mathematics

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Inspection—Corruption Game of Illegal Logging and Other Violations: Generalized Evolutionary Approach

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Abstract: Games of inspection and corruption are well developed in the game-theoretic literature. However, there are only a few publications that approach these problems from the evolutionary point of view. In previous papers of this author, a generalization of the replicator dynamics of the evolutionary game theory was suggested for inspection modeling, namely the pressure and resistance framework, where a large pool of small players plays against a distinguished major player and evolves according to certain myopic rules. In this paper, we develop this approach further in a setting of the two-level hierarchy, where a local inspector can be corrupted and is further controlled by the higher authority (thus combining the modeling of inspection and corruption in a unifying setting). Mathematical novelty arising in this investigation involves the analysis of the generalized replicator dynamics (or kinetic equation) with switching, which occurs on the "efficient frontier of corruption". We try to avoid parameters that are difficult to observe or measure, leading to some clear practical consequences. We prove a result that can be called the "principle of quadratic fines": We show that if the fine for violations (both for criminal businesses and corrupted inspectors) is proportional to the level of violations, the stable rest points of the dynamics support the maximal possible level of both corruption and violation. The situation changes if a convex fine is introduced. In particular, starting from the quadratic growth of the fine function, one can effectively control the level of violations. Concrete settings that we have in mind are illegal logging, the sales of products with substandard quality, and tax evasion.

Keywords: inspection; corruption; illegal logging; tax evasion; substandard quality; evolutionary games; pressure and resistance games; dynamic law of large numbers; stable equilibria; approximate Nash equilibria; principle of quadratic fines; efficient frontier of corruption

Citation: Kolokoltsov, V.N. Inspection—Corruption Game of Illegal Logging and Other Violations: Generalized Evolutionary Approach. *Mathematics* **2021**, *9*, 1619. https://doi.org/10.3390/math9141619

Academic Editor: Vladimir Mazalov

Received: 2 June 2021
Accepted: 7 July 2021
Published: 9 July 2021

1. Introduction

Games of inspection and corruption are well developed in the game-theoretic literature; see, e.g., monographs [1–4] and reviews [5,6]. However, there are only a few publications that approach these problems from the evolutionary point of view. Paper [7] offers various insights, including some simple evolutionary models with evolutionarily stable strategies applied to the setting of corruption games. Another evolutionary approach to the games of corruption was suggested in [8], specifically for the setting of illegal logging, where, in order to fit to the classical evolutionary setting of pairwise games, it was assumed that arbitrary pairs of players can hire inspectors who can control their pairwise agreement (but can be corrupted).

In [9,10], a generalized evolutionary approach to various classes of models was developed, including the pressure and resistance framework, where a large pool of small players plays against a distinguished major player and evolves according to certain myopic rules. This approach led to the generalized replicator dynamics (RD) or, in a more physical language, kinetic equations (see details in the next Section), describing the dynamic law of large numbers for games of N players as $N \to \infty$. The remaining points of these equations were shown to represent approximate Nash equilibria for games with a finite N (the games that we are actually interested in).

In this paper, we develop this approach further in a setting of the two-level hierarchy, where a local inspector can be corrupted and is further controlled by the higher authority (thus combining the modeling of inspection and corruption in a unifying setting). Applying the second level of control goes one step in the direction indicated by the famous question, "Who will guard the guardians?" addressed in the Nobel Prize lecture of L. Hurwicz [11].

One of the mathematical novelties arising in the present investigation involves the analysis of the generalized replicator dynamics (or kinetic equation) with switching, which occurs on the "efficient frontier of corruption". Some general aspects of this development were initiated in [2], but there, only trivial examples of such switching were looked upon (like the generalized RD for the minority game).

In this paper, we specifically try to avoid complicated parameters that are difficult to observe or measure, in order to derive some clear practical consequences. In particular, we prove a result that can be called the "principle of quadratic fines", which concerns the major tool of the mechanism design of the controlling authority, the fine function that specifies fines due from a detected violator depending on the level of violation. We show that if the fine for violations (both for criminal businesses and corrupted inspectors) is proportional to the level of violations (which is seemingly a common practice in many regulations), the stable rest points of the dynamics support the maximal possible level of both corruption and violation. The situation changes drastically if a convex fine is introduced. In particular, starting from the quadratic growth of the fine function, one can effectively control the level of violations. The convex fines bear analogy with progressive taxation. Furthermore, in line with the arguments of [7], our analysis suggests that it is difficult to expect the complete elimination of corruption, since full compliance with the regulations can hardly be achieved as a stable equilibrium. However, keeping the size of violations to some controlled values can be achieved well by a regulatory mechanism (without betaking to any draconian methods). Another conclusion arises from the possibility of having several distinct stable equilibria, meaning practically that with the identical regulations, the equilibrium outcomes can be quite different depending on the initial conditions (say, different social norms), and therefore copying the practices of successful peers may not necessarily lead to a success.

