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RESEARCH PAPER

MONTE CARLO ESTIMATION OF THE SOLUTION OF FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS

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Abstract

The paper is devoted to the numerical solutions of fractional PDEs based on its probabilistic interpretation, that is, we construct approximate solutions via certain Monte Carlo simulations. The main results represent the upper bound of errors between the exact solution and the Monte Carlo approximation, the estimate of the fluctuation via the appropriate central limit theorem (CLT) and the construction of confidence intervals. Moreover, we provide rates of convergence in the CLT via Berry-Esseen type bounds. Concrete numerical computations and illustrations are included.

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Key Words and Phrases: numerical solution of fractional PDE; stable process; simulation; Monte-Carlo estimation; central limit theorem; Berry-Esseen bounds

1. **Introduction**

The study of fractional partial differential equations (FPDEs) is a very popular topic of modern research due to their ubiquitous application in natural sciences. In particular, there is an immense amount of literature devoted to numerical solution of FPDEs. However most of them exploit the various kinds of deterministic algorithms (lattice approximation, finite element methods, etc), see e.g. [1, 2, 3, 17] and numerous references therein. However, there are only few papers based on probabilistic methods. For

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instance, [16] exploits the CTRW (continuous time random walk) approximation for solutions to FPDEs, and [12] is based on the exact probabilistic representation.

CTRW approximation to the solutions of FPDEs was developed by physicists more than half a century ago and it became one of the basic stimulus to the modern development of fractional calculus. Exact probabilistic representation appeared a bit later first for fractional equations and then for generalized fractional (e.g. mixed fractional), see e.g. [10, 11, 8, 13] for various versions of this representation. There are now many books with detailed presentation of the basics of fractional calculus, see e.g. [9, 13, 7].

The paper is devoted to the numerical solutions of fractional PDEs based on its probabilistic representation with the main new point being the detailed discussion of the convergence rates. Namely, the main results represent the upper bound of errors between the exact solution and the Monte Carlo approximation, the estimate of the fluctuation via the appropriate central limit theorem and the construction of confidence intervals. Concrete numerical computations and illustrations are included.

We denote $C_\infty\left(\mathbb{R}^d\right) := \{f : \mathbb{R}^d \to \mathbb{R}$ is continuous and vanishes at infinity$\}$. Let $g \in C_\infty\left(\mathbb{R}^d\right)$, consider the problem

$$\begin{aligned} \left(-{}_tD_a + A_x\right) u\left(t,x\right) &= -g\left(x\right), \quad \left(t,x\right) \in (a,b] \times \mathbb{R}^d, \\ u\left(a,x\right) &= \phi\left(x\right), \quad x \in \mathbb{R}^d, \end{aligned} \tag{1.1}$$

where A_x is a generator of a Feller semigroup on $C_\infty\left(\mathbb{R}^d\right)$ acting on x, $\phi \in Dom\left(A_x\right)$, the operator $-{}_tD_a$ is a genetalised differential operator of Caputo type of order less than 1 acting on the time variable $t \in [a,b]$.

The solution $u \in C_\infty\left((-\infty,b] \times \mathbb{R}^d\right)$ of the problem (1.1) exsits and is given by [4]. u has the stochasitc representation (see [4] Equation (4) and Theorem 4.20),

$$u\left(t,x\right) = \mathbb{E}\left[\phi\left(X^x_{T_t}\right) + \int_0^{T_t} g\left(X^x_s\right) ds\right], \tag{1.2}$$

where $\{X^x_s\}_{s\geqslant 0}$ is the stochastic process started at $x \in \mathbb{R}^d$ generated by A_x. Let $\{Y^{a,t}_s\}_{s\geqslant 0}$ be the decreasing $[a,b]$-valued stochastic process started at $t \in [a,b]$ generated by $-{}_tD_a$, $T_t = \inf\{s > 0, Y^{a,t}_s < a\}$. In this paper, we assume $\{Y^{a,t}_s\}$ is a decreasing α-stable Levy process started at t, i.e. $Y^{a,t}_s \overset{d}{=} t - s^{1/\alpha}\eta$, where η is a random variable with α-stable distribution whose Laplace transform is $\mathbb{E}[\mathrm{e}^{-z\eta}] = \mathrm{e}^{-z^\alpha/\cos(\pi\alpha/2)}$(and we denote this by $\eta \sim S_\alpha(1,1,0)$).

Remark 1.1. Given a Levy measure ν on $\mathbb{R}_+$ satisfying

$$\int_0^\infty \min\{1,r\}\nu(\mathrm{d}r) < \infty,$$

the operator $-{}_tD_a$ is defined by

$$-{}_tD_a f(s) := \int_0^{s-a} (f(s-r) - f(s))\nu(\mathrm{d}r) - (f(a) - f(s)) \int_{s-a}^{\infty} \nu(\mathrm{d}r),$$

$t \in (a, b]$.

When $\{X_s^x\}_{s \geqslant 0}$ is Brownian motion, then A_x would be $\frac{1}{2}\Delta$, where $\Delta = \sum_{i=1}^d \left(\frac{\partial}{\partial x_i}\right)^2$. If $\{Y_s^{a,t}\}$ is the deterministic drift, i.e. $-{}_tD_a = -\frac{d}{dt}$ and $g = 0$, then (1.1) becomes

$$\frac{1}{2}\Delta u\,(t, x) = \frac{d}{dt} u\,(t, x)\,, \tag{1.3}$$

the heat equation that we are more familiar with.

We assume $\{X_s^x\}_{s \geqslant 0}$ is isotropic β-stable.(What 'isotropic' means is explained in Section 2, after Lemma 2.1) In this paper we shall investigate some properties of the representation (1.2) and its Monte-Carlo estimator, i.e.

$$u_N^h\,(t, x) = \frac{1}{N} \sum_{k=1}^{N} \left(\phi\left(X_{T_t^k}^{x,k}\right) + \sum_{i=1}^{\lfloor T_t^k/h \rfloor} hg\left(X_{t_i^k}^{x,k}\right) \right), \tag{1.4}$$

where $h > 0$ is the step length, T_t^k are iid samples of T_t, and $t_i^k = (i-1)\,h$. Note that we can sample the stopping time T_t (see Lemma 2.1 below), then sample the isotropic β-stable process $\{X_s^x\}$ and finally simulate the estimator (1.4).

In Section 2 we mainly focus on the situation when $g = 0$, i.e. the estimator now is

$$u_N\,(t, x) = \frac{1}{N} \sum_{k=1}^{N} \phi\left(X_{T_t^k}^{x,k}\right). \tag{1.5}$$

To make central limit theorem and Berry-Esseen bound hold, we only need to estimate the tail of the stable process at some stopping time, i.e. $\mathbb{P}\left[|X_{T_t}^x| > s\right]$ for large s. And we begin with showing that the order of the tail of multidimentional stable distribution has the same order of the tail of each component of itself. In Section 3 we study the property of the Monte-Carlo estimator when the forcing term $g \neq 0$. We estimate the upper bound of the second moment of the estimator and then, the L^2 error between the estimator and the solution. Besides, we use there properties to show that the central limit theorem holds using the triangular arrays. In Section 4 we give numerical examples, demonstrating the performance of our simulation algorithm.

2. Properties of the estimator when the forcing term g=0

In this paper, for function $f, g : \mathbb{R}^d \to \mathbb{R}$, we use the notation $f(x) = O(g(x))$, meaning that $|\frac{f(x)}{g(x)}|$ is bounded as $|x| \to \infty$. Also we use the notation $f(x) \sim g(x)$, meaning that both $|\frac{f(x)}{g(x)}|$ and $|\frac{g(x)}{f(x)}|$ are bounded as $|x| \to \infty$.

In this section, we study the situation when $g(x) = 0$ for all $x \in \mathbb{R}^d$, then the stochastic representation (1.2) becomes

$$u(t,x) = \mathbb{E}\left[\phi\left(X_{T_t}^x\right)\right] \tag{2.1}$$

and the estimator now is defined in (1.5).

Our main results tell us how close $u_N(t,x)$ and $u(t,x)$ are, namely:

THEOREM 2.1. *(i) For all continuous function* $\phi : \mathbb{R}^d \to \mathbb{R}$,

$$u_N(t,x) \stackrel{a.s.}{\to} u(t,x), \text{ as } N \to \infty. \tag{2.2}$$

(ii) Let $S_N(t,x) = \sqrt{N}\left(u_N(t,x) - u(t,x)\right)/\sigma(t,x)$ *and* W *be the standard normal distribution. If* $\phi(x)$ *satisfies* $\phi(x) = O\left(|x|^{\frac{\beta}{2+\delta}}\right)$, *where* $\delta > 0$, *then the central limit theorem holds, i.e. for all bounded uniformly continuous funtion* ψ,

$$\mathbb{E}\left[\psi\left(S_N(t,x)\right)\right] \to \mathbb{E}\left[\psi(W)\right] \text{ as } N \to \infty.$$

(iii) Let $Y(t,x) := \phi\left(X_{T_t}^x\right) - \mathbb{E}\left[\phi\left(X_{T_t}^x\right)\right]$, *denote* $\mathbb{E}\left[Y(t,x)^2\right] = \sigma(t,x)^2$, $\mathbb{E}\left[|Y(t,x)|^3\right] = \rho(t,x)$. *If* $\phi(x)$ *satisfies* $\phi(x) = O\left(|x|^{\frac{\beta}{3+\delta}}\right)$, *where* $\delta > 0$, *then for all* C^3 *functions* $\psi : \mathbb{R} \to \mathbb{R}$,

$$\left|\mathbb{E}\left[\psi\left(S_N(t,x)\right)\right] - \mathbb{E}\left[\psi(W)\right]\right| \leqslant 0.433||\psi'''||_\infty \frac{\rho(t,x)}{\sqrt{N}\sigma(t,x)^3}.$$

Here C^3 *means the space of functions with bounded third derivatives.*

In other words, the central limit theorem can be written using convergence in distribution:

$$\sqrt{N}\left(u_N(t,x) - u(t,x)\right) \stackrel{d}{\to} N\left(0, \sigma(t,x)^2\right) \quad \text{as} \quad N \to \infty.$$

Since the estimator is unbiased, Theorem 2.1(i) holds because of the strong law of large numbers. For (ii), it is the standard central limit theorem and we only need to show that $\mathbb{E}\left[\phi\left(X_{T_t}^x\right)^2\right] < \infty$. For (iii), it is a version of the Berry-Esseen bound and we need to show that $\mathbb{E}\left[|\phi\left(X_{T_t}^x\right)|^3\right] < \infty$.

