ON THE LINEAR QUADRATIC DYNAMIC OPTIMIZATION PROBLEMS WITH FIXED-LEVELS CONTROL FUNCTIONS

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Abstract. This paper deals with a constrained LQ-type optimal control problem (OCP) in the presence of fixed levels input restrictions. We consider control processes governed by linear differential equations with a priori known control switching structure. The set of admissible inputs reflects some important natural engineering applications and moreover, can also be interpreted as a result of a quantization procedure applied to the original dynamic system. We propose a novel implementable algorithm that makes it possible to calculate a (numerically consistent) approximative solution to the constrained LQ-type OCPs under consideration. Our contribution mainly discusses theoretic aspects of the proposed solution scheme and contains an illustrative numerical example.

Keywords: optimal control, systems theory, convex optimization, numerical methods.

1. Introduction

Optimal control methodology is nowadays a mature powerful approach to the practical synthesis of several types of modern switched-type and interconnected dynamic systems (see e.g., [3], [4], [8], [14], [16], [18], [19], [20], [22], [27], [29], [30]). In this context let us also refer to [7], [9], [10], [29], [30], [32], [34], [39], [45] for some examples of specific optimization techniques and concrete real-world appplications. Recently, the problem of effective numerical methods for the constrained LQ based systems optimization has attracted a lot of attention, thus both theoretical results and applications were developed (see, e.g., [5], [6], [25], [22], [26], [27], [28], [31], [32],

[34], [36], [39], [40], [45] and the references therein). Note that handling various types of constraints in practical system design is an important issue in most, if not all, real world applications. It is readily appreciated that the implementable dynamic systems have a corresponding set of constraints; for example, inputs always have maximum and minimum values and states are usually required to lie within certain ranges. Moreover, it is generally true that optimal levels of performance are associated with operating on, or near, constraint boundaries (see [14], [21], [42]).Thus, a control engineer really can not ignore constraints without incurring a performance penalty.

The aim of our contribution is to elaborate a consistent computational algorithm for an LQ-type OCP in systems with piecewise constant control inputs. The given restrictive structure of the admissible control function under consideration is motivated by some important engineering applications (see [5], [6], [12], [13], [22], [26], [25], [27], [28], [30], [36], [40], [42]) as well as by application of common quantization procedures to the original dynamics (see e.g., [17], [31], [34]). Note that quadratic optimal control of piecewise linear systems was addressed earlier in [28], [36]. The treatment there was based on the backward solutions of Riccati differential equations, and the optimum had to be recomputed for each new final state. Computation of non-linear gain using the Hamilton-Jacobi-Bellman equation and the convex optimization techniques has also been done in [36]. Let us also refer to a sophisticated solution techniques for non-linear OCPs proposed in [11]. This approach is based on a newly developed relaxation procedure. On the other hand, the common optimization approaches to linear constrained and switched-type systems are not sufficiently advanced to LQ-type problems for linear systems with fixed levels controls. In our paper we propose a new numerical method that includes a specific relaxation scheme in combination with the classic projection approach. Moreover, it should be noted already at this point that the optimization algorithm we propose can be effectively used as a part of a concrete control design procedure for some classes of dynamic systems with switched nature.

Recall that the general switched (and hybrid) systems constitute formal framework of systems where two types of dynamics are present, continuous and discrete event behaviour (see, e.g., [16], [31]). Evidently, a dynamic model with fixed levels control inputs constitutes a simple example of a switched system. In order to understand how these systems can be operated efficiently, both aspects of the dynamics/controls have to be taken into account during the system optimization phase. The non-stationary linear systems we study in this paper include a particular family of switched systems with the time-driven location transitions. We refer to [1], [4], [15], [18], [19], [20], [23], [32], [38], [41], [43], [45] for some relevant examples and abstract concepts.

The remainder of our paper is organized as follows: Section 2 contains a problem statement, the necessary preliminary facts and basic concepts. Section 3 deals with a simple relaxation scheme of the initial constrained LQ-type OCP. We propose a projected gradient method for the concrete numerical treatment of the OCPs under consideration (see e.g. [4], [24], [33]). In Section 4 we discuss a specific controllability condition for the concrete dynamic system under consideration. Section 5 is devoted to the numerical aspects of the proposed algorithm and contains an illustrative example. Section 6 summarizes the paper.

2. Problem formulation and some basic facts

Consider the following linear non-stationary system with a switched control structure

$$\dot{x}(t) = A(t)x(t) + B(t)u(t),\ t \in [t_0, t_f],\ x(t_0) = x_0, \tag{1.1}$$

where $A(\cdot) \in \mathbb{L}^\infty[t_0,t_f;\mathbb{R}^{n\times n}]$, $B(\cdot) \in \mathbb{L}^\infty[t_0,t_f;\mathbb{R}^{n\times m}]$. Here $\mathbb{L}^\infty[t_0,t_f;R^{n\times n}]$ and $\mathbb{L}^\infty[t_0,t_f;\mathbb{R}^{n\times m}]$ are the standard Lebesgue spaces of the essentially bounded matrix-functions defined on a bounded time interval $[t_0,t_f]$. Similarly to the classic LQR (the Linear Quadratic Regulator) theory it is desired to minimize the following quadratic cost functional associated with (1.1)

$$J(u(\cdot)) = \frac{1}{2}\int_{t_0}^{t_f} (\langle Q(t)x(t),x(t)\rangle + \langle R(t)u(t),u(t)\rangle)dt + \frac{1}{2}\langle Gx(t_f),x(t_f)\rangle, \tag{1.2}$$

where $G \in \mathbb{R}^{n\times n}$ and the matrix-functions $Q(\cdot)$ and $R(\cdot)$ are assumed to be integrable. Following the conventional LQR theory we next introduce the standard regularity/ positivity hypothesis: $G \geq 0$, $Q(t) \geq 0$, $R(t) \geq \delta I$, $\delta > 0\ \forall t \in [t_0,t_f]$. It is well known that the classic LQ optimal control strategy $u^{opt}(\cdot)$ does not incorporate any additional (state or control) restrictions into the resulting design procedure. Let us recall here the explicit formula for $u^{opt}(\cdot)$ (see e.g., [21], [27])

$$u^{opt}(t) = -R^{-1}(t)\left[B^T(t)P(t)\right]x^{opt}(t), \tag{1.3}$$

where $P(\cdot)$ is the matrix-function, namely, the solution to the classic differential matrix Riccati equation associated with the LQ problem (1.1)-(1.2). In the above-mentioned conventional case (1.1)-(1.2) the optimization problem is formally studied in the full space $\mathbb{L}^2[t_0,t_f;\mathbb{R}^m]$ of square integrable control functions. In contrast to the classic case, we consider system (1.1) in combination with the specific piecewise constant admissible inputs $u(\cdot)$ of the following type (see Fig. 1).

