problem: states and control histories obtained by the presented method with $n = 20$.

Table II. Goddard problem: the obtained values of switching times, final time, and performance index for various values *n*.

n	t_1	t_2	t_f	$\mathcal{J}$
4	3.89629843	**4**8.85670263	**20**1.25447	**15**2130.409
6	**4**.06034572	**46**.71383731	**20**5.22985	**15**6498.763
8	**4.1**1259805	**46**.06282708	**206**.44124	**157**254.022
10	**4.120**08101	**45.9**7098992	**206.6**6684	**1573**39.818
15	**4.1209**0929	**45.96082**620	**206.69**136	**157347.3**43
20	**4.1209109**2	**45.9608275**0	**206.6930**4	**157347.357**
25	**4.1209109**0	**45.9608275**1	**206.6930**2	**157347.357**

$$D(v,h) = \alpha v^2 exp(-\beta h), \qquad g(h) = g_0.$$

The problem data are taken from [10] as

$$\alpha = 0.01227, \quad \beta = 0.000145, \quad g_0 = 9.81, \quad c = 2060,$$
$$m_0 = 214.839, \quad m_f = 67.9833, \quad {}^{max} = 9.52551.$$

According to [10–12], it is known that the control function is singular with the following structure:

$$u(t) = \begin{cases} u^{max}, & \text{if } 0 \le t \le t_1, \\ u^{sing}(x(t)), & \text{if } t_1 \le t \le t_2, \\ 0, & \text{if } t_2 \le t \le t_f. \end{cases}$$

Following [11, 12], we show that the problem order is equal to $\kappa = 1$ and the singular control on the second interval $[t_1, t_2]$ is obtained as

$$u^{sing}(h,v,m) = \frac{D}{c} + m\frac{(c-v)D_h + (D_v + cD_{vv})g + c(mg_h - D_{vh}v)}{D + 2cD_v + c^2 D_{vv}}.$$

After applying the present method with various values of n, the resulted switching times and the objective function for this problem are summarized in Table II. Also, for a better vision, the state and control functions are plotted in Figure 2 with $n = 20$.

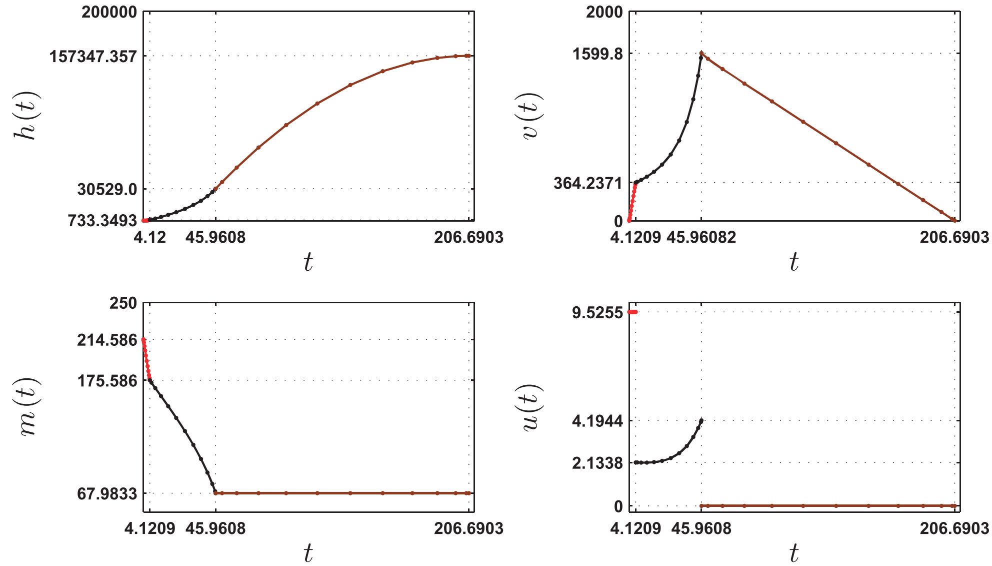

Figure 2. Goddard problem: states and control histories obtained by the presented method with $n = 12$.

6. Conclusion

In the present work, a modified LR pseudospectral procedure has been developed for obtaining the optimal solution of a family of singular optimal control problems. The main idea is to use an LR pseudospectral approach, in which the control function is considered as feedback form and the state variable is approximated by a piecewise continuous polynomial. The computational technique is illustrated on two benchmark problems. The results show that the present method obtains accurate solution and can capture switching points very accurately. We believe that the proposed approach can be extended to solve a broad class of singular optimal control problems.

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