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Optimal fixed-levels control for nonlinear systems with quadratic cost-functionals

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SUMMARY

This paper is devoted to general optimal control problems (OCPs) associated with a family of nonlinear continuous-time switched systems in the presence of some specific control constraints. The stepwise (fixed-level type) control restrictions we consider constitute a common class of admissible controls in many real-world engineering systems. Moreover, these control restrictions can also be interpreted as a result of a quantization procedure appglied to the inputs of a conventional dynamic system. We study control systems with a priori given time-driven switching mechanism in the presence of a quadratic cost functional. Our aim is to develop a practically implementable control algorithm that makes it possible to calculate approximating solutions for the class of OCPs under consideration. The paper presents a newly elaborated linear quadratic-type optimal control scheme and also contains illustrative numerical examples. Copyright © 2015 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Optimization of sophisticated constrained dynamic models is, nowadays, a mature and relative simple design methodology for the practical development of several types of modern controllers (see, e.g., [1–19]). It is readily appreciated that the real-world control systems have a corresponding set of constraints; for example, inputs always have maximum and minimum values, and states are usually required to lie within certain ranges. Of course, one could proceed by ignoring these constraints and hope that no serious consequences result from this approach. This simple procedure may be sufficient at times. On the other hand, it is generally true that optimal levels of a suitable performance are associated with operating on, or near, constraint boundaries (see [9, 20–27] and the references therein). Thus, a control engineer really cannot ignore constraints without incurring a performance penalty.

Recently, the problem of effective numerical methods for constrained systems optimization has attracted a lot of attention, thus both theoretical results and applications were developed (see, e.g., [3, 4, 6, 8, 14, 19, 21, 27–30]). The handling constraints in practical systems design is an important issue in most, if not all, real world applications. The main aim of our contribution is to elaborate a consistent computational algorithm for a general optimal control problem (OCP) in the presence of piecewise constant control inputs. We study a class of OCPs with quadratic costs functionals. The given structure of the admissible control function under consideration is mainly motivated by

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some important practical control applications (see [8, 21, 30–32]) as well as by the wide applicable modern quantization procedure associated with the original system dynamics (see, e.g., [33–35]). Note that some classes of optimal control with piecewise constraints was studied earlier. We refer to [7, 36] for the interesting self-closed results on the piecewise linear dynamics. The formal treatment there was based on the backward solutions of the Riccati differential equations, and the optimum had to be recomputed for each new final state. Computation of nonlinear gains using the Hamilton–Jacobi–Bellman equation and the convex optimization techniques has also been performed in [36]. On the other hand, the aforementioned optimization approaches to the linear constrained systems are not sufficiently advanced to the general types of OCPs governed by nonlinear switched systems. In our paper, we propose a new analytic and numerical methods based on a combination of the classic convex relaxation scheme and the first-order projection approach. And, it should be noted already at this point that a computational algorithm we propose can be effectively used in a concrete control synthesis phase associated with a practical engineering design of a switched dynamic system.

Recall that the general switched systems constitute a class of models where two types of dynamics are present, continuous and discrete event dynamic behavior (see, e.g., [2, 37, 38]). In order to understand how these systems can be operated efficiently, both aspects of the actual dynamics have to be considered and taken into account during the optimal control design procedure. The nonlinear systems we study can be interpreted as a particular family of the general switched systems with the time-driven location transitions. We refer to [5, 19, 37–39] for the basic concepts and some technical details of the generic switched systems theory.

The remainder of our paper is organized as follows: Section 2 contains the problem statement and some preliminary theoretic facts. Section 3 deals with some specific relaxation schemes for the initial OCP. Moreover, we also apply a projected gradient method for the concrete numerical treatment of the relaxed and initial OCPs. In Section 4, we study the specific implementability conditions associated with the gradient-based algorithm we propose. Section 5 is devoted to the computational aspects of the proposed numerical scheme and also contains some computational examples. Section 5 summarizes our paper.

2. PROBLEM FORMULATION AND SOME PRELIMINARY FACTS

Consider the following nonlinear system with a switched control structure

$$\begin{aligned} \dot{x}(t) &= f(t, x(t), u(t)),\ t \in [t_0, t_f], \\ x(t_0) &= x_0, \end{aligned} \tag{1}$$

where $f(\cdot, \cdot, u)$ is a Caratheodory function (see [25, 40] for theoretic details), that is, a function measurable in t and continuous in x. We next assume that $f(t, x, \cdot)$ is a continuously differentiable function. Let us specify the set of admissible control inputs of switched nature. In this paper, we study the piecewise-constant control functions associated with the dynamic system (1). For each component $u_k(\cdot)$ of a feasible control input $u(\cdot) = [u_1(\cdot), \dots, u_m(\cdot)]^T$, we introduce a finite set of bounded values that represent admissible fixed-levels controls mentioned earlier:

$$\mathscr{Q}^{(k)} := \left\{ q_j^{(k)} \in \mathbb{R}, j = 1, \dots, M_k \right\},\ M_k \in \mathbb{N}, \quad k = 1, \dots, m.$$

In general, all the given sets $\mathscr{Q}^{(k)}$ are different (contain different levels) and have various numbers of elements. We assume that each $\mathscr{Q}^{(k)}$ possesses the following strict order property

$$q_1^{(k)} < q_2^{(k)} < \dots < q_{M_k}^{(k)}.$$

Let us now introduce the set of switching times associated with an admissible control function

$$\mathscr{T}^{(k)} := \left\{ t_i^{(k)} \in \mathbb{R}_+, i = 0, \dots, N_k \right\},\ N_k \in \mathbb{N}, \quad k = 1, \dots, m.$$

All the introduced sets $\mathscr{T}^{(k)}$ are determined for the corresponding control components $u_k(\cdot)$, $k = 1, \dots, m$, where $\mathbb{R}_+$ denotes a non-negative real semi-axis. Let $t_0^{(k)} < t_1^{(k)} < \dots < t_{N_k}^{(k)}$. For each

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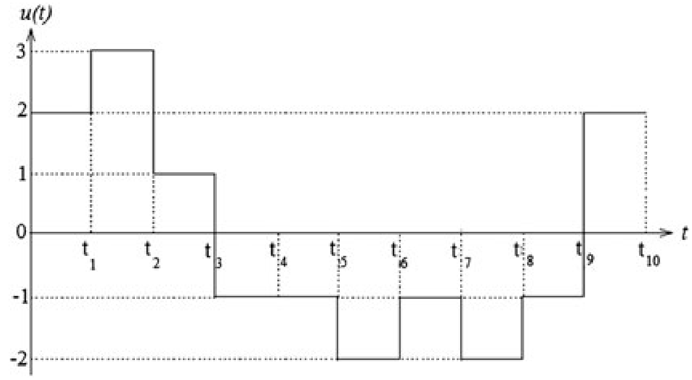

Figure 1. The admissible switched-type control inputs $u(\cdot)$.

$\mathscr{T}^{(k)}$, we determine $t^{(1)}_{N_1} = \ldots = t^{(m)}_{N_m} = t_f$. Using the feasible control level sets $\mathscr{Q}^{(k)}$ defined earlier and the switching times $\mathscr{T}^{(k)}$, the set of admissible control functions $\mathscr{S}$ can now be specified as the Cartesian product

$$\mathscr{S} = \mathscr{S}_1 \times \ldots \times \mathscr{S}_m, \tag{2}$$

where $\mathscr{S}_k, k = 1, \ldots, m$ is defined as follows:

$$\mathscr{S}_k := \left\{ v : [t_0, t_f] \to \mathbb{R} | v(t) = \sum_{i=1}^{N_k} I_{[t^{(k)}_{i-1}, t^{(k)}_i)}(t) q^{(k)}_{j_i} \right\}, \; q^{(k)}_{j_i} \in \mathscr{Q}^{(k)}, \; j_i \in \mathbb{Z}[1, M_k], \; t^{(k)}_i \in \mathscr{T}^{(k)}.$$

By $\mathbb{Z}[1, M_k]$, we denote here the set of all integers from the interval $[1, M_k]$ and $I_{[t^{(k)}_{i-1}, t^{(k)}_i)}(t)$ is a characteristic function of the interval $[t^{(k)}_{i-1}, t^{(k)}_i)$. Evidently, the set of admissible control inputs $\mathscr{S}$ can be qualitatively interpreted as the set of all functions $u : [t_0, t_f] \to \mathbb{R}^m$, such that each component $u_k(\cdot)$ of $u(\cdot)$ attains a constant level value $q^{(k)}_{j_i} \in \mathscr{Q}^{(k)}$ for $t \in [t^{(k)}_{i-1}, t^{(k)}_i)$. Moreover, the level changes of the control components occur only at the prescribed time instants $t^{(k)}_i \in \mathscr{T}^{(k)}, i = 1, \ldots, N_k$.

Recall that the general existence/uniqueness theory for nonlinear ordinary differential equations implies that for every $u(\cdot) \in \mathscr{S} \subset \mathbb{L}^2[t_0, t_f; \mathbb{R}^m]$, the initial value problem (1) has a unique absolutely continuous solution $x(\cdot)$. Here, $\mathbb{L}^2\{[t_0, t_f]; \mathbb{R}^m\}$ denotes the Lebesgue space of all square integrable functions $u : [t_0, t_f] \to \mathbb{R}^m$. We refer to [22, 25, 26, 40] for necessary existence and uniqueness results.