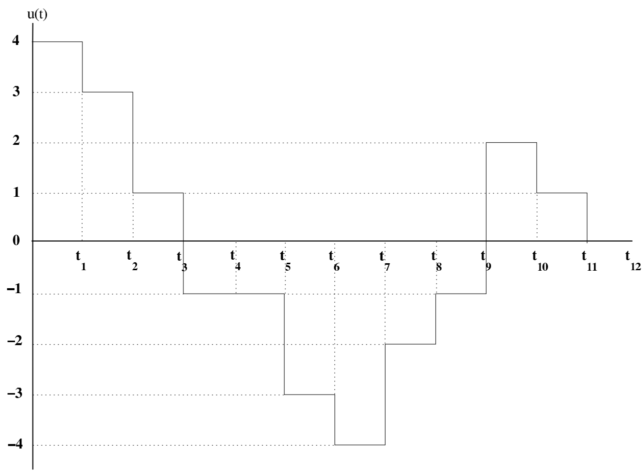

Figure 1: The admissible switched-type control inputs $u(\cdot)$

Resulting from the admissibility assumption the main minimization problem for the linear system (1.1) can be interpreted as a restricted LQ optimization problem. For example, the control signal $u(\cdot)$ showed in Fig. 1 can take a value (level) within the finite set $\mathbb{Q} = \{-4,-3,-2,-1,0,1,2,3,4\}$ during the time interval $[t_{i-1},t_i)$, $i = 1,\dots,12$. In addition the control signal here is only allowed to change its value at the times $t_0,t_1,\dots,t_f$ being fixed between these times.

Let us now specify formally the set of admissible piecewise constant control functions for system (1.1) in a general case. For each component $u^{(k)}(\cdot)$ of the feasible control input $u(\cdot) = [u^{(1)}(\cdot),\dots,u^{(m)}(\cdot)]^T$ we introduce the following finite set of feasible value levels: $\mathbb{Q}^k := \{q_j^{(k)} \in \mathbb{R},\ j = 1,\dots,M_k\},\ M_k \in \mathbb{N}, \quad k = 1,\dots,m$. In general, all the sets $\mathbb{Q}^k$ are different (contains different levels) and have various numbers of elements. In addition, each $\mathbb{Q}^k$ possesses a strict order property

$$q_1^{(k)} < q_2^{(k)} < \ldots < q_{M_k}^{(k)}.$$

We now introduce the set of switching times associated with an admissible control function $\mathbb{T}^k := \{t_i^{(k)} \in \mathbb{R}_+, i = 1,\dots,N_k\}, \quad N_k \in \mathbb{N}, \quad k = 1,\dots,m$. The sets $\mathbb{T}^k$ are assumed to be defined for each control component $u^{(k)}(\cdot)$, $k = 1,\dots,m$, where $\mathbb{R}_+$ denotes a non-negative semi-axis. Let us consider an ordered sequence of time instants: $t_0 < t_1^{(k)} < \ldots < t_{N_k}^{(k)}$. For the final time instants associated with each set $\mathbb{T}^k$ we put $t_{N_1}^{(1)} = \ldots = t_{N_m}^{(m)} = t_f$. Using the notation of the level sets $\mathbb{Q}^k$ and the fixed switching times $\mathbb{T}^k$ introduced above, the set of admissible controls $\mathscr{S}$ can now be easily specified by the Cartesian product

$$\mathscr{S} := \mathscr{S}_1 \times \ldots \times \mathscr{S}_m, \tag{1.4}$$

where each set $\mathscr{S}_k$, $k = 1,...,m$ is defined as follows

$$\mathscr{S}_k := \{v(\cdot) \mid v(t) = \sum_{i=1}^{N_k} I_{[t_{i-1}^{(k)},t_i^{(k)})}(t) q_{j_i}^{(k)}; q_{j_i}^{(k)} \in \mathbb{Q}^k; j_i \in \mathbb{Z}[1,M_k];\ t_i^{(k)} \in \mathbb{T}^k\}.$$

By $\mathbb{Z}[1,M_k]$ we denote here the set of all integers into the interval $[1,M_k]$ and $I_{[t_{i-1}^k,t_i^k)}(t)$ is the characteristic function of the interval $[t_{i-1}^k,t_i^k)$. Evidently, the set of admissible control inputs $\mathscr{S}$ can be qualitatively interpreted as the set of all the possible functions $u : [t_0,t_f] \to \mathbb{R}^m$, such that each component $u^{(k)}(\cdot)$ of $u(\cdot)$ attains a constant level value $q_{j_i}^{(k)} \in \mathbb{Q}^k$ for $t \in [t_{i-1}^k,t_i^k)$. Moreover, the component level changes occur only at the prescribed times $t_i^k \in \mathbb{T}^k$, $i = 1,...,N_k - 1$. The clear combinatorial character of the examined control functions associated with the initial system (1.1) can be illustrated by a simple example.