The major concrete setting that we have in mind represents illegal logging. An extensive discussion of the models for illegal logging can be found in [8], starting from the observation that "corruption is one of the most serious obstacles for ecosystem management and biodiversity conservation". Illegal harvesting of forest trees has devastating consequences in many regions on Earth, from tropical rain forests of Brasil to conifer boreal forests of Russia. Legal logging is usually linked with additional investments that can sustainably provide ecosystem services, say, by new planting. Another setting, well fitted to our modeling, provides the selling of sub-standard-quality products making illegal savings on production. Examples are numerous. For instance, one can mention selling meat produced under non-hygienic condition in Europe (one can recall the scandal around the 2 Sisters Food Group in the UK in 2018) or selling substandard quality ice creams in some regions of Russia. Yet another group of examples comes from the problem of tax evasion. This is one of the most standard examples in the literature on corruption and inspection games. Here one analyzes the situations when, for a certain bribe, corrupted inspectors can accept false reports on the tax return.

The further content of the paper is as follows. In the next section, we briefly describe the generalized replicator dynamics arising in the framework of the pressure and resistance games, which we exploit here. Section 3 describes our game theoretic model introducing, in particular, the key notion of an efficient corruption frontier that effectively controls the possible spread of corruption. In Section 4, we analyze the switching kinetic equations in the case of a linear fine function, leading to our first major conclusion that linear fines are not effective: they support an infinite growth of illegal gains from violation and corruption. In Section 5, we derive our major conclusion: the principle of quadratic fines as explained above. In Section 6, some possible further developments are briefly outlined.

2. Preliminaries

We propose here a generalized evolutionary model that belongs to the general framework of pressure and resistance games developed in [10] (with preliminary results in [12] and additional details and examples in [2]). The pressure and resistance framework describes the game of many small players or agents, say N, with a principle (big player) I. It is supposed that small players are identical in the sense that each one of them has the same finite collection of strategies $\{1, \cdots, d\}$ and they choose their strategies according to the same rule, such that the state of the group of small players is fully described by a collection $(n_1, \cdots, n_d)$ of d non-negative integers with $N = n_1 + \cdots + n_d$, where n_i denotes the number of players adhering to the strategy i. We denote $x_i = n_i/N$ and $x = (x_1, \cdots, x_d)$. It is assumed that the strategy of the big player I can be specified by a certain number b taking values from a subset of real numbers. One defines the game Γ_N of $N+1$ players by specifying certain payoffs $B(x, b, N)$ to the principle I and $R_i(x, b)$ to each small player that adheres to the strategy i.

On the other hand, one can consider the process of myopic adjustment of the behavior of small players as a Markov chain on the set of the states $(n_1, \cdots, n_d)$, where a clock is attached to each pair of agents (i, j) with a $\varkappa/N$-exponential random time (with some constant $\varkappa > 0$), so that when the bell rings, the player i with a lower payoff $R_i(x, b)$ can change to the better strategy j with a higher payoff $R_j(x, b)$ with a probability proportional to $R_j(x, b) - R_i(x, b)$. It is proven in [10] that, if

$$b^*(x, N) = arg \max B(x, b, N)$$

is well defined and $b^*(x, N) \to b^*(x)$, as $N \to \infty$, and the functions $R(x, b)$, $R(x, b^*(x))$ are Lipschitz continuous functions of their variables, then this Markov chain converges weakly, as $N \to \infty$, to the deterministic process described by the following kinetic Equation (or generalized replicator dynamics RD):

$$\dot{x}_j = \varkappa x_j[R_j(x, b^*(x)) - \bar{R}(x, b^*(x))] = x_j[R_j(x, b^*(x)) - \sum_i x_i R_i(x, b^*(x))]. \tag{1}$$

It is also shown that, when functions $R_j(x, b^*(x))$ are just continuous (not necessarily Lipschitz, so that evolution (1) may not be well defined), the fixed points of these dynamics define approximate ϵ-Nash equilibria for the game Γ_N. When functions $R_j(x, b^*(x))$ are Lipschitz, then ϵ is of order $1/N$. In [2], a generalization of this result was considered for some cases of piecewise continuous R_j, where, in the points of discontinuity of R_j, Equation (1) generalizes to the corresponding differential inclusions.

Of course kinetic, Equation (1) (as well as a classical RD) has a clear intuitive meaning independent of any Markov approximation. Therefore, equations of this kind are often used without explicit reference to the prelimiting Markov model, assuming (often tacitly) that they make sense only for a large population. However, we stress that we are eventually interested in the situation with a finite number of players and results from [10] provide quantitative estimates for the deviations from the law of large number limits.

It is seen that Equation (1) are generalized versions of the standard replicator dynamics (RD) of the evolutionary game theory (essentially, the latter are obtained when $R(x, b^*(x))$ are linear functions in x). Similar equations can be obtained when considering generalized RD for games with a field; see, e.g., [13] for the latter.

Numerous examples of modeling under the pressure and resistance framework were given in [2,10] including modeling of inspection, corruption, counterterrorism, cybersecurity, evolutionary coalition building, optimal allocations, and others.

For the general notions of game theory used throughout this paper, we refer to any text on the basic theory of game, see, e.g., [14,15].

3. Main Model

We assume that there is a large number M of firms or agents, which can violate the rules, or places, where a violation can occur. Violation can occur on several levels, resulting with the illegal (criminal) gain r_j, $j = 1, \cdots, J$, so that $r_1 < r_2 < \cdots < r_J$. We also set $r_0 = 0$, which corresponds to the strategy "refrain from violation". We denote by V the number of violating agents, so that $v = V/M$ is the fraction of violating agents. The state of the system can be described by the vector $(x_0, x_1, \cdots, x_J)$, where x_j is the fraction of agents adhering to the level of illegal gain r_j, so that

$$\sum_{j=0}^{J} x_j = 1, \quad \sum_{j=1}^{J} x_j = v.$$

The average level of violation is

$$\bar{r} = \mathbf{E}r = \sum_{j=1}^{J} r_j x_j.$$

Here and everywhere below, $\mathbf{E}$ denotes the expectation.