These facts are evident if $\phi(x)$ is bounded. To deal with unbounded $\phi(x)$, let us recall the following fact: for any random variable U,

$$\mathbb{E}\left[U^2\right] = \int_0^\infty \mathbb{P}\left[U^2 > t\right] dt. \tag{2.3}$$

It is finite if $\mathbb{P}\left[|U| > t\right] = O\left(t^{-(2+\delta)}\right)$, where δ is a positive constant. Now let us look back at our problems. Once we know the tail of $X^x_{T_t}$ and the growth rate of $\phi(x)$, the tail of $\phi\left(X^x_{T_t}\right)$ would be clear as well as the finiteness of the moments of $\phi\left(X^x_{T_t}\right)$.

Luckily, we have the following result:

Proposition 2.1. *Assume that $\{X_s\}_{s\geqslant 0}$ is a β stable process, then $\mathbb{P}\left[|X^x_{T_t}| > u\right] = O\left(u^{-\beta}\right)$.*

To prove Proposition 2.1, we need a little lemma telling us that the distribution of T_t is analytically accessible:

Lemma 2.1. *Denote $\bar{a} := t - a$, then $T_t \overset{d}{=} \left(\frac{\bar{a}}{\eta}\right)^\alpha$ where $\eta \sim S_\alpha(1,1,0)$.*

P r o o f. Note that $Y^{a,t}_s \overset{d}{=} t - s^{1/\alpha}\eta$. $\{T_t > s\} = \{Y^{a,t}_s > a\}$, since τ has monotone paths. Hence

$$\mathbb{P}\left[T_t > s\right] = \mathbb{P}\left[t - s^{1/\alpha}\eta > a\right] = \mathbb{P}\left[s^{\frac{1}{\alpha}}\eta < \bar{a}\right] = \mathbb{P}\left[s < (\bar{a}/\eta)^\alpha\right]$$

□

Together with the facts that X^x_s is β stable and Lemma 2.1, we have

$$X^x_{T_t} - x \overset{d}{=} T_t^{\frac{1}{\beta}} X_1 \overset{d}{=} \left(\frac{\bar{a}}{\eta}\right)^{\frac{\alpha}{\beta}} X_1. \tag{2.4}$$

Also we need Lemma 2.2 and Lemma 2.3 given below. Before that let us explain what 'isotropic' means in our assumption of $\{X_s\}_{s\geqslant 0}$.

For d-dim β-stable random variable $U = \left(U_{(1)}, ..., U_{(d)}\right)$ on $\mathbb{R}^d$, there are a finte measure λ on sphere S and γ in $\mathbb{R}^d$ such that the characteristic function of U satisfies

$$\hat{U}(z) := \mathbb{E}[\mathrm{e}^{i\langle z,U\rangle}] = \exp\left[-\int_S |\langle z,\xi\rangle|^\beta \left(1 - \mathrm{i}\tan\frac{\pi\beta}{2}\mathrm{sgn}\langle z,\xi\rangle\right)\lambda(\mathrm{d}\xi) + \mathrm{i}\langle\gamma, z\rangle\right],$$

for $\beta \neq 1$ and vice versa. Hence each component of U is 1-dim stable random variable and the stability index is still β. Besides, for 1-dim β-stable random variable V whose characteristic function has form

$$\hat{V}(z) = \mathbb{E}[\mathrm{e}^{iVz}] = \exp(-\sigma^\beta|z|(1 - i\rho(\mathrm{sign}z)\tan(\pi\beta/2) + i\mu z),$$

we use the notation $V \sim S_\beta(\sigma, \rho, \mu)$. We say a d-dim stable random variable U is **isotropic** if its coordinates have the same distribution, i.e. $U_{(i)} \sim S_\beta(\sigma, \rho, \mu)$ $i = 1, ..., d$. We say a process $\{X_s\}_{s\geqslant 0}$ is isotropic stable if X_1 is an isotropic stable random variable.

LEMMA 2.2. *Let $U = (U_{(1)}, ..., U_{(d)})$ be an isotropic d-dim β-stable random variable, and $U_{(i)} \sim S_\beta(\sigma, \rho, \mu)$, then $\mathbb{P}[|U| > s] \sim s^{-\beta}$ as $s \to \infty$.*

P r o o f. Since $\{|U| = \sqrt{U_{(1)}^2 + ... + U_{(d)}^2} > s\} \supset \{|U_{(1)}| > s\}$, we have

$$\mathbb{P}[|U| > s] \geqslant \mathbb{P}[|U_{(1)}| > s].$$

Since $\{|U| > s\} \subset \{\max_{1\leqslant i\leqslant d} |U_{(i)}| > s/\sqrt{d}\} \subset \cup_{i=1}^d \{|U_{(i)}| > s/\sqrt{d}\}$, we have

$$\mathbb{P}[|U| > s] \leqslant \sum_{i=1}^{d} \mathbb{P}[U_{(i)} > s/\sqrt{d}].$$

Now recall the well known result of the tail of 1-dim stable random variable: if $V \sim S_\beta(\sigma, \rho, \mu)$, then

$$\lim_{s\to\infty} s^\beta \mathbb{P}[|V| > s] = C_\beta \sigma^\beta, \tag{2.5}$$

where $C_\beta = \left(\int_0^\infty x^{-\beta} \sin x dx\right)^{-1} = \frac{1-\beta}{\Gamma(2-\beta)\cos(\pi\beta/2)}$(see [15], Property 1.2.15). Hence for any $\epsilon > 0$, there exists some M, such that for all $s > M$ and $i = 1, ..., d$,

$$(C_\beta \sigma^\beta - \epsilon)s^{-\beta} \leqslant \mathbb{P}[|U_{(i)}| > s] \leqslant \left(C_\beta \sigma^\beta + \epsilon\right) s^{-\beta}.$$

Hence for $s > \sqrt{d}M$,

$$\mathbb{P}[|U| > s] \leqslant \sum_{i=1}^{d} \mathbb{P}[|U_{(i)}| > s/\sqrt{d}] \leqslant d^{1+\beta/2}\left(\epsilon + C_\beta \sigma^\beta\right) s^{-\beta}. \tag{2.6}$$

Therefore $\mathbb{P}[|X| > s] \sim s^{-\beta}$ as $s \to \infty$. □

LEMMA 2.3. *Let U, V be positive random variables such that*

$$\lim_{t\to\infty} t^\alpha \mathbb{P}[U > t] \geqslant C_1, \lim_{t\to\infty} t^\alpha \mathbb{P}[V > t] \leqslant C_2,$$

where $C_1 > C_2$, then

$$\mathbb{P}[U - V > t] = O\left(t^{-\alpha}\right) \quad \textit{for} \quad t \to \infty.$$

P r o o f. Given a positive number M, there exsits T and $\epsilon > 0$, such that for all $t > T$,

$$\begin{aligned}\mathbb{P}[U-V>t] &\geqslant \mathbb{P}[U>(M+1)\,t]-\mathbb{P}[V>Mt]\\ &\geqslant \frac{C_1+\epsilon}{(M+1)^{\alpha}}t^{-\alpha}-\frac{C_2-\epsilon}{M^{\alpha}}t^{-\alpha}\\ &\geqslant \frac{1}{M^{\alpha}}\left((C_1+\epsilon)\left(\frac{M}{M+1}\right)^{\alpha}-(C_2-\epsilon)\right)t^{-\alpha}.\end{aligned}$$

If we pick M big enough, we have $\mathbb{P}[U-V>t]\geqslant Ct^{-\alpha}$ for some constant C. On the other hand, for large t,

$$\begin{aligned}\mathbb{P}[U-V>t] &= \int_{V>0}\mathbb{P}[U-v>t]\mathbb{P}[V\in \mathrm{d}v]\\ &\leqslant \int_{V>0}\mathbb{P}[U>t]\mathbb{P}[V\in dv]\\ &\leqslant \int C_1t^{-\alpha}\mathbb{P}[V\in \mathrm{d}v]\\ &\leqslant C_1t^{-\alpha}.\end{aligned}$$

Therefore $\mathbb{P}[U-V>t]\sim t^{-\alpha}$ as $s\to\infty$. □

Lemma 2.2 tells us the order of tail of high dimentional stable process. Lemma 2.3 shows the order of the difference between certian random variables and we can apply it to the logarithm of (2.4), i.e. $\log|X_1|+\frac{\alpha}{\beta}\log\bar{a}-\frac{\alpha}{\beta}\log\tau_1$.

Now we can come back to the proof of Proposition 2.1.

P r o o f o f P r o p o s i t i o n 2.1. Now let us estimate the tail of $X^x_{T_t}$. For large $u>0$,

$$\begin{aligned}\mathbb{P}\left[|X^x_{T_t}|>u\right] =&\mathbb{P}\left[|\left(\frac{\bar{a}}{\eta}\right)^{\frac{\alpha}{\beta}}X_1+x|>u\right]\leqslant \mathbb{P}\left[\left(\frac{\bar{a}}{\eta}\right)^{\frac{\alpha}{\beta}}|X_1|>u-|x|\right]\\ =&\mathbb{P}\left[\log|X_1|-\frac{\alpha}{\beta}\log\eta>\log\left(u-|x|\right)-\frac{\alpha}{\beta}\log\bar{a}\right]\\ =&\mathbb{P}[A-B>r,A>0,B>0]+\mathbb{P}[A-B>r,A>0,B<0]+\\ &\mathbb{P}[A-B>r,A<0,B<0],\end{aligned}\tag{2.7}$$

where $A:=\log|X_1|$, $B:=\frac{\alpha}{\beta}\log(\eta)$, $r:=\log(u-|x|)-\frac{\alpha}{\beta}\log\bar{a}$. (Note that for large u we have $r>0$).