Example 1.1 Suppose $u(t) \in \mathbb{R}^2$ and $\mathbb{Q}^1 = \{0,1,2\}$, $\mathbb{Q}^2 = \{0,-1\}$. Furthermore, the set of switching times for each control component is assumed to be given by $\mathbb{T}^1 = \{0,0.5,1\}$, $\mathbb{T}^2 = \{0,0.33,0.66,1\}$. Resulting from the above definitions, the set $\mathscr{S}$ in (1.4) can be written as $\mathscr{S} = \mathscr{S}_1 \times \mathscr{S}_2$, where:

$$\mathscr{S}_1 = \{v : [t_0, t_f] \to \mathbb{R} \mid v(t) = I_{[0,0.5)}(t) q^{(1)}_{j_1} + I_{[0.5,1)}(t) q^{(1)}_{j_2},\ q^{(1)}_{j_i} \in \mathbb{Q}^1\};$$
$$\mathscr{S}_2 = \{w : [t_0, t_f] \to \mathbb{R} \mid w(t) = I_{[0,0.33)}(t) q^{(2)}_{j_1} +$$
$$I_{[0.33,0.66)}(t) q^{(2)}_{j_2} + I_{[0.66,1)}(t) q^{(2)}_{j_3}, q^{(2)}_{j_i} \in \mathbb{Q}^2\}$$

In that concrete case we evidently have: $M_1 = 3$, $M_2 = 2$, $N_1 = 2$ and $N_2 = 3$.

The cardinality of the control set $\mathscr{S}$ is given as follows: $|\mathscr{S}| = 3^2 \cdot 2^3 = 72$. In other words, we have 72 admissible control inputs, among which we must find the one that minimizes the quadratic performance criterion.

In general, the cardinality of the set $\mathscr{S}$ of admissible controls $u(\cdot)$ with $u(t) \in \mathbb{R}^m$ can be expressed as follows

$$|\mathscr{S}| = \prod_{l=1}^{m} M_l^{N_l}. \tag{1.5}$$

Motivating from various engineering applications, we now can formulate the following specific constrained LQ-type OCP

$$\begin{aligned} &\text{minimize } J(u(\cdot)) \\ &\text{subject to } u(\cdot) \in \mathscr{S}, \end{aligned} \tag{1.6}$$

where $J(\cdot)$ is the costs functional defined in (1.2). Note that $\mathscr{S}$ constitutes a nonempty subset of the space $\mathbb{L}^2[t_0, t_f; \mathbb{R}^m]$. However, the classically LQ-optimal control input $u^{opt}(\cdot)$ in (1.3) does not belong to the introduced specific set $\mathscr{S}$. Due to the highly restrictive condition $u(\cdot) \in \mathscr{S}$, the main optimization problem (1.6) can not be generally solved by a direct application of the classic Pontryagin Maximum Principle. A possible application of a suitable hybrid version of the conventional Maximum principle from [4], [16], [23], [38], [41], [45] is also complicated by a non-standard structure of the simple control inputs under consideration. Let us additionally note that the value of an exponentially growing cardinality $|\mathscr{S}|$ in (1.5) exacerbates crucially a possible application of some combinatorial and various state/control discretization based numerical algorithms for OCPs (see e.g., [7], [8], [13], [24], [33], [34], [35], [38], [42], [45] and the references therein).

The aim of this contribution is to propose a relative simple implementable computational procedure for a consistent numerical treatment of the constrained OCP (1.6). We use a simple relaxation technique associated with the main OCP (1.6) in combination with a gradient based algorithm for this purpose. We first obtain an optimal solution of a convex relaxed OCP. Next we use it in a constructive solution procedure for the original problem (1.6).

3. The gradient-based approach to the relaxed optimal control problem

In this section we propose a constructive computational scheme for the constrained LQ-type OCP (1.6) formulated above. The proposed approach incorporates a simply relaxed OCPs associated with the initial problem (1.6). Let us first recall a necessary auxiliary result from the classic convex analysis (see [3], [33], [37]): it is a well known fact that a composition of two convex functionals is not necessarily convex. In the following we will need a basic result providing conditions that ensure convexity of the composition (see e.g., [3], [37]).

Lemma 3.1 *Let $g^1 : \mathscr{W} \to \mathbb{R}$ be a convex functional determined on a convex set $\mathscr{W} \subseteq \mathbb{R}^p$ and $g^2 : \mathscr{V} \to \mathscr{W}$ be an affine mapping defined on a convex subset $\mathscr{V}$ of a real* Hilbert *space H. Then the composed functional $g : \mathscr{V} \to \mathbb{R}$, $g(\cdot) := g^1(g^2(\cdot))$ is convex.*

Let now $x^u(\cdot)$ be a solution to the initial value problem (1.1) generated by an admissible control $u(\cdot) \in \mathscr{S}$. Evidently, every component of $x^u(\cdot)$ is an affine function (functional) of $u(\cdot)$

$$x(t,u) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau. \tag{3.1}$$

Here $\Phi(\cdot,\tau)$ is the fundamental solution matrix associated with (1.1). Let us note that set of admissible controls $\mathscr{S}$ constitutes a non-convex set. This fact is due to the originally combinatorial structure of $\mathscr{S}$ determined in (1.4).

Example 3.2 Under assumptions of Example 1.1 we have $\mathscr{S} := \mathscr{S}_1 \times \mathscr{S}_2$ and moreover,

$$\begin{aligned}
&\mathscr{S}_1 = \{(0 \times I_{[0,0.5)}(t) + 1 \times I_{[0.5,1)}(t));\ (1 \times I_{[0,0.5)}(t) + 0 \times I_{[0.5,1)}(t));\\
&(0 \times I_{[0,0.5)}(t) + 2 \times I_{[0.5,1)}(t));\ (2 \times I_{[0,0.5)}(t) + 0 \times I_{[0.5,1)}(t));\\
&(1 \times I_{[0,0.5)}(t) + 2 \times I_{[0.5,1)}(t));\ (2 \times I_{[0,0.5)}(t) + 1 \times I_{[0.5,1)}(t));\ (0);\ (1);\ (2)\}\\
&\mathscr{S}_2 = \{(0 \times I_{[0,0.33)}(t) + (-1) \times I_{[0.33,0.66)}(t) + (-1) \times I_{[0.66,1)}(t));\\
&(0 \times I_{[0,0.33)}(t) + (-1) \times I_{[0.33,0.66)}(t) + 0 \times I_{[0.66,1)}(t));\\
&(0 \times I_{[0,0.33)}(t) + 0 \times I_{[0.33,0.66)}(t) + (-1) \times I_{[0.66,1)}(t));\\
&((-1) \times I_{[0,0.33)}(t) + (-1) \times I_{[0.33,0.66)}(t) + 0 \times I_{[0.66,1)}(t));\\
&((-1) \times I_{[0,0.33)}(t) + 0 \times I_{[0.33,0.66)}(t) + 0 \times I_{[0.66,1)}(t));\\
&((-1) \times I_{[0,0.33)}(t) + 0 \times I_{[0.33,0.66)}(t) + (-1) \times I_{[0.66,1)}(t));\ (0);\ (-1)\}
\end{aligned}$$