Similarly to the classic LQ (linear quadratic) optimization theory, we now introduce a quadratic cost functional associated with the dynamic system (1)

$$J(u(\cdot)) = \frac{1}{2} \int_{t_0}^{t_f} (\langle Q(t)x(t), x(t) \rangle + \langle R(t)u(t), u(t) \rangle)\, dt + \frac{1}{2} \langle Gx(t_f), x(t_f) \rangle,$$

where $G \in \mathbb{R}^{n \times n}$ is a symmetric positive defined matrix and $Q(\cdot)$, $R(\cdot)$ are integrable matrix-functions that satisfy standard symmetry and positivity hypothesis $Q(t) \geqslant 0$, $R(t) \geqslant \delta I$, $\delta > 0$ for all $t \in [t_0, t_f]$. Recall that in the case of a linear variant of system (1) and in the absence of the (fixed-levels) control restrictions, we obtain the standard LQ-type OCP. In this special case, the optimization problem is formally stated in the full space $\mathbb{L}^2[t_0, t_f; \mathbb{R}^m]$ of square integrable control functions. In contrast to the classic case, we consider system (1) in combination with the given specific piecewise-constant control inputs $u(\cdot)$ (Figure 1).

The control signal $u(\cdot)$ here can only take some prescribed fixed-level values within a finite feasibility set $\mathscr{Q}$ (here, we have $\mathscr{Q} = \{-2, -1, 0, 1, 2, 3\}$) during the time interval $[t_{i-1}, t_i), i = 1, \ldots, 10$. Moreover, a change of the constant control values can only occur at the specific time

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instants $t_0, t_1, \dots, t_f$. The evident combinatorial character of the considered OCP associated with the continuous-time system (1) can be illustrated by the following simple example.

Example 1
Suppose $u(t) \in \mathbb{R}^2$ and $\mathscr{Q}^{(1)} = \{0, 1, 2\}$, $\mathscr{Q}^{(2)} = \{0, -1\}$. Furthermore, the set of switching times for each control component is given as follows

$$\mathscr{T}^{(1)} = \{0, 0.5, 1\},\ \mathscr{T}^{(2)} = \{0, 0.33, 0.66, 1\}.$$

Resulting from the above formal definitions, the set $\mathscr{S}$ in (2) has the natural representation $\mathscr{S} = \mathscr{S}_1 \times \mathscr{S}_2$, where

$$\mathscr{S}_1 = \left\{ v : [0,1] \to \mathbb{R} | v(t) = I_{[0,0.5)}(t) q^{(1)}_{j_1} + I_{[0.5,1)}(t) q^{(1)}_{j_2},\ q^{(1)}_{j_i} \in \mathscr{Q}^1 \right\},$$

$$\mathscr{S}_2 = \left\{ w : [0,1] \to \mathbb{R} | w(t) = I_{[0,0.33)}(t) q^{(2)}_{j_1} + I_{[0.33,0.66)}(t) q^{(2)}_{j_2} + I_{[0.66,1)}(t) q^{(2)}_{j_3},\ q^{(2)}_{j_i} \in \mathscr{Q}^2 \right\}.$$

We evidently have $M_1 = 3,\ M_2 = 2,\ N_1 = 2,\ N_2 = 3$. The cardinality of the control set $\mathscr{S}$ in our example can be calculated as follows: $|\mathscr{S}| = 3^2 \cdot 2^3 = 72$. In other words, we have here 72 feasible control values.

Note that in general, the cardinality of the set of admissible controls $\mathscr{S}$ can be expressed as follows:

$$|\mathscr{S}| = \prod_{k=1}^{m} M_k^{N_k}$$

Evidently, $|\mathscr{S}|$ grows exponentially if N_k increases. Motivated from various engineering applications, we now can formulate the following constrained OCP associated with the dynamic system (1)

$$\begin{aligned} &\text{minimize } J(u(\cdot)) \\ &\text{subject to } (1),\ u(\cdot) \in \mathscr{S}, \end{aligned} \tag{3}$$

where $J(\cdot)$ is the quadratic cost functional introduced earlier. Note that $\mathscr{S}$ constitutes a non-empty subset of the real Hilbert space $\mathbb{L}^2[t_0, t_f; \mathbb{R}^m]$. Because of the highly restrictive nonlinear control constraint $u(\cdot) \in \mathscr{S}$, the obtained optimization problem (3) can not be solved by a direct application of the classic Pontryagin Maximun Principle. Recall that the conventional versions of the celebrated Maximum Principle make it possible to specify an optimal solution in a full (non-restricted) space $\mathbb{L}^2[t_0, t_f; \mathbb{R}^m]$ of square integrable control functions. Otherwise, a possible computational treatment of the specific variants of the Maximum Principle for the restricted OCPs constitutes a numerically sophisticated task (see, e.g., [9, 20, 22]). Motivated from that fact, our main contribution is with a development of a relative simple and consistent computational procedure for the generic OCP (3). We use a novel relaxation technique in combination with a variant of the gradient-type algorithm for this purpose. We first obtain an optimal solution of the weakly relaxed OCP. Next, we use it in a concrete constructive numerical solution procedure for the original problem (3).

3. THE GRADIENT-BASED COMPUTATIONAL APPROACH TO RELAXED OPTIMAL CONTROL PROBLEMS

In this section, we propose a constructive computational scheme for the OCP (3) formulated earlier. The method we consider incorporates a simple relaxed OCPs associated with the initial problem. ‘Relaxing a problem’ has various meanings in mathematics, depending on the areas where it is defined, depending also on what one relaxes (a functional, the underlying space, etc.). In the context of an OCP of the type (3), when dealing with the minimization of $J(u(\cdot))$, the most general way of looking at relaxation is to consider the lower-semicontinuous hull of $J(u(\cdot))$ determined on a

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convexification of the set of admissible controls in (3). We refer to [41–47] for some modern relaxation procedures in optimal control.

First, let us note that under basic assumptions from Section 2, the following control-state mapping

$$x^u(t) : \mathbb{L}^2\{[t_0, t_f]; \mathbb{R}^m\} \to \mathbb{R}^n$$

is Fréchet differentiable for every $t \in [t_0, t_f]$. Therefore, the quadratic cost functional $J(\cdot)$ in (3) is also Fréchet differentiable. We refer to [20, 22, 24, 48] for the classic differentiability concepts. The original set $\mathscr{S}$ of admissible controls in (3) is a non-convex set. This is an immediate consequence of the combinatorial structure of $\mathscr{S}$ given by (2). Motivated from this fact, we next consider the following polytope

$$\operatorname{conv}(\mathscr{S}) := \left\{ v(\cdot) | v(t) = \sum_{s=1}^{|\mathscr{S}|} \lambda_s u_s(t),\ \sum_{s=1}^{|\mathscr{S}|} \lambda_s = 1,\ \lambda_s \geqslant 0,\ u_s(\cdot) \in \mathscr{S},\ s = 1, \ldots, |\mathscr{S}| \right\}.$$

From the definition of $\mathscr{S}$, we conclude that the convex set $\operatorname{conv}(\mathscr{S})$ is closed and bounded. We can also give an easy alternative characterization of $\operatorname{conv}(\mathscr{S})$:

$$\operatorname{conv}(\mathscr{S}) = \operatorname{conv}(\mathscr{S}_1) \times \ldots \times \operatorname{conv}(\mathscr{S}_m),$$

where $\operatorname{conv}(\mathscr{S}_k)$ is a convex hull of the partial set $\mathscr{S}_k$, $k = 1, \ldots, m$. Because $\operatorname{conv}(\mathscr{Q}_k) \equiv \left[q_1^{(k)}, q_{M_k}^{(k)}\right]$, we obtain

$$\operatorname{conv}(\mathscr{S}_k) := \left\{ v(\cdot) | v(t) = \sum_{i=1}^{N_k} I_{\left[t_{i-1}^{(k)}, t_i^{(k)}\right)}(t) q_{j_i}^{(k)},\ q_{j_i}^{(k)} \in \left[q_1^{(k)}, q_{M_k}^{(k)}\right],\ j_i \in \mathbb{Z}[1, M_k],\ t_i^{(k)} \in \mathscr{T}^{(k)} \right\}.$$

Roughly speaking, $\operatorname{conv}(\mathscr{S})$ contains all the piecewise constant functions $u(\cdot)$ such that the corresponding constant value $u_k(t)$ belongs to the interval $\left[q_1^{(k)}, q_{M_k}^{(k)}\right]$ for all $t \in \left[t_{i-1}^{(k)}, t_i^{(k)}\right)$. Let us note that in contrast to the initially considered control set $\mathscr{S}$, the convex hull $\operatorname{conv}(\mathscr{S})$ introduced earlier is an infinite dimensional space.

Let us consider the construction of the closed convex hull $c\overline{o}\{J(u(\cdot))\}$ of the objective $J(u(\cdot))$. We refer to [49] for the exact concept of $c\overline{o}\{J(u(\cdot))\}$. Note that biconjugate $J^{**}(u(\cdot))$ of $J(u(\cdot))$ (determined on $\mathbb{L}^2\{[t_0, t_f]; \mathbb{R}^m\}$) is equal to $c\overline{o}\{J(u(\cdot))\}$ (see [49] for the formal proof). For a given objective functional $J(\cdot)$, getting its closed convex hull is a complicated, but at the same time fascinating, operation. Note that the objective functional $J(u(\cdot))$ in (3) is a composite functional. This fact can be easily stated taking into account the mapping $x^u(t)$ considered at the beginning of this section. This situation (a composite cost functional) is a typical attribute of the general optimal control processes governed by ordinary differential equations. The numerical determination of $c\overline{o}\{J(u(\cdot))\}$ is not discussed in our paper. Let us note that a novel computational method for a practical constructive characterization of $c\overline{o}\{J(u(\cdot))\}$ is proposed in [44, 46, 47]. This approach constitutes a specific generalization of the celebrated McCormic relaxation scheme in the case of a composite functional associated with an OCP. The closed convexification of such a functional can be realized by solving an 'auxiliary control system' (see [46] for details).