If detected, the illegal gain is supposed to be taken away from the agent, and the fine $f(r_j)$ is to be paid. A local inspector I is given some budget that allows her to detect each violation with some probability p. If an inspector is honest (strategy H), then a detected violator is fined. If an inspector is corrupted (strategy C), she just demands for herself the part αr_j of the illegal gain.

Additionally, there is a central authority A, which eventually makes independent inspections in local areas. These inspections are of smaller scale. For simplicity, we assume that A makes one random check among the M firms, and hence A detects a violation with probability $v = V/M$. Under the condition that A detects a violation, the probability for any particular violator to be the one detected is about $1/V = \epsilon/v$, where ϵ is small of order $1/M$.

Remark 1. *To get the limiting kinetic equation, we pass to the limit $M \to \infty$. However, eventually, we are interested in the case of a finite M. Thus, with the aim of deriving from the limiting case the results for finite M, we can keep ϵ fixed of order $1/M$ with a given M of interest.*

If A detects an agent that violates the rules but was not detected by the local inspector I, then the agent just pays the corresponding fine, but nothing changes with I. If, however, A finds a violator that was previously detected by I and paid the bribe αr_j to I in order to avoid paying the full fine $f(r_j)$, then the detected agent pays the fine $f(r_j)$, but I is also punished. Namely, in accordance with the most standard modeling of corruption games (see [5,6]), I loses the illegal profit αr_j and pays the fine $F(\alpha r_j)$ to A. Moreover, I has some standard salary w, but if detected in corruption, she gets a lower-level job with some reserve salary w_0.

Let us now write the corresponding table of payoffs for the game of inspector I with an agent (possible violator) under two scenarios: A does not find a violation and otherwise. Note that the strategies of an agent include Refrain R from a violation and violations V_j on various levels r_j. Under the first condition, which occurs with probability $1 - v$, we have the table

		Violator	
		Refrain R	Violate V_j
I	H	$w, 0$	$w, (1-p)r_j - pf(r_j)$
	C	$w, 0$	$w + \alpha p \bar{r}, r_j(1 - p\alpha)$

because $(1-p)r_j + p(1-\alpha)r_j = r_j(1 - p\alpha)$.

Under the second condition, which occurs with probability v, we have the table

Violator

I		Refrain R	Violate V_j
	H	$w, 0$	$w, -(\epsilon/v)f(r_j) + (1-\epsilon/v)[(1-p)r_j - pf(r_j)]$
	C	$w, 0$	$p(w_0 - \mathbf{E}F(\alpha r)) + (1-p)w, -(\epsilon/v)f(r_j) + (1-\epsilon/v)r_j(1-p\alpha)$

Here, $\mathbf{E}F(\alpha r) = \alpha \sum_{j=1}^{J} x_j r_j$ is the expectation of the gain of an inspector arising from the corrupted behavior.

Thus, the total overall table of payoffs is

	R	Violate V_j
H	$w, 0$	$w, (1-\epsilon)[(1-p)r_j - pf(r_j)] - \epsilon f(r_j)$
C	$w, 0$	$(1-v)(w + \alpha p\bar{r}) + v[p(w_0 - \mathbf{E}F(\alpha r)) + (1-p)w], (1-\epsilon)r_j(1-p\alpha) - \epsilon f(r_j)$

The probability p can be modelled differently. It seems natural that the budget for local crime detection must be increased when the overall crime level increases, so that one can set

$$p = p_0 + \beta v \tag{2}$$

with some nonnegative constants p_0 and β such that $p_0 + \beta \leq 1$, which we adopt from now on.

Remark 2. *The result below will not be changed drastically if one assumes a simpler condition that p is a given constant, or more generally that p is an arbitrary increasing function of v.*

The corruption is profitable for inspectors I if the average illegal surplus is positive. As is seen from the table, this is the case whenever

$$(1-v)(w + \alpha p\bar{r}) + v[p(w_0 - \mathbf{E}F(\alpha r)) + (1-p)w] \geq w,$$

that is, inside the *corruption domain* (which does not depend on p)

$$M_c = \{(x_1, \cdots, x_J) \in \mathbf{R}_+^J : \alpha\bar{r}(1-v) - v\mathbf{E}F(\alpha r) \geq v(w - w_0)\}, \tag{3}$$

which is bounded by the hyperplanes of the orthant $\mathbf{R}_+^J$ and the *effective corruption frontier*

$$\Gamma_c = \{(x_1, \cdots, x_J) \in \mathbf{R}_+^J : \alpha\bar{r}(1-v) - v\mathbf{E}F(\alpha r) = v(w - w_0)\} \setminus \{0\}. \tag{4}$$

4. Linear Fines

4.1. Corruption Frontier

The fine functions $F(r)$ and $f(r)$ represent the mechanism design of the central authority A. Let us start with the simplest linear functions

$$f(r) = fr, \quad F(r) = Fr$$

with some constants f and F.