The combinatorial structure of $\mathscr{S}$ is evident. Recall that a combinatorial set is a non-convex set. The convex hull $\mathrm{conv}(\mathscr{S})$ of the original set $\mathscr{S}$ has a simple expression:

$$\begin{aligned}
&\mathrm{conv}(\mathscr{S}) = \{(C_1 \times I_{[0,0.5)}(t), C_2 \times I_{[0.5,1)}(t))\} \times\\
&\{(D_1 \times I_{[0,0.33)}(t) + D_2 \times I_{[0.33,0.66)}(t) + D_3 \times I_{[0.66,1)}(t))\}
\end{aligned}$$

where $C_1, C_2 \in [0,2]$ and $D_1, D_2, D_3 \in [0,-1]$.

Motivated from the above facts let us consider the convex hull $\mathrm{conv}(\mathscr{S})$ associated with $\mathscr{S}$

$$\mathrm{conv}(\mathscr{S}) := \{v(\cdot) \mid v(t) = \sum_{s=1}^{|\mathscr{S}|} \lambda_s u_s(t),\ \sum_{s=1}^{|\mathscr{S}|} \lambda_s = 1,$$

where $\lambda_s \geq 0,\ u_s(\cdot) \in \mathscr{S},\ s = 1,\dots,|\mathscr{S}|\}$. From the definition of $\mathscr{S}$ we conclude that the convex set $\text{conv}(\mathscr{S})$ is closed and bounded. Using (1.4), we also can give the alternative characterization: $\text{conv}(\mathscr{S}) = \text{conv}(\mathscr{S}_1) \times \dots \times \text{conv}(\mathscr{S}_m)$, where $\text{conv}(\mathscr{S}_k)$ is a convex hull of $\mathscr{S}_k$ $k = 1,...,m$. Since $\text{conv}(\mathbb{Q}^k) \equiv [q_1^{(k)}, q_{M_k}^{(k)}]$, we have

$$\text{conv}(\mathscr{S}_k) := \{v(\cdot) \mid v(t) = \sum_{i=1}^{N_k} I_{[t_{i-1}^{(k)},t_i^{(k)})}(t) q_{j_i}^{(k)};\ q_{j_i}^{(k)} \in [q_1^{(k)}, q_{M_k}^{(k)}];\ j_i \in \mathbb{Z}[1,M_k];\ t_i^{(k)} \in \mathbb{T}^k\}.$$

Roughly speaking $\text{conv}(\mathscr{S})$ contains all the piecewise constant functions $u(\cdot)$ such that the constant value $u^{(k)}(t)$ belongs to the interval $[q_1^{(k)}, q_{M_k}^{(k)}]$ for all $t \in [t_{i-1}^{(k)}, t_i^{(k)})$. Let us note that in contrast to the initial set $\mathscr{S}$, the corresponding convex hull $\text{conv}(\mathscr{S})$ is an infinite dimensional space. Using the above convex construction, we can formulate the following auxiliary OCP

$$\begin{aligned} &\text{minimize } J(u(\cdot)) \\ &\text{subject to } u(\cdot) \in \text{conv}(\mathscr{S}). \end{aligned} \tag{3.2}$$

The problem (3.2) formulated above is in fact a simple convex relaxation of the initial OCP (1.6). We will study this problem and use it for a constructive numerical treatment of (1.6). Let us firstly formulate the following key property of the auxiliary OCP (3.2).

Theorem 3.3 *The cost functional* $J : \text{conv}(\mathscr{S}) \to \mathbb{R}$

$$J(u(\cdot)) = \frac{1}{2}\int_{t_0}^{t_f} [\langle Q(t)x^u(t), x^u(t)\rangle + \langle R(t)u(t), u(t)\rangle] dt + \frac{1}{2}\langle Gx^u(t_f), x(t_f)\rangle$$

is convex and the auxiliary OCP (3.2) *constitutes a convex optimization problem in the* Hilbert *space* $\mathbb{L}^2[t_0,t_f;\mathbb{R}^m]$.

Proof. Evidently, $\text{conv}(\mathscr{S})$ is a bounded closed and convex subset of $\mathbb{L}^2[t_0,t_f;\mathbb{R}^m]$. The cost functional $J(\cdot)$ is in fact a sum of two functionals:

$$J(u(\cdot)) = J_1(u(\cdot)) + J_2(u(\cdot)),\ J_1(u(\cdot)) := \frac{1}{2}\int_{t_0}^{t_f} [\langle R(t)u(t), u(t)\rangle] dt,$$

$$J_2(u(\cdot)) := \frac{1}{2}\int_{t_0}^{t_f} [\langle Q(t)x^u(t), x^u(t)\rangle] dt + \frac{1}{2}\langle Gx^u(t_f), x^u(t_f)\rangle.$$

The first one, namely, the functional $J_1(\cdot)$ is convex (recall that its Hessian is positive definite matrix). Moreover, $J_2(\cdot)$ is a composition of a convex (quadratic) functional of $x^u(\cdot)$, where $x^u(\cdot)$ is an affine mapping with respect to $u(\cdot)$ (see (3.1)). Applying Lemma 3.1, we now easily deduce the convexity of $J_2(\cdot)$. Since the sum of two convex functions is convex, we obtain the desired convexity result for $J(\cdot)$. The proof is completed. ∎

