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SURFACE LAMINATIONS AND CHAOTIC DYNAMICAL SYSTEMS

Institute of Computer Science

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Contents

To Olga, Nataliya, Pavel, Liza, Matvei, Masha and Ilya

and

to our wives Tatyana and Tatyana

for persistent support and patient tolerance

Preface

Beginning with Sir Isaac Newton, Differential Equations have many applications in natural science and engineering. Mainly, mathematical models of this applications lead to nonlinear differential equations that required a qualitative approach to find solutions. After Henry Poincaré, nonlinear part of Differential Equations led to the branch called Dynamical Systems with rich mathematical subject.

This book concerns to some aspects of **Qualitative Theory of Dynamical Systems** originated by Henry Poincaré and Aleksander Lyapunov, and developed by Aleksander Andronov, Ivar Bendixon and George Birkhoff. Beginning with works by Marston Morse, Qualitative Theory of Dynamical Systems was enriched by methods of Topology and Geometry. Therefore, this theory is now often called **Geometric Theory of Dynamical Systems**.

The modern Geometric Theory of Dynamical Systems was based by Steve Smale, Dmitrii Anosov, Jakov Sinai, and his numerous progeny. It is not impossible to imagine the modern Theory of Dynamical Systems without the hyperbolic ideology when a most interesting dynamics endowed with some types of hyperbolicity. A hyperbolicity implies the existence of smooth or continuous families of invariant manifolds which often looks locally like a family of straight lines or parallel hyperplanes. For such a family, we refer to as a **local lamination**. This notion includes foliations and laminations. To be precise, a hyperbolicity implies the existence of a two-web consisting of local laminations. Clearly that the study of dynamical systems is intimately connected with the study of local laminations.

So, we consider dynamical systems with additional lamination structures. Therefore, the big part of the book concerns to Theory of Local Laminations. In our opinion, an acquaintance with Theory of Dynamical Systems and Local Laminations (including, Foliations) must begin with low dimensional spaces when a dimension of support space does not exceed 2. In this case, the Reader get the complete and maximal presentation on goals, methods, and deep results of this Theories. The another reason to begin with low dimensional cases is a minimal knowledge that allows to young researchers to make first steps independently.

This book is meant to be a graduate or undergraduate textbook and partly a monograph on the subject. The book is virtually divided into three parts. First part are Chapters 1 and 2. This part is intended to be an introduction in geometric aspects of dynamical systems. Second part are Chapters 3, 4, and 5. Here, we focus on surface local laminations. At last, in the third part which are Chapter 6, Chapter 7, Chapter 8, and Chapter 9 studies surface dynamical systems (flows and diffeomorphisms) with invariant one-dimensional local laminations, and 2-webs, and one-dimensional basic sets.

Each chapter is ended by Bibliographic Notes and Panoramas where we give some historical remarks, overviews, and results (in the frame of our knowledge).

The authors wrote series of surveys [16,17,19,20,31,32,90,97,172] some of them were written together with Samuil Aranson and Dmitrii Anosov who are the moral coauthors of this book. Anosov unfortunately already dead in 2014. He was advisor and friend, and we miss him. Our common teacher, friend and often co-authors Samuil Aranson did much for us, both in science and in life.

The book summarizes the many years researches by the authors in Theory of Dynamical Systems. Both authors belong to the famous scientific school founded by Aleksandr Andronov in Nizhny Novgorod. As students, authors began to study the Theory of Dynamical Systems in Department of Differential Equations, which was headed by Evgenia Alexandrovna Leonovich-Andronova (wife of Andronov).

It is our pleasure to thank people who helped us in many ways. Among those are G. Belitskii, M. Brin, E. Gurevich, M. Malkin, N. G. Markley, S. Matsumoto, V. Medvedev, Ya. Pesin, Ya. Sinai, A. Zorich.

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Chapter 1

Foundations

Here we introduce the first notation concerning Dynamical Systems. Any dynamical system is defined on a space that often is a manifold. Therefore, in Section 1.1, we introduce briefly manifolds, mainly low-dimensional. A good introduction to low-dimensional Topology are the books [53, 210, 211]. In Section 1.2, we give the definition of a dynamical system with continuous and discrete time. Since the general definition was induced by properties of solutions of ordinary differential equations, we recommend to get acquainted with some principles of Ordinary Differential Equations. Good sources are [119, 151]. In Section 1.3, we introduce the basic notation of Dynamical Systems that traditionally attributes to Topological Dynamics.

The notion of hyperbolicity is essential in modern Theory of Dynamical Systems. In Section 1.4, we recall some results on linear maps and introduce canonical hyperbolic fixed points. In Section 1.5, one formulates Grobman–Hartman Theorem that is a keystone of Hyperbolic Dynamics. In Section 1.6, we present classical example of Smale Horseshoe and introduce the notion of a hyperbolic set. At last, in Section 1.7, one considers basic results concerning hyperbolic sets.

1.1. Manifolds

Roughly speaking, a topological n-manifold is a second-countable Hausdorff topological space M^n locally homeomorphic to Euclidean space $\mathbb{R}^n$ or half-space $\mathbb{R}^n_+$. The points corresponding to the boundary of $\mathbb{R}^n_+$ form the boundary ∂M^n of M^n. If M^n is compact and $\partial M^n = \varnothing$, M^n is a *closed manifold*. To consider differentiable dynamical systems one needs differentiable manifolds M^n. At first time, it suffices to think of a differentiable manifold M^n as an n-dimensional differentiable surface or submanifold in $\mathbb{R}^N$, where $N > n$ (see John Lee's *Introduction to Topological manifolds*). The most part of the book concerns to 1-manifolds and 2-manifolds (surfaces). Therefore, let us give more details for such manifolds.

One-dimensional and two-dimensional manifolds

1-manifolds are intervals (compact or open) of the real line $\mathbb{R}$. A unique closed 1-manifold is a circle S^1. Mainly, we consider the unit circle $S^1 = [0; 1]/(0 \sim 1)$, where $0 \sim 1$ indicates that 0 and 1 are identified. We'll also consider S^1 as $S^1 = \left\{z \in \mathbb{C}\colon |z| = \frac{1}{2\pi}\right\}$ in complex plane $\mathbb{C}$, so-called multiplicative circle. The two notations are related by $z = \frac{1}{2\pi}e^{2\pi ix}$, $x \in [0; 1]/(0 \sim 1)$. The first representation endowed S^1 with the natural cyclic coordinate.

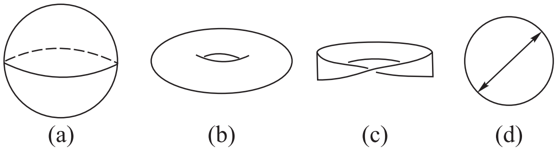


Figure 1.1. 2-sphere (a); torus (b); Möbius band (c); projective plane (d).

Surface is a 2-manifold. At first, one can think of surfaces in a primitive manner. Fig. 1.1 is a description of two oriented (two-sided) surfaces, a sphere and torus, and two non-oriented (one-sided) surfaces, a Möbius band and projective plane (on the right, each pair of antipodal points of the circle are supposed to be identified). Note that the torus (Fig. 1.1 (b)) can be considered as the product $S^1 \times S^1$.

The simplest example of a surface is Euclidean plane $\mathbb{R}^2$. The next simple example is a *plane domain* or 2-*sphere with n holes* that obtained by deleting from 2-sphere n (possibly, $n = 0$) disjoint disks. If the removing disks are open we get a compact plane domain. If the removing disks are closed, one obtains an open (noncompact) plane domain. In the last case some disks are accepted to be points. This points and the boundaries of removing closed disks one considers as an *ideal boundary*.

By a *handle* we mean a surface obtained by deleting from a torus the interior of a 2-cell, Fig. 1.2 (a). If $0 \leqslant p \leqslant n$ handles are attached to the boundary of the 2-sphere with n holes, the resulting surface is an *oriented surface of genus p with $b = n - p$ holes* denoted by $M^2_{p,b}$, Fig. 1.2 (b). Note that historically, a genus was introduced as a "measure of connectivity" defined

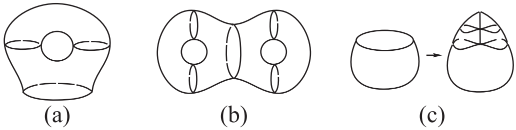


Figure 1.2. A handle (a); the surface M_2^2 of genus 2 (b); an attaching a Möbius band (c).

by the maximum number of disjoint closed curves which can be drawn on the surface without separating.

A *non-oriented surface of genus* p *with* $b = n - p$ *holes* or a *sphere with* p *cross-caps and* $b = n - p$ *holes* is a surface obtained by starting with a 2-sphere with n holes and then attaching a Möbius band to the boundary of each of p holes, Fig. 1.2 (c). Such a surface is denoted by $N_{p,b}^2$. Clearly, $M_{p,0}^2 = M_p^2$ and $N_{p,0}^2 = N_p^2$ are closed surfaces. Note that any non-orientable surface has genus $\geqslant 1$.

Non-branched covering projections

One of the methods to construct dynamical systems starting with simple models is their lifting on covering surfaces or projecting from covering surfaces. The advantage is that sometimes a covering surface, for example, universal covering surface, can be endowed with simple coordinate charts, and therefore a dynamical system can be described directly. We begin with non-branched covering projections.

A map $p\colon X \to Y$ is called a (non-branched) *covering projection* or *covering map* (and X is called a *covering space* of Y) if each point $y \in Y$ has a neighborhood U such that $p^{-1}(U)$ is a nonempty disjoint union of sets U_α (sheets, which are components of $p^{-1}(U)$) on which $p|_{U_\alpha}$ is a homeomorphism $U_\alpha \to U$. We always will assume that p is surjective. If the cardinality $|p^{-1}(m)|$ of $p^{-1}(m)$ does not depend on a point $m \in Y$ and is finite, one says that p is a $|p^{-1}(m)|$*-sheeted covering projection* or $|p^{-1}(m)|$*-sheeted covering map*. If the space X is simply connected, p is called *universal* and X is a *universal covering space*. The simplest example of universal covering projection is

$$\mathbb{R} \to S^1 = [0;\,1]/(0 \sim 1) \quad \text{where } x \mapsto x \bmod 1.$$

This example produces the universal covering projection

$$\pi\colon \mathbb{R} \times \mathbb{R} \to S^1 \times S^1, \quad \pi(x,\, y) = (x \bmod 1,\; y \bmod 1) \tag{1.1}$$

of $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ onto the torus $\mathbb{T}^2 = S^1 \times S^1$.

For 2-manifolds X and Y, we can assume that the neighborhood U above is homeomorphic to an open disk or $\mathbb{R}^2$. Well known that every surface possesses universal covering surfaces, see, for example [210].

Let M^2 be a surface and $\overline{M}^2$ its covering surface with the covering projection $\pi\colon \overline{M}^2 \to M^2$. A *deck transformation* is a homeomorphism $\gamma\colon \overline{M}^2 \to \overline{M}^2$ such that $\pi \circ \gamma = \pi$. All such mappings γ form a group called the *covering group* and denoted by Γ_π or $\Gamma(M^2)$. Two points of $\overline{M}^2$ are *congruent* if they belong to the same orbit under the covering group. Certainly, two congruent points have the same projection on M^2. If conversely, any two points lying over the same point of M^2 are congruent, the covering group is said to be *transitive*. Later on, ***we consider only transitive covering groups***. Any deck transformation of non-branched covering surface has no fixed points, except the identity mapping.

Let X be a topological space and G be a group of transformations of X. An *action* of G on X is a map $G \times X \to X$, where the image of (f, x) will be denoted by $f(x)$, such that $(fh)(x) = f(h(x))$, $x \in X$, $f, h \in G$. The orbit of x is the set $O(x) = \{f(x) \mid f \in G\}$. It is easy to see that two orbits are either disjoint or identical. The space of orbits is denoted by X/G, with the quotient topology from the map $X \to X/G$ taking x to $O(x)$, and is called the *quotient space* or *orbit space*. An action of G on X is said to be *properly discontinuous* if for any point $x \in X$ the fact that point x has a neighborhood U such that $f(U) \cap U \neq \varnothing$ implies $f = e$, the identity map [53], III.7.1. Given a covering map $\overline{M}^2 \to M^2$, if the covering group Γ is properly discontinuous then the quotient space $\overline{M}^2/\Gamma$ is homeomorphic to M^2. Studying quotient spaces one usually finds a convenient *fundamental domain*, the closure of set containing a unique point of each Γ-orbit. Let us consider some examples.

Example 1.1 *2-sheeted (non-branched) covering projection* $S^2 \to P\mathbb{R}^2$.

Consider a 2-sphere S^2 embedded in $\mathbb{R}^3$ such that S^2 is invariant under the central symmetry $\mathscr{S}$ with respect to the origin $O \in \mathbb{R}^3$, Fig. 1.3. The symmetry $\mathscr{S}$ is an involution. Hence, the group generated by $\mathscr{S}$ is $\mathbb{Z}_2$. One can take the fundamental domain to be a half-sphere, say S^2_+. $\mathscr{S}$ identifies each pair of antipodal points of ∂S^2_+. Therefore, the quotient space $S^2/\mathbb{Z}_2$ is a projective plane $P\mathbb{R}^2$, and the projection $S^2 \to S^2/\mathbb{Z}_2$ is a 2-sheeted (non-branched) covering projection $S^2 \to P\mathbb{R}^2$. $\diamondsuit$

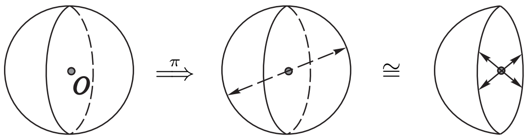


Figure 1.3. The 2-sheeted (non-branched) cover $S^2 \to P\mathbb{R}^2$.

Example 1.2 *The universal covering space of torus T^2 and Klein bottle K^2, and 2-sheeted (non-branched) covering projection $T^2 \to K^2$.*

Take the Euclidean plane $\mathbb{R}^2$ endowed with the standard coordinates x, y. For the torus T^2, one can take as Γ the group $\mathbb{Z}^2$ of integer translations S_{nk}:

$$(x,\, y) \to (x+n,\, y+k), \quad n,\, k \in \mathbb{Z}. \tag{1.2}$$

For the Klein bottle K^2, the group Γ consists of transformations

$$(x,\, y) \to \left(x + \frac{n}{2},\, (-1)^n y + k\right), \quad n,\, k \in \mathbb{Z}. \tag{1.3}$$

Let us check that the quotient space $\mathbb{R}^2/\Gamma$ is a corresponding surface. Since the group $\mathbb{Z}^2$ consists of integer translations S_{nk}, the square $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$ can be taken to be the fundamental domain of $\mathbb{Z}^2$ that identifies the opposite sides of this square. Hence, $\mathbb{R}^2/\mathbb{Z}^2$ is T^2, and the natural projection

$$\pi\colon \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2 = T^2$$

is a universal covering projection defined by (1.1) with the group of deck transformations $\Gamma(T^2) = \mathbb{Z}^2$, Fig. 1.4.

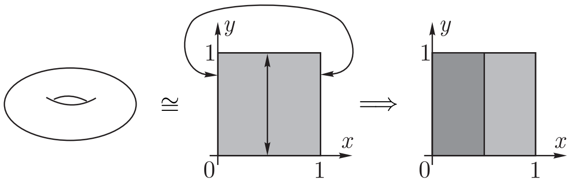


Figure 1.4. The 2-sheeted (non-branched) cover $T^2 \to K^2$.

Denote by Γ' the group consisting of transformations (1.3). Similarly, one can check that the quadrangle $0 \leqslant x \leqslant \frac{1}{2}$, $0 \leqslant y \leqslant 1$ is fundamental domain of Γ', and the quotient space $\mathbb{R}^2/\Gamma'$ is K^2. Thus, $\Gamma' = \Gamma(K^2)$ is a group of deck transformations, and $\mathbb{R}^2 \to \mathbb{R}^2/\Gamma' = K^2$ is a universal covering projection.

Since the double iteration of the map $(x, y) \to \left(x + \frac{1}{2}, -y\right)$ gives the map $(x, y) \to (x+1, y)$, $\mathbb{Z}^2$ is a subgroup of $\Gamma(K^2)$ of index 2, $\Gamma(K^2)/\mathbb{Z}^2 = \mathbb{Z}_2$. This implies that

$$T^2 = \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\Gamma(K^2) = K^2$$

is a 2-sheeted (non-branched) covering projection $T^2 \to K^2$. Geometrically, the square $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$ covers the quadrangle $0 \leqslant x \leqslant \frac{1}{2}$, $0 \leqslant y \leqslant 1$ and quadrangle that is congruent to it. ◇

Example 1.2 is a particular case of the following well known fact that any non-orientable surface N^2 has two-sheeted (non-branched) covering orientable surface $\widehat{M}$ [210].

Proposition 1.1 *Given any non-orientable surface N_p^2 of genus p, there is a 2-sheeted (non-branched) covering projection $M_{p-1}^2 \to N_p^2$, where M_{p-1}^2 is an oriented surface of genus $p-1$.*

This result is generalized by the following statement.

Lemma 1.1 *Given any non-orientable closed n-dimensional manifold M^n, there are an oriented closed n-dimensional manifold $\widetilde{M}^n$ and 2-sheeted (non-branched) covering projection $\widetilde{M}^n \to M^n$, where $n \geqslant 2$.*

Proof can be found in “topological” books, exm., [209, 212].

Fundamental groups

A fundamental group was introduced by Poincaré [193]. Suppose X is a topological space and $x_0 \in X$ is a choice of base point. Roughly speaking, the fundamental group $\pi_1(X, x_0)$ is formed by homotopy classes of loops starting and finishing at x_0. A continuous mapping $f\colon X \to Y$ induces the homomorphism $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$. The fundamental group $\pi_1(X, x_0)$ is a topological invariant, i. e., if $f\colon X \to Y$ is a homeomorphism, then the groups $\pi_1(X, x_0)$ and $\pi_1(Y, f(x_0))$ are isomorphic. In particular, if $X = Y$, f_* is an automorphism. For a surface $X = M_p^2$, every automorphism of $\pi_1(M_p^2, x_0)$ is induced by some homeomorphism $(M_p^2, x_0) \to (M_p^2, x_0)$.

Proposition 1.2 *Let $\phi\colon \pi_1(M_p^2, x_0) \to \pi_1(M_p^2, x_0)$ be an automorphism. Then there is a homeomorphism $f\colon M_p^2 \to M_p^2$ defined uniquely up to homotopy such that $f_* = \phi$.*

1.2. Definition of a Dynamical System

Theory of Dynamical Systems studies the long-time behavior of evolving of different type systems. Mainly one considers a discrete time and continuous time. Respectively, one considers a discrete-time dynamical system and continuous-time dynamical system.

Continuous-time dynamical systems

The motivation for the definition of dynamical systems (DS) with continuous time are autonomous differential equations on closed manifolds. Recall that an n-manifold M^n is locally Euclidean space $\mathbb{R}^n$ endowed with coordinates $(x_1, \ldots, x_n)$. The autonomous system of first order differential equations is

$$\dot{x}_i = F_i(x_1, \ldots, x_n), \quad i = 1, \ldots, n, \tag{1.4}$$

where $\dot{x}$ denotes the derivative of a function x with respect to a single variable t that often traditionally is called a *time*. If the functions F_i are smooth enough, the existence and uniqueness theorem for ordinary differential equations guarantees that given any $p_0 = (x_1^0, \ldots, x_n^0) \in M^n$ and $t_0 \in \mathbb{R}$, there are functions $(\varphi_1(t), \ldots, \varphi_n(t))$ such that

$$\varphi_i(t_0) = x_i^0 \text{ and } \left.\frac{d}{dt}\varphi_i(t)\right|_{t=t_0} \equiv F_i\left(\varphi_1(t), \ldots, \varphi_n(t)\right), \quad i = 1, \ldots, n \; \forall t.$$

The vector-function $f(t) = (\varphi_1(t), \ldots, \varphi_n(t))\colon \mathbb{R} \to M^n$ is called a *solution* satisfying the initial condition (p_0, t_0). The graph of the function $f(t)$ is the curve

$$\widehat{l} = \{(t, f(t))\colon -\infty < t < +\infty\} \subset \mathbb{R}^{n+1}.$$

Let $P\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the natural projection $(t, x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n)$.

Since the right hand of (1.4) does not depend on t, it follows that $f(t+C)$ is also a solution of (1.4) for any constant $C \in \mathbb{R}$. Therefore, $\widehat{l}$ is projected by P onto the curve $l = P(\widehat{l})$ that is either a simple closed curve or non-closed curve with no self-intersections, Fig. 1.5 (a). This curve l is called *integral*. An integral curve endowed with the (positive) direction corresponding to the increasing t is called a *trajectory*.

Let $f(\cdot, t)\colon M^n \to M^n$ be a shift along integral curves by t: $p_0 \mapsto f_0(t)$, where f_0 is a solution satisfying the initial condition (p_0, t_0). Clearly that the consecutive shifts by t_1 and t_2 is equivalent to the shift by $t_1 + t_2$. The theorem

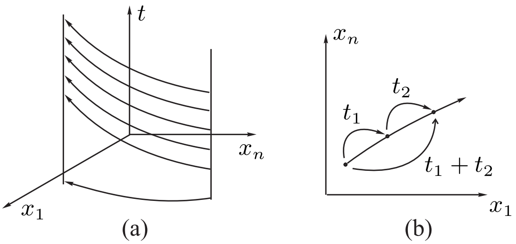


Figure 1.5. The curves $\widehat{l}$ and $l = P(\widehat{l})$.

of continuous dependence of solutions on initial conditions implies that shifts satisfy to the following properties:

- $f(\cdot, t)\colon M^n \to M^n$ is a diffeomorphism for any $t \in \mathbb{R}$;
- $f(p, t+s) = f(f(p, t), s)$ for any $p \in M^n$ and all $s, t \in \mathbb{R}$, Fig. 1.5 (b);
- $$\frac{d}{dt} f_i(p, t)\bigg|_{t=0} = F_i(p) \quad \text{for each } p \in M^n, \quad i = 1, \ldots, n.$$

This properties motivate the general definition of a dynamical system with continuous time (sometimes, one says *continuous dynamical system*). Often, DS with continuous time is called shortly a *flow*. Recall that a map $f\colon M^n \to M^n$ is a C^r, $r \geqslant 1$, diffeomorphism if f is invertible and the both f and f^{-1} have the continuous derivatives of the orders $1, \ldots, r$. C^0 diffeomorphism means a homeomorphism. When the dimension of a manifold M^n is not important we shall denote M^n by M.

A C^r *flow* (or a *continuous-time C^r dynamical system*) is a one-parameter family $\{f^t\}$ of C^r diffeomorphisms $f^t\colon M^n \to M^n$, where $t \in \mathbb{R}$ is a parameter, $r \geqslant 0$, such that

1) $f^{t_1+t_2} = f^{t_1} \circ f^{t_2}$; $t_1, t_2 \in \mathbb{R}$,

2) $f^0 = id$.

With no confusing, we often denote a flow by f^t.

The *positive* (*negative*) *semi-trajectory* of the point $p \in M^n$ is the set

$$l^+(p) = \{f^t(m)\colon t \geqslant 0\} \quad (\text{resp., } l^-(m) = \{f^t(m)\colon t \leqslant 0\}).$$

The set $l(p) = l^+(p) \cup l^-(p)$ is called a *trajectory* through the point p.

The trajectory $l(p) = p$ that is a unique point is called a *fixed point* or *singularity*. If $p \in M^n$ is not a fixed point then it is called a *regular point* and $l(p)$ will be called a *one-dimensional trajectory*. If $l(p)$ is homeomorphic to S^1, then $l(p)$ is a *closed trajectory* or *periodic trajectory*. A trajectory $l(p)$ is *non-closed* if $l(p)$ is neither a fixed point nor a periodic trajectory.

Example 1.3 *Constant parallel flow on* $\mathbb{R}^n$.

Let $\vec{V}_0$ be a nonzero vector in $\mathbb{R}^n$. Then $f^t(p) = p_t$, where p_t is the end of vector $\vec{Op} + t\vec{V}_0$ different from the original O, $p \in \mathbb{R}^n$, defines an analytic *constant parallel flow* on $\mathbb{R}^n$, Fig. 1.6, (b). Every trajectory is a straight line parallel to the vector $\vec{V}_0$. $\diamondsuit$

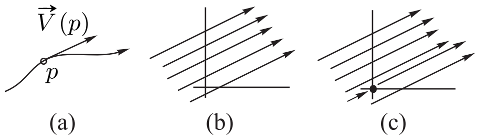


Figure 1.6. Vector field $\vec{V}$ (a), constant parallel flow (b), constant parallel flow with a fake saddle.

Example 1.4 *Constant flow on a torus.*

We represent the two-dimensional torus T^2 as the product $S^1 \times S^1$ of two unit circles $S^1 = [0; 1]/(0 \sim 1)$. Fix two numbers $\alpha \neq 0$ and β. Let f^t be the composition of the rotation of T^2 along the meridian $\{\cdot\} \times S^1$ through the angle βt and the rotation along the parallel $S^1 \times \{\cdot\}$ through the angle αt. To be precise,

$$f^t(x, y) = (x + \alpha t \bmod 1, \; y + \beta t \bmod 1).$$

The family $\{f^t\}$, $t \in \mathbb{R}$, forms an analytic *constant* flow denoted by $f^t_{\alpha,\beta}$. $\diamondsuit$

Let p be a regular point of flow f^t and $U(p)$ be a neighborhood of p. Given any trajectory l, a connected component of $l \cap U(p)$ (if non-empty) is called *trajectory arc* in $U(p)$.

The following Theorem describes the local structure of a flow in a neighborhood of regular point (the proof can be found in series of monographs on Differential Equations, see for example [44], ch. 2).

Theorem 1.1 *Let p be a regular point of C^r flow f^t generated by C^r vector field on a manifold M^n. Then there are a neighborhood $U(p)$ of p, and a C^r diffeomorphism $\psi\colon U(p) \to \mathbb{R}^n$, and a constant parallel flow f^t_0 such that ψ takes every trajectory arc in $U(p)$ to a trajectory of f^t_0, Fig. 1.7.*

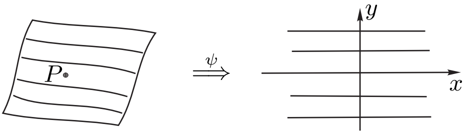


Figure 1.7. Rectifying diffeomorphism.

Keeping the notation of theorem 1.1, the neighborhood $U(p)$ is called a *neighborhood with the structure of constant flow*, and the diffeomorphism ψ is called a *rectifying diffeomorphism*.

Theorem 1.1 gives a complete description of topological structure of flow in a neighborhood of a regular point. Actually, it says that this structure is trivial.

Vector fields and ordinary differential equations

Given a point $p \in M^n$, $T_p(M^n)$ denotes the vector space of all tangent vectors to M^n at p. The union

$$\bigcup_{p\in M^n} T_p(M^n) = \{(p, \vec{v})\colon \vec{v} \in T_p(M^n)\} \stackrel{\text{def}}{=} T(M^n)$$

is a tangent space of M^n, a smooth $2n$-dimensional manifold. It is well-known that the natural projection $\xi\colon T(M^n) \to M^n$ is a tangent bundle with the fiber $\mathbb{R}^n$. A *vector field* $\vec{V}$ on M^n is a cross section of ξ, that is, a map $\vec{V}\colon M^n \to T(M^n)$ such that $\xi \circ \vec{V} = id$. C^r *smoothness* of $\vec{V}$ means that $\vec{V}\colon M^n \to T(M^n)$ is C^r smooth.

Each point $p \in M^n$ is covered by a local chart $\mathscr{U}$ in which a coordinate system is given by a mapping $(x_1, \ldots, x_n)\colon \mathscr{U} \to \mathbb{R}^n$. To define a vector field $\vec{V}$ in $\mathscr{U}$ we must specify n functions $v_1, \ldots, v_n\colon \mathscr{U} \to \mathbb{R}$, the components of $\vec{V}$ in $\mathscr{U}$. So, the field $\vec{V}$ broken up into its coordinates is just a set of n functions. This field defines at each local chart $\mathscr{U}$ the system of differential equations:

$$\dot{x}_1 = v_1(x_1, \ldots, x_n), \quad \ldots, \quad \dot{x}_n = v_n(x_1, \ldots, x_n).$$

The smoothness of $\vec{V}$ equals the ones of $v_1, \ldots, v_n$.

If this functions are C^1, the Theorem of Existence and Uniqueness of Solutions implies the existence of (local) flow in some neighborhood of p [168, 198].

Example 1.5 *Constant flow with a fake saddle on* $\mathbb{R}^n$.

Let $\vec{V}_0 = (\mu_1, \ldots, \mu_n)$ be a nonzero vector in $\mathbb{R}^n$. The system of differential equations $\dot{x}_1 = \mu_1, \ldots, \dot{x}_n = \mu_n$ defines a constant parallel flow denoted by $\overline{f}^t_\mu$ on $\mathbb{R}^n$, see Example 1.3. The system of differential equations

$$\dot{x}_1 = \mu_1(x_1^2 + \cdots + x_n^2), \quad \ldots, \quad \dot{x}_n = \mu_n(x_1^2 + \cdots + x_n^2)$$

defines the flow denoted by $\overline{f}^t_{\mu,0}$ with the fixed point at $(0; \ldots; 0)$ that is called a *passable* fixed point or a *passable singularity*. One says that $\overline{f}^t_{\mu,0}$ is obtained from $\overline{f}^t_\mu$ by *putting* a passable fixed point at $(0; \ldots; 0)$.

For $n = 2$, $(0; \ldots; 0) = (0; 0)$ is often called a *fake saddle*, Fig. 1.6, (c). $\diamondsuit$

For a closed M^n, a C^r vector field defines C^r flow. A vector field is usually given as a system of differential equations in local charts. Conversely, a C^r flow f^t defines at each point $p \in M^n$ the tangent vector

$$\vec{V}(p) = \frac{d}{dt} f^t(p)\bigg|_{t=0} \in T_p(m)$$

to the trajectory passing through p. Here, the components are

$$v_1(q) = \frac{d}{dt} x_1\left(f^t(q)\right)\bigg|_{t=0}, \quad \ldots, \quad v_n(q) = \frac{d}{dt} x_n\left(f^t(q)\right)\bigg|_{t=0},$$
$$q(x_1, \ldots, x_n) = q \in \mathscr{U}.$$

The correspondence $p \to \vec{V}(p)$, where p runs the whole M^n, is a C^{r-1} *vector field induced by the flow* f^t, see Fig. 1.6, (a).

Discrete-time dynamical systems

Let $f\colon M^n \to M^n$ be a C^r diffeomorphism, $r \geqslant 0$. The iterations f^k of f satisfy the obvious conditions $f^{k_1+k_2} = f^{k_1} \circ f^{k_2}$, $f^0 = id$, that is particular one defining a continuous-time dynamical system. It is natural the family $\{f^k\}_{k\in\mathbb{Z}}$ call a *discrete-time dynamical system* or *cascade*. We even simply call f itself a discrete-time dynamical system.

Let f^t be a C^r flow on M^n. For each real number t_0, the iterations $(f^{t_0})^k$ of the *time-*t_0 *map* $f(\cdot, t_0)$ form a discrete-time dynamical system. In particular, the diffeomorphism $f(\cdot, 1)$ is called the *time-one map*.

The set $\{f^k(x)\}_{-\infty}^{\infty}$ is called the *orbit* of $x \in M^n$ under f, denoted by $O(x, f)$, or simply $O(x)$, or $\mathrm{Orb}(x)$. The *forward orbit* $O^+(x)$ is the set $\{f^k(x)\colon k \geqslant 0\}$. The *backward orbit* $O^-(x)$ is the set $\{f^k(x)\colon k \leqslant 0\}$. Sometimes, we call a forward (backward) orbit a positive (negative) semi-orbit. We call x a *periodic point* if there is $q \in \mathbb{N}$ such that $f^q(x) = x$. The such minimal positive q is called a *period*. Denote the set of periodic points by $\mathrm{Per}(f)$. The orbit $O(x)$ of periodic point consists of finitely many points and is called a *periodic orbit*. A periodic point of period 1 is a *fixed point*. We denote the set of fixed points of f by $\mathrm{Fix}(f)$.

Example 1.6 *Rotations of the circle. Represent S^1 as factor space $\mathbb{R}/G$, where G is the group of motions $\mathbb{R} \to \mathbb{R}$ such that each element $\gamma \in G$ is given by formula $\gamma(x) = x + m_\gamma$, where $m_\gamma \in \mathbb{Z}$. Denote by π natural projection $\mathbb{R} \to S^1$. Then natural distance on $[0; 1]$ induces a distance d on S^1 by formula*

$$d(s_1, s_2) = \min(|x_1 - x_2|, 1 - |x_1 - x_2|),$$

where $s_1, s_2 \in S^1$ and $x_1, x_2 \in [0, 1]$, $\pi(x_1) = s_1$, $\pi(x_2) = s_2$.

For $\alpha \in \mathbb{R}$, let R_α be the rotation of S^1 by value α i. e.,

$$R_\alpha(x) = x + \alpha \bmod 1. \tag{1.5}$$

For the multiplicative circle, R_α is the rotation by angle $2\pi\alpha$, $R_\alpha(z) = z \exp 2\pi\alpha$. $\diamond$

Proposition 1.3 *Let $R_\alpha\colon S^1 \to S^1$ be a circle rotation defined by* (1.5). *Then* 1) *if α is rational, every orbit of R_α is periodic*; 2) *if α is irrational, every orbit of R_α is dense.*

Proof. If $\alpha = \frac{p}{q}$ is rational then $R_\alpha^q = Id$. Hence, every point is periodic. Consider the case of α is irrational. Let us show that any forward orbit is dense. Take any point $x \in S^1$ and a number $\varepsilon > 0$. There is $k \in \mathbb{N}$ such that $\frac{1}{k} < \varepsilon$. Since α is an irrational number, the points $R_\alpha^i(x)$, $i = 0, \ldots, k$, divide S^1 into $k+1$ intervals. Hence there are $i_1, i_2 \in \{0, 1, \ldots, k\}$, $i_1 \neq i_2$, such that $d(R_\alpha^{i_1}(x), R_\alpha^{i_2}(x)) < \varepsilon$. For definiteness, suppose that $i_1 < i_2$. Then $R_\alpha^{i_2 - i_1}(R_\alpha^{i_1}(x)) = R_\alpha^{i_2}(x)$. Since R_α is a rigid rotation, $R_\alpha^{i_2 - i_1}$ moves any point of S^1 no further than ε. It follows that the forward orbit of x under R_α comes through the ε-neighborhood of every point of S^1. Since ε is arbitrary, any forward orbit is dense. $\square$

Suspension and cross-section

There are natural constructions connecting continuous-time and discrete-time dynamical systems. At first, let us consider the particular (but the most important) case of suspension over a circle homeomorphism $f\colon S^1 \to S^1$. Consider the quotient space

$M_f = S^1 \times [0;\,1]/\!\sim$, where $\sim$ is the equivalence relation $(x;\,1) \sim (f(x);\,0)$.

Actually, f pasts in sense two boundary components of the closed cylinder $S^1 \times$ $\times\, [0;\,1]$, Fig. 1.8.

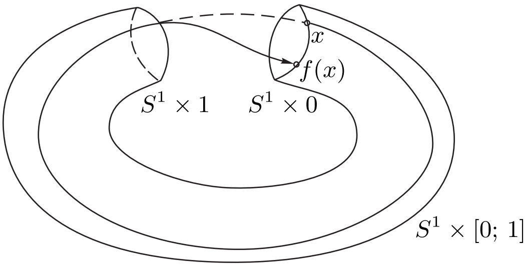


Figure 1.8. Suspension over a circle homeomorphism.

Since a preserving orientation circle homeomorphism is isotopic to the identity, M_f is a torus provided f preserves orientation. Otherwise, M_f is a Klein bottle. The *suspension over* f is the flow $\mathrm{sus}^t(f)$ given by

$$\mathrm{sus}^t(x;\,\tau) = (f^{[t+\tau]}(x);\,\{t+\tau\}),$$

where $[z]$ (resp., $\{z\}$) denotes the integer (resp., fractional) part of a number z. In other words, a trajectory of the flow goes to $(x;\,1)$ along $\{x\} \times [0;\,1)$, then jumps to $(f(x);\,0)$, and continues along $f(x) \times [0;\,1)$, and so on, see Fig. 1.8. The proof of the property $\mathrm{sus}^{t_1+t_2} = \mathrm{sus}^{t_1} \circ \mathrm{sus}^{t_2}$ is left to the reader. The following statement holds.

Lemma 1.2 *Let $f\colon S^1 \to S^1$ be a circle homeomorphism. Then* 1) *If f has a dense orbit, then* sus^t *has a dense trajectory*; 2) *If f has a nowhere dense orbit, then* sus^t *has a nowhere dense trajectory*; 3) *If f has a periodic orbit, then* sus^t *has a periodic trajectory. The converse assertions are also valid.*

Proof follows from the definition of sus^t and the fact that a trajectory through $(x;\,0)$ passes through the points $(f^n(x);\,0)$, $n \in \mathbb{Z}$. □

Note that if f is orientation reversing, f has a periodic orbit and has no dense orbits.

Example 1.7 *A constant flow on the torus revised.*

Take a rotation $R_\mu\colon S^1 \to S^1$. Then $\mathrm{sus}^t(R_\mu)$ is a constant flow on the torus $\mathbb{T}^2$, see Example 1.4.

Example 1.8 *Irrational and rational torus flows.*

The trajectories of $\mathrm{sus}^t(R_\mu)$ are either all closed (periodic) or non-closed in dependence on the ratio $\frac{\beta}{\alpha} = \mu$. If μ is rational, then the trajectories are closed while μ is irrational, then the trajectories are all non-closed, and every trajectory is dense. For $\frac{\beta}{\alpha}$ is irrational (rational), $f^t_{\alpha,\beta}$ is called the *irrational* (*rational*) *torus flow*. ♢

Now we present the general construction. Let $f\colon M^n \to M^n$ be a homeomorphism and $\varphi\colon M^n \to \mathbb{R}^+$ a positive function. Consider the quotient space

$$M^{n+1}_{f,\varphi} = \{(x,\,t) \in M^n \times \mathbb{R}^+ : 0 \leqslant t \leqslant \varphi(x)\}/\,(x,\,\varphi(x) \sim (f(x),\,0)).$$

The *suspension over f with ceiling function* φ is denoted by $\mathrm{sus}^t_{f,\varphi}$ and defined as follows. The trajectories are formed from the lines $\{x\} \times [0;\,\varphi(x)]$, $x \in M^n$. A current point moves along $\{x\}\times[0;\,\varphi(x)]$ to $(x,\,\varphi(x))$, then jumps to $(f(x),\,0)$ and continues along $\{f(x)\} \times [0;\,\varphi(f(x))]$, and so on. Sometimes $\mathrm{sus}^t_{f,\varphi}$ is called a *suspension flow under a function* φ. The suspension flow sus^t constructed above is actually $\mathrm{sus}^t_{f,1}$.

Let f^t be a flow on M^n and $C \subset M^n$ be a codimension one submanifold. Suppose that there is a nonempty subset $C_0 \subset C$ such that given any point $x \in C_0$, the set $T_x = \{t \in \mathbb{R}^+\colon f^t(x) \in C\}$ is a nonempty discrete subset of $\mathbb{R}^+$. The number $\tau(x) = \min T_x$ is called a *return time* to C. The map $P\colon C_0 \to C$ given by $P(x) = f^{\tau(x)}(x)$ is called a *first return map*. This map is often called a *Poincaré map* also.

Hyperbolic diffeomorphisms

We begin with the famous example of hyperbolic diffeomorphism, so-called toral automorphism. This example was a source of Hyperbolic Dynamics.

Example 1.9 *Toral automorphisms.*

Let $A = (a_{ij})$ be an $n \times n$ matrix with $a_{ij} \in \mathbb{Z}$ and $\det A = \pm 1$. Let $\overline{L}_A\colon \mathbb{R}^n \to \mathbb{R}^n$ be the induced linear map. Recall that $\pi\colon \mathbb{R}^n \to \mathbb{T}^n$ is the universal covering projection which takes a point $(x_1, \ldots, x_n)$ to the point in the torus $\mathbb{T}^n$ by taking each component modulo 1, see (1.1). Because A has all integers entries, $\overline{L}_A$ takes the integer lattice (points with all integer coordinates) into itself. Moreover, any congruent points in $\mathbb{R}^n$ are mapped to congruent points. Hence, $\overline{L}_A$ induces a map $L_A\colon \mathbb{T}^n \to \mathbb{T}^n$ such that $L_A \circ \pi = = \pi \circ \overline{L}_A$. Since $\det A = \pm 1$, the inverse A^{-1} is again a matrix with integer entries, so L_A is a diffeomorphism with $L_A^{-1} = L_{A^{-1}}$. Suppose in addition that the matrix A has no eigenvalues of absolute value 1. Such diffeomorphisms are called *hyperbolic toral automorphisms*. They are prototypes of the more general class of hyperbolic dynamical systems. The main feature of these automorphisms is the existence of mutually transversal invariant families of hyperplanes.

For simplicity, we consider the concrete map for $n = 2$. Let $\overline{L}_A\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus, $\overline{L}_A$ is defined by the following coordinate formulas

$$\overline{x} = x + y, \quad \overline{y} = x. \tag{1.6}$$

It is easy to check that the eigenvalues of A are

$$\lambda^s = \frac{1 - \sqrt{5}}{2}, \quad \lambda^u = \frac{1 + \sqrt{5}}{2}, \quad |\lambda^s| < 1, \quad |\lambda^u| > 1,$$

and the corresponding eigenvectors are

$$\vec{v}^s = \begin{pmatrix} 2 \\ -1 - \sqrt{5} \end{pmatrix}, \quad \vec{v}^u = \begin{pmatrix} 2 \\ -1 + \sqrt{5} \end{pmatrix}.$$

Denote by $\overline{\mathscr{F}}^s$ (resp., $\overline{\mathscr{F}}^u$) the family of straight lines parallel to $\vec{v}^s$ (resp., $\vec{v}^u$). Since $\vec{v}^s$ and $\vec{v}^u$ are eigenvectors and $\overline{L}_A$ is a linear map, the both $\overline{\mathscr{F}}^s$ and $\overline{\mathscr{F}}^u$ are invariant under $\overline{L}_A$. The invariantness of $\overline{\mathscr{F}}^s$ means that if l^s is a line of $\overline{\mathscr{F}}^s$ then $\overline{L}_A(l^s)$ is also a line of $\overline{\mathscr{F}}^s$. Similarly, $\overline{\mathscr{F}}^u$.

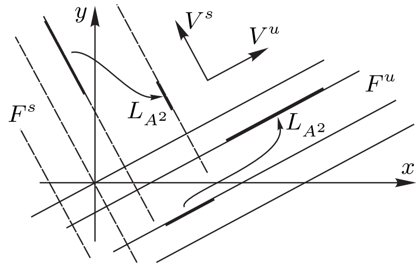


Figure 1.9

The map $\overline{L}_A$ contracts by the factor λ^s in the direction of the eigenvector $\vec{v}^s$. This is a reason to call $\overline{\mathscr{F}}^s$ a *stable foliation* of $\overline{L}_A$. Similarly, $\overline{\mathscr{F}}^u$ is called an *unstable foliation* because $\overline{L}_A$ expands by the factor λ^u in the direction of the eigenvector $\vec{v}^u$, Fig. 1.9.

The linear map $\overline{L}_{A^2} \colon \mathbb{R}^2 \to \mathbb{R}^2$ induced by the matrix

$$A^2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

has the same eigenvectors with $\overline{L}_A$, but $\overline{L}_A$ reverses orientation and has one negative eigenvalue λ^s, while $\overline{L}_{A^2}$ preserves orientation and has two positive eigenvalues $(\lambda^s)^2$, $(\lambda^u)^2$. Similarly to $\overline{L}_A$, $\overline{L}_{A^2}$ induces a map $L_{A^2} \colon \mathbb{T}^2 \to \mathbb{T}^2$ such that $L_A \circ \pi = \pi \circ \overline{L}_A$.

Stable and unstable foliations of $\overline{L}_A$ are families of integral curves of constant flow on $\mathbb{R}^2$, see Example 1.3. They project to so called stable and unstable foliations $\mathscr{F}^s$, $\mathscr{F}^u$ respectively of the diffeomorphism $L_A \colon T^2 \to T^2$. The both $\mathscr{F}^s$ and $\mathscr{F}^u$ are invariant under L_A. Notice that both eigenvectors $\vec{v}^s$, $\vec{v}^u$ have irrational slope. Taking in mind Example 1.4, we see that the foliations $\mathscr{F}^s$, $\mathscr{F}^u$ are irrational. Thus, each of hyperbolic toral automorphisms L_A, L_{A^2} has invariant mutually transversal foliations. One of this foliation is stable in the sense that an automorphism contracts every leaf, while another foliation is unstable meaning that the automorphism expands every leaf. $\diamond$

Proposition 1.4 *Let $L_A \colon \mathbb{T}^2 \to \mathbb{T}^2$ be a linear hyperbolic toral automorphism defined by an integer unimodular matrix A. Then the periodic points of L_A are dense in $\mathbb{T}^2$.*

Proof. For fixed natural k, let $\mathrm{Ratio}(k)$ be the rational points in $\mathbb{T}^2$ with denominators k:

$$\mathrm{Ratio}(k) = \left\{\pi\left(\frac{m}{k};\ \frac{n}{k}\right),\ (m,\,n) \in \mathbb{Z}^2\right\} \subset \mathbb{T}^2.$$

Then $L_A(\mathrm{Ratio}(k)) \subset \mathrm{Ratio}(k)$ because of the matrix A is integer. Since the rational on the segment $[0;\,1)$ with denominators k equals k, the set $\mathrm{Ratio}(k)$ consists of k^2 points. Clearly, the restriction $L_A|_{\mathrm{Ratio}(k)}$ is one-to-one because of L_A is a diffeomorphism. Hence, $L_A|_{\mathrm{Ratio}(k)}$ is a permutation of $\mathrm{Ratio}(k)$ and every point in this set is periodic. Finally, the union $\bigcup\limits_{k\in\mathbb{N}} \mathrm{Ratio}(k)$ is dense in $\mathbb{T}^2$, and the periodic points are dense. □

Topological conjugacy and semi-conjugacy

Two maps $f\colon M \to M$, $g\colon M \to M$ are called *topologically semi-conjugate* (or simply, semi-conjugate) if there is a continuous map $h\colon M \to M$ such that $h \circ f = g \circ h$. One frequently writes this relation in the form of a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & M \\ \downarrow h & & \downarrow h \\ M & \xrightarrow{g} & M. \end{array}$$

The map h is called a *semi-conjugacy map* (or simply, semi-conjugacy) between f and g. Sometimes one says that h semi-conjugates the map f into g.

If h is a homomorphism then f and g are called *topologically conjugate* (or simply, conjugate). Roughly, such f and g differ by homeomorphic change of coordinates.

Lemma 1.3 *Any orientation preserving homeomorphisms of* $[0;\,1]$ *without fixed points in* $(0;\,1)$ *are topologically conjugate.*

Proof. Let f, g are orientation preserving homeomorphisms of $[0;\,1]$ without fixed points in $(0;\,1)$. Suppose that $f(x) > x$ for some (hence, any) $x \in (0;\,1)$. Then $\vartheta(x) = 1 - x$ is a conjugacy map between f and the homeomorphism

$$f' = \vartheta \circ f \circ \vartheta^{-1} = \vartheta \circ f \circ \vartheta$$

with $f'(x) < x$ for any $x \in (0;\,1)$. Indeed, $f'(x) = 1 - f(1 - x)$. Since $f(1 - x) > 1 - x$, $f'(x) < x$. Therefore, without loss of generality, we can assume that $f(x) < x$ and $g(x) < x$ for any $x \in (0;\,1)$.

Take any $x_0 \in (0;\, 1)$ any homeomorphism $h_0\colon [x_0;\, f(x_0)] \to [x_0;\, g(x_0)]$. For every $n \in \mathbb{Z}$, put by definition

$$h_n|_{[f^n(x_0);\, f^{n+1}(x_0)]} = \\ = g^n \circ h_0 \circ f^{-n}\colon [f^n(x_0);\, f^{n+1}(x_0)] \to [g^n(x_0);\, g^{n+1}(x_0)].$$

Clearly that these h_n glue together to give a homeomorphism $h\colon [0;\, 1] \to [0;\, 1]$ such that $h \circ f = g \circ h$. □

The notion of structural stability

Let $\mathrm{Diff}^1(M^n)$ be the space of C^1 diffeomorphisms on M^n endowed with the uniform C^1 topology [115]. A diffeomorphism f is said to be *structurally stable* if there exists a neighborhood U of f in $\mathrm{Diff}^1(M^n)$ such that every $g \in U$ is conjugate to f.

Action of a group on a manifold

A generalization of the notion of DS with continuous and discrete time is an action of a group. Let G be a (multiplicative) Lie group and M be a manifold. Suppose that any $g \in G$ corresponds a homeomorphism $H_g\colon M \to M$ such that the map $G \times M \to M$, $(g,\, x) \mapsto H_g(x)$ is continuous. The family $\{H_g\colon g \in G\}$ is called an *action of* G *on* M (or simply G-action) if

- $H_{g_1 g_2} = H_{g_1} \circ H_{g_2}$ for $\forall g_1,\, g_2 \in G$;
- $H_{g^{-1}} = H_g^{-1}$ for $\forall g \in G$.

Usually, we denote a G-action by $\mathscr{A}_G$ or $\mathscr{A}(G)$. A flow (continuous-time DS) can be interpreted as an $\mathbb{R}$-action while a cascade (discrete-time DS) a $\mathbb{Z}$-action. It is easy to see that **if the group G is commutative then the mappings H_g commute**.

Given any G-action $\mathscr{A}_G$ on M, the set $\{H_g(x)\colon g \in G\}$ is called the *orbit* of $x \in M$ under $\mathscr{A}_G$, denoted by $O(x,\, \mathscr{A}_G)$, or simply $O(x)$, or $\mathrm{Orb}(x) = \mathrm{Orb}_G(x)$. Denote by G_x the *isotropy group of* x (i. e., the subgroup of G that stabilizes x) which is a closed subgroup of G. Then the orbit $\mathrm{Orb}_G(x)$ is the quotient space G/G_x. Note that the orbits may not all have the same dimension, in general. However, there is a natural class of actions those orbits are locally homeomorphic to G.

G-action on M is *locally free*, if given any $x \in M$ the isotropy group G_x is discrete. In this case, there is a neighborhood V of the unity $e \in G$ such that the map $V \to M$, $g \mapsto H_g(x)$, is injective. Therefore, the orbits are locally homeomorphic to G (here, we assume that G is a connected Lie group, for example $\mathbb{R}^n$).

Often we shall omit the letter H in H_g writing simply $g\colon M \to M$ instead of $H_g\colon M \to M$. Below, we shall assume that actions are smooth enough.

1.3. Elements of Topological Dynamics

Topological Dynamics studies *topological dynamical systems*. This means that $f^t\colon M \to M$ or $f\colon M \to M$ is a homeomorphism in the case of continuous-time DS (flow) or discrete-time DS respectively. Often, M is just a topological manifold. Topological Dynamics studies properties that are invariant under semi-conjugacy and conjugacy maps. We restrict ourself mainly by discrete-time DS since notions are often similar for continuous-time DS.

Limit Sets

Let $f\colon M \to M$ be a homeomorphism of topological manifold M endowed with a metric d. A point $a \in M$ is an *ω-limit point* of $x \in M$ if there is a sequence of natural numbers n_k such that $d(a,\, f^{n_k}(x)) \to 0$ as $n_k \to \infty$ ($k \to \infty$). The *ω-limit set* of x denoted by $\omega_f(x) = \omega(x)$ is the union of all ω-limit points of x. Sometimes, we'll apply the following notations

$$\omega(x) = \omega(x,\, f) = L_\omega(x) = L_\omega(x,\, f) = L^+(x,\, f).$$

Similarly, a point $b \in M$ is an *α-limit point* of x if there is a sequence of integers n_k such that

$$d(a,\, f^{n_k}(x)) \to 0 \quad \text{as } n_k \to -\infty\ (k \to \infty).$$

The *α-limit set* of x denoted by

$$\alpha_f(x) = \alpha(x) = \alpha(x,\, f) = L_\alpha(x) = L_\alpha(x,\, f) = L^-(x,\, f)$$

is the union of all α-limit points of x. Clearly, $\alpha_f(x) = \omega_{f^{-1}}(x)$. It is easy to check that if x is a periodic point then $\omega(x) = \alpha(x) = O(x)$.

A subset $N \subset M$ is called *forward invariant* if $f(N) \subset N$, and *backward invariant* if $f^{-1}(N) \subset N$. A subset N is *invariant* if it is both forward and backward invariant. The simplest example of invariant subset is a periodic orbit.

Lemma 1.4 *An ω- and α-limit sets of a point are closed invariant sets. The both sets are nonempty provided M is compact.*

Proof. Since $\alpha_f(x) = \omega_{f^{-1}}(x)$, it is enough to consider only the ω-limit set of a point $x \in M$. The compactness of M implies $\omega(x) \neq \varnothing$ because the sequence $\{f^k(x)\}_0^\infty$ has a converge subsequence. Now one assumes that $\omega(x) \neq \varnothing$. Take $y \in \omega(x)$. Then there is a sequence $n_k \to \infty$ such that $f^{n_k}(x) \to y$. Then $f^{n_k+1}(x) \to f(y)$. Hence, $f(\omega(x)) \subset \omega(x)$. Similarly, $f^{-1}(\omega(x)) \subset \omega(x)$. Thus, $\omega(x)$ is invariant.

Suppose the sequence of $y_i \in \omega(x)$ converges to y_*. Given any y_i, there is the sequence $n_k^{(i)} \to \infty$ such that $f^{n_k^{(i)}}(x) \to y_i$ as $k \to \infty$. Then there is a sequence $j_i \to \infty$ such that $f^{n_{j_i}^{(i)}}(x) \to y_*$ as $i \to \infty$. Hence, $y_* \in \omega(x)$, and $\omega(x)$ is closed. □

The *full limit set* $L(f)$ of f is the topological closure of the union of all limit points,

$$L(f) = \operatorname{clos}\left(\bigcup_{x\in M} \left(L^+(x,\, f) \cup L^-(x,\, f)\right)\right).$$

Proposition 1.5 *A full limit set is closed and invariant.*

Proof is omitted. We leave it to the Reader. □

Lemma 1.5 *If N is invariant then so are* $\operatorname{clos} N$, $\operatorname{int} N$, *and* ∂N.

Proof. Since f is a homeomorphism,

$$f(\operatorname{clos} N) \subset \operatorname{clos} f(N) = N \quad \text{and} \quad f^{-1}(\operatorname{clos} N) \subset \operatorname{clos} f^{-1}(N) = N.$$

Hence, $f(\operatorname{clos} N) = \operatorname{clos} N$. The other two are similar. □

Non-wandering points

A point $p \in M$ is called *non-wandering point* of homeomorphism $f\colon M \to$ $\to M$ if given any neighborhood U of p and $N_0 \in \mathbb{N}$, there is $n \geqslant N_0$ such that $f^n(U) \cap U \neq \varnothing$. A point is *wandering* if it is not non-wandering. Denote by $NW(f)$ the union of all non-wandering points called a *non-wandering set*.

Lemma 1.6 *The set $NW(f)$ is closed and invariant.*

Proof. By definition, a point is wandering if it has a neighborhood V such that $V \cap f^n(V) = \varnothing$ for any $n \in \mathbb{Z}$. Therefore, the set of wandering points is open. Hence $NW(f)$ is closed. Since f and f^{-1} are continuous, $NW(f)$ is invariant. □

Lemma 1.7 *Every point of full limit set is non-wandering, $L(f) \subset NW(f)$.*

Proof. By Lemma 1.6, $NW(f)$ is closed. Therefore, it is sufficient to prove that every $\omega(\alpha)$-limit point is non-wandering. Given any ω-limit point $a \in M$ there are point $x \in M$ and a sequence of natural numbers n_k such that $d(a, f^{n_k}(x)) \to 0$ as $n_k \to \infty$ $(k \to \infty)$. For any neighborhood U of the point a, the sequence $f^{n_k}(x)$ belongs to U beginning with some index n_k. Since $n_k \to \infty$, $a \in NW(f)$. If a is an α-limit point, the argument is similar. □

Since every periodic point is an $\omega(\alpha)$-limit one,

$$\operatorname{clos}(\operatorname{Per}(f)) \subset L(f) \subset NW(f).$$

Chain Recurrent Sets

Let $f\colon M \to M$ be a homeomorphism and $\varepsilon > 0$. An *ε-chain of length* $n \in \mathbb{N}$ from a point $x \in M$ to a point $y \in M$ is a finite sequence $\{x = x_0, x_1, \ldots, x_n = y\}$ such that

$$d(f(x_{i-1}), x_i) < \varepsilon \quad \text{for all } 1 \leqslant i \leqslant n, \text{ Fig. 1.10.}$$

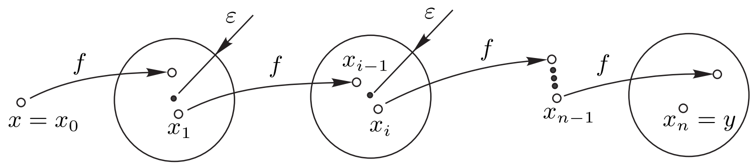


Figure 1.10

A *chain recurrent set* of f is the set $\mathfrak{R}(f)$ of points $x \in M$ such that there is an ε-chain of arbitrary large length from x to x. Since an ε-chain can be a part of real orbit, $NW(f) \subset \mathfrak{R}(f)$.

Lemma 1.8 *The chain recurrent set $\mathfrak{R}(f)$ is closed and invariant.*

Proof is similar to the proof of Lemma 1.6. We leave the details to the Reader. □

One can easily check that

$$L(f) \subset NW(f) \subset \mathfrak{R}(f).$$

Now we define a relation $\sim$ on $\mathfrak{R}(f)$ as follows. Set $x \sim y$ if for all $n \in \mathbb{N}$ and $\varepsilon > 0$ there are ε-chains of lengths greater than n from x to y and from y to x. Two such points $x, y \in \mathfrak{R}(f)$ are called *chain equivalent*. It is clear that this is an equivalence relation. The equivalence classes are called the *chain components*.

Shift Maps

Here, we present the useful class of dynamical systems which helps to study many brilliant examples of dynamical systems. Fixing a positive integer $k \geqslant 2$, denote by $\Sigma_k = \{0, \ldots, k\}^{\mathbb{N}}$ the set of all sequences taking values $\{0, \ldots, k\}$ indexed by $\mathbb{Z}$. Let us introduce a metric d on Σ_k. If the sequences $\{x_n\}_{n\in\mathbb{Z}}$, $\{y_n\}_{n\in\mathbb{Z}}$ are different, set $D = \min\{i \geqslant 0 \mid x_i \neq y_i,$ or $x_{-i} \neq y_{-i}\}$. Put by definition,

$$d\left(\{x_n\}_{n\in\mathbb{Z}}, \{x_n\}_{n\in\mathbb{Z}}\right) = \left(\frac{1}{2}\right)^D.$$

The metric d induces the corresponding topology in Σ_k. For example, the 2^{-l}-neighborhood U_l of a $X = \{x_n\}_{n\in\mathbb{Z}}$ is the set

$$U_l = \left\{X' = \{x'_n\}_{n\in\mathbb{Z}} \in \Sigma_k \mid d(X, X') < \frac{1}{2^l}\right\}.$$

This implies that $x_i = x'_i$ and $x_{-i} = x'_{-i}$ for every $|i| \leqslant l$.