Let $X_1=(X_{(1)},...,X_{(d)})$ and $X_{(i)}\sim S_\beta(\sigma,\rho,\mu)$, $i=1,...,d$. By the Proof of Lemma 2.2, for any $\epsilon>0$, there exists some M, such that for all $s>M$ and $i=1,...,d$,

$$\mathbb{P}[|X_{(i)}| > s] \leqslant \left(C_\beta \sigma^\beta + \epsilon\right) s^{-\beta}.$$

Hence for $s > \sqrt{d}M$,

$$\mathbb{P}[|X_1| > s] \leqslant \sum_{i=1}^{d} \mathbb{P}[|X_{(i)}| > s/\sqrt{d}] \leqslant d^{1+\beta/2} \left(\epsilon + C_\beta \sigma^\beta\right) s^{-\beta},$$

and for $t > \log(\sqrt{d}M)$,

$$\mathbb{P}[\log|X_1| > t] = \mathbb{P}[|X_1| > \mathrm{e}^t] \leqslant d^{1+\beta/2} \left(\epsilon + C_\beta \sigma^\beta\right) \mathrm{e}^{-\beta t}.$$

Now let us discuss (2.7) in three conditions. For $r > \log(\sqrt{d}M)$:

(1) When $A > 0, B > 0$, we have

$$\begin{aligned} \mathbb{P}[A - B > r, A > 0, B > 0] &\leqslant \mathbb{P}[A > r] = \mathbb{P}[|X_1| > \mathrm{e}^s] \\ &\leqslant d^{1+\beta/2} \left(\epsilon + C_\beta \sigma^\beta\right) \mathrm{e}^{-\beta r}. \end{aligned} \tag{2.8}$$

(2) When $A > 0$, $B < 0$, pick integer $k = \lfloor r/S \rfloor$, and we divide the event $\{A + (-B) > r\}$ into k parts:

$$\begin{aligned} \{A + (-B) > r\} = &\bigcup_{i=1}^{k-1} \{A + (-B) > r, -B \in \left(\frac{i-1}{k} r, \frac{i}{k} r\right]\} \\ &\bigcup \{A + (-B) > r, -B > \frac{k-1}{k} r\} \\ \subset &\bigcup_{i=1}^{k-1} \{A > \frac{k-i}{k} r, -B \in \left(\frac{i-1}{k} r, \frac{i}{k} r\right]\} \bigcup \{-B > \frac{k-1}{k} r\} \\ \subset &\bigcup_{i=1}^{k} \{A > \frac{k-i}{k} r, -B > \frac{i-1}{k} r\}. \end{aligned} \tag{2.9}$$

Hence,

$$\begin{aligned} \mathbb{P}[A + (-B) > r, A > 0, B < 0] &\leqslant \sum_{i=1}^{k} \mathbb{P}[A > \frac{k-i}{k} r, -B > \frac{i-1}{k} r] \\ &= \sum_{i=1}^{k} \mathbb{P}[A > \frac{k-i}{k} r] \mathbb{P}[-B > \frac{i-1}{k} r]. \end{aligned} \tag{2.10}$$

Recall that

$$\mathbb{P}[\log|X_1| > \frac{k-i}{k} r] \leqslant d^{1+\beta/2} \left(\epsilon + C_\beta \sigma^\beta\right) \mathrm{e}^{-\frac{k-i}{k}\beta r}. \tag{2.11}$$

Using the result (3.7) that we shall mention later, we have

$$\mathbb{E}[\eta^{-2\alpha}] = \left(\cos\left(\frac{\pi\alpha}{2}\right)\right)^2 \frac{2}{\Gamma\left(1+2\alpha\right)}. \tag{2.12}$$

By the Markov inequality,

$$\mathbb{P}[\frac{\alpha}{\beta}\log\left(\eta^{-1}\right) \geqslant \frac{i-1}{k}r] = \mathbb{P}[\eta^{-1} > \mathrm{e}^{\frac{\beta}{\alpha}\frac{i-1}{k}r}] \leqslant \frac{\mathbb{E}[\tau_1^{-2\alpha}]}{\left(\mathrm{e}^{\frac{\beta}{\alpha}\frac{i-1}{k}r}\right)^{2\alpha}} \leqslant 2\mathrm{e}^{-2\frac{i-1}{k}\beta r}. \tag{2.13}$$

Combining (2.10), (2.11) and (2.13), we have

$$\begin{aligned}
\mathbb{P}[A+(-B)>r, A>0, B<0] &\leqslant \sum_{i=1}^{k} d^{1+\beta/2}\left(\epsilon + C_\beta\sigma^\beta\right)\mathrm{e}^{-\frac{k-i}{k}\beta r}2\mathrm{e}^{-2\frac{i-1}{k}\beta r} \\
&= 2d^{1+\beta/2}\left(\epsilon + C_\beta\sigma^\beta\right)\sum_{i=1}^{k}\mathrm{e}^{-\frac{i-2}{k}\beta r}\mathrm{e}^{-\beta r} \\
&\leqslant 2d^{1+\beta/2}\left(\epsilon + C_\beta\sigma^\beta\right)\frac{\mathrm{e}^{\beta r/k}}{1-\mathrm{e}^{-\beta r/k}}\mathrm{e}^{-\beta r} \\
&\leqslant 2d^{1+\beta/2}\left(\epsilon + C_\beta\sigma^\beta\right)\frac{\mathrm{e}^{2\beta S}}{\mathrm{e}^{\beta S}-1}\mathrm{e}^{-\beta r}.
\end{aligned} \tag{2.14}$$

(3) When $A<0$, $B<0$, then

$$\begin{aligned}
\mathbb{P}[A-B>r, A<0, B<0] &\leqslant \mathbb{P}[A<0, -B>r] \leqslant \mathbb{P}[-B>r] \\
&= \mathbb{P}[\frac{\alpha}{\beta}\log\left(\eta^{-1}\right) > r] = \mathbb{P}[\eta^{-1} > \mathrm{e}^{\frac{\beta}{\alpha}r}] \\
&\leqslant \frac{\mathbb{E}[\eta^{-\alpha}]}{\left(\mathrm{e}^{\frac{\beta}{\alpha}r}\right)^{\alpha}} \leqslant \mathrm{e}^{-\beta r}.
\end{aligned} \tag{2.15}$$

Combining the three conditions above, we know that for large u,

$$\begin{aligned}
\mathbb{P}[|X^x_{T_t}|>u] &\leqslant \left(\left(1+2\frac{\mathrm{e}^{2\beta S}}{\mathrm{e}^{\beta S}-1}\right)d^{1+\beta/2}\left(\epsilon + C_\beta\sigma^\beta\right)+1\right)\mathrm{e}^{-\beta r} \\
&= O\left(\mathrm{e}^{-\beta\left(\log(u-|x|)-\frac{\alpha}{\beta}\log\bar{a}\right)}\right) = O\left(u^{-\beta}\right).
\end{aligned} \tag{2.16}$$

□

Now let us finish the proof of our main result. First let us see this for Theorem 2.1(ii).

P r o o f o f T h e o r e m 2.1

$$\mathbb{P}[\phi\left(X^x_{T_t}\right)>u] = O\left(u^{-(2+\delta)}\right), \tag{2.17}$$

and by (2.3) $\mathbb{E}\left[\left(\phi\left(X_{T_t}^x\right)\right)^2\right]$ is finite. $\square$

For the proof of Theorem 2.1(iii), $\mathbb{E}[|\phi\left(X_{T_t}^x\right)|^3]$ is finite because of the similar argument. For the rest of proof, see [14], page 356, Variant Berry-Esseen Theorem.

Remark 2.1.

(1) If $C_\alpha < C_\beta \sigma^\beta$, by Lemma 2.3, we have

$$\mathbb{P}[A - B > r] \geqslant \mathbb{P}[A - B > r, A > 0, B > 0] \geqslant Ct^{-\beta}, \tag{2.18}$$

where C is a constant that can be chosen from the proof of Lemma 2.3. This result means the order $t^{-\beta}$ is the best one.

(2) In the proof of Proposition 2.1 we need $r = \log(u - |x|) - \frac{\alpha}{\beta}\log \bar{a}$ and $r > \log(\sqrt{d}M)$. Hence there exists some constant M_0 such that for $u > M_0$, (2.16) holds and M_0 has order $d^{1/2}$.

Besides, we can roughly give the upper bound of $\mathbb{E}[\phi\left(X_{T_t}^x\right)^2]$.

Example 2.1. If $\phi(x)$ satisfies $\phi(x) \leqslant |x|^{\frac{\beta}{\delta+2}}$, where $\delta > 0$, then from Remark 2.1 we know that there exists some M_0 such that for all $t > M_0$,

$$\begin{aligned}\mathbb{P}[|X_{T_t}^x| > t] &\leqslant \left(\left(1 + \frac{\mathrm{e}^{2\beta S}}{\mathrm{e}^{\beta S} - 1}\right) d^{1+\beta/2}\left(\epsilon + C_\beta \sigma^\beta\right) + 1\right)\mathrm{e}^{-\beta r} = M^{(1)}\mathrm{e}^{-\beta r} \\ &= M^{(1)}\mathrm{e}^{-\beta\log(t-|x|) - \frac{\alpha}{\beta}\log \bar{a}} \leqslant M^{(2)} t^{-\beta},\end{aligned} \tag{2.19}$$

where $M^{(1)} = \left(1 + \frac{\mathrm{e}^{2\beta S}}{\mathrm{e}^{\beta S}-1}\right) d^{1+\beta/2}\left(\epsilon + C_\beta\sigma^\beta\right) + 1$, $M^{(2)} = 2\bar{a}^{-\frac{\alpha}{\beta}} M^{(1)}$. Hence,

$$\begin{aligned}\mathbb{E}[\phi\left(X_{T_t}^x\right)^2] &= \int_0^\infty \mathbb{P}[\phi\left(X_{T_t}^x\right)^2 > t]dt \leqslant M_0 + \int_{M_0}^\infty \mathbb{P}[|X_{T_t}^x| > \sqrt{t}^{\frac{2+\delta}{\beta}}]dt \\ &\leqslant M_0 + M^{(2)}\int_{M_0}^\infty \sqrt{t}^{\frac{2+\delta}{\beta}\cdot(-\beta)} dt = M_0 + 2M^{(2)} M_0^{-\delta/2}/\delta.\end{aligned} \tag{2.20}$$

Note that M_0 has order $d^{1/2}$ and $M^{(2)}$ has order $d^{1+\beta/2}$, This upper bound has order $d^{1+\beta/2}$.

3. Properties of the estimator when g is not 0

In this section we want to clarify the Monte-Carlo estimator of the stochastic representation in Section 1. Here we assume that g satisfies the condition $|g(x) - g(y)| \leqslant L|x - y|_\gamma$, where $|x|_\gamma = \sum_{i=1}^d |x_{(i)}|^\gamma$, $x_{(i)}$ is the coordinate of x, $0 < \gamma < \beta/2$.

Our main results in this section are as follows.

THEOREM 3.1. *Assume* $|\phi(x)| = O(|x|^{\frac{\beta}{2+\delta}})$ *for* $|x| \to \infty$, *where* $\delta > 0$.

(i) $\mathbb{E}[\left(u_N^h(t,x) - u(t,x)\right)^2] \to 0$ *as* $N \to \infty$, $h \to 0$.