Using the aforementioned convex constructions of the objective functional and of the convexified set of admissible control functions, we are now ready to formulate the (auxiliary) relaxed OCP:

$$\begin{aligned} &\text{minimize } c\overline{o}\{J(u(\cdot))\} \\ &\text{subject to } u(\cdot) \in \operatorname{conv}(\mathscr{S}) \end{aligned} \tag{4}$$

The problem (4) is in fact a simple relaxation of the initial OCP (3). The optimization procedure in (4) is determined over a convex set of admissible control inputs $\operatorname{conv}(\mathscr{S})$. We will study this problem and consider it for a constructive numerical treatment of the initial problem (3). Note that OCP (4) constitutes a convex minimization problem in a real Hilbert space. We refer to [1, 5, 27]

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for the general theory of convex OCPs with ordinary differential equations. The approximability property of the fully relaxed OCP (4) can be expressed as follows ([49]):

$$\inf_{u(\cdot)\in\mathbb{L}^2\{[t_0,t_f];\mathbb{R}^m\}} J(u(\cdot)) = \inf_{u(\cdot)\in\mathbb{L}^2\{[t_0,t_f];\mathbb{R}^m\}} c\overline{o}\{J(u(\cdot)).$$

This is simply a consequence of the following simple facts:

$$\begin{aligned} \inf_{u(\cdot)\in\mathbb{L}^2\{[t_0,t_f];\mathbb{R}^m\}} J(u(\cdot)) &= -J^*(0), \\ J^*(u(\cdot)) = c\bar{o}\{J(u(\cdot))\}, \end{aligned}$$

where $J^*(u(\cdot))$ is a conjugate of $J(u(\cdot))$. Let us now introduce the auxiliary variable x_{n+1} and the extended state vector $\tilde{x} := (x^T, x_{n+1})^T$ such that

$$\begin{aligned} &\dot{x}_{n+1}(t) = \frac{1}{2}(\langle Q(t)x(t), x(t)\rangle + \langle R(t)u(t), u(t)\rangle),\ t \in [t_0, t_f], \\ &x_{n+1}(t_0) := 0. \end{aligned}$$

The given OCPs (4) can now be equivalently rewritten using the modified terminal costs functional

$$J(u(\cdot)) = \phi(\tilde{x}(t_f)) := x_{n+1}(t_f) + \frac{1}{2}\langle Gx(t_f), x(t_f)\rangle. \tag{5}$$

The formal Hamiltonian associated with (4) for the given dynamic system (1) extended by the aforementioned additional differential equation has the following form:

$$H(t, x, u, p, p_{n+1}) = \langle p, f(t, x, u)\rangle + \frac{1}{2}p_{n+1}\left(\langle Q(t)x, x\rangle + \langle R(t)u, u\rangle\right), \tag{6}$$

where $p \in \mathbb{R}^n$, $p_{n+1} \in \mathbb{R}$ are adjoint variables. Assume that $p_{n+1} \neq 0$. We next use the following notation $\tilde{p} := (p^T, p_{n+1})^T$. Note that the Hamiltonian introduced earlier does not depend on the auxiliary state variable x_{n+1}.

Assume that $u^*(\cdot) \in \text{conv}(\mathscr{S}) \subset \mathbb{L}^2\{[t_0, t_f]; \mathbb{R}^m\}$ is an optimal solution of (4). By $x^*(\cdot)$, we denote the corresponding optimal trajectory (solution) of (1) generated by $u^*(\cdot)$. Because $J(u(\cdot))$ is a continuously Fréchet differentiable functional, the closed convex hull $c\overline{o}\{J(u(\cdot))\}$ is also Fréchet differentiable (see [49, 50] for the formal proof). The convex structure of (4) makes it possible to apply powerful numerical approaches from convex programming [24, 49]. In this paper, we use a variant of the projected gradient method in combination with the celebrated multiple shooting method (see, e.g., [15, 17]) for the concrete numerical treatment of (4). The projected gradient method for (4) can now be expressed as follows:

$$u_{(l+1)}(\cdot) = \gamma_l \mathscr{P}_{\text{conv}(\mathscr{S})}\left[u_{(l)}(\cdot) - \alpha_l \nabla c\overline{o}\{J(u_{(l)}(\cdot))\}\right] + (1-\gamma_l)u_{(l)}(\cdot),\ l \in \mathbb{N}, \tag{7}$$

where $\mathscr{P}_{\text{conv}(\mathscr{S})}$ is the operator of projection on the convex set $\text{conv}(\mathscr{S})$, $\{\alpha_l\}$ and $\{\gamma_l\}$ are sequences of some suitable step sizes associated with the method. By ∇, we denote here the Fréchet derivative of the convexified functional $\overline{co}\{J(u_{(l)}(\cdot))\}$. The generic projection operator $\mathscr{P}_{\text{conv}(\mathscr{S})}$ is defined as follows:

$$\mathscr{P}_{\text{conv}(\mathscr{S})}[u(\cdot)] := \text{Arg}\min_{v(\cdot)\in\text{conv}(\mathscr{S})}\left(\| v(\cdot) - u(\cdot) \|_{\mathbb{L}^2\{[t_0,t_f];\mathbb{R}^m\}}\right).$$

Note that the projection here is determined in the real Hilbert space $\mathbb{L}^2\{[t_0, t_f]; \mathbb{R}^m\}$.

Several choices are possible for the step sizes α_l and γ_l. Let us describe briefly the main strategies of the step sizes selection.

- The constant step size: $\gamma_l = 1$ and $\alpha_l = \alpha > 0$ for all $l \in \mathbb{N}$.
- Armijo line search along the boundary of $\text{conv}(\mathscr{S})$: $\gamma_l = 1$, for all $l \in \mathbb{N}$ and α_l is determined by

$$\alpha_l := \bar{\alpha}\theta^{\chi(l)}$$

for some $\bar{\alpha} > 0, \theta,\ \delta \in (0, 1)$, where

$$\chi(l) := \min\{\chi \in \mathbb{N} | c\overline{o}\{J(\mathscr{P}_{\text{conv}(\mathscr{S})}\left[u_{(l,s)}(\cdot)\right])\} \leq c\overline{o}\{J(u_{(l)}(\cdot))\} - \delta\langle\nabla c\overline{o}\{J(u_{(l)}(\cdot))\}, u_{(l)}(\cdot) - \mathscr{P}_{\text{conv}(\mathscr{S})}[u_{(l,s)}(\cdot)]\rangle\}$$

and

$$u_{(l,s)}(\cdot) := u_{(l)}(\cdot) - \bar{\alpha}\theta^{\chi}\nabla c\overline{o}\{J(u_{(l)}(\cdot))\}.$$

- Armijo line search along the feasible direction: $\{\alpha_l\} \subset [\bar{\alpha}, \hat{\alpha}]$ for some $\bar{\alpha} < \hat{\alpha} < \infty$ and γ_l is deftermined by the following Armijo rule $\gamma_l := \theta^{\chi(l)}$, for some $\theta,\ \delta \in (0, 1)$, where

$$\chi(l) := \min\{\chi \in \mathbb{N} | c\overline{o}\{J(u_{(l,s)}(\cdot))\} \leqslant c\overline{o}\{J(u_{(l)}(\cdot))\} - \theta^{\chi}\delta\langle\nabla c\overline{o}\{J(u_{(l)}(\cdot))\}, u_{(l)}(\cdot) - \mathscr{P}_{\text{conv}(\mathscr{S})}[w_l(\cdot)]\rangle\}$$

and $u_{(l,s)}(\cdot) := \theta^{\chi}\mathscr{P}_{\text{conv}(\mathscr{S})}[w_l(\cdot)] + (1 - \theta^{\chi})u_{(l)}(\cdot)$.
- Exogenous step size before projecting: $\gamma_l = 1$ for all $l \in \mathbb{N}$ and α_l given by

$$\alpha_l := \frac{\delta_l}{||\nabla c\overline{o}\{J(u_{(l)}(\cdot))\}||_{\mathbb{L}^2\{[t_0,t_f];\mathbb{R}^m\}}},\ \sum_{l=0}^{\infty}\delta_l = \infty,\ \sum_{l=0}^{\infty}\delta_l^2 < \infty.$$

Recall that under some non-restrictive assumptions, the projected gradient iterations (7) generates a minimizing sequence for the relaxed optimization problem (4). Many useful and mathematically exact convergence theorems for the gradient-iterations (7) can be found in [14, 24, 51]. A comprehensive discussion of the weakly and strongly convergent variants of the basic gradient method can be found in [48, 52]. We also refer to [3–5, 16, 27, 28, 53] for some specific convergence results obtained for the gradient-based schemes applied to hybrid and switched OCPs.