The set Σ_k endowed with the metric d becomes a topological compact space. Sometimes Σ_k is called a *full shift space*. We define the maps $\sigma_l\colon \Sigma_k \to \Sigma_k$, $\sigma_r\colon \Sigma_k \to \Sigma_k$ by

$$\{\sigma_l(X)\}_n = x_{n+1}, \ \{\sigma_l(X)\}_n = x_{n-1}, \ \forall n \in \mathbb{Z}, \text{ where } X = \{x_n\}_{n\in\mathbb{Z}} \in \Sigma_k,$$

$$\sigma_l\colon (\ldots, x_{-1}, \underbrace{x_0}_{\text{zero place}}, x_1, \ldots) \mapsto (\ldots, x_0, \underbrace{x_1}_{\text{zero place}}, x_2, \ldots),$$

$$\sigma_r\colon (\ldots, x_{-1}, \underbrace{x_0}_{\text{zero place}}, x_1, \ldots) \mapsto (\ldots, x_{-2}, \underbrace{x_{-1}}_{\text{zero place}}, x_0, \ldots).$$

Since σ_l and σ_r shift sequences by one place, they are called the *left shift map* and the *right shift map* respectively.

Lemma 1.9 *Each shift map σ_l and σ_r is a homeomorphism $\Sigma_k \to \Sigma_k$.*

Proof. The letter τ means either l or r. If $d(X, Y) \leqslant \left(\frac{1}{2}\right)^D$, where $X, Y \in \Sigma_k$, then

$$d(\sigma_\tau(X), \sigma_\tau(Y)) \leqslant \left(\frac{1}{2}\right)^{D-1} = \frac{1}{2}d(X, Y).$$

Hence, σ_τ is continuous. The inverse σ_τ^{-1} is a shift map. It is continuous by the same sort of argument as above. □

Lemma 1.10 *Let* $\sigma\colon \Sigma_k \to \Sigma_k$ *be a shift map* (*left or right*). *Then*

- *periodic points are dense in* Σ_k;
- *transitive orbits are dense in* Σ_k;
- *every point of* Σ_k *is non-wandering.*

Proof. Take arbitrary $X = \{x_n\}_{n\in\mathbb{Z}} \in \Sigma_k$ and 2^{-l}-neighborhood U_l of a X. Denote by B the finite block $\langle x_{-l}, \ldots, x_{-1}, x_0, x_1, \ldots, x_l\rangle$. Then the point

$$Y = \{y_n\}_{n\in\mathbb{Z}} = \{\ldots, \sigma_r^{-i(2l+1)}(B), \ldots,$$
$$\ldots, \sigma_r^{(2l+1)}(B), B, \sigma_r^{(2l+1)}(B), \ldots, \sigma_r^{i(2l+1)}(B), \ldots\} \in \Sigma_k$$

is periodic and $Y \in U_l$. This follows that periodic points are dense in Σ_k.

Let $B_1, B_2, \ldots$ the family of all possible finite blocks consisting of the values $\{0, \ldots, k\}$. Then the point

$$Q = \{\ldots, B_i, \ldots, B_2, B_1, B, B_1, B_2, \ldots, B_i, \ldots\}$$

is transitive. Moreover, $Q \in U_l$. Hence, transitive orbits are dense in Σ_k.

The last statement follows from the each of the previous statements. □

Birkhoff Transitivity Theorem

Let f be a homeomorphism of a complete metric space M with countable basis. Recall that a subset N is *residual* if there is a countable number of open dense sets V_i such that $N = \bigcap_i V_i$. The Baire category theorem states that any residual set $N \subset M$ is dense in M.

A set $A \subset M$ is *transitive* if there is a point $x \in A$ such that the orbit $O_f(x)$ is dense in A, $A = \operatorname{clos} O_f(x)$. Sometimes, we say that f is *topologically transitive* on A, or simply *transitive*. If $A = M$, f is (topologically) *transitive*.

Theorem 1.2 *Suppose that given any open sets V_1 and $V_2 \subset M$, there is $k \in \mathbb{N}$ such that $f(V_1) \cap V_2 \neq \varnothing$. Then there is a residual subset $\mathscr{R} \subset M$ such that the orbit of every point $p \in \mathscr{R}$ is dense in M. In particular, f is transitive.*

Proof. Let $\{w_j\}_{j \in \mathbb{Z}}$ be a countable basis of M. By condition, the orbit $O(w_j) = \bigcup_{n \in \mathbb{Z}} f^n(w_j)$ of every w_j is an open and dense set. Therefore, the set

$$\mathscr{R} = \bigcap_{j \in \mathbb{Z}} O(w_j)$$

is residual. Given any $x \in \mathscr{R}$, $O(x) \cap w_j \neq \varnothing$ for all $j \in \mathbb{Z}$. Hence, $O(x)$ is dense in M. □

A dynamical system with the property that every its orbit is dense is called *minimal*. Obviously, a minimal dynamical system is transitive. A circle rotation R_α is minimal provided α is irrational.

Mixing

A homeomorphism $f\colon M \to M$ is called *mixing* if for any (nonempty) open sets $U, V \subset M$ there is $n_0 \in \mathbb{N}$ such that $f^n(U) \cap V \neq \varnothing$ for all $n \geqslant n_0$.

Theorem 1.2 implies that a mixing homeomorphism is transitive but not vice versa: the rotation R_α of the circle S^1 by an irrational α is a transitive (see Example 1.6) but not mixing.

Example 1.10 *A hyperbolic toral automorphism $A\colon \mathbb{T}^2 \to \mathbb{T}^2$ defined by $\overline{x} = x + y \bmod 1$, $\overline{y} = x \bmod 1$ is mixing.*

We keep the notion of Example 1.9. Every leaf of the unstable foliation $\overline{\mathscr{F}}^u$ is parallel to the eigenvector $\vec{v}^u$. Therefore, the slope of every leaf equals $\alpha = \frac{-1+\sqrt{5}}{2}$ since the corresponding eigenvalue is $\lambda^u = \frac{1+\sqrt{5}}{2}$. Obviously, α is irrational. By Lemma 1.2, any leaf, say W^u, of $\overline{\mathscr{F}}^u$ is dense in $\mathbb{T}^2$. Given any $\varepsilon > 0$, the family of balls of radius ε centered at points of W^u covers $\mathbb{T}^2$. Because of $\mathbb{T}^2$ is compact, a finite collection of these balls also covers $\mathbb{T}^2$. Hence, there are $l > 0$ and a compact segment $K \subset W^u$ of the length l such that the ε-neighborhood of K covers $\mathbb{T}^2$ i. e., K is ε-dense in $\mathbb{T}^2$.

A rigid translation of $\mathbb{T}^2$ is an isometry. Since the unstable foliation $\overline{\mathscr{F}}^u$ is invariant under any rigid translation, every segment of the length l belonging to a leaf of $\overline{\mathscr{F}}^u$ is ε-dense in $\mathbb{T}^2$.

Take now any nonempty open sets U, $V \subset \mathbb{T}^2$. A point $x \in V$ has an ε-neighborhood $U_\varepsilon(x)$ such that $U_\varepsilon(x) \subset V$. Choose a segment $d \subset U$ of length $\delta > 0$ belonging to some W^u. Since $|\lambda^u| > 1$, there exists $n_0 \in \mathbb{N}$ such that $|\lambda^u|^{n_0}\delta \geqslant l$. This follows that the length of the segment $A^n(d)$ is at least l for every $n \geqslant n_0$. As a consequence, $A^n(d)$ intersects $U_\varepsilon(x)$. Hence, $A^n(U) \cap V \neq \varnothing$ for all $n \geqslant n_0$. $\diamond$

Similarly, one can prove the following statement (we omit the proof).

Lemma 1.11 *A hyperbolic toral automorphism $A\colon T^n \to T^n$ is mixing.*

Lemma 1.12 *The full one- and two-sided shifts are mixing.*

Proof. Let $U, V \subset \Sigma_k$ be any non-empty open sets. Take points X, $X'\Sigma_k$ with those $\frac{1}{2^{l_1}}$- and $\frac{1}{2^{l_2}}$-neighborhoods respectively denoted by N_1 and N_2 such that $N_1 \subset U$, $N_2 \subset V$. Recall that

$$N_1 = \left\{Y' = \{y'_n\}_{n\in\mathbb{Z}} \in \Sigma_k \mid d(X, Y') < \frac{1}{2^{l_1}} \cong y'_n = x_n \text{ for } |n| \leqslant l_1\right\},$$

$$N_2 = \left\{Z = \{z_n\}_{n\in\mathbb{Z}} \in \Sigma_k \mid d(X', Z) < \frac{1}{2^{l_2}} \cong x'_n = z_n \text{ for } |n| \leqslant l_2\right\}.$$

By definition, elements of $\sigma_r^m(N_1)$ are sequences with specified values in the places $m - l_1, \ldots, m + l_1$. Therefore, when $m - l_1 > l_2$, the intersection $\sigma_r^m(N_1) \cap N_2$ is non-empty. So, $\sigma_r^m(U) \cap V \neq \varnothing$ for all $m \geqslant l_1 + l_2 + 1$.

The proof for σ_l is similar (one can also use that $\sigma_l = \sigma_r^{-1}$). $\square$

Topological Entropy

Topological entropy characterizes the complexity of the orbit structure of a DS. Roughly speaking, it shows the exponential rate of the growth of essentially different orbits. For a continuous map $f\colon M \to M$ of a metric space M that is endowed with a metric d, we introduce the metric

$$d_n(x, y) = \max_{0\leqslant m\leqslant n} d\left(f^m(x), f^m(y)\right) \quad \text{for each } n \in \mathbb{N}.$$

The distance $d_n(x, y)$ is the maximum distance between the first n iterations of x and y. Obviously, $d \leqslant d_n \leqslant d_{n+1}$.

Fix $\varepsilon > 0$. A subset $A \subset M$ is (n, ε)-*spanning* if for every $p \in M$ there is $q \in A$ such that $d_n(p, q) < \varepsilon$. Thus, a family of points $\{q_1, \ldots, q_r\}$ is (n, ε)-spanning, if given any point $p \in M$, the n-segment orbit $\{p, f(p), \ldots, f^n(p)\}$ of p visits at least one ε-ball $B_\varepsilon(q_i)$. If the space M is compact, there are finite (n, ε)-spanning sets. Denote by $\operatorname{span}(n, \varepsilon)$ the minimum cardinality of an (n, ε)-spanning set.

A subset $B \subset M$ is called (n, ε)-*separated* if any two different points in B are at least ε apart in the metric d_n. Since M is compact, any (n, ε)-separated set is finite. Denote by $\operatorname{sep}(n, \varepsilon)$ the maximum cardinality of an (n, ε)-separated set.

Let B be an (n, ε)-separated set with the maximal cardinality $\operatorname{sep}(n, \varepsilon)$. By the maximality, given any point $p \in M$, the n-segment orbit $\{p, f(p), \ldots, \ldots, f^n(p)\}$ of p visits at least one ε-ball centered at the points of B. It follows that

$$\operatorname{span}(n, \varepsilon) \leqslant \operatorname{sep}(n, \varepsilon). \tag{1.7}$$

As usual, the diameter of a set is the supremum of distances between pairs of points in the set. Denote by $\operatorname{cov}(n, \varepsilon)$ the minimum cardinality of ε-balls of d_n-diameter that cover M. By compactness of M, $\operatorname{cov}(n, \varepsilon)$ is finite. Obviously, $\operatorname{sep}(n, \varepsilon) \leqslant \operatorname{cov}(n, \varepsilon)$. One can prove that $\operatorname{cov}(n, 2\varepsilon) \leqslant \leqslant \operatorname{span}(n, \varepsilon)$. Taking in mind (1.7), one gets

$$\operatorname{cov}(n, 2\varepsilon) \leqslant \operatorname{span}(n, \varepsilon) \leqslant \operatorname{sep}(n, \varepsilon) \leqslant \operatorname{cov}(n, \varepsilon). \tag{1.8}$$

Lemma 1.13 *The sequence* $a_n = \log \operatorname{cov}(n, \varepsilon)$ *is subadditive i. e.* $a_{n+m} \leqslant \leqslant a_n + a_m$.

Proof. Take a set U with d_n-diameter less than $\varepsilon > 0$, and a set V with d_m-diameter less than ε. By the definition of d_{n+m} metric, $U \cap f^{-n}(V)$ has d_{n+m}-diameter less than ε. Then

$$\operatorname{cov}(n + m, \varepsilon) \leqslant \operatorname{cov}(n, \varepsilon) \cdot \operatorname{cov}(m, \varepsilon).$$

It follows the result. □

Lemma 1.14 *Suppose that the sequence* $\{a_n\}_{n=1}^{\infty}$, $a_n \geqslant 0$, *is subadditive. Then*

$$\lim_{n\to\infty} \frac{a_n}{n} = \inf_n \frac{a_n}{n}.$$

Proof. Since the sequence $\{a_n\}_{n=1}^{\infty}$ is subadditive, $a_{km} \leqslant k a_m$. Let us fix m. Given any $n > 0$, $n = km + j$ where $0 \leqslant j < m$. Then

$$\frac{a_n}{n} = \frac{a_{km+j}}{km + j} \leqslant \frac{a_{km}}{km} + \frac{a_j}{km} \leqslant \frac{k a_m}{km} + \frac{a_j}{km} = \frac{a_m}{m} + \frac{a_j}{km}.$$

Putting $n \to \infty$, $k \to \infty$, one gets

$$\limsup_{n\to\infty} \frac{a_n}{n} \leqslant \frac{a_m}{m}, \quad \text{and hence,} \quad \limsup_{n\to\infty} \frac{a_n}{n} \leqslant \inf_m \frac{a_m}{m}.$$

Since

$$\limsup_{n\to\infty} \frac{a_n}{n} \geqslant \liminf_{n\to\infty} \frac{a_n}{n} \geqslant \inf_m \frac{a_m}{m},$$

it follows that $\lim\limits_{n\to\infty} \frac{a_n}{n}$ exists and equals to $\inf\limits_n \frac{a_n}{n}$. □

As a consequence of Lemmas 1.13, 1.14, one gets that the following limit exists and is finite

$$h_\varepsilon = \lim_{n\to\infty} \frac{\log \operatorname{cov}(n, \varepsilon)}{n}.$$

Obviously that $\operatorname{cov}(n, \varepsilon)$ increases provided $\varepsilon \to 0$. Therefore, h_ε increases monotonically as well. Hence, the limit

$$h_{top} = \lim_{\varepsilon\to 0+} h_\varepsilon = \lim_{\varepsilon\to 0+} \lim_{n\to\infty} \frac{\log \operatorname{cov}(n, \varepsilon)}{n} \tag{1.9}$$

exists, and is called the *topological entropy* of f.

It follows from (1.8) that

$$h_{top} = \lim_{\varepsilon\to 0+} \lim_{n\to\infty} \frac{\log \operatorname{cov}(n, \varepsilon)}{n} = \lim_{\varepsilon\to 0+} \lim_{n\to\infty} \frac{\log \operatorname{sep}(n, \varepsilon)}{n} =$$
$$= \lim_{\varepsilon\to 0+} \lim_{n\to\infty} \frac{\log \operatorname{span}(n, \varepsilon)}{n}. \tag{1.10}$$

We see that the topological entropy is either a finite non-negative number or an infinity.

Expansiveness

A homeomorphism $f\colon M \to M$ is *expansive* if there exists $\delta > 0$ such that for any distinct points $x, y \in M$

$$d\left(f^{n_0}(x),\, f^{n_0}(y)\right) \geqslant \delta \quad \text{for some } n_0 \in \mathbb{Z}.$$

It is easy to check that a circle rotation is not expansive. One can prove that the full one- and two-sided shifts, and the hyperbolic toral automorphism are expansive.

The similar definition of expansiveness holds for non-invertible maps. In this case $n_0 \in \mathbb{N}$, and f is called *positively expansive*. One can prove that the expanding circle endomorphism is positively expansive.

Any number $\delta > 0$ with the above property is called an *expansiveness constant*.

For expansive homeomorphisms, the topological entropy can be calculated as follows.

Lemma 1.15 *Let $f\colon M \to M$ be an expansive homeomorphism with expansive constant δ. Then*

$$h(f) = h_\varepsilon(f) \quad \text{for any } \varepsilon < \delta.$$

Proof is omitted. See e. g., Proposition 2.5.7 [54].

Algebraic Entropy

Let G be an abelian group with the finite system of generators $\{\gamma_1, \ldots, \gamma_s\}$. An element $\gamma \in G$ has a unique representation $\gamma = i_1\gamma_1 + \cdots + i_s\gamma_s$ where $i_1, \ldots, i_s \in \mathbb{Z}$. Denote by $|\gamma| = \sum_{j=1}^{s} |i_j|$ the *length* of γ.

Let $A\colon G \to G$ be an endomorphism. Put by definition,

$$l_n = \max_{1 \leqslant i \leqslant s} |A^n(\gamma_i)|, \quad n \in \mathbb{N}.$$

The limit

$$h_{alg}(A) = \lim_{n\to\infty} \frac{\log l_n}{n} \tag{1.11}$$

is called the *algebraic entropy* of A. Since $l_{n+m} \leqslant l_n \cdot l_m$, the limit (1.11) exists. One can prove that this definition does not depend on the choice of a system of generators.

Let $f\colon \mathbb{T}^d \to \mathbb{T}^d$ be a continuous map of d-torus $\mathbb{T}^d$, $d \geqslant 1$. Then f induces the linear mapping $f_* = f_{*1}\colon H_1(\mathbb{T}^d, \mathbb{Z}) \to H_1(\mathbb{T}^d, \mathbb{Z})$ of the first (integer) homology group $H_1(\mathbb{T}^d, \mathbb{Z})$ that can be identified with $\mathbb{Z}^d$. The entropy $h_{alg}(f_*)$ is called the *homological-group entropy* of f. The mapping f_* is the linear part of f, and can be defined by the integer matrix denoted by $A(f)$. The linear mapping $A(f) = f_*$ can be considered as a linear simplification of f. Therefore, one can prove that

$$h_{alg}(f_*) \leqslant h_{top}(f). \tag{1.12}$$

See details in [124], Theorem 8.1.1.

The mapping f_* can be naturally extended to the linear map $\mathbb{R}^d \to \mathbb{R}^d$ that is defined by the same matrix $A(f)$. Moreover, this is the mapping $f_* = f_{*1}\colon H_1(\mathbb{T}^d, \mathbb{R}) \to H_1(\mathbb{T}^d, \mathbb{R})$ of the first homology group $H_1(\mathbb{T}^d, \mathbb{R})$ over $\mathbb{R}$ induced by f. Let $\operatorname{Spec}(f_*) = \{\lambda_1, \ldots, \lambda_k\}$ be the eigenvalues of f_*, and suppose that

$$|\lambda_{\max}| = |\lambda_1| \geqslant |\lambda_2| \geqslant \cdots \geqslant |\lambda_s| > 1 > |\lambda_{s+1} \geqslant \cdots \geqslant |\lambda_k|.$$

Then calculations give

$$h_{alg}(f_*) = \sum_{i=1}^{s} \log|\lambda_i| \geqslant \log|\lambda_{\max}|. \tag{1.13}$$

As a consequence of (1.12) and (1.13), one gets

Corollary 1.1 *Let $f\colon \mathbb{T}^d \to \mathbb{T}^d$ be a continuous map of d-torus $\mathbb{T}^d$, and $f_* = f_{*1}\colon H_1(\mathbb{T}^d, \mathbb{R}) \to H_1(\mathbb{T}^d, \mathbb{R})$ be the mapping of the first homology group $H_1(\mathbb{T}^d, \mathbb{R})$ over $\mathbb{R}$ induced by f. Let $\operatorname{Spec}(f_*) = \{\lambda_1, \ldots, \lambda_k\}$ be the eigenvalues of f_*, and suppose that $|\lambda_{\max}| = |\lambda_1| \geqslant |\lambda_2| \geqslant \cdots \geqslant |\lambda_s| > 1 > |\lambda_{s+1} \geqslant \cdots \geqslant |\lambda_k|$. Then*

$$\log|\lambda_{\max}| \leqslant h_{top}(f).$$

1.4. Linear maps

Denote by $\mathbf{L}(\mathbb{R}^k, \mathbb{R}^m)$ the set of linear maps $\mathbb{R}^k \to \mathbb{R}^m$. Given bases $\{\vec{e}_i\}_{i=1}^k$ of $\mathbb{R}^k$ and $\{\vec{r}_j\}_{j=1}^m$ of $\mathbb{R}^m$, a linear map $L \in \mathbf{L}(\mathbb{R}^k, \mathbb{R}^m)$ determines a $k \times m$ matrix $A = (a_{ij})$ by

$$L(\vec{V}) = L\left(\sum_{j=1}^{k} x_j \vec{e}_j\right) = \sum_{i=1}^{m}\left(\sum_{j=1}^{k} a_{ij} x_j\right)\vec{r}_i,$$

where $x_1, \ldots, x_k$ are coordinates of the vector $\vec{V}$ in the base $\{\vec{e}_i\}_{i=1}^k$. Vice versa, a $k \times m$ matrix $A = (a_{ij})$ determines the corresponding linear map $L = L_A$. If the bases are fixed, we often identify such a linear map L_A with this $k \times m$ matrix A. With this identification, the linear map L_A is given by

$$A\begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix},$$

where $y_1, \ldots, y_m$ are coordinates of the vector $L_A(\vec{V})$ in the base $\{\vec{r}_j\}_{j=1}^m$. It follows from the formula above that $y_i = \sum_{j=1}^{k} a_{ij} x_j$.

From dynamical point of view, we are interesting in positive and negative iterations of a mapping. Therefore, one supposes $k = m$ and L_A is invertible. Thus, $\det A \neq 0$, and the invert A^{-1} exists, and $L_A^{-1} = L_{A^{-1}}$. Denote by $GL(\mathbb{R}^m)$ the space of invertible linear maps of $\mathbb{R}^m$.

In the simplest case $m = 1$, the space $GL(\mathbb{R}^1)$ is the set of the maps $A(x) = kx$, $k \neq 0$.

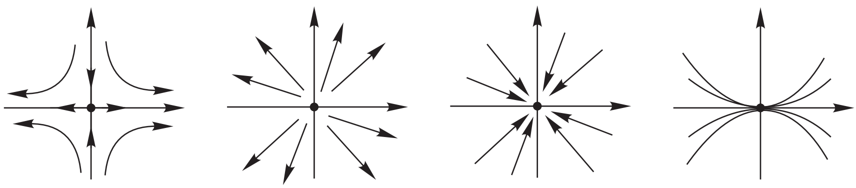
Below, we are considering some examples of maps from $GL(\mathbb{R}^2)$. The base is canonical: $\vec{e}_1 = (1; 0)$, $\vec{e}_2 = (0; 1)$. It is easy to extend these examples to $GL(\mathbb{R}^m)$, $m \geqslant 3$.

Example 1.11 *Canonical hyperbolic saddle.*

Let $L_A \in GL(\mathbb{R}^2)$ is defined by the diagonal matrix

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

where $|\lambda_1| > 1$, $|\lambda_2| < 1$. Since L_A is invertible, $\lambda_1\lambda_2 \neq 0$. The point $(0; 0)$ is fixed under L_A, $L_A(0; 0) = (0; 0)$. Given any $(x_0; y_0) \in \mathbb{R}^2$, $L_A^n(x_0; y_0) = (\lambda_1^n x_0; \lambda_2^n y_0)$. Because of $\lambda_1^n \to \pm\infty$ and $\lambda_2^n \to 0$ as $n \to +\infty$, we see that the map L_A expands along $\vec{e}_1$, and contracts along $\vec{e}_2$, Fig. 1.11 (a), so the fixed point $(0; 0)$ is called a *canonical hyperbolic saddle*. Note that if $\lambda_1\lambda_2 > 0$, L_A preserves orientation. Otherwise, L_A reverses orientation. $\diamond$



(a) (b) (c) (d)

Figure 1.11

Remark that if $\lambda_1 = 1$ or $\lambda_2 = 1$, $(0; 0)$ is not isolated fixed point. If $\lambda_1 = -1$ or $\lambda_2 = -1$, $(0; 0)$ is not isolated periodic point.

Example 1.12 *Canonical hyperbolic node.*

Let us consider linear maps defined by the following matrices

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad B = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \quad C = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix},$$

where $|\lambda| \neq 1$, and either $|\lambda_1| < 1$, $|\lambda_2| < 1$ or $|\lambda_1| > 1$, $|\lambda_2| > 1$. Since L_A, and L_B, and L_C are invertible, $\lambda_1 \lambda_2 \lambda \neq 0$. The study of dynamics is similar to study of the hyperbolic saddle above, so the details are left to the reader. The dynamics of L_A is shown in Fig. 1.11 (b) for $|\lambda_1| > 1$, $|\lambda_2| > 1$, and in Fig. 1.11 (c) for $|\lambda_1| < 1$, $|\lambda_2| < 1$ (L_B is a particular case of L_A). The dynamics of L_C is shown in Fig. 1.11 (d). $\diamond$

Example 1.13 *Canonical focus and center.*

Let $L_A \in GL(\mathbb{R}^2)$ is defined by the matrix

$$A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix},$$

where $\beta \neq 0$. For simplicity, assume $\beta = 1$. To study the dynamics we pass to the polar coordinates:

$$x = \rho \cos \varphi, \quad y = \rho \sin \varphi.$$

Then L_A becomes

$$\begin{cases} \overline{\rho} \cos \overline{\varphi} = \alpha \rho \cos \varphi + \rho \sin \varphi \\ \overline{\rho} \sin \overline{\varphi} = -\rho \cos \varphi + \alpha \rho \sin \varphi \end{cases}$$

It follows

$$\overline{\rho}^2 = \rho^2 \big(\alpha^2 \cos^2 \varphi + 2\alpha \sin \varphi \cos \varphi + \sin^2 \varphi + \\ + \cos^2 \varphi - 2\alpha \sin \varphi \cos \varphi + \alpha^2 \sin^2 \varphi \big).$$

Hence,

$$\overline{\rho} = \rho \sqrt{1 + \alpha^2}. \tag{1.14}$$

Further,

$$\tan \overline{\varphi} = \frac{-\cos \varphi + \alpha \sin \varphi}{\alpha \cos \varphi + \sin \varphi} = \frac{\tan \varphi - \frac{1}{\alpha}}{1 + \frac{1}{\alpha} \tan \varphi}.$$

Hence

$$\tan\overline{\varphi} = \tan(\varphi - \theta), \tag{1.15}$$

where $\tan\theta = \frac{1}{\alpha}$.

Taking in mind (1.14) and (1.15), we see that the dynamics of L_A is given by Fig. 1.12 (a) for $\alpha \neq 0$ and by Fig. 1.12 (c) for $\alpha = 0$. In the last case, the fixed point $(0; 0)$ is called a *canonical center*. If $\alpha \neq 0$, the fixed point $(0; 0)$ is called a *canonical* (*hyperbolic*) *focus*. $\diamond$

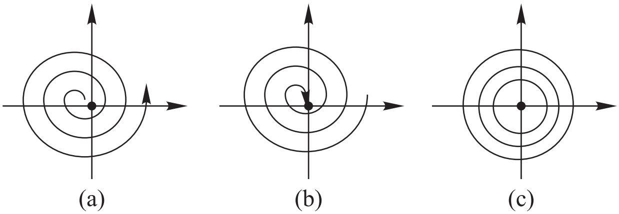


Figure 1.12

Remark 1.1 *Note that the dynamics could be represented by Fig.* 1.12 (*b*)*, when* $|\beta| < 1$.

This example shows that to study the dynamics of linear map it is natural to find a base such that the linear map is represented by a matrix of simplest type, similar in sense to diagonal.

Real Jordan canonical form

Let now A be a real $m \times m$ matrix. Fix a base$\{\vec{e}_i\}_{i=1}^m$ of $\mathbb{R}^m$, so A defines the linear map $L_A\colon \mathbb{R}^m \to \mathbb{R}^m$.

Recall that a linear map A has a (scalar) eigenvalue $\lambda \in \mathbb{R}$ if there is a nonzero eigenvector $\vec{v} \in \mathbb{R}^m$ such that $A(\vec{v}) = \lambda \cdot \vec{v}$. If A has m pairwise different real eigenvalues $\lambda_1, \ldots, \lambda_m$ then there are m linear independent eigenvectors $\vec{v}_1, \ldots, \vec{v}_m$ respectively. Thus, the eigenvectors form a base such that the linear map L_A above is defined by the diagonal matrix

$$\operatorname{diag}(\lambda_1, \ldots, \lambda_m) = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_m \end{pmatrix}.$$

In general, one can find a base such that a linear map is described by a matrix with so-called Jordan blocks,

$$A = \mathrm{diag}(\lambda_i), \quad B = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix},$$

$$C = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \quad D = \begin{pmatrix} C & E_2 \\ 0 & C \end{pmatrix},$$

where E_2 is the 2×2 identity matrix. One can prove (see [77, 119]) that given any invertible linear map $L\colon \mathbb{R}^m \to \mathbb{R}^m$, there is a base such that L determines the $m \times m$ matrix M, $L = L_M$, of the type

$$M = \mathrm{diag}(A_1 \dots A_{n_1} B_1 \dots B_{n_2} C_1 \dots C_{n_3} D_1 \dots D_{n_4}),$$

where A_i, B_i, C_i, and D_i are Jordan blocks described above.

As a consequence, the dynamics of invertible linear map $\mathbb{R}^m \to \mathbb{R}^m$ splits into corresponding dynamics. This means the following. Each Jordan block corresponds to the invariant linear subspace $E_{1n_1}, \dots, E_{n_1 n_1}, E_{1n_2}, \dots, E_{n_2 n_2}, E_{1n_3}, \dots, E_{n_3 n_3}, E_{1n_4}, \dots, E_{n_4 n_4}$ such that their direct sum equals $\mathbb{R}^m$. The restrictions of L on the invariant subspaces are defined by the blocks. Given any point $P \in \mathbb{R}^m$, one can take the projections of P on E_{jn_i}. The behavior of the projection under L is defined by the restriction $L_{E_{jn_i}}$. Since the dynamics of $L_{E_{jn_i}}$ are independent, one can easily conclude the picture of global behavior of P under L.

Definition 1.1 *An invertible linear map $L_A\colon \mathbb{R}^m \to \mathbb{R}^m$ is **hyperbolic** if non of the eigenvalues of the matrix A have modulus* 1.

The dynamics of hyperbolic linear map splits into dynamics of Examples 1.11–1.13 excluding the center fixed point. As a consequence, one gets the following description.

Lemma 1.16 *Let $L_A\colon \mathbb{R}^m \to \mathbb{R}^m$ be a linear hyperbolic map with s eigenvalues of modulus less than* 1 *and u eigenvalues with modulus greater than* 1*, $s + u = m$. Then there is a splitting $\mathbb{R}^m = E^s \oplus E^u$ into invariant subspaces E^s, E^u, where $\dim E^s = s$ and $\dim E^u = u$, such that*

- $|L_A^n(\vec{v})| \to 0$ *as* $n \to +\infty$ *if* $\vec{v} \in E^s$;
- $|L_A^{-n}(\vec{v})| \to 0$ *as* $n \to +\infty$ *if* $\vec{v} \in E^u$.

1.5. Grobman – Hartman Theorem

In this section, we introduce the notion of hyperbolic periodic point for diffeomorphisms and fixed point for flows. One needs to define carefully the notion of derivation for diffeomorphisms of manifolds.

Derivative of a map

Roughly speaking, a manifold is a set that is locally homeomorphic to Euclidean space. To be precise, an d-manifold M^d is a second countable metric space covered by open subsets $\{U_\alpha\}$ such that 1) given any α, there is a homeomorphism $\varphi_\alpha \colon \mathbb{R}^d \to U_\alpha$; 2) if $U_\alpha \cap U_\beta \neq \varnothing$ then

$$\varphi_{\alpha,\beta} = \varphi_\beta^{-1} \circ \varphi_\alpha \colon \varphi_\alpha^{-1}(U_\alpha \cap U_\beta) \subset \mathbb{R}^d \to \varphi_\beta^{-1}(U_\alpha \cap U_\beta) \subset \mathbb{R}^d$$

is a C^∞ diffeomorphism between open subsets of $\mathbb{R}^d$. The domains U_α are called *coordinate charts*. We see that any point of M^d is endowed with Euclidean coordinates that are smoothly recalculated if the point belongs to two coordinate charts.

Let $f \colon U \subset \mathbb{R}^k \to f(U) \subset \mathbb{R}^m$ be a C^r map. This means that all partial derivatives of r order exist and are continuous. Given any point $p \in U$, the derivative of f at p i. e., the matrix Df_p consisting of the partial derivatives of the first order,

$$Df_p = \left(\frac{\partial f_i}{\partial x_j}(p) \right)$$

should actually be thought of as a linear map $\mathbf{L}(\mathbb{R}^k, \mathbb{R}^m)$. If f is a diffeomorphism (hence, $k = m = d$), Df_p belongs to $GL(\mathbb{R}^d)$.

It is convenient the domain $\mathbb{R}^k$ of Df_p to think of the copy of Euclidean plane with the origin at p. Such $\mathbb{R}^k$ is usually considered as the tangent space at p denoted by $T_p(\mathbb{R}^k)$. The similar is referred to the point $f(p)$. From this point of view, D_pf is a linear map taking the tangent space $T_p(\mathbb{R}^k)$ into the tangent space $T_{f(p)}(\mathbb{R}^m)$.

Let M and N be two manifolds. One says that $f \colon M \to N$ is a C^r map provided for each point $p \in M$ and coordinate charts $\varphi_\alpha \colon \mathbb{R}^k \to U_\alpha$ and $\varphi_\beta \colon \mathbb{R}^m \to U_\beta$ at p and $f(p)$ respectively, the map $\varphi_\beta^{-1} \circ f \circ \varphi_\alpha$ is a C^r mapping. It maps $\mathbb{R}^k$ into $\mathbb{R}^m$ where $k = \dim M$, $m = \dim N$. Put by definition, the derivative of f at p is the linear map

$$Df_p = D(\varphi_\beta^{-1} \circ f \circ \varphi_\alpha(p)) \colon T_pM \to T_{f(p)}N.$$

One can prove that this definition does not depend on the choice of coordinate charts.

Hyperbolic periodic points

Let $f\colon M^d \to M^d$ be a diffeomorphism with a fixed point $p = f(p)$. This point is called *hyperbolic* if no eigenvalues of $Df_p\colon TM_p^d \to TM_p^d$ belong to the unit circle $|z| = 1$. It is easy to check that the fixed points in Examples 1.11–1.13 are hyperbolic. A periodic point q of f of period k is called *hyperbolic* if q is a hyperbolic fixed point of f^k. The following statement is a part of Grobman – Hartman theorem for diffeomorphisms.

Theorem 1.3 *Let $f\colon \mathbb{R}^d \to \mathbb{R}^d$ be a C^r diffeomorphism with a hyperbolic fixed point p. Then there are neighborhoods U of p and V of O and a homeomorphism $h\colon V \to U$ such that*

$$f \circ h(x) = h \circ Df_p(x)$$

for all $x \in V$.

Let f^t be a flow with a fixed point x_0 on a d-manifold M. Let $\varphi\colon \mathbb{R}^d \to U$ be a coordinate chart containing x_0. In the right coordinates, the smoothly enough flow f^t is defined by the system of differential equations

$$\begin{cases} \dot{x}_1 = F_1(x_1, \ldots, x_d) = a_{11}x_1 + \ldots + a_{1d}x_d + \psi_1(x_1, \ldots, x_d) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \dot{x}_d = F_d(x_1, \ldots, x_d) = a_{d1}x_1 + \ldots + a_{dd}x_d + \psi_d(x_1, \ldots, x_d) \end{cases}$$

where $\psi_j(x_0) = 0 = D\psi(x_0)$. The point x_0 is called *hyperbolic* if none of the eigenvalues of the matrix

$$\begin{pmatrix} a_{11} & \cdots & a_{1d} \\ \vdots & \ddots & \vdots \\ a_{d1} & \cdots & a_{dd} \end{pmatrix}$$

has zero real part. One can prove that this definition does not depend on the choice of coordinate chart. The following statement is a part of Grobman – Hartman theorem for flows.

Theorem 1.4 *Let x_0 be a hyperbolic fixed point of a flow f^t defined by the differential equation $\dot{x} = F(x)$ near the point x_0. Let $O \in \mathbb{R}^d$ be the fixed point of the flow F^t defined by the linear differential equation $\dot{x} = Ax$ where $A = DF_{x_0}$. Then f^t is locally equivalent at x_0 to F^t at the point O.*

We restrict ourself by the proof in the simplest case for diffeomorphisms when $d = 1$. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a diffeomorphism with the hyperbolic fixed point x_0. The hyperbolicity means that $|Df(x_0)| \neq 1$. To be definite, assume that $|Df(x_0)| < 1$. If $|Df(x_0)| > 1$, the proof is similar. Let U be a neighborhood of x_0 such that $|Df(x)| < 1$ for any $x \in U$. To prove Grobman–Hartman theorem for f it is sufficient to prove that the restriction of f on some segment belonging to U conjugates to the restriction of the linear map $L\colon x \to kx$ on some segment belonging to the origin 0, where $Df(x_0) = k$.

Take a segment $[x_0 - \alpha;\, x_0 + \alpha] \subset U$. Since $|Df(x)| < 1$ for any $x \in \in [x_0 - \alpha;\, x_0 + \alpha]$,

$$|f(x) - x_0| = |f(x) - f(x_0)| = |Df(q)| \cdot |x - x_0| < |x - x_0|,$$
$$\text{where} \quad x \in [x_0 - \alpha;\, x_0 + \alpha], \quad |x - q| \leqslant |x_0 - x|.$$

Hence, $f([x_0 - \alpha;\, x_0 + \alpha]) \subset [x_0 - \alpha;\, x_0 + \alpha]$. Moreover, $f^n(x) \to x_0$ as $n \to +\infty$.

Take any $\beta > 0$. Because of $|k| < 1$, $L([-\beta;\, \beta]) \subset [-\beta;\, \beta]$. Moreover, $L^n(x) \to 0$ as $n \to +\infty$.

There are two possibilities: (1) f is order preserving, $0 < Df(x_0) < 1$; (2) f is order reversing, $-1 < Df(x_0) < 0$. In the first case, let h_0 be any orientation preserving homeomorphism that takes $[f(x_0+\alpha);\, x_0+\alpha]$ to $[k\beta;\, \beta]$ and $[x_0 - \alpha;\, f(x_0 - \alpha)]$ to $[-\beta;\, -k\beta]$ such that $h_0(x_0 + \alpha) = \beta$ and $h_0(x_0 - - \alpha) = -\beta$. Clearly, $h_0(f(x_0 + \alpha)) = k\beta$ and $h_0(f(x_0 - \alpha)) = -k\beta$. So,

$$h_0 \circ f(x_0 + \alpha) = L \circ h_0(x_0 + \alpha), \quad h_0 \circ f(x_0 - \alpha) = L \circ h_0(x_0 - \alpha). \tag{1.16}$$

Given any $x_0 < x \leqslant x_0 + \alpha$, there are unique $n \in \mathbb{Z}_+$ and $z \in (f(x_0 + + \alpha);\, x_0 + \alpha]$ such that $x = f^n(z)$. Similarly given any $x_0 - \alpha \leqslant x < 0$, there are unique $n \in \mathbb{Z}_+$ and $z \in [x_0 - \alpha;\, f(x_0 + \alpha))$ such that $x = f^n(z)$. Put by definition

$$h(x) = L^n \circ h_0 \circ f^{-n}(x) = L^n \circ h_0(s).$$

As a composition of homeomorphisms, h is a homeomorphism at $x \neq f^n(x_0 + + \alpha)$, $n \in \mathbb{N}$. It follows from (1.16) that h is homeomorphism at any $x = = f^n(x_0 + \alpha)$, $n \in \mathbb{N}$. Since $L^n(x) \to 0$ and $f^n(x) \to x_0$ as $n \to +\infty$,

$$H(x) = \begin{cases} h(x), \text{ if } x \neq x_0 \\ 0, \text{ if } x = x_0 \end{cases}$$

is a homeomorphism $[x_0 - \alpha;\, x_0 + \alpha] \to [-\beta;\, \beta]$. Moreover,

$$H \circ f(x) = h \circ f(x) = L^{n+1} \circ h_0 \circ f^{-(n+1)}(x) =$$
$$= L \circ L^n \circ h_0 \circ f^{-n} \circ f^{-1} \circ f(x) = L \circ h(x), \quad \text{for } x \neq x_0.$$

Obviously, $H \circ f(x_0) = H(x_0) = 0 = L(0) = L \circ H(x_0)$. Hence, H is a conjugacy map between $f|_{[x_0-\alpha;\, x_0+\alpha]}$ and $L|_{[-\beta;\, \beta]}$.

In the second case, when f is order reversing ($-1 < Df(x_0) = k < 0$), denote by I_f the segment $[f^2(x_0 + \alpha);\, x_0 + \alpha]$ and denote by I_0 the segment $[k^2\beta;\, \beta]$. Now L is the linear map $x \to -kx$. Let h_0 be any orientation preserving homeomorphism that takes I_f to I_0 such that $h_0(x_0 + \alpha) = \beta$ and $h_0(f^2(x_0 + \alpha)) = k^2\beta$. Note that

$$\bigcup_{n\geqslant 0} f^n(I_f) = [f(x_0 + \alpha);\, x_0) \cup (x_0;\, x_0 + \alpha],$$

$$\bigcup_{n\geqslant 0} L^n(I_0) = [-k\beta;\, 0) \cup (0;\, \beta].$$

Given any $x \in [f(x_0 + \alpha);\, x_0) \cup (x_0;\, x_0 + \alpha]$, there are unique $n \in \mathbb{Z}_+$ and $z \in (f^2(x_0 + \alpha);\, x_0 + \alpha]$ such that $x = f^n(z)$. Put by definition $h(x) = L^n \circ h_0 \circ f^{-n}(x)$. One can check that H above again is a conjugacy map between $f|_{[f(x_0+\alpha);\, x_0+\alpha]}$ and $L|_{[-k\beta;\, \beta]}$. This completes the proof of Grobman – Hartman Theorem for diffeomorphisms when $d = 1$.

Local Stable and Unstable Manifolds

Lemma 1.16 shows that if $L_A \colon \mathbb{R}^m \to \mathbb{R}^m$ is a linear hyperbolic map then there is a splitting $\mathbb{R}^m = E^s \oplus E^u$ into invariant subspaces E^s, E^u such that

$$|L_A^n(\vec{v})| \to 0 \text{ for } \vec{v} \in E^s, \quad \text{and} \quad |L_A^{-n}(\vec{v})| \to 0 \text{ for } \vec{v} \in E^u \quad \text{as } n \to +\infty.$$

One says that E^s and E^u are *stable and unstable manifolds* of O respectively. The origin O is an attracting fixed point for the restriction $L_A|_{E^s}$ of L_A on E^s. Similarly, O is a repelling fixed point for $L_A|_{E^u}$.

Grobman – Hartman Theorem says that the dynamics near a hyperbolic fixed point of a diffeomorphism is locally the same with the dynamics near a hyperbolic fixed point of the corresponding linear map. This implies the existence of so-called local stable and unstable manifolds of hyperbolic fixed point for arbitrary diffeomorphism $f\colon M^m \to M^m$. To be precise, for some $\varepsilon > 0$, the sets

$$W^s_\varepsilon(x_0) = \{x \in M^m \mid d(f^n(x); x_0) < \varepsilon \ \forall n \in \mathbb{N}, \text{ and } f^n(x) \to x_0 \text{ as } n \to +\infty\},$$

$$W^u_\varepsilon(x_0) = \{x \in M^m \mid d(f^{-n}(x); x_0) < \varepsilon \ \forall n \in \mathbb{N}, \text{ and } f^{-n}(x) \to x_0 \text{ as } n \to +\infty\},$$

are smoothly embedded manifolds

$$J_s\colon \mathbb{R}^{\dim E^s} \to W^s_\varepsilon(x_0) \subset M^m, \quad J_u\colon \mathbb{R}^{\dim E^u} \to W^u_\varepsilon(x_0) \subset M^m$$

respectively. They are called *local stable* and *local unstable* manifolds respectively of the fixed point x_0 of diffeomorphism $f\colon M^m \to M^m$, Fig. 1.13.

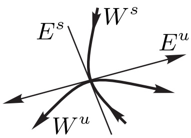


Figure 1.13. Local stable and unstable manifolds.

As a consequence, one can prove that given any hyperbolic periodic point, there are local stable and unstable manifolds (for some $\varepsilon > 0$). The sum of dimensions of these manifolds equals to the dimension of the ambient manifold M^m, $\dim E^s + \dim E^u = \dim M^m$. Below, we represent the more general result on the existence of stable and unstable manifolds for arbitrary hyperbolic set.

1.6. Hyperbolic Sets

Hyperbolicity of invariant set is characterized by local expansion and contraction in complementary directions at each point of the invariant set. This notion is an extension of hyperbolicity of periodic point to arbitrary invariant set.

Let f be a C^∞ diffeomorphism of a closed manifold M^d endowed with some Riemannian metric. A closed f-invariant set $\Lambda \subset M^d$ is called *hyperbolic* (one says also, Λ *has a hyperbolic structure*) if the tangent bundle $T_\Lambda M^d$ is the Whitney sum of two Df-invariant bundles,

$$T_\Lambda M^d = \mathbb{E}^s_\Lambda \oplus \mathbb{E}^u_\Lambda,$$

and there are constants $C_s > 0$, $C_u > 0$, and $0 < \lambda < 1$ such that

$$\|Df^i(v)\| \leqslant C_s \lambda^i \|v\|,\ v \in \mathbb{E}^s_\Lambda, \quad \|Df^{-i}(w)\| \leqslant C_u \lambda^i \|w\|,\ w \in \mathbb{E}^u_\Lambda, \quad i > 0.$$

The subspace $\mathbb{E}^u_x$ (respectively, $\mathbb{E}^s_x$) is called the *unstable* (respectively, *stable*) subspace at $x \in \Lambda$. The family $\{\mathbb{E}^u_x\}_{x\in\Lambda}$ ($\{\mathbb{E}^s_x\}_{x\in\Lambda}$) forms the *unstable* (*stable*) bundle $\mathbb{E}^u_\Lambda$ ($\mathbb{E}^s_\Lambda$).

Lemma 1.17 *Let Λ be a hyperbolic set of diffeomorphism f. The subspaces $\mathbb{E}^u_x$, $\mathbb{E}^s_x$ depend continuously on $x \in \Lambda$.*

Proof. Let $x_j \in \Lambda$ be a sequence converging to $x_\infty \in \Lambda$. Passing to a subsequence, one can assume that $\dim \mathbb{E}^s_{x_j}$ is constant. Let $v_j \in \mathbb{E}^s_{x_j}$ be a sequence converging to $v_\infty \in \mathbb{E}^s_{x_\infty}$. Passing to the limit as $j \to +\infty$ in

$$\|Df^n(v_j)\| \leqslant C_s \lambda^n \|v_j\|,$$

we see that $\dim \mathbb{E}^s_{x_\infty} \geqslant \dim \mathbb{E}^s_{x_j}$. A similar argument shows that $\dim \mathbb{E}^u_{x_\infty} \geqslant$ $\geqslant \dim \mathbb{E}^u_{x_j}$. It follows from

$$\dim \mathbb{E}^s_{x_\infty} + \dim \mathbb{E}^u_{x_\infty} = \dim \mathbb{E}^s_{x_j} + \dim \mathbb{E}^u_{x_j}$$

that $\dim \mathbb{E}^s_{x_\infty} = \dim \mathbb{E}^s_{x_j}$ and $\dim \mathbb{E}^u_{x_\infty} = \dim \mathbb{E}^u_{x_j}$. The result is proved. □

Example 1.14 *Linear hyperbolic transformation of Euclidean space.*

Let $\overline{L}_A = \mathbb{R}^m \to \mathbb{R}^m$ be an invertible linear transformation defined by a real $m \times m$ matrix $A = (a_{ij})$. Let $\operatorname{Spec}(\overline{L}_A) = \{\lambda_1, \ldots, \lambda_m\}$ be the eigenvalues of L_A, and suppose that

$$|\lambda_1| \leqslant \cdots \leqslant |\lambda_s| < 1 < |\lambda_{s+1}| \leqslant \cdots \leqslant |\lambda_m|.$$

Then there are vector spaces $\mathbb{E}^s$, $\mathbb{E}^u$ of dimensions s and $m - s$ respectively such that $\mathbb{R}^m = \mathbb{E}^s \oplus \mathbb{E}^u$ where $\mathbb{E}^s$ is spanned by generalized eigenvectors corresponding to $\lambda_1, \ldots, \lambda_s$, and $\mathbb{E}^u$ is spanned by generalized eigenvectors corresponding to $\lambda_{s+1}, \ldots, \lambda_m$. We see that the restriction $\overline{L}_A\big|_{\mathbb{E}^s}$ is uniformly attractive while $\overline{L}_A\big|_{\mathbb{E}^u}$ is uniformly expanding. Since the tangent space at each point of $\mathbb{R}^m$ is canonically isomorphic to $\mathbb{R}^m$ itself, $\mathbb{R}^m$ is a hyperbolic set of $\overline{L}_A$. ♢

Example 1.15 *Anosov linear torus automorphism.*

We keep the notation of Example 1.14. Suppose that the matrix A is integer and unimodular, $a_{ij} \in \mathbb{Z}$ and $\det A = \pm 1$. Then $\overline{L}_A$ induces the hyperbolic torus automorphism $L_A\colon \mathbb{T}^m \to \mathbb{T}^m$, see Example 1.9. It follows from Example 1.14 that $\mathbb{T}^m$ is a hyperbolic set of L_A. Often, L_A is called an *Anosov linear torus automorphism.* $\diamondsuit$

The Horseshoe

Here we represent the famous example of nontrivial hyperbolic set discovered by Smale [207, 208], so-called Smale horseshoe. The Smale horseshoe map $f\colon D^2 = [0;\, 1]^2 \to \mathbb{R}^2$ is defined by contracting the x direction and expanding the y direction, and then twist the result around as indicated in Fig. 1.14.

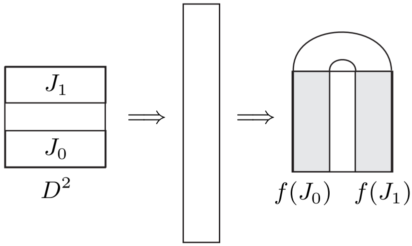


Figure 1.14

The map f can be easily extended to the diffeomorphism of $\mathbb{R}^2$ or the sphere S^2. Since the most interesting dynamics is on D^2, we only describe this part of f analytically. Set

$$J_0 = [0;\, 1] \times \left[0;\, \frac{1}{3}\right], \quad J_1 = [0;\, 1] \times \left[\frac{2}{3};\, 1\right]$$

and define

$$f|_{J_0}\colon J_0 \to f(J_0), \quad (x;\, y) \mapsto \left(\frac{1}{3}x;\, 3y\right), \tag{1.17}$$

respectively

$$f|_{J_1}\colon J_1 \to f(J_1), \quad (x;\, y) \mapsto \left(1 - \frac{1}{3}x;\, 3(1-y)\right). \tag{1.18}$$

The set $D^2 \cap f(D^2)$ consists of two vertical rectangles $f(J_0) \overset{\text{def}}{=} R_0$ and $f(J_1) \overset{\text{def}}{=} R_1$ of the width $\frac{1}{3}$. The set $D^2 \cap f(D^2) \cap f^2(D^2)$ consists of for vertical rectangles R_{00}, $R_{01} \subset R_0$, R_{10}, $R_{11} \subset R_1$ of the width $\frac{1}{3^2}$. Here,

$$R_{00} = R_0 \cap f(R_0), \quad R_{01} = R_0 \cap f(R_1),$$
$$R_{10} = R_1 \cap f(R_0), \quad R_{11} = R_1 \cap f(R_1).$$

For any finite sequence $\{i_0, i_1, \ldots, i_n\}$ of zeros and ones,

$$R_{i_0, i_1, \ldots, i_n} = R_{i_0} \cap f(R_{i_1}) \cap \cdots \cup f^n(R_{i_n})$$

is a vertical rectangle of width $\frac{1}{3^n}$. Moreover, $D^2 \cap f^n(D^2)$ is the union of 2^n rectangles of width $\frac{1}{3^n}$, see the left part of Fig. 1.15.

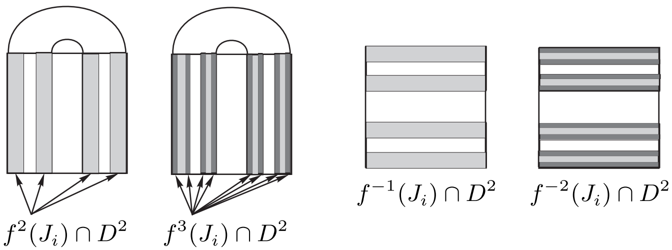


Figure 1.15

For an infinite sequence $\mathscr{I} = \{i_0, i_1, \ldots, i_n, \ldots\}$ of zeros and ones,

$$R_{\mathscr{I}} = \bigcap_{n=0}^{+\infty} f^n(R_{i_n})$$

is a vertical segment through the classical Cantor set $C^+ \subset [0; 1] \times \{0\}$. So, the set

$$\Lambda^+ = \bigcap_{n=0}^{+\infty} f^n(D^2) = \bigcup_{\mathscr{I} \in \{0, 1\}^{\mathbb{N}}} R_{\mathscr{I}}$$

is the product of a vertical segment of length one and the Cantor set C^+. Clearly, that all negative iterations of a point of Λ^+ stay in D^2.

We can now construct, in a similar way, a set Λ^- using pre-images. We saw that $D^2 \cap f^{-1}(D^2)$ consists of two horizontal rectangles J_0 and J_1 of height $\frac{1}{3}$. The set $D^2 \cap f^{-1}(D^2) \cap f^{-2}(D^2)$ consists of four rectangles J_{00}, $J_{01} \subset J_0$, J_{10}, $J_{11} \subset J_1$ of the height $\frac{1}{3^2}$, see the right part of Fig. 1.15. Here,

$$J_{00} = J_0 \cap f^{-1}(J_0), \quad J_{01} = J_0 \cap f^{-1}(J_1),$$
$$J_{10} = J_1 \cap f^{-1}(J_0), \quad J_{11} = J_1 \cap f^{-1}(J_1).$$

For any finite sequence $\{j_n, j_{n-1}, \dots, j_1\}$ of zeros and ones,

$$J_{j_n, j_{n-1}, \dots, j_1} = J_{j_n} \cap f^{-1}(J_{j_{n-1}}) \cap \dots \cup f^{-n}(J_{j_0})$$

is a horizontal rectangle of width $\frac{1}{3^n}$. For an infinite sequence $\mathscr{J} = \{\dots, j_n, j_{n-1}, \dots, j_1\}$ of zeros and ones,

$$J_{\mathscr{J}} = \bigcap_{n=0}^{+\infty} f^{-n}(J_{j_n})$$

is a horizontal segment through the classical Cantor set $C^- \subset \{0\} \times [0;\,1]$. The set

$$\Lambda^- = \bigcap_{n=0}^{+\infty} f^{-n}(D^2) = \bigcup_{\mathscr{J} \in \{0,\,1\}^{\mathbb{N}}} J_{\mathscr{J}}$$

is the product of a horizontal segment of length one and the Cantor set C^-. Note that all positive iterations of a point of Λ^- stay in D^2.

The *horseshoe set* $\Lambda = \Lambda^+ \cap \Lambda^-$ is the product of two Cantor sets C^+, C^-. Obviously, Λ is closed and f-invariant. It follows from (1.17) and (1.18) that Λ is a hyperbolic set.

Proposition 1.6 *The restriction $f|_\Lambda$ of the Smale horseshoe map f on the horseshoe set Λ conjugates to the full two-sided shift Σ_2.*

Proof. By construction, given any infinite sequence $\{\dots, j_n, j_{n-1}, \dots, j_1, i_0, i_1, \dots, i_n, \dots\} = \omega \in \{0,\,1\}^{\mathbb{N}}$, one corresponds a unique point $\varphi(\omega) = J_{\mathscr{J}} \cap R_{\mathscr{I}}$, and vise versa. Clearly that this correspondence φ is a homeomorphism $\Sigma_2 \to \Lambda$. Moreover,

$$f \circ \varphi(\omega) = f\left(\bigcap_{n=0}^{+\infty} f^{-n}(J_{j_n}) \bigcap_{n=0}^{+\infty} f^{n}(R_{i_n})\right) =$$
$$= \bigcap_{n=1}^{+\infty} f^{-n}(J_{j_n}) \bigcap_{n=0}^{+\infty} f^{n+1}(R_{i_n}) \cup D^2 = \varphi \circ \sigma_r(\omega),$$

where σ_r is the right shift in Σ_2. Thus, φ is a conjugacy map between $f|_\Lambda$ and (Σ_2, σ_r). □

It follows from Lemma 1.10 and Proposition 1.6 the following corollary.

Corollary 1.2 *Let $f|_\Lambda$ be the restriction of the Smale horseshoe map f on the horseshoe set Λ. Then*

- *periodic points are dense in Λ;*
- *transitive orbits are dense in Λ;*
- *every point of Λ is non-wandering.*

In particular, $f|_\Lambda$ is transitive.

Stable and Unstable Manifolds

Here, omitting proofs, we represent two important and essential results of Hyperbolic Dynamics on the existence of local and global stable and unstable manifolds for a hyperbolic set. Suppose that a manifold M admits a Riemannian structure, and so, M endowed with the corresponding metric $d\colon M \times M \to \mathbb{R}_+$. Similarly to the case of hyperbolic periodic points, there are local stable and unstable manifolds of points of hyperbolic set.

Theorem 1.5 *Let $f\colon M \to M$ be a C^1 diffeomorphism and $\Lambda \subset M$ a hyperbolic set. Then there is $\varepsilon > 0$ such that for every point $x \in \Lambda$*

- *the sets*

$$W^s_\varepsilon(x) = \{y \in M\colon d(f^m(x),\, f^m(y)) < \varepsilon \textit{ for all } m \in \mathbb{N}\},$$
$$W^u_\varepsilon(x) = \{y \in M\colon d(f^{-m}(x),\, f^{-m}(y)) < \varepsilon \textit{ for all } m \in \mathbb{N}\}$$

  are C^1 embedded Euclidean planes $\mathbb{R}^{\dim \mathbb{E}^s}$, $\mathbb{R}^{\dim \mathbb{E}^u}$ respectively;
- $T_x W^s_\varepsilon(x) = \mathbb{E}^s$, $T_x W^u_\varepsilon(x) = \mathbb{E}^u$;
- $f\left(W^s_\varepsilon(x)\right) \subset W^s_\varepsilon(fx)$, $f^{-1}\left(W^u_\varepsilon(x)\right) \subset W^u_\varepsilon(f^{-1}x)$.