(ii) *(CLT with a bias correction) Let* $h_N = N^{-\frac{2\beta}{\gamma}}$, $u(t,x) = \mathbb{E}Z(t,x)$ *where*

$$Z(t,x) = \phi\left(X_{T_t}^x\right) + \int_0^{T_t} g\left(X_s^x\right) ds,$$

and W *be the standard normal distribution, then for all bounded uniformly continuous function* ψ,

$$\mathbb{E}\left[\psi\left(\sqrt{N}\left(u_N^{h_N}(t,x) - u(t,x)\right)/\sqrt{VarZ(t,x)}\right)\right] \to \mathbb{E}[\psi(W)] \text{ as } N \to \infty.$$

Let

$$Y_h(t,x) = \phi\left(X_{T_t}^x\right) + \sum_{i=1}^{\lfloor T_t/h \rfloor} hg\left(X_{t_i}^x\right)$$

be the approximation of $Z(t,x)$. And let

$$u_N^h(t,x) = \frac{1}{N}\sum_{k=1}^N Y_h^k(t,x),$$

where $Y_h^k(t,x) = \phi\left(X_{T_t^k}^{x,k}\right) + \sum_{i=1}^{\lfloor T_t^k/h \rfloor} hg\left(X_{t_i^k}^{x,k}\right)$, $k = 1, ..., N$. $Y_h^k(t,x)$ are the iid copies of $Y_h(t,x)$. Note that for random variable U, let V be its approximation and V^k, $k = 1, ..., N$ be the iid copies of V. The L^2 error satisfies

$$\mathbb{E}\left[\left(\mathbb{E}U - \frac{1}{N}\sum_{k=1}^N V^k\right)^2\right] = \frac{1}{N}varV + (\mathbb{E}U - \mathbb{E}V)^2. \tag{3.1}$$

Therefore, to estimate the L^2 error $\mathbb{E}[(u(t,x) - u_N(t,x))^2]$, we only need to study $varY_h(t,x)$ and $\mathbb{E}Z(t,x) - \mathbb{E}Y_h(t,x)$, and the following propositions answer these questions.

PROPOSITION 3.1. *There exists a constant* $M_{t,x}^1$ *(depending on* t,x*) such that* $VarY_h(t,x) \leqslant M_{t,x}^1$.

PROPOSITION 3.2. *There exists a constant* $M_{t,x}^2$ *(depending on* t,x*) such that* $\mathbb{E}[|Z(t,x) - Y_h(t,x)|] \leqslant M_{t,x}^2 h^{\frac{\gamma}{\beta}}$.

PROPOSITION 3.3. *There exists a constant $M^3_{t,x}$ (depending on t, x) such that* $\mathbb{E}[|Z(t,x) - Y_h(t,x)|^2] \leqslant M^3_{t,x} h^{\frac{2\gamma}{\beta}}$.

Sections 3.1 and 3.2 give proofs of these propositions. Section 3.3 is the proof of our CLT.

REMARK 3.1.

- Non-asymptotic confidence interval: Combining (3.1), Proposition 3.1 and Proposition 3.2 we have
$$\mathbb{E}[\Big(u(t,x) - u^h_N(t,x)\Big)^2] \leqslant \frac{1}{N} M^1_{t,x} + \left(M^2_{t,x}\right)^2 h^{\frac{2\gamma}{\beta}}, \tag{3.2}$$
where $u(t,x)$ is the solution of problem (1.1). Now we can construct the confidence interval using the Markov inequality:
$$\begin{aligned} \mathbb{P}[|u(t,x) - u^h_N(t,x)| > r] &\leqslant \mathbb{E}[\Big(u(t,x) - u^h_N(t,x)\Big)^2]/r^2 \\ &\leqslant \frac{1}{r^2}\left(\frac{1}{N} M^1_{t,x} + \left(M^2_{t,x}\right)^2 h^{\frac{2\gamma}{\beta}}\right). \end{aligned} \tag{3.3}$$
Hence we can pick suitable N and h such that $\mathbb{P}[|u(t,x) - u^h_N(t,x)| > r] < 1 - \epsilon$ for some small ϵ.
- Asymptotic confidence interval: We can use CLT in Theorem 3.1 to get the asymptotic confidence interval. In other words, the central limit theorem can be written using convergence in distribution:
$$\sqrt{N}\left(u^{h_N}_N(t,x) - u(t,x)\right) \xrightarrow{d} N(0, VarZ(t,x)) \quad \text{as} \quad n \to \infty. \tag{3.4}$$
Once we have the upper bound $M(t,x)$ of $\sqrt{VarZ(t,x)}$ (e.g. see Example 2.1), it is easy to see that it yields a $100(1-\alpha)\%$ asymptomic confidence interval $u^{h_N}_N \pm \frac{M(t,x)}{\sqrt{N}} z(\alpha/2)$ for $u(t,x)$, where $z(t)$ satisfies $\Phi(z(t)) = 1 - t$ and Φ is the distribution function of the standard normal distribution. See Section 4.3 for a simple example.

Before the calculation, we need the following results (see [6], page 162):

(1) For constants $c > 0$, $\eta \in (-1, \beta)$ and a symmetric β-stable 1-dim process U_t with $\mathbb{E}[e^{izU_t}] = e^{-tc|z|^\beta}$, we have
$$\mathbb{E}[|U_t|^\eta] = (tc)^{\eta/\beta} \frac{2^\eta \Gamma\left(\frac{1+\eta}{2}\right) \Gamma\left(1 - \frac{\eta}{\beta}\right)}{\sqrt{\pi}\Gamma\left(1 - \frac{\eta}{2}\right)}. \tag{3.5}$$
Recall that each component of $X_1 - X_0$, denoted by $X_{(j)}$, is symmetric with $\mathbb{E}[e^{izX_{(j)}}] = e^{-c|z|^\beta}$ and $c > 0$.

(2) If $0 < \alpha < 1$ and $\{X_t\}$ is a stable subordinator with $\mathbb{E}[e^{-uX_t}] = e^{-tc'u^\alpha}$, where c' is some constant, then for $-\infty < \eta < \alpha$,

$$\mathbb{E}[X_t^\eta] = \left(tc'\right)^{\eta/\alpha} \frac{\Gamma\left(1-\frac{\eta}{\alpha}\right)}{\Gamma\left(1-\eta\right)}. \tag{3.6}$$

Since $\eta \sim S_\alpha(1,1,0)$, we have $\mathbb{E}[e^{-u\eta}] = \exp\{-\frac{1}{\cos\left(\frac{\pi\alpha}{2}\right)}u^\alpha\}$ (see [15], Proposition 1.2.12). Hence,

$$\mathbb{E}[\eta^\eta] = \left(\frac{1}{\cos\left(\frac{\pi\alpha}{2}\right)}\right)^{\eta/\alpha} \frac{\Gamma\left(1-\frac{\eta}{\alpha}\right)}{\Gamma\left(1-\eta\right)}. \tag{3.7}$$

3.1. **Estimation of variance of the approximation.** In this section we estimate $Var(Y_h)$.

Denote $\lfloor T_t/h \rfloor$ by n. Note that the variance does not change when added some constant, and denote $g\left(X_{t_i}\right) - g\left(X_0\right) = g_i$. We have

$$\begin{aligned} VarY_h(t,x) &= Var\left(\phi\left(X_{T_t}^x\right) + h\sum_{i=1}^{n} g\left(X_{t_i}^x\right)\right) \\ &= Var\left(\phi\left(X_{T_t}^x\right) + h\sum_{i=1}^{n}\left(g\left(X_{t_i}^x\right) - g\left(X_0\right)\right)\right) \\ &\leqslant \mathbb{E}\left(\phi\left(X_{T_t}^x\right) + h\sum_{i=1}^{n} g_i\right)^2 \\ &\leqslant \mathbb{E}[\phi\left(X_{T_t}^x\right)^2] + h^2\mathbb{E}[\left(\sum_{i=1}^{n} g_i\right)^2] + 2h\mathbb{E}[\phi\left(X_{T_t}^x\right)\sum_{i=1}^{n} g_i]. \end{aligned} \tag{3.8}$$

Denote the upper bound of $\mathbb{E}\left[\phi\left(X_{T_t}\right)^2\right]$ by M_1. Next,

$$\begin{aligned} \mathbb{E}\left[g_i^2|T_t\right] &\leqslant L^2\mathbb{E}[|X_{t_i} - X_0|_\gamma^2|T_t] \\ &= L^2\mathbb{E}[\left(\sum_{j=1}^{d}|X_{t_i,(j)} - X_{0,(j)}|^\gamma\right)^2|T_t] \\ &\leqslant dL^2\mathbb{E}[\sum_{j=1}^{d}|X_{t_i,(j)} - X_{0,(j)}|^{2\gamma}|T_t] \\ &= dL^2 t_i^{\frac{2\gamma}{\beta}}\mathbb{E}[\sum_{j=1}^{d}|X_{1,(j)} - X_{0,(j)}|^{2\gamma}]. \end{aligned} \tag{3.9}$$

Using the result of (3.5), for $j = 1, ..., d$, we have

$$\mathbb{E}[|X_{1,(j)} - X_{0,(j)}|^{2\gamma}] = c^{2\gamma/\beta} \frac{2^{2\gamma}\Gamma\left(\frac{1+2\gamma}{2}\right)\Gamma\left(1-\frac{2\gamma}{\beta}\right)}{\sqrt{\pi}\Gamma(1-\gamma)}. \tag{3.10}$$

Let us denote

$$M_2 := \sum_{j=1}^{d} \mathbb{E}[|X_{1,(j)} - X_{0,(j)}|^{2\gamma}] = dc^{2\gamma/\beta} \frac{2^{2\gamma}\Gamma\left(\frac{1+2\gamma}{2}\right)\Gamma\left(1-\frac{2\gamma}{\beta}\right)}{\sqrt{\pi}\Gamma(1-\gamma)}. \tag{3.11}$$

Then (3.9) becomes

$$\mathbb{E}[g_i^2|T_t] \leqslant dL^2 t_i^{\frac{2\gamma}{\beta}} M_2, \tag{3.12}$$

and for $i \neq j$,

$$\mathbb{E}[g_i g_j|T_t] \leqslant \left(\mathbb{E}[g_i^2|T_t]\mathbb{E}[g_j^2]|T_t\right)^{\frac{1}{2}} \leqslant dL^2 t_i^{\frac{\gamma}{\beta}} t_j^{\frac{\gamma}{\beta}} M_2. \tag{3.13}$$

Therefore,

$$\mathbb{E}[\left(\sum_{i=1}^{n} g_i\right)^2 |T_t] \leqslant \sum_{i=1}^{n} dL^2 t_i^{\frac{2\gamma}{\beta}} M_2 + \sum_{i\neq j} 2dL^2 t_i^{\frac{\gamma}{\beta}} t_j^{\frac{\gamma}{\beta}} M_2 = dL^2 M_2 \left(\sum_{i=1}^{n} t_i^{\frac{\gamma}{\beta}}\right)^2. \tag{3.14}$$

Recall that $t_i = ih$ and $n = \lfloor T_t/h \rfloor$, hence

$$\begin{aligned}\sum_{i=1}^{n} t_i^{\frac{\gamma}{\beta}} = h^{\frac{\gamma}{\beta}} \sum_{i=1}^{n} i^{\frac{\gamma}{\beta}} \leqslant h^{\frac{\gamma}{\beta}} \int_0^{n+1} x^{\frac{\gamma}{\beta}} dx &= \frac{1}{1+\frac{\gamma}{\beta}} h^{\frac{\gamma}{\beta}} (n+1)^{1+\frac{\gamma}{\beta}} \\ &\leqslant \frac{1}{1+\frac{\gamma}{\beta}} h^{\frac{\gamma}{\beta}} (T_t/h+1)^{1+\frac{\gamma}{\beta}}.\end{aligned} \tag{3.15}$$