In this case the corruption domain becomes

$$M_c = \{v \geq 0 : \alpha\bar{r}(1 - v - vF) \geq v(w - w_0)\}. \tag{5}$$

Thus, the condition of corruption simplifies to the condition

$$v(1+F) \leq 1 - \frac{v(w - w_0)}{\alpha\bar{r}}. \tag{6}$$

and the *effective corruption frontier* becomes the surface (a hyperboloid)

$$\Gamma_c = \{(x_1, \cdots, x_J) \in \mathbf{R}_+^J : \alpha\bar{r}(1 - v - vF) - v(w - w_0) = 0\} \setminus \{0\}, \tag{7}$$

It is seen that for small v, the corruption is profitable, so the ideal point (no violation, no corruption) cannot be a stable equilibrium.

4.2. Partial Kinetic Equations

We shall now analyze separately the behavior of kinetic equations in two domains of interest: corruptive and non-corruptive inspector.

Recall that the state of the set M of agents is described by the vector $(x_0, x_1, \cdots, x_J)$, specifying the fraction of agents using the strategies with violations $r_0 = 0, r_1, \cdots, r_J$.

Inside the domain of corruption kinetic, Equation (1) takes the form

$$\dot{x}_j = \varkappa x_j (r_j - \bar{r})[(1-\epsilon)(1-\alpha p) - \epsilon f]$$

Since ϵ is negligibly small, we can write approximately 1 instead of $1-\epsilon$, so that the dynamics simplifies to

$$\dot{x}_j = \varkappa x_j (r_j - \bar{r})(1 - \alpha p - \epsilon f) = \varkappa x_j (r_j - \bar{r})(1 - \alpha p_0 - \alpha\beta v - \epsilon f). \tag{8}$$

Remark 3. *Keeping $1-\epsilon$ would just make the following formulas more lengthy, not change anything essential. Of course ϵf is also small, but we keep it here, because ϵf can be comparable with other small terms like αp_0.*

For the analysis, the following three cases must be considered:

$$\begin{aligned} &(i) \quad 1 - \alpha p_0 - \epsilon f < 0, \\ &(ii) \quad 0 < 1 - \alpha p_0 - \epsilon f < \alpha\beta, \\ &(iii) \quad 1 - \alpha p_0 - \epsilon f > \alpha\beta. \end{aligned} \tag{9}$$

The case (i) is included here for completeness of the argument. Effectively we are interested in cases (ii), (iii). Since ϵ is of order $1/M$, to make ϵf of order 1 (and to obtain (i)), an extremely high f should be chosen. These f will increase as M increases, and thus they cannot be chosen as the universal rule.

Remark 4. *We will not consider some (not interesting) additional details arising in the non-generic cases of equalities in cases (9).*

Dynamics (8) has, as usual, the pure strategy fixed points X_j (the corners of the simplex of probability measures on $J+1$ points) with $x_j = 1$ and other coordinate zeros. Additionally, in case (ii), it has the hyperplane of rest points with

$$v = \frac{1 - \alpha p_0 - \epsilon f}{\alpha\beta}. \tag{10}$$

Around the point X_j, we can choose J independent coordinates x_i with $i \neq j$ and get

$$\bar{r} = r_j + \sum_{k \neq j} x_k (r_k - r_j),$$

so that

$$\dot{x}_i = x_i [r_i - r_j - \sum_{k \neq j} x_k (r_k - r_j)](1 - \alpha p_0 - \alpha\beta v s - \epsilon f).$$

If $j \neq 0$, in the linear approximation of small x_i, this equation turns into the equation

$$\dot{x}_i = x_i [r_i - r_j](1 - \alpha p_0 - \alpha\beta - \epsilon f). \tag{11}$$

In the case tht $j = 0$, the linear approximation becomes

$$\dot{x}_i = x_i r_i (1 - \alpha p_0 - \epsilon f). \quad (12)$$

It is seen that whenever r_i is neither maximum r_J nor zero, this is a saddle point, so that for the search for stable rest points, these intermediate values can be always dismissed.

Lemma 1. *In case (iii), the point X_J is the only stable rest point for (8). In case (ii), the points X_0 and X_J are both repulsive, but the hyperplane of the rest points (10) is the global attractor for (8): starting from all points except X_j, the solutions of (8) will tend to this hyperplane. In case (i), the rest point X_0 of honest behavior is the only stable point.*

Proof. In case (iii), $(1 - \alpha p_0 - \alpha\beta - \epsilon f) > 0$ and the r.h.s. of (11) is negative for $j = J$ and all i (and only for this j). Hence, X_J is stable. Similarly, in case (i), X_0 is the only stable rest point. In case (ii), both X_0 and X_J are repulsive. Let us show that the hyperplane (10) is attracting by proving that the function

$$L = (1 - \alpha p_0 - \epsilon f - v\alpha\beta)^2$$

can be taken as a Lyapunov function (that decreases along the trajectories). In fact,

$$\frac{d}{dt}L = -2(1 - \alpha p_0 - \epsilon f - v\alpha\beta)\alpha\beta\frac{d}{dt}v = -2\varkappa(1 - \alpha p_0 - \epsilon f - v\alpha\beta)^2\alpha\beta x_0\bar{r} = -2\varkappa x_0\bar{r}\alpha\beta L,$$

as required. □

Outside the corruptive domain, the kinetic Equation (1) takes the form

$$\dot{x}_j = \varkappa x_j(r_j - \bar{r})[(1-\epsilon)(1 - p(1+f)) - \epsilon f]$$

$$= \varkappa x_j(r_j - \bar{r})[(1-\epsilon)(1 - p_0(1+f)) - \epsilon f - (1-\epsilon)\beta v(1+f)].$$