After local manifolds, it is natural to introduce (global) stable and unstable manifolds

$$W^s(x) = \bigcup_{m=0}^{\infty} f^{-m} W^s_\varepsilon(f^m(x)), \quad W^u(x) = \bigcup_{m=0}^{\infty} f^m W^u_\varepsilon(f^{-m}(x)), \quad x \in \Lambda$$

that satisfy the following statement.

Theorem 1.6 *Let $f\colon M \to M$ be a C^r diffeomorphism and Λ a hyperbolic set. Then given any $p \in \Lambda$, there are C^r immersions*

$$J_s\colon \mathbb{R}^{s_0} \to M, \quad J_u\colon \mathbb{R}^{u_0} \to M,$$

where $s_0 = \dim \mathbb{E}^s_p$ and $u_0 = \dim \mathbb{E}^u_p$, such that

1) *$J_s(\mathbb{R}^{s_0}) \overset{\text{def}}{=} W^s(p)$ and $J_u(\mathbb{R}^{u_0}) \overset{\text{def}}{=} W^u(p)$ are embedded Euclidean spaces $\mathbb{R}^{s_0}$ and $\mathbb{R}^{u_0}$ respectively (so, the both $J_s\colon \mathbb{R}^{s_0} \to W^s(p)$ and $J_u\colon \mathbb{R}^{u_0} \to W^u(p)$ are one-to-one).*
2) *$W^s(p)$ is tangent to $\mathbb{E}^s_p$, and $W^u(p)$ is tangent to $\mathbb{E}^u_p$,*

$$T_P W^s(p) = \mathbb{E}^s_p, \quad T_P W^u(p) = \mathbb{E}^u_p.$$

3) *Given any $q \in W^s(p)$ the distance $d(f^n(p), f^n(q))$ tends exponentially to 0 as $n \to +\infty$; similarly, given any $q \in W^u(p)$ the distance $d(f^n(p), f^n(q))$ tends exponentially to 0 as $n \to -\infty$.*
4) *The both family $W^s(\Lambda) = \{W^s(x)\}_{x\in\Lambda}$, $W^u(\Lambda) = \{W^u(x)\}_{x\in\Lambda}$ are invariant under f:*

$$f(W^s(x)) = W^s(f(x)), \quad f(W^u(x)) = W^u(f(x)) \quad \textit{for any } x \in \Lambda.$$

5) *Given any $x, y \in \Lambda$, the sets $W^{s(u)}(x)$, $W^{s(u)}(y)$ are either coincide or disjoint.*

Proof can be found in [116, 117, 198].

The set above $W^{s(u)}(p)$ is called *stable* (*stableunstable*) *stablemanifold* of a point $p \in \Lambda$. Clearly that the unstable manifold $W^u(p)$ is a stable one for the diffeomorphism f^{-1}.

Local Product Structure

Let Λ be a hyperbolic set. By Lemma 1.17, the subspaces $\mathbb{E}^s_x$, $\mathbb{E}^u_x$ depend continuously on $x \in \Lambda$. Moreover, given any point $x \in \Lambda$, $\mathbb{E}^s_x \cap \mathbb{E}^u_x = \{0\}$. Therefore, ***the local stable and unstable manifolds*** of x ***intersect at*** x ***transversally***, $W^s_\varepsilon(x) \pitchfork W^u_\varepsilon(x) = \{x\}$. As a consequence, one gets

Lemma 1.18 *Let Λ be a hyperbolic set of f. Then for every $\varepsilon > 0$ there is $\delta > 0$ such that if $x, y \in \Lambda$ and $d(x, y) < \delta$ then the intersection $W^s_\varepsilon(x) \cap W^u_\varepsilon(x)$ is transverse and consists of a unique point denoted by $[x, y]$, Fig.* 1.16 (*a*). *Moreover, $[x, y]$ depends continuously on x and y, Fig.* 1.16 (*b*).

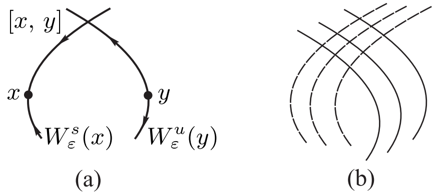


Figure 1.16. Intersections of local manifolds.

Note that the point $[x, y]$ from Lemma 1.18 might does not belong to Λ. Hyperbolic sets with a such feather form the special class of hyperbolic sets. We say that a hyperbolic set Λ has a *local product structure* if there are $\varepsilon > 0$ and $\delta > 0$ such that

- for all $x, y \in \Lambda$, the intersection $W^s_\varepsilon(x) \cap W^u_\varepsilon(y)$ consists of at most one point which belongs to Λ;
- for all $x, y \in \Lambda$ with $d(x, y) < \delta$, the intersection $W^s_\varepsilon(x) \cap W^u_\varepsilon(y)$ consists of exactly one point, which belongs to Λ, and the intersection is transverse.

The Smale horseshoe gives the example of hyperbolic set with a local hyperbolic structure. Here is an example by Rufus Bowen that shows that there is a compact part of Horseshoe with no a local product structure.

Example 1.16 *Bowen example of compact hyperbolic set with no local product structure.*

Let Λ be the hyperbolic set of the Smale horseshoe. We see that Λ one can thought as the full two-sided shift Σ_2 i. e., the space of sequences of zeros and ones with the shift map σ_r. Consider the subset Λ_0 of Λ, and correspondingly of Σ_2,

$$\Lambda_0 = \{\omega \in \Sigma_2 \colon \text{any finite maximal string of 0's is of even length}\}.$$

Obviously, Λ_0 is a compact σ_r-invariant subset of Λ. Denote by $\sigma_j \in \Lambda_0$, $j \in \mathbb{N}$, the sequence containing the maximal string $[-j, j]$ of zeros,

$$\omega_{-j-1} = 1,\ \omega_{-j} = 0,\ \ldots,\ \omega_{-1} = 0,\ \omega_0 = 0,\ \ldots,\ \omega_{j-1} = 0,\ \omega_j = 1.$$

Then $d(\sigma_j, \sigma_{j+1})$ is arbitrarily small when j is large. Take $\sigma_* \in \Lambda$ that is σ_j with the unique difference $\omega_{-j-1} = 0$. Then $\sigma_* \in \Lambda$ but $\sigma_* \notin \Lambda_0$. Given any $\varepsilon > 0$, there is j such that $d(\sigma_j, \sigma_*) < \varepsilon$ and $d(\sigma_*, \sigma_{j+1}) < \varepsilon$. Moreover,

$$d\left(\sigma_r^k(\sigma_*), \sigma_r^k(\sigma_{j+1})\right) \to 0, \text{ and } d\left(\sigma_r^{-k}(\sigma_*), \sigma_r^{-k}(\sigma_j)\right) \to 0 \text{ as } k \to +\infty.$$

It follows that $\sigma_* \in W^s_\varepsilon(\sigma_{j+1})$ and $\sigma_* \in W^u_\varepsilon(\sigma_j)$. Hence, $\sigma_* \in W^s_\varepsilon(\sigma_{j+1}) \cap \cap W^u_\varepsilon(\sigma_j)$. We see that $\sigma_* = [\sigma_j, \sigma_{j+1}]$, but $\sigma_* \notin \Lambda_0$. $\diamondsuit$

Locally Maximal Hyperbolic Sets

Let $U = U(\Lambda)$ be some neighborhood of hyperbolic set Λ. Denote by $I(U)$ the points of U whose orbits belong to U i. e.,

$$I(U) = \left\{ x \in U : O(x) = \bigcup_{m \in \mathbb{Z}} f^m(x) \subset U \right\}.$$

Clearly, $I(U)$ is invariant and equals

$$I(U) = \bigcap_{m \in \mathbb{Z}} f^m(U).$$

Moreover, $I(U)$ contains Λ and is the maximal invariant set belonging to $U(\Lambda)$.

A hyperbolic set Λ is called *locally maximal* if there exists a neighborhood U of Λ such that

$$\Lambda = \bigcap_{m \in \mathbb{Z}} f^m(U) = I(U).$$

One can say that Λ is locally maximal if there is a neighborhood U of Λ such that any invariant set contained entirely inside of U is a subset of Λ.

The Horseshoe is an example of a locally maximal hyperbolic set. The set Λ from Example 1.16 is not locally maximal. This can be checked directly or due to the following result.

Theorem 1.7 *A hyperbolic set is locally maximal if and only if it has a local product structure.*

The proof can be found in [54], Proposition 5.9.3.

1.7. Basic Theorems for Hyperbolic Sets

Here, we present a list of basic results of Hyperbolic Dynamics. There are numerous books and monographs with careful proofs. Some of these proofs are technical. Therefore, we omit detailed proofs restricting ourselves by ideas and discussions, and references.

Inclination Lemma (λ-lemma)

Let p be a hyperbolic fixed point of the saddle type, and $W^\tau(p)$ be an invariant manifold of p, where τ is either u or s. Since $W^\tau(p)$ is an C^1 embedded manifold, it is endowed with the natural interior metric d_τ. Denote by $D_r^\tau(p) \subset W^\tau(p)$ the τ-disk of radius $r > 0$ and the center at p. The following statement usually is called the Inclination Lemma or Lambda Lemma.

Theorem 1.8 *Let p be a hyperbolic fixed point of a diffeomorphism $f: M \to M$, and $q \in W^s(p)$. Suppose that $D \ni q$ is a C^1 disk of dimension $u = \dim W^u(p)$ intersecting $W^s(p)$ transversally at q. Then given any $r > 0$, there are a neighborhood U of p and $n_0 \in \mathbb{N}$ such that the components of the intersections*

$$D_1 = f(D) \cap U, \quad D_2 = f(D_1) \cap U, \quad \ldots, \quad D_n = f(D_{n-1}) \cap U, \quad \ldots$$

form the sequence of u-disks for all $n \geqslant n_0$ that converges uniformly to $D_r^u(p)$ in terms of the C^1 topology.

The proof can be found in [181].

The important consequence of the Inclination Lemma is the existence of so-called lamination structure (see below the exact notation) formed by stable or unstable manifolds of the same dimension. Denote by $\mathscr{W}_d^u$ the union of d-dimensional unstable manifolds of hyperbolic points.

Theorem 1.9 *Let $f: M^n \to M^n$ be a diffeomorphism with hyperbolic nonwandering set $NW(f)$ that equals* $\operatorname{clos}(\operatorname{Per}(f))$. *Then given any $1 \leqslant d \leqslant n - 1$ and a point $x \in W^u(y) \subset \mathscr{W}_d^u$, there is a ball-neighborhood U of x such that the components of intersections $U \cap W^u(y)$, $W^u(y) \subset \mathscr{W}_d^u$, are locally equivalent to the family of parallel d-hyperplanes. This means that there is a homeomorphism $\psi: U \to \mathbb{R}^d$ such that every component of $U \cap W^u(y)$ is mapped by ψ onto a d-hyperplane defined by the equations*

$$x_1 = c_1, \quad \ldots, \quad x_{n-d} = c_{n-d} \quad \textit{for some constants } c_1, \ldots, c_{n-d}.$$

The similar theorem holds for stable invariant manifolds $\mathscr{W}_d^u$ of fixed dimension d. Another consequence of the Inclination Lemma is the following statement on the continuous dependence of invariant manifolds under initial conditions on compact disks.

Theorem 1.10 *Stable* (*unstable*) *invariant manifolds of the same fixed dimension depend continuously under initial conditions on compact disks.*

Birkhoff – Smale Theorem

Theorem 1.11 *Let p be a periodic hyperbolic point of a diffeomorphism $f\colon M^d \to M^d$, and q be a transversal homoclinic point for p. Then for each neighborhood U of the pair $\{p, q\}$, there is $n \in \mathbb{N}$ such that f^n has a nontrivial hyperbolic invariant set $\Lambda \subset U$ with $p, q \in \Lambda$ and the restriction $f^n|_\Lambda$ is topologically conjugate to the two-sided shift map on $k \geqslant 2$ symbols, σ on Σ_k. In addition,*

$$\Lambda \subset \mathrm{clos}(\mathrm{Per}(f)) \subset NW(f).$$

We give the *idea of the proof.* Without loss of generality, one can assume that p is a fixed point. Take a compact disk $B^s \subset W^s(p)$ with the dimension $\dim B^s = \dim W^s(p) = s$ such that $p, q \in B^s$. By the extended version of Grobman – Hartman Theorem, there is a compact neighborhood D^d of B^s such that the restriction $f|_{D^d}$ conjugates to the linear diffeomorphism. We can assume that D^d is a tubular neighborhood of B^s that is diffeomorphic to the product $B^s \times B^u$ where the disk $B^u \subset W^u(p)$ has the dimension $\dim B^u = = \dim W^u(p) = u$, Fig. 1.17 (a).

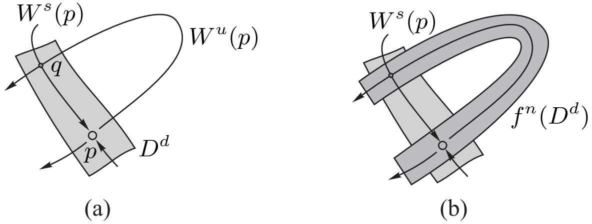
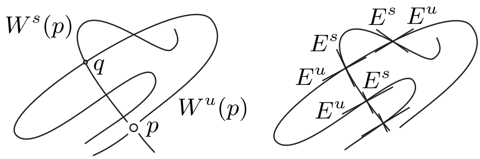


Figure 1.17

Actually, one can think of D^d as a curvilinear high dimensional parallelogram in $\mathbb{R}^d = \mathbb{R}^s \times \mathbb{R}^u$ whose sides are parallel to $\mathbb{R}^s \times \{0\}$ and $\{0\} \times \mathbb{R}^u$ respectively. Moreover, the linear map contracts the parallelogram in $\mathbb{R}^s \times \{0\}$ direction and expands in $\{0\} \times \mathbb{R}^u$ direction.

Because of the existence of the homoclinic point q, there is $n \in \mathbb{N}$ such that $f^n(D^d)$ intersects D^d at least at two components, Fig. 1.17 (b). One of them contains p and an another contains q. We see that $f^n|_{D^d}$ is Smale horseshoe map. By Proposition 1.6, there is a nontrivial invariant locally maximal set Λ belonging to $D^d \cup f^n(D^d)$. Note that the union $D^d \cup f^n(D^d)$ contains the orbit $O(q)$ of the point q, Fig. 1.18 (a), since p is a fixed point.

Denote $\{p\} \cup O(q)$ by Λ_q. Clearly, Λ_q is invariant under f, and $\Lambda_q \subset \Lambda$. We introduce the splitting of the tangent space $T(\Lambda_q)$ into invariant bundles as follows. For p, set $T_p(M^d) = T_x(W^s(p)) \oplus T_x(W^u(p))$. For a point $x \in \in O(q) \cap W^s(p)$, set $E^s_x = T_x(W^s(p))$ and $E^u_x = T_x(W^u(p))$, Fig. 1.18 (b).



(a) (b)

Figure 1.18

Since q is a point of transversal intersection of $W^s(p)$ and $W^u(p)$, $T_x(M^d) = E^s \oplus E^u$. Because of $f(W^s(p)) = W^s(p)$ and $f(W^u(p)) = = W^u(p)$,

$$Df\,(T_x(W^s(p))) = T_{f(x)}W^s(p), \quad Df\,(T_x(W^u(p))) = T_{f(x)}W^u(p).$$

Hence, the splitting $E^s \oplus E^u|_{\Lambda_q}$ is invariant under Df. One can extend the hyperbolic structure to Λ. Now the inclusions $\Lambda \subset \mathrm{clos}(\mathrm{Per}(f)) \subset NW(f)$ follow from Corollary 1.2. $\square$

Spectral Decomposition Theorem

Let $f\colon M \to M$ be a diffeomorphism having hyperbolic periodic points. Denote by $H(f)$ the set of all hyperbolic periodic points of f. We say that a point $p \in H(f)$ is *heteroclinically related to* $q \in H(f)$, or p is *h-related* to q, if $W^u(O(p))$ has a nonempty transverse intersection with $W^s(O(q))$ and $W^u(O(q))$ has a nonempty transverse intersection with $W^s(O(p))$, Fig. 1.19, (a). We also will write $p \overset{h}{\sim} q$.

Lemma 1.19 *If* $p \overset{h}{\sim} q$ *then*

$$\dim W^u_p = \dim W^u_q, \quad \dim W^s_p = \dim W^s_q.$$

Proof. If $\dim W^u_p < \dim W^u_q$ then $\dim W^u_p + \dim W^s_q = \dim W^u_p + + n - \dim W^u_q < n$. This contradicts to the transversality of the intersection $W^u(O(p)) \cap W^s(O(q))$. Therefore, $\dim W^u_p = \dim W^u_q$. Similarly, $\dim W^s_p = \dim W^s_q$. $\square$

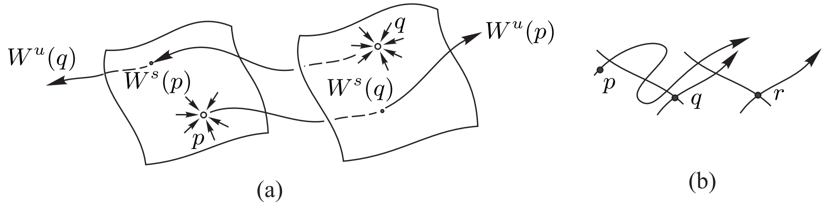


Figure 1.19

It follows from the Inclination Lemma (Theorem 1.8) that being h-related is an equivalence relation, see Fig. 1.19, (b). The set

$$H_p = \{q \in H(f) \colon p \overset{h}{\sim} q\}$$

is called the h-class of $p \in H(f)$.

Lemma 1.20 *Cloud Lemma. Suppose* $x \in W^s(p) \cap W^u(q)$ *and* $y \in W^u(p) \cap \cap W^s(q)$ *where* $p \overset{h}{\sim} q$. *Then* $x, y \in \mathrm{clos}(\mathrm{Per}(f))$. *In particular,* $x, y \in \in NW(f)$.

Proof. By Theorem 1.8, $W^u(p)$ intersects transversally $W^s(p)$ arbitrary close to x. Theorem 1.11 implies that $x \in \mathrm{clos}(\mathrm{Per}(f))$. Similarly, $y \in \in \mathrm{clos}(\mathrm{Per}(f))$.

There is another way to prove the result (in sense, classical) that gives the name Cloud Lemma. Let U be a neighborhood of $x \in W^s(p) \pitchfork W^u(q)$. We take m so large that p, q are fixed points of f^m. Figure 1.20 shows that some iteration $f^{km}(U)$ must re-intersect U eventually. Hence, $x \in NW(f)$. Similarly, $y \in NW(f)$. □

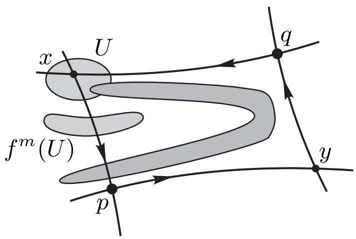


Figure 1.20. $f^{km}(U)$ must re-intersect U.

Lemma 1.21 *If* $p \in H(f)$ *then*

- *the both* H_p *and* $\operatorname{clos}(H_p)$ *are invariant*;
- *the set* $\operatorname{clos}(H_p)$ *is transitive.*

Proof. Obviously, H_p is invariant. By Lemma 1.5, $\operatorname{clos}(H_p)$ is also invariant.

To prove that $\operatorname{clos}(H_p)$ is transitive, take any relative open nonempty sets $\widehat{U}_1$, $\widehat{U}_2$ in $\operatorname{clos}(H_p)$, i.e. there are open sets U_1, U_2 in M such that $\widehat{U}_1 = U_1 \cap \operatorname{clos}(H_p)$ and $\widehat{U}_2 = U_2 \cap \operatorname{clos}(H_p)$. Taking in mind the Birkhoff Transitive Theorem (Theorem 1.2), it is sufficient to show that $f^k(\widehat{U}_1) \cap \widehat{U}_2 \neq \varnothing$ for some $k \in \mathbb{N}$. Since the sets $\widehat{U}_1$, $\widehat{U}_2$ are open, there are points $x_1 \in U_1 \cap H_p$, $x_2 \in U_2 \cap H_p$. Because of $x_1 \overset{h}{\sim} x_2$, $W^u(O(x_1))$ has a nonempty transverse intersection with $W^s(O(x_2))$ at some point, say y_1, and $W^u(O(x_2))$ has a nonempty transverse intersection with $W^s(O(x_1))$ at some point, say y_2. By Theorem 1.11, the both y_1 and y_2 are in the closure of periodic points which must be heteroclinically related to x_1 and x_2, since stable and unstable manifolds depend continuously on initial conditions (Theorem 1.10). Therefore, $y_1, y_2 \in \operatorname{clos}(H_p)$. It follows that $f^k(\widehat{U}_1) \cap \widehat{U}_2 \neq \varnothing$ for some $k \in \mathbb{N}$. □

Theorem 1.12 *Let* $f\colon M \to M$ *be a diffeomorphism of a compact manifold* M. *Suppose that the non-wandering set* $NW(f)$ *is hyperbolic and periodic points are dense in* $NW(f)$. *Then* $NW(f)$ *is a finite union*

$$NW(f) = \Omega_1 \cup \cdots \cup \Omega_k$$

of pairwise disjoint f*-invariant closed sets* Ω_i *such that*

- *every restriction* $f|_{\Omega_i}$ *is topologically transitive,* $i = 1, \ldots, k$;
- *every* Ω_i *is the closure of some* h*-class,* $i = 1, \ldots, k$;
- *every* Ω_i *has a local product structure,* $i = 1, \ldots, k$.

Proof. If $\operatorname{clos}(H_p) \cap \operatorname{clos}(H_q) \neq \varnothing$, there is a point $r \in \operatorname{clos}(H_p) \cap \operatorname{clos}(H_q)$ that must be a limit point for periodic points $P_i \in H_p$ and $Q_j \in H_q$. Since $NW(f) = \operatorname{clos}(\operatorname{Per}(f))$ is endowed with a hyperbolic structure, the points P_i and Q_j are h-related beginning with some $i \geqslant i_0$, $j \geqslant j_0$. It follows that P_i and Q_j belong to the same h-class. Hence, $\operatorname{clos}(H_p) = \operatorname{clos}(H_q)$. So, $NW(f)$ is the union of pairwise disjoint h-classes $\operatorname{clos}(H_n)$. By Lemma 1.21, all $\operatorname{clos}(H_n)$ are invariant and transitive.

Let us prove that there are only finitely many h-classes. Suppose the contrary. Then there is a limit point for periodic points from infinitely many (different) h-classes. Again, the hyperbolic structure on $\mathrm{clos}(\mathrm{Per}(f))$ implies that these periodic points must belong to the same h-class. This contradiction proves the result.

Denote by $\Omega = \mathrm{clos}(H_p)$, where $p \in \mathrm{Per}(f)$. It remains to prove that Ω has a local product structure. Let $x, y \in \Omega$. There are sequences $x_k \to x$, $y_k \to y$ with $x_k, y_k \in H_p$. For sufficiently small $\varepsilon > 0$, each intersection $W^u_\varepsilon(x_k) \cap W^s_\varepsilon(y_k)$ and $W^u_\varepsilon(y_k) \cap W^s_\varepsilon(x_k)$ has at most one point, say μ_k and ν_k respectively. For sufficiently close x and y, these μ_k and ν_k are single points at corresponding transverse intersection. By Lemma 1.20, $\mu_k, \nu_k \in \mathrm{clos}(\mathrm{Per}(f))$. There is $\delta > 0$ such that if $d(x, y) < \delta$, then $\mu_k, \nu_k \in \Omega$, because of the closures of h-classes are a finite distance apart. By the continuity of invariant manifolds (Theorem 1.10), μ_k approach $\mu \in W^u(x) \cap W^s(y)$ and ν_k approach $\nu \in W^u(y) \cap W^s(x)$. Since Ω is compact, $\mu, \nu \in \Omega$ and μ, ν are points of corresponding transverse intersections. This completes the proof. □

These Ω_i are called the *basic sets* of f. The dimension of unstable manifold $W^u(x)$, where $x \in \Omega_i$, is called the *index* of Ω_i. Sometimes, $\dim W^u(x)$ is called *Morse index*, or *unstable index* of Ω_i. By Lemma 1.17, this definition does not depend on the choice of point $x \in \Omega_i$.

Following Smale [208], we call Theorem 1.12 the Spectral Decomposition Theorem. The conditions of Theorem 1.12 form so called an Axiom A introduced by Smale [208]. So if $NW(f)$ is hyperbolic and the periodic points are dense in $NW(f)$, f is called an *A-diffeomorphism*. Thus, the non-wandering set of A-diffeomorphism is the union of basic sets.

Isolated periodic (in particular, fixed) point gives the example of *trivial* basic set. A basic set is *nontrivial* if it is not a periodic isolated orbit.

Neighborhoods of hyperbolic sets

Here, we consider the asymptotic behavior of points near a compact basic set Λ of an A-diffeomorphism. Theorem 1.6 states that if $p \in W^s(q)$ for a point $q \in \Lambda$, then p is asymptotic with q, and hence, with Λ. Assume that we know that the positive orbit of a point p stays near Λ. The following theorem says that p is asymptotic with a point of Λ.

Theorem 1.13 *Let $f\colon M \to M$ be an A-diffeomorphism and $\Omega_1, \ldots, \Omega_k$ the basic sets of f, $NW(f) = \Omega_1 \cup \cdots \cup \Omega_k$. Then*

$$M = \bigcup_{i=1}^{k} W^s(\Omega_i) = \bigcup_{i=1}^{k} W^u(\Omega_i),$$

where

$$W^s(\Omega_i) = \bigcup_{x\in\Omega_i} W^s(x), \quad W^u(\Omega_i) = \bigcup_{x\in\Omega_i} W^u(x), \quad i = 1, \ldots, k.$$

Proof. It follows from Lemma 1.7, that $f^j(p)$ tends to $NW(f)$ as $j \to \to \pm\infty$ for any point $p \in M$. Hence, $M = \bigcup_{i=1}^{k} W^s(\Omega_i) = \bigcup_{i=1}^{k} W^u(\Omega_i)$. Since the basic sets $\Omega_1, \ldots, \Omega_k$ are disjoint, $f^j(p)$ tends to some Ω_i as $j \to +\infty$ and to some Ω_r as $j \to -\infty$. So, $p \in W^s(\Omega_i)$ and $p \in W^u(\Omega_r)$.

We have to prove that $p \in W^s(q_1)$ and $p \in W^u(q_2)$ where $q_1 \in \Omega_i$ and $q_2 \in \Omega_r$. It is enough to consider only the case when $p \in W^s(\Omega_i)$. Denote Ω_i by Λ. Since Λ is invariant, it is sufficient to prove that

$$f^m(p) \in W^s_\varepsilon(\Lambda) = \bigcup_{x\in\Lambda} W^s_\varepsilon(x) \quad \text{for some } m \in \mathbb{N},\ \varepsilon > 0.$$

The existence of local product structure implies that for sufficiently small $\varepsilon > 0$, the set

$$D^u = \operatorname{clos}\left(W^s_\varepsilon(\Lambda) \setminus f(W^s_\varepsilon(\Lambda))\right)$$

is a proper fundamental domain for $W^u_\varepsilon(\Lambda)$ see Fig. 1.21 (a), i. e., a compact set such that

$$W^u_\varepsilon(\Lambda) \setminus \Lambda \subset \bigcup_{m\geqslant 0} f^{-m}(D^u), \quad D^u \cap \Lambda = \varnothing.$$

Let U be a neighborhood of D^u, Fig. 1.21, (b). One can assume that $f^k(U) \cap U = \varnothing$ for $k \geqslant 2$. Then

$$V = W^s_\varepsilon(\Lambda) \bigcup_{m\geqslant 0} f^{-m}(U)$$

is a neighborhood of Λ. Note that by construction, $f^k(U) \cap V = \varnothing$ for $k \geqslant 2$. Therefore, if $f^m(p) \in V$ for all sufficiently large m then $f^m(p)$ have to belong to $W^s_\varepsilon(\Lambda)$. $\square$

Spectral decomposition of basic set

Here, we prove that every nontrivial basic set Ω has a spectral decomposition itself. To be precise, the following statement holds.

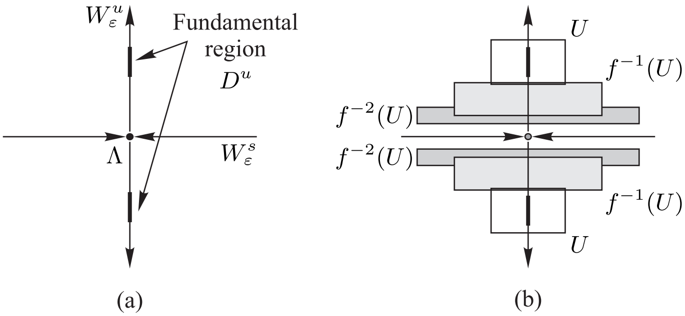


Figure 1.21

Theorem 1.14 *Let Ω be a nontrivial basic set of an A-diffeomorphism. Then Ω is a union of pairwise disjoint closed subsets $\Omega_{c1}, \ldots, \Omega_{ch}$ such that*

$$f^h(\Omega_{cj}) = \Omega_{cj}, \quad f(\Omega_{cj}) = \Omega_{c,j+1},$$
$$\textit{where} \quad \Omega_{c,h+1} = \Omega_{c1} \quad (1 \leqslant j \leqslant h),$$

and each intersection $W^s_x \cap \Omega_{cj}$, $W^u_x \cap \Omega_{cj}$ is dense in Ω_{cj} for any periodic point $x \in \Omega_{cj}$.

Proof. Take a periodic point $p \in \Omega$ and put by definition $\Omega_p = \operatorname{clos}\big(W^u_p \cap NW(f)\big)$. It is convenient to prove the theorem in steps.

Step 1.1 *$\Omega_p \subset NW(f)$ and $f^{\operatorname{Per}(p)}(\Omega_p) = \Omega_p$, where* Per$(p)$ *is a period of* p*.*

Proof of Step 1.1. Let the sequence $x_i \in \Omega_p = \operatorname{clos}\big(W^u_p \cap NW(f)\big)$ converge to x as $i \to \infty$. Then there exists a sequence $y_i \in W^u_p \cap NW(f)$ that converges to x as well. In particular, $y_i \in NW(f)$. Since $NW(f)$ is closed, $x \in NW(f)$. The equality $f^{\operatorname{Per}(p)}(\Omega_p) = \Omega_p$ is obvious because of $f^{\operatorname{Per}(p)}(W^u_p) = W^u_p$. $\diamondsuit$

Step 1.2 *The set Ω_p is relatively open in $NW(f)$.*

Proof of Step 1.2. Indeed, assume that $\delta > 0$ satisfies the condition of Theorem 1.7 on existence of local coordinates, i. e. if $x, y \in NW(f)$ and $d(x, y) < \delta$ then the intersection $W^s_\varepsilon(x) \cap W^u_\varepsilon(y)$ consists of a unique point from $NW(f)$ for some $\varepsilon > 0$. Let $U(\Omega_p)$ be the

$\frac{\delta}{2}$-neighborhood of Ω_p. Take a periodic point $q \in U_\delta(\Omega_p)$. Since $\Omega_p = \operatorname{clos}\left(W_p^u \cap NW(f)\right)$, there is a point $x \in W_p^u \cap NW(f)$ such that $d(x, q) < \delta$. Hence, the intersection $W_\varepsilon^u(x) \cap W_\varepsilon^s(q) \subset W_p^u \cap W_q^s$ consists of a unique point $y \in NW(f)$. Since W_P^u and W_q^s are invariant under $f^{\operatorname{Per}(p)\cdot\operatorname{Per}(q)}$, $q \in \operatorname{clos}\left(W_p^u \cap NW(f)\right) = \Omega_p$ because of

$$f^{k\cdot\operatorname{Per}(p)\cdot\operatorname{Per}(q)}(y) \to q \quad \text{as } k \to +\infty.$$

Since periodic points are dense in $NW(f)$, it follows that any non-wandering point from $U(\Omega_p)$ belongs to Ω_p. This concludes that Ω_p is relatively open in $NW(f)$. $\diamondsuit$

Step 1.3 $\Omega_p \subset \operatorname{clos}(H_p)$.

Proof of Step 1.3. Take $z \in \Omega_p$. First, suppose that $z \in W_p^u \cap NW(f)$. If $z \in \operatorname{clos}(H_q)$ for some periodic point q, then $\operatorname{clos}(H_p) = \operatorname{clos}(H_q)$ because of $f^{k\cdot\operatorname{Per}(p)}(z) \to p$ as $k \to +\infty$, and h-related classes are closed and pairwise disjoint. Hence, $z \in \operatorname{clos}(H_p)$. If $z \in \Omega_p \setminus \left(W_p^u \cap NW(f)\right)$ then there is a sequence $x_i \in W_p^u \cap NW(f)$ converging to z. By previous considerations, $x_i \in \operatorname{clos}(H_p)$. Therefore, $z \in \operatorname{clos}(H_p)$ as well. $\diamondsuit$

Step 1.4 *If* $q \in \Omega_p$ *then* $\Omega_p = \Omega_q$.

Proof of Step 1.4. By Step 1.3, the points p, q are h-related. Lemma 1.19 implies $\dim W_p^u = \dim W_q^u$ and $\dim W_p^s = \dim W_q^s$. It follows from the Inclination Lemma (Theorem 1.8) that $\Omega_q \subset \Omega_p$. Similarly, $\Omega_p \subset \Omega_q$. $\diamondsuit$

As a consequence of Steps 1.2, 1.4, one gets that the sets Ω_p, Ω_q are either coincident or disjoint, where p, q are periodic points. Indeed, if $x \in \Omega_p \cap \Omega_q$ then there is a periodic point $r \in \Omega_p \cap \Omega_q$, since periodic points are dense in $NW(f)$. By Step 1.4, $\Omega_p = \Omega_r = \Omega_q$.

Now, given any periodic point $q \in \Omega = \operatorname{clos}(H_p)$, there exists $k \in \mathbb{Z}$ such that $f^k(p)$ is h-related to q. Then $q \in \Omega_{f^k p}$. By Step 1.4, $\Omega_{f^k p} = \Omega_q$. It follows from Step 1.1 that Ω is the union of pairwise disjoint sets Ω_p, $f(\Omega_p)$, $\dots$, $f^{P_0-1}(\Omega_p)$ for some $P_0 \in \mathbb{N}$.

Take a periodic point $q \in \Omega_p$. By Step 1.3, W_p^s intersects W_q^u. Hence, $q \in \operatorname{clos}(W_p^s)$. Therefore, the intersection $W_p^s \cap \Omega_p$ is dense in Ω_p. Similarly, $W_q^s \cap \Omega_p$ is dense in Ω_p. $\square$

These Ω_{cj} are called *C-dense components* of Ω. We will also call them *periodic components* of Ω. If Ω has a unique C-dense component, then we will say that Ω is a *C-dense basic set*.

1.8. Nontrivial attractors and repellers

Later on, we study A-diffeomorphisms of compact manifold M^n (possibly with boundary) with a nontrivial basic set Λ in the interior of M^n. Recall that $W^s_{x,\varepsilon}$ ($W^u_{x,\varepsilon}$) is an ε-neighborhood of the point $x \in \Lambda$ on the manifold W^s_x (W^u_x) in the inner metric d^s (d^u). Due to Theorem 1.7 (existence of local product structure), every point $x \in \Lambda$ has the canonical neighborhood $V_x \subset \Lambda$ homeomorphic to the direct product $\check{W}^s_{x,\varepsilon} \times \check{W}^u_{x,\varepsilon}$, where

$$\check{W}^s_{x,\varepsilon} = W^s_{x,\varepsilon} \cap \Lambda, \quad \check{W}^u_{x,\varepsilon} = W^u_{x,\varepsilon} \cap \Lambda.$$

Lemma 1.22 *A nontrivial basic set Λ of a diffeomorphism $f\colon M^n \to M^n$ coincides with its stable* (*unstable*) *manifold if and only if there is a point $x \in \Lambda$ such that*

$$\dim \check{W}^s_x = \dim W^s_x \quad (\dim \check{W}^u_x = \dim W^u_x).$$

Proof. The necessity is evident. We now prove the sufficiency. To be definite let $\dim \check{W}^u_x = \dim W^u_x$. From the definition of the unstable manifold it follows that $\Lambda \subset W^u_\Lambda$. Therefore to prove $W^u_\Lambda = \Lambda$ it is sufficient to show that $W^u_y \subset \Lambda$ for every $y \in \Lambda$. Since $\dim \check{W}^u_x = \dim W^u_x$, there is a point $z \in \Lambda$ and there is $\delta > 0$ such that $W^u_{z,\delta} \subset \Lambda$. Since the set of the periodic points is dense in Λ there is a periodic point p in the neighborhood $V = W^u_{z,\delta} \times \check{W}^s_{z,\delta} \subset \Lambda$. The structure of the direct product in V gives us that there is $\eta > 0$ such that $W^u_{p,\eta} \subset \Lambda$. Therefore $W^u_p \subset \Lambda$ and $W^u_{\mathcal{O}_p} \subset \Lambda$. Hence, $\operatorname{clos} W^u_{\mathcal{O}_p} = \Lambda$.

Let $K \subset W^u_y$ be a compact neighborhood of a point y which contains the point w. Then since the set $W^u_{\mathcal{O}_p}$ is dense in Λ and since the unstable manifolds are C^1-close on compact sets, the set K is the topological limit of the compact subsets $W^u_{\mathcal{O}_p}$. Then $K \subset \Lambda$ and therefore $w \in \Lambda$. □

Lemma 1.23 *Let Λ be a nontrivial basic set of a diffeomorphism $f\colon M^n \to M^n$ such that for a point $x \in \Lambda$ $\dim \check{W}^s_x = 0$ ($\dim \check{W}^u_x = 0$). Then $\operatorname{clos} \check{W}^s_{x,\varepsilon}$ ($\operatorname{clos} \check{W}^u_{x,\varepsilon}$) is the Cantor set and $\dim \Lambda = \dim \check{W}^u_x$ ($\check{W}^s_x$).*

Proof. To be definite let $\dim \check{W}^s_x = 0$. To prove that $\operatorname{clos} \check{W}^s_{x,\varepsilon}$ is the Cantor set it suffices to show that $\operatorname{clos} \check{W}^s_{x,\varepsilon}$ has no isolated points in W^s_x. Assume the contrary: there is an isolated point $y \in \operatorname{clos} \check{W}^s_{x,\varepsilon}$ in W^s_x. Then there is a neighborhood V of it in the set Λ which coincides with the set $\check{W}^u_{y,\delta}$

for some $\delta > 0$. Since the periodic points are dense the neighborhood $V = \check{W}^u_{y,\delta}$ contains at least two periodic points and this contradicts the property of an unstable manifold.

We now show that $\dim \Lambda = \dim \check{W}^u_x$. The local structure of the basic set Λ implies that $\dim \Lambda = \dim(\check{W}^s_{x,\varepsilon} \times \check{W}^u_{x,\varepsilon})$, therefore $\dim \Lambda \geqslant \dim \check{W}^u_{x,\varepsilon}$. On the other hand,

$$\dim(\check{W}^s_{x,\varepsilon} \times \check{W}^u_{x,\varepsilon}) \leqslant \dim \check{W}^s_{x,\varepsilon} + \dim \check{W}^u_{x,\varepsilon}.$$

Then from $\dim \check{W}^s_x = 0$ we get that $\dim(\check{W}^s_{x,\varepsilon} \times \check{W}^u_{x,\varepsilon}) \leqslant \dim \check{W}^u_{x,\varepsilon}$. Thus, $\dim \Lambda = \dim \check{W}^u_x$. $\square$

Theorem 1.15 *Let $f\colon M^n \to M^n$ be an A-diffeomorphism having an n-dimensional basic set Λ. Then $\Lambda = M^n$, and f is an Anosov diffeomorphism.*

Proof. Since $\dim \Lambda = n$, there is an open n-ball U in the set Λ [121]. Then for some point $x \in U$ and some $\delta > 0$ the inclusions $W^u_{x,\delta} \subset \Lambda$ and $W^s_{x,\delta} \subset \Lambda$ hold. Then by Lemma 1.22, $W^u_y \subset \Lambda$ and $W^s_y \subset \Lambda$ for every point $y \in \Lambda$.

Since $\Lambda = W^s_\Lambda = W^u_\Lambda$, the canonical neighborhood V_x of every point $x \in \Lambda$ is homeomorphic to $W^s_{x,\varepsilon} \times W^u_{x,\varepsilon}$. Then the point x has a neighborhood which is open in M^n and therefore the set Λ is open. Since Λ is closed, it coincides with the entire manifold M^n. Since every point $x \in \Lambda$ has a neighborhood homeomorphic to $\mathbb{R}^n$ we have that M^n is a manifold without boundary. Since $\Lambda = M^n$ it follows that f is an Anosov diffeomorphism. $\square$

Definition 1.2 *A basic set Λ is called an attractor for f if there is a compact neighborhood U of Λ such that*

$$f(U) \subset \operatorname{int} U, \qquad \bigcap_{k \geqslant 0} f^k(U) = \Lambda.$$

A basic set Λ is called repeller if it is an attractor for f^{-1}.

Now, we give criteria for a basic set Λ to be an attractor (repeller).

Theorem 1.16 *A basic set Λ of a diffeomorphism $f\colon M^n \to M^n$ is an attractor (repeller) if and only if $W^u_\Lambda = \Lambda$ ($W^s_\Lambda = \Lambda$).*

Proof. To be definite let Λ be an attractor.

Necessity. From the definition of an attractor it follows that there is a compact neighborhood U_Λ such that $f(U_\Lambda) \subset \operatorname{int} U_\Lambda$ and $\Lambda = \bigcap_{k\geqslant 0} f^k(U_\Lambda)$. Then there is $\varepsilon > 0$ such that $\bigcup_{x\in\Lambda} W^u_{x,\varepsilon} \subset U_\Lambda$. Hence there is $0 < \delta < \varepsilon$ such that $W^u_x = \bigcup_{j\geqslant 0} f^j(W^u_{f^{-j}(x),\delta})$. Then $W^u_\Lambda \subset \bigcup_{j\geqslant 0} f^j(\bigcup_{x\in\Lambda} W^u_{x,\delta})$. Since $\bigcup_{x\in\Lambda} W^u_{x,\delta} \subset U_\Lambda$ and $\bigcup_{j\geqslant 0} f^j(U_\Lambda) \subset U_\Lambda$ we have $W^u_\Lambda \subset U_\Lambda$. Then $\bigcap_{k\geqslant 0} f^k(W^u_\Lambda) \subset \bigcap_{k\geqslant 0} f^k(U_\Lambda)$. Since $W^u_\Lambda = \bigcap_{k\geqslant 0} f^k(W^u_\Lambda)$ and $\Lambda = \bigcap_{k\geqslant 0} f^k(U_\Lambda)$ we have $W^u_\Lambda \subset \Lambda$. From $\Lambda \subset W^u_\Lambda$ it follows that $W^u_\Lambda = \Lambda$.

Sufficiency. Hyperbolicity of the set Λ implies that there is $\delta > 0$ such that $\operatorname{cl} f(W^s_{x,\delta}) \subset \operatorname{int} W^s_{f(x),\delta}$ for every point $x \in \Lambda$. Let $U_\Lambda = \operatorname{cl}\left(\bigcup_{x\in\Lambda} W^s_{x,\delta}\right)$. From $\Lambda = W^u_\Lambda$ it follows that U_Λ is a compact neighborhood of Λ in the manifold M^n. By construction $f(U_\Lambda) \subset \operatorname{int} U_\Lambda$ and $\Lambda = \bigcap_{k\geqslant 0} f^k(U_\Lambda)$. Therefore Λ is an attractor. □

Theorem 1.17 *Every codimension one basic set Λ of a diffeomorphism $f\colon M^n \to M^n$ is either an attractor or a repeller.*

Proof. Let $x \in \Lambda$ and let V_x be the canonical neighborhood of the point x. It follows from $\dim V_x = n-1$ that $\dim \check{W}^s_{x,\varepsilon} + \dim \check{W}^u_{x,\varepsilon} \geqslant \dim \check{W}^s_{x,\varepsilon} \times \check{W}^u_{x,\varepsilon} = n-1$.

Since $\dim \check{W}^u_{x,\varepsilon} \leqslant \dim W^u_{x,\varepsilon}$, $\dim \check{W}^s_{x,\varepsilon} \leqslant \dim W^s_{x,\varepsilon}$ and $\dim W^s_{x,\varepsilon} + \dim W^u_{x,\varepsilon} = n$ we have that either $\dim \check{W}^u_{x,\varepsilon} = \dim W^u_{x,\varepsilon}$ or $\dim \check{W}^s_{x,\varepsilon} = \dim W^s_{x,\varepsilon}$. Indeed, if we suppose that $\dim \check{W}^u_{x,\varepsilon} < \dim W^u_{x,\varepsilon}$ and $\dim \check{W}^s_{x,\varepsilon} < \dim W^s_{x,\varepsilon}$ then we get $\dim \check{W}^s_{x,\varepsilon} + \dim \check{W}^u_{x,\varepsilon} \leqslant \dim W^s_{x,\varepsilon} - 1 + \dim W^u_{x,\varepsilon} - 1 = n-2$ and this contradicts $\dim \check{W}^s_{x,\varepsilon} + \dim \check{W}^u_{x,\varepsilon} \geqslant n-1$. Now the result follows from Lemma 1.22 and Theorem 1.16. □

A nontrivial basic set Λ, which is an attractor, is called *expanding* if the topological dimension of Λ is equal to the dimension of W^u_x, $x \in \Lambda$. A nontrivial basic set Λ, which is a repeller, is called *contracting* if it is an expanding attractor for the diffeomorphism f^{-1}.

Example 1.17 *Prove that every 1-dimensional basic set of a diffeomorphism $f\colon M^2 \to M^2$ is either an expanding attractor or a contracting repeller.*

Theorem 1.18 *Every expanding attractor* (*contracting repeller*) Λ *has the local structure of the direct product of the* k*-dimensional Euclidean space and the Cantor set, where* k *is the topological dimension of* Λ.

Proof. To be definite let Λ be an attractor. Let $x \in \Lambda$ and V_x be the canonical neighborhood of the point x. Theorem 1.16 implies that $W^u_{x,\varepsilon} = \check{W}^u_{x,\varepsilon}$. From [121] it follows that

$$\dim W^u_{x,\varepsilon} = \dim V_x \leqslant \dim W^u_{x,\varepsilon} + \dim \check{W}^s_{x,\varepsilon} \tag{1.19}$$

If $\dim \check{W}^s_{x,\varepsilon} \geqslant 1$ then we pick a subset $W \subset \check{W}^s_{x,\varepsilon}$ of dimension 1 and as a corollary of Hurewicz theorem [121] we get that $\dim V_x \geqslant \dim W^u_{x,\varepsilon} + 1$, and this contradicts (1.19). Therefore $\dim \check{W}^s_{x,\varepsilon} = 0$. By Lemma 1.23 $\check{W}^s_x$ has the local structure of the Cantor set and from this the hypothesis of the theorem follows. □

Topological dimension and type of basic sets

We state some important properties of the basic sets in relation to their type and dimension. These properties are used for the topological classification of the basic sets (including expanding attractors and contracting repellers) as well as of important classes of structurally stable diffeomorphisms.

The pair (a, b) where $a = \dim W^u_x$, $b = \dim W^s_x$, $x \in \Lambda$ is called the *type of the basic set* Λ. Notice that $(1, 1)$ is the type of every nontrivial basic set of a surface diffeomorphism.

Definition 1.3 *Let* Λ *be a basic set of type* $(n-1, 1)$ *or* $((1, n-1))$ *of an A-diffeomorphism* $f\colon M^n \to M^n$, *let* $x \in \Lambda$ *and let* Λ_x *be the periodic component of the set* Λ *containing the point* x.

- *We say a connected component of the set* $W^s_x \setminus x$ $(W^u_x \setminus x)$ *to be* densely situated in Λ_x *if it contains a set which is dense in* Λ_x.
- *We say a point* x *to be s-dense* (*u-dense*) *if both connected components of the set* $W^s_x \setminus x$ $(W^u_x \setminus x)$ *are densely situated in* Λ_x.
- *We say a point* x *to be an s-boundary point* (*u-boundary point*), *if one of the connected components of the set* $W^s_x \setminus x$ $(W^u_x \setminus x)$ *is disjoint from* Λ.

Lemma 1.24 *If* Λ *is a basic set of type* $(n-1, 1)$ *then for every point* $x \in \Lambda$ *at least one connected component of the set* $W^s_x \setminus x$ *is densely situated in* Λ_x.

Proof. Introduce a parameter $t \in \mathbb{R}$ on the manifold W^s_x such that $W^s_x(0) = x$. Let W^{s+}_x, W^{s-}_x denote the connected components of the curve $W^s_x \setminus x$ for $t > 0$ and $t < 0$ respectively. Due to a local product structure, W^s_x contains a dense in Λ_x set. Therefore there is a sequence $t_n \to \infty$ such that $z_n = W^s_x(t_n) \in \Lambda_x$. One can assume that $t_n \to +\infty$ (or $t_n \to -\infty$) and that the sequence z_n converges to a point $z \in \Lambda_x$, $z \notin W^x_s$ (otherwise one considers a subsequence of t_n). To be definite let $t_n \to +\infty$. We now show that W^{s+}_x contains a set dense in Λ_x.

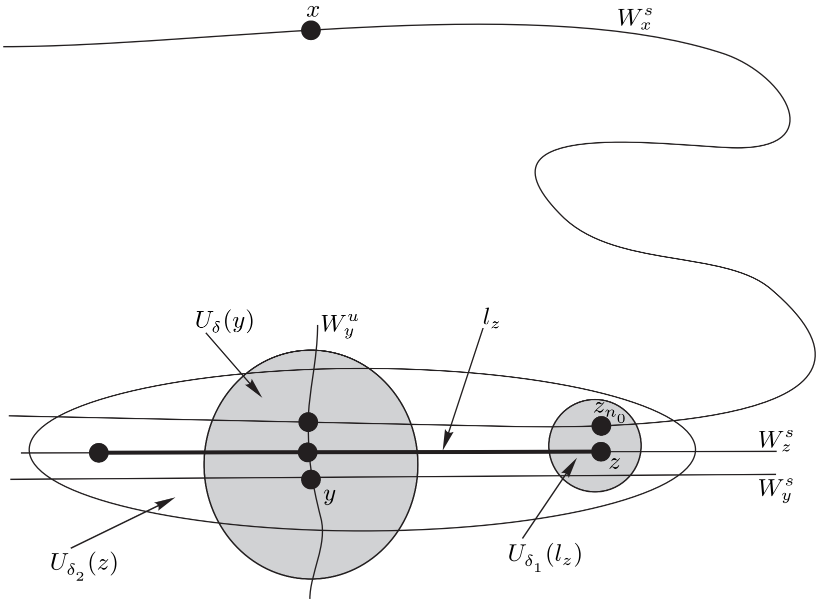


Figure 1.22. An illustration to the proof of Lemma 1.24

Let y be an arbitrary point of Λ_x distinct from x and let $U_\delta(y)$ be a δ-neighborhood of the point y in M^n. Since W^s_z contains a dense in Λ_x set there is a segment l_z of the manifold W^s_z which contains the point z and which intersects $U_\delta(y)$. On the other hand the fact that stable manifolds are C^1-close on compact sets for any $\delta_1 > 0$ implies that there is $\delta_2 > 0$ such that any manifold intersecting δ_2-neighborhood $U_{\delta_2}(z)$ of the point z, while staying in δ_1-neighborhood $U_{\delta_1}(l_z)$ of the segment l_z, intersects the neighborhood $U_\delta(y)$, see Figure 1.22. Now pick δ_1 such that $x \notin U_{\delta_1}(l_z)$, then pick n_0 such that $z_{n_0} \in U_{\delta_2}(z)$. Then there is a segment $l_{z_{n_0}}$ of the component W_x^{s+} which passes through the point z_{n_0} and which intersects the neighborhood $U_\delta(y)$. The local product structure implies that for δ small enough the intersection $l_{z_{n_0}} \cap W_y^u$ contains a point of periodic component Λ_x in the neighborhood $U_\delta(y)$. □

The similar result holds for a basic set of type $(1, n-1)$. For this type, the set $W_x^u \setminus x$ is densely situated in Λ_x.

Lemma 1.25 *Let Λ be a nontrivial basic set of a diffeomorphism $f\colon M^n \to$ $\to M^n$. If $f^m(W_x^u) = W_x^u$ ($f^m(W_x^s) = W_x^s$) for some point $x \in \Lambda$ and some $m \in \mathbb{N}$ then the map $f^m|_{W_x^u}$ ($f^m|_{W_x^s}$) has a periodic point which is a hyperbolic source (sink) for the restriction $f^m|_{W_x^u}$ ($f^m|_{W_x^s}$).*

Proof. To be definite assume $f^m(W_x^s) = W_x^s$. Without loss of generality we assume $m = 1$ (otherwise the same arguments hold for the diffeomorphism f^m). From the definition of a stable manifold it follows that $d(f^k(x), f^{k+1}(x)) \to 0$ for $k \to +\infty$. A hyperbolic structure implies that there are $\varepsilon > 0$ and $\mu \in (0, 1)$ such that from $x_1, x_2 \in W^s_{f^k(x),\varepsilon}$ it follows that $d^s(f(x_1), f(x_2)) < \mu d^s(x_1, x_2)$. Let $k_0 \in \mathbb{N}$ be such that $d^s(f^{k_0}(x), f^{k_0+1}(x)) < \frac{\varepsilon}{2}(1-\mu)$. Then for any point $y \in \mathrm{cl}\left(W^s_{f^{k_0}(x), \frac{\varepsilon}{2}}\right)$ it holds

$$d^s(f^{k_0}(x), f(y)) \leqslant d^s(f^{k_0}(x), f^{k_0+1}(x)) + d^s(f(y), f(f^{k_0}(x))) < \\ < \frac{\varepsilon}{2}(1-\mu) + \frac{\varepsilon}{2}\mu = \frac{\varepsilon}{2},$$

that is $f\left(\mathrm{cl}\left(W^s_{f^{k_0}(x), \frac{\varepsilon}{2}}\right)\right) \subset \mathrm{int}\, W^s_{f^{k_0}(x), \frac{\varepsilon}{2}}$. Therefore for the segment $I =$ $= \mathrm{cl}\left(W^s_{f^{k_0}(x), \frac{\varepsilon}{2}}\right)$ we have $f(I) \subset \mathrm{int}\, I$ and from $x_1, x_2 \in I$ it follows that $d^s(f(x_1), f(x_2)) < \mu d^s(x_1, x_2)$, that is $f|_I$ is a contraction. By the Contraction Mapping Principle $f|_I$ has a unique fixed point p such that $\lim\limits_{k\to+\infty} d^s(f^k(x), p) = 0$ for every point $x \in I$. Since the basic set is closed p is hyperbolic. □

The following statement describes the structure of a basic set of types $(n-$ $-1, 1)$ and $(1, n-1)$.

Theorem 1.19 *Let Λ be a basic set of type $(n-1, 1)$ ($(1, n-1)$) for an A-diffeomorphism $f\colon M^n \to M^n$. Then*

1) *if Λ has s-boundary (u-boundary) points then there are finitely many of them and all of them are periodic*;
2) *a point $x \in \Lambda$ is s-dense (u-dense) if and only if the manifold W^s_x (W^u_x) contains no s-boundary (u-boundary) points*;
3) *if W^u_x (W^s_x) contains no s-boundary (u-boundary) points for some point $x \in \Lambda$ then for every $\delta > 0$ both connected components of the set $W^s_{x,\delta} \setminus x$ ($W^u_{x,\delta} \setminus x$) intersect Λ.*

Proof. To be definite assume the basic set Λ to be of type $(n-1, 1)$.

(1) Let Λ have an s-boundary point x. We now show that x is a periodic point. Assume the contrary: x is not periodic. Consider two cases: 1) for each pair of integers $i > j$ the relation $W^u_{f^i(x)} \cap W^u_{f^j(x)} = \varnothing$ holds; 2) there are integers $i > j$ such that $W^u_{f^i(x)} \cap W^u_{f^j(x)} \neq \varnothing$.

In the case 1) there is a sequence $k_n \to +\infty$ such that the subsequence $\{x_n = f^{k_n}(x)\}$ converges to some point $y \in \Lambda_x$ and $x_n \notin W^u_y$ for every $n \in \mathbb{N}$. In the case 2) $W^u_{f^i(x)} = W^u_{f^j(x)}$ and therefore $f^{i-j}(W^u_x) = W^u_x$. By Lemma 1.25 the map $f^{i-j}|_{W^u_x}$ has a fixed hyperbolic source p. Then there is an increasing sequence of integers $m_1, m_2, \ldots$ such that the subsequence $\{x_n = f^{m_n(i-j)}(x)\}$ converges to some point $y \in \Lambda_x$. We now show that there are $\eta > 0$ and $n_0 \in \mathbb{N}$ such that $x_n \notin W^u_{y,\eta}$ for $n > n_0$. Assume the contrary: for every $\eta > 0$ and every $n_0 \in \mathbb{N}$ there is $x_n \in W^u_{y,\eta}$ for $n > n_0$. Since p is a hyperbolic source there is $\delta > 0$ such that the diffeomorphism $f|_{W^u_{p,\delta}}$ is topologically conjugate to a linear expansion. Pick $k \in \mathbb{N}$ such that $f^{-k}(y) \in W^u_{p,\delta}$. Then the sequence $\{f^{-k}(x_n)\}$ converges to the point $z = f^{-k}(y)$. But by the assumption there is a neighborhood $U(z) \subset W^u_{p,\delta}$ of the point z which contains infinitely many positive iterations of the point x by the diffeomorphism f and this contradicts the dynamics of the diffeomorphism $f|_{W^u_{p,\delta}}$.

In both cases let V_y be the canonical neighborhood of the point y. Then for every point $w \in V_y$ there is a unique pair of points $w^s \in \check{W}^s_{y,\varepsilon}$, $w^u \in \check{W}^u_{y,\varepsilon}$ such that $w = W^u_{w^s} \cap W^s_{w^u}$. Since the subsequence x_n converges to the point y and $x_n \notin W^u_{y,\varepsilon}$ for n greater then some N, there are $n_1 > N$, $n_2 > N$ such that $x_{n_1}, x_{n_2} \in V_y$, $x^s_{n_1}, x^s_{n_2}$ belong to the same connected component of the set $W^s_y \setminus y$ and $x^s_{n_2} \in (y, x^s_{n_1})^s$ (see Figure 1.23). Then $W^s_{x_{n_2}} \cap W^u_{x_{n_1}} \neq \varnothing$, $W^s_{x_{n_2}} \cap W^u_y \neq \varnothing$ and therefore both connected components of the set $W^s_{x_{n_2}} \setminus x_{n_2}$ intersect Λ. Since x_{n_2} is an iteration by the diffeomorphism f of the s-boundary point it itself is an s-boundary point and we have a contradiction.