First strategy presented earlier was analyzed in [51], and its weak convergence was proved under Lipschitz continuity of $\nabla c\overline{o}\{J(\cdot)\}$. The main difficulty here is the necessity of taking $\alpha \in (0, 2/L)$, where L is the Lipschitz constant for $\nabla c\overline{o}\{J(\cdot)\}$ (see also [48]).

Note that the second gradient-based strategy requires one projection onto conv($\mathscr{S}$) for each step of the inner loop resulting from the Armijo line search. Therefore, many projections might be performed for each iteration l, making second strategy inefficient when the projection onto the set conv($\mathscr{S}$) cannot be computed explicitly. On the other hand, third strategy demands only one projection for each outer step, that is, for each iteration l. Second and third optimization strategies from the variants presented earlier are the constrained versions of the line search proposed in [54] for solving unconstrained optimization problems. Under existence of minimizers and some convexity assumptions for the minimization problem under consideration, it is possible to prove, for the second and third strategies, convergence of the whole sequence to a solution in finite dimensional spaces ([55]).

Last strategy from the approaches presented earlier, as its counterpart in the unconstrained case, fails to be a descent method. Furthermore, it is easy to show that this approach implies

$$||u_{(l+1)}(\cdot) - u_{(l)}(\cdot)|| \leqslant \delta_l$$

for all l, with δ_l given earlier. This reveals that convergence of the sequence of points generated by this exogenous approach can be very slow (stepsizes are small). Note that the second and third strategies allow for occasionally long step sizes because both strategies employ all information available at each l-iteration. Moreover, the last strategy does not take into account the values of the objective functional for determining the 'best' step sizes. These characteristics, in general, entail poor computational performance. The basic convergence results for the obtained approximating sequence $\{u_{(l)}(\cdot)\}$ can be stated as follows.

Theorem 1
Assume that all hypotheses from Section 2 are satisfied and $p_{n+1} \neq 0$. Consider a sequence $\{u_{(l)}(\cdot)\}$ generated by method (7) with a constant step size α. Then, for an admissible initial point $u_{(0)}(\cdot) \in$ conv($\mathscr{S}$), the resulting sequence $\left\{u_{(l)}(\cdot)\right\}$ is a minimizing sequence for (4), that is,

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$$\lim_{l\to\infty} c\overline{o}\{J(u_{(l)}(\cdot))\} = c\overline{o}\{J(u^*(\cdot))\}.$$

Additionally, assume that $\partial f(t,x,u)/\partial u$ is Lipschitz continuous with respect to (x,u):

$$\left\| \frac{\partial f(t,x_1,u_1)}{\partial u} - \frac{\partial f(t,x_2,u_2)}{\partial u} \right\| \leqslant L_x||x_1-x_2|| + L_u||u_1-u_2||,$$

and $\alpha \in (0, 2/L)$, where

$$L := (L_x l + L_u) + \lambda,\ l := \max_{t\in[t_0,t_f]}\{l_t(t)\},\ \lambda := \max_{t\in[t_0,t_f]}\{\lambda^R_{\max}(t)\},$$

and $\lambda^R_{\max}(t)$ is the maximal eigenvalue of the matrix $R(t)$. Here, $l_t(t)$ are Lipschitz constants of the control-state mapping $x^u(t)$, for $t\in[t_0,t_f]$. Then $\{u_{(l)}(\cdot)\}$ converges $\mathbb{L}^2\{[t_0,t_f];\mathbb{R}^m\}$ – weakly to a solution $u^*(\cdot)$ of (4).

Proof

As mentioned earlier, the convexified cost functional $J(\cdot)$ in (4) is Frechet differentiable. The property of $\{u_{(l)}(\cdot)\}$ to be a minimizing sequence for (4) is an immediate consequence of [24, 48]. Following [1, 2, 5, 27, 56], the reduced gradient $\nabla c\overline{o}\{J(\cdot)\}$ of the modified costs functional $c\overline{o}\{J(\cdot)\}$ at $u(\cdot)\in \text{conv}(\mathscr{S})$ can be computed from the following Hamilton-type boundary value problem

$$\begin{aligned}
\nabla c\overline{o}\{J(u(\cdot))\}(t) &= -\frac{\partial H(t,x(t),u(t),p(t),p_{n+1}(t))}{\partial u}\\
&= -\left(\frac{\partial f(t,x(t),u(t))}{\partial u}\right)^T p(t) - R(t)u(t)p_{n+1}(t),\\
\frac{d\tilde{p}(t)}{dt} &= -\frac{\partial H(t,x(t),u(t),p(t),p_{n+1}(t))}{\partial \tilde{x}}\\
&= -\left(\left(\frac{\partial H(t,x(t),u(t),p(t),p_{n+1}(t))}{\partial x}\right)^T, 0\right)^T,\\
\tilde{p}(t_f) &= \frac{\partial(c\overline{o}\{\phi(\tilde{x}(t_f))\})}{\partial\tilde{x}} = \left(-\left(c\overline{o}\{Gx(t_f)\}\right)^T, -1\right)^T,\\
\frac{d\tilde{x}(t)}{dt} &= \frac{\partial H(t,x(t),u(t),p(t),p_{n+1}(t))}{\partial\tilde{p}},\\
\tilde{x}(t_0) &= \left(x_0^T, 0\right)^T,
\end{aligned} \tag{8}$$

where $x(\cdot)$ and $\tilde{p}(\cdot)$ are state and adjoint variables associated with an admissible control function $u(\cdot)\in\text{conv}(\mathscr{S})$. The differentiability of $x^u(t)$ implies the Lipschitz continuity of this control-state mapping on the bounded convex set $\text{conv}(\mathscr{S})$. Because $\partial f(t,x,u)/\partial u$ is also assumed to be Lipschitz continuous and the composition of two Lipschitz continuous mappings possesses the same property, we next can establish the Lipschitz continuity of the derivative $\nabla \bar{co} J(u(\cdot))(t)$ uniformly in $t\in[t_0,t_f]$. From (8), we easily deduce $p_{n+1}(t)\equiv -1$ for all $t\in[t_0,t_f]$. Using the explicit expression for the gradient, namely, the first relation in (8), the Lipschitz constant for $\nabla \bar{co}\{J(u(\cdot))\}(t)$ can be computed as follows:

$$L = (L_x l + L_u) + \lambda.$$

The weak convergence to $u^*(\cdot)$ of the sequence $\{u_l(\cdot)\}$ generated by (7) with a constant step size $\alpha\in(0,2/L)$ follows now from the main result [51]. The proof is completed. □

The proposed gradient-type method (7) provides a well-defined numerical basis for the computational treatment of (4). The concrete calculation of the functional gradient is comprehensively

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discussed in the next section. We use the classic version of the multiple shooting method (see, e.g., [15, 17]) and apply it for solving the boundary value problem in (8). Using the obtained optimal solution $u^*(\cdot) \in \text{conv}(\mathscr{S})$ of the auxiliary convex OCP (4), we can determine a suitable numerical treatment of the original OCP (3).

From the computational point of view, the fully (globally) convexified OCP (4) is related with a mathematically sophisticated procedure, namely, with the calculation of a convex envelope of a composite functional in Hilbert space. Motivating from this fact, we now consider an alternative relaxation approach for the original optimal control problem (3). This new idea is related to the 'local convexification' procedure and to the infimal (prox) convolution

$$J_\lambda(u(\cdot)) + \frac{\lambda}{2}||u(\cdot)||^2_{\mathbb{L}^2\{[t_0,t_f];\mathbb{R}^m\}}$$

determined for the original objective functional $J(u(\cdot))$. We adapt here the general abstract concepts from [42, 57] to our concrete OCP (3) studied in the real Hilbert space $\mathbb{L}^2\{[t_0,t_f];\mathbb{R}^m\}$.

Definition 1
We say that $J(u(\cdot))$ is locally para-convex around $u(\cdot) \in \mathbb{L}^2\{[t_0,t_f];\mathbb{R}^m\}$ if the infimal convolution $J_\lambda(u(\cdot))$ is convex and continuous on a δ-ball $\mathcal{B}_\delta(u(\cdot))$ around $u(\cdot)$ for some $\delta > 0$ and $\lambda > 0$.

Note that the infimal convolution $J_\lambda(u(\cdot))$ from Definition 1 is a locally convex functional at $u(\cdot) \in \mathbb{L}^2\{[t_0,t_f];\mathbb{R}^m\}$.