Again, writing approximately 1 instead of $1 - \epsilon$, we get the dynamics

$$\dot{x}_j = \varkappa x_j(r_j - \bar{r})[1 - p_0(1+f) - \epsilon f - \beta v(1+f)]. \quad (13)$$

Similarly to the above, the dynamics distinguishes the following cases:

$$\begin{aligned} &(i) \quad 1 - (1+f)p_0 - \epsilon f < 0, \\ &(ii) \quad 0 < 1 - (1+f)p_0 - \epsilon f < (1+f)\beta, \\ &(iii) \quad 1 - (1+f)p_0 - \epsilon f > (1+f)\beta. \end{aligned} \quad (14)$$

Analogously to Lemma 1, we get the following result.

Lemma 2. *In case (iii), the point X_J is the only stable rest point for (13). In case (ii), the points X_0 and X_J are both repulsive, but the hyperplane of the rest points*

$$v = \frac{1 - (1+f)p_0 - \epsilon f}{(1+f)\beta} \quad (15)$$

is the global attractor for (13) (outside the points X_j). In case (i), the rest point X_0 of honest behavior is the only stable point.

4.3. Full Switching Dynamics for $J = 1$

Let us now look at the full switching dynamics, which evolves according to (8) inside the corruption domain M_c and according to (13) outside M_c.

Remark 5. *We do not bother with the exact description of the dynamics exactly on the switching corruption frontier because it does not affect our results in any substantial way. To make a full description of the dynamics, one can either smooth it in an arbitrary small neighborhood of the frontier (and thus avoid all problems with nonsmoothness), or work in the framework of differential inclusions (as suggested in [2] in a similar setting), where the r.h.s. of a differential equation in a point of discontinuity is substituted by the convex hull of all left and right limits.*

Let us first look at the simplest case of only one level of violation, that is, when $J = 1$, $r_1 = r$ and $v = x_1$. In fact, the case already captures the main qualitative results that we shall point out in this paper.

In this case, the corruption domain becomes

$$M_c = \{v \geq 0 : \alpha r(1 - v(1 + F)) \geq (w - w_0)\} \in \mathbf{R}_+. \tag{16}$$

It is given by the condition

$$v \leq v_c = \frac{1}{1+F}\left(1 - \frac{w - w_0}{r\alpha}\right), \tag{17}$$

so that the frontier turns to one point $v = v_c$.

In particular, corruption is possible only if

$$\alpha r > w - w_0, \tag{18}$$

which is quite natural (the illegal gain of inspectors exceeds possible losses) and indicates already that corruption is profitable for large illegal profits.

The remaining hyperplanes of kinetic equations in corruptive and non-corruptive (honest) domains, given by Lemmas 1 and 2, turn correspondingly to the points

$$v_{rc} = \frac{1 - \alpha p_0 - \epsilon f}{\alpha\beta}, \quad v_{rh} = \frac{1 - (1+f)p_0 - \epsilon f}{(1+f)\beta}. \tag{19}$$

We note that

$$v_{rh} < v_{rc} \Longleftrightarrow \epsilon f < 1.$$

The total switching dynamics is given by (8) and (13) in the corruptive domain M_c and outside it, respectively. Its behavior is determined by the relative positions of the points v_c, v_{rc}, v_{rh} as described by the following result.

We shall discuss in detail only the most interesting case when (ii) holds in (9) and (14). The case (i) was already dismissed as a not very relevant one. Furthermore, in case (iii), the corner point $X_J = X_1$ takes the role of the corresponding rest points v_{rc} or v_{rh}.

Theorem 1. *Assume that* (18) *and the cases (ii) in both* (9) *and* (14).

(i) If $v_{rc} < v_c < v_{rh}$, then there are two stable rest points of the switching dynamics, namely v_{rc} and v_{rh}, which are attracting for the starting points in the domains $0 < v < v_c$ and $1 > v > v_c$, respectively.

(ii) If $v_c < v_{rc} < v_{rh}$, then there is only one stable rest point of the switching dynamics, namely v_{rh}, which is the global attractor (for all starting points except the rest corners $v = 0$ and $v = 1$).

(iii) If $v_{rh} < v_c$, then there is only one stable rest point of the switching dynamics, namely $\min(v_c, v_{rc})$, which is the global attractor (for all starting points except the rest corners $v = 0$ and $v = 1$).

Proof. Case (i) is straightforward. In case (ii), all points starting at $v < v_c$ will move right in the direction of v_{rc} until they reach v_c. At this point, the dynamics will be switched to the non-corruptive domain and will move further toward v_{rh}. In case (iii), for $v < v_c$, two cases must be distinguished: $v_{rc} < v_c$ and $v_c < v_{rc}$. In the first case, the points $v < v_c$ will move towards v_{cr} according to the corruptive branch of the dynamics, and all points $v > v_c$ will first move left towards v_r according to the non-corruptive branch of the dynamics until they reach v_c; then, they switch to the corruptive branch and continue moving towards v_{rc}. If $v_c < v_{rc}$, then all points $v < v_c$ will move towards v_c by the corruptive branch, and all points $v > v_c$ will move towards v_c by the non-corruptive branch. □

The key conclusion is that the rest points do not depend essentially on the level of corruption r (the point v_c converges to $1/(1+F)$, as $r \to \infty$). This means that the mechanism design with linear fine functions supports infinite growth rv_c of losses due to violations and/or corruption.