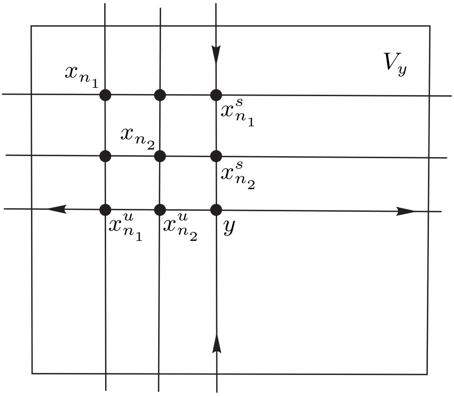


Figure 1.23. The local structure of the basic set

We now show that the set of the s-boundary points of the set Λ is finite. Indeed, if we assume the contrary we get a sequence $p_n \in \Lambda$ of s-boundary points converging to some point $z \in \Lambda$. Applying the same arguments as above to the canonical neighborhood V_z we then prove the existence of a point p_{n_2} in the sequence p_n for which both connected components of the set $W^s_{p_{n_2}} \setminus p_{n_2}$ intersect Λ, and this contradicts the assumption.

(2) *Necessity.* Let $x \in \Lambda$ be an s-dense point. We now show that W^s_x contains no s-boundary points. Assume the contrary: there is an s-boundary point $p \in W^s_x$. Denote by J_x the connected component of the set $W^s_x \setminus x$ which contains the point p and denote by J_p the connected component of the set $W^s_p \setminus p$ disjoint from Λ. Then $J_p \subset J_x$ and the segment $I = \mathrm{cl}(J_x \setminus J_p)$ contains a subset which is dense in Λ. Hence $\Lambda \subset I$ which is impossible because that means that a connected component of the set $W^s_x \setminus x$ distinct from J_x contains a dense in Λ subset.

Sufficiency. Let W^s_x contain no s-boundary points for $x \in \Lambda$. We now show that x is a s-dense point.

Introduce a parameter $t \in \mathbb{R}$ on the manifold W^s_x such that $W^s_x(0) = x$. Denote by W^{s+}_x, W^{s-}_x the connected components of the curve $W^s_x \setminus x$ for $t > 0$, $t < 0$ respectively. By Lemma 1.24 at least one of the connected components of the set $W^s_x \setminus x$ contains a set which is dense in Λ_x. To be definite let it be W^{s-}_x (otherwise we change the parameter t). We now show that the component W^{s+}_x contains a set which is dense in Λ_x as well. Let $t_n \to +\infty$ ($n = 0, 1, 2, \ldots,$ $t_0 = 0$) be a sequence of values of the parameter t such that $z_n = W^s_x(t_n) \in \Lambda_x$

and the sequence of points z_n converges to a point $z \in \Lambda_x$. Consider two cases: a) $z \notin W^s_x$, b) z belongs to W^s_x.

In the case a) applying the arguments analogues to that of Lemma 1.24 we get that the component W^{s+}_x contains a set dense in Λ_x. In the case b) two subcases are possible: b1) $z \notin W^{s-}_x$, b2) z belongs to W^{s-}_x. Consider the subcase b1) (for the subcase b2 the proof is similar). Denote by W^{s-}_z the component of the set $W^{s-}_z \setminus z$ such that $W^{s-}_z \cap W^{s-}_x \neq \varnothing$. Since W^{s-}_x contains a set dense in Λ_x the component W^{s-}_z contains a set dense in Λ_x as well.

Let now y be an arbitrary point from Λ_x and let $\delta > 0$ be arbitrary small. Since the component W^{s-}_z contains a dense in Λ_x set there is a segment l_z of the component W^{s-}_z which passes through the point z and which intersects the δ-neighborhood of the point y. The component W^{s-}_z contains finitely many points $z_0 = x$, z_1, …, z_k of the sequence $\{z_n\}$. Since stable manifolds are C_1-close on compact sets we have that for any $\delta_1 > 0$ there is $\delta_2 > 0$ such that the stable manifold which passes through δ_2-neighborhood of the point z intersects the δ-neighborhood of the point y while staying in the δ_1-neighborhood of the segment l_z. Pick δ_1 such that the δ_1-neighborhood l_z does not contain the point z_{k+1} and pick $n_0 > k+1$ such that z_{n_0} belongs to the δ_2-neighborhood of the point z. Then there is a segment $l_{z_{n_0}}$ of the component W^{s+}_x which passes through the point z_{n_0} and intersects $W^u_{y,\delta}$.

(3) Let a point $x \in \Lambda$ be such that W^u_x contains no s-boundary points. We now show that for every $\delta > 0$ both connected components of the set $W^s_{x,\delta} \setminus x$ intersect Λ. Consider two cases: 1) x is a periodic point; 2) x is not periodic. In the case 1) the hypothesis of the lemma follows from item (2). In the case 2) analogously to the proof of the item (1) there are a sequence $x_{k_q} = f^{-k_q}(x)$ and $N \in \mathbb{N}$, $\eta > 0$ such that for all $n > N$ both connected components of the set $W^s_{x_{k_n},\eta} \setminus x_{k_n}$ intersect Λ. Consider $f^{k_n}(W^s_{x_{k_n},\eta})$ for all $n > N$. Since $\lim\limits_{n\to+\infty} \operatorname{diam} f^{k_n}(W^s_{x_{k_n},\eta}) = 0$, for every $\delta > 0$ both connected components of the set $W^s_{x,\delta}$ intersect Λ. □

Theorem 1.20 *A basic set* Λ *of type* $(n-1,\,1)$ $((1,\,n-1))$ *contains an* s*-boundary* (u*-boundary*) *point if and only if* Λ *is not a repeller* (*attractor*).

Proof. To be definite let the basic set Λ be of type $(n-1,\,1)$.

Necessity. Let Λ contain an s-boundary point p; then W^s_p does not belong to Λ. Therefore $W^s_\Lambda \neq \Lambda$ and by Theorem 1.16 Λ is not a repeller.

Sufficiency. Let Λ be not a repeller. We now show that Λ contains at least one s-boundary point. Suppose the contrary: for every point $x \in \Lambda$ both connected components of the set $W^s_x \setminus x$ intersect Λ. By Theorem 1.16 $W^s_\Lambda \neq \neq \Lambda$. Then by Lemma 1.22 there is a point $z \in \Lambda$ such that $\dim \check{W}^s_z = 0$ and by Theorem 1.18 for every $\varepsilon > 0$ the set $\operatorname{cl} \check{W}^s_{z,\varepsilon}$ is a Cantor set. We now show that every point $x \in \check{W}^s_{z,\varepsilon}$ is a limit point for the points from Λ. Moreover both connected components of any neighborhood of the point x on the curve W^s_x contain points from Λ and this would contradict the structure of a Cantor set.

If x is a periodic point then both connected components of $W^s_x \setminus x$ evidently intersect Λ. Now let x be not periodic. Pick an increasing sequence of integers $n_1 < n_2 < \ldots$ and a point $y \in \Lambda$ such that $f^{-n_q}(x) \to y$ for $q \to +\infty$. Since both connected components of the set $W^s_y \setminus y$ intersect Λ there is a segment $l_y \subset W^s_y$ for which the point y is interior. Denote by $y_1, y_2 \in \Lambda$ the boundary points of the segment l_y. Since the stable manifolds are C^1-close on compact sets and since Λ has the local structure of the product it follows that there is $r > 0$ and there is a sequence of segments l_{n_q} with the boundary points $y_1^{n_q}, y_2^{n_q} \in \Lambda$ such that:

(1) $l_{n_q} \subset W^s_{f^{n_q}(x)}$;

(2) $f^{n_q}(x) \in \operatorname{int} l_{n_q}$;

(3) $y_1^{n_q} \to y_1$, $y_2^{n_q} \to y_2$ for $q \to +\infty$;

(4) $d^s(y_1^{n_q}, y_2^{n_q}) < r$, where d^s is the interior metric on the curve $W^s_{f^{n_q}(x)}$.

Since $d^s(f^{n_q}(y_1^{n_q}), f^{n_q}(y_2^{n_q})) \to 0$ for $q \to +\infty$ we have $f^{n_q}(y_1^{n_q}) \to x$, $f^{n_q}(y_2^{n_q}) \to x$. Therefore the point x is a limit point for the points from Λ and both connected components of any neighborhood of the point x on the curve W^s_x contain points from Λ. $\square$

Corollary 1.3 *Let $f\colon M^2 \to M^2$ be a surface A-diffeomorphism which has a nontrivial basic set Λ. Then*

0) *if $\dim \Lambda = 0$ then by Theorem 1.16 Λ is neither an attractor nor a repeller and by Theorem 1.20 Λ necessarily has both s-boundary and u-boundary points at the same time.*

1) *if $\dim \Lambda = 1$ then by Exercise 1.17 Λ is either an expanding attractor or a contracting repeller. Then by Theorem 1.20 Λ has s-boundary points only in the case of the attractor and u-boundary points only in the case of the repeller.*

2) *if $\dim \Lambda = 2$ then by Theorem 1.15 $\Lambda = M^2$ and f is an Anosov diffeomorphism. By Theorem 1.16 Λ is both an attractor and a repeller, therefore Λ has neither s-boundary points nor u-boundary points by Theorem 1.20.*

Proof. The first item follows from Theorem 1.16 and Theorem 1.20. The second item follows from Exercise 1.17 and Theorem 1.20. The third item follows from Theorem 1.15, and Theorem 1.16, and Theorem 1.20. □

1.9. Expansiveness, mixing, and shadowing

Expansiveness of basic sets

Lemma 1.26 *Let Ω be a basic set of f. Then the restriction $f_\Omega\colon \Omega \to \Omega$ is expansive.*

Proof. Recall that Ω is locally maximal hyperbolic set and hence, Ω has a local product structure i. e., there are $0 < \delta < \varepsilon$ such that for all $x,\ y \in \Omega$ with $d(x,\ y) < \delta$, the intersection $W^s_\varepsilon(x) \cap W^u_\varepsilon(y)$ consists of exactly one point, which belongs to Ω, see Theorem 1.7. We shall prove that δ is an expansiveness constant for f_Ω. Assume the contrary. Then there are $x,\ y \in \Omega$ such that $d(f^n(x),\ f^n(y)) \leqslant \delta$ for all $n \in \mathbb{Z}$. The inequality $d(f^n(x),\ f^n(y)) \leqslant \delta$ for all $n \in \mathbb{N}$ implies that $y \in W^s_\varepsilon(x)$. The inequality $d(f^n(x),\ f^n(y)) \leqslant \delta$ for all negative integers implies that $y \in W^u_\varepsilon(x)$. Hence, $y \in W^s_\varepsilon(x) \cap W^u_\varepsilon(x)$. From the other hand, by Theorem 1.7, the intersection $W^s_\varepsilon(x) \cap W^u_\varepsilon(x)$ consists of a unique point x because of $d(x,\ y) \leqslant \delta$. Therefore, $x = y$. □

Mixing of C-dense basic sets

Theorem 1.21 *Let Ω be a C-dense basic set of f. Then the restriction $f|_\Omega\colon \Omega \to \Omega$ is mixing.*

Proof. Let $U,\ V \subset \Omega$ be relatively open sets. Since periodic points are dense in Ω, there are periodic points $p \in U$ and $q \in V$. Denote $m = \mathrm{Per}(q)$. Because of Ω is a C-dense basic set, q is h-related to $f^k(p)$ for every $0 \leqslant k \leqslant$ $\leqslant m - 1$. Therefore, there exists a non-wandering point $x_k \in f^k(V) \cap W^s_q$ so that $f^{mj}(x_k) \in U$ for all $j \geqslant j_k$ beginning with some j_k. It follows that $f^{tm}\left(f^k(U)\right) \cap V \neq \varnothing$ for all $t \geqslant j_k$.

Denote $N_0 = \max\{j_1,\ \ldots,\ j_{m-1}\}$. Any $n \geqslant (m+1)N_0$ can be represented as $n = tm + k$, where $0 \leqslant k \leqslant m - 1$. Since $t \geqslant N_0$,

$$f^n(U) \cap V = f^{tm+k}(U) \cap V = f^{tm}\left[f^k(U)\right] \cap V \neq \varnothing.$$

This concludes the proof. □

Corollary 1.4 *Let Ω be a basic set and Ω_c be a C-dense component of Ω. Then there is $n \in \mathbb{N}$ such that $f^n(\Omega_c) = \Omega_c$ and the restriction $f^n|_{\Omega_c} : \Omega_c \to \to \Omega_c$ is mixing.*

Shadowing

Let $f: M \to M$ be a homeomorphism, and $\{x_i\}_{i_1}^{i_2}$ be a δ-chain of length $i_2 - i_1$. Recall that this means that

$$d(f(x_{i_1+j}), x_{i_1+j+1}) < \delta \quad \text{for all } 0 \leqslant j \leqslant i_2 - i_1 - 1.$$

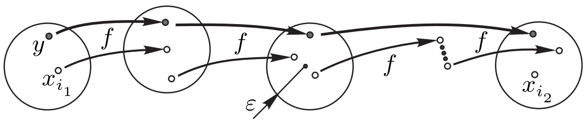


Figure 1.24

We say that a point y *ε-shadows* $\{x_i\}_{i_1}^{i_2}$ if $d(x_i, f^{i-i_1}y) < \varepsilon$ for all $i_1 \leqslant \leqslant i \leqslant i_2$, Fig. 1.24. Similar notation one holds for $i_1 = -\infty$ or $i_2 = +\infty$ or both.

Obviously, every μ-chain $\{x_0, x_1\}$ of length 1 is μ-shadowed by the point x_0. Moreover,

Lemma 1.27 *Let $f: M \to M$ be a homeomorphism of compact manifold M. Then given a fixed length $N \in \mathbb{N}$ and $\nu > 0$, there is $\mu > 0$ such that every μ-chain $\{x_0, x_1, \ldots, x_N\}$ of length N is ν-shadowed by the orbit of the point x_0 of length N.*

Proof. Denote by $B_\kappa(x)$ the ball of radius $\kappa > 0$ and the center at the point $x \in M$. Because of M is compact, there is $0 < \mu < \frac{\nu}{N}$ such that

$$f\left(B_\mu(x)\right) \subset B_{\frac{\nu}{N}}(f(x)), \quad f^2\left(B_\mu(x)\right) \subset B_{\frac{\nu}{N}}\left(f^2(x)\right), \quad \ldots,$$
$$f^N\left(B_\mu(x)\right) \subset B_{\frac{\nu}{N}}\left(f^N(x)\right) \quad \text{for any } x \in M.$$

In particular,

$$d(f^{N-k+1}x_i, f^{N-k}x_{i+1}) = d\left(f^{N-k}(fx_i), f^{N-k}x_{i+1}\right) < \frac{\nu}{N}$$

since $d(fx_i, x_{i+1}) < \mu$ by definition of μ-chain $\{x_0, x_1, \ldots, x_N\}$. Then

$$\begin{aligned} d(f^{N-j}x_0, x_{N-j}) \leqslant d(f^{N-j}x_0, f^{N-j-1}x_1)+ \\ + d(f^{N-j-1}x_1, f^{N-j-2}x_2) + \cdots + d(f^2 x_{N-j-2}, fx_{N-j-1})+ \\ + d(fx_{N-j-1}, x_{N-j}) < N \cdot \frac{\nu}{N} = \nu \end{aligned}$$

for every $0 \leqslant j \leqslant N-1$. This completes the proof. □

Theorem 1.22 *Let Ω be a basic set of diffeomorphism $f\colon M \to M$ of compact manifold M. Then given any $\beta > 0$, there is $\alpha > 0$ such that for any α-chain $\{x_i\}_{i_1}^{i_2}$ in Ω (i. e., every $x_i \in \Omega$) there is $y \in \Omega$ which β-shadows $\{x_i\}_{i_1}^{i_2}$.*

Proof. Recall that a hyperbolicity of Ω implies that given any $q \in W^s(p)$, the distance $d(f^n(p), f^n(q))$ tends exponentially to 0 as $n \to +\infty$. Similarly, given any $q_1 \in W^u(p_1)$, $d(f^n(p_1), f^n(q_1))$ tends exponentially to 0 as $n \to -\infty$. Formally, there exists $0 < \lambda < 1$ such that

$$d(f^n(p), f^n(q)) \leqslant \lambda^n d(p, q) \quad \text{if} \quad q \in W^s(p), \tag{1.20}$$

and

$$d(f^n(p_1), f^n(q_1)) \leqslant \lambda^{-n} d(p_1, q_1) \quad \text{if} \quad q_1 \in W^s(p_1) \quad \text{for } n \in \mathbb{N}.$$

Since Ω is locally maximal hyperbolic set, Ω has a local product structure i. e., there is $0 < \delta < \varepsilon$ such that for all $x, y \in \Omega$ with $d(x, y) < \delta$, the intersection $W^s_\varepsilon(x) \cap W^u_\varepsilon(y)$ consists of exactly one point, which belongs to Ω, see Theorem 1.7. Take ε and δ so small that

$$\frac{\varepsilon}{1-\lambda} + \frac{\delta}{2} < \beta. \tag{1.21}$$

Choose $N \in \mathbb{N}$ such that

$$\lambda^N \varepsilon < \frac{\delta}{2}. \tag{1.22}$$

Because of Ω is compact, there exists $\alpha > 0$ such that any α-chain $\{z_i\}_0^N$ is $\frac{\delta}{2}$-shadowed by the orbit of z_0 of length N, where $z_i \in \Omega$ (see Lemma 1.27). This means that

$$\text{if } d(f(z_{i-1}), z_i) < \alpha \text{ then } d(z_i, f^i(z_0)) < \frac{\delta}{2} \text{ for } 1 \leqslant i \leqslant N. \tag{1.23}$$

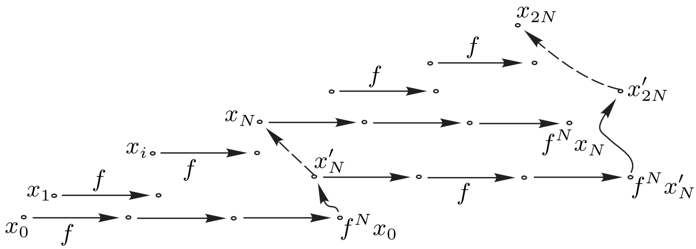


Figure 1.25. α-chain of length $2N$

First of all, we consider α-chains of length kN, $k \in \mathbb{N}$, Fig. 1.25. Given fixed kN-chain $\{x_i\}_0^{kN}$ in Ω, we introduce the sequence of points x'_0, x'_N, x'_{2N}, ..., x'_{kN} as follows:

$$\begin{gathered} x'_0 = x_0, \quad x'_N = W^s_\varepsilon(x_N) \cap W^u_\varepsilon(f^N x_0) = W^s_\varepsilon(x_N) \cap W^u_\varepsilon(f^N x'_0), \\ x'_{2N} = W^s_\varepsilon(x_{2N}) \cap W^u_\varepsilon(f^N x'_N), \quad \dots \\ x'_{jN} = W^s_\varepsilon(x_{jN}) \cap W^u_\varepsilon(f^N x'_{(j-1)N}), \quad \dots \\ x'_{kN} = W^s_\varepsilon(x_{kN}) \cap W^u_\varepsilon(f^N x'_{(k-1)N}). \end{gathered}$$

This sequence is well defined. Indeed, by (1.23), $d(x_N, f^N(x_0)) < \frac{\delta}{2}$. It follows from Theorem 1.7 on the existence of a local product structure, there is a unique point $x'_N = W^s_\varepsilon(x_N) \cap W^u_\varepsilon(f^N x'_0)$. In particular, $d(x_N, x'_N) < \varepsilon$. By (1.22), $d(f^N x_N, f^N x'_N) \leqslant \lambda^N \varepsilon < \frac{\delta}{2}$. Therefore,

$$d(x_{2N}, f^N x'_N) \leqslant d(x_{2N}, f^N x_N) + d(f^N x_N, f^N x'_N) < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

Hence, there exists a unique point $x'_{2N} = W^s_\varepsilon(x_{2N}) \cap W^u_\varepsilon(f^N x'_N)$. In particular, $d(x_{2N}, x'_{2N}) < \varepsilon$. Continuing by induction, one gets $d(x_{jN}, x'_{jN}) < \varepsilon$, $j \geqslant 2$. By (1.22), $d(f^N x_{jN}, f^N x'_{jN}) \leqslant \lambda^N \varepsilon < \frac{\delta}{2}$. By (1.23), $d(x_{(j+1)N}, f^N x_{jN}) < \frac{\delta}{2}$ (recall that $\{x_{jN}, \dots, x_{(j+1)N}\}$ is an α-chain). We see that

$$d(x_{(j+1)N}, f^N x'_{jN}) \leqslant d(x_{(j+1)N}, f^N x_{jN}) + d(f^N x_{jN}, f^N x'_{jN}) < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

Hence, there exists a unique point $x'_{(j+1)N} = W^s_\varepsilon(x_{(j+1)N}) \cap W^u_\varepsilon(f^N x'_{jN})$. This follows that the sequence x'_0, x'_N, x'_{2N}, ..., x'_{kN} is well defined.

Our first goal is to prove that the point $f^{-kN}(x'_{kN}) \overset{\text{def}}{=} x$ β-shadows the α-chain $\{x_i\}_0^{kN}$. Note that the point $f^{(k-1)N}(x)$ β-shadows the

α-chain $\{x_{(k-1)N}\}_0^{kN}$. Indeed, taking in mind (1.20)–(1.23) and $x'_{(k-1)N} \in W^s(x_{(k-1)N})$, $x'_{kN} \in W^s_{f^{-1}}(f^N(x'_{(k-1)N}))$, one gets for any $0 \leqslant j \leqslant N$

$$d\left(f^j \circ f^{(k-1)N}(x),\, x_{(k-1)N+j}\right) = d\left(f^{(k-1)N+j}(x),\, x_{(k-1)N+j}\right) \leqslant$$
$$\leqslant d\left(f^{(k-1)N+j}(x),\, f^j(x'_{(k-1)N})\right) + d\left(f^j(x'_{(k-1)N}),\, f^j(x_{(k-1)N})\right) +$$
$$+ d\left(f^j(x_{(k-1)N}),\, x_{(k-1)N+j}\right) <$$
$$< d\left(f^{-N+j} \circ f^{kN}(x),\, f^{-N+j} \circ f^N(x'_{(k-1)N})\right) +$$
$$+ \lambda^j d(x'_{(k-1)N}, x_{(k-1)N}) + \frac{\delta}{2} < \varepsilon\lambda^{N-j} + \lambda^j\varepsilon + \frac{\delta}{2} \leqslant \frac{\varepsilon}{1-\lambda} + \frac{\delta}{2} < \beta.$$

Similarly, the point $f^{(k-2)N}(x)$ β-shadows the α-chain $\{x_{(k-2)N}\}_0^{kN}$. It is enough to check that $f^{(k-2)N}(x)$ β-shadows the α-chain $\{x_{(k-2)N}\}_0^{(k-1)N}$. Note that above estimate for $j = 0$ gives

$$d\left(f^{(k-1)N}(x),\, x'_{(k-1)N}\right) < \lambda^N\varepsilon$$
$$\text{and hence,}\quad d\left(f^{(k-1)N}(x),\, f^N(x'_{(k-2)N})\right) < \lambda^N\varepsilon + \varepsilon.$$

Again, taking in mind (1.20)–(1.23) and $x'_{(k-2)N} \in W^s(x_{(k-2)N})$, $x'_{(k-1)N} \in W^s_{f^{-1}}(f^N(x'_{(k-2)N}))$, one gets for any $0 \leqslant j \leqslant N$

$$d\left(f^j \circ f^{(k-2)N}(x),\, x_{(k-2)N+j}\right) = d\left(f^{(k-2)N+j}(x),\, x_{(k-2)N+j}\right) \leqslant$$
$$\leqslant d\left(f^{(k-2)N+j}(x),\, f^j(x'_{(k-2)N})\right) + d\left(f^j(x'_{(k-2)N}),\, f^j(x_{(k-2)N})\right) +$$
$$+ d\left(f^j(x_{(k-2)N}),\, x_{(k-2)N+j}\right) <$$
$$< d\left(f^{-N+j} \circ f^{(k-1)N}(x),\, f^{-N+j} \circ f^N(x'_{(k-2)N})\right) +$$
$$+ \lambda^j d(x'_{(k-2)N}, x_{(k-2)N}) + \frac{\delta}{2} <$$
$$< \lambda^{N-j} d\left(f^{(k-1)N}(x),\, f^N(x'_{(k-2)N}\right) + \lambda^j\varepsilon + \frac{\delta}{2} < \lambda^{N-j}(\lambda^N\varepsilon + \varepsilon) + \varepsilon + \frac{\delta}{2} <$$
$$< \lambda^{2N-j}\varepsilon + \lambda^{N-j}\varepsilon + \varepsilon + \frac{\delta}{2} \leqslant \frac{\varepsilon}{1-\lambda} + \frac{\delta}{2} < \beta.$$

Continuing similarly with the induction method, we see that

$$d\left(f^{(k-i)N}(x),\, x'_{(k-i-1)N}\right) < \lambda^{iN}\varepsilon + \lambda^{(i-1)N}\varepsilon + \cdots + \lambda^{N}\varepsilon + \varepsilon.$$

Here, $0 \leqslant i \leqslant k$. Therefore,

$$\begin{aligned}
d\left(f^{j} \circ f^{(k-i-1)N}(x),\, x_{(k-i-1)N+j}\right) &= \\
&= d\left(f^{(k-i-1)N+j}(x),\, x_{(k-i-1)N+j}\right) \leqslant \\
&\leqslant d\left(f^{(k-i-1)N+j}(x),\, f^{j}(x'_{(k-i-1)N})\right) + \\
&+ d\left(f^{j}(x'_{(k-i-1)N}),\, f^{j}(x_{(k-i-1)N})\right) + \\
&+ d\left(f^{j}(x_{(k-i-1)N}),\, x_{(k-i-1)N+j}\right) < \\
&< d\left(f^{-N+j} \circ f^{(k-i)N}(x),\, f^{-N+j} \circ f^{N}(x'_{(k-i-1)N})\right) + \\
&+ \lambda^{j} d(x'_{(k-i-1)N},\, x_{(k-i-1)N}) + \frac{\delta}{2} < \\
&< \lambda^{N-j} d\left(f^{(k-i)N}(x),\, f^{N}(x'_{(k-i-1)N}\right) + \lambda^{j}\varepsilon + \frac{\delta}{2} < \\
&< \lambda^{N-j}(\lambda^{iN}\varepsilon + \cdots + \lambda^{N}\varepsilon + \varepsilon) + \varepsilon + \frac{\delta}{2} < \\
&< \lambda^{(i+1)N-j}\varepsilon + \lambda^{iN-j}\varepsilon + \cdots + \varepsilon + \frac{\delta}{2} \leqslant \frac{\varepsilon}{1-\lambda} + \frac{\delta}{2} < \beta.
\end{aligned}$$

It follows that the point $f^{-kN}(x'_{kN}) = x$ β-shadows the α-chain $\{x_i\}_0^{kN}$.

Any finite α-chain can be extended (for example, by real orbit) to the α-chain $\{x_i\}_0^{kN}$ for some sufficiently large k. This proves the theorem for finite α-chains. Now, let $\{x_i\}_{-\infty}^{+\infty}$ be the infinite α-chain. Given any $m \in \mathbb{N}$, there is $y_m \in \Omega$ that β-shadows the α-chain $\{x_i\}_{-m}^{m}$. Then the limit point $y \in \Omega$ of the sequence y_m β-shadows the α-chain $\{x_i\}_{-\infty}^{+\infty}$. This completes the proof. □

Corollary 1.5 *Let Ω be a basic set of diffeomorphism $f\colon M \to M$ of compact manifold M. Then given any $\beta > 0$, there is $\alpha > 0$ such that given any periodic α-chain*

$$\{\ldots,\, x_0,\, x_1,\, \ldots,\, x_{p-1},\, x_p = x_0,\, x_1,\, \ldots\}$$

in Ω there is a periodic point $z \in \Omega$ that β-shadows this α-chain i. e., $d(f^{j}(z),\, x_j) \leqslant \beta$ for $0 \leqslant j \leqslant p$ and $f^{p}(z) = z$.

Proof. Without loss of generality, we can assume that 2β is an expansiveness constant for Ω, see Lemma 1.26. By Theorem 1.22, there is $\alpha > 0$ and $z \in \Omega$ such that the orbit $\mathrm{Orb}(z)$ β-shadows this α-chain i. e.,

$$d\left(f^{p+kp}(z),\, x_0\right) \leqslant \beta, \quad \text{and} \quad d\left(f^{j+kp}(z),\, x_j\right) \leqslant \beta,$$
$$\text{for all } 0 \leqslant j \leqslant p-1,\ k \in \mathbb{Z}.$$

This means that given any $n \in \mathbb{Z}$, there is $0 \leqslant i_n \leqslant p-1$ such that $d(f^n(z),\, x_{i_n}) \leqslant \beta$ and $d(f^{n+p}(z),\, x_{i_n}) \leqslant \beta$. One holds

$$d\left(f^n(z),\, f^n(f^p(z))\right) \leqslant d\left(f^n(z),\, x_{i_n}\right) + d\left(x_{i_n},\, f^n(f^p(z)\right) \leqslant 2\beta$$

for all $n \in \mathbb{Z}$. Because of 2β is an expansiveness constant for Ω, $f^p(z) = z$. □

Corollary 1.6 *Let Ω be a basic set of diffeomorphism $f\colon M \to M$ of compact manifold M. Then given any $\beta > 0$, there is $\alpha > 0$ such that if $d(f^p(x),\, x) < \alpha$ for $x \in \Omega$, then there is $x' \in \Omega$ such that*

$$f^p(x') = x', \quad \text{and} \quad d\left(f^j(x),\, f^j(x')\right) \leqslant \beta \quad \text{for all } 0 \leqslant j \leqslant p.$$

Proof. It is easy to see that the sequence

$$\ldots,\, x,\, f(x),\, \ldots,\, f^{p-1}(x),\, x,\, f(x),\, \ldots,\, f^{p-1}(x),\, x,\, f(x),\, \ldots$$

is a periodic α-chain. The statement follows from Corollary 1.5. □

Now we formulate omitting the proof the generalization of Theorem 1.22.

Theorem 1.23 *Let Λ be a compact hyperbolic invariant set of diffeomorphism f. Then given any $\varepsilon > 0$, there are $\delta > 0$ and $\eta > 0$ such that if $\{x_j\}_{j_1}^{j_2}$ is a δ-chain that belongs to the η-neighborhood of Λ then there exists the point y which ε-shadows $\{x_j\}_{j_1}^{j_2}$. Moreover, if $\{x_j\}_{j_1}^{j_2}$ is periodic, y is periodic. If $j_1 = -\infty$, $j_2 = +\infty$, and Λ is an isolated invariant set (or has a local product structure), then y is unique and $y \in \Lambda$.*

Proof see in [198].

Growth of number of periodic orbits

Later on, in this section, we consider nontrivial basic set Ω of A-diffeomorphism f. We shall show that the number of periodic orbits grows exponentially with respect to periods.

Lemma 1.28 *Let Ω be a basic set of diffeomorphism $f\colon M \to M$ of compact manifold M, and $\delta > 0$ be the expansiveness constant of the restriction $f_\Omega\colon \Omega \to \Omega$. Suppose that $\mathrm{Orb}(x)$, $\mathrm{Orb}(y) \subset \Omega$ are distinct periodic orbits with the same period. Then there are points $x' \in \mathrm{Orb}(x)$, $y' \in \mathrm{Orb}(y)$ such that $d(x', y') \geqslant \delta$.*

Proof. Assume the contrary. Then $\mathrm{Orb}(y)$ belongs to the δ-neighborhood of $\mathrm{Orb}(x)$. The definition of expansiveness constant implies that $\mathrm{Orb}(x) = \mathrm{Orb}(y)$. □

Denote by N_m the number of points of $\mathrm{Fix}(f^m|_\Omega)$ i. e., N_m is the number of periodic points of period $m \geqslant 1$ that belong to Ω. Because of compactness of M and Lemma 1.28, N_m is finite.

Lemma 1.29 *Let Ω be a C-dense basic set of $f\colon M \to M$. Then there is $l \in \mathbb{N}$ such that*

$$N_m \leqslant N_{m+l} \quad \text{for all } m \in \mathbb{N}. \tag{1.24}$$

Proof. Let $\delta > 0$ be the expansiveness constant of the restriction $f_\Omega\colon \Omega \to \Omega$. By Theorem 1.22 and Corollary 1.5, there is $\alpha > 0$ such that for any (periodic) α-chain $\{x_0, x_1, \ldots, x_{k-1}, x_k = x_0\}$ in Ω there is a real periodic point $z \in \Omega$ that $\frac{\delta}{3}$-shadows this α-chain.

Since Ω is compact, there are finitely many $\frac{\alpha}{2}$-balls $V_1, \ldots, V_s \subset \Omega$ that cover Ω. By Theorem 1.21, the restriction $f_\Omega\colon \Omega \to \Omega$ is mixing. Therefore, given any V_k and V_j, there is $l \in \mathbb{N}$ such that $f^l(V_k) \cap V_j \neq \varnothing$.

Take $x \in \mathrm{Fix}(f^m)$. Then $x \in V_\nu$ for some $1 \leqslant \nu \leqslant s$. There is a point $x_\nu \in V_\nu$ such that $f(x_\nu) \in V_\nu$. We construct the periodic α-chain A_x as follows

$$\{\ldots, x, f(x), \ldots, f^{m-1}(x), x_\nu, f(x_\nu), \ldots, f^{l-1}(x_\nu), x, \ldots\}.$$

By Corollary 1.5, there is the real periodic point z_x of period $m+l$ that $\frac{\delta}{3}$-shadows A_x. If $y \in \mathrm{Fix}(f^m)$ is a periodic point different from x, then there are points $x' \in \mathrm{Orb}(x)$, $y' \in \mathrm{Orb}(y)$ such that $d(x', y') \geqslant \delta$ (see Lemma 1.28). By construction, the orbits $\mathrm{Orb}(x)$, $\mathrm{Orb}(y)$ belong to A_x, A_y respectively. Therefore, there are points $x'' \in A_x$, $y'' \in A_y$ such that $d(x', x'') \leqslant \frac{\delta}{3}$ and $d(y', y'') \leqslant \frac{\delta}{3}$. Hence,

$$d(x'', y'') \geqslant d(x', y') - 2\frac{\delta}{3} \geqslant \frac{\delta}{3}.$$

It follows that $A_x \neq A_y$. We see that the correspondence $\mathrm{Fix}(f^m) \to$ $\to \mathrm{Fix}(f^{m+l})$ defined by $x \to A_x$ is an immersion. The proof is complete. □

Corollary 1.7 *Let Ω be a basic set of $f\colon M \to M$. Then there is $l \in \mathbb{N}$ such that*

$$N_m \leqslant N_{m+l} \quad \text{for all } m \in \mathbb{N}. \tag{1.25}$$

Proof. By Theorem 1.14, Ω is the union of finitely many C-dense components $\Omega_{c1}, \ldots, \Omega_{ch}$. Given any Ω_{ci}, there is l_i that satisfies (1.24). Then $l =$ $= \max\{l_1, \ldots, l_h\}$ satisfies (1.25) because of any Ω_{ci} is a C-dense component. □

Similarly, one can prove the following statement.

Lemma 1.30 *Let Ω be a basic set of $f\colon M \to M$. Then there is $l \in \mathbb{N}$ such that*

$$N_i N_j \leqslant N_{i+j+2l} \quad \text{for all } i, j \in \mathbb{N}, \tag{1.26}$$

$$N_{i+j} \leqslant N_{i+l} N_{l+l} \quad \text{for all } i, j \in \mathbb{N}. \tag{1.27}$$

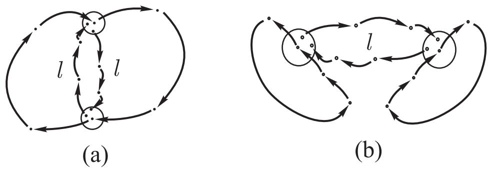


Figure 1.26

Proof. We illustrate the ideas by Fig. 1.26 (a), (b). The details are omitted to the Reader. □

Theorem 1.24 *Let Ω be a nontrivial basic set of $f\colon M \to M$, and N_m be the number of points of $\mathrm{Fix}(f^m|_\Omega)$. Then the following holds*:

1) *There are constants $C > 0$, $H > 0$ such that $N_m \leqslant Ce^{Hm}$ for all $m \in \mathbb{N}$.*
2) *There are constants $c > 0$, $h > 0$ such that $ce^{hm_k} \leqslant N_{m_k}$ for some sequence $\{m_k\}_1^\infty$ where $m_k \to +\infty$ as $k \to \infty$.*

Proof. (1). Take l satisfying Corollary 1.7 and (1.27) from Lemma 1.30. Without loss of generality, one can assume that $N_{1+l} \geqslant 2$ since Ω is the union of C-dense components. By (1.25) and (1.27), $N_2 \leqslant N_{1+l} N_{1+l} = N_{1+l}^2$.

Continuing by induction, one gets

$$N_m = N_{1+(m-1)} \leqslant N_{1+l}N_{(m-1)+l} \leqslant N_{1+l}N_{1+l}^{m-1} = N_{1+l}^m.$$

Hence, $N_m \leqslant N_{1+l}^m$ for all $m \in \mathbb{N}$. This follows the first assertion.

(2). Suppose l satisfies (1.26) from Lemma 1.30. Taking in mind (1.26), one gets

$$N_{2+2l} = N_{1+1+2l} \geqslant N_1^2, \quad N_{3+4l} = N_{1+(2+2l)+2l} \geqslant N_1 N_{2+2l} \geqslant N_1^3.$$

Continuing by induction, one gets

$$N_{n+(2n-2)l} = N_{1+(n-1)+[2(n-1)-2]l+2l} \geqslant \\ \geqslant N_1 N_{(n-1)+[2(n-1)-2]l} \geqslant N_1 N_1^{n-1} = N_1^n.$$

Hence, $N_{m+(2m-2)l} \geqslant N_1^m$ for all $m \in \mathbb{N}$. This follows the second estimate. □

Bibliographic Notes and Panoramas

Chapter 1. The modern Geometric Theory of Dynamical Systems applies the power of many fields of Mathematics such as Topology, Geometry, Theory of Foliations, and Theory of Complex Functions, and so on. A nice introduction to Modern Dynamical Systems are the books [54, 123, 181, 198]. A good introduction to low dimensional Dynamical Systems are the books [3, 26, 183].

(1.1). Bernhard Riemann was first who motivated the idea of a manifold by an intuitive process of varying a given object in a new direction, and presciently described the role of coordinate systems and charts. A modern definition of a 2-dimensional manifold was given by Hermann Weyl in his 1913 book on Riemann surfaces [217]. The widely accepted general definition of a manifold in terms of an atlas is due to Hassler Whitney [219], see also [154, 164].

(1.2). The first general definition of a dynamical system is by Birkhoff [47], see also [168].

(1.3). A good introduction to Topological Dynamics is the book [2]. The modern formulation of the non-wandering set appeared in [47].

(1.4). A system of differential equations can be approximated by a system of first degree differential equations for the first-step approaching to investigate the solutions of the system. Therefore, it is natural to approximate a differentiable dynamical system by a linear dynamical system. In particular, it is natural to start the study of discrete-time dynamical system with linear mappings.

(1.5). First, the Grobman – Hartman Theorem was proved for a continuous-time dynamical system. In 1959, Grobman [98] announced this result. The complete proof was published in [99]. In 1960, Hartman [109] proved the result for the more strong conditions on the smoothness than Grobman's ones. Namely, Philip Hartman required the smoothness C^2. He wrote that it was raised by M. M. Peixoto's question. In [110], the complete proof was represented with the weaker smoothness conditions. For diffeomorphisms (discrete-time dynamical systems), the Grobman – Hartman Theorem was proved by Pugh [194].

(1.6). Roughly speaking, a hyperbolicity means that the linear part of a transformation has complementary expanding and contracting directions. The notion of hyperbolicity was introduced by Anosov [4, 5] and Smale [207, 208].

(1.7). Theorem 1.12 was proved by S. Smale in [208], Theorem 1.13 was proved in [118], and Theorem 1.14 was proved in [6] and [52].

(1.8). Theorems 1.16, 1.17, 1.18 was proved in [185]. Lemmas 1.24, 1.25 and Theorems 1.19, 1.20 was proved in [82] and [96].

CHAPTER 2

Dynamics of Degree One Circle Maps

The biggest part of this chapter concerns to discrete-time dynamical systems generated by orientation preserving homeomorphisms of a circle. This chapter can be considered as a careful introduction to One-Dimensional Dynamics. Excellent exposition of One-Dimensional Dynamics is in [161], see also [180], ch. 1.

Section 2.1 is devoted to an elementary introduction to the qualitative theory of degree one circle mappings. In Section 2.2 one studies Poincaré rotation number. This notion turns out to be very useful in Theory of Dynamical Systems. Poincaré rotation number is considered for monotone circle mapping of degree 1. This map allows points of discontinuity and segments transforming to points. In Section 2.3, we consider mainly circle degree one mappings and homeomorphisms with irrational rotation numbers.

2.1. Degree of circle maps

Here we consider degree one maps of the unit circle $S^1 = [0;\,1]/(0 \sim 1)$. We'll also consider S^1 as $S^1 = \left\{z \in \mathbb{C}\colon |z| = \frac{1}{2\pi}\right\}$ in $\mathbb{C}$, so-called multiplicative circle. The two notations are related by $z = \frac{1}{2\pi}e^{2\pi i x}$, $x \in [0;\,1]/(0 \sim 1)$.

Degree of cover transformations

To specify circle maps, it is convenient to use cover transformations of the universal covering space of S^1, which is the real line $\mathbb{R}$. The natural projection

$$\pi\colon \mathbb{R} \to S^1 = [0;\,1]/(0 \sim 1), \quad \pi(x) = x \bmod 1,$$

is the universal covering map. For the multiplicative circle, $\pi(x) = \frac{1}{2\pi}e^{2\pi i x}$.

Two sets of $\mathbb{R}$ that are mapped under π to the same set are called *congruent*. Given any set $A \subset \mathbb{R}$ and $n \in \mathbb{Z}$, the set $B = A + n = \{x + n\colon x \in A\}$ is congruent to A.

Suppose that $\overline{f}\colon \mathbb{R} \to \mathbb{R}$ is a map (not necessary continuous) that takes congruent points to congruent points. Then, given any point $x \in \mathbb{R}$, there is $k(x) \in \mathbb{Z}$ such that

$$\overline{f}(x+1) = \overline{f}(x) + k(x). \tag{2.1}$$

Definition 2.1 *Let $f\colon S^1 \to S^1$ be a mapping of the circle. Then $\overline{f}\colon \mathbb{R} \to \mathbb{R}$ is called the **cover transformation** for f or a **lift** of f, if*

$$\pi \circ \overline{f} = f \circ \pi\colon \quad \begin{array}{ccc} \mathbb{R} & \xrightarrow{\overline{f}} & \mathbb{R} \\ \downarrow \pi & & \downarrow \pi \\ S^1 & \xrightarrow{f} & S^1 \end{array}$$

*One says that $\overline{f}$ **induces** f or f is a **projection** of $\overline{f}$ under π.*

Proposition 2.1 *If $\overline{f}\colon\mathbb{R}\to\mathbb{R}$ satisfies* (2.1) *then $\overline{f}$ induces some map $f\colon S^1\to S^1$ that is $\overline{f}$ is a cover mapping for f.*

Proof. Given any $x \in S^1$ and $\overline{x} \in \pi^{-1}(x)$, put by definition $f(x) = = \pi \circ \overline{f}(\overline{x})$. Due to (2.1), $f(x)$ does not depend on the choice of $\overline{x} \in \pi^{-1}(x)$. Indeed, take $\overline{x} + l \in \pi^{-1}(x)$, $l \in \mathbb{Z}$. Then

$$\pi(\overline{f}(\overline{x}+l)) = \pi(\overline{f}(\overline{x}) + k(\overline{x}) + \ldots + k(\overline{x} + (l-1)) = \pi(\overline{f}(\overline{x})),$$

since $k(\overline{x}) + \ldots + k(\overline{x} + (l-1)) \in \mathbb{Z}$. $\square$

Definition 2.2 *A map $\overline{f}\colon \mathbb{R} \to \mathbb{R}$ is said to be of **degree** $k \in \mathbb{Z}$ if $\overline{f}(\overline{x}+1) = = \overline{f}(\overline{x}) + k$ for any $\overline{x} \in \mathbb{R}$.*

Example 2.1 *Examples.*

- The map $\overline{E}_k\colon \mathbb{R} \to \mathbb{R}$ given by the formula $\overline{E}_k(\overline{x}) = k\overline{x}$ has degree $k \in \mathbb{N}$. For $k > 1$ this map is cover mapping for non-invertible expanding endomorphism $E_k\colon S^1 \to S^1$ given by the formula $E_k(x) = k\overline{x} \bmod 1$ where $\overline{x} \in [0, 1)$ and $\pi(\overline{x}) = x$. This endomorphism k-multiply covers S^1 by itself;
- The map $\overline{R}_\alpha\colon \mathbb{R} \to \mathbb{R}$ given by the formula $\overline{R}_\alpha(\overline{x}) = \overline{x} + \alpha$ has degree 1; this is cover mapping for the rotation $R_\alpha\colon S^1 \to S^1$ by α and given by the formula $R_\alpha(x) = \overline{x} + \alpha \bmod 1$;
- The map $\overline{f}_{\omega,\varepsilon}$ given by the formula $\overline{f}_{\omega,\varepsilon}(\overline{x}) = \overline{x} + \omega + \varepsilon \sin 2\pi\overline{x}$ has degree 1; this map can be considered as a perturbation of $\overline{R}_\alpha$.

Proposition 2.2 *If $\overline{f}$ is a lift of f, then $\overline{f}^n$ is a lift of f^n for any $n \in \mathbb{N}$.*

Proof. Since $\pi \circ \overline{f} = f \circ \pi$, $\pi \circ \overline{f}^n = (\pi \circ \overline{f}) \circ \overline{f}^{n-1} = f \circ (\pi \circ \overline{f}^{n-1}) = = \ldots = f^n \circ \pi$. □

Lemma 2.1 *Given any continuous map $f\colon S^1 \to S^1$, there is a unique $k \in \mathbb{Z}$ such that f has a lift $\overline{f}\colon \mathbb{R} \to \mathbb{R}$ of degree k, and $\overline{f}$ is continuous. Moreover, given any cover transformations $\overline{f}_1$, $\overline{f}_2$ for f there is $n \in \mathbb{Z}$ such that $\overline{f}_2(\overline{x}) = = \overline{f}_1(\overline{x}) + n$ for $\forall \overline{x} \in \mathbb{R}$. Vice versa, if $\overline{f}$ is a lift of f, then $\overline{f} + n$ is also a lift of f.*

Proof. Take an arbitrary point $\overline{x}_0$ from $\pi^{-1}(f(\pi(0))$ which is a lift of $f(\pi(0))$. The image $f(\pi([0;\,1))$ of $\pi([0;\,1)) \subseteq S^1$ can be considered as a path starting at $f(\pi(0))$. Since π is a universal covering map, there is a unique path $\overline{P}$ starting at $\overline{s}$ and covering $f(\pi([0;\,1)))$. Denote by $\overline{x}_1$ the endpoint of $\overline{P}$ different from $\overline{x}_0$ (note that $\overline{s}_1$ could be geometrically coincident with $\overline{x}_0$). Since f is continuous, $\overline{x}_1$ is congruent to $\overline{x}_0$. Taking in mind that π is a local homeomorphism, we see that there exists a continuous map $\overline{f}_0\colon [0;\,1) \subset \mathbb{R} \to \operatorname{clos} \overline{P}$ such that $\pi \circ \overline{f}_0|_{[0,\,1)} = f \circ \pi|_{[0,\,1)}$. Let $\overline{x}_1 - - \overline{x}_0 = k$. Put by definition, $\overline{f}(\overline{x}) = \overline{f}_0(\overline{x} - m) + km$ provided $\overline{x} \in [m;\, m+1)$, $m \in \mathbb{Z}$. It is easy to check that $\overline{f}$ is continuous and lift of f. Moreover, $\overline{f}(\overline{x} + + 1) = \overline{f}_0(\overline{x} + 1 - (m+1)) + k(m+1) = \overline{f}_0(\overline{x} - m) + k(m+1) = \overline{f}(\overline{x}) - - km + k(m+1) = \overline{f}(\overline{x}) + k$. Hence $\overline{f}$ is of degree k.

Let $\overline{f}_2$, $\overline{f}_1$ be two lifts. Then the function $\overline{f}_2(\overline{x}) - \overline{f}_1(\overline{x})$ is continuous and value-integer for any $\overline{x}$. Hence, $\overline{f}_2(\overline{x}) - \overline{f}_1(\overline{x})$ is some integer for all $\overline{x} \in \mathbb{R}$. Obviously, $\overline{f} + n$ is a lift for any $n \in \mathbb{Z}$. □

Lemma 2.1 allows us to introduce a *degree* of a continuous map $f\colon S^1 \to S^1$.

Definition 2.3 *The degree of a continuous map $f\colon S^1 \to S^1$ is called the number which is equal to the degree of any lift $\overline{f}\colon \mathbb{R} \to \mathbb{R}$ for f.*

Corollary 2.1 *Any continuous map $f\colon S^1 \to S^1$ is homotopy to E_k, where $k \in \mathbb{Z}$ is the degree of f.*

Proof. Take lifts $\overline{f}$ and $\overline{E}_k$ for f and E_k respectively. Put $\overline{f}_t(\overline{x}) = (1 - - t)\overline{f}(\overline{x}) + t\overline{E}_k(\overline{x})$, where $0 \leqslant t \leqslant 1$, $\overline{x} \in \mathbb{R}$. Then

$$\begin{aligned}\overline{f}_t(\overline{x}+1) &= (1-t)\overline{f}(\overline{x}+1) + t\overline{E}_k(\overline{x}+1) = \\ &= (1-t)\overline{f}(\overline{x}) + (1-t)k + t(k\overline{x}+k) = (1-t)\overline{f}(\overline{x}) + k = \overline{f}_t(\overline{x}) + k.\end{aligned}$$

Therefore, $\overline{f}_t$ induces $f_t\colon S^1 \to S^1$ that realizes the homotopy between $f = f_0$ and $E_k = f_1$. □

The orientation of the real line $\mathbb{R}$ induces the orientation of S^1 by the projection π. So, one can say about orientation preserving and orientation reversing homeomorphisms of S^1. A particular case of Lemma 2.1 is the following statement.

Lemma 2.2 *Any homeomorphism of S^1 has a lift that is a homeomorphism of $\mathbb{R}$. Moreover, if a homeomorphism f preserves (reverses) orientation, f is of degree 1 (-1).*

Proof. Clearly that if $\overline{f}$ is a lift of a homeomorphism $f\colon S^1 \to S^1$, then $\overline{f}$ is a homeomorphism. If f preserves orientation, then $\overline{f}$ preserves orientation also and

$$\overline{f}(\overline{x}+1) = \overline{f}(\overline{x}) + 1. \tag{2.2}$$

This implies

$$\overline{f}(\overline{x}+n) = \overline{f}(\overline{x}) + n, \quad \forall n \in \mathbb{Z}. \tag{2.3}$$

If f reverses orientation, then

$$\overline{f}(\overline{x}+1) = \overline{f}(\overline{x}) - 1. \tag{2.4}$$

This implies

$$\overline{f}(\overline{x}+n) = \overline{f}(\overline{x}) - n, \quad \forall n \in \mathbb{Z}. \tag{2.5}$$

□

Monotone transformations

Denote by $MT_+(\mathbb{R})$ the set of maps $\overline{f}\colon \mathbb{R} \to \mathbb{R}$, which satisfy the following conditions:

$$\overline{f}(\overline{x}+1) = \overline{f}(\overline{x}) + 1, \quad \overline{x} \in \mathbb{R} \quad (\overline{f} \text{ is of degree } 1). \tag{2.6}$$

$$\overline{f}(\overline{x}_1) \leqslant \overline{f}(\overline{x}_2), \quad \text{if} \quad \overline{x}_1 < \overline{x}_2 \quad \text{(monotonicity)}. \tag{2.7}$$

Let us note that a monotone transformation from the class $MT_+(\mathbb{R})$ is not necessarily continuous. We see that $MT_+(\mathbb{R})$ are degree 1 monotone transformations of $\mathbb{R}$ each of which induces a circle map by Proposition 2.1. Denote by $MT_+(S^1)$ the set of these circle maps (of degree 1). It immediately follows from definition of $MT_+(\mathbb{R})$ that given any $\overline{f} \in MT_+(\mathbb{R})$,

$$\overline{f}^n(\overline{x}+k) = \overline{f}^n(\overline{x}) + k \tag{2.8}$$

$$\overline{f}^n(\overline{x}_1) \leqslant \overline{f}^n(\overline{x}_2), \quad \text{for any } \overline{x}_1 < \overline{x}_2, \tag{2.9}$$

where $n \in \mathbb{N}$, $k \in \mathbb{Z}$. From (2.2), one gets

Lemma 2.3 *Any $\overline{f} \in MT_+(\mathbb{R})$ can be represented as $\overline{f}(\overline{x}) = \overline{x} + \overline{\varphi}(\overline{x})$, where $\overline{\varphi}(\overline{x})$ is a 1-periodic function, $\overline{\varphi}(\overline{x}+1) = \overline{\varphi}(\overline{x})$.*

In $MT_+(S^1)$, there are maps which transform S^1 to a point. We shall call such maps *trivial*. For nontrivial maps, there is the following generalization of Lemma 2.1.

Lemma 2.4 *Let $\overline{f}_1, \overline{f}_2 \in MT_+(\mathbb{R})$ be cover transformations for the same nontrivial $f \in MT_+(S^1)$. Then there is $k \in \mathbb{Z}$ such that $\overline{f}_2(\overline{x}) = \overline{f}_1(\overline{x}) + k$ for any $\overline{x} \in \mathbb{R}$.*

Proof. Clearly, $\overline{f}_2(\overline{x}) - \overline{f}_1(\overline{x}) = k(\overline{x}) \in \mathbb{Z}$. We have to prove that $k(\overline{x}) = \text{const}$. Suppose the contrary. Then $k(\overline{x})$ has the point $\overline{x}_0$ of discontinuity with the jump $k_0 \in \mathbb{Z}$, $k_0 \neq 0$. Therefore there are $\overline{x}$ arbitrary close to $\overline{x}_0$ such that $k(\overline{x}) = k_0 + k(\overline{x}_0)$. Without loss of generality, one can assume that $\overline{x} > \overline{x}_0$. Then

$$k(\overline{x}) = \overline{f}_2(\overline{x}) - \overline{f}_1(\overline{x}) = k_0 + \overline{f}_2(\overline{x}_0) - \overline{f}_1(\overline{x}_0),$$
$$\text{or} \quad \overline{f}_2(\overline{x}) - \overline{f}_2(\overline{x}_0) = k_0 + \overline{f}_1(\overline{x}) - \overline{f}_1(\overline{x}_0).$$

The last equality is impossible because $|k_0| \geqslant 1$ and, due to the nontriviality, (2.6) and (2.7) we have $0 \leqslant \overline{f}_i(\overline{x}) - \overline{f}_i(\overline{x}_0) < 1$, $\overline{f}_i(\overline{x})$, $i = 1, 2$. □

Later on, we shall assume any $\overline{f}(x) \in MT_+(\mathbb{R})$ is nontrivial.

Spaces of monotone transformations

Let $\mathrm{Diff}^r_+(S^1)$, $r \geqslant 1$, be the set of preserving orientation C^r diffeomorphisms, and $\mathrm{Diff}^\omega_+(S^1)$ the set of preserving orientation analytic diffeomorphisms, and $\mathrm{Homeo}_+(S^1) \overset{\text{def}}{=} \mathrm{Diff}^0_+(S^1)$ the set of preserving orientation homeomorphisms of S^1. Obviously,

$$\mathrm{Diff}^\omega_+(S^1) \subset \mathrm{Diff}^\infty_+(S^1) \subset \ldots \subset \mathrm{Diff}^1_+(S^1) \subset \mathrm{Diff}^0_+(S^1) \subset MT_+(S^1).$$

Denote by $\mathrm{CDiff}_+^\nu(\mathbb{R})$ ($\nu = r \geqslant 0$, $\nu = \infty$ or $\nu = \omega$) the set of cover transformations for $\mathrm{Diff}_+^\nu(S^1)$. Here,

$$\mathrm{CDiff}_+^\omega(\mathbb{R}) \subset \mathrm{CDiff}_+^\infty(\mathbb{R}) \subset \ldots \subset \mathrm{CDiff}_+^1(\mathbb{R}) \subset \mathrm{CDiff}_+^0(\mathbb{R}) \subset MT_+(\mathbb{R}).$$

Let us introduce the metric on $\mathrm{CDiff}_+^\nu(\mathbb{R})$. Given any $\overline{f}$, $\overline{g} \in \mathrm{CDiff}_+^\nu(\mathbb{R})$, put by definition

$$\overline{d}_0(\overline{f}, \overline{g}) = \max_{x\in\mathbb{R}} |\overline{f}(x) - \overline{g}(x)|.$$

It follows from Lemma 2.3 that

$$\overline{d}_0(\overline{f}, \overline{g}) = \max_{x\in[0;\,1]} |\overline{f}(x) - \overline{g}(x)|.$$

Therefore, $\overline{d}_0$ is really a metric that makes the set $\mathrm{CDiff}_+^\nu(\mathbb{R})$ a metric (hence, topological) space. This metric defines the metric d_0 on $\mathrm{CDiff}_+^\nu(S^1)$.

2.2. The Poincaré rotation number

The rotation number, in sense, is the limit of average rotation of a point. For a large class of circle transformations, this limit does not depend on the choosing of a point and characterizes the transformation itself. We give two proofs of the existence of rotation number. The first one called conventionally classical is for degree 1 monotone transformations $MT_+(S^1)$, which includes homeomorphisms as well as transformations with discontinuity points. The second proof conventionally called ergodic is for homeomorphisms.

Classical proof

It is methodologically convenient to prove the existence of rotation number for degree 1 covering transformations $MT_+(\mathbb{R})$ instead of $MT_+(S^1)$.

Lemma 2.5 *Let $\overline{f} \in MT_+(\mathbb{R})$, and suppose the following inequalities hold*

$$r < \overline{f}^k(\overline{x}_0) - \overline{x}_0 < r + 1 \tag{2.10}$$

for some $k \in \mathbb{N}$, $r \in \mathbb{Z}$ and $\overline{x}_0 \in \mathbb{R}$. If $\overline{f}^k(\overline{x}) - \overline{x} \notin \mathbb{Z}$ for all $\overline{x} \in \mathbb{R}$, then (2.10) *holds for every $\overline{x} \in \mathbb{R}$.*

Proof. Denote $\mathscr{D} \stackrel{\text{def}}{=} \{\overline{x} \in \mathbb{R} \colon r < \overline{f}^k(\overline{x}) - \overline{x}\}$. By condition, $\overline{x}_0 \in \mathscr{D}$. Let us show that if $\overline{x}_1 \in \mathscr{D}$, then $[\overline{x}_1; \overline{x}_2] \in \mathscr{D}$ for some $\overline{x}_2 \neq \overline{x}_1$. Indeed, due to (2.9), if $\overline{x}_1 \leqslant \overline{x} \leqslant \overline{f}^k(\overline{x}_1) - \overline{x}_1$, then $\overline{x} + r \leqslant \overline{f}^k(\overline{x}_1) \leqslant \overline{f}^k(\overline{x})$. Since $\overline{f}^k(\overline{x}) - \overline{x} \notin \mathbb{Z}$, the strong inequality $\overline{x} + r < \overline{f}^k(\overline{x})$ holds i.e., the interval $[\overline{x}_1; \overline{f}^k(\overline{x}_1) - \overline{x}_1] \subset \mathscr{D}$.