Hence

$$h^2\mathbb{E}[\left(\sum_{i=1}^{n} g_i\right)^2] = h^2\mathbb{E}[\mathbb{E}[\left(\sum_{i=1}^{n} g_i^2\right)|T_t]] \leqslant \frac{dL^2 M_2}{\left(1+\frac{\alpha}{\beta}\right)^2}\mathbb{E}[(T_t+1)^{2\left(1+\frac{\gamma}{\beta}\right)}]. \tag{3.16}$$

Note that

$$\mathbb{E}[(T_t+1)^{2\left(1+\frac{\gamma}{\beta}\right)}] \leqslant \mathbb{E}[(T_t+1)^3] = \mathbb{E}[T_t^3] + 3\mathbb{E}[T_t^2] + 3\mathbb{E}[T_t] + 1. \tag{3.17}$$

And by (3.7), we know that for $k = 1, 2, 3$,

$$\mathbb{E}[T_t^k] = \bar{a}^{k\alpha}\mathbb{E}[\eta^{-k\alpha}] = \bar{a}^{k\alpha}\left(\cos\left(\frac{\pi\alpha}{2}\right)\right)^k \frac{\Gamma(1+k)}{\Gamma(1+k\alpha)}, \tag{3.18}$$

implying the upper bound of $h^2\mathbb{E}[(\sum_{i=1}^{n} g_i)^2]$. By the Cauchy-Schwarz inequality,

$$h\mathbb{E}[\phi\left(X_{T_t}\right)\sum_{i=1}^{n} g_i] \leqslant \left(h^2\mathbb{E}[\left(\sum_{i=1}^{n} g_i\right)^2]\mathbb{E}[(\phi\left(X_{T_t}\right))^2]\right)^{\frac{1}{2}} \tag{3.19}$$

and hence we get the upper bound of $VarY_h(t,x)$ using (3.8):

$$\begin{aligned} VarY_h(t,x) \leqslant & \mathbb{E}[\phi\left(X_{T_t}\right)^2] + \frac{dL^2M_2}{\left(1+\frac{\alpha}{\beta}\right)^2}\mathbb{E}[(T_t+1)^{2\left(1+\frac{\gamma}{\beta}\right)}] + ... \\ & + \left(\mathbb{E}[\phi\left(X_{T_t}\right)^2]\frac{dL^2M_2}{\left(1+\frac{\alpha}{\beta}\right)^2}\mathbb{E}[(T_t+1)^{2\left(1+\frac{\gamma}{\beta}\right)}]\right)^{\frac{1}{2}}. \end{aligned} \tag{3.20}$$

Remark 3.2. By Example 2.1, we know the upper bound of $\mathbb{E}[\phi\left(X_{T_t}\right)^2]$ has order $d^{1+\frac{\beta}{2}}$. By (3.11), M_2 has order d. By (3.16), the upper bound of $h^2\mathbb{E}[(\sum_{i=1}^{n} g_i)^2]$ has order d^2. Hence the upper bound of $VarY_h$ has order d^2.

3.2. **Estimation of EZ-EY.** Similarly, we begin with the estimation the conditional expectation.

Conditioning on T_t, we write

$$\begin{aligned} & \mathbb{E}[Z(t,x)] - \mathbb{E}[Y_h(t,x)|T_t] \\ =& \mathbb{E}\left[\left(\phi\left(X_{T_t}\right) + \int_0^{T_t} g\left(X_s\right)ds - \left(\phi\left(X_{T_t}\right) + \sum_{i=1}^{\lfloor T_t/h \rfloor} hg\left(X_{t_i}\right)\right)\right)|T_t\right] \\ =& \mathbb{E}\left[\int_0^{T_t} g\left(X_s\right)ds - \sum_{i=1}^{\lfloor T_t/h \rfloor} hg\left(X_{t_i}\right)|T_t\right] \\ =& \mathbb{E}\left[\left(\sum_{i=1}^{\lfloor T_t/h \rfloor}\int_{t_i}^{t_i+h} \left(g\left(X_s\right) - g\left(X_{t_i}\right)\right)ds + \int_{\lfloor T_t/h \rfloor h}^{T_t} g\left(X_s\right)ds\right)|T_t\right]. \end{aligned} \tag{3.21}$$

We have

$$
\begin{aligned}
&\mathbb{E}[|\sum_{i=1}^{\lfloor T_t/h\rfloor}\int_{t_i}^{t_i+h}\left(g\left(X_s\right)-g\left(X_{t_i}\right)\right)ds|\big|T_t] \\
&\leqslant\mathbb{E}[\sum_{i=1}^{\lfloor T_t/h\rfloor}\int_{t_i}^{t_i+h}L|X_s-X_{t_i}|_\gamma ds\big|T_t] \\
\text{(stationarity of increments) }&=\mathbb{E}[\sum_{i=1}^{\lfloor T_t/h\rfloor}\int_0^h L|X_s-X_0|_\gamma ds\big|T_t] \\
&=\mathbb{E}[\sum_{i=1}^{\lfloor T_t/h\rfloor}\int_0^h L\sum_{j=1}^{d}|X_{s,(j)}-X_{0,(j)}|^\gamma ds\big|T_t] \\
(X_t\text{ is }\beta\text{-stable}) &=\mathbb{E}[\lfloor T_t/h\rfloor L\int_0^h s^{\frac{\gamma}{\beta}}\sum_{j=1}^{d}|X_{1,(j)}-X_{0,(j)}|^\gamma ds|T_t] \\
&=C_0\lfloor T_t/h\rfloor h^{1+\frac{\gamma}{\beta}}\leqslant C_0T_th^{\frac{\gamma}{\beta}},
\end{aligned}
\tag{3.22}
$$

where $C_0=\frac{1}{1+\frac{\gamma}{\beta}}L\sum_{j=1}^d\mathbb{E}|X_{1,(j)}-X_{0,(j)}|^\gamma$.

From (3.5), we know that

$$
\mathbb{E}[|X_{1,(j)}-X_{0,(j)}|^\gamma]=c^{\frac{\gamma}{\beta}}\frac{2^\gamma\Gamma\left(\frac{1+\gamma}{2}\right)\Gamma\left(1-\frac{\gamma}{\beta}\right)}{\sqrt{\pi}\Gamma\left(1-\frac{\gamma}{2}\right)}. \tag{3.23}
$$

Next, it is clear that

$$
g\left(X_s\right)\leqslant g\left(X_{T_t}\right)+\left|g\left(X_{T_t}\right)-g\left(X_s\right)\right|, \tag{3.24}
$$

Thus

$$
\begin{aligned}
\mathbb{E}g\left(X_s\right)&\leqslant\mathbb{E}\left(g\left(X_{T_t}\right)+\left|g\left(X_{T_t}\right)-g\left(X_s\right)\right|\right) \\
&\leqslant\mathbb{E}[g\left(X_{T_t}\right)+L|X_{T_t}-X_s|_\gamma] \\
&=\mathbb{E}[g\left(X_{T_t}\right)+L\left(T_t-s\right)^{\frac{\gamma}{\beta}}\left|X_1-X_0\right|],
\end{aligned}
\tag{3.25}
$$

and

$$
\begin{aligned}
&\mathbb{E}[\int_{\lfloor T_t/h\rfloor h}^{T_t}g\left(X_s\right)ds|T_t] \\
\leqslant&\mathbb{E}[\int_{\lfloor T_t/h\rfloor h}^{T_t}\mathbb{E}[g\left(X_{T_t}\right)+\mathbb{E}L\left(T_t-s\right)^{\frac{\gamma}{\beta}}|X_1-X_0|_\gamma]ds|T_t] \\
\leqslant&\mathbb{E}[h\mathbb{E}[g\left(X_{T_t}\right)+\frac{1}{1+\frac{\gamma}{\beta}}Lh^{1+\frac{\gamma}{\beta}}\mathbb{E}|X_1-X_0|_\gamma]|T_t].
\end{aligned}
\tag{3.26}
$$

We have showed that if $g\left(x\right)=O\left(x^{\frac{\beta}{1+\delta}}\right)$, where $\delta>0$, then

$$\mathbb{E}[g\left(X_{T_t}\right)] < M_3 < \infty,$$

with some M_3.

Therefore, by (3.26),

$$\mathbb{E}[\mathbb{E}[\int_{\lfloor T_t/h \rfloor h}^{T_t} g\left(X_s\right) ds | T_t]] \leqslant M_2 h + \frac{1}{1+\frac{\gamma}{\beta}} L\mathbb{E}|X_1 - X_0|_\gamma h^{1+\frac{\gamma}{\beta}}. \tag{3.27}$$

Combining (3.21), (3.22) and (3.27) we have

$$\begin{aligned}
|\mathbb{E}[Z-Y]| &= \mathbb{E}[\mathbb{E}[Z-Y|T_t]] \\
&\leqslant M_3 h + Ldc^{\frac{\gamma}{\beta}} \frac{2^\gamma \Gamma\left(\frac{1+\gamma}{2}\right)\Gamma\left(1-\frac{\gamma}{\beta}\right)}{\left(1+\frac{\gamma}{\beta}\right)\sqrt{\pi}\Gamma\left(1-\frac{\gamma}{2}\right)} h^{1+\frac{\gamma}{\beta}} \\
&\quad + Ldc^{\frac{\gamma}{\beta}} \frac{2^\gamma \Gamma\left(\frac{1+\gamma}{2}\right)\Gamma\left(1-\frac{\gamma}{\beta}\right)}{\left(1+\frac{\gamma}{\beta}\right)\sqrt{\pi}\Gamma\left(1-\frac{\gamma}{2}\right)} h^{\frac{\gamma}{\beta}} \bar{a}^\alpha \frac{\Gamma(2)}{\Gamma(1+\alpha)} \\
&= O\left(h^{\frac{\gamma}{\beta}}\right).
\end{aligned} \tag{3.28}$$

Remark 3.3. (1) With similar argument, we can see $\mathbb{E}[Z^2] < \infty$: Since $\mathbb{E}[\phi\left(X_{T_t}^x\right)^2] < \infty$, we only need $\mathbb{E}\left[\left(\int_0^{T_t} g\left(X_s\right) ds\right)^2\right] < \infty$.