4.4. A Numeric Example

Assume we have $M = 20$ agents in a local area, so that $\epsilon = 1/20$. Assume that, by the "social norms" of corruption, a corrupted inspector requires $\alpha = 1/4$ of the illegal profit of a violator. Let the routine inspections in the area detect violations with the basic probability $p_0 = 1/2$, which increases to $p = p_0 + v/2$, when the rate of violation v is expected. Let the difference in the salaries of an inspector and the reserve salary be $w - w_0 = 5000$ dollars.

The mechanism design of the government consists in choosing the coefficients f and F. As is shown below, the value of f is not very essential. Namely, assume that f lies somewhere in the interval $[1, 10]$. Then

$$v_{rc} = \frac{1 - \frac{1}{8} - \frac{f}{20}}{\frac{1}{8} \cdot \frac{1}{2}} > 1, \quad v_{rh} = \frac{1 - \frac{1}{2}(1+f) - \frac{f}{20}}{\frac{1}{2}(1+f)} < 0,$$

so that, according to Theorem 1 (iii), the only stable point of the dynamics is the effective frontier of corruption

$$v_c = \frac{1}{1+F}\left(1 - \frac{20000}{r}\right),$$

if this v_c belongs to $[0, 1]$. This occurs when the amount of illegal profit exceeds 20 thousand.

Therefore, the corruption becomes profitable for the inspector when $r > 20{,}000$, in which case the illegal gains, due to violations and corruption, equal rv_c in equilibrium, supporting the maximal available level of violation r.

According to a result from [10] mentioned after Equation (1), the equilibrium value v_c^M for the actual game of $M = 20$ players differs from the equilibrium of the limiting evolution v_c by amount of order $1/M$, that is, by 5%, which is not very large.

4.5. Full Switching Dynamics for Arbitrary J

Let us see what modifications arise for an arbitrary J. Taking into account the condition for corruption (18), it is natural to assume that

$$\alpha r_j > w - w_0, \quad j > 0. \tag{20}$$

Remark 6. *One sees that if* (20) *is reversed for all* j*, then corruption again becomes unprofitable. If inequalities* (20) *hold for some* j*, the analysis of corruption can be performed, showing that the levels of* r_j *that do not satisfy this condition do not essentially influence the behavior of the system.*

Under (20), the corruption frontier (7) is separated from the origin and becomes the part of the hyperboloid

$$\alpha \sum_{j>1} r_j x_j \left(1 - (1+F)\sum_{j>1} x_j\right) - \sum_{j>1} x_j (w - w_0) = 0, \tag{21}$$

lying in $\mathbf{R}^J_+ \setminus \{0\}$ that intersects the coordinate axes of $\mathbf{R}^J$ in the points Y_j that have zero coordinates of indices $i \neq j$ and the jth coordinate

$$y_j = \frac{1}{1+F}\left(1 - \frac{w - w_0}{\alpha r_j}\right).$$

Notice that $y_1 < y_2 < \cdots < y_J$.

Equation (21) can be also written as expressing $\bar{r}$ in terms $v = \sum_{j>0} x_j$:

$$\bar{r} = \frac{v(w - w_0)}{1 - (1+F)v}. \tag{22}$$

The key property of this hyperboloid is that the function (22) is a strictly increasing function of v. This allows us to get the multidimensional version of Theorem 1. Of course, there are more cases on the various relationships between the three key surfaces: the corruption frontier Γ_c and the remaining hyperplanes v_{rh} and v_{rc} (with some abuse of notation, we denote by the same letter, say v_{rh}, both the corresponding value of v and the hyperplane $\{v : v = v_{rh}\}$).

The most straightforward cases arise when the points v_{rc} and v_{rh} are separated from the interval (y_1, y_J). Then we get following result.

Theorem 2. *Assume* (20) *holds.*

(i) If $v_{rc} \leq y_1 < y_J \leq v_{rh}$, then there are two stable hyperplanes of rest points of the switching dynamics, namely v_{rc} and v_{rh}, which are attracting for the starting points in the corruption domains M_c and in $\mathbf{R}^J_+ \setminus M_c$ (that is, from below and from above the efficient frontier), respectively.

(ii) If $y_J \leq v_{rc} < v_{rh}$, then there is only one stable hyperplane of rest points of the switching dynamics, namely v_{rh}, which is the global attractor (for all starting points except the rest corners X_j).

(iii) If $v_{rh} \leq y_1 < y_J \leq v_{rc}$, then the corruption frontier Γ_c is the global attractor for the switching dynamics (for all starting points except the rest corners).

The case (iii) with $J = 2$ and $v_{rh} < y_1 < y_2 = v_{rc}$ is illustrated in Figure 1. The corruption frontier is given by the curve (actually a part of a hyperbola) connecting the points $(0, y_2)$ and $(y_1, 0)$.

When various intersections occur, they are also treated as in Theorem 1. Let us consider just one of the cases.

Theorem 3. *Assume* (20) *holds, and let $v_{rh} \leq y_1 < v_{rc} < y_J$. Then the global attractor is the boundary of the set*

$$M_c \cap \{(x_1, \cdots x_J) : v \leq v_{rc}\},$$

more precisely, the part of this boundary where either $v = v_{rc}$ or $(x_1, \cdots, x_J) \in \Gamma_c$.