Put $\overline{x}_\infty = \sup\{\overline{x} \in \mathscr{D} \colon [\overline{x}_0; \overline{x}] \subset \mathscr{D}\}$. Assume that $\overline{x}_\infty < +\infty$. Then $\overline{x}_\infty \notin \mathscr{D}$ and $\overline{x}_\infty > \overline{f}^k(\overline{x}_\infty) - r$. However, for $\overline{x} \in \mathscr{D}$ satisfying $\overline{f}^k(\overline{x}_\infty) - - r \leqslant \overline{x} < \overline{x}_\infty$, one gets $\overline{f}^k(\overline{x}) \leqslant \overline{f}^k(\overline{x}_\infty) < \overline{x} + r$. This contradicts to $\overline{x} \in \mathscr{D}$. Therefore, $\overline{x}_\infty = +\infty$.

It follows from (2.8) that $\overline{x}_0 + n \in \mathscr{D}$ for all negative integers n. Hence, $\mathscr{D} = \mathbb{R}$. The similar proof is for $\overline{f}^k(\overline{x}) - \overline{x} < r + 1$. $\square$

Theorem 2.1 *Given any $\overline{f} \in MT_+(\mathbb{R})$ and $\overline{x} \in \mathbb{R}$, the limit*

$$\lim_{n\to\infty} \frac{\overline{f}^n(\overline{x})}{n} \stackrel{\text{def}}{=} \operatorname{rot}(\overline{f}) \tag{2.11}$$

exists and does not depend on $\overline{x}$. Moreover, if $\overline{f}^k(\overline{x}_0) = \overline{x}_0 + r$ for some point $\overline{x}_0 \in \mathbb{R}$ and numbers $k \in \mathbb{N}$, $r \in \mathbb{Z}$, then $\operatorname{rot}(\overline{f}) = \frac{r}{k} \in \mathbb{Q}$.

Proof. First, we prove that the limit (2.11) does not depend on a point $\overline{x}$. Suppose (2.11) exists for some point $\overline{x}_1 \in \mathbb{R}$. Given any point $\overline{x}_2 \in \mathbb{R}$, there is $s \in \mathbb{Z}$ such that $\overline{x}_1 + s \leqslant \overline{x}_2 < \overline{x}_1 + s + 1$. It follows from (2.8), (2.9) that

$$\overline{f}^n(\overline{x}_1) + s = \overline{f}^n(\overline{x}_1 + s) \leqslant \overline{f}^n(\overline{x}_2) \leqslant \overline{f}^n(\overline{x}_1 + s + 1) = \overline{f}^n(\overline{x}_1) + s + 1.$$

Hence,

$$\left| \frac{\overline{f}^n(\overline{x}_1) - \overline{f}^n(\overline{x}_2)}{n} \right| \leqslant \frac{|s| + 1}{n} \longrightarrow 0 \quad n \to \infty.$$

It remains to prove the existence of (2.11). First, suppose that $\overline{f}^k(\overline{x}_0) = \overline{x}_0 + r$ for some point $\overline{x}_0 \in \mathbb{R}$ and numbers $k \in \mathbb{N}$, $r \in \mathbb{Z}$. Any natural number n can be expressed as $n = qk + s$, where $0 \leqslant s < k$. By (2.8), we have

$$\overline{f}^{qk}(\overline{x}_0) = \underbrace{\overline{f}^k \circ \ldots \circ \overline{f}^k}_{q}(\overline{x}_0) = \underbrace{\overline{f}^k \circ \ldots \circ \overline{f}^k}_{q-1}(\overline{x}_0) + k = \ldots = \overline{x}_0 + qr.$$

Next, it follows from $\overline{f}^n(\overline{x}_0) = \overline{f}^s \circ \overline{f}^{qk}(\overline{x}_0) = \overline{f}^s(\overline{x}_0 + qr) = \overline{f}^s(\overline{x}_0) + qr$ that

$$\frac{\overline{f}^n(\overline{x}_0)}{n} = \frac{\overline{f}^s(\overline{x}_0) + qr}{n} = \frac{\overline{f}^s(\overline{x}_0)}{n} + \frac{qr}{qr+s} = \frac{\overline{f}^s(\overline{x}_0)}{n} + \frac{r}{k + \frac{s}{q}} \to \frac{r}{k}$$

because the set $\{\overline{f}^s(\overline{x}_0) \colon 0 \leqslant s < k\}$ is bounded and $q \to \infty$ as $n \to \infty$.

Now consider the case when $\overline{f}^k(\overline{x}) - \overline{x} \notin \mathbb{Z}$ for all $\overline{x} \in \mathbb{R}$ and $k \in \mathbb{N}$. Then given any $\overline{x} \in \mathbb{R}$ and $k \in \mathbb{N}$, there is $q_k(\overline{x}) \in \mathbb{Z}$ such that $q_k(\overline{x}) < \overline{f}^k(\overline{x}) - - \overline{x} < q_k(\overline{x}) + 1$. By Lemma 2.5, $q_k(\overline{x}) = q_k$ does not depend on $\overline{x}$,

$$q_k < \overline{f}^k(\overline{x}) - \overline{x} < q_k + 1, \quad \overline{x} \in \mathbb{R}. \tag{2.12}$$

Applying (2.12) for the points $\overline{x} = 0, \overline{f}^k(0), \ldots, \overline{f}^{k(n-1)}(0)$, we have

$$q_k < \overline{f}^k(0) < q_k + 1, \quad q_k < \overline{f}^{2k}(0) - \overline{f}^k(0) < q_k + 1, \quad \ldots$$
$$\ldots, \quad q_k < \overline{f}^{nk}(0) - \overline{f}^{(n-1)k}(0) < q_k + 1.$$

After adding together and dividing by nk, we get

$$\frac{q_k}{k} < \frac{\overline{f}^{nk}(0)}{nk} < \frac{q_k}{k} + \frac{1}{k}. \tag{2.13}$$

Dividing $q_k < \overline{f}^k(0) < q_k + 1$ by k, one has

$$\frac{q_k}{k} < \frac{\overline{f}^k(0)}{k} < \frac{q_k}{k} + \frac{1}{k}. \tag{2.14}$$

It follows from (2.13), (2.14) that

$$\left| \frac{\overline{f}^{nk}(0)}{nk} - \frac{\overline{f}^k(0)}{k} \right| < \frac{1}{k}. \tag{2.15}$$

Since n, k were arbitrary, we can repeat formulas above replacing n, k. Then one gets

$$\left| \frac{\overline{f}^{nk}(0)}{nk} - \frac{\overline{f}^n(0)}{n} \right| < \frac{1}{n}. \tag{2.16}$$

It follows from (2.15), (2.16) that

$$\left| \frac{\overline{f}^n(0)}{n} - \frac{\overline{f}^k(0)}{k} \right| < \frac{1}{k} + \frac{1}{n}.$$

This means that the sequence $\left\{ \frac{\overline{f}^n(0)}{n} \right\}_1^\infty$ is a Cauchy sequence. Therefore the limit (2.11) exists at $\overline{x} = 0$, and hence at any $\overline{x} \in \mathbb{R}$. □

Definition 2.4 *The number*

$$\lim_{n\to\infty} \frac{\overline{f}^n(\overline{x})}{n} \stackrel{\text{def}}{=} \operatorname{rot}(\overline{f}), \quad \overline{x} \in \mathbb{R},$$

is called the ***rotation number*** *of* $\overline{f} \in MT_+(\mathbb{R})$.

If $\overline{f} \in MT_+(\mathbb{R})$, then $\overline{f}_1(\overline{x}) = \overline{f}^k(\overline{x}) + r$ belongs to $MT_+(\mathbb{R})$ for any fixed numbers $k \in \mathbb{N}$ and $r \in \mathbb{Z}$.

Lemma 2.6 $\operatorname{rot}(\overline{f}^k + r) = k \operatorname{rot}(\overline{f}) + r$.

Proof.

$$\operatorname{rot}(\overline{f}^k + r) = \lim_{n\to\infty} \frac{(\overline{f}^k + r)^n}{n} = \lim_{n\to\infty} \frac{\overline{f}^{nk} + nr}{n} =$$
$$= k \lim_{n\to\infty} \frac{\overline{f}^{nk}}{nk} + r = k \operatorname{rot}(\overline{f}) + r. \quad \square$$

Definition 2.5 *For* $f \in MT_+(S^1)$*, take its lift* $\overline{f} \in MT_+(\mathbb{R})$*. The number*

$$\operatorname{rot}(\overline{f}) \bmod 1 = \lim_{n\to\infty} \frac{\overline{f}^n(\overline{x})}{n} \bmod 1 \stackrel{\text{def}}{=} \operatorname{rot}(f), \quad \textit{where } \overline{x} \in \mathbb{R},$$

is called the ***rotation number*** *of* f.

If $\overline{f}_1, \overline{f}_2 \in MT_+(\mathbb{R})$ are lifts of f, then $\overline{f}_2 = \overline{f}_1 + r$ for some $r \in \mathbb{Z}$. Due to Lemma 2.6, $\operatorname{rot}(\overline{f}_2) = \operatorname{rot}(\overline{f}_1) + r$. Therefore, definition 2.5 is correct.

Ergodic proof

Here we give the so-called ergodic proof of the existence of rotation number for $f \in \operatorname{Homeo}_+(S^1)$. It is well known that such f has (not necessarily unique) invariant normalized ($\mu(S^1) = 1$) measure μ. Then this measure can be lifted to the invariant measure $\overline{\mu}$ of any cover map $\overline{f} \in \mathrm{CDiff}_+^0(\mathbb{R})$. Since $\mu(S^1) = 1$, $\overline{\mu}(I) = 1$ for any unit interval I of $\mathbb{R}$.

Proposition 2.3 *Let $\overline{f} \in \mathrm{CDiff}_+^0(\mathbb{R})$ be a cover map for $f \in \mathrm{Homeo}_+(S^1)$. Then given any $k \in \mathbb{N}$,*

$$\overline{f}^k(\overline{x}) = \overline{x} + \sum_{i=0}^{k-1} \overline{\varphi} \circ \overline{f}^i(\overline{x}),$$

where $\overline{\varphi}(\overline{x}) = \overline{f}(\overline{x}) - \overline{x}$ is a 1-periodic continuous function (*see Lemma* 2.3).

Proof.

$$\overline{f}^k(\overline{x}) = \overline{f}(\overline{f}^{k-1}(\overline{x})) = \overline{f}^{k-1}(\overline{x}) + \overline{\varphi} \circ \overline{f}^{k-1}(\overline{x}) = \ldots =$$
$$= \overline{x} + \overline{\varphi}(\overline{x}) + \ldots + \overline{\varphi} \circ \overline{f}^{k-1}(\overline{x}). \quad \square$$

Lemma 2.7 *Let $\overline{f}(\overline{x}) = \overline{x} + \overline{\varphi}(\overline{x}) \in \mathrm{CDiff}_+^0(\mathbb{R})$ and*

$$M = \max_{\overline{x} \in \mathbb{R}} \overline{\varphi}(\overline{x}), \quad m = \min_{\overline{x} \in \mathbb{R}} \overline{\varphi}(\overline{x}).$$

Then $M - m < 1$.

Proof. Since $\overline{\varphi}(\overline{x})$ is a continuous 1-periodic function, there are points $\overline{x}_M$, $\overline{x}_m$ such that $M = \overline{\varphi}(\overline{x}_M)$, $m = \overline{\varphi}(\overline{x}_m)$. If $\overline{\varphi}(\overline{x}) = \mathrm{const}$, nothing to prove. Suppose $\overline{\varphi}(\overline{x}) \neq \mathrm{const}$, so $m \neq M$, $\overline{x}_m \neq \overline{x}_M$. Since $\overline{\varphi}(\overline{x})$ is 1-periodic, we can assume $\overline{x}_M \in [\overline{x}_m; \overline{x}_m + 1]$. It follows from $\overline{f}(\overline{x} + 1) = = \overline{f}(\overline{x}) + 1$ and the monotonicity of $\overline{f}$ that $\overline{f}(\overline{x}_M) - \overline{f}(\overline{x}_m) < 1$. Taking in mind $\overline{f}(\overline{x}_M) = \overline{x}_M + M$ and $\overline{f}(\overline{x}_m) = \overline{x}_m + m$, one gets

$$M - m = \overline{f}(\overline{x}_M) - \overline{f}(\overline{x}_m) - (\overline{x}_M - \overline{x}_m) < 1 - (\overline{x}_M - \overline{x}_m) < 1. \quad \square$$

In the following lemma, $\overline{\mu}$ is the invariant measure of $\overline{f}$ that is a lift measure of the invariant measure μ of f.

Lemma 2.8 *Given any $k \in \mathbb{N}$,*

$$\int_0^1 (\overline{f}^k(\overline{x}) - \overline{x})\, d\overline{\mu} = k \int_0^1 \overline{\varphi}(\overline{x})\, d\overline{\mu}$$

Proof. We have $\int_0^1 \overline{\varphi}(\overline{x})\, d\overline{\mu} = \int_0^1 \overline{\varphi} \circ \overline{f}^i(\overline{x})\, d\overline{\mu}$ for any $i \in \mathbb{N}$. By Proposition 2.3,

$$\int_0^1 (\overline{f}^k(\overline{x}) - \overline{x})\, d\overline{\mu} = \sum_{i=0}^{k-1} \int_0^1 \overline{\varphi} \circ \overline{f}^i(\overline{x})\, d\overline{\mu} = k \int_0^1 \overline{\varphi}(\overline{x})\, d\overline{\mu}. \quad \square$$

Put by definition,

$$\overline{\mu}(\overline{\varphi}) = \int_0^1 \overline{\varphi}(\overline{x})\, d\overline{\mu}, \quad \overline{\varphi}_k(\overline{x}) \stackrel{\text{def}}{=} \overline{f}^k(\overline{x}) - \overline{x} - k\overline{\mu}(\overline{\varphi}).$$

Corollary 2.2 *The function $\overline{\varphi}_k(\overline{x})$ has at least one root on the interval* $[0; 1]$.

Proof. By Lemma 2.8, $\overline{\varphi}_k(\overline{x})$ must take positive and negative values because of

$$\int_0^1 \overline{\varphi}_k(\overline{x})\, d\overline{\mu} = \int_0^1 (\overline{f}^k(\overline{x}) - \overline{x} - k\overline{\mu}(\overline{\varphi}))\, d\overline{\mu} = 0. \quad \square$$

Lemma 2.9 $\max\limits_{\overline{x}\in\mathbb{R}} |\overline{f}^k(\overline{x}) - \overline{x} - k\overline{\mu}(\overline{\varphi})| < 1$ *for any* $k \in \mathbb{N}$.

Proof. Since the function $\overline{\varphi}_k(\overline{x})$ is periodic, $\max\limits_{\overline{x}\in\mathbb{R}} |\overline{\varphi}_k(\overline{x})| = \max\limits_{\overline{x}\in[0;\,1]} |\overline{\varphi}_k(\overline{x})|$. By Lemma 2.7, the difference between the maximal and minimal values of $\overline{\varphi}_k(\overline{x})$ is less than 1. By Corollary 2.2, $\overline{\varphi}_k(\overline{x})$ has a root. As a consequence, $\max\limits_{\overline{x}\in\mathbb{R}} |\overline{\varphi}_k(\overline{x})| < 1$. $\square$

The following result implies the existence of rotation number.

Theorem 2.2 *Given any* $\overline{f} \in \mathrm{CDiff}^0_+(\mathbb{R})$, $\left|\frac{\overline{f}^n(\overline{x}) - \overline{x}}{n}\right|$ *converges uniformly to*

$$\int_0^1 (\overline{f}(\overline{x}) - \overline{x})\, d\overline{\mu} \stackrel{\text{def}}{=} \mathrm{rot}(\overline{f}),$$

as $n \to \infty$.

Proof. Indeed, by Lemma 2.9,

$$\max_{\overline{x}\in\mathbb{R}}\left|\frac{\overline{f}^{n}(\overline{x})-\overline{x}}{n}-\overline{\mu}(\overline{\varphi})\right|<\frac{1}{n}.$$

This implies the statement. □

After Theorem 2.2, we can generalize Corollary 2.2 as follows.

Corollary 2.3 *The function* $\overline{\varphi}_k(\overline{x})=\overline{f}^{k}(\overline{x})-\overline{x}-k\,\mathrm{rot}(\overline{f})$ *has at least one root on* $[0;\,1]$.

Now we give a list of properties concerning a rotation number.

The keeping of order. The following lemma can be applied for an approximate calculation of rotation number.

Lemma 2.10 *Let* $\overline{f}\in MT_{+}(\mathbb{R})$. *Then*

1) *If* $r\leqslant\overline{f}^{k}(\overline{x}_0)-\overline{x}_0\leqslant r+1$ *for some numbers* $k\in\mathbb{N}$, $r\in\mathbb{Z}$ *and a point* $\overline{x}_0\in\mathbb{R}$, *then*
$$\frac{r}{k}\leqslant\mathrm{rot}(\overline{f})\leqslant\frac{r+1}{k}. \tag{2.17}$$

2) *If* $\overline{f}$ *is continuous and* $\frac{r}{k}<\mathrm{rot}(\overline{f})<\frac{r+1}{k}$, *then there is* $\eta>0$ *such that*
$$r+\eta<\overline{f}^{k}(\overline{x})-\overline{x}<r+1-\eta\quad\text{for all }\overline{x}\in\mathbb{R}. \tag{2.18}$$

Proof. 1). By (2.8) and (2.9), $nr+\overline{x}\leqslant\overline{f}^{nk}(\overline{x})\leqslant n(r+1)+\overline{x}$. Hence,

$$\frac{r}{k}\leqslant\lim_{n\to\infty}\frac{nr+\overline{x}}{nk}\leqslant\lim_{n\to\infty}\frac{\overline{f}^{nk}(\overline{x})}{nk}=\mathrm{rot}(\overline{f})\leqslant\lim_{n\to\infty}\frac{n(r+1)+\overline{x}}{nk}=\frac{r+1}{k}.$$

2). By (2.17), $r\leqslant\overline{f}^{k}(\overline{x})-\overline{x}$ for all $\overline{x}\in\mathbb{R}$. If one assumes $r=\overline{f}^{k}(\overline{x}_0)-\overline{x}_0$, then $\mathrm{rot}(\overline{f})=\frac{r}{k}$, due to Theorem 2.11. Therefore, $r<\overline{f}^{k}(\overline{x})-\overline{x}$ for all $\overline{x}\in\mathbb{R}$. Since the function $\overline{f}^{k}(\overline{x})-\overline{x}$ is periodic and continuous, there is $\eta_1>0$ such that $r+\eta_1<\overline{f}^{k}(\overline{x})-\overline{x}$ for all $\overline{x}\in\mathbb{R}$. Similarly, there is $\eta_2>0$ such that $\overline{f}^{k}(\overline{x})-\overline{x}<r+1-\eta_2$. One can take $\eta=\min\{\eta_1,\,\eta_2\}$. □

The continuous dependence of rotation number. Let us consider $\mathrm{rot}(\overline{f})$, the rotation number of $\overline{f}$, as a function $\mathrm{rot}\colon\ \mathrm{CDiff}^0_+(\mathbb{R}) \to \mathbb{R}$. Recall that $\mathrm{CDiff}^0_+(\mathbb{R})$ is a space endowed with C^0 topology induced by metric

$$\overline{d}_0(\overline{f}, \overline{g}) = \max_{\overline{x}\in\mathbb{R}} |\overline{f}(\overline{x}) - \overline{g}(\overline{x})|, \quad \overline{f}, \overline{g} \in \mathrm{CDiff}^0_+(\mathbb{R}).$$

Theorem 2.3 *The function* $\mathrm{rot}\colon\ \mathrm{CDiff}^0_+(\mathbb{R}) \to \mathbb{R}$ *is continuous.*

Proof. Take $\overline{f}_0 \in \mathrm{CDiff}^0_+(\mathbb{R})$ and $\varepsilon > 0$. Choose integers $k > \frac{2}{\varepsilon}$ and r such that $\frac{r}{k} < \mathrm{rot}(\overline{f}_0) < \frac{r+1}{k}$. By Lemma 2.10, there is $\eta > 0$ such that $r + \eta < \overline{f}_0^k(x) - \overline{x} < r + 1 - \eta$ for all $\overline{x} \in \mathbb{R}$. Since any homeomorphism from $\mathrm{CDiff}^0_+(\mathbb{R})$ is uniformly continuous, there is $\delta > 0$ such that $\overline{d}_0(\overline{f}_0, \overline{g}) < \delta$ implies $\overline{d}_0(\overline{f}_0^k, \overline{g}^k) < \eta$. Then $r < \overline{g}^k(\overline{x}) - \overline{x} < r + 1$. By Lemma 2.10,

$$\frac{r}{k} \leqslant \mathrm{rot}(\overline{g}) \leqslant \frac{r+1}{k}, \quad \text{hence} \quad |\mathrm{rot}(\overline{f}_0) - \mathrm{rot}(\overline{g})| \leqslant \frac{2}{k} < \varepsilon. \quad \square$$

Invariantness under semi-conjugacy. Recall that $\overline{f} \in MT_+(\mathbb{R})$ is *semi-conjugate* to $\overline{f}_0 \in MT_+(\mathbb{R})$, if there is a continuous map $\overline{h} \in MT_+(\mathbb{R})$ such that

$$\overline{h} \circ \overline{f} = \overline{f}_0 \circ \overline{h}. \tag{2.19}$$

Another words, the diagram

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\overline{f}} & \mathbb{R} \\ \downarrow \overline{h} & & \downarrow \overline{h} \\ \mathbb{R} & \xrightarrow{\overline{f}_0} & \mathbb{R} \end{array}$$

is commutative. Sometimes, we say that $\overline{f}$ is semi-conjugate to $\overline{f}_0$ by $\overline{h}$. If $\overline{h}$ is an orientation preserving homeomorphism, one gets the definition of the *conjugacy*. The similar definitions hold for circle maps.

Theorem 2.4 *If* $\overline{f} \in MT_+(\mathbb{R})$ *is semi-conjugate to* $\overline{f}_0 \in MT_+(\mathbb{R})$*, then*

$$\mathrm{rot}(\overline{f}) = \mathrm{rot}(\overline{f}_0).$$

Proof. Let $\overline{f}$ be semi-conjugate to $\overline{f}_0$ by $\overline{h}$. Due to (2.19), $\overline{h} \circ \overline{f}^n = \overline{f}_0^n \circ \overline{h}$. Therefore,

$$\mathrm{rot}(\overline{f}) = \lim_{n\to\infty} \frac{\overline{f}^n(\overline{h}(\overline{x}))}{n} = \lim_{n\to\infty} \frac{\overline{h}(\overline{f}_0^n(\overline{x}))}{n}.$$

By Lemma 2.3, any $\overline{h} \in MT_+(\mathbb{R})$ can be represented as $\overline{h}(\overline{x}) = \overline{x} + + \overline{\varphi}(\overline{x})$, where $\overline{\varphi}(\overline{x})$ is a 1-periodic continuous function. Obviously, $|\overline{\varphi}(\overline{x})| \leqslant \leqslant N_0 < +\infty$. Then

$$\operatorname{rot}(\overline{f}) = \lim_{n\to\infty} \frac{\overline{h}(\overline{f}_0^n(\overline{x}))}{n} = \lim_{n\to\infty} \frac{\overline{f}_0^n(\overline{x}) + \overline{\varphi}(\overline{f}_0^n(\overline{x}))}{n} = \\ = \lim_{n\to\infty} \frac{\overline{f}_0^n(\overline{x})}{n} = \operatorname{rot}(\overline{f}_0). \quad \square$$

Corollary 2.4 *If $f,\ f_0 \in MT_+(S^1)$ are semi-conjugate, then* $\operatorname{rot}(f) = \operatorname{rot}(f_0)$.

Proof. Suppose that f is semi-conjugate to f_0 by h. Take the covers $\overline{f}$, $\overline{f}_0$, $\overline{h}$ for f, f_0, h respectively. Since $\overline{h} \circ \overline{f}$ covers $h \circ f$, and $\overline{f}_0 \circ \overline{h}$ covers $f_0 \circ h$,

$$\overline{f}_0 \circ \overline{h} = \overline{h} \circ \overline{f} + k = \overline{h} \circ (\overline{f} + k)$$

for some $k \in \mathbb{Z}$, due to Lemma 2.4. It follows from Theorem 2.4 that $\operatorname{rot}(\overline{f}_0) = = \operatorname{rot}(\overline{f} + k) = \operatorname{rot}(\overline{f}) + k$. Hence, $\operatorname{rot}(f) = \operatorname{rot}(f_0)$. $\square$

Periodic points. Recall that $x_0 \in S^1$ is a *periodic point* of $f\colon S^1 \to S^1$ if $f^q(x_0) = x_0$ for some $q \in \mathbb{N}$. If $q = 1$, x_0 is a *fixed point*. If $q \geqslant 2$ and $f^i(x_0) \neq x_0$ for $i = 1, \ldots, q-1$, then q is a *minimal period*. Clearly that if $\overline{f}(x)\colon \mathbb{R} \to \mathbb{R}$ is a covering map for $f\colon S^1 \to S^1$, then f has a periodic point $x_0 \in S^1$ of period $q \in \mathbb{N}$ if and only if for any point $\overline{x}_0 \in \pi^{-1}(x_0)$ there is $p \in \mathbb{Z}$ such that $\overline{f}^q(\overline{x}_0) = \overline{x}_0 + p$.

Theorem 2.5 *$f \in \operatorname{Homeo}_+(S^1)$ has a periodic point, say $x_0 \in S^1$, of period $k \in \mathbb{N}$ if and only if* $\operatorname{rot}(f) = \frac{r}{k} \in \mathbb{Q}$. *Moreover, k is a minimal period if and only if the numbers r, k are relatively prime.*

Proof. It follows from Theorem 2.1 that if f has a periodic point $x_0 \in S^1$ of period $k \in \mathbb{N}$, then $\operatorname{rot}(f) = \frac{r}{k} \in \mathbb{Q}$. Moreover, if the period k is minimal, the fraction $\frac{r}{k}$ is noncancellable and r, k are relatively prime.

Suppose now that $\operatorname{rot}(f) = \frac{r}{k} \in \mathbb{Q}$. By Corollary 2.3, there is $\overline{x}_0 \in \mathbb{R}$ such that

$$\overline{f}^k(x_0) - x_0 - k\operatorname{rot}(\overline{f}) = \overline{f}^k(x_0) - x_0 - r = 0.$$

Hence, $\pi(\overline{x}_0)$ is a periodic point of f. If r, k are relatively prime, the fraction $\frac{r}{k}$ is noncancellable. Hence, k is a minimal period. $\square$

Corollary 2.5 *$f \in \operatorname{Homeo}_+(S^1)$ has no periodic points if and only if the rotation number* $\operatorname{rot}(f)$ *is irrational.*

For maps $MT_+(S^1)$ that are not necessarily homeomorphisms, the following statement holds.

Corollary 2.6 *If $f \in MT_+(S^1)$ has a periodic point, then $\operatorname{rot}(f)$ is rational. If $\operatorname{rot}(f)$ is irrational, $f \in MT_+(S^1)$ has no periodic points.*

2.3. Circle maps with irrational rotation number

According to Corollary 2.6, if $f \in MT_+(S^1)$ has an irrational rotation number $\operatorname{rot}(f)$, then f has no periodic points. For f being a homeomorphism, we have the more precise statement. By Corollary 2.5, a homeomorphism f has no periodic points if and only if the rotation number $\operatorname{rot}(f)$ is irrational. By Example 1.6, the rotation $R_\alpha \colon S^1 \to S^1$ with $\operatorname{rot}(f) = \alpha \in \mathbb{R} \setminus \mathbb{Q}$ is minimal and, in particular, transitive.

Semi-conjugacy and conjugacy with minimal rotation

Let $A \subset \mathbb{R}$. Denote by $A \oplus \mathbb{Z}$ the set of points $x + m$, where $x \in A$, $m \in \mathbb{Z}$.

Lemma 2.11 *Let $\overline{f}_0 \in MT_+(\mathbb{R})$ has an irrational $\operatorname{rot}(\overline{f}) = \alpha$. Given any point $x_0 \in \mathbb{R}$, the map*

$$\overline{h}\colon O(x_0, \overline{f}) \oplus \mathbb{Z} \to O(0, \overline{R}_\alpha) \oplus \mathbb{Z}$$

that takes a point $\overline{f}^n(x_0) + m$ to $n \cdot \operatorname{rot}(\overline{f}) + m$ is one-to-one and monotone, where $n, m \in \mathbb{Z}$.

Proof. Since $\operatorname{rot}(\overline{f})$ is irrational, $\overline{h}$ is one-to-one. Let us show that $\overline{h}$ is monotone. Suppose that $\overline{f}^{n_1}(x_0) + m_1 < \overline{f}^{n_2}(x_0) + m_2$. For definiteness assume that $n_1 \leqslant n_2$. Then $y_0 + m_1 < \overline{f}^{n_2 - n_1}(y_0) + m_2$, where $y_0 = \overline{f}^{n_1}(x_0)$. By Theorem 2.1, $\overline{f}^{n_2-n_1}(y_0) - y_0 \notin \mathbb{Z}$ because of $\operatorname{rot}(\overline{f})$ is irrational. It follows from Lemma 2.5 that the inequality $y + m_1 < \overline{f}^{n_2-n_1}(y) + m_2$ holds for all $y \in \mathbb{R}$. If $n_1 = n_2$, $m_1 < m_2$ and so $n_1 \cdot \operatorname{rot}(\overline{f}) + m_1 < n_2 \cdot \operatorname{rot}(\overline{f}) + m_2$. If $n_1 < n_2$, it follows from Lemma 2.10 that $\frac{m_1 - m_2}{n_2 - n_1} \leqslant \operatorname{rot}(\overline{f})$. Hence, $n_1 \cdot \operatorname{rot}(\overline{f}) + m_1 \leqslant n_2 \cdot \operatorname{rot}(\overline{f}) + m_2$. Since $\operatorname{rot}(\overline{f})$ is irrational, the last inequality must be strict. □

Since $\operatorname{rot}(\overline{f}) = \alpha$ is irrational, the set $O(0, \overline{R}_\alpha) \oplus \mathbb{Z}$ is dense in $\mathbb{R}$. Therefore,

$$\operatorname{clos}\big(O(x_0, \overline{f}) \oplus \mathbb{Z}\big) = \mathbb{R}. \tag{2.20}$$

Lemma 2.12 *Let the condition of Lemma* 2.11 *holds. Then the map* $\overline{h}$ *has a continuous extension to a monotone mapping* (*which we again denote by* $\overline{h}$)

$$\overline{h}\colon \operatorname{clos}\bigl(O(x_0, \overline{f}) \oplus \mathbb{Z}\bigr) \to \mathbb{R}$$

such that

$$\overline{h} \circ \overline{f}|_{\operatorname{clos}\left(O(x_0, \overline{f})\oplus\mathbb{Z}\right)} = \overline{R}_\alpha \circ \overline{h}|_{\operatorname{clos}\left(O(x_0, \overline{f})\oplus\mathbb{Z}\right)}, \tag{2.21}$$

$$\overline{h}(x+1) = \overline{h}(x) + 1, \quad x \in \operatorname{clos}\bigl(O(x_0, \overline{f}) \oplus \mathbb{Z}\bigr). \tag{2.22}$$

Moreover,

1) *if* $\operatorname{clos}\bigl(O(x_0, \overline{f}) \oplus \mathbb{Z}\bigr) = \mathbb{R}$ *then* $\overline{h}$ *is a homeomorphism*;
2) *if* $\operatorname{clos}\bigl(O(x_0, \overline{f}) \oplus \mathbb{Z}\bigr) \neq \mathbb{R}$ *and* (a, b) *is an interval such that* $(a, b) \subset \subset \mathbb{R} \setminus \operatorname{clos}\bigl(O(x_0, \overline{f}) \oplus \mathbb{Z}\bigr)$ *with* $a, b \in \operatorname{clos}\bigl(O(x_0, \overline{f}) \oplus \mathbb{Z}\bigr)$, *then* $\overline{h}(a, b)$ *is a point such that* $\overline{h}(a, b) = \overline{h}(a) = \overline{h}(b)$.

Proof. Given any point $x \in \operatorname{clos}\bigl(O(x_0, \overline{f}) \oplus \mathbb{Z}\bigr)$, there exists a sequence $x_n \in O(x_0, \overline{f}) \oplus \mathbb{Z}$ converging to x. Since $\overline{h}$ is monotone, the sequence x_n is bounded and has at least one accumulation point, say y. We claim that this accumulation point is unique. Suppose the contrary. By (2.20) there are $y_1, y_2 \in O(0, \overline{R}_\alpha) \oplus \mathbb{Z}$, $y_1 \neq y_2$. Because of monotonicity, two points $\overline{h}^{-1}(y_1)$, $\overline{h}^{-1}(y_2)$ divide the sequence x_n. This contradicts to x_n being convergent. It can be shown similarly that the point $y = \lim_{n\to\infty} \overline{h}(x_n)$ does not depend on the choice of the sequence x_n converging to x.

Put by definition, $\overline{h}_{exten}(x) = y$. The monotonicity of $\overline{h}$ implies that the extension $\overline{h}_{exten}$ of $\overline{h}$ is monotone. As a consequence, $\overline{h}_{exten}$ is continuous.

Given any point $\overline{f}^n(x_0) + m$, $\overline{f}\left(\overline{f}^n(x_0) + m\right) = \overline{f}^{n+1}(x_0) + m$. Therefore,

$$\begin{aligned}\overline{h} \circ \overline{f}\left(\overline{f}^n(x_0) + m\right) &= \overline{h}\left(\overline{f}^{n+1}(x_0) + m\right) = \\ &= (n+1)\cdot \operatorname{rot}(\overline{f}) + m = n \cdot \operatorname{rot}(\overline{f}) + m + \operatorname{rot}(\overline{f}) = \\ &= \overline{h}\left(\overline{f}^n(x_0) + m\right) + \alpha = \overline{R}_\alpha \circ \overline{h}\left(\overline{f}^n(x_0) + m\right).\end{aligned}$$

By continuity, (2.21) holds for any $x \in \operatorname{clos}\bigl(O(x_0, \overline{f}) \oplus \mathbb{Z}\bigr)$.

For $x = \overline{f}^n(x_0) + m \in O(x_0, \overline{f}) \oplus \mathbb{Z}$, $\overline{h}(x) = n \cdot \mathrm{rot}(\overline{f}) + m$. Hence, $\overline{h}(x + 1) = \overline{h}(x) + 1$. By continuity, (2.22) holds for any $x \in \mathrm{clos}\,\big(O(x_0, \overline{f}) \oplus \mathbb{Z}\big)$. Below $\overline{h}_{exten}$ is denoted by $\overline{h}$.

Consider two cases: 1) $\mathrm{clos}(O(x_0, \overline{f}) \oplus \mathbb{Z}) = \mathbb{R}$; 2) $\mathrm{clos}(O(x_0, \overline{f}) \oplus \mathbb{Z}) \neq \mathbb{R}$. In the case 1), $\overline{h}$ is a homeomorphism because of the set $O(0, \overline{R}_\alpha) \oplus \mathbb{Z}$ is dense in $\mathbb{R}$. In the case 2), the monotonicity of $\overline{h}$ implies that $\overline{h}(a, b) = \overline{h}(a) = \overline{h}(b)$. □

Corollary 2.7 *Let $f \in MT_+(S^1)$ has an irrational rotation number α. Then there is a continuous monotone mapping $h\colon S^1 \to S^1$ such that $h \circ f = R_\alpha \circ h$.*

Corollary 2.8 *Suppose that a homeomorphism f has an irrational rotation number α. If $f\colon S^1 \to S^1$ is transitive, then there is a conjugacy map between f and R_α. If f is not transitive, then there is a semi-conjugacy map between f and R_α.*

Proof. We keep the notation of Lemmas 2.11 and 2.12. By (2.22), $\overline{h}$ is a lift of a continuous and monotone circle map $h\colon S^1 \to S^1$. It follows from Lemma 2.12 that f is either semi-conjugate or conjugate with a circle rotation R_α. □

Transitive and minimal homeomorphisms

Here we show that a rotation number of a transitive (and minimal) circle homeomorphism is a complete conjugacy invariant. Recall that a map f is transitive if f has a dense orbit. If any orbit of f is dense, f is minimal.

First, we study the centralizer of minimal circle rotation $R_\alpha\colon S^1 \to S^1$, $R_\alpha(x) = x + \alpha \bmod 1$. The following lemma says that the centralizer of a such R_α consists of circle rotations.

Lemma 2.13 *Suppose that the minimal circle rotation $R_\alpha\colon S^1 \to S^1$ commutes with an order preserving homeomorphism $h\colon S^1 \to S^1$, $h \circ R_\alpha = R_\alpha \circ h$. Then $h = R_\beta$ for some β.*

Proof. Take a point $x_0 \in S^1$. Put by definition, $\beta = h(x_0) - x_0$. Then $R_\beta(x_0) = x_0 + \beta = h(x_0)$. Taking in mind that $h \circ R_\alpha^n = R_\alpha^n \circ h$ for every $n \in \mathbb{Z}$, one gets

$$h\left(R_\alpha^n(x_0)\right) = R_\alpha^n \circ h(x_0) = R_\alpha^n \circ R_\beta(x_0) = R_\beta\left(R_\alpha^n(x_0)\right).$$

Since the orbit $O(x_0, R_\alpha)$ is dense (see Example 1.6), $R_\beta(x) = h(x)$ for all $x \in S^1$. □

Theorem 2.6 *Let $f\colon S^1 \to S^1$ be a transitive homeomorphism with an irrational rotation number* $\mathrm{rot}(f) = \alpha$. *Then*

- *f and R_α are conjugate via an orientation preserving homeomorphism.*
- *f is minimal.*
- *If h_1, h_2 are orientation preserving conjugacy maps between f and R_α, then $h_2 = R_\beta \circ h_1$.*

Proof. Let $\overline{f}\colon \mathbb{R} \to \mathbb{R}$ be a lift of f. Since f is transitive, there is a point $x_0 \in S^1$ such that the orbit $O(x_0, f)$ is dense in S^1. Properties of the universal covering map $\pi\colon \mathbb{R} \to S^1$ imply that $\pi^{-1}(O(x_0, f)) = O(\overline{x}_0, \overline{f}) \oplus \mathbb{Z}$, where $x_0 = \pi(\overline{x}_0)$. Since $O(x_0, f)$ is dense, the set $(O(x_0, f)) = O(\overline{x}_0, \overline{f}) \oplus \mathbb{Z}$ is dense in $\mathbb{R}$, $\mathrm{clos}\,(O(x_0, f)) = O(\overline{x}_0, \overline{f}) \oplus \mathbb{Z} = \mathbb{R}$. By Lemma 2.12, there is an order preserving conjugacy map $\overline{h}\colon \mathbb{R} \to \mathbb{R}$ that conjugates $\overline{f}$ with $\overline{R}_\alpha$. By (2.22), $\overline{h}$ is the lift of order preserving homeomorphism $h\colon S^1 \to S^1$ that obviously conjugates f with R_α.

The second item follows from the first one because of R_α is minimal, see Example 1.6.

To prove the third item notice that the relations $h_1 \circ f = R_\alpha \circ h_1$, $h_2 \circ f = R_\alpha \circ h_2$ imply

$$R_\alpha \circ (h_2 \circ h_1^{-1}) = h_2 \circ f \circ h_1^{-1} = (h_2 \circ h_1^{-1}) \circ R_\alpha.$$

It follows from Lemma 2.13 that $h_2 \circ h_1^{-1} = R_\beta$ for some β. Hence, $h_2 = R_\beta \circ h_1$. □

Theorem 2.7 *Let f_1, f_2 be minimal circle homeomorphisms. Then f_1, f_2 are conjugate via an order preserving homeomorphism if and only if*

$$\mathrm{rot}(f_1) = \mathrm{rot}(f_2).$$

Proof. By corollary 2.4, if f_1 and f_2 are conjugate via an order preserving homeomorphism, then $\mathrm{rot}(f_1) = \mathrm{rot}(f_2)$.

Suppose that $\mathrm{rot}(f_1) = \mathrm{rot}(f_2) = \alpha$. Because of minimality, α is irrational. By Theorem 2.6, there is an order preserving homeomorphism $h_i\colon S^1 \to S^1$ such that $h_i \circ f_i = R_\alpha \circ h_i$ for every $i = 1, 2$. One gets

$$f_1 \circ \left(h_1^{-1} \circ h_2\right) = \left(f_1 \circ h_1^{-1}\right) \circ h_2 = h_1^{-1} \circ R_\alpha \circ h_2 = \left(h_1^{-1} \circ h_2\right) \circ f_2.$$

Thus, the order preserving homeomorphism $h_1^{-1} \circ h_2$ is a conjugacy mapping between f_1 and f_2. □

Denjoy homeomorphisms

Example 1.6 gives us a transitive (so, minimal) circle homeomorphism. The natural question now is the existence of nontransitive circle homeomorphism with an irrational rotation number. Such a homeomorphism must have every orbit being non-periodic and nowhere dense. This question relates to the second possibility in Corollary 2.8, that corresponds to the second possibility $\operatorname{clos}\big(O(x_0,\, \overline{f}) \oplus \mathbb{Z}\big) \neq \mathbb{R}$ in Lemma 2.12. Let us show that such homeomorphisms exist.

It is convenient to represent the circle $S^1(1+a)$, $a > 0$, of the length $1+a$ as the segment $[0,\, 1+a]$ with the identified endpoints,

$$S^1(1+a) = [0,\, 1+a]/(0 \sim 1+a).$$

So, $S^1 = S^1(1)$. The natural orientation of $[0,\, 1+a]$ induces the orientation of $S^1(1+a)$.

Lemma 2.14 *Let λ_n, $n \in \mathbb{Z}$, be positive numbers such that the sum*

$$\sum_{n\in\mathbb{Z}} \lambda_n = a$$

is finite. Then given any orbit $O(x,\, R_\alpha)$, $x \in S^1$, of circle rotation R_α with an irrational α, there exists a family of disjoint open intervals $I_n \subset S^1(1+$ $+\, a)$, $n \in \mathbb{Z}$, which are ordered in the same way in $S^1(1+a)$ as $R_\alpha^n(x)$ such that $|I_n| = \lambda_n$ for all $n \in \mathbb{Z}$ and such that $\bigcup\limits_{n\in\mathbb{Z}} I_n$ is dense in $S^1(1+$ $+\, a)$. Moreover, there is a monotone continuous map $h\colon S^1(1+a) \to S^1$ and a homeomorphism $f\colon S^1(1+a) \to S^1(1+a)$ such that

$$f(I_n) = I_{n+1},\ \forall n \in \mathbb{Z}, \quad h \circ f = R_\alpha \circ h. \tag{2.23}$$

Proof. Without loss of generality, one can assume that the point $0 \sim 1 \in$ $\in S^1$ does not belong to the orbit $O(x,\, R_\alpha)$. Denote $x_n = R_\alpha^n(x)$. Let J_{0l} be the set of (integer) indices of points $x_\mu \in O(x,\, R_\alpha)$ that belong to $(0,\, x_0)$. To be precise, $\mu \in J_{0l}$ iff $0 < x_\mu < x_0$. Denote

$$a_0 = \sum_{\mu\in J_{0l}} \lambda_\mu, \quad b_0 = a_0 + \lambda_0.$$

Put by definition, $I_0 = (a_0,\, b_0) \subset S^1(1+a)$. Note that $b_0 = 1 + a - \sum\limits_{\nu\in J_{0r}} \lambda_\nu$, where J_{0r} is the set of indices of points $x_\nu \in O(x,\, R_\alpha)$ that belong to $(x_0,\, 1)$, since $0 \notin J_{0l} \cup J_{0r}$.

There are two cases: 1) $x_1 \in (0, x_0)$; 2) $x_1 \in (x_0, 1)$. We only consider case 1) because case 2) is similar. By previous construction, $1 \in J_{0l}$. Hence, $a_0 = \sum_{\mu \in J_{0l}} \lambda_\mu > \lambda_1$. Let J_{1l} be the set of indices of points $x_\mu \in \in O(x, R_\alpha)$ that belong to $(0, x_1)$, and $a_1 = \sum_{\mu \in J_{1l}} \lambda_\mu$. Put by definition, $I_1 = = (a_1, b_1) \subset S^1(1+a)$, where $b_1 = a_1 + \lambda_1$. Continuing by similar way with points $x_{-1}, x_2, x_{-2}, \ldots$, one gets the family of disjoint open intervals $I_n \subset S^1(1+a)$, $|I_n| = \lambda_n$, $n \in \mathbb{Z}$, which are ordered in the same way in $S^1(1+a)$ as $R_\alpha^n(x)$.

Given any $n \in \mathbb{Z}$, take any orientation preserving homeomorphism $f_n\colon I_n \to I_{n+1}$. Let $h_0\colon \bigcup_{n \in \mathbb{Z}} I_n \to S^1$ be defined by $h_0(I_n) = x_n$ and $f_0\colon \bigcup_{n \in \mathbb{Z}} I_n \to \bigcup_{n \in \mathbb{Z}} I_n$ by $f_0(x) = f_n(x)$ if $x \in I_n$. Since

$$R_\alpha(x_n) = R_\alpha \circ h_0(I_n) = x_{n+1} = h_0(I_{n+1}),$$

$h_0 \circ f_0 = R_\alpha \circ h_0$. By construction, the both f_0 and h_0 are monotone maps. Since $O(x, R_\alpha)$ is dense, h_0 is continuous and has the continuous extension $h\colon \operatorname{clos}\left(\bigcup_{n \in \mathbb{Z}} I_n\right) \to \operatorname{clos} O(x, R_\alpha) = S^1$. It follows from the condition $\sum_{n \in \mathbb{Z}} \lambda_n = a$ and the density of $O(x, R_\alpha)$ in S^1 that the union $\bigcup_{n \in \mathbb{Z}} I_n$ is also dense in $S^1(1+a)$. Therefore, f_0 extends continuously to the circle homeomorphism $f\colon S^1(1+a) \to S^1(1+a)$ such that $h \circ f = R_\alpha \circ h$. $\square$

Let f be a homeomorphism satisfying Lemma 2.14. By corollary 2.4, f has the irrational rotation number $\operatorname{rot}(f) = \alpha$. Hence, f has no periodic orbits. By (2.23), the orbit of any interior point, say x, of the interval I_0 intersects I_0 at x. As a consequence, the orbit $O(x, f)$ is nowhere dense. Therefore, f is not minimal and is not transitive.

A homeomorphism $f\colon S^1 \to S^1$ is called *Denjoy homeomorphism* if the rotation number $\operatorname{rot}(f)$ is irrational and f is not minimal.

Let $f\colon S^1 \to S^1$ be a Denjoy homeomorphism and $h\colon S^1 \to S^1$ a semiconjugacy map between f and the rotation $R_\alpha\colon S^1 \to S^1$ where $\alpha = \operatorname{rot}(f)$. The set

$$\chi(f, h) = \{x \in S^1\colon h^{-1}(x) \text{ contains more than one point}\}$$

is called a *characteristic set* of f with respect to h.

Lemma 2.15 *Let $\chi(f, h)$ be a characteristic set of Denjoy homeomorphism f with respect to the semi-conjugacy h between f and R_α, where $\alpha = \mathrm{rot}(f)$. Then*

- *given any $x \in \chi(f, h)$, $h^{-1}(x)$ is a closed interval.*
- $S^1 \setminus \bigcup\limits_{x\in\chi(f,h)} \mathrm{int}\, h^{-1}(x) = NW(f) = \mathrm{Lim}(f) = \mathrm{Lim}^+(f) = \mathrm{Lim}^-(f)$ *is a Cantor set.*
- *$\chi(f, h)$ is an invariant set that consists of at most countable family of orbits.*
- *$h\,(NW(f)) = S^1$.*
- *the restriction*

$$h|_{S^1\setminus \bigcup\limits_{x\in\chi(f,h)} h^{-1}(x)} \colon S^1 \setminus \bigcup_{x\in\chi(f,h)} h^{-1}(x) \to S^1 \setminus \chi(f, h)$$

is a homeomorphism.

Proof. Because of h is monotone and continuous, $h^{-1}(x)$ is a closed interval for any $x \in \chi(f, h)$.

Since the rotation R_α is minimal, $NW(R_\alpha) = \mathrm{Lim}(R_\alpha) = \mathrm{Lim}^+(R_\alpha) = = \mathrm{Lim}^-(R_\alpha)$. It follows from semi-conjugacy between f and R_α that $S^1 \setminus \bigcup\limits_{x\in\chi(f,h)} \mathrm{int}\, h^{-1}(x) = NW(f) = \mathrm{Lim}(f) = \mathrm{Lim}^+(f) = \mathrm{Lim}^-(f)$, since any point of $\mathrm{int}\, h^{-1}(x)$ is wandering. Clearly, $S^1 \setminus \bigcup\limits_{x\in\chi(f,h)} \mathrm{int}\, h^{-1}(x)$ is a closed and nonempty set. Again, semi-conjugacy between f and R_α implies that $S^1 \setminus \bigcup\limits_{x\in\chi(f,h)} \mathrm{int}\, h^{-1}(x)$ is a perfect set, as a consequence, a Cantor set.

Since $NW(f)$ is invariant, $S^1 \setminus \bigcup\limits_{x\in\chi(f,h)} \mathrm{int}\, h^{-1}(x)$ is also invariant. Hence, $\chi(f, h)$ is an invariant set. The complement to a Cantor set is the union of a countable family of open intervals. Therefore, $\chi(f, h)$ consists of at most countable family of orbits. The last items follow from the previous ones. □

By Lemma 2.13, if h_1, h_2 are semi-conjugacy maps between f and R_α then $\chi(f, h_2) = R_\beta\,(\chi(f, h))$ for some β. This motivates the following definition of equivalence relation. Two sets $\chi_1, \chi_2 \subset S^1$ are called *rotation equivalent*, we write $\chi_1 \equiv \chi_2$, if $\chi_2 = R_\beta(\chi_1)$ for some $\beta \in \mathbb{R}$. So a characteristic set is uniquely defined up to the rotation equivalence. Taking in mind this equivalence, we denote by $\chi(f)$ the characteristic set of Denjoy homeomorphism f.

The following theorem shows that a characteristic set together with a rotation number is a complete topological invariant for Denjoy homeomorphisms.

Theorem 2.8 *Let f_1, f_2 be Denjoy homeomorphisms. Suppose that h_i is a semi-conjugacy map between f_i and $R_{\mathrm{rot}(f_i)}$, $i = 1, 2$. Then f_1 and f_2 are conjugate via an orientation preserving homeomorphism if and only if* $\mathrm{rot}(f_1) = \mathrm{rot}(f_2)$ *and* $\chi(f_1, h_1) \equiv \chi(f_2, h_2)$.

Proof. Suppose that f_1, f_2 are conjugate, and h is a monotone conjugacy map between f_1 and f_2, $h \circ f_1 = f_2 \circ h$. By corollary 2.4, $\mathrm{rot}(f_1) = = \mathrm{rot}(f_2) = \alpha$. It remains to prove that $\chi(f_1, h_1) \equiv \chi(f_2, h_2)$. Take a point $x \in \chi(f_1, h_1)$. By Lemma 2.15, the closed interval $h_1^{-1}(x)$ intersects $NW(f_1)$ at its endpoints only. Then $h\left(h_1^{-1}(x)\right)$ is a closed interval intersecting $NW(f_2)$ at its endpoints only because $h\left(NW(f_1)\right) = = NW(f_2)$. By Lemma 2.15, h_2 takes $h\left(h_1^{-1}(x)\right)$ into a point. Thus, $h_2\left(h\left(h_1^{-1}(x)\right)\right) \in \chi(f_2, h_2)$. For $x \in S^1 \setminus \chi(f_1, h_1)$, $h_1^{-1}(x)$ is a single point, so does $h_2\left(h\left(h_1^{-1}(x)\right)\right)$. It follows that the mapping $h_0 = = h_2 \circ h \circ h_1^{-1}$ is well defined and $h_0(\chi(f_1, h_1)) \subset \chi(f_2, h_2)$. This mapping is continuous because h_1, h_2 and h are monotone and continuous. Similarly, $h_0^{-1} = h_1 \circ h^{-1} \circ h_2^{-1}$ is well defined, and continuous, and $h_0^{-1}(\chi(f_2, h_2)) \subset \chi(f_1, h_1)$. Consequently, h_0 is a homeomorphism taking $\chi(f_1, h_1)$ onto $\chi(f_2, h_2)$. Given any $x \in S^1 \setminus \chi(f_1, h_1)$, one gets

$$R_\alpha \circ h_0(x) = R_\alpha \circ h_2 \circ h \circ h_1^{-1}(x) = h_2 \circ f_2 \circ h \circ h_1^{-1}(x) = \\ = h_2 \circ h \circ f_1 \circ h_1^{-1}(x) = h_2 \circ h \circ h_1^{-1} \circ R_\alpha(x) = h_0 \circ R_\alpha(x).$$

By continuity, $R_\alpha \circ h_0(x) = h_0 \circ R_\alpha(x)$ for all $x \in S^1$. Due to Lemma 2.13, $h_0 = R_\beta$ for some β. Hence, $\chi(f_1, h_1) \equiv \chi(f_2, h_2)$.

Suppose now that $\mathrm{rot}(f_1) = \mathrm{rot}(f_2) = \alpha$ and $h_0(\chi(f_1, h_1)) = \chi(f_2, h_2)$, where $h_0 = R_\beta$ for some β. Denote by Ω_1° the set $S^1 \setminus h_1^{-1}\left(\chi(f_1, h_1)\right))$. This set is exactly the Cantor set $NW(f_1)$ without endpoints of the adjacent intervals that are mapped to $\chi(f_1, h_1)$. By definition of $\chi(f_1, h_1)$, the restriction $h_1|_{\Omega_1^\circ} \colon \Omega_1^\circ \to h_1(\Omega_1^\circ)$ is one-to-one. Similarly, $\Omega_2^\circ = = S^1 \setminus h_2^{-1}\left(\chi(f_2, h_2)\right))$. Since $h_i(\Omega_i^\circ) = S^1 \setminus \chi(f_i, h_i)$, where $i = 1, 2$, and $\chi(f_2, h_2) = R_\beta\left(\chi(f_1, h_1)\right)$, the mapping $h_2^{-1} \circ R_\beta \circ h_1|_{\Omega_1^\circ} \colon \Omega_1^\circ \to \Omega_2^\circ$ is well defined and one-to-one.

Let us show that this mapping is uniformly continuous. Take $\varepsilon > 0$. There is a finite collection of adjacent intervals $J_1, \ldots, J_k$ from $S^1 \setminus \Omega_2$ such that the length of any interval of $S^1 \setminus \bigcup_{i=1}^{k} J_k$ is less than ε. Since $h_2\left(\bigcup_{i=1}^{k} J_k\right) \subset \subset \chi(f_2, h_2)$ and $h_0(\chi(f_1, h_1)) = \chi(f_2, h_2)$, there are adjacent intervals $G_1, \ldots, G_k$ from $S^1 \setminus \Omega_1$ such that $h_0 \circ h_1\left(\bigcup_{i=1}^{k} G_k\right) = h_2\left(\bigcup_{i=1}^{k} J_k\right)$. Take a number $0 < \delta < \frac{1}{2}$ to be less than the length of each G_i, $i = 1, \ldots, k$. In this case, if the length of the interval (x_1, x_2), where $x_1, x_2 \in \Omega_1^\circ$, is less than δ then (x_1, x_2) does not contain any of the intervals G_i. Hence, one of the arcs with endpoints $h_2^{-1} \circ h_0 \circ h_1(x_1)$, $h_2^{-1} \circ h_0 \circ h_1(x_2)$ does not contain any of the intervals J_i. Therefore,

$$|h_2^{-1} \circ h_0 \circ h_1(x_2) - h_2^{-1} \circ h_0 \circ h_1(x_1)| < \varepsilon.$$

This proves that $h_2^{-1} \circ h_0 \circ h_1|_{\Omega_1^\circ}$ is uniformly continuous.

Similarly, one proves that $h_1^{-1} \circ h_0^{-1} \circ h_2|_{\Omega_2^\circ}$ is also uniformly continuous.

Since $h_2^{-1} \circ h_0 \circ h_1|_{\Omega_1^\circ}$ is uniformly continuous, it can be extended to a continuous map $h\colon \operatorname{clos}(\Omega_1^\circ) = \Omega_1 \to \Omega_2 = \operatorname{clos}(\Omega_2^\circ)$. Because of $h_2^{-1} \circ h_0 \circ \circ h_1|_{\Omega_1^\circ}$ is monotone, h is also monotone. It follows from the density of Ω_1° in Ω_1 that the restriction $h|_{\Gamma(\Omega_1)}\colon \Gamma(\Omega_1) \to \Gamma(\Omega_2)$ is one-to-one, where $\Gamma(\Omega_i)$ is the set of endpoints of all adjacent intervals from $S^1 \setminus \Omega_i$. Hence, $h|_{\Gamma(\Omega_1)}$ is a monotone homeomorphism $\Omega_1 \to \Omega_2$.

Let us now extend h to a homeomorphism of the whole circle S^1. First, one breaks up the adjacent intervals from $S^1 \setminus \Omega_1$ into equivalence classes: two adjacent intervals G', G'' are equivalent if there is $n \in \mathbb{Z}$ such that $G'' = f_1(G')$. From each equivalence class we choose a representative, say $G_1, G_2, \ldots$. Actually, an equivalence class is an orbit of some G_i. Given any G_i, we extend h to a homeomorphism $h_i\colon G_i \to S^1 \setminus \Omega_2$ on the whole interval G_i by natural way. For $x \in f_1^n(G_i)$, one sets $h_{i,n}(x) = f_2^n \circ h_i \circ f_1^{-n}(x)$. Denote the extension of h by the mappings $h_{i,n}$ again by h. It follows from construction that h is a homeomorphism.

Let us check that h is a conjugacy map between f_1 and f_2. If $x \in \Omega_1^\circ$, then

$$\begin{aligned} h \circ f_1(x) = h_2^{-1} \circ h \circ h_1 \circ f_1(x) = \\ = h_2^{-1} \circ R_\beta \circ R_\alpha \circ h_1(x) = \\ = f_2 \circ h_2^{-1} \circ R_\beta \circ h_1(x) = f_2 \circ h(x). \end{aligned}$$

By continuity, $h \circ f_1(x) = f_2 \circ h(x)$ for any $x \in \Omega_1$. If $x \in f_1^n(G_i)$, then

$$h \circ f_1(x) = h_{i,n+1} \circ f_1(x) = f_2^{n+1} \circ h_i \circ f_1^{-(n+1)} \circ f_1(x) = \\ = f_2 \circ f_2^n \circ h_i \circ f_1^{-n}(x) = f_2 \circ h(x). \quad \square$$

Theorem 2.8 shows that the characteristic set is an invariant of topological conjugacy up to a rotation for Denjoy homeomorphism. The next theorem shows that the characteristic set is a complete invariant (recall that a characteristic set is a countable invariant set of minimal rotation, Lemma 2.15).