Definition 2
We say that $J(u(\cdot))$ is is prox-regular at $\hat{u}(\cdot) \in \mathbb{L}^2\{[t_0,t_f];\mathbb{R}^m\}$ if there exist $\epsilon > 0$ and $r > 0$ such that

$$J(u_1(\cdot)) > J(u_2(\cdot)) + \langle \nabla J(\hat{u}(\cdot)), u_1(\cdot) - u_2(\cdot)\rangle_{\mathbb{L}^2\{[t_0,t_f];\mathbb{R}^m\}} - \frac{r}{2}||u_1(\cdot) - u_2(\cdot)||^2_{\mathbb{L}^2\{[t_0,t_f];\mathbb{R}^m\}}$$

for all $u_1(\cdot)$ from a ϵ-ball $\mathcal{B}_\epsilon(\hat{u}(\cdot))$ around $\hat{u}(\cdot)$ whenever $u_2(\cdot) \in \mathcal{B}_\epsilon(\hat{u}(\cdot))$ and

$$|J(u_1(\cdot)) - J(\hat{u}(\cdot))| < \epsilon.$$

In the case of problem (3) with the quadratic cost $J(u(\cdot))$, we evidently have

$$J_\lambda(u(\cdot)) = \frac{1}{2}\int_{t_0}^{t_f} (\langle Q(t)x(t), x(t)\rangle + \langle (R(t) + \lambda I)u(t), u(t)\rangle)\, dt + \frac{1}{2}\langle Gx(t_f), x(t_f)\rangle,$$

where I is a unit matrix. Consider now the following infimal convolution-based OCP

$$\begin{aligned} &\text{minimize } J_\lambda(u(\cdot)) \\ &\text{subject to } u(\cdot) \in \text{conv}(\mathscr{S}) \end{aligned} \tag{9}$$

and assume that it possesses an optimal solution $u_\lambda^{opt}(\cdot)$. The corresponding optimal trajectory is denoted by $x_\lambda^{opt}(\cdot)$. Similar to the fully relaxed case, we next introduce the auxiliary variable x_{n+1} as follows

$$\begin{aligned} &\dot{x}_{n+1}(t) = \frac{1}{2}(\langle Q(t)x(t), x(t)\rangle + \langle (R(t) + \lambda I)u(t), u(t)\rangle), \quad t \in [t_0, t_f], \\ &x_{n+1}(t_0) := 0. \end{aligned}$$

The infimal convolution-based OCPs (9) is now equipped with the objective functional of type (5). The corresponding Hamiltonian can be written as follows:

$$H(t, x, u, p, p_{n+1}) = \langle p, f(t, x, u)\rangle + \frac{1}{2}p_{n+1}(\langle Q(t)x, x\rangle + \langle (R(t) + \lambda I)u, u\rangle).$$

Assume $p_{n+1} \neq 0$. Evidently, (9) constitutes a partial relaxation (convexification) of the initial OCP (3). Note that in the case $\lambda = 0$, the auxiliary OCP (9) represents a relaxed variant of the initial OCP

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(3) with an original objective $J(u(\cdot))$ and with the convexified control set. The optimal pair for this specific problem is denoted by $(u_0^{opt}(\cdot), x_0^{opt}(\cdot))$.

Recall that under some week assumptions, the prox-regularity of a functional in a Hilbert space implies para-convexity of the corresponding infimal convolution ([57]). Using this fact, we finally can prove a local convergence result for the sequence $\{u_l(\cdot)\}$ generated by the basic gradient method

$$u_{(l+1)}(\cdot) = \gamma_l \mathscr{P}_{\text{conv}(\mathscr{S})}\left[u_{(l)}(\cdot) - \alpha_l \nabla J_\lambda(u_{(l)}(\cdot))\right] + (1-\gamma_l)u_{(l)}(\cdot),\ l \in \mathbb{N}. \tag{10}$$

Note that (10) is applied to the infimal convolution-based OCP (9).

Theorem 1
Assume that all hypotheses from Section 2 are satisfied, $p_{n+1} \neq 0$ and $u_0^{opt}(\cdot) \in \text{int}\{\text{conv}(\mathscr{S})\}$. Consider a sequence $\{u_{(l)}(\cdot)\}$ generated by method (10) with a constant step size α. Then there exists an initial point $u_{(0)}(\cdot) \in \text{conv}(\mathscr{S})$ such that

$$\lim_{\lambda\to 0}\lim_{l\to\infty} J_\lambda(u_{(l)}(\cdot)) = \min_{\text{conv}(\mathscr{S})} J(u(\cdot)) = J(u_0^{opt}(\cdot)). \tag{11}$$

Proof
Consider a ϵ-ball $\mathcal{B}_\epsilon(u_0^{opt}(\cdot))$ around the optimal control function $u_0^{opt}(\cdot)$ and let us estimate the difference $J(u_2(\cdot)) - J(u_1(\cdot))$ for some $u_1(\cdot),\ u_2(\cdot) \in \mathcal{B}_\epsilon(u_0^{opt}(\cdot))$. First, note that the gradient $\nabla J(u(\cdot))(\cdot)$ in problem (9) with a $\lambda \geq 0$ can be calculated similar to (8)

$$\begin{aligned}
&\nabla J_\lambda(u(\cdot))(t) = -\frac{\partial H(t, x(t), u(t), p(t), p_{n+1}(t))}{\partial u} = -\frac{\partial f(t, x(t), u(t))}{\partial u}^T p(t) - \\
&(R(t) + \lambda I)u(t)p_{n+1}(t),\\
&\frac{d\tilde{p}(t)}{dt} = -\frac{\partial H(t, x(t), u(t), p(t), p_{n+1}(t))}{\partial \tilde{x}} = -\left(\left(\frac{\partial H(t, x(t), u(t), p(t), p_{n+1}(t))}{\partial x}\right)^T, 0\right)^T,\\
&\tilde{p}(t_f) = -\frac{\partial(\phi(\tilde{x}(t_f)))}{\partial \tilde{x}} = \left(-\left(Gx(t_f)\right)^T, -1\right)^T,\\
&\frac{d\tilde{x}(t)}{dt} = \frac{\partial H(t, x(t), u(t), p(t), p_{n+1}(t))}{\partial \tilde{p}},\\
&\tilde{x}(t_0) = \left(x_0^T, 0\right)^T.
\end{aligned} \tag{12}$$

Applying the weak version of the Pontryagin Maximum Principle to OCP (9) for

$$\lambda = 0,\ \ u_0^{opt}(\cdot) \in \text{int}\{\text{conv}(\mathscr{S})\},$$

we next obtain

$$\frac{\partial H\left(t, x_0^{opt}(t), u_0^{opt}(t), p(t), p_{n+1}(t)\right)}{\partial u} = 0$$

([22]). Therefore,

$$\left\langle \nabla J\left(u_0^{opt}(\cdot)\right), u_1(\cdot) - u_2(\cdot)\right\rangle_{\mathbb{L}^2\{[t_0,t_f];\mathbb{R}^m\}} = 0 \tag{13}$$

for all admissible $u_1(\cdot),\ u_2(\cdot)$. From the Lipschitz continuity of the control-state mapping $x^u(\cdot)$ and also taking into account the boundedness of the control set $\text{conv}(\mathscr{S})$, we easily deduce the following inequality

$$J(u_2(\cdot)) - J(u_1(\cdot)) < \frac{r}{2}||u_1(\cdot) - u_2(\cdot)||^2_{\mathbb{L}^2\{[t_0,t_f];\mathbb{R}^m\}}. \tag{14}$$

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here, $r > 0$ is a suitable constant. The combination of (13) and (14) implies the prox-regularity property of the functional $J(u(\cdot))$ at $u_0^{opt}(\cdot)$ (Definition 2). Because $J(u(\cdot))$ is continuous, the prox-regularity property implies the para-convexity of the infimal convolution $J_\lambda(u(\cdot))$ of $J(u(\cdot))$ in a neighborhood of the optimal solution $u_0^{opt}(\cdot)$. Using the convergence results for the gradient method in the case of a locally convex function determined on a convex set ([24, 48]), we conclude that

$$\lim_{l\to\infty} J_\lambda(u_{(l)}(\cdot)) = J_\lambda(u_\lambda^{opt}(\cdot)). \tag{15}$$

Using the continuity property of the infimal convolution $J_\lambda(u(\cdot))$ and (15), we obtain (11). The proof is completed. □

As one can see, Theorem 2 establishes the well approximating property of the infimal-based OCP (9) for the 'weakly relaxed' variant of (9) with $\lambda = 0$. We next use this weak relaxation for a constructive numerical treatment of the initial OCP (3).

4. REGULARITY CONDITIONS AND REALIZABILITY OF THE GRADIENT - BASED ALGORITHM

The implementability issue of the proposed gradient-based methods (7) and (10) strongly depends on the constructive expression for the functional gradients determined as solutions of the boundary value problems (8) and (12), respectively. On the other side the existence of the nontrivial (absolutely continuous) adjoint variables

$$\left(p^T(\cdot), p_{n+1}\right)^T \in \mathbb{R}^{n+1} \setminus \{0\}$$

in (8) and in (12) that satisfy the corresponding boundary value problems is closely related to the so called regularity (Lagrange regularity) conditions in optimal control (see, e.g., [9, 22, 58, 59]). When solving conventional optimal control problems based on some necessary conditions for optimality, one is often faced with two possible technical difficulties: the irregularity of the Lagrange multipliers associated with the given constraints (see, e.g., [22, 60]) and the degeneracy phenomenon (see, e.g., [58, 59]). Various supplementary conditions, namely, constraint qualifications, have been proposed under which it is possible to assert that the Lagrange multiplier rule holds in a 'usual' constructive form ([42]). Examples of the regularity conditions are the well known: the Slater regularity condition for classic convex programming and the Mangasarian–Fromovitz regularity conditions for general nonlinear optimization problems. Let us also note here the celebrated Kurcyusz–Robinson–Zowe regularity conditions for some classes of abstract optimal control problems ([60, 61]). We also refer to [16, 22, 56, 58, 60] for some additional facts and mathematical details. Note that some regularity conditions for OCPs can be formulated as controllability conditions for the linearized system [60].

