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Vyacheslav V. Chistyakov

# From Approximate Variation to Pointwise Selection Principles

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Vyacheslav V. Chistyakov
Department of Informatics, Mathematics,
and Computer Sciences
National Research University Higher
School of Economics
Nizhny Novgorod, Russia

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*To Sveta, Dasha, and Vasilisa*

# Preface

This book addresses the minimization of special lower semicontinuous functionals over (closed) balls in metric spaces, called the approximate variation. The new notion of approximate variation contains more information about the bounded variation functional and has the following features: the infimum in the definition of approximate variation is not attained in general and the total Jordan variation of a function is obtained by a limiting procedure as a parameter tends to zero. By means of the approximate variation, we are able to characterize regulated functions in a generalized sense and provide powerful compactness tools in the topology of pointwise convergence, conventionally called pointwise selection principles.

The aim of this book is to present a thorough self-contained study of the approximate variation. We illustrate this new notion by a large number of appropriate examples designed specifically for this contribution. Moreover, we elaborate on the state-of-the-art pointwise selection principles applied to functions with values in metric spaces, normed spaces, reflexive Banach spaces, and Hilbert spaces. Although we study the minimization of only one functional (namely, the Jordan variation), the developed methods are of a quite general nature and give a perfect intuition of what properties the minimization procedure of lower semicontinuous functionals over metric balls may have. The content is accessible to students with some background in real and functional analysis, general topology, and measure theory.

The book contains new results that were not published previously in book form. Among them are properties of the approximate variation: semi-additivity, change of variable formula, subtle behavior with respect to uniformly and pointwise convergent sequences of functions, and the behavior on proper metric spaces. These properties are crucial for pointwise selection principles in which the key role is played by the limit superior of the approximate variation. Interestingly, pointwise selection principles may be regular, treating regulated limit functions, and irregular, treating highly irregular functions (e.g., Dirichlet-type functions), in which a significant role is played by Ramsey's Theorem from formal logic.

In order to present our approach in more detail, let $X$ be a metric space with metric $d_X$ (possibly taking infinite values) and $B_\varepsilon[x] = \{y \in X : d_X(x, y) \le \varepsilon\}$ be a closed ball in $X$ of radius $\varepsilon > 0$ centered at $x \in X$. Given a functional $V : X \to [0, \infty]$ on $X$, not identical to $\infty$, consider the following family of minimization problems:

$$V_\varepsilon(x) := \inf_{B_\varepsilon[x]} V \equiv \inf\{V(y) : y \in B_\varepsilon[x]\}, \quad \varepsilon > 0, \ \ x \in X. \tag{1}$$

In other words, if $\operatorname{dom} V = \{y \in X : V(y) < \infty\} \subset X$ is the effective domain of $V$, then the previous formula can be rewritten, for $\varepsilon > 0$ and $x \in X$, as

$$V_\varepsilon(x) = \inf\{V(y) : V(y) < \infty \text{ and } d_X(x, y) \le \varepsilon\} \quad (\inf \varnothing := \infty). \tag{2}$$

The family $\{V_\varepsilon\}_{\varepsilon>0}$ of functionals $V_\varepsilon : X \to [0, \infty]$, $\varepsilon > 0$, is said to be the *approximate family of* $V$. In this generality, the family has only few properties.

Clearly, for each $x \in X$, the function $\varepsilon \mapsto V_\varepsilon(x) : (0, \infty) \to [0, \infty]$ is *nonincreasing* on $(0, \infty)$, and so, we have inequalities (for one-sided limits)

$$V_{\varepsilon+0}(x) \le V_\varepsilon(x) \le V_{\varepsilon-0}(x) \ \text{ in } \ [0, \infty] \ \text{ for all } \ \varepsilon > 0$$

and the property (since $x \in B_\varepsilon[x]$ for all $\varepsilon > 0$)

$$\lim_{\varepsilon \to +0} V_\varepsilon(x) = \sup_{\varepsilon>0} V_\varepsilon(x) \le V(x) \ \text{ in } \ [0, \infty]. \tag{3}$$

(Note also that if $y \in X$ and $V(y) = 0$, then $V_\varepsilon(x) = 0$ for all $\varepsilon > 0$ and $x \in B_\varepsilon[y]$.)