Theorem 2.9 *Given any minimal rotation R_α and a countable invariant set χ, there is a Denjoy homeomorphism f such that $\chi = \chi(f, h)$, where h is a semi-conjugacy map between f and R_α.*

Proof is similar to the construction of the homeomorphism h in Lemma 2.14. So, we give only a scheme of the proof. Given any orbit O_l from χ, take a sequence $\{a_n^{(l)}\}_{n\in\mathbb{Z}}$ of numbers $a_n^{(l)} > 0$ such that

$$\sum_{n,l} a_n^{(l)} = a < +\infty.$$

For example, one can take $a_n^{(l)} = (|n| + l + 1)^{-1}(|n| + l + 2)^{-1}$. Then, similarly to the proof of Lemma 2.14, one can construct a family of disjoint open intervals $I_n^{(l)} \subset S^1(1+a)$, which are ordered in the same way in $S^1(1+a)$ as χ such that $|I_n^{(l)}| = a_n^{(l)}$ for all n, l and such that $\bigcup_{n,l} I_n^{(l)}$ is dense in $S^1(1+$ $+\, a)$. Moreover, there is a monotone continuous map $\widetilde{h}\colon S^1(1+a) \to S^1$ and a homeomorphism $\widetilde{f}\colon S^1(1+a) \to S^1(1+a)$ such that $\widetilde{f}(I_n^{(l)}) = I_{n+1}^{(l)}$ for all $n \in \mathbb{Z}$ and $\widetilde{h} \circ \widetilde{f} = R_\alpha \circ \widetilde{h}$. Let $S^1(1+a) \to S^1(1+a)$ be any linear mapping. Then this mapping induces a Denjoy homeomorphism f and a semi-conjugacy map h between f and R_α such that $\chi = \chi(f, h)$. $\square$

The semi-conjugacy h above is a so-called *blowing down* operation that sends the closure of each interval $I_n^{(l)}$ to a point of χ. The inverse operation is called a *blowing up* operation, one thinks of a blowing up of each point of χ to a closed interval.

Let $\chi(f)$ be the characteristic set of Denjoy homeomorphism f. The cardinality of $\chi(f)$ denoted by $\mathrm{card}(f)$ is called a *characteristic* of f. Clearly that $\mathrm{card}(f)$ is either a natural number $\geqslant 1$ or denumerable.

Theorem 2.10 *Let* f_1, f_2 *be Denjoy homeomorphisms with* $\mathrm{card}(f_1) = \mathrm{card}(f_2) = 1$. *Then* f_1 *and* f_2 *are conjugate via an orientation preserving homeomorphism if and only if* $\mathrm{rot}(f_1) = \mathrm{rot}(f_2)$.

Proof. By corollary 2.4, it is enough to prove that if $\mathrm{rot}(f_1) = \mathrm{rot}(f_2)$ then f_1 and f_2 are conjugate via an orientation preserving homeomorphism. One can easily check that if O_1, O_2 are orbits of the minimal circle rotation R_α, where $\alpha = \mathrm{rot}(f_1) = \mathrm{rot}(f_2)$, then there exists a $\beta \in \mathbb{R}$ such that $R_\beta(O_1) = O_2$. This follows that $\chi(f_1) \equiv \chi(f_2)$. By Theorem 2.8, one gets the result. □

Theorem 2.11 *Given any irrational* $\alpha \in (0, 1)$ *and natural number* $k \geqslant 2$, *there is a continual set of pairwise non-conjugate Denjoy homeomorphisms* $\{f_\mu\}$ *with* $\mathrm{rot}(f_\mu) = \alpha$ *and* $\mathrm{card}(f_\mu) = k$.

Proof. It is enough to prove the theorem for $k = 2$. By Theorems 2.8, 2.9, it suffices to show that there is a continual set of pairwise non-equivalent sets, each of which consists of two orbits of the rotation R_α. Let us fix an orbit O_1 of R_α and consider the set $\chi_\mu = O_1 \cup O_\mu$, where O_μ is an orbit of R_α different from O_1. Suppose χ_{μ_1} is equivalent to χ_{μ_2}. Then $R_\beta(\chi_{\mu_1}) = \chi_{\mu_2}$ for some $\beta \in \mathbb{R}$. Since $R_\beta \circ R_\alpha = R_\alpha \circ R_\beta$, R_β takes an orbit of R_α onto an orbit of R_α. It follows that there are only two possibilities: 1) $R_\beta(O_1) = O_1$, 2) $R_\beta(O_1) = O_{\mu_2}$. In case 1), $R_\beta(O_{\mu_1}) = O_{\mu_1}$ because of any rotations commute. Hence, $O_{\mu_1} = O_{\mu_2}$. In case 2), $R_\beta(O_{\mu_1}) = O_1$. As a consequence, given any $\chi_\mu = O_1 \cup O_\mu$, there is a unique $\chi_{\mu'} = O_1 \cup O_{\mu'}$ that is equivalent to χ_μ and different from χ_μ. Certainly, there is a continual set of distinct sets χ_μ. Therefore, there is a continual set of pairwise non-equivalent sets of the type χ_μ. □

Bibliographic Notes and Panoramas

Chapter 2. A circle is a unique one-dimensional closed manifold. Therefore, the circle was a natural place for basic and first investigations of dynamics for various classes of dynamical systems. Maier [140] and Peixoto [182] proved that the Morse–Smale circle C^1-diffeomorphisms form the open and dense set in the space $\mathrm{Diff}^1(S^1)$ of circle C^1-diffeomorphisms. Moreover, the Morse–Smale circle C^1-diffeomorphisms are completely classified [140].

(2.1). The advantage to consider cover transformations is the existence of universal coordinates on a covering space. The universal covering space for the circle S^1 is $\mathbb{R}$ with a unique coordinate. Thus, cover transformations can be defined with using 1-periodic functions.

(2.2). A rotation number for circle homeomorphisms was introduced by Poincaré [192] to study fixed-point free flows on a torus.

It is a nontrivial task to extend the concept of Poincaré rotation number to the study of higher dimensional dynamics. Generalizations in this direction, in different forms, were introduced in [201] where in rough terms, one used rotation vectors to describe the asymptotic motion of orbits in the homology classes. The notion of rotation number was generalized to the class of continuous circle mappings of degree one in [171]. In this case one gets a rotation interval. The idea to generalize the notion of Poincaré rotation number to many-dimensional cases appears in the papers [113, 128, 137]. In [163] one introduced the rotation set $\rho(F)$ for a lift F of a homeomorphism $f\colon \mathbb{T}^d \to \mathbb{T}^d$, which is homotopic to the identity, as the set of accumulation points of the subset

$$\left\{ \frac{F^n(x) - x}{n} \mid x \in \mathbb{R}^d,\ n \in \mathbb{N} \right\}$$

One proved that $\rho(F)$ is convex. The following statement that is similar to Theorem 2.5 was proved in [73] (see also [137]).

Theorem 2.12 *Suppose $f\colon \mathbb{T}^2 \to \mathbb{T}^2$ is a homeomorphism homotopic to the identity map, and let $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a lift. If a vector v lies in the interior of the rotation set $\rho(F)$ and has both coordinates rational, then there is a periodic point $x \in \mathbb{T}^2$ with the property that*

$$\frac{F^q(x_0) - x_0}{q} = v$$

where $x_0 \in \mathbb{R}^2$ is any lift of x and q is the least period of x.

Further developments for area preserving surface diffeomorphisms are in [74].

(2.3). The circle homeomorphism in Lemma 2.14 with a Cantor minimal set and without periodic orbits was constructed by Poincaré [192]. Such C^1 diffeomorphism was first constructed by Bohl [48] and independently by Denjoy [66]. Beginning with the paper [66], a circle homeomorphism with a Cantor minimal set and without periodic orbits is called a Denjoy type homeomorphism (or in short, Denjoy homeomorphism). Hermann [113] constructed a $C^{1+\alpha}$ Denjoy diffeomorphism. Hall [107] constructed a C^∞ Denjoy homeomorphism. This smooth homeomorphism has two critical points and so, it is not a diffeomorphism. However, Yoccoz [225] proved that there are no analytic Denjoy homeomorphisms. See also [159, 160].

Theorem 2.8 was proved in [149].

Denjoy type multi-dimensional mappings $\mathbb{T}^k \to \mathbb{T}^k$ was constructed in [155, 156].

A rotation functional introduced independently in [122] and [184] is an extension of Poincaré rotation number when one passes from a $\mathbb{Z}$-action to a $\mathbb{Z}^d$-action, $d \geqslant 2$. Recall that if $\overline{f}\colon \mathbb{R} \to \mathbb{R}$ is a lift of degree one homeomorphism $f\colon S^1 \to S^1$ then $\overline{f}$ has a rotation number, i.e. (2.11) holds. One can represent (2.11) as follows

$$\overline{f}^n(x) = x + \operatorname{rot}(\overline{f})n + \alpha(n, x), \quad \text{where } \lim_{n\to\infty} \frac{\alpha(n, x)}{n} = 0 \text{ uniformly under } x.$$

Indeed,

$$\lim_{n\to\infty} \frac{\overline{f}^n(x) - x}{n} = \operatorname{rot}(\overline{f}) \quad \leftrightarrow \quad \lim_{n\to\infty} \left[\frac{\overline{f}^n(x) - x}{n} - \operatorname{rot}(\overline{f})\right] = 0 \quad \leftrightarrow$$
$$\leftrightarrow \quad \lim_{n\to\infty} \left[\frac{\overline{f}^n(x) - x - \operatorname{rot}(\overline{f})n}{n}\right] = 0.$$

Denote $\overline{f}^n(x) - x - \operatorname{rot}(\overline{f})n = \alpha(n, x)$. By Theorem 2.2, $\frac{\alpha(n,x)}{n}$ converge to 0 uniformly under x. The number $\operatorname{rot}(\overline{f})$ (respectively $\operatorname{rot}(f) \bmod 1$) is a rotation number of $\overline{f}$ (resp. f).

Let $\overline{\mathscr{A}}(\mathbb{Z}^d)$ be a $\mathbb{Z}^d$-action on $\mathbb{R}$ i.e., given any $\gamma \in \mathbb{Z}^d$, one corresponds a degree one homeomorphism $\overline{H}_\gamma\colon \mathbb{R} \to \mathbb{R}$ such that

- $H_{\gamma_1\gamma_2} = H_{\gamma_1} \circ H_{\gamma_2}$ for $\forall \gamma_1, \gamma_2 \in \mathbb{Z}^d$;
- $H_{\gamma^{-1}} = H_\gamma^{-1}$ for $\forall \gamma \in \mathbb{Z}^d$.

Since $\mathbb{Z}^d$ is a commutative group, any mappings H_g commute. To simplify matters, denote $\overline{H}_\gamma(x)$ by γ_x.

A linear functional $\overline{\lambda}\colon \mathbb{Z}^d \to \mathbb{R}$ is called *rotation functional* of $\overline{\mathscr{A}}(\mathbb{Z}^d)$ if given any $\gamma \in \mathbb{Z}^d$,

$$\gamma_x = x + \overline{\lambda}(\gamma) + \alpha(\gamma, x), \tag{2.24}$$

$$\text{where } \lim_{||\gamma||\to\infty} \frac{\alpha(\gamma, x)}{||\gamma||} = 0 \text{ converges uniformly under } x.$$

Here $||\gamma||$ is a distance between $\gamma = (k_1, \ldots, k_d)$ and $(0, \ldots, 0)$, $||\gamma|| = = \sqrt{k_1^2 + \cdots + k_d^2}$. One can prove the following statement

Theorem 2.13 *Let* $\overline{\mathscr{A}}(\mathbb{Z}^d)$ *be a* $\mathbb{Z}^d$*-action on* $\mathbb{R}$ *such that given any* $\gamma \in \mathbb{Z}^d$*, the corresponding mapping* $\overline{H}_\gamma \colon \mathbb{R} \to \mathbb{R}$ *is a degree one lift of a circle homeomorphism. Then* $\overline{\mathscr{A}}(\mathbb{Z}^d)$ *has a unique rotation functional.*

Similarly to Theorem 2.5, one can proved the following theorem (see details in [2] and [3]).

Theorem 2.14 *A* $\mathbb{Z}^d$*-action* $\mathscr{A}$ *on* S^1 *has a periodic point if and only if the rotation functional is rational.*

CHAPTER 3

Introduction to Local Laminations

Foliations and laminations are objects arising naturally in Topology, Geometry and Hyperbolic Dynamics. For example, fibers of locally trivial fiber bundles form special foliations considered in Differential Topology. Hyper-surfaces of constant curvature form families of special laminations considered in Geometry. Invariant manifolds and invariant sets appear like foliations and laminations in Hyperbolic Dynamics. One can represent huge amount of other examples. The uniformizing notation for a foliation and lamination is a local lamination. This chapter is devoted to the introduction to the Theory of Local Laminations.

In Section 3.1, we formulate basic definitions illustrated by simple examples. In Section 3.2, one represents some methods of construction of local laminations (including foliations and laminations) on surfaces. We also introduce some basic examples. In Section 3.3, one treats the notion of limit set for a curve with no self-intersections. Section 3.4 is devoted to properties of orientable and non-orientable local laminations. In Section 3.5, one gives conditions of the existence of closed transversals for local laminations. In Section 3.6, we represent classical results concerning indexes of singularities and closed curves under foliations. In Section 3.7, we introduce the notions of minimal, quasiminimal and Maier quasiminimal sets for local laminations. Here, one considers basic properties of this invariant sets of local laminations. Section 3.8 is devoted to geodesic laminations which are important in constructing of invariants of dynamical systems considering in this book.

3.1. First notations and examples

Let M^2 (in this part, sometimes denoted simply by M) be a surface. An *infinite curve* l on M^2 is the image $m(\mathbb{R})$ under a local homeomorphism

$$m\colon \mathbb{R} \to m(\mathbb{R}) = l \subset M^2$$

that defines the parametrization $t \to m(t) \in l$ ($t \in \mathbb{R}$). The infinite curve l is *simple*, if $m\colon \mathbb{R} \to l$ is one-to-one. In this case, the parametrization m is injective. Sometimes one says that l *has no self-intersections*. A *closed curve* l on M^2 is the image $m(S^1)$ under a local homeomorphism $m\colon S^1 \to M^2$. The closed curve l is called *simple*, if $m\colon S^1 \to m(S^1) = l$ is a homeomorphism.

Recall that we consider S^1 as the interval $[0;\,1]$ whose endpoints are identified, $S^1 = [0;\,1]/(0 \approx 1)$. Put by definition, $\pi(x) = x \bmod 1$, $x \in \mathbb{R}$. For a simple closed curve $l = m(S^1)$, we'll sometimes use a *periodic parametrization*

$$m \circ \pi\colon \mathbb{R} \to l \subset M^2.$$

Such parametrization naturally appears when one considers periodic trajectories. A *curve* is either an infinite curve or a closed curve. A curve l is C^r*-smooth*, if the corresponding map m is locally a C^r-map, $r \geqslant 1$.

Foliated boxes and simplest examples

The motivation for the definition of local lamination is Theorem 1.1 that says that trajectories locally look like parallel straight lines beyond of singularities. At first, we give the definition of a local lamination for a surface M^2 without boundary, $\partial M^2 = \varnothing$. As usual, one assumes that the Euclidean plane $\mathbb{R}^2$ is equipped with Cartesian coordinates $(x,\, y)$. By a C^0*-diffeomorphism*, we mean a homeomorphism. Fix integer numbers $0 \leqslant l \leqslant r \leqslant \infty$.

Definition 3.1 *Let* $\mathscr{M} \subset M^2$ *be a subset of* M^2 (*which may coincide with* M^2) *that contains some closed subset* $S \subset \mathscr{M}$. *Let* $\mathscr{M}$ *be a union* $S \bigcup\limits_{\alpha} L_\alpha$, *where* L_α *are pairwise disjoint* C^r*-smooth simple curves* (α *runs through some set of indices*). *We say that the family* $\{L_\alpha\}$ *forms a* $C^{r,l}$-***local lamination***, *if for any point* $P \in \mathscr{M} - S$, *there exists a neighborhood* $U(P)$ *of* P *and a* C^l*-diffeomorphism* $\psi\colon U(P) \to \mathbb{R}^2$, $\psi(P) = (0,\, 0)$, *such that any connected component of the intersection* $U(P) \cap L_\alpha$ (*provided that this intersection is nonempty*) *is mapped by* ψ *onto the line* $y = \mathrm{const}$ *and the restriction* $\psi|_{U(P)\cap L_\alpha}$ *is a* C^r*-diffeomorphism onto its image, Fig.* 1.7.

Roughly speaking, $\{L_\alpha\}$ is a family pairwise disjoint simple curves locally homeomorphic to a family of parallel straight lines. For simplicity, we refer to the family $\{L_\alpha\}$, denoted by $\mathscr{D}$, as a $C^{r,l}$ *local lamination with the set of singularities* $S \stackrel{\text{def}}{=} \operatorname{Sing}(\mathscr{D})$. The set

$$\mathscr{M} = \Big(\bigcup_{\alpha} L_\alpha\Big) \cup S \stackrel{\text{def}}{=} \operatorname{supp} \mathscr{D}$$

itself is called a *support of the local lamination* $\mathscr{D}$. The curves L_α are called *leaves*. Each point of the set $\operatorname{Sing}(\mathscr{D})$ is called a *singularity*. A point that is not a singularity is called *regular*.

Thus, any collection of parallel straight lines on $\mathbb{R}^2$ gives us a simplest and trivial example of a local lamination, and by definition, every local lamination is locally homeomorphic to such simplest example beyond the set of singularities. A few generalization of this simplest local lamination gives the following example.

Example 3.1 *A family of parallel graphics.*

Let $f\colon \mathbb{R}^1 \to \mathbb{R}$ be a C^r function. Given any $\alpha \in \mathbb{R}$, the set

$$L_\alpha = \{(x,\, y) \mid y = f(x) + \alpha,\ x \in \mathbb{R}\}$$

is a graph of the function $f + \alpha$. The homeomorphism $(x,\, y) \to (x,\, y + f(x))$ takes L_α to the line $y = \alpha$. Hence, given any set $B \subset \mathbb{R}$, the collection $\mathscr{L} = \{L_\alpha \mid \alpha \in B\}$ forms a local lamination without singularities. If B is closed, $\mathscr{L}$ is a lamination, see below exact definitions. If $B = \mathbb{R}$, $\mathscr{L}$ is both a lamination and a foliation with no singularities on $\mathbb{R}^2$. $\diamondsuit$

Keeping the notation of Definition 3.1, the neighborhoods $U(P)$ where $P \in \operatorname{supp}\mathscr{D} - \operatorname{Sing}(\mathscr{D})$ are called *neighborhoods with the structure of a linear local lamination* and the diffeomorphisms ψ are called *rectifying diffeomorphisms*. The pre-image $\psi^{-1}([-1;\, +1] \times [-1;\, +1])$ is called a *closed trivially foliated box*. The interior of this box is an *open trivially foliated box*. Actually, foliated boxes are neighborhoods with the structure of a linear local lamination, but they are bounded in part by transversal segments. Given any leaf L_α, a connected component of the intersection of L_α with an open trivially foliated box is called a *local leaf*. Local leaves form a base of topology on each L_α. We call this topology the *intrinsic* (*or interior*) *topology of the leaf* L_α. Taking in mind this topology, we speak on compactness of a leaf or that a leaf is homeomorphic to some 1-dimensional manifold, for example, $\mathbb{R}$, S^1 and so on.

Let Σ be a segment (the image of the unit interval $[0;\, 1]$ under an embedding of $[0;\, 1]$ into M^2) through the regular point P. If there is the rectifying diffeomorphism $\psi\colon U(P) \to \mathbb{R}^2$ such that $\Sigma \subset U(P)$ and ψ maps Σ into the line $x = 0$, then Σ is called *locally transversal segment* at P. The segment Σ is called *transversal* if it is locally transversal at each point of $\operatorname{supp}\mathscr{D} \cap \Sigma$. A closed simple curve C is called a *closed transversal* if every arc-wise part of C is a transversal segment.

The concept of a local lamination generalizes the classical concepts of lamination and foliation. If $\operatorname{supp}\mathscr{D}$ is closed and $\operatorname{Sing}(\mathscr{D}) = \varnothing$, then $\mathscr{D}$ is called a $C^{r,l}$*-lamination*. An important example of a lamination is a geodesic lamination. We will usually denote a lamination by $\mathscr{L}$. Note that a local $C^{r,l}$-lamination without singularities is not always a lamination. If $\operatorname{supp}\mathscr{D} = M^2$, then $\mathscr{D}$ is called a $C^{r,l}$*-foliation*. We usually denote a foliation by $\mathscr{F}$. One may say that a local lamination with singularities is a “foliation” (with singularities) on a some subset. If this subset is closed and there are no singularities, then we obtain a lamination. If this subset coincides with a manifold (and there may be some singularities), then the local lamination is a foliation. It follows from the aforesaid that the concept of a local lamination is a quite general concept, which includes, as particular cases, the concepts of lamination and foliation. Therefore,

all assertions and definitions that are valid for a local lamination are also valid for laminations and foliations.

In Theory of Foliations, a union of leaves and singularities of some foliation is traditionally called a *saturated set*. In Theory of Dynamical Systems, the union of orbits or trajectories is called *invariant*. With no confusing, we’ll use the both notions for the union of leaves and singularities of local lamination.

If $r = l$, then we write C^r instead of $C^{r,r}$. If $r = l = 0$ or the smoothness is inessential, we simply say “local lamination”.

Example 3.2 *Reeb foliations, Fig.* 3.1.

On the strip $[-1; +1] \times \mathbb{R} \subset \mathbb{R}^2$, consider a family of curves

$$l_\alpha = \{\alpha + \phi(t) \colon -1 < t < +1\},$$
$$\text{where} \quad \phi(t) = \exp\frac{t^2}{1-t^2}, \; -\infty < \alpha < +\infty,$$
$$l_{-1} = \{(-1, y) \colon -\infty < y < +\infty\},$$
$$l_1 = \{(+1, y) \colon -\infty < y < +\infty\}.$$

These curves form a C^∞ foliation $\mathscr{F}_{Reeb}(S)$, which is called a *Reeb foliation on an infinite strip*. The leaves are invariant under the translation along the axis Y. Performing the identification $(x, y) \sim (x, y + 1)$, we obtain a foliation $\mathscr{F}_{Reeb}(A)$ in the annulus $[-1; +1] \times S^1$, which is called a *Reeb foliation on a closed annulus* or a *Reeb component on a closed annulus*, Fig. 3.1, (b). Both boundary components of the annulus $[-1; +1] \times S^1$ are

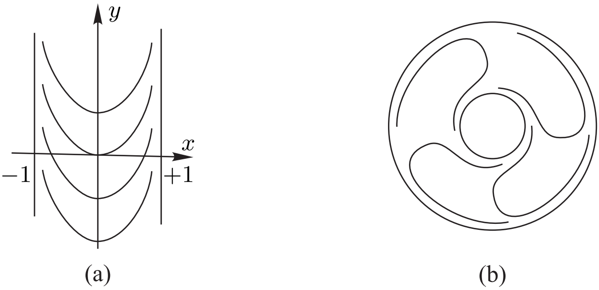


Figure 3.1. The Reeb foliations on an infinite strip (a), and a closed annulus (b).

leaves of $\mathscr{F}_{Reeb}(A)$. If we glue two boundary components of the annulus by an orientation-reversing homeomorphism, we obtain a *Reeb foliation on the two-dimensional torus* $\mathbb{T}^2$. ♢

If one deletes a saturated set of some foliation or lamination, we get a local lamination. In this way, one can produce many examples of local laminations starting with foliations and laminations. Therefore, later on, we mainly consider examples of foliations and laminations.

Often, the structure of local lamination in a neighborhood of $\mathrm{Sing}(\mathscr{D})$ should be described separately for each class of local laminations. Let us consider some special singularities of a foliation.

Example 3.3 *Saddle type singularities.*

A saddle type singularities are shown on Fig. 3.2. Roughly speaking, any such singularity has a family of special leaves, so-called separatrices, that go to the singularity and divide a neighborhood of the singularity into sectors in which leaves looks like hyperbolas. Later on, we shall give a precise definition.

A saddle with a single separatrix is called a *thorn* (*or needle*); with two separatrices, a *fake saddle*; with three separatrices, a *tripod*; and with four separatrices, a *standard* (*ordinary*) *saddle*. ♢

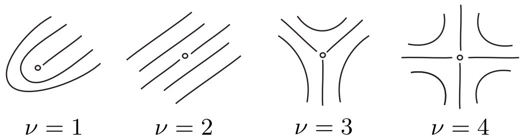


Figure 3.2. Saddle type singularities with ν separatrices.

Below, the notion of a separatrix will be generalized for a singularity which is not necessary a saddle type singularity (certainly, the separatrices of saddle type singularities shall be separatrices in general sense).

Example 3.4 *Level sets of a submersion.*

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a smooth submersion, i. e. $Df \neq 0$. Well known that, for each point $(x, y) \in \mathbb{R}^2$, there is a coordinate neighborhood U, $(u, v)\colon U \to \mathbb{R}^2$, of this point and a coordinate neighborhood V, $u\colon V \to \mathbb{R}$, of $f(x, y)$, relative to which the formula for $f|_U$ becomes $f(u, v) = u$. It follows that the level sets $f^{-1}(z)$, $z \in \mathbb{R}$, are properly embedded curves that locally looks like parallel copies of $\mathbb{R}$ in $\mathbb{R}^2$. Thus, the connected components of the nonempty level sets are the leaves of a foliation. ♢

If $\partial M^2 \neq \varnothing$, each component of ∂M^2 is homeomorphic to a circle. In this case the behavior of leaves near ∂M^2 is also described separately. We'll usually assume that a component of ∂M^2 either is transversal (in the topological sense) to leaves L_α (Fig. 3.3, (c)) or consists of leaves and singularities (Fig. 3.3, (a), (b)).

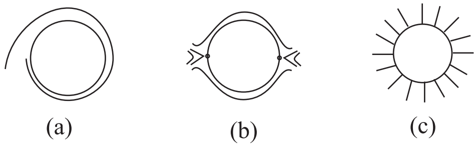


Figure 3.3. The structure of a foliation which is possible near a boundary.

A compact leaf is homeomorphic either to a compact interval with endpoints lying on the boundary ∂M^2 or to a circle. In the latter case, the leaf is said to be *closed*.

Example 3.5 *Flows.*

Let f^t be a C^r flow with a set of fixed points $\mathrm{Fix}(f^t)$ on M^2. Then the trajectories of f^t form a C^r foliation $\mathscr{F}$ on M^2 with the set of singularities $\mathrm{Sing}(\mathscr{F}) = \mathrm{Fix}(f^t)$. If f^t is induced by a C^r vector field, then f^t is a $C^{r+1,r}$ foliation [168, 198]. ♢

Orientable and non-orientable foliations

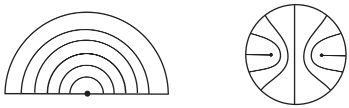
Thus, every flow gives us an example of a foliation, which belongs to a class of so-called, orientable foliations. Let us give a precise definition.

Definition 3.2 *A foliation that can be embedded into a flow is called* ***orientable****; otherwise,* ***non-orientable****.*

By an *embedding* of a foliation $\mathscr{F}$ into a flow f^t, we mean the following: each one-dimensional leaf of the foliation $\mathscr{F}$ becomes a one-dimensional trajectory of the flow f^t, and $\mathrm{Sing}(\mathscr{F}) = \mathrm{Fix}(f^t)$. Sometimes we use the term "oriented" instead of "orientable". Reeb foliations on an infinite strip and on a closed annulus are orientable. The Reeb foliation on the two-dimensional torus $\mathbb{T}^2$ is non-orientable.

On an orientable surface, the orientability of a foliation $\mathscr{F}$ is equivalent to the so-called *transversal orientability*. This means the existence of a flow $f_\perp^t$ with $\mathrm{Sing}(\mathscr{F}) = \mathrm{fix}(f_\perp^t)$ such that the trajectories of $f_\perp^t$ transversally intersect the leaves of $\mathscr{F}$ on the set $M^2 - \mathrm{Sing}(\mathscr{F})$.

Example 3.6 *The simplest foliations on a half-disk and disk.*



(a) (b)

Figure 3.4. The simplest foliations $\mathscr{F}_0$, $\mathscr{F}_{\mathrm{st}}$ on a half-disk and disk respectively.

We begin with the simplest foliation on the half-disk

$$D^+ = \{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 \leqslant 1,\ y \geqslant 0\}.$$

Obviously, the family of semicircles $x^2 + y^2 = c$, $0 < c \leqslant 1$, defines a foliation on D^+, which we denote by $\mathscr{F}_0$, Fig. 3.4, (a). The point $(0, 0) = \mathrm{Sing}(\mathscr{F}_0)$ is a singularity of this foliation. On the interval $[-1; 1] \subset D^+$, the foliation $\mathscr{F}_0$ induces the map $H(x) = -x$. Now one glues together two copies of the half-disk D^+ with the simplest foliations $\mathscr{F}_0$ along the semicircle $x^2 + y^2 = 1$. It is clear that gluing together two half-disks D^+ yields a disk. In order to distinguish the type of the singularities, we glue to the boundary of the disk a ring with a trivial foliation by segments that are transversal to the boundary. The foliation $\mathscr{F}_{\mathrm{st}}$ thus obtained is called the *simplest foliation on a disk*. This foliation is transversal to the boundary of the disk and has two singularities (thorns), Fig. 3.4, (b). $\diamondsuit$

A continuous dependence on initial conditions

We formulate here the fundamental Theorem 3.2 on continuous dependence of compact arcs of leaves on initial conditions, which plays an important role in the whole theory. Let's start with the following simple lemma that demonstrates an intimating connection between a local lamination and foliation.

Lemma 3.1 *Let $\mathscr{D}$ be a $C^{r,l}$ local lamination on M^2. Then any point of a leaf has a neighborhood homeomorphic to an open trivially foliated box. Moreover, the local lamination $\mathscr{D}$ can be extended to a C^l foliation in this neighborhood.*

Proof follows immediately from Definition 3.1. □

This lemma allows one to extend a local lamination to a foliation not only in a neighborhood of a regular point but also in some neighborhood of arbitrary compact subset of a leaf. Namely, the following theorem holds.

Theorem 3.1 *Let $\mathscr{D}$ be a $C^{r,l}$ local lamination on M^2. Then, for any connected compact subset K of any leaf l_α, there exists a neighborhood of K in M^2 that is homeomorphic to an open trivially foliated box. Moreover, the local lamination $\mathscr{D}$ can be extended to a C^l foliation in this neighborhood.*

Proof. Due to a compactness, K can be covered by a finite family of neighborhoods each of which is homeomorphic to an open trivially foliated box by Lemma 3.1. Then, the proof repeats, with inessential modifications, the proof of Theorem 5.1 from [213], where one constructs a subcovering for a given covering by trivially foliated boxes (such neighborhoods are called marked coordinate neighborhoods in [213]) in such a way that (new) intersecting coordinate neighborhoods have a common rectifying diffeomorphism. The idea of the proof is illustrated by Fig. 3.5. This means that K has a neighborhood in which the local lamination is homeomorphic to a family of straight lines on $\mathbb{R}^2$. Hence, we obtain the required result. □

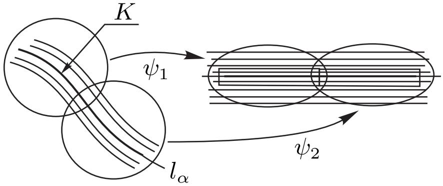


Figure 3.5

In the Theory of Differential Equations, there is the theorem ***on continuous dependence of solutions on initial conditions on compact arcs*** (see [3], thm 4) that can be reformulated for local laminations as follows.

Let $A_0 = [a_0; b_0]$ ***be an arc on a leaf*** l_0***. Then given any*** $\varepsilon > 0$***, there is*** $\delta > 0$ ***such that for every leaf*** l ***through*** $U_\delta(a_0)$ ***there is the arc*** $A = [a; b] \subset l$ ***with*** $a \in U_\delta(a_0)$***,*** $b \in U_\delta(b_0)$***, and*** $A \in U_\delta(A_0)$.

As a corollary to Theorem 3.1, we get such statement for local laminations. In view of the importance of this result, we formulate it as a theorem.

Theorem 3.2 *On compact arcs, the leaves of a local lamination depend continuously on the initial conditions.*

Corollary 3.1 *Let* N *be a saturated set of a local lamination* $\mathscr{D}$*. Then the topological closure* $\operatorname{clos} N$*, and the boundary* ∂N*, and the interior* $\operatorname{int} N$ *are saturated sets.*

Proof. We prove only that $\operatorname{clos} N$ is saturated. Take $p \in \operatorname{clos} N$. If p is a singularity, nothing to prove. Assume that p belongs to a (one-dimensional) leaf l. Take any $q \in l$. Then the arc K_{pq} of l with endpoints p, q is compact. Applying Theorem 3.2 to K_{pq}, one get that $q \in \operatorname{clos} N$. It follows that $l \subset \subset \operatorname{clos} N$. Hence, $\operatorname{clos} N$ is saturated. The remain statements are proved in the similar way. □

The Poincaré mapping

First of all, following [101], let's introduce a Σ-arc for a simple curve. Suppose that an infinite simple curve l intersects transversally an arc Σ at two points a and b. The arc $\widehat{ab}$ of the curve l with endpoints a and b is called a *Σ-arc* if $\Sigma \cap \widehat{ab} = a \cup b$. In other words, $\widehat{ab}$ intersects Σ only at the endpoints. One can generalize this notion as follows. Suppose that l intersects transversally two arcs Σ_1 and Σ_2. Then the arc $\widehat{ab}$ of the curve l with endpoints $a \in \Sigma_1$ and $b \in \Sigma_2$ is called a *$\Sigma_1\Sigma_2$-arc* if $\widehat{ab}$ intersects $\Sigma_1 \cup \Sigma_2$ only at the endpoints, $(\Sigma_1 \cup \Sigma_2) \cap \widehat{ab} = a \cup b$.

Let $l^+(z_1)$ be a positive semileaf of a local lamination $\mathscr{D}$ that intersects a transversal segment Σ_2 at z_2, $z_2 \neq z_1$. Take a transversal segment Σ_1 through z_1, and suppose that the arc $l^+(z_1, z_2)$ of $l^+(z_1)$ with endpoints z_1, z_2 is a $\Sigma_1\Sigma_2$-arc: $l^+(z_1, z_2) \cap (\Sigma_1 \cup \Sigma_2) = \{z_1, z_2\}$. One can consider z_2 as an image of z_1 under some mapping P induced by $\mathscr{D}$. Suppose for the moment that $\mathscr{D}$ is orientable local lamination. Theorem 3.1 allows to extend P to some neighborhood $U(z_1) \subset \Sigma_1$ of z_1 on Σ_1 such that for any point $\widetilde{z}_1 \in$ $\in U(z_1) \cap \operatorname{supp} \mathscr{D}$ the positive semileaf $\widetilde{l}^+ = l^+(\widetilde{z}_1)$ intersects Σ_2. Denote by $\widetilde{z}_2 \in \Sigma_2$ the first point after $\widetilde{z}_1$ where $\widetilde{l}^+$ intersects Σ_2, see Fig. 3.6, (a). Certainly, we assume that $\widetilde{z}_2 \to z_2$ as $\widetilde{z}_1 \to z_1$.

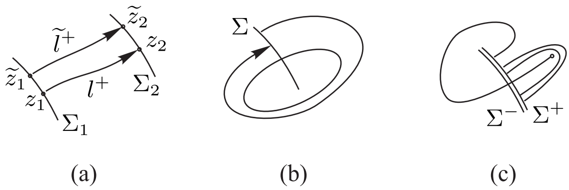


Figure 3.6. The Poincaré mappings.

The mapping $P^+_{\Sigma_1,\Sigma_2} \colon (\operatorname{supp} \mathscr{D}) \cap U(z_1) \to \Sigma_2$ that sends $\widetilde{z}_1$ to $\widetilde{z}_2$ according to the rule above is called the *Poincaré mapping* or, to be precise, *forward Poincaré map* induced by $\mathscr{D}$. For a negative semileaf $l^-(z_1)$, one gets a *back Poincaré map* $P^-_{\Sigma_1,\Sigma_2}$.

In Theory of Foliations, a Poincaré mapping is often called a *total holonomy* [60]. Clearly that if $\mathscr{D}$ is a $C^{r,l}$ foliation, then the Poincaré mappings are C^l diffeomorphisms.

Suppose now that $\Sigma_1 = \Sigma_2 \overset{\text{def}}{=} \Sigma$, see Fig. 3.6, (b). By the similar way, one gets the map $P_\Sigma \colon (\operatorname{supp} \mathscr{D}) \cap U(z_1) \to \Sigma$ called the *first return map*. Often, this map is also called the *Poincaré map* in honor of H. Poincaré who introduced such maps to study the behavior of flows near periodic trajectories.

In the case of non-orientable local lamination when there are no natural orientations of leaves, there exist two ways to introduce a Poincaré mapping. The first way is to fix an arc $l(z_1, z_2)$ of a semileaf $l(z_1)$ and consider a map induced by arcs sufficiently close to $l(z_1, z_2)$. This way is usual in the theory of high dimensional foliations when one considers the so called holonomy map along a fixed path on a leaf. In the second way, one needs to indicate a side of transversal where semileaves start. To be precise, if C is a transversal segment or closed transversal having a cylinder neighborhood, then one can introduce a positive C^+ and negative C^- sides of C. The sides are often defined by introducing a continuous unit vector field $\vec{n}$ on C that is transversal to C. We can consider C^+ and C^- as copies of C endowed with some sign. If a current point on $l(z_1)$ moves coherently in sense with $\vec{n}$ near z_1, then $z_1 \in C^+$. Otherwise, $z_1 \in C^-$. Similarly, $z_2 \in C^+$ or $z_2 \in C^-$. Thus, the Poincaré mapping P

is defined on a subset of $C^+ \cup C^-$, $P\colon \operatorname{Dom} P \subset C^+ \cup C^- \to C^+ \cup C^-$, where $z \in \operatorname{Dom} P$, provided there exists a semileaf $l(z)$ intersecting C after z, Fig. 3.6, (c).

Lemma 3.2 *Let C_1, C_2, C_3 be transversal segments of an orientable foliation $\mathscr{F}$ on a compact M. Suppose that there are points $m_{12} \in C_1$, $m_{13} \in C_1$ such that the positive semileaves $l^+(m_{12})$, $l^+(m_{13})$ intersect C_2, C_3 respectively as shown in Fig. 3.7, (a): the C_1C_2-arc $m_{12}P^+_{C_1C_2}(m_{12})$ does not intersect C_3, and the C_1C_3-arc $m_{13}P^+_{C_1C_3}(m_{13})$ does not intersect C_2. Then given any $j = 2, 3$, there is a point $m \in C_1$ between m_{12} and m_{13} such that either the semileaf $l^+(m)$ passes through an endpoint of C_j or $l^+(m)$ does not intersect $C_2 \cup C_3$. If C_2 or C_3 is a closed transversal, $l^+(m)$ does not intersect $C_2 \cup C_3$.*

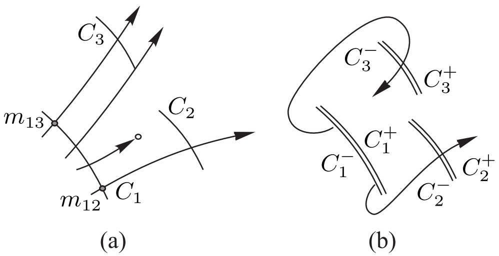


Figure 3.7

Proof. Consider only $j = 2$ because for $j = 3$, the proof is similar. If C_2 is a transversal segment and there is $m \in C_1$ such that $l^+(m)$ passes through an endpoint of C_2, then there is nothing to prove. Suppose that such a point does not exist. Then there is the maximal interval $(m_1, m_2) \subset C_1$ containing m_{12} such that $(m_1, m_2) \subset \operatorname{Dom} P^+_{C_1C_2}$, and given any $m \in (m_1, m_2)$, the C_1C_2-arc $mP^+_{C_1C_2}(m)$ does not intersect C_3. Assume for definiteness that m_1 is between m_{12} and m_{13}. Since (m_1, m_2) is a maximal interval, $l^+(m_1)$ does not intersect the interior of C_2. If we assume that $l^+(m_1)$ intersects the interior of C_3, Theorem 3.1 implies that there are $m \in (m_1, m_2)$ sufficiently close to m_1 such that the C_1C_3-arc $mP^+_{C_1C_3}(m)$ intersects C_3 without meeting C_2. This is impossible because by the inclusion $(m_1, m_2) \subset$ $\subset \operatorname{Dom} P^+_{C_1C_2}$, $l^+(m)$ must first intersect C_2. Thus, $l^+(m_1)$ does not intersect $C_2 \cup C_3$. $\square$

Lemma 3.2 has the following extension to non-orientable foliations.

Lemma 3.3 *Let* C_1*,* C_2*,* C_3 *be transversal segments of a foliation* $\mathscr{F}$ *on a compact* M*. Suppose that there are points* $m_{12} \in C_1$*,* $m_{13} \in C_1$ *such that the semileaves* $l(m_{12})$*,* $l(m_{13})$ *intersect* C_2*,* C_3 *respectively as shown in Fig.* 3.7*,* (*b*)*: the both starting points* m_{12} *and* m_{13} *of* $l(m_{12})$*,* $l(m_{13})$ *belong to the same side of* C_1 (*either* C_1^+ *or* C_1^-)*. Then given any* $j = 2, 3$*, there is a point* $m \in C_1^{\pm}$ *between* m_{12} *and* m_{13} *such that either the semileaf* $l(m)$ *passes through an endpoint of* C_j *or* $l(m)$ *does not intersect* $C_2 \cup C_3$*. If* C_2 *or* C_3 *is a closed transversal,* $l^+(m)$ *does not intersect* $C_2 \cup C_3$*.*

Proof is similar to the proof of Lemma 3.2, and we omit it. □

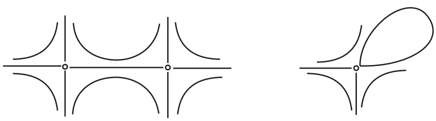
Transitive and highly transitive foliations

Actually, we've met transitive and highly transitive foliations as an irrational linear foliation on the torus $\mathbb{T}^2$, see Example 1.8.

Definition 3.3 *A foliation on a surface is called* ***transitive*** *if it has at least one everywhere dense leaf. A foliation is called* ***highly transitive*** *if every* (*one-dimensional*) *leaf is dense on a surface.*

Obviously, any highly transitive foliation is a transitive one. One can prove that if a transitive foliation has only isolated singularities, then each singularity is of saddle type.

A leaf that is a separatrix in the both positive and negative directions is called a *separatrix connection*, see Fig. 3.8. Sometimes, a separatrix connects different topological saddles, Fig. 3.8, (a), but sometimes the connecting saddles coincide. In the last case, a separatrix forms a loop, Fig. 3.8, (b).



(a) (b)

Figure 3.8. Separatrix connections.

In general, a transitive foliation can have separatrix connections, while a highly transitive foliation have no separatrix connections (obviously, a separatrix connection can't be dense). A highly transitive foliation can have fake saddles the number of whose could be arbitrary with no connection with the topology of supporting surface. In this sense, fake saddles are artificial. Therefore, it is natural to distinguish transitive foliations without separatrix connections and fake saddles.

Definition 3.4 *A highly transitive foliation with no fake saddles is called* ***irrational*** *if it has only isolated singularities.*

An irrational linear foliation on $\mathbb{T}^2$ is irrational. Later on, we show that an irrational foliation can have singularities only being saddles. Now we represent another examples of such foliations.

Example 3.7 *Irrational foliations on a disk.*

Now, take three half-disks D_1, D_2, and D_3 with simplest foliations $\mathscr{F}_1$, $\mathscr{F}_2$, and $\mathscr{F}_3$, respectively, and glue these half-disks together along their diameters as shown in Fig. 3.9. Denote the obtained foliation by $\mathscr{F}_\alpha$, and the set obtained by gluing together three half-disks by D. The foliation $\mathscr{F}_\alpha$ has three singularities (thorns) at the points $\frac{\alpha}{2}$, $\frac{1}{2}$, $\frac{\alpha+1}{2} \in [0; 1]$ inside D and one singularity (tripod) at the point $\alpha \in [0; 1]$ on the boundary of D. It turns out that the "dynamical" properties of the foliation $\mathscr{F}_\alpha$ depend on the arithmetic properties of the number α. Assume, for definiteness, that $\alpha < \frac{1}{2}$.

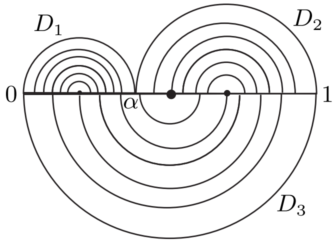


Figure 3.9. The foliation $\mathscr{F}_\alpha$.

One can easily verify that the foliations $\mathscr{F}_1$, $\mathscr{F}_2$, and $\mathscr{F}_3$ induce the Poincaré maps

$$\begin{aligned} H_1(x) &= \alpha - x, && 0 \leqslant x \leqslant \alpha; \\ H_2(x) &= 1 + \alpha - x, && \alpha \leqslant x \leqslant 1, \\ H_3(x) &= 1 - x, && 0 \leqslant x \leqslant 1, \end{aligned}$$

respectively. Then

$$H_3 \circ H_1(x) = x - \alpha + 1, \qquad 0 \leqslant x \leqslant \alpha,$$
$$H_3 \circ H_2(x) = x - \alpha, \qquad \alpha \leqslant x \leqslant 1.$$

If we represent a circle S^1 as $[0; 1]$ with identified endpoints, then the mappings $H_3 \circ H_1$ and $H_3 \circ H_2$ induce the rotation $R_{-\alpha}$ by $-\alpha$ of S^1. According to Example 1.6, for an irrational α, all semi-orbits of $R_{-\alpha}$ are everywhere dense on S^1. Therefore, any leaf inside D intersects $[0; 1] \subset D$ in an everywhere dense set of points (note that the semi-orbits of the rotation $R_{-\alpha}$ give only "half" of the intersection points). Hence, for an irrational α, each leaf of the foliation $\mathscr{F}_\alpha$ is everywhere dense in D. $\diamondsuit$

Similar arguments involving the rotation $R_{-\alpha}$ show that for a rational α, the foliation $\mathscr{F}_\alpha$ has a separatrix connection that connects the thorns and a separatrix connection that connects a tripod with a thorn. The remaining leaves inside D are closed (homeomorphic to a circle).

By analogy with Example 1.8, we'll call $\mathscr{F}_\alpha$ with irrational α an *irrational foliation on a disk.*

Irrational foliations on a disk and torus are highly transitive foliations. An irrational foliation on a torus is orientable while an irrational foliation on a disk is non-orientable because there are locally non-orientable singularities, namely, thorns and tripods. Note also that by virtue of the Poincaré–Bendixson Theorem (see Theorem 4.1), there are no flows with everywhere dense trajectories and semitrajectories on a disk.

Example 3.8 *The foliation* $\mathscr{F}(\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q)$.

Generalizing the above scheme, we construct a foliation on a disk for the set of numbers

$$\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q, \quad 0 < \alpha_i < 1, \quad 0 < \beta_i < 1,$$

by gluing $p + 1$ half-disks with simplest foliations to the segment $[0; 1]$ from above and $q + 1$ half-disks with simplest foliations from below (as is shown in Fig. 3.10). Denote the foliation obtained by $\mathscr{F}(\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q)$. $\diamondsuit$

Example 3.9 *Irrational foliations on a sphere.*

There are several methods for constructing transitive foliations on the sphere S^2. Consider some of them. Take the simplest foliation $\mathscr{F}_{\rm st}$ on a disk D.

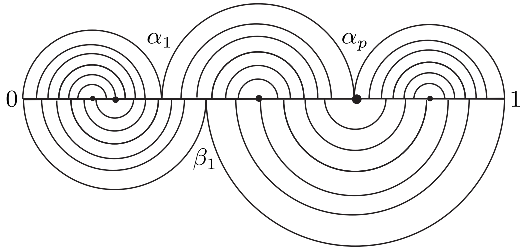


Figure 3.10. The foliation $\mathscr{F}(\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q)$.

Without loss of generality, we may assume that the boundary of the disk D is a unit circle S^1, which we represent as the unit segment $[0; 1]$ with identified endpoints. This representation defines a cyclic coordinate on S^1. Again, without loss of generality, we may assume that the map induced by $\mathscr{F}_{\mathrm{st}}$ on S^1 has the form $H_{\mathrm{st}}(x) = 1 - x \pmod 1$. This means that the separatrices of thorns intersect S^1 at the points $x = 0$ and $x = 0{,}5$. Let us glue together two copies of disks D by the rotation of a circle,

$$R_\alpha \colon S^1 \to S^1, \quad R_\alpha(x) = x + \alpha \bmod 1.$$

This gluing yields the sphere S^2 with the foliation denoted by $F_\alpha(S^2)$. The singularities of $F_\alpha(S^2)$ are four thorns.

Let us prove that for irrational α, $F_\alpha(S^2)$ is irrational. The boundary of the disks D forms the closed transversal C on S^2 that divides S^2 into two disks D_1 and D_2. The restriction of $F_\alpha(S^2)$ onto each of these disks D_i is topologically equivalent to the standard foliation and induces the Poincaré mapping (along leaves on D_i) $H_{\mathrm{st},i}(x) = 1 - x \bmod 1$. Taking into account the gluing $R_\alpha(x) = x + \alpha \bmod 1$, we have

$$H_{\mathrm{st},1} \circ R_\alpha \circ H_{\mathrm{st},2}(x) = x - \alpha \bmod 1 = R_{-\alpha}(x).$$

Hence, for an irrational α, each (one-dimensional) leaf of $F_\alpha(S^2)$ is everywhere dense on S^2. $\diamondsuit$

3.2. Basic specification methods

Here, we represent some methods to construct local laminations (including foliations and laminations) on surfaces. At the same time, one introduces some basic examples.

Lifting and projecting

One of the methods to construct local laminations starting with simple models is their lifting on covering surfaces or projecting from covering surfaces. The advantage is that sometimes a covering surface, for example, universal covering surface, can be endowed with simple coordinate charts, and therefore a local lamination can be described directly. We begin with non-branched covering projections.

Recall that S_{nk} is a group of integer translations $(x, y) \to (x+n, y+k)$, $n, k \in \mathbb{Z}$, of the Euclidean plane $\mathbb{R}^2$. A system of differential equations

$$\dot{x} = f_1(x, y), \quad \dot{y} = f_2(x, y) \tag{3.1}$$

defines the vector field $\vec{v} = (f_1(x, y), f_2(x, y))$ on $\mathbb{R}^2$. Since the Jacobian of any integer translation S_{nk} is the identity, $\vec{v}$ is invariant under S_{nk} if and only if the functions $f_1(x, y)$, $f_2(x, y)$ are invariant under the group $\mathbb{Z}^2$. This means that $f_1(x, y)$, $f_2(x, y)$ are 1-periodic in both arguments:

$$f_i(x+n, y+k) \equiv f_i(x, y) \ \ \forall (x, y) \in \mathbb{R}^2, \ \ (n, k) \in \mathbb{Z}^2, \ \ i = 1, 2. \tag{3.2}$$

Then $\vec{v}$ is projected into a vector field on $\mathbb{T}^2$. For the Klein bottle K^2, $\vec{v}$ is invariant under the group $\Gamma(K^2)$ if and only if

$$f_1\left(x + \frac{n}{2}, y+k\right) \equiv f_1(x, y), \quad f_2\left(x + \frac{n}{2}, y+k\right) \equiv (-1)^n f_2(x, y) \tag{3.3}$$

$\forall (x, y) \in \mathbb{R}^2$, $(n, k) \in \mathbb{Z}^2$. This arguments are true in general.

Lemma 3.4 *Let $\pi\colon \overline{M} \to M$ be a covering projection and $\vec{v}$ a vector field on $\overline{M}$. If $\vec{v}$ is invariant under the group of deck transformations, $\vec{v}$ induces the vector field $\pi(\vec{v})$ on M.*

Sometimes one says that $\pi(\vec{v})$ is a projection of $\vec{v}$ under π.

Example 3.10 *Constant flows on the torus $\mathbb{T}^2$ and Klein bottle K^2 revised.*

The constant flow $\overline{f}_\mu^t$ on $\mathbb{R}^2$ defined by the differential equations $\dot{x} = 1$, $\dot{y} = \mu$ is a covering flow for the constant flow f_μ^t on $\mathbb{T}^2$, see example 1.8. The system of differential equations

$$\dot{x} = \sin^2 \pi x + \sin^2 \pi y, \quad \dot{y} = \mu(\sin^2 \pi x + \sin^2 \pi y)$$

defines the covering flow for the flow $f^t_{0,\mu}$ with the fake saddle at the point $\pi(0;\,0)$ on $\mathbb{T}^2$. One says that $f^t_{0,\mu}$ is obtained from f^t_μ by *putting a fake saddle at* $\pi(0;\,0)$.

By (3.3), $\overline{f}^t_\mu$ is a lift of linear flow on K^2 if and only if $\mu = 0$. In this case, all trajectories on K^2 are periodic. ♢

Hyperbolic (Lobachevsky) plane and group of isometries

There have been many books devoted to hyperbolic (Lobachevsky) geometry and Riemannian surfaces [46, 63, 125, 211]. Here, we mention only some of the results that are used in the text. There are two models of hyperbolic (Lobachevsky) plane, $\mathbb{H}^2$-model when one uses the upper half-plane

$$\mathbb{H}^2 = \{z = x + iy\colon\, y > 0\} \text{ with the metric generated by the differential } ds = \frac{|dz|}{y}$$

and Δ-model when we use the unit disk

$$\Delta = \{z \in \mathbb{C}\colon\, |z| < 1\} \text{ with the metric generated by the differential } ds = \frac{|dz|}{1 - |z|^2}.$$

In both cases, denote by $\overline{d}_{\mathrm{NE}}$ the metric generated by ds. In both models, one introduces the *absolute*, or the *circle at infinity* (sometimes, one says *circle composed of infinitely remote points*):

$$\begin{aligned} S_\infty &= \mathbb{R} \cup \{\infty\} && \text{in the case of the } \mathbb{H}^2 \text{ model,} \\ S_\infty &= \{z \in \mathbb{C}\colon\, |z| = 1\} && \text{in the case of the } \Delta \text{ model.} \end{aligned}$$

These points do not belong to the hyperbolic plane; however, they play a very important role in hyperbolic geometry. The geodesics are arcs of Euclidean circles and straight lines that orthogonal to S_∞. We will suppose that endpoints of geodesics, *ideal endpoints*, belong to S_∞.

The group $I(\mathbb{H}^2)$ of isometries of $\mathbb{H}^2$ consists of the following transformations:

$$z \mapsto \frac{az + b}{cz + d}, \qquad \text{where} \quad a,\, b,\, c,\, d \in \mathbb{R}, \quad ad - bc = 1, \tag{3.4}$$

$$z \mapsto \frac{-a\overline{z} + b}{-c\overline{z} + d}, \qquad \text{where} \quad a,\, b,\, c,\, d \in \mathbb{R}, \quad ad - bc = 1. \tag{3.5}$$

The group $I(\Delta)$ of isometries of Δ consists of the following transformations:

$$z \mapsto \frac{az+b}{\overline{b}z+\overline{a}}, \quad \text{where} \quad a, b \in \mathbb{C}, \quad |a|^2 - |b|^2 = 1, \tag{3.6}$$

$$z \mapsto \frac{a\overline{z}+b}{\overline{b}\overline{z}+\overline{a}}, \quad \text{where} \quad a, b \in \mathbb{C}, \quad |a|^2 - |b|^2 = 1. \tag{3.7}$$

Isometries of the type (3.6), (3.4) preserve orientation, and of the type (3.7), (3.5) reverse orientation. Thus, each isometry is a transformation that is naturally extended to the circle at infinity S_∞. Any isometry has a type defined by the property of being orientation-preserving or orientation-reversing and by the number of fixed points in the extended hyperbolic plane $\Delta \cup S_\infty = \overline{\Delta}$. According to [220], any non-identity transformation of the group $I(\Delta)$ has one of the following types:

- an *elliptic transformation*; this transformation preserves orientation and has exactly one fixed point in $\overline{\Delta}$, which lies in Δ;
- a *parabolic transformation*; this transformation preserves orientation and has exactly one fixed point in $\overline{\Delta}$, which belongs to S_∞;
- a *hyperbolic transformation*; this transformation preserves orientation and has exactly two fixed points in $\overline{\Delta}$, which belong to S_∞; one fixed point is attracting (sink) while another fixed point is repelling (source);
- a *glide reflection*; this transformation reverses orientation and has exactly two fixed points in $\overline{\Delta}$, which belong to S_∞; the square of a glide reflection is a hyperbolic transformation with same fixed points;
- a *reflection with respect to a geodesic*; this transformation reverses orientation and has a continuum of fixed points in $\overline{\Delta}$, which form the indicated geodesic and its ideal endpoints on S_∞.

For fixed hyperbolic transformation and glide reflection there is a unique geodesic that is invariant and joins fixed points on S_∞. This geodesic is called an *axis*.

The following statement (the proof can be found in [46], Theorem 5.1.2) gives a sufficient condition of a group $\Gamma \subset I(\Delta)$ being not properly discontinuous.

Lemma 3.5 *Let g_1 be a hyperbolic map. Suppose maps $g_1, g_2 \in I(\Delta)$ have only one common fixed point. Then the group (g_1, g_2) generated by g_1 and g_2 is not discrete* (*as a consequence, is not properly discontinuous*).

Hyperbolic surfaces

A surface $M^2 = M$ is called *hyperbolic* if it is locally isometric to the hyperbolic plane. Such a closed surface can be obtained as the quotient space $M = \Delta/\Gamma$ for a properly discontinuous $\Gamma \subset I(\Delta)$. In general case, M is the quotient space $\overline{M}/\Gamma$, where $\overline{M}$ is a convex subset of Δ. The natural projection $\pi\colon \overline{M} \to M = \Delta/\Gamma$ is a universal covering map that induces a geometric structure on M such that π is a local isometry. Following [220], we call a properly discontinuous group $\Gamma \subset I(M)$ *crystallographic*. Obviously, that the ***limit set of a crystallographic group belongs to the circle at infinity***.

Clearly, that a crystallographic group, in general, consists of glide reflections, and hyperbolic, and parabolic maps. A crystallographic group consisting of orientation preserving maps is called *Fuchsian*.