As we can see, (3.2) is a convex relaxation of the initial OCP (1.6). The proved convexity of OCP (3.2) makes it possible to apply the powerful numerical convex programming approaches to this auxiliary optimization problem. In this paper, we use a variant of the projected gradient method for a concrete numerical treatment of (3.2). Note that under the basic assumptions introduced in Section the following mapping $x^u(t) : \mathbb{L}^2[t_0,t_f;\mathbb{R}^m] \to \mathbb{R}^n$ is Fréchet differentiable for every $t \in [t_0,t_f]$ (see [17, 23]). Therefore, the quadratic costs functional $J(\cdot)$ in (3.2) is also Fréchet differentiable. We refer to [23, 29] for the corresponding differentiability concept. Assume $u^*(\cdot) \in \text{conv}(\mathscr{S})$ is an optimal solution of (3.2). The existence of an optimal input $u^*(\cdot)$ is guaranteed in the convex problem (3.2) (see e.g., [33]). By $x^*(\cdot)$ we next denote the corresponding optimal trajectory (solution) of (1.1) generated by $u^*(\cdot)$. The projected gradient method for problem (3.2) can now be expressed as follows:

$$u_{l+1}(\cdot) = \mathscr{P}_{\text{conv}(\mathscr{S})}\left[u_l(\cdot) - \alpha_l \nabla J(u_l(\cdot))\right] \tag{3.3}$$

where $\mathscr{P}_{\text{conv}(\mathscr{S})}$ is the operator of projection on to convex set $\text{conv}(\mathscr{S})$ and $\{\alpha_l\}$ is a sequence of step sizes. The conventional projection operator $\mathscr{P}_{\text{conv}(\mathscr{S})}$ is defined as usual:

$$\mathscr{P}_{\text{conv}(\mathscr{S})}(u(\cdot)) := \text{Argmin}_{v(\cdot)\in\text{conv}(\mathscr{S})}\left(||v(\cdot) - u(\cdot)||_{\mathbb{L}^2[t_0,t_f;\mathbb{R}^m]}\right)$$

Recall that the projected gradient iterations (3.3) generate a minimizing sequence for the convex optimization problem (3.2). Some useful mathematically exact convergence theorems for iterations (3.3) can be found in [11], [24], [33], [37]. We also refer to [9], [10], [19] for the related results. In the context of OCP (3.2) and method (3.3) the basic convergence result from [33], [37] can be reformulated as follows.

Theorem 3.4 *Assume that all the hypotheses from* Section *are satisfied. Consider a sequence of control functions generated by* (3.3)*. Then there exists an admissible initial data* $(u^0(\cdot),x^0(\cdot))$ *and a sequence of the step-sizes* $\{\alpha_l\}$ *such that* $\{u_l(\cdot)\}$ *is a minimizing sequence for* (3.2)*, i.e.,* $\lim_{l\to\infty} J(u_l(\cdot) = J(u^*(\cdot))$.

The proposed gradient-type method (3.3) provides a basis for the computational approach to (3.2). Using an optimal solution $u^*(\cdot) \in \text{conv}(\mathscr{S})$ we next need to determine a suitable approximation for a solution to the original OCP (1.1). In the next sections we propose a constructive numerical procedure for this purpose.

4. On the controllability condition for the linear system with a switched control structure

The study of OCPs with piecewise constant controls also involves a question of the general interest. Consider the initial dynamic system (1.1) determined on the given set of admissible controls $\mathscr{S}$ and reformulate the classical controllability question associated with the specific control set of piecewise constant inputs: system (1.1) on $\mathscr{S}$ is said to be controllable if for any initial state $x(t_0)$ and any final state $x(t_f)$, there exist an admissible function $u(\cdot) \in \mathscr{S}$ that transfers $x(t_0)$ to $x(t_f)$ in finite time. It is necessary to stress, that there are some (expectable) examples of non-controllable linear system involving the piecewise constant controls. In connection with this observation we formulate here a simple controllability criterion for the specific case of constant system/control matrices $A(t) \equiv A$, $B(t) \equiv B$ and unified switching times $N_k \equiv N$ $\mathbb{T}^k \equiv \mathbb{T}$ for all $k = 1, \dots, m$.

Theorem 4.1 *Consider the stationary variant of the linear system* (1.1) *for* $u(\cdot) \in \mathscr{S}$ *and assume* $N_k \equiv N, \ \ \mathbb{T}^k \equiv \mathbb{T}, \ \ k = 1, \dots, m$. *Let*

$$W(N) := \sum_{i=1}^{N} \left[\int_{t_{i-1}}^{t_i} e^{-A\tau} d\tau BB^T \int_{t_{i-1}}^{t_i} e^{-A^T\tau} d\tau \right], \ t_i \in \mathbb{T}.$$

and assume that

$$-B^T \int_{t_{i-1}}^{t_i} e^{-A^T\tau} d\tau W(N)^{-1} \left(x(t_0) - e^{-At_f} x(t_f) \right) \in \mathbb{Q}, \tag{4.1}$$

where $\mathbb{Q} := \mathbb{Q}^1 \times \dots \times \mathbb{Q}^m$. *Then system* (1.1) *is controllable if and only if matrix* $W(N)$ *is non-singular.*

Proof. Let $W(N)$ be non-singular. Then

$$x(t_f) = e^{At_f} x(t_0) + \int_{t_0}^{t_f} e^{A(t_f - \tau)} Bu(\tau) d\tau,$$

or equivalently

$$x(t_f) = e^{At_f} \left[x(t_0) + \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} e^{-A\tau} d\tau Bu^i \right], \tag{4.2}$$

where $u^i \in R^m$ is a constant vector associated with the interval $[t_{i-1}, t_i)$. The resulting input value u^i such that $u(t) = u^i$ for $t \in [t_{i-1}, t_i)$ and $x(t_0)$, $x(t_f)$ belongs to the corresponding trajectory of (1.1) generated by $u(\cdot)$ is given by

$$u^i = -B^T \int_{t_{i-1}}^{t_i} e^{-A^T\tau} d\tau W(N)^{-1} \left(x(t_0) - e^{-At_f} x(t_f) \right).$$

From (4.1) it follows $u^i \in Q$. Substituting the obtained expression in (4.2), we next obtain

$$x(t_f) = e^{At_f} [x(t_0) - W(N)W(N)^{-1}(x(t_0) - e^{-At_f} x(t_f))] = x(t_f).$$

We conclude that the given system is controllable under piecewise constant inputs.