This section is devoted to the specific regularity conditions for the auxiliary infimal-based OCP (9) that guarantee existence of the nontrivial Lagrange multipliers for these optimization problems (Lagrange regularity). We restrict our consideration to the case of a stationary control system in (1) and correspondingly assume $f(t, x, u) = f(x, u)$. The Lagrange regularity determines the consistence of the main boundary value problems in (8) and finally implies the realizability of the proposed gradient-based methods in the concrete constructive form (10) – (12).

In parallel to the initially given nonlinear dynamic model in (1), we next introduce the conventional linearized system. The linearization is considered over an optimal pair $(u_\lambda^{opt}(\cdot), x_\lambda^{opt}(\cdot))$ associated with the infimal-based OCP (9)

$$\begin{aligned} \dot{y}(t) &= \frac{\partial f(x_\lambda^{opt}(t), u_\lambda^{opt}(t))}{\partial x} y(t) + \frac{\partial f(x_\lambda^{opt}(t), u_\lambda^{opt}(t))}{\partial u} v(t), \\ y(t_0) &= 0, \end{aligned} \tag{16}$$

where $v(t) \in \mathbb{R}^m$ is a control input for (16).

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Theorem 2
Assume that all hypotheses from Section 2 are satisfied, and moreover, the linearized system (16) is controllable. Then the infimal-based OCP (9) is Lagrange regular.

Proof
Let us introduce the following system operator

$$F : \mathbb{L}^2\left\{[t_0, t_f]; \mathbb{R}^m\right\} \times \mathbb{W}^{1,\infty}\left\{[t_0, t_f]; \mathbb{R}^n\right\} \to \mathbb{W}_n^{1,\infty}\left\{[t_0, t_f]; \mathbb{R}^n\right\},$$
$$F(u(\cdot), x(\cdot)) := x(\cdot) - x_0 - \int_{t_0}^{\cdot} f(x(t), u(t)),$$

where $\mathbb{W}^{1,\infty}\{[t_0, t_f]; \mathbb{R}^n\}$ denotes a Sobolev space of all absolutely continuous functions with essentially bounded derivatives. Under the basic assumptions of Section 2, the system operator is Fréchet differentiable ([22]). From the main result of [61], it follows that the optimization problem (9) is Lagrange regular, if the Fréchet derivative $DF(u_\lambda^{opt}(\cdot), x_\lambda^{opt}(\cdot))$ of the system mapping $F(u(\cdot), x(\cdot))$ at $(u_\lambda^{opt}(\cdot), x_\lambda^{opt}(\cdot))$ is surjective.

Let $z(\cdot) \in \mathbb{W}^{1,\infty}\{[t_0, t_f]; \mathbb{R}^n\}$ be an arbitrary function. The integral equation

$$y(t) = \int_{t_0}^{t} \frac{\partial f\left(x_\lambda^{opt}(t), u_\lambda^{opt}(t)\right)}{\partial x} y(t)dt + z(t),\ t \in [t_0, t_f]$$

is a linear Volterra equation of the second kind. This equation has a solution $y(\cdot) := \zeta(\cdot)$ from $\mathbb{W}^{1,\infty}\{[t_0, t_f]; \mathbb{R}^n\}$ [62]. We now put this specific function $y(\cdot)$ into the linearized system (16). The controllability assumption implies the existence of a pair

$$(\tilde{v}(\cdot), \tilde{y}(\cdot)) \in \mathbb{L}^2\{[t_0, t_f]; \mathbb{R}^m\} \times W^{1,\infty}\{[t_0, t_f]; \mathbb{R}^n\}$$

that satisfies the initial value problem (16). Therefore,

$$\begin{aligned} &DF\left(u_\lambda^{opt}(\cdot), x_\lambda^{opt}(\cdot)\right)[(\tilde{y}(\cdot) + \zeta(\cdot), \tilde{v}(\cdot))] = \tilde{y}(\cdot) + \zeta(\cdot) - \\ &\int_{t_0}^{\cdot} \left[\frac{\partial f\left(x_\lambda^{opt}(t), u_\lambda^{opt}(t)\right)}{\partial x} (\tilde{y}(t) + \zeta(\cdot)) + \frac{\partial f\left(x_\lambda^{opt}(t), u_\lambda^{opt}(t)\right)}{\partial u} \tilde{v}(t) \right] dt = \tilde{y}(\cdot) + z(\cdot) \\ &- \int_{t_0}^{\cdot} \left[\frac{\partial f\left(x_\lambda^{opt}(t), u_\lambda^{opt}(t)\right)}{\partial x} \tilde{y}(t) + \frac{\partial f\left(x_\lambda^{opt}(t), u_\lambda^{opt}(t)\right)}{\partial u} \tilde{v}(t) \right] dt = z(\cdot). \end{aligned} \quad (17)$$

The final relation (17) characterizes the surjectivity property of $DF(u_\lambda^{opt}(\cdot), x_\lambda^{opt}(\cdot))$ that implies the Lagrange regularity of (9). The proof is completed. □

Evidently, the controllability property of system (16) considered on the originally given control space $\mathscr{S}$ implies the assumption of the basic Theorem 3 for $v(t) \in \mathbb{R}^m$. Let us now present a simple controllability conditions for the linearized system (16) determined on the originally given control space, namely, on the space of the fixed-levels controls $v(\cdot) \in \mathscr{S}$. We restrict here our consideration to a specific stationary case $A := \partial f(x_\lambda^{opt}(t), u_\lambda^{opt}(t))/\partial x$ and $B := \partial f(x_\lambda^{opt}(t), u_\lambda^{opt}(t))/\partial u$ for $u(\cdot) \in \mathscr{S}$.

Theorem 3
Consider the linearized system (16) for $v(\cdot) \in \mathscr{S}$ and $N_k \equiv N,\ \mathscr{T}^k \equiv \mathscr{T},\ k = 1, ..., m$. Assume that

$$-B^T \int_{t_{i-1}}^{\cdot} e^{-A^T\tau} d\tau W(N)^{-1} \left(y(t_0) - e^{-At_f} y(t_f)\right) \in \mathscr{S}. \quad (18)$$

Then, system (16) is controllable if and only if the following matrix

$$W(N) := \sum_{i=1}^{N} \left[\int_{t_{i-1}}^{t_i} e^{-A\tau} d\tau BB^T \int_{t_{i-1}}^{t_i} e^{-A^T\tau} d\tau \right],\ t_i \in \mathscr{T}.$$

is nonsingular.

Proof
Let $W(N)$ be nonsingular. Then

$$y(t_f) = e^{At_f} y(t_0) + \int_{t_0}^{t_f} e^{A(t_f-\tau)} Bv(\tau) d\tau,$$

or equivalently

$$y(t_f) = e^{At_f} \left[y(t_0) + \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} e^{-A\tau} d\tau Bv^i \right], \tag{19}$$

where $v^i \in \mathbb{R}^m$ is a constant vector associated with the interval $[t_{i-1}, t_i)$. The resulting input value v^i such that $v(t) = v^i$ for $t \in [t_{i-1}, t_i)$ and $y(t_0)$, $y(t_f)$ belongs to the corresponding trajectory of (16) generated by $v(\cdot)$ is given by

$$v^i = -B^T \int_{t_{i-1}}^{t_i} e^{-A^T\tau} d\tau W(N)^{-1} \left(y(t_0) - e^{-At_f} y(t_f) \right) \in \mathscr{S}.$$

Substituting the last expression in (19), we next obtain

$$y(t_f) = e^{At_f} [y(t_0) - W(N)W(N)^{-1}(y(t_0) - e^{-At_f} y(t_f))] = y(t_f).$$

We finally conclude that the given system is controllable under piecewise constant inputs.

Let the initial system (16) be controllable by piecewise constant controls $v(\cdot)$ from $\mathscr{S}$. Assume that the symmetric matrix $W(N)$ is not a (strictly) positive definite matrix. This hypothesis implies the existence of a nontrivial vector $w \in \mathbb{R}^n$ such that $w^T W(N) w = 0$, or equivalently

$$0 = w^T \sum_{i=1}^{N} \left[\int_{t_{i-1}}^{t_i} e^{-A\tau} d\tau BB^T \int_{t_{i-1}}^{t_i} e^{-A^T\tau} d\tau \right] w = \sum_{i=1}^{N} ||w^T \int_{t_{i-1}}^{t_i} e^{-A\tau} d\tau B||^2.$$

The last fact implies the following

$$w^T \int_{t_{i-1}}^{t_i} e^{-A\tau} d\tau B = 0\ \forall i = 1, \dots, N.$$

Because the controllability of the system (16) for $v(\cdot) \in \mathscr{S}$ is assumed, there exist a sequence of values $\{v^i\}$ such that the state $y(t_0) \equiv w$ can be transferred into $y(t_f) \equiv 0$. Therefore, we next deduce

$$0 = e^{At_f} \left[w + \sum_{i=1}^{N} \left(\int_{t_{i-1}}^{t_i} e^{-A\tau} d\tau \right) Bv^i \right]. \tag{20}$$

Evidently, (20) holds if and only if

$$0 = w + \sum_{i=1}^{N} \left(\int_{t_{i-1}}^{t_i} e^{-A\tau} d\tau \right) Bv^i. \tag{21}$$

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We now multiply (21) by w^T and obtain the contradiction with the non-triviality hypothesis $w \neq 0$:

$$0 = w^T w + \sum_{i=1}^{N} w^T \left(\int_{t_{i-1}}^{t_i} e^{-A\tau} d\tau \right) B v^i = w^T w$$

Therefore, $W(N)$ is a positive definite symmetric matrix, and the existence of the inverse $W(N)^{-1}$ follows immediately. The proof is completed. □

Using Theorem 2 and Theorem 3, we now can easily obtain the following specific implementability result associated with the proposed gradient-based algorithm (10) – (12).