The case with $J = 2$ and $v_{rh} < y_1 < v_{rc} < y_2$ is illustrated in Figure 2. The global attractor consists of the part of a line $v = v_{rc}$ joining the points $(0, v_{rc})$ and A and the part of the corrupted frontier joining the points A and $(y_1, 0)$.

The main conclusion remains the same: linear fine functions support infinite growth of illegal gains on violations and/or corruption.

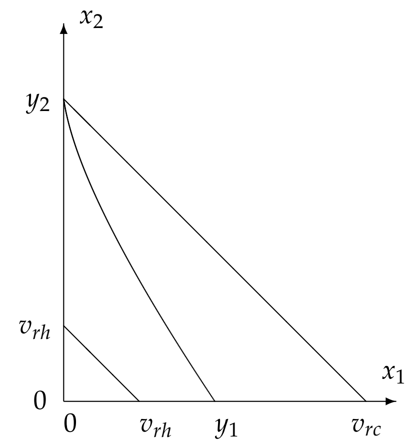

Figure 1. Corruption frontier for Theorem 2(iii).

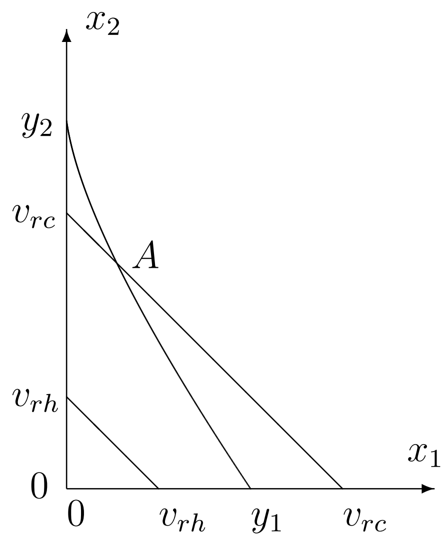

Figure 2. Global attractor for Theorem 3.

5. Power-Type Fines

Let us assume now that the fine functions are of power type:

$$f(r) = fr^\gamma, \quad F(r) = Fr^\gamma,$$

with some positive constants f, F, γ. We shall also reduce attention to the case $\gamma > 1$; that is, to the case of convex power functions.

For simplicity, we reduce the discussion to the case of one level of violation only: $J = 1$, $r_1 = r$, and $v = x_1$.

Remark 7. *As for the linear fines, the general case creates more complicated (quite curious in fact) geometry of the efficient frontier, but our major qualitative conclusion remains the same. The new feature of nonlinear fines for $J > 1$ is that the rest points never form continuous surfaces (like hyperplanes) and are concentrated on two strategies (with only two x_i non-vanishing). In a slightly different setting, this effect was discussed in [9].*

In this case, the corruption domain becomes

$$M_c = \{v \geq 0 : \alpha vr(1-v) - v^2 F\alpha^\gamma r^\gamma \geq v(w - w_0)\}, \tag{23}$$

so that it is determined by the inequality

$$v \le v_c = \frac{1}{1+F(\alpha r)^{\gamma-1}}\left(1 - \frac{w-w_0}{r\alpha}\right). \tag{24}$$

Therefore, the corruption frontier v_c decreases as r grows, unlike the linear case.

The kinetic equation in the corruption domain becomes (again in our usual approximation with 1 instead of $1-\epsilon$)

$$\dot{v}. = v(1-v)r[1 - p_0\alpha - \alpha\beta v - \epsilon f r^{\gamma-1}] \tag{25}$$

and outside the corruptive domain, it is

$$\dot{v} = v(1-v)r[1 - p_0 - fr^{\gamma-1}(p_0+\epsilon) - v\beta(1+fr^{\gamma-1})]. \tag{26}$$

The analogs of (9) now depend on r. We must consider the cases

$$\begin{aligned} &(i) \quad 1 - \alpha p_0 - \epsilon f r^{\gamma-1} < 0, \\ &(ii) \quad 0 < 1 - \alpha p_0 - \epsilon f r^{\gamma-1} < \alpha\beta, \\ &(iii) \quad 1 - \alpha p_0 - \epsilon f r^{\gamma-1} > \alpha\beta. \end{aligned} \tag{27}$$

for the corruption branch of the dynamics and the cases

$$\begin{aligned} &(i) \quad 1 - p_0 - fr^{\gamma-1}(p_0+\epsilon) < 0, \\ &(ii) \quad 0 < 1 - p_0 - fr^{\gamma-1}(p_0+\epsilon) < \beta(1+fr^{\gamma-1}), \\ &(iii) \quad 1 - p_0 - fr^{\gamma-1}(p_0+\epsilon) > \beta(1+fr^{\gamma-1}). \end{aligned} \tag{28}$$

for the non-corruption branch of the dynamics.

The corresponding analogs of Lemmas 1 and 2 are valid:

Lemma 3. *For the evolution (25) in the corrupted domain, the following holds true. In case (iii) of (27), the point X_1 is the only stable rest point. In case (ii), the points X_0 and X_J are both repulsive, but the rest point*

$$\frac{1-\alpha p_0 - \epsilon f r^{\gamma-1}}{\alpha\beta}$$

is the global attractor. In case (i), the rest point X_0 of honest behavior is the only stable point.