Let now the initial system (1.1) be controllable by piecewise constant controls from $\mathscr{S}$. Assume that the symmetric matrix $W(N)$ is not a (strictly) positive definite matrix. This hypothesis implies the existence of a non-trivial vector $v \in \mathbb{R}^n$ such that $v^T W(N) v = 0$, or equivalently:

$$0 = v^T \sum_{i=1}^{N} \left[\int_{t_{i-1}}^{t_i} e^{-A\tau} d\tau BB^T \int_{t_{i-1}}^{t_i} e^{-A^T\tau} d\tau \right] v = \sum_{i=1}^{N} ||v^T \int_{t_{i-1}}^{t_i} e^{-A\tau} d\tau B||^2.$$

The last fact evidently implies the following $v^T \int_{t_{i-1}}^{t_i} e^{-A\tau} d\tau B = 0 \; \forall i = 1,\dots,N$. Since the controllability of the system (1.1) for $u(\cdot) \in \mathscr{S}$ is assumed, there exist a sequence of values $\{u^i\}$ such that the state $x(t_0) \equiv v$ can be transferred into $x(t_f) \equiv 0$. Therefore, we deduce the next consequence

$$0 = e^{At_f} \left[v + \sum_{i=1}^{N} \left(\int_{t_{i-1}}^{t_i} e^{-A\tau} d\tau \right) Bu^i \right]. \tag{4.3}$$

Evidently, (4.3) holds if and only if

$$0 = v + \sum_{i=1}^{N} \left(\int_{t_{i-1}}^{t_i} e^{-A\tau} d\tau \right) Bu^i. \tag{4.4}$$

We now multiply (4.4) by v^T

$$0 = v^T v + \sum_{i=1}^{N} v^T \left(\int_{t_{i-1}}^{t_i} e^{-A\tau} d\tau \right) Bu^i = v^T v$$

and obtain the contradiction with the non-triviality hypothesis $v \neq 0$. Therefore, $W(N)$ is a positive definite symmetric matrix and the existence of the inverse $W(N)^{-1}$ follows immediately. The proof is completed. ∎

Note that Theorem 4.1 makes it possible to establish the existence of an optimal solution to the restricted OCP of the type (1.6) with additional terminal constraint $x(t_f) = x_f$, where $x_f \in \mathbb{R}^n$ is a prescribed final state. We refer to [21], [44] for the classic result and for the corresponding regularity conditions in some classes of constrained OCPs.

5. A relaxation based numerical method for the initial optimal control problem

Theorem 3.4 and the classic gradient-type iterations (3.3) provide an analytic basis for a consistent computational approach to the initial OCP (1.6). Recall that in contrast to the relaxed optimization problem (3.2) the original OCP (1.6) does not possesses any convexity property. However, the simple relaxed OCP (3.2) can be effectively used for an approximative numerical treatment of the original problem (1.6). Let us introduce the formal Hamiltonian associated with problems (1.6) and (3.2)

$$H(t,x,u,p) = \langle p, A(t)x + B(t)u \rangle - \frac{1}{2} \left(\langle Q(t)xt, x \rangle + \langle R(t)u, u \rangle \right).$$

where $p \in \mathbb{R}^n$ is the adjoint variable. By $\hat{u}(\cdot) \in \mathscr{S}$ we now denote an optimal solution to the initial OCP (1.6). Using the explicit representation of the gradient $\nabla J(u_l(\cdot))$ in OCPs with ordinary differential equations (see e.g., [3], [4], [20], [33], [38], [42]), we can propose a conceptual computational scheme for (1.6).

Conceptual Algorithm 1

(0) *Set the initial condition for the iterative scheme* $u_{(0)}(\cdot) := \mathscr{P}_{\mathrm{conv}(\mathscr{S})}(u^{opt}(\cdot))$, *where* $u^{opt}(\cdot)$ *is the optimal control input* (1.3) *from the classic* LQ *problem. Calculate the corresponding trajectory* $x_{(l)}(\cdot)$ *of* (1.1) *and put the iterations index* $l := 0$.

(1) *Calculate* $\nabla J(u_{(l)})(\cdot)$ *as* (see [3], [4], [7], [33], [42])

$$\nabla J(u_{(l)})(t) = -\frac{\partial H(t, x_{(l)}(t), u_{(l)}(t), p(t))}{\partial u} = -B^T(t)p(t) + R(t)u_{(l)},$$

where the adjoint variable $p(\cdot)$ *is a solution to the usual boundary value problem*

$$\dot{p}(t) = -\frac{\partial H(t, x_{(l)}(t), u_{(l)}(t), p)}{\partial x} = -A^T(t)p(t) + Q(t)x_{(l)}(t),$$
$$p(t_f) = -Gx_{(l)}(t_f).$$

(2) *Calculate the projection of* $u_{(l)}(\cdot) - \alpha_{(l)}\nabla J(u_{(l)}(\cdot))$ *on the convex* (*relaxed*) *restriction set* $\mathrm{conv}(\mathscr{S})$ *and determine* $\bar{u}_{(l+1)}(\cdot) := \mathscr{P}_{\mathrm{conv}(\mathscr{S})}(\bar{u}_{(l)}(\cdot))$.

(3) *Evaluate the* $(l+1)$ *iteration of the control function given by components*

$$u^{(k)}_{(l+1)}(t) = \sum_{i=1}^{N_k} I_{[t^{(k)}_{i-1}, t^{(k)}_i)}(t)\bar{q}^{(k)}_{i,n} \quad \forall k = 1, \dots, m,$$

where:

$$\bar{q}^{(k)}_{i,l} := \begin{cases} q^{(k)}_1, & \bar{\bar{q}}^{(k)}_{i,l} < q^{(k)}_1 \\ \bar{\bar{q}}^{(k)}_{i,l}, & q^{(k)}_1 \leq \bar{\bar{q}}^{(k)}_{i,l} \leq q^{(k)}_{M_k} \\ q^{(k)}_{M_k}, & q^{(k)}_{M_k} \leq \bar{\bar{q}}^{(k)}_{i,l} \end{cases} , i = 1, \dots, N_k.$$

and $q^{(k)}_j \in \mathbb{Q}^k$, $\forall j = 1, \dots, M_k$, $\bar{\bar{q}}^{(k)}_{i,l} := \frac{1}{\Delta_i}\int_{t_{i-1}}^{t_i} \bar{u}^{(k)}_{(l)}(t)dt$, $\Delta_i := t_i - t_{i-1}$.