Corollary 1
Assume that all hypotheses from Section 2 and condition (18) are satisfied, and moreover, the matrix $W(N)$ associated with the linearized system (16) of the stationary original system (1) with $N_k \equiv N,\ \mathscr{T}^k \equiv \mathscr{T},\ k = 1, ..., m$ is nonsingular. Then the auxiliary infimal-based OCP (9) is Lagrange regular, and the projected gradient method for (9) can be implemented in the constructive form (10) – (12).

Note that controllability conditions for a linearized system (16) equipped with controls $v(\cdot) \in \mathscr{S}$ evidently imply the controllability property of the same system considered for $v(\cdot) \in \text{conv}(\mathscr{S})$. This fact is a simple consequence of the inclusion $\mathscr{S} \subset \text{conv}(\mathscr{S})$. Therefore, the realizability conditions for the gradient-based algorithm (10) – (12) proposed for the auxiliary OCP (9) with $u(\cdot) \in \text{conv}(\mathscr{S})$ can be formulated using the controllability conditions for the original dynamic system (1) determined on the initially given fixed-level control set $\mathscr{S}$.

5. NUMERICAL TREATMENT OF THE INITIAL OPTIMAL CONTROL PROBLEM

Theorem 1 and the classic gradient-based iterations (10) provide an theoretic basis for a relative simple computational approach to the initial OCP (3). Recall that in contrast to the relaxed optimization problem (4), the original OCP (3) does not possesses the desired convexity property. The obtained Theorem 1 provides a consistent approximation concept for the solution of the weakly relaxed OCP (9) (for $\lambda = 0$).

Using Theorem 1, the explicit representation (12) of the gradient $\nabla J(\cdot)$ and the additional projection techniques, we can propose an implementable computational scheme for the concrete numerical treatment of the initial problem (3). By $\hat{u}(\cdot)$, we next denote an approximative solution to the original OCP (3), and $\hat{x}(\cdot)$ denotes the corresponding quasi-optimal trajectory in (3).

Conceptual Algorithm 1

1. Select a sufficiently small $\lambda > 0$ and a constant step size $\alpha > 0$. Set an admissible initial condition $u_{(0)}(\cdot) \in \text{conv}(\mathscr{S})$. Calculate the corresponding trajectory $\tilde{x}_l(\cdot)$ generated by the extended control system and put $l := 0$.
2. Calculate $\nabla J_\lambda(u_{(l)}(\cdot))(\cdot)$ from the boundary value porоblem (12).
3. Using $\nabla J_\lambda(u_{(l)}(\cdot))(\cdot)$ and (10), calculate $u_{(l+1)}(\cdot)$.
4. Evaluate the $(l+1)$-iteration $v_{(l+1)}$ for the optimal control function in problem (9)

$$v_{(l+1)}^{(k)}(t) = \sum_{i=1}^{N_k} I_{\left[t_{i-1}^{(k)}, t_i^{(k)}\right)}(t) \bar{q}_{i,n}^{(k)} \quad \forall k = 1, \dots, m$$

where

$$\bar{q}_{i,l}^{(k)} = \begin{cases} q_1^{(k)}, & \bar{\bar{q}}_{i,l}^{(k)} < q_1^{(k)} \\ \bar{\bar{q}}_{i,l}^{(k)}, & q_1^{(k)} \leqslant \bar{\bar{q}}_{i,l}^{(k)} \leqslant q_{M_k}^{(k)} \\ q_{M_k}^{(k)}, & q_{M_k}^{(k)} \leqslant \bar{\bar{q}}_{i,l}^{(k)} \end{cases}$$

Here, $i = 1, \dots, N_k$,

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$$q_j^{(k)} \in \mathcal{Q}^k, \; \forall j = 1, \ldots, M_k$$

and

$$\bar{\bar{q}}_{i,l}^{(k)} := \frac{1}{\Delta_i} \int_{t_{i-1}}^{t_i} v_{(l)}^{(k)}(t)dt, \; \Delta_i := t_i - t_{i-1}.$$

1. Calculate

$$\Delta := |J_\lambda(v_{(l+1)}(\cdot)) - J_\lambda(v_{(l)}(\cdot))|.$$

If $\Delta \leq \varepsilon$ for a prescribed accuracy $\varepsilon > 0$, then put

$$u^{opt}(\cdot) \equiv v_{(l+1)}(\cdot)$$

(an approximative optimal solution to (9)). STOP.
2. Else, update the iteration register $l := l + 1$ and go to Step (2).
3. Using the evaluated function $u^{opt}(\cdot)$, the approximative optimal control for (3) $\hat{u}(\cdot) \in \mathcal{S}$ is finally calculated by components

$$\hat{u}^{(k)}(\cdot) = \sum_{i=1}^{N_k} I_{\left[t_{i-1}^{(k)}, t_i^{(k)}\right)}(t)\hat{q}_i^{(k)} \quad \forall k = 1, \ldots, m$$

where

$$\hat{q}_i^k := \text{Arg} \min_{v \in \mathcal{Q}^k} \left| v - \bar{q}_{i,l+1}^{(k)} \right|.$$

Close system (1) by the obtained quasi-optimal control input $\hat{u}(\cdot)$ and calculate the corresponding approximative optimal trajectory $\hat{x}(\cdot)$. STOP.

The proposed method (10) and the presented Conceptual Algorithm I constitute a consistent (convergent) solution procedure for a numerical treatment of the open-loop system optimization problem. In this paper, we do not consider an optimal feedback control setting. Because of the highly restrictive nonlinear control constraint $u(\cdot) \in \mathcal{S}$, the initial optimization problem (3) cannot be solved by a direct application of the Pontryagin Maximun Principle as well as by a numerical method based on the Bellman optimality condition (for example, adaptive dynamic programming approach [63, 64] or fuzzy optimal control methodology [65, 66]). Let us recall that the conventional and advanced versions of the celebrated Maximum Principle and of the Bellman optimality approach make it possible to specify an optimal solution that belongs to the full (non-restricted) control functions space. Motivated from that fact, our main contribution is with a development of a relative simple and at the same time a convergent computational procedure for the generic OCP (3). We use a novel weakly relaxed technique in combination with an implementable gradient-type algorithm for this purpose. The obtained optimal solution of the weakly relaxed OCP provides a conceptual numerical base for a concrete computational solution procedure for the original problem (3). Moreover, the proposed method involves the existence of an optimal solution to the weakly relaxed OCP under consideration. Note that we do not use here any numerical scheme that involves a specific optimality condition for classic OCPs, namely, the Hamilton–Jacobi–Bellman approach or Pontryagin Maximum Principle. Furthermore, in contrast to the greedy heuristic dynamic programming solution method (see, e.g., [63]), we study the generally nonlinear (non-affine) dynamic system. This fact makes it difficult to determine so-called 'expected control' that constitutes a necessary formal step of the greedy heuristic dynamic programming approach.

The computational scheme presented earlier consists of two main subroutines. The first one constitutes a numeric tool for solving the boundary value problem (12) (Step 2). We use here the well-known multiple shooting method for this purpose (see [15] for details). We next apply the celebrated Armijo's algorithm in the second main subroutine (the gradient method (10)). The proposed algorithm finally returns a quasi-optimal control input $u^{opt}(\cdot) \in \text{conv}(\mathcal{S})$ in the context of the infimal-based (relaxed) problem (9).

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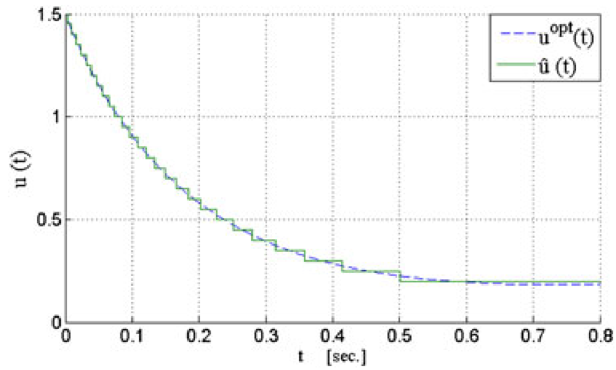

Figure 2. Quasioptimal control input $\hat{u}(t)$.

We now study two computational examples in order to illustrate the numerical effectiveness of the computational approach I proposed in this section.

Example 2
Consider the following model of a continuous stirred-tank chemical reactor described in [23]

$$\dot{x}_1(t) = -2[x_1(t) + 0.25] + [x_2(t) + 0.5] \exp\left(\frac{25x_1(t)}{x_1(t)+2}\right) - [x_1(t) + 0.25]u(t)$$
$$\dot{x}_2(t) = 0.5 - x_2 - [x_2(t) + 0.5] \exp\left(\frac{25x_1(t)}{x_1(t)+2}\right)$$
$$x(0) = \begin{bmatrix} 0.05 & 0 \end{bmatrix}^T, \; t_0 = 0, \; t_f = 0.78.$$

The control variable $u(t)$ represents a flow-rate in the cooling fluid. The state variable $x_1(t)$ describes the deviation from steady-state temperature, and $x_2(t)$ characterizes the deviation from the steady-state concentration. The cost functional is given by

$$J(u(\cdot)) = \frac{1}{2}\int_0^{0.78} \left(x_1^2(t) + x_2^2(t) + 0.1u^2(t)\right) dt.$$

The 'qualitative' optimal control requirement here can be described as follows: to maintain the temperature and concentration close to some prescribed steady-state values and to guarantee the minimal system / controller energy use. We next put

$$\mathcal{Q} = \{0, 0.05, 0.1, 0.15, 0.2, \dots, 1.95, 2\}$$

Applying Conceptual Algorithm I, we can compute $\hat{u}(\cdot) \in \mathcal{S}$. The corresponding numerical results are presented on Figure 2.