Like (3.21), we have

$$\begin{aligned}
&\mathbb{E}[\left(\int_0^{T_t} g\left(X_s\right) ds\right)^2] \leqslant \mathbb{E}[\left(\int_0^{T_t} |g\left(X_s\right)| ds\right)^2] \\
\leqslant &\mathbb{E}[\left(\int_0^{T_t} |g\left(X_0\right)| + |g\left(X_0\right) - g\left(X_s\right)| ds\right)^2] \\
= &\mathbb{E}[\mathbb{E}[\left(\int_0^{T_t} |g\left(X_0\right)| + |g\left(X_0\right) - g\left(X_s\right)| ds\right)^2 |T_t]] \\
\leqslant &2\mathbb{E}[\mathbb{E}[\left(\int_0^{T_t} |g\left(X_0\right)| ds\right)^2 |T_t] + \mathbb{E}[\left(\int_0^{T_t} |g\left(X_0\right) - g\left(X_s\right)| ds\right)^2 |T_t]].
\end{aligned} \tag{3.29}$$

Note that

$$\mathbb{E}[\mathbb{E}[\left(\int_0^{T_t} |g\left(X_0\right)| ds\right)^2 |T_t]] = \mathbb{E}[T_t^2]|g\left(X_0\right)|^2 < \infty, \tag{3.30}$$

and

$$\begin{aligned}
&\mathbb{E}[\left(\int_0^{T_t} |g(X_0) - g(X_s)|ds\right)^2 |T_t] \\
\leqslant&\mathbb{E}[\left(\int_0^{T_t} L|X_0 - X_s|_\gamma ds\right)^2 |T_t] \\
=&L^2\mathbb{E}[\frac{1}{T_t}\int_0^{T_t} |X_0 - X_s|_\gamma^2 ds|T_t] \\
=&L^2\frac{1}{T_t}\mathbb{E}[\int_0^{T_t}\left(\sum_{j=1}^d |X_{0,(j)} - X_{s,(j)}|^\gamma\right)^2 ds|T_t] \\
\leqslant&L^2 d\frac{1}{T_t}\mathbb{E}[\int_0^{T_t}\sum_{j=1}^d |X_{0,(j)} - X_{s,(j)}|^{2\gamma} ds|T_t] \\
\leqslant&L^2 d\frac{1}{T_t}\mathbb{E}[\int_0^{T_t}\sum_{j=1}^d s^{\frac{2\gamma}{\beta}}|X_{0,(j)} - X_{1,(j)}|^{2\gamma} ds|T_t] \\
=&L^2 d\frac{1}{T_t}\frac{1}{1+\frac{2\gamma}{\beta}}T_t^{1+\frac{2\gamma}{\beta}}\mathbb{E}[\sum_{j=1}^d |X_{0,(j)} - X_{1,(j)}|^{2\gamma}].
\end{aligned}$$

Hence

$$\begin{aligned}
&\mathbb{E}[\mathbb{E}[\left(\int_0^{T_t} |g(X_0) - g(X_s)|ds\right)^2 |T_t]] \\
\leqslant&L^2 d\frac{1}{1+\frac{2\gamma}{\beta}}\mathbb{E}[T_t^{\frac{2\gamma}{\beta}}]\mathbb{E}[\sum_{i=1}^d |X_{0,(j)} - X_{1,(j)}|^{2\gamma}] < \infty.
\end{aligned} \tag{3.31}$$

Combining (3.30) and (3.31), we have

$$\mathbb{E}[\left(\int_0^{T_t} g(X_s)\, ds\right)^2] < \infty, \tag{3.32}$$

and therefore $\mathbb{E}[Z^2] < \infty$.

(2) With the same condition, we can also show that $\mathbb{E}[|Y - Z|^2]$ has order $h^{\frac{2\gamma}{\beta}}$ using similar argument:

$$\begin{aligned}
&\mathbb{E}[|Y-Z|^2] \\
=&\mathbb{E}\left[\left(\sum_{i=1}^{\lfloor T_t/h\rfloor}\int_{(i-1)h}^{ih}(g(X_s) - g(X_{(i-1)h}))ds + \int_{\lfloor T_t/h\rfloor h}^{T_t} g(X_s)\, ds\right)^2\right].
\end{aligned} \tag{3.33}$$

And,

$$
\begin{aligned}
&\mathbb{E}[|\int_{\lfloor T_t/h \rfloor}^{T_t} g\left(X_s\right) ds|^2|T_t] \\
\leqslant &\mathbb{E}[(T_t - \lfloor T_t/h \rfloor h) \int_{\lfloor T_t/h \rfloor h}^{T_t} g\left(X_s\right)^2 ds] \\
\leqslant &\mathbb{E}[h \int_{\lfloor T_t/h \rfloor h}^{T_t} \left(g\left(X_{T_t}\right) + |g\left(X_{T_t}\right) - g\left(X_s\right)|\right)^2 ds] \\
\leqslant &\mathbb{E}[2h \int_{\lfloor T_t/h \rfloor}^{T_t} g\left(X_{T_t}\right)^2] + 2h\mathbb{E}[\int_{\lfloor T_t/h \rfloor h}^{T_t} \left(g\left(X_{T_t}\right) - g\left(X_s\right)\right)^2 ds],
\end{aligned}
\tag{3.34}
$$

$$
\begin{aligned}
&\mathbb{E}[\int_{\lfloor T_t/h \rfloor h}^{T_t} \left(g\left(X_{T_t}\right) - g\left(X_s\right)\right)^2 ds|T_t] \\
\leqslant &L^2\mathbb{E}[\int_{\lfloor T_t/h \rfloor h}^{T_t} |X_{T_t} - X_s|_\gamma^2 ds|T_t] \\
\leqslant &L^2 d\mathbb{E}[\int_{\lfloor T_t/h \rfloor h}^{T_t} \left(\sum_{j=1}^{d} |X_{T_t,(j)} - X_{s,(j)}|^{2\gamma}\right) ds|T_t] \\
= &L^2 d\mathbb{E}[\int_{\lfloor T_t/h \rfloor h}^{T_t} (T_t - s)^{\frac{2\gamma}{\beta}} \sum_{j=1}^{d} |X_{1,(j)} - X_{0,(j)}|^{2\gamma} ds|T_t] \\
\leqslant &L^2 d\mathbb{E}[\sum_{j=1}^{d} |X_{1,(j)} - X_{0,(j)}|^{2\gamma} \frac{1}{\frac{2\gamma}{\beta} + 1} h^{1+\frac{2\gamma}{\beta}},
\end{aligned}
\tag{3.35}
$$

$$
\begin{aligned}
&\mathbb{E}[\left(\sum_{i=1}^{\lfloor T_t/h \rfloor} \int_{(i-1)h}^{ih} g\left(X_s\right) - g\left(X_{(i-1)h}\right) ds\right)^2 |T_t] \\
\leqslant &\mathbb{E}[\lfloor T_t/h \rfloor \sum_{i=1}^{\lfloor T_t/h \rfloor} \left(\int_{(i-1)h}^{ih} g\left(X_s\right) - g\left(X_{(i-1)h}\right) ds\right)^2 |T_t] \\
\leqslant &\lfloor T_t/h \rfloor L^2 \sum_{i=1}^{\lfloor T_t/h \rfloor} \mathbb{E}[\left(\int_{(i-1)h}^{ih} |X_s - X_{(i-1)h}|_\gamma ds\right)^2 |T_t] \\
\leqslant &\lfloor T_t/h \rfloor L^2 \sum_{i=1}^{\lfloor T_t/h \rfloor} \mathbb{E}[h \int_{(i-1)h}^{ih} |X_s - X_{(i-1)h}|_\gamma^2 ds|T_t]
\end{aligned}
$$

$$
\begin{aligned}
&=\lfloor T_t/h\rfloor L^2 \sum_{i=1}^{\lfloor T_t/h\rfloor} \mathbb{E}[h\int_{(i-1)h}^{ih}\left(\sum_{j=1}^{d}|X_{s,(j)}-X_{(i-1)h,(j)}|^{\gamma}\right)^2 ds|T_t]\\
&\leqslant\lfloor T_t/h\rfloor L^2 \sum_{i=1}^{\lfloor T_t/h\rfloor} \mathbb{E}[h\int_{(i-1)h}^{ih} d\sum_{j=1}^{d}|X_{s,(j)}-X_{(i-1)h,(j)}|^{2\gamma} ds|T_t]\\
&=\lfloor T_t/h\rfloor^2 L^2 \mathbb{E}[h\int_0^h s^{\frac{2\gamma}{\beta}}\sum_{j=1}^{d}|X_{1,(j)}-X_{0,(j)}|^{2\gamma} ds|T_t]\\
&\leqslant T_t^2 L^2\mathbb{E}[\sum_{j=1}^{d}|X_{1,(j)}-X_{0,(j)}|^{2\gamma}]h^{\frac{2\gamma}{\beta}}.
\end{aligned}
$$

Hence $\mathbb{E}[|Y-Z|^2]\leqslant M_{t,x}^3 h^{\frac{2\gamma}{\beta}}$ where $M_{t,x}^3$ is a constant that only depends on t and x.

3.3. **Proof of Central Limit Theorem with a bias correction.** Recall the definition of **null array**. By this we mean a triangular array of random variables $(\xi_{nj}), 1\leqslant j\leqslant m_n, n, m_n\in\mathbb{N}$, such that the ξ_{nj} are independent for each n and satisfy

$$\sup_j \mathbb{E}[|\xi_{nj}|\wedge 1]\to 0. \tag{3.36}$$

The following result is well-known (see [5], Theorem 5.15).

THEOREM 3.2. *Let (ξ_{nj}) be a null array of random variables, then $\sum_{j=1}^{m_n}\xi_{nj}\xrightarrow{d} N(b,c)$ iff these conditions hold:*

(i) $\sum_{j=1}^{m_n}\mathbb{P}[|\xi_{nj}|>\epsilon]\to 0$ for all $\epsilon>0$ as $n\to\infty$;
(ii) $\sum_{j=1}^{m_n}\mathbb{E}[\xi_{nj};|\xi_{nj}|\leqslant 1]\to b$ as $n\to\infty$, where $\mathbb{E}[X;A]=\mathbb{E}[X\mathbb{I}_A]$,
(iii) $\sum_{j=1}^{m_n}Var[\xi_{nj};|\xi_{nj}|\leqslant 1]\to c$ as $n\to\infty$, where $Var(X;A):=Var(X\mathbb{I}_A)$.

Again denote

$$Z:=\phi\left(X_{T_t}^x\right)+\int_0^{T_t} g\left(X_s^x\right)ds,$$

$$Y_h^k:=\phi\left(X_{T_t^k}^x\right)+\sum_{i=1}^{\lfloor T_t^k/h\rfloor} hg\left(X_{ih}^x\right),$$

where T_t^k are independent samples of the stopping time. Now we will apply Theorem 3.2 to prove Theorem 3.1(ii).