(4) *Calculate the difference* $|J(u_{(l+1)}(\cdot)) - J(u_{(l)}(\cdot))|$. *If it is less than a prescribed accuracy* $\varepsilon > 0$, *then we put* $u^*(\cdot) \equiv u_{(l+1)}(\cdot)$ *(an approximating optimal solution to* (3.2)) *and* Stop. *Else, update the iteration register and go to* Step (1).

(5) *Using the evaluated function* $u^*(\cdot)$ *the approximating optimal control* $\hat{u}(\cdot) \in \mathscr{S}$ *can finally be calculated by components*

$$\hat{u}^{(k)}(\cdot) = \sum_{i=1}^{N_k} I_{[t^{(k)}_{i-1}, t^{(k)}_i)}(t)\hat{q}^{(k)}_i \quad \forall k = 1, \dots, m. \tag{5.1}$$

where $\hat{q}^k_i := \mathrm{Arg\,min}_{v \in \mathbb{Q}^k} |v - \bar{q}^{(k)}_{i,l+1}|$. *Solve* (1.1) *with the obtained control input* $\hat{u}(\cdot) \in \mathscr{S}$ *and obtain the approximating optimal trajectory* $\hat{x}(\cdot)$. Stop.

Using Theorem 3.4 and the continuity property of the objective functional one can establish the convergence of the proposed Conceptual Algorithm 1. Note that this type of convergence is determined as a ”convergence in functional”. To put it another way, the control sequence $\{u_l(\cdot)\}$ generated by Steps (0)-(4) of the above Algorithm is a minimizing sequence (see Theorem 3.4). By implementation and taking into consideration the continuity of the objective functional, we finally can establish the convergence property (”in functional”) of the resulting sequences $\{\hat{u}^{(k)}(\cdot)\}$, $k = 1,\dots,m$ obtained in Step (5) of Algorithm.

We now illustrate the effectiveness of the proposed Conceptual Algorithm 1 and consider two simple examples.

Example 5.1 Consider the following linear system

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -x_1(t) + u(t) \end{bmatrix},\ t \in [0,5],\ x(0) = (1,-1)^T$$

associated with the quadratic cost functional

$$J(u(\cdot)) = \frac{1}{2}\int_0^5 \left(x_1^2(t) + 10x_2^2(t) + u^2(t)\right) dt,$$

Let $\mathbb{Q} = \{0, 0.25, 0.5, 0.75, 1, 1.25, 1.5, \dots, 5\}$ be the given finite set of constant control values. Assume $N_k = 10$, $k = 1$ and the set $\mathbb{T}$ is not given a priory. The classic LQ optimal control $u^{opt}(\cdot)$ can be here easily calculated. Applying the proposed Conceptual Algorithm 1, we now compute the approximating optimal control $\hat{u}(\cdot)$ (see Fig. 2).

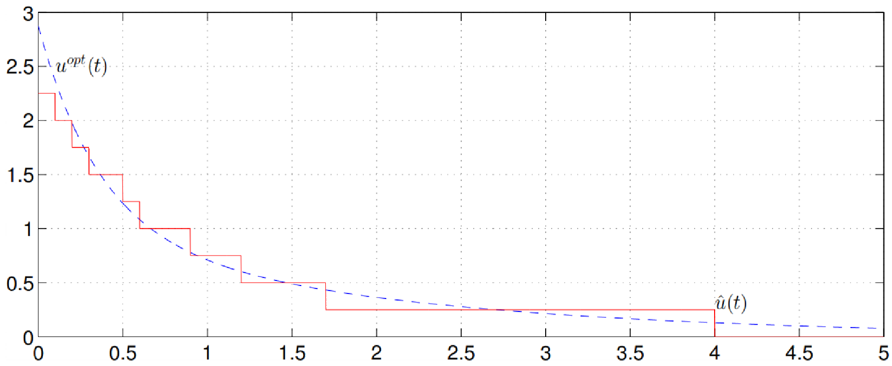

Figure 2: Control inputs $u^{opt}(t)$ and $\hat{u}(t)$

The associated trajectory $\hat{x}(\cdot)$ is indicated on Fig. 3. The calculated cost in problem (1.6) associated with our example was evaluated as follows: $J(\hat{u}(\cdot)) = 7.5362$. Evidently, this value is higher in comparison with the optimal cost in the conventional (non-restricted) LQ problem.

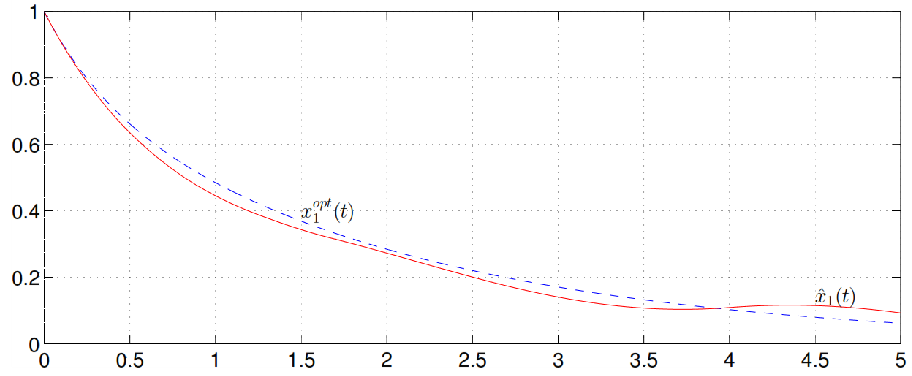

Figure 3: First components of the optimal trajectories $x_1^{opt}(t)$ and $\hat{x}_1(t)$

Example 5.2 We now consider (1.1) for $n = 3$

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -0.875x_2(t) - 20x_3(t) \\ -50x_3(t) + 50u(t) \end{bmatrix},$$
$$x(0) = (1, 0, -1)^T,$$

where $t \in [0,1]$. The quadratic cost functional in problem (1.6) associated with our example has been given in the following concrete form

$$J(u(\cdot)) = \frac{1}{2}\int_0^1 \left(3x_1^2(t) + x_2^2(t) + 2x_3^2(t) + u^2(t)\right) dt,$$

We next assume

$$\mathbb{Q} = \{-5, -4.5, -4, -3.5, \ldots, 3.5, 4, 4.5, 5\},\ N_k = 3,\ k = 1.$$

The set $\mathbb{T}$ is not given a priory. Application of the proposed numerical solution procedure, namely, of Conceptual Algorithm 1 leads to the computational results for the quasi-optimal control $\hat{u}(\cdot)$ and the corresponding trajectory $\hat{x}(\cdot)$). These numerically optimal functions are indicated on Fig. 4 and Fig. 5, respectively.