A particular case of the proposition known as the Killing–Hopf theorem states that ***any connected complete hyperbolic surface*** M ***is a quotient space*** Δ/Γ***, where*** Γ ***is a some crystallographic group***. Later on, we consider only such surfaces, mainly restricted ourself by closed ones. Since $\Gamma \subset I(\Delta)$ does not contain elliptic elements and reflections, the ***hyperbolic surface has no boundary***. A hyperbolic surface $M = \Delta/\Gamma$ is closed if and only if Γ has no parabolic elements. A hyperbolic surface $M = \Delta/\Gamma$ is orientable if and only if Γ has no glide reflections i. e., Fuchsian. The famous Uniformization Theorem (see [130, 191]) asserts that given any closed orientable surface M of negative Euler characteristic, there is a finitely generated Fuchsian group $\Gamma \subset I(\Delta)$ such that $M = \Delta/\Gamma$.

Example 3.11 *Polar Morse–Smale flow on a hyperbolic surface.*

Take a hyperbolic surface $M = \Delta/\Gamma$ to be a closed orientable surface of genus $p \geqslant 2$. One can construct a fundamental domain Φ of Γ to a $4p$-polygon, see Fig. 3.11, (a), where $p = 2$. If $f(z)$ is an automorphic function with respect to Γ i. e., $f(\gamma(z)) = f(z)$ for all $\gamma \in \Gamma$, then the vector field $\vec{V} = \operatorname{grad} f$ is invariant under Γ, and hence to projected to the vector field $\pi(\vec{V})$ on M. One can choose $f(z)$ to get a polar Morse-Smale vector field $\pi(\operatorname{grad} f)$ that has a unique sink, unique source, and $2p$ saddles, see Fig. 3.11, (b), where the vertices of $4p$-gon are identified under Γ and project to the source on M. $\diamondsuit$

Lemma 3.6 *Let* $M = \Delta/\Gamma$ *be a hyperbolic surface* (*possibly, noncompact and non-orientable*). *Suppose that for some element* $\gamma \in \Gamma$ *there is a point* $\overline{z} \in \Delta$ *such that* $\gamma^n(\overline{z}) \to \sigma \in S_\infty$ *as* $n \to \pm\infty$. *If* $\overline{z}$ *is not a fixed point of* γ

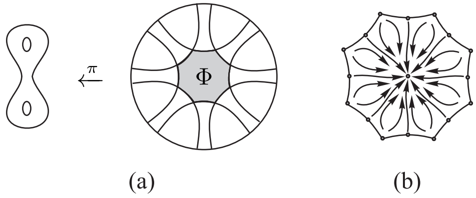


Figure 3.11. The fundamental 8-gon Φ of M_2^2 (a); the phase diagram of polar-flow (b).

then γ *is parabolic* (*and, hence, the surface is noncompact*). *Moreover, if a curve* $\overline{C} \subset \Delta$ *is invariant under* γ, *has ideal endpoints that coincide with* σ (*so that* $\overline{C} \cup \sigma$ *is homeomorphic to a circle*), *and is projected onto a simple closed curve* $\pi(\overline{C})$, *then* $\pi(\overline{C})$ *bounds a disk on* M *punctured at one point.*

Proof. Since $\gamma^n(\overline{z}) \to \sigma$, $\gamma(\sigma) = \sigma$. Because of $\gamma^n(\overline{z}) \to \sigma$ both as $n \to +\infty$ and as $n \to -\infty$, γ is parabolic. Denote by $\overline{D}$ the domain bounded by the curve $\overline{C} \cup \sigma$. The point σ cannot be a fixed point of a hyperbolic element or of a glide reflection from Γ because the group Γ is properly disconnected (proposition 3.5). Since γ preserves orientation, $\gamma(\overline{D}) = \overline{D}$. This, combined with the fact that the curve $\pi(\overline{C})$ is simple, implies that there are no congruent points in $\overline{D}$ with respect to elements of the group Γ that are different from the transformations of the form γ^k, $k \in \mathbb{Z}$. Then, $\pi(\overline{D}) = \overline{D}/\{\gamma^n : k \in \mathbb{Z}\}$. It is easily seen that the quotient space $\overline{D}/\{\gamma^n : k \in \mathbb{Z}\}$ is a disk punctured at one point. □

One of the methods for constructing Riemannian and hyperbolic surfaces consists in the formation of the quotient space with respect to a free action of a (totally) disconnected group. Recall that a surface is called *hyperbolic* if it is locally isometric to the hyperbolic plane [211]. A group G of transformations of the plane Δ is said to be *disconnected* if the orbit of any point under the action of the group G has no accumulation points in Δ. The group G is said to be *freely acting* if any transformation γ from G has no fixed points.

A particular case of the proposition known as the Killing–Hopf theorem states that any connected complete hyperbolic surface is a quotient space of the form Δ/Γ, where Γ is a certain disconnected group of isometries that freely acts on Δ. The natural projection $\pi\colon \Delta \to \Delta/\Gamma$ is a local isometry and a universal covering of the hyperbolic surface $M = \Delta/\Gamma$. Since the group $\Gamma \subset I(\Delta)$ of deck transformations acts freely, it does not contain elliptic elements or reflections. In particular, a hyperbolic surface has no boundary. We will need the following proposition.

Lemma 3.7 *Let $M = \Delta/\Gamma$ be a hyperbolic surface* (*possibly, noncompact and non-orientable*). *Suppose that for a certain element $\gamma \in \Gamma$ and a point $\overline{z} \in \Delta$, we have $\gamma^n(\overline{z}) \to \sigma \in S_\infty$ as $n \to \pm\infty$. Then the element γ is parabolic* (*and, hence, the surface is noncompact*). *Moreover, if a curve $\overline{C} \subset \Delta$ is invariant under γ, has ideal endpoints that coincide with σ* (*so that $\overline{C} \cup \sigma$ is homeomorphic to a circle*), *and is projected onto a simple closed curve $\pi(\overline{C})$, then $\pi(\overline{C})$ bounds a disk on M punctured at one point.*

Proof. Since $\gamma^n(\overline{z}) \to \sigma$, we have $\gamma(\sigma) = \sigma$. Since $\gamma^n(\overline{z}) \to \sigma$ both as $n \to +\infty$ and as $n \to -\infty$, the element γ is parabolic. Denote by $\overline{D}$ the domain bounded by the curve $\overline{C} \cup \sigma$. The point σ cannot be a fixed point of a hyperbolic element or of a glide reflection from Γ because the group Γ is disconnected [46, Theorem 5.1.2]. Since γ preserves orientation, $\gamma(\overline{D}) = \overline{D}$. This combined with the fact that the curve $\pi(\overline{C})$ is simple, implies that there are no congruent points in $\overline{D}$ with respect to elements of the group Γ that are different from the transformations of the form γ^k, $k \in \mathbb{Z}$. Then, $\pi(\overline{D}) = \overline{D}/\{\gamma^n \colon k \in \mathbb{Z}\}$. It is easily seen that the quotient space $\overline{D}/\{\gamma^n \colon k \in \mathbb{Z}\}$ is a disk punctured at one point. □

Let Γ be the group of deck transformations of a hyperbolic surface $M = \Delta/\Gamma$. According to [46], any cyclic subgroup of the group Γ is discrete (Theorem 8.4.1), and there are no parabolic and hyperbolic transformations with a common fixed point (Theorem 5.1.2). Moreover, the stabilizer of any point in a Fuchsian group (i. e., a group consisting of orientation-preserving transformations) is a cyclic subgroup generated by either a hyperbolic or a parabolic element (Theorem 8.1.2). The following lemma shows that one more situation is possible for a non-orientable hyperbolic surface.

Lemma 3.8 *Let $\Gamma_\sigma \subset \Gamma$ be a nontrivial stabilizer of a point $\sigma \in S_\infty$. Then, the following cases are possible*:

1) Γ_σ *is a cyclic group generated by one hyperbolic element*;
2) Γ_σ *is a cyclic group generated by one parabolic element*;
3) Γ_σ *is a cyclic group generated by one glide reflection.*

Proof. If Γ_σ consists only of orientation-preserving transformations, then the first two cases are realized [46, Theorem 8.1.2]. Suppose that Γ_σ contains a glide reflection η. Then, η^2 is a hyperbolic transformation, and according to [46], Γ_σ is a cyclic group generated by a glide reflection. □

For points a, b, $c \in S_\infty$, denote by (a, b, c) the open interval with endpoints a and c that lies on the absolute S_∞ and contains b.

Corollary 3.2 *Suppose that the hypotheses of Lemma* 3.8 *hold, and let* $\omega \in S_\infty$ *be a point such that* $\Gamma_\omega \cap \Gamma_\sigma = \varnothing$. *Then there exist transformations* $\varepsilon_1, \varepsilon_2 \in \Gamma_\sigma$ *such that*

$$\gamma(\omega) \notin (\varepsilon_1(\omega), \omega, \varepsilon_2(\omega))$$

for any nonidentity transformation $\gamma \in \Gamma_\sigma$.

A C^r surface in Euclidean space $\mathbb{R}^3$

The specification of a local lamination on a surface depends on and is closely connected with the representation of the surface. For example, the surface can be represented as a set of points in $\mathbb{R}^3$, or it can be represented as a two-dimensional manifold by means of local charts with compatible coordinates. A surface can also be represented as a quotient space of universal covering space with respect to some group of transformation. These and other representations enable one to define local laminations in various ways.

Here, we assume that $\mathbb{R}^3$ is endowed with Cartesian coordinates (x, y, z) that define an orthnormal chart relative to the inner product $\langle \cdot, \cdot \rangle$. Sometimes the inner product of $\vec{a}$ and $\vec{b}$, we denote as the scalar product $\vec{a} \cdot \vec{b}$.

A surface M^2 is locally defined by an equation $\psi(x, y, z) = 0$, where ψ is a real function of class C^r, $r \geqslant 1$, whose gradient $\nabla\psi = (\psi_x, \psi_y, \psi_z)$ does not vanish on M^2. The surfaces in this book, with the exception of rare occasions, will be of class C^∞, and will be referred as smooth. For a such smooth surface M^2, at each point $p \in M^2$, can be defined the unit normal vector $\vec{n}_p = \frac{\nabla\psi(p)}{|\nabla\psi(p)|}$. The tangent bundle

$$T(M^2) = \{(p, \vec{v}) \mid p \in S \subset \mathbb{R}^3, \ \vec{v} \in \mathbb{R}^3, \ \langle \vec{n}_p, \vec{v} \rangle = 0\}$$

will be endowed with the inner (scalar) product induced from $\langle \cdot, \cdot \rangle$.

Example 3.12 *The projection method.*

Suppose that the equation $\phi_p(x, y) = 0$ defines $p+1$ disjoint ovals on $\mathbb{R}^2$, where p of them have disjoint interiors and lie inside the biggest one. For example,

$$\phi_p(x, y) = (x^2 + y^2 - 16)[(x-2)^2 + y^2 - 1][(x+2)^2 + y^2 - 1] = 0$$

defines $p + 1 = 3$ such ovals, see Fig. 3.12 (a), when $p = 2$. Then the equation $\psi(x, y, z)\phi_p(x, y) + z^2 = 0$ determines a closed orientable surface M_p^2 of genus $p \geqslant 0$, Fig. 3.12 (b).

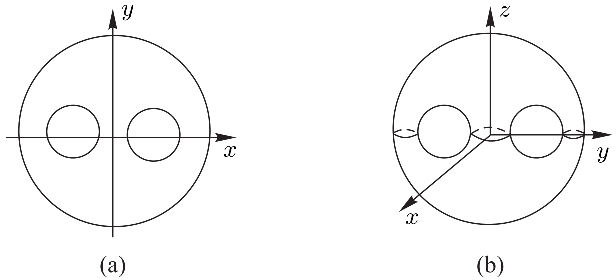


Figure 3.12. 3 ovals (a), and the corresponding surface of genus 2, the pretzel (b).

Given any smooth vector field $\vec{V}$ on $\mathbb{R}^3$, one can get the vector field

$$\vec{v}_{tan} = \vec{V} - (\vec{n} \cdot \vec{V})\vec{n},$$

where $\vec{n} = \frac{\nabla\psi}{|\nabla\psi|}$ is a unit vector that is orthogonal to M_p^2. Since $\vec{n} \cdot \vec{v}_{tan} = (\vec{n} \cdot \vec{V})\vec{n}^2 = 0$, $\vec{v}_{tan}$ is a vector field on M_p^2. Actually, $\vec{v}_{tan}$ is a projection of $\vec{V}$ on the tangent space of M_p^2.

If $\vec{V}$ is given by the system of differential equations

$$\dot{x} = P(x,\, y,\, z), \quad \dot{y} = Q(x,\, y,\, z), \quad \dot{z} = R(x,\, y,\, z),$$

then $\vec{v}_{tan}$ is given by the system

$$\begin{cases} \dot{x} = P(x,\, y,\, z) - |\nabla\psi|^{-1}\langle \vec{V},\, \nabla\psi\rangle \\ \dot{y} = Q(x,\, y,\, z) - |\nabla\psi|^{-1}\langle \vec{V},\, \nabla\psi\rangle \\ \dot{z} = R(x,\, y,\, z) - |\nabla\psi|^{-1}\langle \vec{V},\, \nabla\psi\rangle. \end{cases}$$

1) If we project the vector field $\vec{V} = (0;\, 0;\, -1)$ on the sphere $S^2\colon x^2+y^2+z^2 = 1$, then the vector field $\vec{v}_{tan}$ induces a so-called "north-south" flow f_{ns}^t on S^2 with two fixed points, at the "north pole" $(0;\, 0;\, +1)$ and "south pole" $(0;\, 0;\, -1)$. All one-dimensional trajectories flow down from the north pole to the south pole along meridians, see Fig. 3.13, (a).

2) Let $\vec{V}$ is given by the system of differential equations $\dot{x} = -y$, $\dot{y} = x$, $\dot{z} = 0$. Easy to see that every point $(0;\, 0;\, z) \in \mathbb{R}^3$ is a fixed point, and the remaining trajectories are closed. The sphere S^2 and the torus T^2 defined by

the equations

$$x^2+y^2+z^2=1, \quad (x^2+y^2-1)\left(x^2+y^2-\frac{1}{4}\right)+z^2=0$$

respectively are integral surfaces of the field $\vec{V}$. Therefore, $\vec{v}_{tan}=\vec{V}$. The corresponding flows are represented in Fig. 3.13, (b), (c). ♢

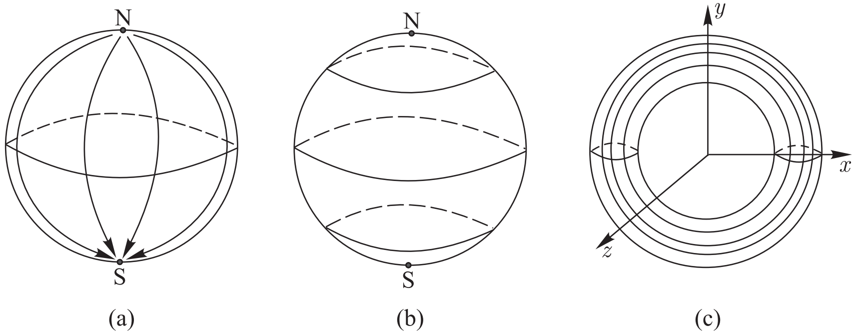


Figure 3.13. Flows on the sphere (a), (b), and torus (c).

Example 3.13 *"North–South" flow on the sphere S^2 revised.*

For S^2 here, we give a structure of 2-manifolds via charts. For this let us take the sphere of radius 1 in $\mathbb{R}^3$, $x^2+y^2+z^2=1$. Thus the north pole is the point $N(0,\,0,\,1)$ and the south pole is $S(0,\,0,\,-1)$. We map $S^2 - (0,\,0,\,1)$ to the plane $z=0\cong\mathbb{R}^2$ by "stereographic" projection as follows. Take the line from $(0,\,0,\,1)$ to another point of S^2 and produce it until it intersects the $(x,\,y)$-plane $\mathbb{R}^2$. The map taking that point on the sphere to that intersection point gives the coordinate system on the chart $\mathscr{U}_N = S^2 - (0,\,0,\,1)$, $(x,\,y)\colon \mathscr{U}_N \to \mathbb{R}^2$. For the second chart $\mathscr{U}_S = S^2-(0,\,0,\,-1)$, we similarly take the stereographic projection from the point $(0,\,0,\,-1)$ to $\mathbb{R}^2$. The coordinate transition $\mathbb{R}^2-\{0\}\to\mathbb{R}^2-\{0\}$ of these charts is given by

$$(x,\,y)\to\frac{1}{\sqrt{x^2+y^2}}(x,\,y) \tag{3.8}$$

as the Reader can calculate. In the charts $\mathscr{U}_N$ and $\mathscr{U}_S$, one considers the systems of differential equations

$$\begin{cases} \dot{x} = \dfrac{x}{1+x^2+y^2} \\ \dot{y} = \dfrac{y}{1+x^2+y^2}, \end{cases} \qquad \begin{cases} \dot{x} = \dfrac{x(x^2+y^2)}{1+x^2+y^2} \\ \dot{y} = \dfrac{y(x^2+y^2)}{1+x^2+y^2} \end{cases}$$

It is not hard to verify that these systems define the "north – south" flow in Fig. 3.13, (a). $\diamondsuit$

Branched covering projections

There are the following generalizations of the definitions of covering space, covering projection etc. Denote by δ_k the mapping $z \to z^k$ of $\mathbb{C}$, where $k \in \mathbb{N}$. A map $\widetilde{\pi}\colon \widetilde{M}^2 \to M^2$ is called a *branched covering projection* or *branched covering map* and $\widetilde{M}^2$ is called the *branched covering surface* if any point $m \in$ $\in M^2$ has a neighborhood U such that $\widetilde{\pi}^{-1}(U)$ is a nonempty union of disjoint sets $\{V_i\}_{i\geqslant 1}$ on which the restriction $\widetilde{\pi}|_{V_i}\colon V_i \to U$ is conjugate to δ_k, $k \in \mathbb{N}$. The number k is called the *branched index* of the point $z_i \in V_i$ denoted by $k(z_i)$. The number $k(z_i) - 1$ is called the *multiplicity* of z_i. A point z_i with positive multiplicity is a *critical point*, and a point $m \in M^2$ with at least one preimage a critical point is a *branched point*. Denote the set of branched points by $\Sigma_{\widetilde{\pi}}$ called the *branched set*. If $\Sigma_{\widetilde{\pi}} = \varnothing$, $\widetilde{\pi}$ is a non-branched covering projection.

If all points from $\widetilde{\pi}^{-1}(m)$, $m \in \Sigma_{\widetilde{\pi}}$, have the same branched index, this branched index is said to be a *branched order* of m. We shall consider only *regular* branched covers i. e., all points from $\widetilde{\pi}^{-1}(m)$ have the same branched index for all $m \in \Sigma_{\widetilde{\pi}}$. If m_1, $m_2 \in M^2$ are not branched points, then $|\widetilde{\pi}^{-1}(m_1)|$, the cardinality of $\widetilde{\pi}^{-1}(m_1)$, equals $|\widetilde{\pi}^{-1}(m_2)|$ provided M^2 is connected. If $|\widetilde{\pi}^{-1}(m)|$ is finite, we say that $\widetilde{\pi}$ is a $|\widetilde{\pi}^{-1}(m)|$-sheeted covering projection.

Example 3.14 *The simplest branched* 2*-sheeted covering projection is*

$$\mathbb{R}^2 \to \mathbb{R}^2/((x,\, y) \simeq (-x,\, -y)) \cong \mathbb{R}^2,$$

where $\mathbb{R}^2/((x,\, y) \simeq (-x,\, -y))$ is the quotient space of $\mathbb{R}^2$ under the group generated by the central symmetry $C\colon (x,\, y) \mapsto (-x,\, -y)$. Indeed, one can prove that the restrictions of C and $\delta_2\colon z \mapsto z^2$ on the upper half-plane $\mathbb{R}^2_+$ are conjugated. $\diamondsuit$

Example 3.15 *2-sheeted branched covering map.*

In $\mathbb{R}^3$, let us consider a closed orientable surface M_{pq} of genus $p \geqslant 0$ that is symmetric with respect to the axis Oz and the plane $y = 0$. Suppose that the intersection of M_{pq} with this plane consists of $q \geqslant 1$ circles, see Fig. 3.14 (a), where $p = 4$, $q = 3$. Note that because of symmetry under the plane $y = 0$, $p - q$ is odd.

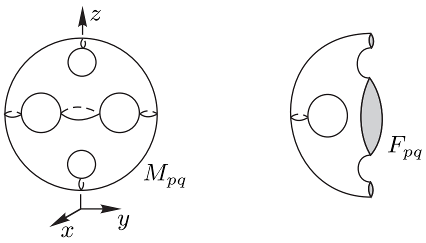


Figure 3.14. The surfaces M_{pq}, (a), and F_{pq}, (b), where $p = 4$, $q = 3$.

Denote by $\mathscr{S}_z$ the symmetry $(x, y, z) \mapsto (-x, -y, z)$. Since M_{pq} is invariant under $\mathscr{S}_z$, the factor-space $M_{pq}/\mathscr{S}_z \overset{\text{def}}{=} \mathscr{M}_{pq}$ is a closed orientable surface of genus $\left[\frac{p-q}{2}\right] + 1$. The surface $\mathscr{M}_{pq}$ can be obtained as follows. Take the part of M_{pq} lying in the half-space $y \leqslant 0$. This part is the surface, say F_{pq}, with $\left[\frac{p-q}{2}\right]+1$ handles and nonempty boundary, and it is a fundamental domain for $\mathscr{S}_z$, Fig. 3.14 (b). Therefore, $\mathscr{M}_{pq} = F_{pq}/\mathscr{S}_z$. The action of $\mathscr{S}_z$ on the boundary of F_{pq} implies the identification of boundary points and looks like a pasting of the holes of F_{pq}. As a result, we get the closed orientable surface $\mathscr{M}_{pq}$ of genus $\left[\frac{p-q}{2}\right]+1$. Taking in mind example 3.14, we see that the natural projection $M_{pq} \to M_{pq}/\mathscr{S}_z = \mathscr{M}_{pq}$ is a 2-sheeted branched covering map with the branched set $F_{pq} \cap Oz = \mathrm{Fix}(\mathscr{S}_z)$, consisting of $2q$ branched points each of branched order 2. $\diamondsuit$

Example 3.15 gives us the constructive proof of the following statement (for references, it is convenient to replace $\mathscr{M}$ with M, and vice versa).

Lemma 3.9 *Given any closed orientable surface M^2 and a finite set $\aleph \subset M^2$ consisting of $2q$ points ($q \geqslant 1$), there is the 2-sheeted branched covering map $\tau\colon \mathscr{M}^2 \to M^2$ with the branched set $\aleph = \Sigma_\tau$ each point of whose has* *the branched order* 2*. The covering surface* $\mathscr{M}^2$ *is a closed orientable surface of genus* $p = 2p_0 + q - 1$*, where* p_0 *is the genus of* M^2*. Moreover, there is the involution* $\theta\colon \mathscr{M}^2 \to \mathscr{M}^2$ *such that* $\operatorname{Fix}\theta = \tau^{-1}(\Sigma_\tau)$*,* $\mathscr{M}^2/\theta = M^2$*.*

Proof. Obviously, one can divide $\aleph$ into q pairs and connect the points in each pair by disjoint arcs. Cutting M^2 by these arcs, we get the surface F of genus p_0 with q holes. Now, embed F into $\mathbb{R}^3$ similarly F_{pq} of example 3.15 such that F belongs to the half-space $y \leqslant 0$ and the intersection $\partial F \cap \{y = 0\}$ is invariant under $\mathscr{S}_z$. Then one can easily to check that the union $F \cup \mathscr{S}_z(F)$ is $\mathscr{M}^2$, $\theta = \mathscr{S}_z$. $\square$

Covering and projecting local laminations

Let $\pi\colon \overline{M} \to M$ be a covering (branched or non-branched) map and $\mathscr{D}$ be a local lamination on M. The local lamination $\overline{\mathscr{D}}$ on $\overline{M}$ is called *covering* for $\mathscr{D}$ if π maps every leaf onto a leaf and $\pi(\operatorname{Sing}\overline{\mathscr{D}}) = \operatorname{Sing}\mathscr{D}$. As a consequence of this definition, one gets that if N is a saturated (i. e., invariant) set of $\mathscr{D}$ then $\pi^{-1}(N)$ is a saturated set of $\overline{\mathscr{D}}$, and vice versa, if $\overline{N}$ is a saturated set of $\overline{\mathscr{D}}$ then $\pi(N)$ is a saturated set of $\mathscr{D}$.

Lemma 3.10 *Let* $\mathscr{D}$ *be a local lamination on a* (*possibly, non-oriented or noncompact*) *surface* N^2 *and* $\widehat{\pi}\colon \widehat{M}^2 \to N^2$ *be a* (*possibly, universal or finitely-sheeted*) *non-branched covering map. Then for* $\mathscr{D}$*, there is the covering local lamination* $\widehat{\mathscr{D}}$ *on* $\widehat{M}^2$*. Moreover, if* $\mathscr{D}$ *is oriented then* $\widehat{\mathscr{D}}$ *is also oriented.*

Proof. Put by definition, any point of $\widehat{\pi}^{-1}(\operatorname{Sing}\mathscr{D})$ is a singularity. The existence of the structure of a (linear) local lamination in neighborhood of every other point is a consequence of the fact that every (non-branched) cover map is a local homeomorphism. Suppose that $\widehat{\mathscr{D}}$ is not oriented while $\mathscr{D}$ is oriented. Then there exist an arbitrarily small transversal segment Σ and Σ-arc $\widehat{ab}$ of $\widehat{\mathscr{D}}$ that intersects Σ non-orientably. Without loss of generality, one can assume that Σ belongs to a neighborhood U such that the restriction $\pi|_U$ is a homeomorphism. Hence, $\pi(\Sigma)$ is a transversal segment of $\mathscr{D}$, and $\pi(\widehat{ab})$ intersects $\pi(\Sigma)$ non-orientably. This contradicts the orientability of $\mathscr{D}$. $\square$

Lemma 3.11 *Let* $\widehat{\mathscr{D}}$ *be a local lamination on a* (*possibly, non-oriented or noncompact*) *surface* $\widehat{M}^2$ *and* $\widehat{\pi}\colon \widehat{M}^2 \to N^2$ *be a* (*possibly, universal or finitely-sheeted*) *non-branched covering map. If* $\widehat{\mathscr{D}}$ *is invariant under the covering group, then there exists the local lamination* $\mathscr{D}$ *on* N^2 *such that* $\widehat{\mathscr{D}}$

is a covering local lamination for $\mathcal{D}$. Moreover, if $\widehat{\mathcal{D}}$ is orientable and every covering map preserves orientation of leaves then $\mathcal{D}$ is also orientable.

Proof. The result follows directly from the fact that N^2 is the factor-space $\widehat{M}^2/\Gamma$, where Γ is the covering group of $\widehat{\pi}$. $\square$

The local lamination $\mathcal{D}$ denoted by $\pi(\widehat{\mathcal{D}})$ is called the *projection* of $\widehat{\mathcal{D}}$.

A covering surface $\widehat{M}^2$ can be endowed with the structure of smooth manifold such that the corresponding cover map $\widehat{\pi}\colon \widehat{M}^2 \to N^2$ preserves this structure (in particular, if N^2 and $\widehat{M}^2$ are analytic manifolds, $\widehat{\pi}$ is an analytic map) provided the covering group keeps this structure. Therefore, a cover local lamination has the same smoothness with the covering local lamination.

Lemma 3.12 *Let $\mathcal{D}$ be a local lamination on a non-oriented surface N^2 and $\widehat{\mathcal{D}}$ be the covering local lamination under two-sheeted (non-branched) cover $\widehat{\pi}\colon \widehat{M}^2 \to N^2$, where $\widehat{M}^2$ is an oriented surface. Suppose that l is a nonclosed semileaf of $\mathcal{D}$, and $\widehat{l}$ is a nonclosed semileaf of $\widehat{\mathcal{D}}$. Then*

1) *$\widehat{\pi}^{-1}(l)$ consists of two nonclosed leaves.*
2) *If $m \in N^2$ belongs to the limit set of l, then each point of $\widehat{\pi}^{-1}(m)$ belongs to the limit set of some leaf from $\widehat{\pi}^{-1}(l)$.*
3) *If l belongs to the limit set of l_*, then each semileaf of $\widehat{\pi}^{-1}(l)$ belongs to the limit set of some semileaf from $\widehat{\pi}^{-1}(l_*)$.*
4) *If $\widehat{l}$ is a nontrivially recurrent semileaf of $\widehat{\mathcal{D}}$, then $\widehat{\pi}(\widehat{l})$ is a nontrivially recurrent semileaf of $\mathcal{D}$.*
5) *If l is nontrivially recurrent semileaf of $\mathcal{D}$, then $\widehat{\pi}^{-1}(l)$ consists of two nontrivially recurrent semileaves of $\widehat{\mathcal{D}}$.*

Proof of items 1)–4) follows from the fact that a covering map is a local homeomorphism. Let's prove the last item. To be definite, assume that l^+ is a positive semileaf, l^+ endowed with the injective parametrization $t\colon [0; +\infty] \to l^+$. We already know that $\widehat{\pi}^{-1}(l^+)$ consists of two nonclosed semileaves $\widehat{l}_1^+$, $\widehat{l}_2^+$ endowed with the parametrizations $\widehat{t}\colon [0; +\infty] \to \widehat{l}_i^+$, $i = 1, 2$, induced by t. If one of the semileaves, say $\widehat{l}_1^+$, does not belong to the limit set of other semileaf, $\widehat{l}_2^+$, then $\widehat{l}_1^+$ is nontrivially recurrent because of item 2). Assume that $\widehat{l}_1^+ \subset \lim(\widehat{l}_2^+)$. Then there is a sequence of points $\widehat{l}_2^+(t_j)$ tending to some point of $\widehat{l}_1^+$. Since $\widehat{\pi}\colon \widehat{M}^2 \to N^2$ is two-sheeted, the covering group is isomorphic to $\mathbb{Z}_2$. Hence, the unique non-identity cover transformation is an involution. Therefore there is the involution $\theta\colon \widehat{M}^2 \to \widehat{M}^2$ that is a cover transformation such that $\theta(\widehat{l}_1^+) = \widehat{l}_2^+$ and $\theta(\widehat{l}_2^+) = \widehat{l}_2^+$. It implies that the sequence $\widehat{l}_1^+(t_j)$ tends to some point of $\widehat{l}_2^+$. It follows that $\widehat{l}_2^+ \subset \lim(\widehat{l}_1^+)$. Hence, the both $\widehat{l}_1^+$, $\widehat{l}_2^+$ are nontrivially recurrent. □

The following example shows that in general an oriented foliation can be a covering foliation for non-oriented one.

Example 3.16 *A non-oriented foliation that has the oriented covering one under non-branched* 2*-sheeted covering map.*

On the torus $\mathbb{T}^2$, consider the foliation $\mathscr{F}$ consisting of two Reeb foliations on a closed ring, see example 3.2. One can construct $\mathscr{F}$ as shown in fig. 3.15 so that it is invariant under the covering group of some 2-sheeted (non-branched) cover $\mathbb{T}^2 \to \mathbb{T}^2$.

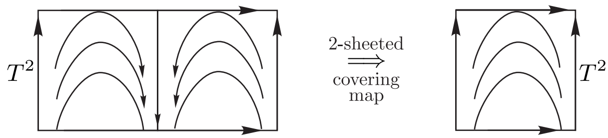


Figure 3.15. A non-oriented foliation can have oriented covering foliation.

One can check that the foliation $\mathscr{F}$ is oriented while its projecting on T^2 is not oriented. ♢

Now we consider a lifting and projecting of local laminations under branched covering maps.

Lemma 3.13 *Let* $\tau\colon \widehat{M}^2 \to M^2$ *be a branched covering map with the branched set* Σ_τ*. Let* $\mathscr{D}$ *be a local lamination on* M^2 *such that* $\Sigma_\tau \subset \operatorname{Sing}\mathscr{D}$*. Then for* $\mathscr{D}$*, there is the covering local lamination* $\widehat{\mathscr{D}}$ *on* $\widehat{M}^2$*. Moreover, if* $\mathscr{D}$ *is oriented then* $\widehat{\mathscr{D}}$ *is also oriented.*

Proof. The restriction $\tau|_{\widehat{M}^2-\tau^{-1}(\Sigma_\tau)}$ of τ on $\widehat{M}^2 - \tau^{-1}(\Sigma_\tau) = \mathscr{M}_0^2$ is a (non-branched) covering map. Since $\Sigma_\tau \subset \operatorname{Sing}\mathscr{D}$, the restriction of $\mathscr{D}$ on $M^2 - \Sigma_\tau$ is a local lamination. Hence, by Lemma 3.10, there is the covering local lamination $\widehat{\mathscr{D}}$ on $\mathscr{M}_0^2$. But $\mathscr{M}_0^2$ is a union of finitely many points. Therefore, one can extend $\widehat{\mathscr{D}}$ to $\widehat{M}^2$ by declaring any point of the set $\tau^{-1}(\Sigma_\tau)$ to be a singularity. Word for word with the proof of Lemma 3.10, one can prove the orientability of $\widehat{\mathscr{D}}$. □

Lemma 3.14 *Let $\widehat{\mathscr{D}}$ be a local lamination on a (possibly, non-oriented or noncompact) surface $\widehat{M}^2$ and $\widehat{\pi}\colon \widehat{M}^2 \to N^2$ be a (possibly, universal or finitely-sheeted) branched covering map. Suppose that $\widehat{\mathscr{D}}$ is invariant under the covering group and $\widehat{\pi}^{-1}(\Sigma_{\widehat{\pi}}) \subset \operatorname{Sing}\widehat{\mathscr{D}}$. Then there exists the local lamination $\mathscr{D}$ on N^2 such that $\widehat{\mathscr{D}}$ is a covering local lamination for $\mathscr{D}$. Moreover, if $\widehat{\mathscr{D}}$ is orientable and every covering map preserves orientation of leaves then $\mathscr{D}$ is also orientable.*

Proof. The result follows directly from the fact that N^2 is the factor-space $\widehat{M}^2/\Gamma$, where Γ is the covering group of $\widehat{\pi}$. □

Example 3.17 *Projecting and lifting of simplest foliations on* $\mathbb{R}^2$.

The system of differential equations $\dot{x} = x$, $\dot{y} = -y$ defines the orientable foliation $\mathscr{S}_0$ on $\mathbb{R}^2$ with unique saddle singularity at (0; 0). It is easy to check that $\mathscr{S}$ is invariant under the central symmetry $C\colon (x, y) \mapsto (-x, -y)$. Denote by $\Gamma(C)$ the group generated by C. Due to Example 3.14, the quotient space $\mathbb{R}^2/\Gamma(C)$ is $\mathbb{R}^2$ and the natural projection $\delta\colon \mathbb{R}^2 \to \mathbb{R}^2/\Gamma(C) \cong \mathbb{R}^2$ is a branched 2-sheeted projection. Since C keeps the orientation on the leaves of $\mathscr{S}_0$, $\delta(\mathscr{S}_0) = \mathscr{S}_1$ is an orientable foliation that is defined by the system of differential equations $\dot{x} = x^2 + y^2$, $\dot{y} = 0$. The foliation $\mathscr{S}_1$ has a fake saddle at (0; 0). One can check that $\mathscr{S}_1$ is invariant under C, but C does not keep the orientation on the leaves of $\mathscr{S}_1$. Therefore $\delta(\mathscr{S}_1) = \mathscr{S}_2$ is a non-orientable foliation, see Fig. 3.16. ♢

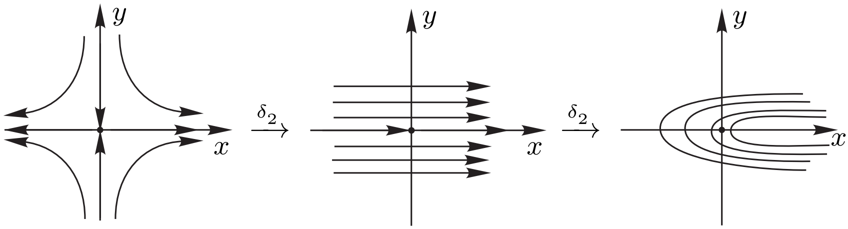


Figure 3.16. A non-oriented foliation can have oriented covering foliation.

Lemma 3.15 *Let $\mathscr{D}$ be a local lamination on an oriented surface M^2 and $\widehat{\mathscr{D}}$ be the cover local lamination under two-sheeted branched cover $\widehat{\pi}\colon \widehat{M}^2 \to M^2$,* *where $\widehat{M}^2$ is an oriented surface. Let l be a nonclosed semileaf of $\mathscr{D}$, and $\widehat{l}$ be a nonclosed semileaf of $\widehat{\mathscr{D}}$. Then*

1) *$\widehat{\pi}^{-1}(l)$ consists of two nonclosed leaves.*
2) *If $m \in N^2$ belongs to the limit set of l, then each point of $\widehat{\pi}^{-1}(m)$ belongs to the limit set of some leaf from $\widehat{\pi}^{-1}(l)$.*
3) *If l belongs to the limit set of l_*, then each semileaf of $\widehat{\pi}^{-1}(l)$ belongs to the limit set of some semileaf from $\widehat{\pi}^{-1}(l_*)$.*
4) *If $\widehat{l}$ is a nontrivially recurrent semileaf of $\widehat{\mathscr{D}}$, then $\widehat{\pi}(\widehat{l})$ is a nontrivially recurrent semileaf of $\mathscr{D}$.*
5) *If l is nontrivially recurrent semileaf of $\mathscr{D}$, then $\widehat{\pi}^{-1}(l)$ consists of two nontrivially recurrent semileaves of $\widehat{\mathscr{D}}$.*

Proof is similar to the proof of Lemma 3.12. We leave details to the reader. Note that a (one-dimensional) leaf of $\mathscr{D}$ does not pass through the branched set $\Sigma_{\widehat{\pi}}$. □

Example 3.18 *Irrational flows on hyperbolic surfaces.*

The simplest way to construct an irrational flow on a closed orientable hyperbolic surface M^2 is a lifting of minimal flow on the torus T^2 under some branched 2-sheeted covering map $M^2 \to T^2$. Take a minimal flow f^t_μ on T^2. If $M^2 = M^2_p$, a closed orientable surface of genus $p \geqslant 2$, choose a set $\Sigma \subset T^2$ consisting of $2q$ points such that any point of Σ belongs to exactly one trajectory of f^t_μ, where $q = p-1$ (recall that any hyperbolic surface is homeomorphic to some M^2_p, see Example 3.2). Put at each point of Σ a fake saddle (see Example 3.10) and denote by $f^t_{0,\mu}$ the flow which is obtained. By Lemma 3.9, there is a 2-sheeted branched covering map $\widetilde{\pi}\colon M^2_p \to T^2$ with the branched set $\Sigma = \Sigma_{\widetilde{\pi}}$. Lemma 3.13 implies that there is the covering flow $\overline{f}^t_{0,\mu}$ on M^2_p. One can prove that if $f^t_{0,\mu}$ is transitive then $\overline{f}^t_{0,\mu}$ is also transitive. Moreover, since the branched points belong to pairwise disjoint trajectories, $\overline{f}^t_{0,\mu}$ is irrational. Note that every fixed point of $\overline{f}^t_{0,\mu}$ is an ordinary saddle because of example 3.17. ◇

Example 3.19 *Irrational foliations on a sphere revised.*

There is another way to construct simplest irrational foliations on a sphere S^2, see example 3.9. It is not hard to verify that the quotient space $\mathbb{R}^2/\Gamma$, where Γ is a group of homeomorphisms of the form

$$x \mapsto (-1)^k x + r, \quad y \mapsto (-1)^k y + s, \quad k,\, r,\, s \in \mathbb{Z},$$

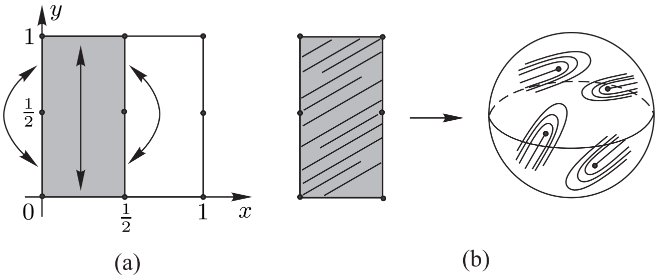


Figure 3.17. The simplest irrational foliation on the sphere.

is homeomorphic to the sphere S^2. Indeed, one can take as a fundamental domain of Γ the rectangle $0 \leqslant x \leqslant \frac{1}{2}$, $0 \leqslant y \leqslant 1$ whose sides are identified under the action of Γ as shown in Fig. 3.17, (a). Denote by $\widetilde{\pi}\colon \mathbb{R}^2 \to \mathbb{R}^2/\Gamma \simeq S^2$ the natural projection which is infinitely-sheeted branched covering map with critical points $\left(\frac{r}{2}, \frac{s}{2}\right)$, $r, s \in \mathbb{Z}$, each of multiplicity 1. The set $\Sigma_{\widetilde{\pi}}$ consists of four branched points $\widetilde{\pi}\left(\frac{r}{2}, \frac{s}{2}\right)$, $r, s \in \mathbb{Z}$.

Take the linear flow $\overline{f}^t_\mu$ on $\mathbb{R}^2$. Put a fake saddle at each critical point and denote by $\overline{f}^t_{0,\mu}$ the flow thus obtained. One can check that this flow is invariant under Γ. By Lemma 3.11, $\widetilde{\pi}(\overline{f}^t_{0,\mu})$ is a foliation $F_\mu(S^2)$, a *linear foliation on the sphere*. Obviously, $\operatorname{Sing} F_\mu(S^2) = \Sigma_{\widetilde{\pi}}$, and every singularity is a thorn. Note that $F_\mu(S^2)$ is non-oriented foliation. If μ is irrational, $F_\mu(S^2)$ is an irrational foliation. If μ is rational, every leaf is closed except two separatrix connections that coincide with thorns. ♢

Blow up and blow down operations

To construct more exotic examples of local laminations, we describe the operation of blowing up a nonclosed leaf or semileaf, and its inverse blowing down operation. First, these operations were used by Poincaré [192] for the construction of a circle nontransitive homeomorphism without periodic points. Poincaré took a transitive circle homeomorphism (for example, a rotation through an angle incommensurable with π) and "stretched" each point of some orbit into an interval so that the total length of all "inserted" intervals was finite. Thus, instead of the orbit of a point, he obtained an orbit of closed intervals. This operation can be described in more formal terms using a special Cantor function that maps each "inserted" interval to a point. Actually, this Cantor mapping is a blowing down operation, and the inverse one is a blowing up operation. On the part of the circle beyond blowing up, Poincaré retained the "old" transitive homeomorphism and then extended the mapping to the homeomorphism of the whole "blown-up" circle. As a result, he obtained a nontransitive circle homeomorphism without periodic points. In this case, the blowing down operation realizes the semiconjugacy between nontransitive and transitive homeomorphisms.

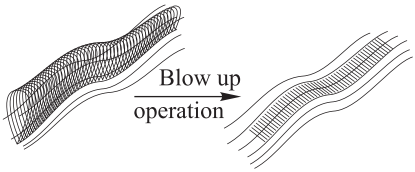


Figure 3.18. Blowing up a leaf.

This idea of Poincaré can be applied to foliations. On a surface M, consider a foliation $\mathscr{F}$ and choose a leaf l of $\mathscr{F}$. Let S be a strip of finite area, for example,

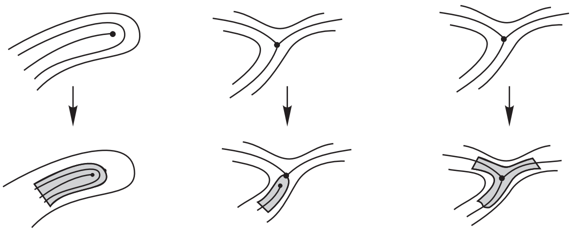
$$S = \left\{(x,\, y) \in \mathbb{R}^2 \colon 0 \leqslant y \leqslant \frac{1}{x^2+1}\right\}.$$

There exists an obvious homeomorphism $\varphi\colon l \to \operatorname{axis} Ox = \{(x,\, 0)\colon x \in \in \mathbb{R}\} \subset S$ (in the intrinsic topology of l). Let us identify the points $(x,\, 0)$ and $\left(x,\, \frac{1}{x^2+1}\right)$ of S and attach the cylinder C thus obtained to l by φ. Roughly speaking, we lay a "pipeline" along l on M. Now, we cut $M \cup_\varphi C$ along the leaf l so that, instead of a point $m \in l$, we obtain the segment $\left\{(\varphi(m),\, y)\colon 0 \leqslant y \leqslant \frac{1}{x^2+1}\right\}$. Since S has finite area and is simply connected, we obtain a surface $M(l)$ that is homeomorphic to the original surface M, Fig. 3.18. There is a family of curves and singularities on the surface $M(l) - S$, which is extended to a foliation on $M(l)$ by the curves $\frac{\alpha}{x^2+1}$, $0 \leqslant \alpha \leqslant 1$, on S. Denote the obtained foliation by $\mathscr{F}(l)$. The above operation is called a *blowing up the leaf* l.

Example 3.20 *Topological Denjoy flows on the torus* $\mathbb{T}^2$ *revised.*

Take a minimal flow f^t on T^2, and choose some finite or countable family F of trajectories of f^t. After the operation of blowing up the every trajectory of F, one gets a topological Denjoy flow on $\mathbb{T}^2$. $\diamondsuit$

Let us return to the blowing up a leaf l defined above. By construction, $\mathrm{Sing}(\mathscr{F}) = \mathrm{Sing}(\mathscr{F}(l))$. The interior of the strip S is a simply connected domain $C^\circ(l)$, which is called an *open cell* of the foliation $\mathscr{F}(l)$. The boundary of the cell $C^\circ(l)$ that is accessible from $C^\circ(l)$ consists of two leaves l_1 and l_2. The strip $C(l) = C^\circ(l) \cup l_1 \cup l_2$ is called a *closed cell* of the foliation $\mathscr{F}(l)$. Obviously, there exists a continuous map $h\colon M(l) \to M$ that takes the leaves of $\mathscr{F}(l)$ to the leaves of $\mathscr{F}$. The mapping h sends the closed cell $C(l)$ to the leaf l; on the complement $M(l) - C(l)$ to $C(l)$, h is one-to-one. This mapping h is called the *blowing down* operation.



(a) (b) (c)

Figure 3.19. Operations of blowing up a separatrix of a thorn, a separatrix of a saddle, and a saddle with separatrices.

One can define similarly a blowing up a separatrix of thorn (Fig. 3.19a), a separatrix of saddle (Fig. 3.19b), and even a saddle with all of its separatrices (Fig. 3.19c). However, when extending a foliation, one should sometimes add one singularity, a thorn, in order that the obtained family of curves forms a foliation. Finally, the blowing up a leaf or a semileaf that is transversal to the boundary of surface is also defined similarly. Note that when blowing up a separatrix of thorn, one formally obtains a thorn and a fake saddle. However, it is clear that the fake saddle can be removed, and no extra singularity arises under blowing up.

When blowing up a complex singularity and separatrices, the singularity may vanish. For example, this case when blowing up a saddle–node and its separatrices, as is shown in Fig. 3.20. It is clear that the operation of blowing up can be defined for a finite, and even a countable, family of leaves. Note that a foliation can be extended into the arising cells in different ways. Therefore, in specific cases, one needs additional information for the operations of blowing up.

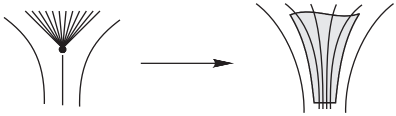


Figure 3.20. Vanishing of a complex singularity.

Example 3.21 *Topological Cherry type foliations and flows on the torus* $\mathbb{T}^2$.

Take a transitive flow f_0^t on T^2, that is obtained from a minimal flow by putting a finitely many fake saddles $O_1, \ldots, O_k$ on disjoint trajectories, see example 1.3. Choose one of two separatrices of O_i for each $i = 1, \ldots, k$. Denote by F_0 the union of all O_i and separatrices which we choose. After the blowing up operation of F_0, one gets a topological Cherry type foliation denoted by $\mathscr{F}_{Ch}$ on T^2. $\diamondsuit$

Black holes and flows on S^2 with Cantor type limit sets

Consider an irrational flow $\mathscr{F}_\alpha$ on a disk D, where α is an irrational number. First, we extend the foliation $\mathscr{F}_\alpha$ to a larger disk $D_1 \supset D$ so as to obtain a foliation $\mathscr{F}'_\alpha$ that is transversal to the boundary ∂D_1 and has an additional singularity, a tripod t_0. At the point α, the foliation $\mathscr{F}'_\alpha$ has a saddle–node. Now, we blow up this saddle–node and its separatrix so that the saddle–node vanishes (Fig. 3.21). Denote the obtained foliation by $\mathscr{H}_\alpha$. This foliation has no compact leaves and separatrix connections. Note that $\mathscr{H}_\alpha$, just as $\mathscr{F}_\alpha$, has three singularities: two thorns and one tripod. Entering the disk D_1 and moving along a leaf, we never leave D_1 (one either reaches the singularity t_0 or is doomed to "perpetual wandering between needles" on a nowhere dense subset of D_1). Due to this property, $\mathscr{H}_\alpha$ is called a *black hole on a disk* or a *black-hole type foliation on a disk.*

Similarly, black-hole type foliations $\mathscr{H}(\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q)$ can be constructed from the transitive foliations $\mathscr{F}(\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q)$.

Let us glue together two copies of the disk D_1 by a certain homeomorphism of the circle $S^1 = \partial D_1$ so that no separatrix connection arises between the tripods. Gluing together two disks along the boundary, we obtain a sphere S^2. The foliations $\mathscr{H}_\alpha$ form a foliation on the sphere, which we denote by $\mathscr{H}_{\text{hole}}(S^2)$. The set of singularities of this foliation consists of six thorns and two tripods. Any leaf different from a separatrix has a Cantor-type limit set both in the positive and negative directions.

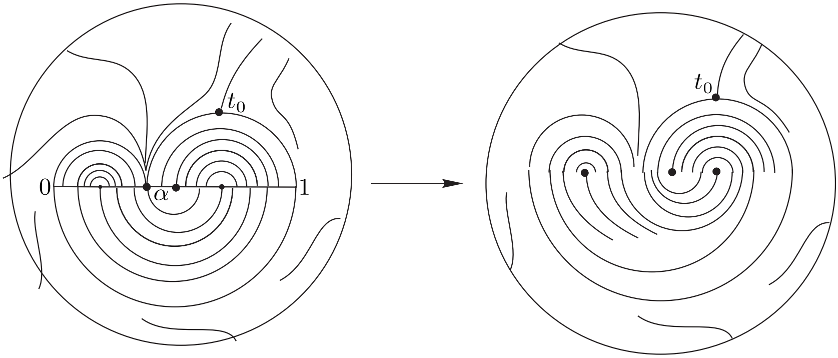


Figure 3.21. Formation of a black hole.

Let us declare that any point of any separatrix is a singularity. Denote the obtained foliation by $\mathscr{H}^*_{\mathrm{hole}}(S^2)$. One can show that the foliation $\mathscr{H}^*_{\mathrm{hole}}(S^2)$ is orientable [25]; i. e., it can be embedded into a flow f^t_{h}. This flow has two Cantor-type limit sets: one repeller and one attractor. In a sense, f^t_{h} has a strange attractor and a strange repeller.

Rosenberg labyrinths

Take the simplest foliation $\mathscr{F}_0$ on the half-disk

$$D^+ = \{(x,\, y) \in \mathbb{R}^2 \colon x^2 + y^2 \leqslant 1,\ y \geqslant 0\}.$$

Let us place a singularity s_0, a fake saddle, on some semicircle $l_0 = \{x^2 + y^2 = c_0\}$, $0 < c_0 \leqslant 1$. This singularity divides the leaf l_0 into two leaves (which were formerly semileaves). Apply the operation of blowing up to one of these leaves so that exactly one singularity, a thorn, arises in the open cell of the obtained foliation $\mathscr{F}'_0$. Following [200], we will call such blowing up the leaf l_0 the *spreading* of l_0. Accordingly, we will say that the *foliation* $\mathscr{F}'_0$ *is obtained* from $\mathscr{F}_0$ *by spreading the leaf* l_0, Fig. 3.22.

A foliation is called a *standard foliation on a half-disk* D^+ if it is obtained from the simplest foliation by successive spreadings of a finite number of leaves.

Now, for numbers $\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q$, $0 < \alpha_i < 1$, $0 < \beta_i < 1$, we construct a foliation

$$\mathscr{F}'(\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q)$$

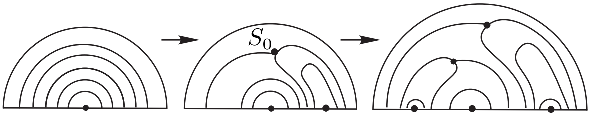


Figure 3.22. Standard foliation on a half-disk D^+.

analogous to $\mathscr{F}(\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q)$ from example 3.8, see Fig. 3.10. Instead of half-disks with simplest foliations, we glue half-disks with standard foliations ($p+1$ half-disks from above and $q+1$ from below) to the interval $I = [0; 1]$. Let us extend the foliation $\mathscr{F}'(\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q)$ to some neighborhood so that saddle–node-type singularities arise at the points $\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q$. Applying the blowing up for all saddle–nodes and their separatrices one gets the foliation $\mathscr{F}_1(\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q)$. This foliation is called a *Rosenberg labyrinth on the segment I*. The foliations $\mathscr{H}(\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q)$ from the preceding subsection are particular cases of the Rosenberg labyrinths. See also [135] about labyrinths.

Denjoy and Cherry foliations

A *Denjoy foliation* is a nontransitive foliation that is topologically equivalent to a foliation obtained from $\mathscr{F}_\mu$ by the operation of blowing up a finite or countable set of leaves. In this case, a standard foliation without singularities is defined in each strip that is obtained in place of the blown up leaf. Note that one can not define a Reeb foliation in these strips because, as a result, one obtains a family of curves on the torus that do not form a foliation. A Denjoy foliation has a unique minimal set, which is nowhere dense and is locally homeomorphic to the product of a segment and a Cantor set. A Denjoy foliation is orientable and is embedded into a flow, called a *Denjoy flow*.

A Denjoy flow (just as a Denjoy foliation) can be defined as a nontransitive flow on the torus without fixed points or periodic orbits (for foliations — without singularities or compact leaves). Although such a flow was known to Poincaré [192], it is named in honor of Denjoy, who constructed such a flow of smoothness class C^1 and proved that there do not exist such flows of smoothness class C^r for $r \geqslant 2$ [66]. Note that Poincaré assumed that there even exists an analytic nontransitive flow without fixed points or periodic orbits on the torus. Denjoy actually refuted this conjecture of Poincaré.

Take a finite family of pairwise different leaves $\{l_\nu\}$ of an irrational foliation $\mathscr{F}_\mu$. If we apply the operation of spreading to the semileaves of the leaves $\{l_\nu\}$, we obtain a nontransitive foliation, which we denote by $\mathscr{F}_{\mathrm{Ch},\mu}$. A foliation that is topologically equivalent to the foliation $\mathscr{F}_{\mathrm{Ch},\mu}$ is called a *Cherry-type foliation on the torus*.

If we place exactly one node-type singularity into each cell obtained upon spreading the semileaves of the leaves $\{l_\nu\}$ and make the starting point of each of the above semileaves of the leaves l_ν into a saddle with four separatrices, then the obtained foliation $\mathscr{F}_{\mu,\mathrm{Ch}}$ (and any foliation that is topologically equivalent to $\mathscr{F}_{\mu,\mathrm{Ch}}$) is called a *Cherry foliation on the torus*. Note that each saddle has exactly one separatrix that necessarily goes to the corresponding node.

A Cherry foliation $\mathscr{F}_{\mu,\mathrm{Ch}}$ is orientable and is embedded into a flow, which is called a *Cherry flow on the torus*. Such a flow of analytic class of smoothness with one cell was constructed by Cherry [65]. Thus, Cherry proved the (weakened) conjecture of Poincaré on the existence of an analytic nontransitive flow on the torus without periodic trajectories.

3.3. Limit sets of a curve and leaf

By a *semi-infinite curve* l^+, we mean the image of the positive half-line $\mathbb{R}^+ = [0;\, +\infty)$ with respect to a locally homeomorphic mapping (parametrization) $m\colon \mathbb{R}^+ \to M$, $l^+ = \{m(t)\colon t \geqslant 0\}$. Since the parameter t takes values in $[0;\, +\infty)$, l^+ is also called an *infinite curve in the positive direction* and is denoted by $l^+(m(0))$, where $m(0)$ is the starting point of l^+.

Definition 3.5 *A limit set* $\lim(l^+)$ *of a semi-infinite curve* l^+ *is the set of points of* M *such that any neighborhood of each of these points contains points of* l^+ *with indefinitely large parameters.*

In other words,

$$\lim l^+ = \big\{x \in M\colon \exists\, t_i \in \mathbb{R}^+ \text{ such that } t_i \to +\infty \\ \text{and } m(t_i) \to x \text{ as } i \to \infty\big\}.$$

There is an equivalent definition of the limit set that can be extended to multidimensional leaves. For an arbitrary monotonically increasing sequence $t_k \in \mathbb{R}^+$, we set

$$\lim l^+ = \bigcap_{k \geqslant 0} \operatorname{clos}\big(l^+ - m([0;\, t_k])\big).$$

One can show that this definition of the limit set is independent of a sequence t_k and coincides with the original Definition 3.5.

Lemma 3.16 $\lim l^+$ *is a closed set. If* M *is compact,* $\lim l^+$ *is connected and nonempty.*

Proof. The closedness of $\lim l^+$ follows immediately from its definition. If we assume that $\lim l^+$ is not connected in the case of a compact M, then there exist open (or closed) disjoint sets A and B such that $\lim(l^+) \subset A \cup B$. Since the semi-infinite curve passes from A to B and vice versa, l^+ must have limit points outside $A \cup B$, which contradicts the inclusion $\lim l^+ \subset A \cup B$. Since M is compact, any sequence contains a converging subsequence. This implies $\lim l^+ \neq \varnothing$. □

One can easily construct an example of a disconnected limit set for a semi-trajectory of a smooth flow in the case of a noncompact M. For example, one can take a smooth vector field in the strip $-\infty < x < +\infty$, $0 \leqslant y \leqslant 1$ with one focus-type singularity, such that the ω-limit set of a trajectory inside the strip consists of two straight lines $y = 0$ and $y = 1$, Fig. 3.23.

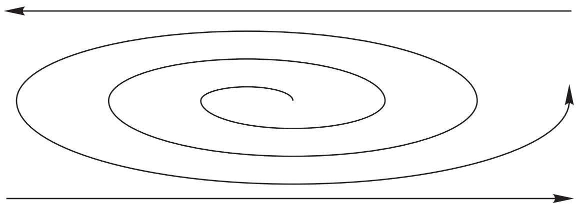


Figure 3.23. The example of disconnected limit set.

For a semi-infinite curve l^- endowed with a negative parametrization $m\colon (-\infty; 0] \to M$, $l^- = \{m(t)\colon t \leqslant 0\}$, the concept of a limit set is similar (with $t_i \to +\infty$ being replaced by $t_i \to -\infty$).