P r o o f o f T h e o r e m 3.1 (ii). Let $\xi_{Nj} = \frac{1}{\sqrt{N}}\left(Y^j_{h_N} - \mathbb{E}Z\right)$, $j = 1, ..., N$, then for any $\epsilon > 0$,

$$
\begin{aligned}
\mathbb{P}[|\xi_{Nj}| > \epsilon] = \mathbb{P}[|Y^j_{h_N} - \mathbb{E}Z| > \sqrt{N}\epsilon] &\leqslant \frac{\mathbb{E}[|Y^j_{h_N} - \mathbb{E}Z|]}{\sqrt{N}\epsilon} \\
&\leqslant \frac{\mathbb{E}[|Y^j_{h_N} - \mathbb{E}[Y^j_{h_N}]|] + \mathbb{E}[|Y^j_{h_N} - Z|]}{\sqrt{N}\epsilon} \\
&\leqslant \frac{var\left(Y^j_{h_N}\right)^{\frac{1}{2}} + \mathbb{E}[|Y^j_{h_N} - Z|]}{\sqrt{N}\epsilon} \leqslant \frac{\sqrt{M^1_{t,x} + M^2_{t,x} h_N^{\frac{\gamma}{\beta}}}}{\sqrt{N}\epsilon} \overset{N\to\infty}{\to} 0,
\end{aligned}
\tag{3.37}
$$

where $M^1_{t,x}, M^2_{t,x}$ are the same as above. This implies that ξ_{Nj} converges to 0 in probability uniformly in N, and therefore (ξ_{Nj}) is a null array.

Denote $A_{Nj} = \{|Y^j_{h_N} - \mathbb{E}Z| \leqslant \sqrt{N}\} = \{|\xi_{Nj}| \leqslant 1\}$. To apply Theorem 3.2, we only need to check that those three conditions hold:

(i) We need to prove that for any $\epsilon > 0$,

$$
\sum_{j=1}^N \mathbb{P}[|\xi_{Nj}| > \epsilon] = \sum_{j=1}^N \mathbb{P}[|Y^j_{h_N} - \mathbb{E}Z| > \sqrt{N}\epsilon] \to 0. \tag{3.38}
$$

Note that

$$
\{|Y^j_{h_N} - \mathbb{E}Z| > \sqrt{N}\epsilon\} \subset \{|Y^j_{h_N} - Z| > \frac{1}{2}\sqrt{N}\epsilon\} \cup \{|Z - \mathbb{E}Z| > \frac{1}{2}\sqrt{N}\epsilon\}.
$$

Hence

$$
\mathbb{P}[|Y^j_{h_N} - \mathbb{E}Z| > \sqrt{N}\epsilon] \leqslant \mathbb{P}[|Y^j_{h_N} - Z| > \frac{1}{2}\sqrt{N}\epsilon] + \mathbb{P}[|Z - \mathbb{E}Z| > \frac{1}{2}\sqrt{N}\epsilon], \tag{3.39}
$$

$$
\begin{aligned}
&\sum_{j=1}^N \mathbb{P}[|Y^j_{h_N} - \mathbb{E}Z| > \sqrt{N}\epsilon] \\
&\leqslant \sum_{j=1}^n \mathbb{P}[|Y^j_{h_N} - Z| > \frac{1}{2}\sqrt{N}\epsilon] + \sum_{j=1}^n \mathbb{P}[|Z - \mathbb{E}Z| > \frac{1}{2}\sqrt{N}\epsilon],
\end{aligned}
\tag{3.40}
$$

$$
\begin{aligned}
\sum_{j=1}^N \mathbb{P}[|Z - \mathbb{E}Z| > \frac{1}{2}\sqrt{N}\epsilon] &= \sum_{j=1}^N \mathbb{E}[1; |Z - \mathbb{E}Z| > \frac{1}{2}\sqrt{N}\epsilon] \\
&\leqslant \sum_{j=1}^N \mathbb{E}[\frac{4|Z - \mathbb{E}Z|^2}{N\epsilon^2}; |Z - \mathbb{E}Z| > \frac{1}{2}\sqrt{N}\epsilon] \\
&= \mathbb{E}[\frac{4|Z - \mathbb{E}Z|^2}{\epsilon^2}; |Z - \mathbb{E}Z| > \frac{1}{2}\sqrt{N}\epsilon] \overset{N\to\infty}{\to} 0,
\end{aligned}
\tag{3.41}
$$

$$\begin{aligned}&\sum_{j=1}^{N}\mathbb{P}[|Y^j_{h_N}-Z|>\frac{1}{2}\sqrt{N}\epsilon]\\ &\leqslant\sum_{j=1}^{N}\mathbb{E}[|Y^j_{h_N}-Z|]/(\frac{1}{2}\sqrt{N}\epsilon)\leqslant 2\sqrt{N}M^1_{t,x}h_N^{\frac{\gamma}{\beta}}/\epsilon \overset{N\to\infty}{\to} 0.\end{aligned}\tag{3.42}$$

Together with (3.41) and (3.42) we know that (3.38) holds.

(ii) We need to prove that

$$\sum_{j=1}^{N}\mathbb{E}[\xi_{Nj};|\xi_{Nj}|\leqslant 1]=\frac{1}{\sqrt{N}}\sum_{j=1}^{N}\mathbb{E}[Y^j_{h_N}-\mathbb{E}Z;A_{Nj}]\to 0 \text{ as } N\to\infty. \tag{3.43}$$

Since

$$\begin{aligned}\frac{1}{\sqrt{N}}\sum_{j=1}^{N}\mathbb{E}[Y^j_{h_N}-\mathbb{E}Z]=\frac{1}{\sqrt{N}}\sum_{j=1}^{N}\mathbb{E}[Y^j_{h_N}-Z]\leqslant\frac{1}{\sqrt{N}}\sum_{j=1}^{N}\mathbb{E}[|Y^j_{h_N}-Z|]\\ \leqslant\frac{1}{\sqrt{N}}\sum_{j=1}^{N}M^2_{t,x}h_N^{\frac{\gamma}{\beta}}\to 0 \text{ as } N\to\infty,\end{aligned}\tag{3.44}$$

we only need to prove

$$\frac{1}{\sqrt{N}}\sum_{j=1}^{N}\mathbb{E}[Y^j_{h_N}-\mathbb{E}Z;A^C_{Nj}]\to 0 \quad\text{as}\quad N\to\infty. \tag{3.45}$$

For a random variable X, we denote $X_+=\max\{X,0\}$. Then

$$\begin{aligned}\frac{\sqrt{N}}{N}\sum_{j=1}^{N}\mathbb{E}[\left(Y^j_{h_N}-\mathbb{E}Z\right)_+;A^C_{Nj}]&=\sqrt{N}\mathbb{E}[\left(Y^1_{h_N}-\mathbb{E}Z\right)_+;A^C_{N1}]\\ (\text{note } A^C_{N1}=\{|Y^1_{h_N}-\mathbb{E}Z|>\sqrt{N}\})&\leqslant\mathbb{E}[\left(Y^1_{h_N}-\mathbb{E}Z\right)_+^2;A^C_{N1}]\\ &\leqslant\mathbb{E}[2\left(Y^1_{h_N}-Z\right)^2+2\left(Z-\mathbb{E}Z\right)^2;A^C_{N1}].\end{aligned}\tag{3.46}$$

By Proposition 3.3,

$$\mathbb{E}[\left(Y^1_{h_N}-Z\right)^2]\leqslant M^3_{t,x}h_N^{\frac{2\gamma}{\beta}}\to 0 \text{ as } N\to\infty. \tag{3.47}$$

Since $\mathbb{P}[A_{N1}]\to 0$ as $N\to\infty$, we have

$$\mathbb{E}[(Z-\mathbb{E}Z)^2;A^C_{N1}]\to 0 \text{ as } N\to\infty. \tag{3.48}$$

Together with (3.47) and (3.48) we have

$$\frac{1}{\sqrt{N}}\sum_{j=1}^{N}\mathbb{E}[\Big(Y^j_{h_N}-\mathbb{E}Z\Big)_+;A^C_{Nj}]\to 0. \tag{3.49}$$

Similarly, denote $X_- := \max\{-X,0\}$. Then,

$$\frac{1}{\sqrt{N}}\sum_{j=1}^{N}\mathbb{E}[\Big(Y^j_{h_N}-\mathbb{E}Z\Big)_-;A^C_{Nj}]\to 0. \tag{3.50}$$

Combining (3.49) and (3.50) we get (3.45) holds.

(iii) We want to prove

$$\sum_{j=1}^{N}Var[\xi_{Nj};|\xi_{Nj}|\leqslant 1]=\sum_{j=1}^{N}\frac{1}{N}Var[Y^j_{h_N}-\mathbb{E}Z;A_{Nj}]\to var\,(Z)\ \text{ as } N\to\infty. \tag{3.51}$$

We have the following equality

$$Var\left(Y^j_{h_N}-\mathbb{E}Z;A_{Nj}\right)=\mathbb{E}[|Y^j_{h_N}-\mathbb{E}Z|^2\mathbb{I}_{A_{Nj}}]-\left(\mathbb{E}[\Big(Y^j_{h_N}-\mathbb{E}Z\Big)\,\mathbb{I}_{A_{Nj}}]\right)^2. \tag{3.52}$$

Note that

$$\begin{aligned}&\frac{1}{N}\sum_{j=1}^{N}\left(\mathbb{E}[\Big(Y^j_{h_N}-\mathbb{E}Z\Big)\,\mathbb{I}_{A_{Nj}}]\right)^2\\
=&\frac{1}{N}\sum_{j=1}^{N}\left(\mathbb{E}[\Big(Y^j_{h_N}-Z+Z-\mathbb{E}Z\Big)\,\mathbb{I}_{A_{Nj}}]\right)^2\\
\leqslant&\frac{2}{N}\sum_{j=1}^{N}\left(\mathbb{E}[\Big(Y^j_{h_N}-Z\Big)\,\mathbb{I}_{A_{Nj}}]^2+\mathbb{E}[(Z-\mathbb{E}Z)\,\mathbb{I}_{A_{Nj}}]^2\right).\end{aligned} \tag{3.53}$$

Since

$$\mathbb{E}[\Big(Y^j_{h_N}-Z\Big)\,\mathbb{I}_{A_{Nj}}]\leqslant\mathbb{E}[|Y^j_{h_N}-Z|]\leqslant M^2_{t,x}h_N^{\frac{\gamma}{\beta}}, \tag{3.54}$$

we have

$$\frac{1}{N}\sum_{j=1}^{N}\mathbb{E}[\Big(Y^j_{h_N}-Z\Big)\,\mathbb{I}_{A_{Nj}}]^2\to 0\ \text{ as }\ N\to\infty. \tag{3.55}$$

Note that $0-\mathbb{E}[(Z-\mathbb{E}Z)\mathbb{I}_{A_{N1}}]=\mathbb{E}[(Z-\mathbb{E}Z)\mathbb{I}_{A^C_{N1}}]$ and $\mathbb{P}[A^C_{N1}]\to 0$ as $N\to\infty$, hence

$$\frac{2}{N}\sum_{j=1}^{N}\mathbb{E}[(Z-\mathbb{E}Z)\,\mathbb{I}_{A_{Nj}}]^2=2\mathbb{E}[(Z-\mathbb{E}Z)\,\mathbb{I}_{A^C_{N1}}]^2\to 0 \text{ as } N\to\infty. \tag{3.56}$$