In this book, in order to have more properties of the family $\{V_\varepsilon\}_{\varepsilon>0}$, we are going to consider the case when $X = M^T$ is the functional space of all functions $x \equiv f : T \to M$ mapping a subset $T \subset \mathbb{R}$ into a metric space $(M, d)$ and equipped with the extended-valued uniform metric $d_X = d_{\infty,T}$ given, for all $f, g \in M^T$, by $d_{\infty,T}(f, g) = \sup_{t\in T} d(f(t), g(t))$. The functional $V : X = M^T \to [0, \infty]$, to be minimized on metric balls in $X$, is the (Jordan) *variation* of $f \in M^T$ defined, as usual, by[1]

$$V(f) \equiv V(f, T) = \sup\left\{\sum_{i=1}^{m} d(f(t_i), f(t_{i-1})) : m \in \mathbb{N} \text{ and } t_0 \le t_1 \le t_2 \le \ldots \le t_m \text{ in } T\right\}.$$

Its effective domain $\operatorname{dom} V$ is the space $\mathrm{BV}(T; M)$ of functions of bounded variation, and it has a number of nice properties such as additivity and (sequential) lower semicontinuity, and a Helly-type pointwise selection principle holds in

[1] There are a number of interesting functionals on $M^T$, which can be studied along the "same" lines, e.g., $V(f) \equiv \operatorname{Lip}(f) = \sup\{d(f(s), f(t))/|s-t| : s, t \in T, \ s \ne t\}$ (the least Lipschitz constant of $f \in M^T$), or functionals of *generalized variation* in the sense of: Wiener–Young [19, 31, 32, 59], Riesz–Medvedev [14–18, 55, 65], Waterman [78, 79], and Schramm [26, 69].

BV$(T; M)$ (see (**V.1**)–(**V.4**) in Chap. 2). Due to this, the approximate family from Eqs. (1) and (2), called the *approximate variation*, is such that

$$\lim_{\varepsilon \to +0} V_\varepsilon(f) = V(f) \quad \text{for all} \quad f \in M^T,$$

and the family of all $f \in M^T$, for which $V_\varepsilon(f)$ is finite for all $\varepsilon > 0$, is exactly the space of all *regulated* functions on $T$ (at least for closed intervals $T = [a, b]$).

Since property (3) holds for the approximate variation, we are able to obtain powerful compactness theorems in the topology of pointwise convergence on $M^T$ generalizing Helly-type pointwise selection principles. As an example, the following holds: *if a sequence of functions* $\{f_j\}_{j=1}^\infty$ *from* $M^T$ *is such that the closure in* $M$ *of the set* $\{f_j(t) : j \in \mathbb{N}\}$ *is compact for all* $t \in T$ *and* $\limsup_{j\to\infty} V_\varepsilon(f_j)$ *is finite for all* $\varepsilon > 0$, *then* $\{f_j\}_{j=1}^\infty$ *contains a subsequence, which converges pointwise on* $T$ *to a bounded regulated function from* $M^T$ (cf. Theorem 4.1).

Additional details on the aim of this book are presented in Chap. 1.

The plan of the exposition is as follows. In Chap. 1, we review a number of well-known pointwise selection theorems in various contexts and note explicitly that there is a certain relationship between such theorems and characterizations of regulated functions. In Chap. 2, we define the notion of approximate variation, study its properties, and show that its behavior is different on proper and improper underlying metric spaces. Chap. 3 is devoted to elaborating a large number of examples of approximate variations for functions with values in metric and normed spaces. Finally, in Chap. 4 we present our main results concerning regular and irregular pointwise selection principles illustrated by appropriate examples.

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Nizhny Novgorod, Russian Federation Vyacheslav V. Chistyakov
July 2021

# Contents

# Acronyms

| | |
|---|---|
| $\mathbb{R} = (-\infty, \infty)$ | Set of all real numbers; $\infty$ means $+\infty$ |
| $[a, b]$ | Closed interval of real numbers |
| $(a, b)$ | Open interval of real numbers |
| $(0, \infty)$ | Set of all positive real numbers |
| $[0, \infty) = \mathbb{R}^+$ | Set of all nonnegative real numbers |
| $[0, \infty]$ | Set of extended nonnegative numbers $[0, \infty) \cup \{\infty\}$ |
| $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{C}$ | Sets of all natural, integer, rational, and complex numbers |
| $\varnothing$ | Empty set; $\inf \varnothing = \infty$, $\sup \varnothing = 0$ |
| $\equiv$ | Identical to; equality by the definition |

Sets of functions from $T \subset \mathbb{R}$ into a metric space $(M, d)$:

| | |
|---|---|
| $M^T$ | Set of all functions $f : T \to M$ (p. 7) |
| $\mathrm{B}(T; M)$ | Set of bounded functions (p. 7) |
| $\mathrm{BV}(T; M)$ | Set of functions of bounded Jordan variation (p. 8) |
| $\mathrm{Reg}(T; M)$ | Set of regulated functions (p. 9 and p. 45) |
| $\mathrm{St}(I; M)$ | Set of step functions (p. 9) |
| $\mathrm{Mon}(T; \mathbb{R})$ | Set of bounded nondecreasing functions $f : T \to \mathbb{R}$ (p. 45) |