The control $u^{opt}(\cdot) \in \mathrm{conv}(\mathcal{S})$ for (9) is indicated here by the dashed line. The solid line represents the approximate solution (quasi-optimal control) $\hat{u}(\cdot)$ of the original OCP (3). The first component of the corresponding quasi-optimal trajectories $\hat{x}_1(t)$ is shown on Figure 3.

The dashed line represents the trajectory x^{opt} generated by $u^{opt}(\cdot)$, and the solid line corresponds to the trajectory $\hat{x}(\cdot)$ of (1) generated by $\hat{u}(\cdot)$. The (graphical) representation of the calculated value of $J(u(\cdot))$ as a function of the number of iterations is given on Figure 4.

The calculated value of the optimal cost in OCP (3) is equal to 0.028 and was obtained after 33 iterations of the proposed algorithm.

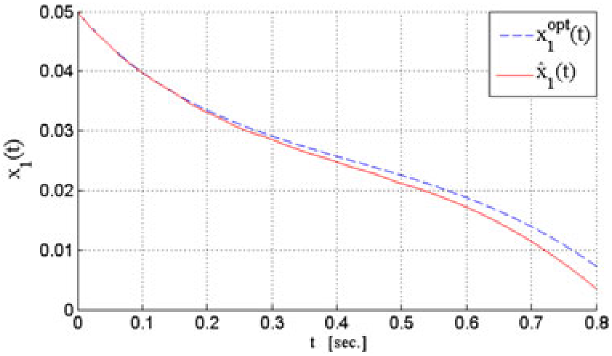

Figure 3. Trajectory component $\hat{x}_1(t)$.

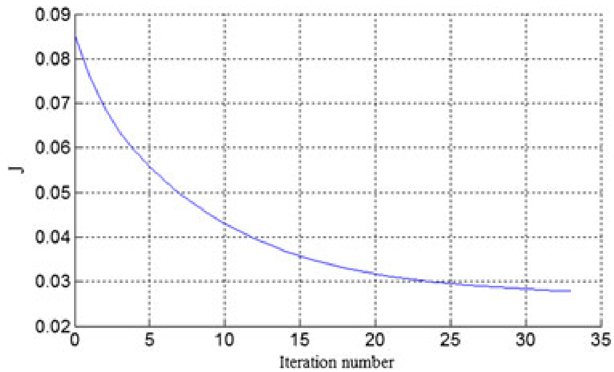

Figure 4. Performance measure evaluation.

Example 3
We now consider the dynamic model of a unicycle robot discussed in [67]

$$\begin{aligned}
\dot{x}_1(t) &= u_1(t)\cos(x_3(t))\\
\dot{x}_2(t) &= u_1(t)\sin(x_3(t))\\
\dot{x}_3(t) &= u_2(t),\\
x(0) &= \left[\, 15 \;\; 15 \;\; 180\,\text{deg} \,\right]^T,\\
t_0 &= 0,\; t_f = 1
\end{aligned}$$

The control variable $u_1(t)$ represents a linear velocity of the vehicle, while the control input $u_2(t)$ determines its orientation. The state variables

$$(x_1(t), x_2(t))$$

denote the coordinates of the robot on the plane, and the additional variable $x_3(t)$ indicates the corresponding orientation. The objective functional associated with the dynamic model under consideration is given as follows:

$$J(u(\cdot)) = \frac{1}{2}\int_0^1 \left(x_1^2(t) + x_2^2(t) + x_2^3(t)\right) dt.$$

Let us put

$$\mathcal{Q} = \{-50, -49, -48, \ldots, 48, 49, 50\}.$$

The numerical results obtained by application of of the basic Conceptual Algorithm I are presented on Figure 5. The control inputs $u_1^{opt}(\cdot)$ and $u_2^{opt}(\cdot)$ in the relaxed OCP (9) are indicated here by the

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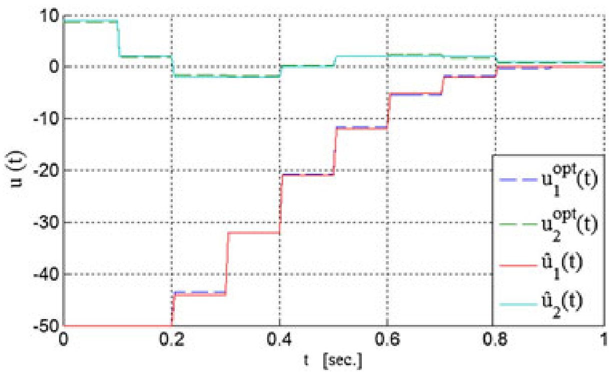

Figure 5. Quasioptimal control inputs $\hat{u}_1(t)$ and $\hat{u}_2(t)$.

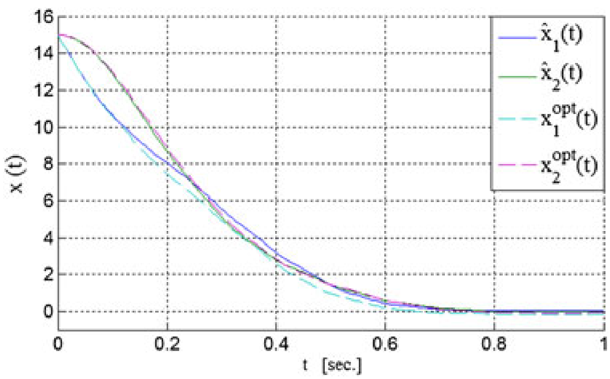

Figure 6. Trajectories components $\hat{x}_1(t)$ and $\hat{x}_2(t)$.

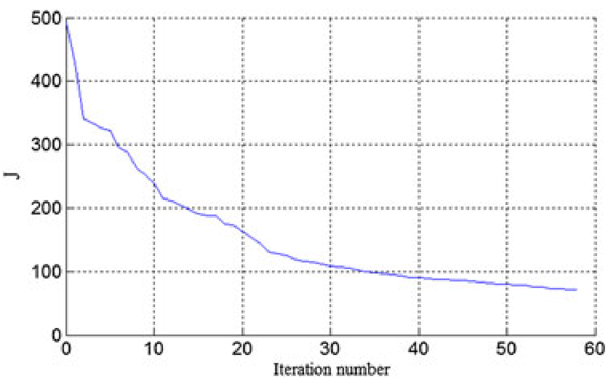

Figure 7. Performance measure evaluation.

dashed lines. The solid lines represent the obtained functions $(\hat{u}_1(\cdot), \hat{u}_2(\cdot))$ (the quasi-optimal controls in the original OCP (3)). The resulting trajectories are shown on Figure 6. The dashed lines corresponds to the trajectory generated by $u^{opt}(\cdot)$, and the solid lines indicates trajectory generated by $\hat{u}(\cdot)$. The calculated values of the objective functional $J(\hat{u}(\cdot))$ in OCP (3) are presented on Figure 7. The calculated controls are obtained after 68 iterations of the numerical algorithm. The calculated optimal cost for the initial OCP (3) is equal to 85.17.

Finally, note that implementations of the numerical algorithm described earlier was carried out using the standard MATLAB packages and the Authors programs.

6. CONCLUDING REMARKS

In this contribution, we propose a new consistent numerical approach to a class of optimal control processes governed by nonlinear ordinary differential equations. The switched dynamic structure under consideration is a consequence of the piecewise-constant nature of the admissible controls. Note that many real-world applications from various engineering disciplines can be studied in the modeling framework that incorporates fixed-level control inputs. The corresponding dynamic models represent an extremely wide range of systems of practical interest (see, e.g., [2, 8, 10, 11, 21, 30–32]) and are accepted as realistic abstractions in industrial electronics, power systems engineering, aircrafts control engineering, and communication theory. Let us also emphasize the aerospace control applications of the mathematical models that include fixed-level control inputs ([8, 31, 32]). In fact, many modern application domains involve complex systems with switched-type control design in which sub-systems interconnections, mode-transitions, and heterogeneous computational devices are presented. Moreover, the given structure of the admissible control functions is also strongly motivated by the widely used quantization procedure associated with the originally continuous system dynamics (see, e.g., [33–35]).

The main result of our contribution constitutes an analytic basis for a new consistent computational algorithm for OCPs associated with the general nonlinear dynamic systems in the presence of piecewise-constant controls. The computational scheme developed in this paper uses a novel relaxation procedure for the initial problem in combination with the first-order gradient techniques. The specific locally convex structure of the obtained relaxed problem makes it possible to take into consideration diverse algorithms from the classic nonlinear programming.

Finally, note that the theoretical and computational approaches presented in this paper can be applied to alternative classes of constrained nonlinear OCPs with switched control structure. The proposed numerical algorithm can also be a constructive consistent part (subroutine) of various numerical methods based on the common a priori discrete approximations of the initially given OCPs.

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