Lemma 4. *For the evolution (26) outside the corrupted domain, the following holds true. In case (iii) of (28), the point X_1 is the only stable rest point. In case (ii), the points X_0 and X_1 are both repulsive, but the rest point*

$$\frac{1-p_0 - fr^{\gamma-1}(p_0+\epsilon)}{\beta(1+fr^{\gamma-1})}$$

is the global attractor (outside the points X_j). In case (i), the rest point X_0 of honest behavior is the only stable point.

Turning to the full switching dynamics, we observe first of all that if

$$r > r_c = \left(\frac{1-\alpha p_0}{\epsilon f}\right)^{1/(\gamma-1)}, \tag{29}$$

then the no-violation rest point $v = 0$ is the unique stable point for the dynamics in corruption domain, and if

$$r > r_h = \left(\frac{1-p_0}{(\epsilon+p_0)f}\right)^{1/(\gamma-1)}, \tag{30}$$

then the no-violation rest point $v = 0$ is the unique stable point for the dynamics in the domain of honest inspectors. It is seen by inspection (as ought to be expected) that $r_c > r_h$. Therefore, r_c is an upper bound for the costs of violation. However, this bound is not universal, because it increases with $\epsilon \to 0$.

Next, for $r < r_c$ it follows that the point

$$v_{rc} = \min\left(1, \frac{1 - \alpha p_0 - \epsilon f r^{\gamma-1}}{\alpha\beta}\right) \tag{31}$$

is the unique stable fixed point for the dynamics (25), and for $r < r_h$, it follows that the point

$$v_{rh} = \min\left(1, \frac{1 - p_0 - f r^{\gamma-1}(p_0 + \epsilon)}{\beta(1 + f r^{\gamma-1})}\right) \tag{32}$$

is the unique stable fixed point for the dynamics (25).

Now we are in the same setting, as in Theorem 1, though with different parameters. Hence we can directly conclude that the following result holds.

Theorem 4. *Assume that $r < r_c$.*

(i) If $r > r_h$, then the unique stable rest point for the total switching dynamics is $\min(v_c, v_{rc})$.

(ii) If $r \le r_h$, then the results of Theorem 1 hold with v_{rc}, v_{rh}, v_c given by (31), (32), (24), respectively.

We can now derive our main conclusion. By (30) (and assuming $p_0 > 0$), the maximal illegal profit under no corruption of inspectors is universally bounded for any $\gamma > 1$, and this bound can be effectively controlled by the choice of f and p_0 (uniformly for all ϵ). The level of corruption cannot go beyond v_c given by (24), which decreases with the growth of illegal profit r for any $\gamma > 1$. However, the bound for the illegal profit rv_c still increases as long as $\gamma < 2$. Furthermore, we are led to the *principle of quadratic fines*: Starting from the power $\gamma = 2$, the illegal profit rv_c becomes universally bounded, and the bound can be effectively controlled by the choice of F.

Under all conditions of the example of Section 4.4, but in the case of a quadratic fine, we find that

$$v_c = \frac{1}{1 + Fr/4}\left(1 - \frac{20000}{r}\right),$$

so that the total loss to society from violation and corruption in the equilibrium point v_c will be

$$rv_c = \frac{4}{F + 4/r}\left(1 - \frac{20000}{r}\right) \le \frac{4}{F},$$

which can be effectively controlled by the choice of F.

6. Further Perspectives

We present our model in the simplest case in order to illustrate the main conclusions in the most transparent way. Let us point out to possible further developments.

Firstly, we considered the parameter α (the proportion of bribes) as an exogenous parameter (kind of "social norm for corruption"). It seems natural to develop second-level kinetic equations describing the evolutionary changes to the behavior of a large pool of inspectors as playing with the authority A (and assuming that the violators adhere to their equilibrium strategies). The various strategies of inspectors can be enriched from the two considered (Honest and Corrupt) to the various levels of corruption specified by various levels of α. I expect that the evolutionary dynamics will drive α to the highest possible levels.

Secondly, more detailed description of the switching kinetic equations for arbitrary J and an arbitrary convex fine functions can be of interest. This investigation can also include the continuous case of levels r taken from some finite interval $[r_{min}, r_{max}]$.

Thirdly, our analysis was based on the stable points of the limiting dynamics, which eventually specify the long-term behavior of this limiting dynamics. Of interest is the long-term behavior of the corresponding prelimiting Markov chain of M players. One can conjecture that, in the large time limit, it will converge to some stationary distribution, with support around the attractors of the kinetic equations.

Fourthly, we assumed that the central authority A makes just one random check in each local area. It is natural to extend the analysis to the case when the number of local checks is any small (compared to M) number.

Fifthly, our inspectors were supposed to act with the best response to instantaneous adjustments. It is, of course, natural to assume that they act strategically, with some planning horizon in mind. This would lead to a more complicated problem of coupled kinetic equations with the optimal policy of inspectors. A general theory of such a coupling was developed in Chapter 3 of [2] and may be applied in the present setting.

The author believes that the principle of quadratic fine is sufficiently robust and will manifest itself in all of these extensions.

Funding: The article was prepared in the framework of a research grant funded by the Ministry of Science and Higher Education of the Russian Federation (grant ID: 075-15-2020-928).

Conflicts of Interest: The author declares no conflict of interest.

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