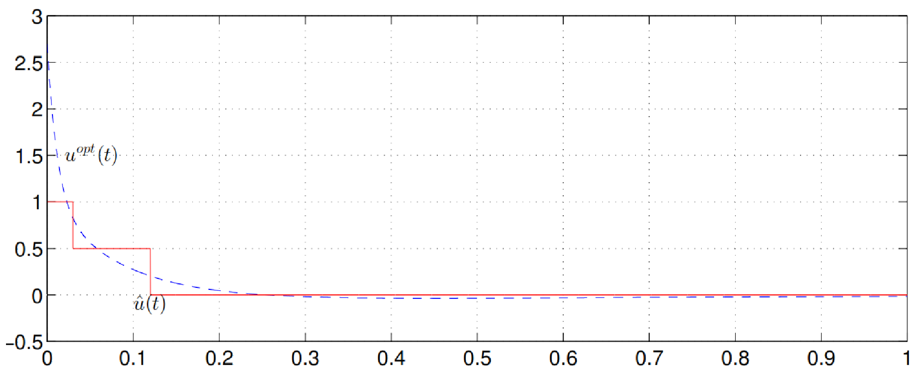

Figure 4: Control inputs $u^{opt}(t)$ and $\hat{u}(t)$

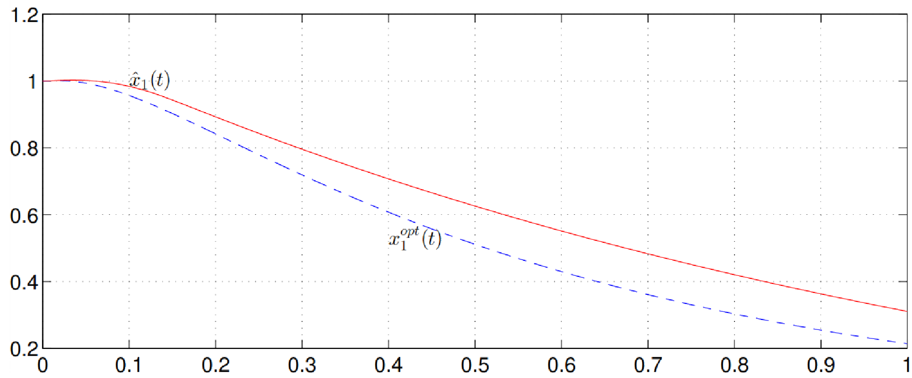

Figure 5: First components of the trajectories $x_1^{opt}(t)$ and $\hat{x}_1(t)$

The calculated cost associated with the initial OCP (1.6) in this example was evaluated as follows: $J(\hat{u}(\cdot)) = 2.0237$. As mentioned above, the calculated optimal cost here has a higher value in comparison with the optimal cost in the conventional LQ problem. This fact is a simple consequence of the evident inclusion $\mathscr{S} \subset \mathbb{L}^2[t_0,t_f;\mathbb{R}^m]$ that constitutes the admissible control set restrictions in the constrained LQ problem under consideration.

Finally, note that implementations of the Conceptual Algorithm 1 presented in Examples 4.1 and 4.2, was carried out, using the standard MATLAB packages and the Authors programs.

6. Conclusion

In this contribution, we have developed a new implementable numerical approach to a constrained LQ-type OCP. This computational method is based on a simple convex relaxation procedure applied to the initial problem in combination with the conventional gradient-based numerical technique. We firstly rewrite the original (non-convex) OCP in a relaxed form and establish the convexity properties. We next use the obtained convex relaxation as an auxiliary tool in a concrete solution scheme for the initial OCP. The convex structure of the auxiliary OCP makes it possible to take into consideration diverse powerful computational algorithms from the classic convex programming. Let us note that various variants of the basic gradient method, namely, Armijo-type gradient schemes can be applied to the obtained relaxed OCP (see [1], [2], [9], [10], [11], [19], [20], [42]). In the presented paper we also discussed the general controllability question associated with the stationary variant of the constrained linear dynamic systems under consideration. The general controllability concept for linear systems of the type (1.1) involves a full theoretic justification of the OCP under consideration and finally, makes it possible to establish the applicability of the gradient method in the presented form (see [10], [11] for details).

It is common knowledge that modern numerical algorithms mainly use specific non-equidistant discretizations with the aim to increase the effectiveness of the resulting algorithm. The specific type of the control functions discussed in our contribution is motivated by the initially given physical nature of the class of controlled processes under consideration. Note that there are various formal models that involve a non-equally spaced inputs grid. The necessary investigation of these types of models and the corresponding engineering applications (mainly from the modern communication science) constitutes an interesting subject of a next contribution. Our paper focuses on equally-spaced controlled models since they are simpler to analyse while capturing the salient features of the newly elaborated control method we propose. Let us also note that the presented Conceptual Algorithm 1 need to be analysed in comparison with some powerful numerical schemes. For example, one needs to compare it with the various implementations of the direct search (see e.g., [27], [42]).

Finally, note that the theoretical and computational approaches presented in this paper can be applied to some alternative classes of constrained OCPs. Let us also note that the proposed numerical algorithm can also constitute a constructive tool of some general numerical techniques based on discretizations and linear approximations associated with the common types of non-linear OCPs.

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