Together with (3.55) and (3.56) we know the right hand side of (3.53) converges to 0, and therefore from (3.52) we only need to prove

$$\frac{1}{N}\sum_{j=1}^{N}\mathbb{E}[|Y^j_{h_N}-\mathbb{E}Z|^2\mathbb{I}_{A_{Nj}}]=\mathbb{E}[|Y^1_{h_N}-\mathbb{E}Z|^2\mathbb{I}_{A_{N1}}]\to Var\,(Z)\,. \qquad (3.57)$$

Note that

$$\begin{aligned}\mathbb{E}[\left(Y^1_{h_N}-\mathbb{E}Z\right)^2]-\mathbb{E}[(Z-\mathbb{E}Z)^2]&=\mathbb{E}[(Y^1_{h_N}+Z-2\mathbb{E}Z)\left(Y^1_{h_N}-Z\right)]\\&\leqslant(\mathbb{E}[(Y^1_{h_N}+Z-2\mathbb{E}Z)^2]\mathbb{E}[(Y^1_{h_N}-Z)^2])^{\frac{1}{2}}.\end{aligned} \qquad (3.58)$$

Since $\mathbb{E}[Z^2]<\infty$ and $\mathbb{E}[|Y_{h_N}-Z|^2]<M^3_{t,x}h^{2\frac{\gamma}{\beta}}$, we know $E[|Y_{h_N}|^2]$ have a uniform upper bound for all N. Hence $\mathbb{E}[(Y^1_{h_N}+Z-2\mathbb{E}Z)^2]$ have a uniform upper bound for all N. Hence the right hand side of (3.58) converges to 0 as $N\to\infty$.

Therefore we only need to show

$$\mathbb{E}[|Y^1_{h_N}-\mathbb{E}Z|^2\mathbb{I}_{A^C_{N1}}]\to 0 \quad\text{as}\quad N\to\infty. \qquad (3.59)$$

In fact, this is true because

$$\mathbb{E}[|Y^1_{h_N}-\mathbb{E}Z|^2\mathbb{I}_{A^C_{N1}}]\leqslant 2\mathbb{E}[|Y^1_{h_N}-Z|^2\mathbb{I}_{A^C_{N1}}]+2\mathbb{E}[|Z-\mathbb{E}Z|^2\mathbb{I}_{A^C_{N1}}]. \qquad (3.60)$$

□

REMARK 3.4. The choice of h_k is not unique. In fact, they only need to satisfy $\sqrt{N}h_N^{\frac{\gamma}{\beta}}\to 0$ as $N\to\infty$.

4. **Simulation and algorithm**

Now we study how the starting level t of the decreasing stable process and the starting point x of the stable process X influence the Monte Carlo estimator (1.4).

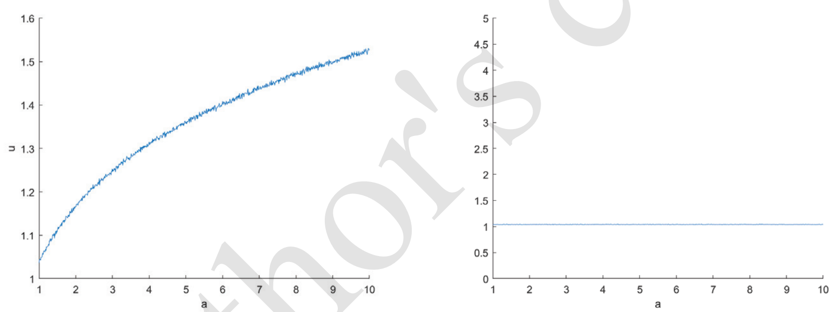
We set $d=1$, $\alpha=1/2$, $\beta=3/2$, and denote $\bar{a}=t-a$. In Sections 4.1 and 4.2 we set $\phi(x)=|x|^{\frac{1}{2}}$.

4.1. **Unbiased FPED.**
Now the estimator is (1.5). Recall that $X^x_{T_t}\overset{d}{=}T_t^{\frac{1}{\beta}}X_1+x\overset{d}{=}\left(\frac{\bar{a}}{\tau_1}\right)^{\frac{\alpha}{\beta}}X_1+x$.

Algorithm 1 Sample $u_N(t,x)$

$u=0$;
for $k=1:N$ **do**
  sample Y_1;
  $T_t=\left(\frac{\bar{a}}{\tau_1}\right)^{\alpha}$;
  sample X_1;
  $X=T_t^{\frac{1}{\beta}}X_1+x$;
  $u=u+\phi(X)$;
end for
$\bar{u}=u/N$;
return $\bar{u}$.

First we set $N=10^5$. Let $x=0$ and $\bar{a}$ increase from 1 to 10.



(a) u_N when $g=0, x=0$ (b) $u_N/\bar{a}^{\frac{1}{6}}$

Fig. 4.1

Figure 4.1 (a) shows that u_N tends to increase as we increase $\bar{a}$. In fact, now we have

$$\mathbb{E}[|X^x_{T_t}|^{\frac{1}{2}}]=\mathbb{E}[\left(\frac{\bar{a}}{\tau_1}\right)^{\frac{\alpha}{2\beta}}|X_1|^{\frac{1}{2}}]=\bar{a}^{\frac{\alpha}{2\beta}}\mathbb{E}[\tau_1^{-\frac{\alpha}{2\beta}}]\mathbb{E}[|X_1|^{\frac{1}{2}}]. \tag{4.1}$$

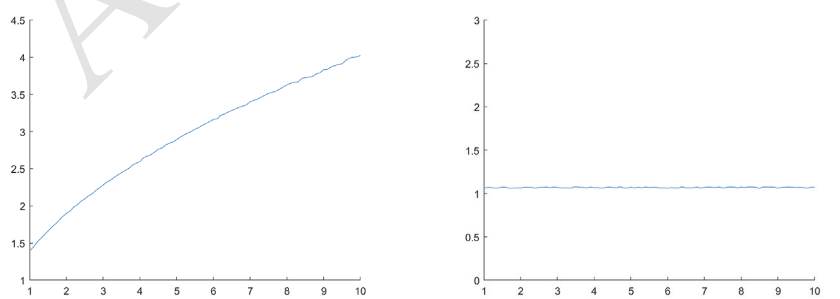
We can check our result by Figure 4.1 (b) '$u_N/(\bar{a})^{\frac{\alpha}{2\beta}}$' above. It is almost a constant, which means our algorithm is correct.

4.2. **FPDE with bias.** We set $g(x) = |x|^{\frac{1}{2}}$.

Algorithm 2 Sample $u_N^h(t, x)$

$u = 0$;
for $k = 1 : N$ **do**
  sample Y_1;
  $T_t = \left(\frac{\bar{a}}{\tau_1}\right)^{\alpha}$;
  sample X_1^j, $j = 1, ..., \lfloor T_t/h \rfloor$;
  $S = 0$;
  $X = x$;
  sample X_1';
  for $j = 1 : \lfloor T_t/h \rfloor$ **do**
    $X = X + h^{\frac{1}{\beta}} X_1^j$;
    $S = S + hg(X)$;
  end for
  $X = X + (T_t - h\lfloor T_t/h \rfloor)^{\frac{1}{\beta}} X_1'$;
  $u = u + \phi(X) + S$;
end for
$\bar{u} = u/N$;
return $\bar{u}$.

Figure 4.2 (c) is the figure of u_N^h when $x = 0, h = 0.01, N = 10^5$ and we change a from 1 to 10.



(c) u_N when $g(t) = |t|^{\frac{1}{2}}, x = 0$ (d) $\frac{1}{N}\sum_{k=1}^{N}\sum_{i=1}^{\lfloor T_t^k/h \rfloor} hg(X_{t_i^k}^k)/(\bar{a})^{2/3}$

Fig. 4.2

Similarly, recall that $T_t \overset{d}{=} \left(\frac{\bar{a}}{\tau_1}\right)^{\alpha}$,

$$\begin{aligned}\mathbb{E}[\int_0^{T_t} g\left(X_s^x\right) ds] &= \mathbb{E}[\mathbb{E}[\int_0^{T_t} |X_s^0|^{\frac{1}{2}} ds | T_t]] = \mathbb{E}[\mathbb{E}[\int_0^{T_t} |X_1^0|^{\frac{1}{2}} s^{\frac{1}{2\beta}} ds | T_t]] \\ &= \mathbb{E}[|X_1^0|^{\frac{1}{2}}]\mathbb{E}[T_t^{1+\frac{1}{2\beta}}] / \left(1 + \frac{1}{2\beta}\right) \\ &= \bar{a}^{\frac{\alpha(1+2\beta)}{2\beta}} \mathbb{E}[|X_1^0|^{\frac{1}{2}}]\mathbb{E}[\tau_1^{-\frac{\alpha(1+2\beta)}{2\beta}}] / \left(1 + \frac{1}{2\beta}\right).\end{aligned} \tag{4.2}$$

As it is shown in Figure 4.2 (d), $\frac{1}{N}\sum_{k=1}^{N}\sum_{i=1}^{\lfloor T_t^k/h \rfloor} hg(X^k_{t_i^k})/(\bar{a})^{2/3}$ is almost a constant.

Below is the figure of u_N^h when we fix $\bar{a} = 5$, and $x = 0 : 0.1 : 10$.

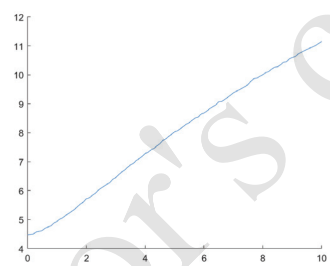


Fig. 4.3: $a = 5, x = 0 : 0.1 : 10$

4.3. **Confidence interval.** For simplicity, we set $\phi\left(x\right) \equiv 1$, $g\left(x\right) = |x|^{\frac{1}{2}}$, $\bar{a} = t - a = 1$, $x = 0$, $h = 10^{-3}$. Recall our discussion in Remark 3.1, we only need the upper bound of $\mathbb{E}\left[Z(t,x)^2\right]$ in this example. In fact, in Remark 3.3, we have already got the computable upper bound of $\mathbb{E}\left[Z(t,x)^2\right]$. Figure 4.4 presents the asymptotic confidence intervals at level 95%.

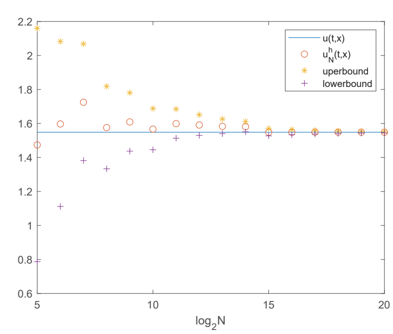


Fig. 4.4: Confidence intervals at level 95%

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