Let l be an infinite curve with a parametrization $m\colon \mathbb{R} \to M$. Recall that a point $m(0)$ divides l into two semi-infinite curves l^+ and l^-. The sets

$$\lim(l^+) \stackrel{\text{def}}{=} \omega(l^+) \quad \text{and} \quad \lim(l^-) \stackrel{\text{def}}{=} \alpha(l^-)$$

are called ω- and α-*limit sets* of l^+ and l^- respectively. Since the parameters t_i in the definition of limit sets go to $\pm\infty$, the sets $\omega(l^+)$ and $\alpha(l^-)$ are independent of the choice of the starting point $m(0)$ and a specific parametrization.

Therefore, one can call them ω- and *α-limit sets of the curve l*, denoting $\omega(l) = \omega(l^+)$ and $\alpha(l) = \alpha(l^-)$.

Let l be a nonclosed (one-dimensional) leaf of local lamination. Then l is homeomorphic to $\mathbb{R}$. An arbitrary point $m_0 \in l$ divides l into the components l_1 and l_2; each of these, augmented with the point m_0, is called a *semileaf* with the starting point m_0. On any semileaf, one can introduce an injective parametrization. Usually, one of the semileaves, say l_1, is endowed with a parametrization $[0; +\infty) \to l_1$, $l_1 = l^+(m_0)$, and such a semileaf is called *positive*, while the second semileaf $l_2 = l^-(m_0)$ is endowed with a parametrization $(-\infty; 0] \to l_2$ and is called *negative*. If l is a one-dimensional trajectory of a certain flow, then the parametrization of the semileaves coincides with the natural "time" parametrization, and a positive (negative) semileaf represents a positive (respectively, negative) semitrajectory.

The proof of the following lemma we leave to the reader as exercise (see Corollary 3.1 and Lemma 3.16).

Lemma 3.17 *Let l be a leaf of local lamination $\mathscr{D}$, and l^+, l^- be the positive and negative semileaves of l respectively. Then the both $\omega(l^+)$ and $\alpha(l^-)$ are closed and saturated. If $\mathscr{D}$ has a compact support, the both $\omega(l^+)$ and $\alpha(l^-)$ are nonempty and connected. Moreover, in any case*

$$\omega(l^+) \subset \omega(l^+) \cup l^+ = \operatorname{clos} l^+, \quad \alpha(l^-) \subset \alpha(l^-) \cup l^- = \operatorname{clos} l^-. \tag{3.9}$$

For a closed curve l, we suppose that l is endowed with a periodic parametrization. In this case, any point $m \in l$ corresponds a two-sided sequence of parameters $\{t_i\}_{i=-\infty}^{\infty}$ such that $\lim\limits_{i \to \pm\infty} t_i = \pm\infty$. Therefore,

$$\omega(l) = \alpha(l) = l$$

provided l is a closed curve. Between various types of limit sets, periodic trajectories and closed leaves occupy a special place.

Example 3.22 *A periodic trajectory as a limit set.*

Let l_0 be a periodic trajectory of a flow f^t. Suppose that the ω-limit set of a trajectory l contains l_0. Then $\omega(l) = l_0$. Indeed, take a transversal segment Σ through a point $m_0 \in l_0$. Then m_0 belongs to a domain $\operatorname{Dom} P_\Sigma^+$ of Poincaré forward mapping P_Σ^+ induced by f^t. To simplify matters, we shall assume that l_0 has a cylinder neighborhood U divided by l_0 into two annuluses, say A_1

and A_2 (in the case when l_0 has a neighborhood homeomorphic to a Möbius band, the proof is similar or can be reduced to our case by passing to a double cover). The point m_0 divides Σ into two segments Σ_1 and Σ_2. Without loss of generality, one can assume that $l \cap A_1 \neq \varnothing$, $\Sigma \subset U$, and $\Sigma_1 \subset A_1$. Since $l_0 \subset \omega(l)$, there is a sequence $m_k \in l \cap \Sigma_1$ such that $m_k \to m_0$ as $k \to \infty$ and the points m_k correspond to the increasing parameters $t_k \to \infty$, $m_k = l(t_k)$.

Take a Σ-arc $m_k P^+(m_k)$ of l. By Theorem 3.2 on a continuous depending of leaves on the initial conditions, the corresponding Σ-loop l_k is homotopy to l_0 for sufficiently large k. We shall assume that this is a case for all $k \geqslant 1$. Then l_k and l_0 bound the annulus A_k. There are two possibilities: 1) $P^+(m_k)$ is between m_k and m_0, Fig. 3.24 (a); 2) m_k is between m_0 and $P^+(m_k)$, Fig. 3.24 (b).

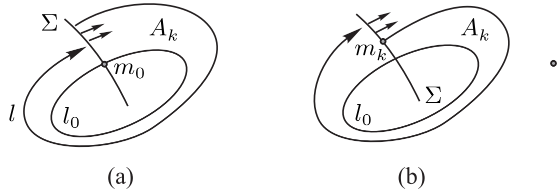


Figure 3.24

In case 2), the vector field induced by f^t does not allow to enter in A_k for l after $P^+(m_k)$. This contradicts to the tending $m_k \to m_0$ as $k \to \infty$. Therefore, case 1) holds. The positive semitrajectory $l^+(P^+(m_k))$ can not intersect Σ between m_k and $P^+(m_k)$. Hence, the points $P^+(m_1)$, $P^+(P^+(m_1))$, $P^+(P^+(P^+(m_1)))$, ... form a monotone sequence tending to m_0. It follows that l tends spirally to l_0, and $\omega(l) = l_0$. $\diamondsuit$

Obviously, one can prove by a similar way that if a periodic trajectory l_0 belongs to the α-limit set $\alpha(l)$ of a trajectory l, then $\alpha(l) = l_0$.

The notion of limit set allows us to give the definition of recurrentness for leaves of local laminations. We give this definition in maximally general form for a curve.

Definition 3.6 *A semi-infinite curve l^+ (respectively, l^-) is said to be* ***nontrivially*** $\omega(\alpha)$-***recurrent*** *if it belongs to its own ω-limit set (respectively, α-limit set),*

$$l^+ \subset \omega(l^+) \quad (\textit{respectively}, \ l^- \subset \alpha(l^-)).$$

*An infinite curve l is said to be **nontrivially recurrent** if both of its semi-infinite curves l^+ and l^- are nontrivially ω- and α-recurrent.*

It is easy to check that if a some positive (negative) leaf of l is nontrivially $\omega(\alpha)$-recurrent, then any positive (negative) semileaf of l is also nontrivially $\omega(\alpha)$-recurrent. Thus, we can say that the leaf itself is nontrivially $\omega(\alpha)$-recurrent. Sometimes one says about nontrivial P^+ and P^- recurrentness [168].

For a simple semi-infinite curve l, the nontrivial recurrentness is equivalent to the fact that the topology induced on $l \subset M^2$ as a subset of M^2 and the interior topology on l do not coincide. Clearly that a positive leaf could be nontrivially recurrent only in the positive direction, i. e. nontrivially ω-recurrent. The similar proposition holds for a negative semileaf. So, we'll often omit the indication on a direction, i. e. ω- or α-. Note that a nontrivially ω-recurrent leaf could be not α-recurrent, and vice versa. For example, a separatrix of a saddle type singularity of irrational foliation.

As a consequence of definitions 3.6 and 3.9, we have

Corollary 3.3 *Let l be a leaf of local lamination $\mathscr{D}$, and l^+, l^- be the positive and negative semileaves of l respectively. Then l^+ (l^-) is nontrivially $\omega(\alpha)$-recurrent if and only if*

$$\operatorname{clos} l^+ = \omega(l^+) \quad (\textit{respectively},\ \operatorname{clos} l^- = \alpha(l^-)).$$

Obviously, any leaf of highly transitive foliation is nontrivially recurrent in at least one direction, i. e. given any leaf, at least one of its semileaves is nontrivially recurrent.

3.4. Orientable and non-orientable local laminations

In Section 3.1, we've defined an orientability for a foliation. Here we extend this concept to a local lamination. By an *arc*, we mean an embedded interval (a homeomorphic image of an interval). Depending on whether this interval is an open or a closed segment, we obtain the definitions of *open* or *closed arcs*, respectively.

Definition 3.7 *Curves (or arcs) A_1 and A_2 on M^2 intersect **transversally** at a point $a \in A_1 \cap A_2$, or, which is the same, a is a **point of single intersection**, if $A_1 \cap A_2$ at the point a is locally homeomorphic to the intersection of the coordinate axes of the Euclidean plane at the origin.*

Let A be an open arc on a (possibly non-orientable) surface M^2. It is well known that A has a neighborhood that is homeomorphic to an open disk which divided by A into two components. Therefore, A is naturally equipped with a *normal orientation*. The components adjoining A define the two-sidedness of the arc A: one component defines a positive side and the other a negative one. For a closed arc, the normal orientation is introduced by means of a slightly larger open arc.

Let l be an infinite curve that intersects the arc A transversally at some point $a \in l \cap A$. Introduce an orientation on l defined by the parametrization: the *positive direction* is defined as the direction corresponding to the increasing parameter.

Definition 3.8 *The* ***intersection index*** $\mathrm{ind}_a(l \cap A)$ *equals to* $+1$ *if the orientation of* l *at the point* a *is consistent with the normal orientation of* A*, i. e., if* l *passes from the negative side of* A *to its positive side at the point* a*. Otherwise, we set* $\mathrm{ind}_a(l \cap A) = -1$*. We say that* l *intersects* A ***orientably*** *if the intersection index is the same* (*either* $+1$ *or* -1) *at all points of* $l \cap A$*.*[1]

Let now C be a simple closed curve on M^2. Suppose that C has a tubular neighborhood $U(C)$ homeomorphic to a ring. Then C divides $U(C)$ into two components. Similar to the previous case with an arc, C is equipped with a some normal orientation: one component defines a positive side and the other, negative one. In the case of a smooth C, its normal orientation can be defined by choosing (and fixing) a continuous field of normals. Now, by analogy with the previous case, we can introduce the index of intersection of the curves C and l at any point of their transversal intersection. We say that l intersects C *orientably* if the intersection index is the same (either $+1$ or -1) at any point of $l \cap C$, Fig. 3.25, (b).

(a) (b)

Figure 3.25. (a) a leaf intersects an arc orientably; (b) a leaf intersects an arc non-orientably.

[1]Thus, we assume that l intersects A transversally at any point.

Let $\mathscr{D}$ be a local lamination on M^2. Introduce some orientation on the leaves of $\mathscr{D}$. This orientation is called *consistent* if, for any leaf l and any point $m \in l$, there exists a transversal segment Σ such that m belongs to the interior of Σ and all leaves intersecting Σ intersect it with the same intersection index.

Definition 3.9 *A local lamination is called* ***orientable*** *if one can introduce a consistent orientation on its leaves. Otherwise, a local lamination is called* ***non-orientable****.*

A non-orientable lamination has a leaf that intersects Σ with different intersection indices, Fig. 3.25, (a).

If a local lamination is an oriented foliation by Definition 3.2 i. e., a flow, then the time-parameter induces naturally the consistent orientation on the leaves. Therefore, an oriented foliation is an oriented local lamination by Definition 3.9. According to [218], if a foliation is orientable by Definition 3.9, then the foliation can be embedded in a flow and, as a consequence, is orientable by Definition 3.2.

Now we consider the particular case of local lamination, a minimal lamination.

Definition 3.10 *A lamination is called* ***minimal*** *if it contains no proper non-empty sub-laminations.*

A unique closed leaf gives us an example of a *trivial minimal lamination*. Other minimal laminations are *nontrivial*. Since a closed leaf forms a sub-lamination, a nontrivial minimal lamination consists of non-closed leaves.

Lemma 3.18 *Let $\mathscr{L}$ be a minimal lamination. Then every leaf of $\mathscr{L}$ is dense in $\mathscr{L}$.*

Proof. Take a leaf l of $\mathscr{L}$. Since a support of lamination is a closed set, $\operatorname{clos}(l) \subset \operatorname{supp} \mathscr{L}$. It follows from corollary 3.1 that the closure $\operatorname{clos}(l)$ consists of leaves of $\mathscr{L}$. Hence, $\operatorname{clos}(l)$ is a lamination. By definition of minimality, $\mathscr{L} = \operatorname{clos}(l)$. $\square$

Now we consider a property of non-orientable minimal laminations we need below. As we saw, an orientability of a lamination means that each leaf can be equipped with an orientation so that, for any point of an arbitrary leaf, there exists a transversal segment that passes through this point and is intersected by the leaves in the same direction. Therefore, the non-orientability of a lamination implies that, irrespective of the orientation of the leaves, there always exists a point (which lies in a certain leaf since the support of the lamination is closed) such that an arbitrary transversal segment passing through this point is intersected by certain leaves in opposite directions. Note that non-orientable minimal lamination is always nontrivial.

Lemma 3.19 *Let $\mathscr{L}$ be a non-orientable minimal lamination and Σ be a transversal segment such that* $\operatorname{int}\Sigma \cap \operatorname{supp}\mathscr{L} \neq \varnothing$. *Then, any leaf $l \in \mathscr{L}$ that is endowed with some orientation intersects Σ in opposite directions.*

Proof. Since, by Lemma 3.18, any leaf is dense in $\mathscr{L}$, it suffices to show that there exists at least one leaf that intersects Σ in opposite directions. Suppose that all leaves that intersect Σ intersect it in the same direction. Introduce an orientation on these leaves assuming that the positive direction corresponds to the positive direction of the transversal orientation of Σ. This orientation introduces the orientation on each leaf from $\mathscr{L}$ because, due to Lemma 3.18, all leaves intersect Σ. Since the lamination $\mathscr{L}$ is non-orientable, there exists the leaf $l_* \in \mathscr{L}$ through a point $x_* \in l_*$ such that any transversal segment Σ_0 through x_* is intersected by some leaf in opposite directions. The leaf l_* intersects Σ, Fig. 3.26. Theorem 3.2 on the continuous dependence of leaves on the initial conditions implies that Σ should also be intersected by certain leaf from $\mathscr{L}$ in opposite directions. The obtained contradiction proves the lemma. □

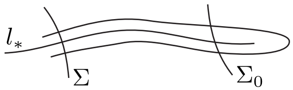


Figure 3.26. Some leaf must intersect Σ in opposite directions.

The following lemma says that a nontrivially recurrent leaf intersects a transversal segment infinitely many times with the same intersection index. Note that a local lamination as well as a surface could be orientable or non-orientable.

Lemma 3.20 *Let $\mathscr{D}$ be a local lamination on a surface M^2 and Σ be a transversal segment. Suppose that a nontrivially recurrent leaf $l = \{l(t)\colon t \in (-\infty; +\infty)\}$ intersects Σ at an internal point $l(t_0)$. Suppose that Σ is equipped with a normal orientation and the intersection index of Σ with l*

at $l(t_0)$ is equal to $+1$. Then, for any $T > 0$, there exist parameters $t_{-1} < -T$ and $t_1 > T$ such that the intersection index of Σ with l at the points $l(t_{-1})$ and $l(t_1)$ also equals $+1$.

Proof. Denote by $l^+(t)$ (respectively, by $l^-(t)$) the positive (respectively, negative) semileaf of the leaf l with the starting point $l(t)$ that corresponds to the parameter t. It is necessary to prove that each of the semileaves $l^+(T)$ and $l^-(-T)$ intersects Σ at least at one point with the intersection index $+1$. Naturally, we may assume that $-T < t_0 < T$. First, we prove the required assertion for one of the semileaves. Suppose the contrary; i. e., let both semileaves $l^+(T)$ and $l^-(-T)$ intersect Σ with the intersection index -1. Then, $l^-(t_0)$ intersects Σ at least at one point, say, at x_0, with the intersection index -1. There exists a foliated box B around the arc $[x_0; l(t_0)] \subset l$ with two opposite sides S_1 and S_2 lying on Σ. Since $l^+(T)$ contains the whole leaf l in its limit set, $l^+(T)$ intersects S_1 and S_2. The fact that the leaf l intersects Σ at the points $l(t_0)$ and x_0 with opposite intersection indices implies that the box B adjoins Σ from the negative side, Fig. 3.27. Therefore, by assumption, the semileaf $l^+(T)$ enters B as the parameter increases. But then it must leave B and intersect Σ with a positive intersection index. The obtained contradiction proves that one of the semileaves intersects Σ with the intersection index $+1$ at least at one point.

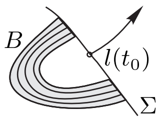


Figure 3.27. The foliated box B.

Note that the following fact can be proved similarly: if there is a finite arc of the leaf l that intersects Σ at its endpoints with opposite intersection indices and adjoins Σ at the endpoints from the negative side, then $l^+(T)$ intersects Σ with the intersection index $+1$ at least at one point. Similarly, if there is a finite arc of the leaf l that intersects Σ at its endpoints with opposite intersection indices and adjoins Σ at the endpoints from the positive side, then $l^-(-T)$ intersects Σ with the intersection index $+1$ at least at one point.

Consider the case when the semileaf $l^+(T)$ intersects Σ with the intersection index $+1$ at least at one point. If $l^+(T)$ intersects Σ with the intersection index -1 at least at one point, then there exists a finite arc of l that intersects Σ at its endpoints with opposite intersection indices and adjoins Σ at the endpoints from the positive side. Then, $l^-(-T)$ intersects Σ with the intersection index $+1$ at least at one point, and the lemma is proved. Suppose that $l^+(T)$ intersects Σ only with the intersection index $+1$. If $l^-(-T)$ intersects Σ only with the intersection index -1, then there exists a finite arc $[x_0; l(t_0)] \subset l$ of the type considered above; hence, there exists an intersection of $l^-(-T)$ with Σ with the index $+1$. But this result contradicts the assumption.

It remains to consider the case when the semileaf $l^-(-T)$ intersects Σ with the intersection index $+1$ at least at one point. If $l^-(-T)$ intersects Σ at least at one point with the intersection index -1, then there exists a finite arc of the leaf l that intersects Σ at its endpoints with opposite intersection indices and adjoins Σ at the endpoints from the negative side. Then, $l^+(T)$ intersects Σ with the intersection index $+1$ at least at one point, and the lemma is proved. Suppose that $l^-(-T)$ intersects Σ only with the intersection index $+1$ and $l^+(T)$ intersects Σ only with the intersection index -1. Then there exists a finite arc $[l(t_0); l(t_*)] \subset l$, $t_* > t_0$, that intersects Σ at its endpoints with opposite intersection indices and adjoins Σ at its endpoints from the positive side. Let us take, around this arc, a foliated box B_* so small that the point $l(-T)$ does not lie in B_*. Since $l^-(-T)$ contains the whole leaf l in its limit set, $l^-(-T)$ intersects B_*. Since $l(-T) \notin B_*$, the arc containing the intersection point of $l^-(-T)$ and B_* belongs to $l^-(-T)$. However, at one of the endpoints of this arc, the semileaf $l^-(-T)$ must intersect Σ with the intersection index -1, which contradicts the assumption that $l^-(-T)$ intersects Σ only with the intersection index $+1$. The lemma is proved. $\square$

3.5. Closed transversals

It follows immediately from Definition 3.1, that through every point of a leaf of a local lamination, one can draw a transversal segment i. e., an arc that transversally intersects the leaves of the local lamination. However, one can not always draw a closed transversal through every point.

Example 3.23 *There are leaves that are not intersected by closed transversals.*

The closed leaf of Reeb foliation on the two-dimensional torus T^2 has no intersections with any closed transversal. $\diamondsuit$

Here we consider some sufficient conditions of existence of closed transversals. A simple closed curve is said to be *two-sided* (*one-sided*) if it has a neighborhood homeomorphic to a cylinder (respectively, to a Möbius band). At first, one can show that if there is a one-sided closed transversal, then there is a two-sided one.

Lemma 3.21 *If a local lamination $\mathscr{D}$ has a one-sided simple closed transversal, say C, then in any neighborhood of C there is a two-sided simple closed transversal for $\mathscr{D}$.*

Proof. Let C be a simple closed transversal with the neighborhood U homeomorphic to a Möbius band. The curve C is a middle closed curve of the Möbius band. Take a simple closed curve that is sufficiently close to C and twice passes around C, Fig. 3.28. Then, one gets a closed transversal for the local lamination $\mathscr{D}$ with a cylindrical neighborhood. Clearly that this two-sided transversal can be constructed in arbitrary neighborhood of C. □

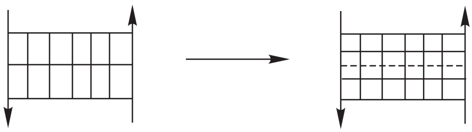


Figure 3.28. One can get a closed transversal with a cylindrical neighborhood.

Recall that an arc $\widehat{ab}$ of the curve l with endpoints a and b is called a Σ-*arc* if $\Sigma \cap \widehat{ab} = a \cup b$. Let $\overline{ab} \subset \Sigma$ be the sub-arc of Σ between the points a and b. It is easily seen that the Σ-arc together with $\overline{ab}$ forms a simple closed curve $\widehat{ab} \cup \overline{ab}$ called Σ-*loop* of l, see Fig. 3.25. A Σ-arc and a Σ-loop in the case when Σ is a closed curve are defined similarly; however, one has to specify how to choose a segment $\overline{ab}$ from the two segments into which Σ is divided by the points a and b. Recall that if we introduce a positive direction on l and endow Σ with a normal orientation, one can speak of orientable or non-orientable intersections of l with Σ.

Theorem 3.3 *Let l be a nonclosed semileaf of a local lamination $\mathscr{D}$ on M, and let there exist a Σ-arc $\widehat{ab} \subset l$ of l, where $a, b \in l \cap \Sigma$. Suppose that one of the following cases takes place:*

1) *the Σ-loop $\widehat{ab} \cup \overline{ab}$ is two-sided, and the Σ-arc $\widehat{ab}$ intersects Σ orientably;*
2) *the Σ-loop $\widehat{ab} \cup \overline{ab}$ is one-sided, and the Σ-arc $\widehat{ab}$ intersects Σ non-orientably.*

Then, there exists a simple closed transversal that intersects l. In the second case, there is such a transversal that has a neighborhood homeomorphic to a Möbius band. In the both cases, there is such a transversal that has a cylindrical neighborhood.

Proof. (1) There is a standard method for constructing a simple closed transversal (see [3, 24, 26]) when a two-sided Σ-arc $\widehat{ab} \subset l$ intersects Σ orientably, Fig. 3.29.

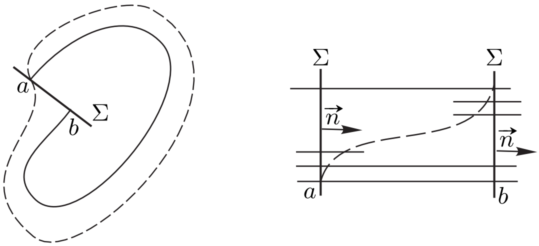


Figure 3.29. Two-sided Σ-arc $\widehat{ab}$ intersects Σ orientably.

According to Theorem 3.1, Σ-arc $\widehat{ab}$ belongs to some open trivially foliated box. Since Σ-arc $\widehat{ab}$ is two-sided, the Σ-loop $\widehat{ab} \cup \overline{ab}$ has a neighborhood, say U, homeomorphic to a cylinder. Taking a smaller U, if it is necessary, one can represent U as a union of two trivially foliated boxes U_1 and U_2 that cover the Σ-arc $\widehat{ab}$ and the arc $\overline{ab}$ respectively. Therefore, starting with the point a, one can draw a transversal line, say λ, along $\widehat{ab}$ to the opposite side of the trivially foliated box U_1 such that the transversal line is tangent to Σ at the endpoints. Since the Σ-arc $\widehat{ab}$ intersects Σ orientably, it is possible to draw λ so that the tangent vectors at the endpoints of λ induce the same orientation of the segment Σ. Therefore, one can carefully modify the transversal line λ to a simple closed transversal. By construction, in this case, the closed transversal has a cylindrical neighborhood U.

(2) Again, applying Theorem 3.1, one can cover the Σ-loop $\widehat{ab} \cup \overline{ab}$ by two trivially foliated boxes U_1 and U_2 that cover the Σ-arc $\widehat{ab}$ and the arc $\overline{ab}$ respectively. In this case, the neighborhood $U = U_1 \cup U_2$ is homeomorphic to a Möbius band, since Σ-arc $\widehat{ab}$ is one-sided, Fig. 3.30. Going by a similar way to first case (1), one can construct a transversal line λ described above. Since the Σ-arc $\widehat{ab}$ intersects Σ non-orientably, it is again possible to draw a transver-

sal line λ so that the tangent vectors at the endpoints of λ induce the same orientation of the segment Σ. Hence, λ produces a simple closed transversal. By construction, in this case, the closed transversal has the neighborhood U homeomorphic to a Möbius band. It follows from Lemma 3.21 that there is a closed transversal with a cylindrical neighborhood in this case also. □

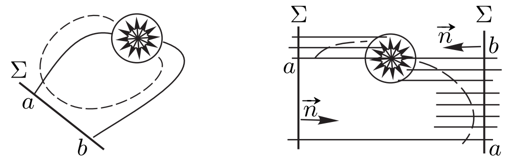


Figure 3.30. One-sided Σ-arc $\widehat{ab}$ intersects Σ non-orientably.

Corollary 3.4 *Let l be a semileaf of an orientable foliation on an orientable surface M, and suppose that l intersects a some transversal segment infinitely many times. Then, there exists a simple closed transversal that intersects l.*

Theorem 3.4 *Let l be a nontrivially recurrent semileaf of a local lamination $\mathscr{D}$ on M. Then, there exists a simple closed transversal for $\mathscr{D}$ that intersects l.*

Proof. There exists a transversal segment Σ that intersects l infinitely many times. Therefore, there exists a Σ-arc $\widehat{ab} \subset l$ of the semileaf l, where $a, b \in \in l \cap \Sigma$. If either the Σ-loop $\widehat{ab} \cup \overline{ab}$ is two-sided and the Σ-arc $\widehat{ab}$ intersects Σ orientably or the Σ-loop $\widehat{ab} \cup \overline{ab}$ is one-sided and the Σ-arc $\widehat{ab}$ intersects Σ nonorientably, then the required assertion follows from Theorem 3.3. Consider the remaining possibilities:

1) the Σ-loop $\widehat{ab} \cup \overline{ab}$ is one-sided and the Σ-arc $\widehat{ab}$ intersects Σ orientably;
2) the Σ-loop $\widehat{ab} \cup \overline{ab}$ is two-sided and the Σ-arc $\widehat{ab}$ intersects Σ nonorientably.

Case (1) is reduced to the previous cases if we consider the next intersection of the leaf l with the segment $\overline{ab} \subset \Sigma$ (for a flow, a similar construction is described in [101]; see also [104]).

In case (2), since l is a nontrivially recurrent semileaf, there exists a Σ-arc $\widehat{bc} \subset l$, where $c \in \overline{ab} \subset \Sigma$. Then, one of the Σ_1-arcs (where $\Sigma_1 \subset \Sigma$ is the corresponding part of Σ) $\widehat{ab}$ or $\widehat{bc}$ is one of the types considered above. □

Corollary 3.5 *Let l be a nontrivially recurrent semitrajectory of a flow f^t on M. Then, there exists a simple closed transversal of f^t that intersects l.*

3.6. Index of closed curve and singularity

Let $\mathscr{F}$ be a foliation and C be a simple closed oriented curve that does not intersect $\mathrm{Sing}(\mathscr{F})$. Suppose that C has a cylindrical neighborhood $U(C)$ that is divided by C into two cylindrical components U_{ext}, U_{int}. A component U_{ext} (respectively, U_{int}) is said to be *external* (respectively, *internal*) if U_{ext} (U_{int}) lies to the right (left) of C when we are moving in the positive direction along C, Fig. 3.31 (a).

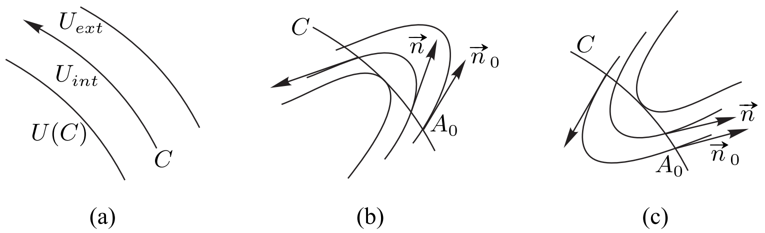


Figure 3.31. The components U_{ext}, U_{int} (a); the internal tangency point (b); the external tangency point (c).

Suppose that there exist points where $\mathscr{F}$ does not tangent to C and take any such point $A_0 \in C$. Let $\vec{n}_0$ be the unit vector at A_0 tangent to $\mathscr{F}$ and oriented from U_{int} to U_{ext}, Fig. 3.31 (b), (c). Given any point $A \in C$, there is a unique unit vector $\vec{n}(A)$ that is tangent to $\mathscr{F}$ and is a continuation of $\vec{n}_0$ when we move along C from A_0 to A in the positive direction. Denote by Θ the total change of the angle that the vector $\vec{n}(A)$ makes with respect to the vector $\vec{n}_0$ when A runs from A_0 to A_0 along C in the positive direction. The number $\mathrm{ind}(C, \mathscr{F}) = \frac{1}{2\pi}\Theta$ is called the *index of C with respect to $\mathscr{F}$*. Obviously that the vector $\vec{n}(A_0)$ that is the continuation of $\vec{n}_0$ after coming back at A_0 is either $\vec{n}_0$ or $-\vec{n}_0$. Hence, $\mathrm{ind}(C, \mathscr{F})$ is either an integer or half-integer respectively. Clearly that the definition of $\mathrm{ind}(C, \mathscr{F})$ does not depend on the choice of the point A_0 and depends continuously on slight deformations of C provided the deformation does not pass through singularities. Since an index is an integer or half-integer, $\mathrm{ind}(C, \mathscr{F})$ does not change at all under such deformations. Therefore, $\mathrm{ind}(C, \mathscr{F})$ ***is invariant under a deformation*** (not necessary small) ***of C unless this deformation passes through singularities***. This observation allows to introduce $\mathrm{ind}(C, \mathscr{F})$ for C tangent to $\mathscr{F}$ at all points because slight deformations of C produce points of transversal intersections $\mathscr{F}$ with C.

If a foliation is orientable and is defined by trajectories of vector field $\vec{v}$, then $\operatorname{ind}(C, \mathscr{F})$ is called the *index of C with respect to $\vec{v}$* or *relative to $\vec{v}$* and is denoted as $\operatorname{ind}(C, \vec{v})$. By orientability, $\operatorname{ind}(C, \mathscr{F})$ is an integer.

Example 3.24 *Index of a curve relative to a vector field on* $\mathbb{R}^2$.

Suppose that a closed simple curve $C \subset \mathbb{R}^2$ does not pass through singularities of C^1 vector field $\vec{v}$ defined by the system of differential equations $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ on $\mathbb{R}^2$. The routine calculation gives

$$\operatorname{ind}(C, \vec{v}) = \frac{1}{2\pi} \oint_C d \arctan \frac{dy}{dx} = \frac{1}{2\pi} \oint_C d \arctan \frac{Q(x, y)}{P(x, y)} = \frac{1}{2\pi} \oint_C \frac{P\, dQ - Q\, dP}{P^2 + Q^2},$$

see details [3], ch. 4. Note that if C bounds a disk with no fixed points, one can deform C to a curve that bounds a disk where the vector field is constant i. e., defined by the system $\dot{x} = 0$, $\dot{y} = 1$. Thus, $\operatorname{ind}(C, \vec{v}) = 0$. $\diamondsuit$

Poincaré [192] suggested another way to calculate an index that often is more convenient. One can slightly deform the curve C to get C in a general position with respect to $\mathscr{F}$ i. e., C has only finitely many points of tangency (in the topological sense) with the leaves of $\mathscr{F}$. Then, the points of tangency of $\mathscr{F}$ with the curve C are naturally divided into *internal and external tangency points*, Fig. 3.31 (b), (c). We see that when a current point $A \in C$ passes an internal (external) tangency point, $\vec{n}(A)$ rotates anti-clock-wisely (clock-wisely) by angle π. Then

$$\operatorname{ind}(C, \mathscr{F}) = \frac{1}{2}(2 - k_{\text{ext}} + k_{\text{int}}) \tag{3.10}$$

where k_{ext} (k_{int}) is the number of points of external (internal) tangency of C with $\mathscr{F}$. As a consequence, one gets the following statement.

Lemma 3.22 *Let $\mathscr{F}$ be a foliation and C be a curve that does not intersect* $\operatorname{Sing}(\mathscr{F})$. *Then*

- *if C bounds a disk without singularities, then* $\operatorname{ind}(C, \mathscr{F}) = 0$, *Fig.* 3.32 (*a*);
- *if C is two-sided and is formed by a transversal segment and an arc of leaf, that intersects the segment with different indices, then* $\operatorname{ind}(C, \mathscr{F}) = \frac{1}{2}$, *Fig.* 3.32 (*b*);

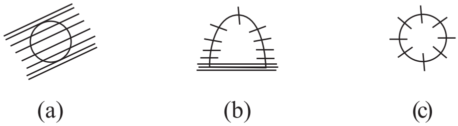


Figure 3.32. The index $\operatorname{ind}(C, \mathscr{F}) = 0$ for (a); $\operatorname{ind}(C, \mathscr{F}) = \frac{1}{2}$ for (b); $\operatorname{ind}(C, \mathscr{F}) =$ $= 1$ for (c).

- *if C is a closed transversal, then* $\operatorname{ind}(C, \mathscr{F}) = 1$*, Fig.* 3.32 (*c*);
- *if C is two-sided and is formed by a transversal segment and an arc of leaf, that intersects the segment with the same indices, then* $\operatorname{ind}(C, \mathscr{F}) =$ $= 1$.

Proof follows from (3.10). We leave details to the Reader. □

Let us point out from [24] omitting a proof a criterion on the orientability (non-orientability) of a foliation on a surface.

Theorem 3.5 *A foliation $\mathscr{F}$ is orientable if and only if* $\operatorname{ind}(C, \mathscr{F})$ *is an integer number for any simple two-sided closed curve C on M^2 that does not pass through singularities.*

The next statement, we distinguish separately from Lemma 3.22 for its importance.

Lemma 3.23 *Let C be a closed leaf of a foliation $\mathscr{F}$. Suppose that C has a cylindrical neighborhood. Then* $\operatorname{ind}(C, \mathscr{F}) = 1$.

Proof. Because of every point of C has a neighborhood that is a trivially foliated box, $k_{\text{ext}} = k_{\text{int}}$ for a slightly deformed curve, Fig. 3.33. Hence, $\operatorname{ind}(C, \mathscr{F}) = 1$. □

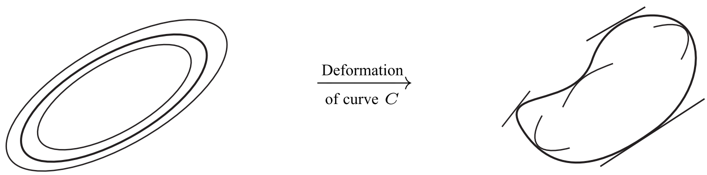


Figure 3.33. $k_{\text{ext}} = k_{\text{int}}$.

Later on in this section, a curve C is assumed to be simple and two-sided i. e., C has a cylindrical neighborhood, and does not pass through singularities of a foliation.

Lemma 3.24 *If C is decomposed into two two-sided simple curves as in Fig.* 3.34 (*a*)*, then* $\mathrm{ind}(C, \mathscr{F}) = \mathrm{ind}(C_1, \mathscr{F}) + \mathrm{ind}(C_2, \mathscr{F})$ *provided C_1 and C_2 do not pass through singularities of $\mathscr{F}$.*

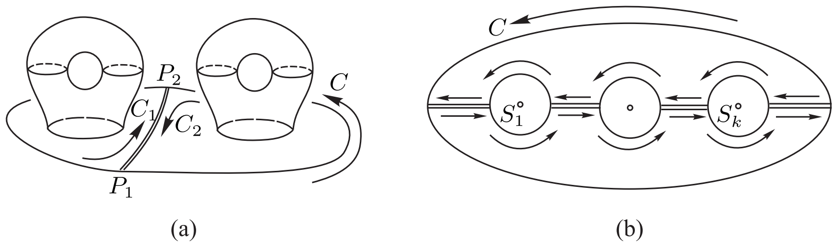


Figure 3.34. A decomposition of C into $C_1 + C_2$ (a); singularities $s_1, \ldots, s_k$ of C (b).

Proof. Keeping the notation above, let us take the starting point A_0 to be p_1. Let $\vec{n}(A)$ be a unit vector that is tangent to $\mathscr{F}$ and depends continuously on a point $A \in C$ when A moves along C in the positive direction from P_1. Obviously that the continuation of the vector $\vec{n}(P_2)$ along the arc P_2P_1 (that not necessary produces $\vec{n}_0$ at P_1) and then back continuation along the arc P_1P_2 give the original vector $\vec{n}(P_2)$. Hence the total change of the angle that the vector $\vec{n}(A)$ makes with respect to $\mathscr{F}$ when A runs the path $P_2P_1 + P_1P_2$ equals to zero. This follows the result. □

Now consider a closed simple curve that bounds a disk (as a consequence, the curve has a cylindrical neighborhood). Such a curve is always equipped with an orientation such that, when moving along C in the positive direction, the disk lies locally to the left. This convention is to define the index of an isolated singularity. For a curve on the 2-sphere, one must indicate to what disk the singularity belongs to.

Let s be an isolated singularity of a foliation $\mathscr{F}$. Then s has a neighborhood homeomorphic to a disk with no singularities different from s.

Definition 3.11 *An index* $\mathrm{ind}\, s$ *of singularity s is the index of any closed curve that encircles only one singularity s and does not contain singularities other than s inside it.*

It follows from (3.10) and Definition 3.11 that the index of a saddle type singularity can be calculated by the number of separatrices ν as follows

$$\mathrm{ind}\, O = 1 - \frac{\nu}{2}. \tag{3.11}$$

Thus, among the saddle type singularities, only a thorn has a positive index that equals to $\frac{1}{2}$. A fake saddle has the index 0, and all the other saddles have negative indices.

Note that an irrational foliation can have thorns, singularities with positive index. Such irrational foliation is obviously non-orientable one. We distinguish irrational foliations with only singularities of negative index.

Definition 3.12 *An irrational foliation with no singularities of positive index is called* ***strongly irrational****.*

Below, we'll see that singularities of irrational foliation are all of saddle type. Therefore, strongly irrational foliation can have singularities being saddles with negative indices. So, one can define a strongly irrational foliation as irrational one with no thorns.

Theorem 3.6 *Let a simple closed curve C bound a domain $D \subset M^2$ homeomorphic to a disk inside of those there are finitely many singularities $s_1, \ldots, s_k$ of a foliation $\mathscr{F}$. Then,*

$$\operatorname{ind}(C, \mathscr{F}) = \sum_{i=1}^{k} \operatorname{ind} s_i.$$

Sketches of the proof in Fig. 3.34 (b) show that the moving along C can be replaced by movings around each singularity. We omit details to the Reader. □

Corollary 3.6 *Let l be a closed leaf of a foliation $\mathscr{F}$ that bounds a domain D on M^2 that is homeomorphic to a disk. Then D contains at least one singularity of positive index in its interior.*

Corollary 3.7 *Let C be a simple closed curve formed by a transversal segment and an arc of some leaf of a foliation $\mathscr{F}$. If C bounds a domain D on M^2 that is homeomorphic to a disk, then D contains at least one singularity of positive index.*

The following lemma generalizes the statement of Example 3.22.

Lemma 3.25 *Let l_0 be a closed leaf of a foliation $\mathscr{F}$. Suppose that the ω-limit set $\omega(l)$ of a positive semileaf l contains l_0. Then $\omega(l) = l_0$.*

Proof. We keep the notation of Example 3.22, where one proved the lemma for an orientable foliation. For a non-orientable foliation, the proof is similar except the possibility of intersection of Σ by $l^{(}P^+(m_k))$ between m_k and $P^+(m_k)$. First of all, we take a neighborhood U so small that there are no singularities in U. This is possible because the set of singularities is closed. If we now assume that $l^{(}P^+(m_k))$ intersects Σ between m_k and $P^+(m_k)$ after $P^+(m_k)$, then it follows that in the cylinder A_k there is a disk D bounded by an arc of $l^{(}P^+(m_k))$ and Σ. By Corollary 3.7, D contains a singularity. This is impossible because $U \supset D$ does not contain singularities. □

Obviously, Lemma 3.25 holds for an α-limit set being a closed leaf: if a closed leaf l_0 belongs to the α-limit set $\alpha(l)$ of a leaf l, then $\alpha(l) = l_0$.

The following remarkable Poincaré – Hopf theorem says that the sum of indices of all singularities does not depend on a foliation but on the topological characteristic of a surface, Euler characteristic. Recall that the Euler characteristic $\chi(M^2)$ of a closed surface M^2 of genus p equals to $2-2p$ if M^2 is orientable and $2-p$ if M^2 is non-orientable.

Theorem 3.7 *Let $\mathscr{F}$ be a foliation with finitely many singularities $s_1, \ldots, s_k$ on a closed surface M^2. Then*

$$\sum_{i=1}^{k} \operatorname{ind} s_i = \chi(M^2).$$

The proof can be found in numerous books, exm., [183].

Index of fixed points for homeomorphisms

Let us introduce an index for an isolated fixed point of a homeomorphism. One can consider a homeomorphism $h\colon \mathbb{R}^2 \to \mathbb{R}^2$ with an isolated fixed point x_0, $h(x_0) = x_0$. The mapping h generates the vector field $\vec{v}_h$ each vector of whose begins at x and finishes at $h(x)$, $x \in \mathbb{R}^2$. Since x_0 is an isolated fixed point, x_0 is an isolated singularity of $\vec{v}_h$. One can define an *index of fixed point* x_0 being the index of the singularity x_0 of the vector field $\vec{v}_h$.

3.7. Minimal and quasiminimal sets

Let $\mathscr{D}$ be a local lamination on M^2. Recall that a set $N \subset M^2$ is called *invariant* (or *saturated*) if N is a union of singularities and leaves of $\mathscr{D}$. A nonempty invariant set is called *minimal* if it is closed and does not contain nonempty proper invariant closed subsets.

A trivial example of a minimal set is a singularity or a compact leaf (in particular, a closed leaf). Such minimal sets are called *trivial*. The following result follows immediately from the definition of minimal set and Corollary 3.1.

Lemma 3.26 *Let N be a minimal set of a local lamination $\mathscr{D}$. Then either $\partial N = N$, $\operatorname{int} N = \varnothing$ or $\partial N = \varnothing$, $\operatorname{int} N = N$.*

Hence, either N is nowhere dense or N is the whole surface M, respectively (in the latter case, one can prove that M is the torus).

It is clear that if a minimal set N contains at least one (one-dimensional) leaf, then N does not contain singularities. If a minimal set N contains at least one noncompact leaf, then every leaf of N is noncompact.

Lemma 3.27 *Let N be a minimal set of a local lamination $\mathscr{D}$. If N contains a noncompact leaf, then every leaf of N is dense in N. If, moreover, $\mathscr{D}$ has a compact support, then every leaf of $\mathscr{D}$ is nontrivially recurrent.*

Proof. Take a leaf l of $\mathscr{D}$. Obviously, $\operatorname{clos}(l)$ is a closed set. Due to Corollary 3.1, $\operatorname{clos}(l)$ is invariant. By definition of minimality, $N = \operatorname{clos}(l)$. Hence, l is dense in N. It remains to prove that l is nontrivially recurrent when $\mathscr{D}$ has a compact support. Take any semileaf $l^{\pm} \subset l$. To be definite, suppose $l^{\pm} = l^{+}$. Because of compactness of $\operatorname{supp} \mathscr{D}$, $\omega(l^{+}) \subset \operatorname{clos}(l^{+})$ is a saturated, closed, and nonempty set, see Lemma 3.17. Again, due to minimality of $N = \operatorname{clos}(l)$, we get $N = \omega(l^{+})$. It follows from Corollary 3.3, that l^{+} is nontrivially recurrent. □

Note that if $\mathscr{D}$ has a noncompact support then, in general, l is not nontrivially recurrent: any straight line of a simplest local lamination on $\mathbb{R}^2$ is a minimal set but, obviously, a straight line is not nontrivially recurrent.

Minimal sets described in Lemma 3.27 are called *nontrivial*, except for one case: the torus T^2 is a minimal set of a linear irrational foliation on T^2 that is traditionally referred to trivial minimal sets. Sometimes nontrivial minimal set is called *exceptional*.

Now we give a description by George Birkhoff [47] of trajectories of minimal sets for flows on any compact manifolds. A trajectory l is called *B-recurrent* or *recurrent in the Birkhoff sense* if for any $\varepsilon > 0$ there exists $T(\varepsilon) > 0$ such that any arc of l of time length $T(\varepsilon)$ approximates the entire trajectory l with an accuracy ε. Obviously, a fixed point and periodic trajectory are B-recurrent.

Lemma 3.28 *Let l be a nonclosed B-recurrent trajectory of a flow on a compact manifold. Then l is nontrivially recurrent.*

Proof. Take any positive semitrajectory $l^{+} = \{l(t) \colon t \geqslant 0\}$ of l and arbitrary point $p \in l^{+}$. By definition, for any $\varepsilon > 0$, there is $T(\varepsilon) > 0$ such that any arc of l of time length $T(\varepsilon)$ approximates the entire trajectory l with an accuracy ε. Given any $T_0 > 0$, take such arc $A \subset l^+$ with endpoints $l(t_1)$, $l(t_2)$, where $T_0 \leqslant t_1 \leqslant t_2 = t_1 + T(\varepsilon)$. Since A approximates l, there is a point $l(t) \in A$ with $t_1 \leqslant t \leqslant t_2$ ε-close to p. It follows that $p \in \omega(l^+)$. Hence, l^+ is nontrivially recurrent. By the similar way, one can prove that any negative semitrajectory of l is also nontrivially recurrent. □

Theorem 3.8 *Let f^t be a flow on a compact manifold. Then any minimal set of f^t consists of B-recurrent trajectories. Moreover, the topological closure of a B-recurrent trajectory is a minimal set.*

Thus, in a compact space, a trajectory l is B-recurrent if and only if its closure $\mathrm{clos}(l)$ is a minimal set. The proof of Theorem 3.8 can be found in [47, 168].

Definition 3.13 *The topological closure of a nontrivially recurrent semileaf l, $\mathrm{clos}(l)$, is called a* ***quasiminimal*** *set.*

According to Lemma 3.27, a nontrivial minimal set of a local lamination with compact support is a quasiminimal set. A support surface of a highly transitive foliation with nonempty set of singularities gives us the Example of a quasiminimal set which is not a nontrivial minimal set (see Examples 3.7, 3.9).

As a consequence of Definition 3.13 and Corollary 3.1, one gets that ***a quasiminimal set is closed and saturated (invariant)***.

Separatrices

Let $l^+(m_0) = l^+ = \{m(t) \colon t \geqslant 0\}$ be a positive semileaf. Suppose that the ω-limit set $\omega(l^+)$ of l^+ consists of exactly one isolated singularity, say s. The semileaf $l^+(m_0)$ is called an *ω-separatrix* if there exist a neighborhood $U(s)$ of the point s and a sequence of points m_k such that

1) $m_0 \notin U(s)$, $m_k \notin U(s)$, $m_k \to m_0$ as $k \to \infty$;
2) through any point m_k, there passes a positive (or negative) semileaf $l_k^+(m_k)$ that first enters $U(s)$ and then necessarily leaves it as the parameter increases (respectively, decreases), Fig. 3.35.

The Theorem 3.2 on the continuous dependence of leaves on the initial conditions implies that if a certain positive semileaf l^+ of a leaf l is an ω-separatrix, then any positive semileaf of l is also an ω-separatrix. Therefore, we can say that l itself is an ω-separatrix if some its positive semileaf is an ω-separatrix.

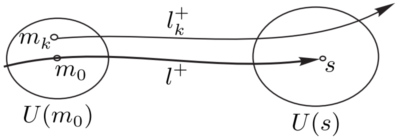


Figure 3.35. An ω-separatrix l^+.

The concept of an α-separatrix for a negative semileaf is introduced similarly. Sometimes, we will speak simply of a separatrix, without explicitly indicating which of the separatrices, ω- or α-, is meant. Usually, this is either clear from the context or inessential.

Take a neighborhood $U(m_0)$ of m_0 homeomorphic to a disk, and assume that the component of $U(m_0) \cap l^+$ through m_0 divides $U(m_0)$ into two parts each of which is homeomorphic to a half-disk. When we move along l^+ in the positive direction, one part is left (right). So, it is natural to call it a left (right) part. If the above-mentioned sequence m_k may locally tend to m_0 from the both left and right parts, the separatrix l^+ is called *two-sided*. If m_k may locally tend to m_0 from the left (resp. right) part, l^+ is called *locally left-sided* (resp. *locally right-sided*). Sometimes, we say that l^+ is *one-sided*. For orientable surface, *left-sided* and *right-sided* separatrices can be naturally defined. For convenience, we will not contrast these concepts; i. e., a left-sided separatrix may be right-sided (hence, two-sided).

Recall that if a leaf is both an ω-separatrix and α-separatrix, it is called a *separatrix connection*, Fig. 3.8, (a). A particular case of a separatrix connection is given by a *separatrix loop*, Fig. 3.8, (b), when the connecting singularities coincide.

Recall that given any transversal segments or closed transversals Σ_1, Σ_2 of an orientable foliation $\mathscr{F}$, $P^+_{\Sigma_1\Sigma_2}$ denotes the forward Poincaré mapping $\operatorname{Dom} P^+_{\Sigma_1\Sigma_2} \cap \Sigma_1 \to \Sigma_2$ induced by $\mathscr{F}$. The following lemma adjusts Lemma 3.2.

Lemma 3.29 *Let C_1, C_2, C_3 be transversal segments of an orientable foliation $\mathscr{F}$ on a compact M. Suppose that there are points $m_{12} \in C_1$, $m_{13} \in C_1$ such that the positive semileaves $l^+(m_{12})$, $l^+(m_{13})$ intersect C_2, C_3 respectively as shown in Fig. 3.7, (a): the C_1C_2-arc $m_{12}P^+_{C_1C_2}(m_{12})$ does not intersect C_3, and the C_1C_3-arc $m_{13}P^+_{C_1C_3}(m_{13})$ does not intersect C_2. Then given any $j = 2, 3$, there is a point $m \in C_1$ between m_{12} and m_{13} such that either the semileaf $l^+(m)$ passes through an endpoint of C_j or $l^+(m)$*

is an ω-separatrix that does not intersect $C_2 \cup C_3$. If C_2 or C_3 is a closed transversal, $l^+(m)$ is an ω-separatrix that does not intersect $C_2 \cup C_3$.

Proof. Consider only $j = 2$ because for $j = 3$, the proof is similar. If C_2 is a transversal segment and there is $m \in C_1$ such that $l^+(m)$ passes through an endpoint of C_2, then there is nothing to prove. Suppose that such a point does not exist. Then by Lemma 3.2, $l^+(m)$ does not intersect $C_2 \cup C_3$. Without loss of generality, one can assume that any positive semileaf through a point between m and m_{12} intersects C_2, so m is a first point where this property fails when a current point moves from m_{12} to m_{13} on C_1.

We have to prove that $\omega(l^+(m))$ is a singularity. Suppose the contrary. Then $\omega(l^+(m))$ contains a regular point, say a. Let Σ be a transversal segment through a. Without loss of generality, we can assume that Σ does not intersect $C_2 \cup C_3$. Since $a \in \omega(l^+(m))$, $l^+(m)$ intersects Σ. Consider two cases: 1) $l^+(m)$ is a closed leaf; 2) $l^+(m)$ is a nonclosed leaf. In case 1), a is a unique point of the intersection of $l^+(m)$ with Σ. Since $l^+(m)$ does not intersect C_2, there is an interval $(m, m_*) \subset C_1$ such that given any point $\widetilde{m} \in (m, m_*)$, the positive semileaf $l^+(\widetilde{m})$ first intersects Σ and after C_2, Fig. 3.36 (a). So, (m, m_*) belongs to the domain of forward Poincaré mapping $P^+_{C_1\Sigma}$. Because of m is a first point such that $l^+(m)$ does not intersect $C_2 \cup C_3$, any positive semileaf through $P^+_{C_1\Sigma}(m, m_*)$ intersects C_2 with no passing the endpoints of C_2. Applying Lemma 3.2 for $C_1 = \Sigma = C_3$, one gets that there is a positive semileaf through a point from the interval $P^+_{C_1\Sigma}(m, m_*)$ with no intersections with $\Sigma \cup C_2$. We get the contradiction because $l^+(\widetilde{m})$ intersects C_2.

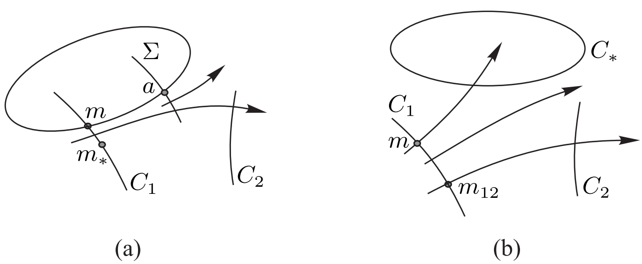


Figure 3.36. $l^+(m)$ is a closed leaf (a); $l^+(m)$ is a nonclosed leaf (b).

In case 2), $l^+(m)$ intersects Σ infinitely many times. By Corollary 3.4, there is a closed transversal C_* through a intersected by $l^+(m)$. Again applying

Lemma 3.2 for $C_3 = C_*$, one gets that there is a positive semileaf through a point from the interval (m, m_{12}) with no intersections with C_2, Fig. 3.36 (b). This is impossible.

Thus, $\omega(l^+(m))$ is a singularity, say s. Take a neighborhood $U(s)$ of s such that $U(s)$ does not intersect $C_2 \cup C_3$. By Theorem 3.2, any semileaf $l^+(\widetilde{m})$ enters $U(s)$ for $\widetilde{m}$ sufficiently close to m before meets $C_2 \cup C_3$. But later on, $l^+(\widetilde{m})$ intersects $C_2 \cup C_3$, and hence, $l^+(\widetilde{m})$ must leave $U(s)$. Therefore, $l^+(m)$ is an ω-separatrix. □

Recall that if C is a transversal of a non-orientable foliation, we modify the notion of Poincaré mapping extending a domain of this mapping to two copies C^+, C^- of C. Lemma 3.29 has the following extension to non-orientable foliations.

Lemma 3.30 *Let C_1, C_2, C_3 be transversal segments of a foliation $\mathscr{F}$ on a compact M. Suppose that there are points $m_{12} \in C_1$, $m_{13} \in C_1$ such that the semileaves $l(m_{12})$, $l(m_{13})$ intersect C_2, C_3 respectively as shown in Fig.* 3.7, (*b*): *the both starting points m_{12} and m_{13} of $l(m_{12})$, $l(m_{13})$ belong to the same side of C_1 (either C_1^+ or C_1^-). Then given any $j = 2, 3$, there is a point $m \in C_1^{\pm}$ between m_{12} and m_{13} such that either the semileaf $l(m)$ passes through an endpoint of C_j or $l(m)$ is a separatrix that does not intersect $C_2 \cup C_3$. If C_2 or C_3 is a closed transversal, $l(m)$ is a separatrix that does not intersect $C_2 \cup C_3$.*

Proof is similar to the proof of Lemma 3.30, and we omit it. □

Let C be a closed transversal of a foliation $\mathscr{F}$. We stress that we think of C^+ and C^- as a disjoint copies of C. So, two intervals $I^+ \subset C^+$, $I^- \subset C^-$ geometrically could be the same, but we consider that they are disjoint.

Lemma 3.31 *Let C be a closed transversal of a foliation $\mathscr{F}$ such that the Poincaré mapping $P_C\colon \mathrm{Dom}(P_C) \subset C^+ \cup C^- \to C^+ \cup C^-$ has a nonempty domain $\mathrm{Dom}(P_C)$. If $\mathscr{F}$ has a finitely many singularities and separatrices, then $\mathrm{Dom}(P_C)$ is a union of finitely many pairwise disjoint intervals.*

Proof. Theorem 3.1 implies that $\mathrm{Dom}(P_C)$ is a union of open disjoint intervals. Due to Lemma 3.30, each endpoint of a maximal interval of $\mathrm{Dom}(P_C)$ is a starting point of a separatrix that after this point does not intersect C. This follows the result because $\mathscr{F}$ has a finitely many singularities and separatrices. □

Lemma 3.32 *Let l be a semileaf that tends to exactly one isolated singularity, say O, and suppose that l belongs to the limit let of a semileaf l_1. If the limit set of l_1 contains a point different from O, then l is a separatrix of the singularity O.*

Proof. Take a neighborhood $U(O)$ of O such that outside of $U(O)$ there are some points $m \in l$ and $m_1 \in \lim(l_1)$. Since $m \in \lim(l_1)$, l_1 passes arbitrarily close to m, so enters to $U(O)$, due to Theorem 3.2. On the other hand, after an entering to $U(O)$, l_1 must leave $U(O)$ because $m_1 \notin U(O)$. This completes the proof. □

Maier quasiminimal set

By definition, a quasiminimal set contains an everywhere dense nontrivially recurrent semileaf. However, it is not necessarily required that any nontrivially recurrent leaf from a quasiminimal set is everywhere dense in this set. Maier [142] proved that the latter property holds for orientable foliations (flows) on surfaces of finite genus. Moreover, Maier showed that any semileaf of a quasiminimal set that does not tend to exactly one fixed point is everywhere dense in the quasiminimal set and, hence, is nontrivially recurrent (the both these Maier's results are formulated and proved below). Taking into account this remarkable result, we give the following definition.

Definition 3.14 *A quasiminimal set Q of a local lamination $\mathscr{D}$ with a closed support* $\operatorname{supp}\mathscr{D}$ *is called a* ***Maier quasiminimal set*** *if any semileaf from Q that does not tend to exactly one singularity is everywhere dense in Q.*

Clearly that a semileaf of Maier quasiminimal set that does not tend to exactly one singularity is nontrivially recurrent. As a consequence of Definition 3.14 and Lemma 3.32, we get the following corollaries.

Corollary 3.8 *Let Q be a Maier quasiminimal set with isolated singularities. Then a semileaf $l \subset Q$ is either nontrivially recurrent or a separatrix* (*so, tends to exactly one singularity*).

Corollary 3.9 *Let Q be a Maier quasiminimal set of a local lamination $\mathscr{D}$ with isolated singularities. Except of singularities, Q can contain only the following* (*one-dimensional*) *leaves*:

- *nontrivially recurrent* (*in both directions*);

- *nontrivially recurrent in one direction and a separatrix in the another direction*;
- *a separatrix connection.*

Moreover, if the local lamination $\mathscr{D}$ has only Maier quasiminimal sets, then any nontrivially recurrent semileaf belongs to unique quasiminimal set.

Thus, ***any leaf of Maier quasiminimal set that is different from a separatrix connection is nontrivially recurrent in at least one direction***.

Corollary 3.10 *Let Q_1, Q_2 be Maier quasiminimal sets. Then*

- *either $Q_1 = Q_2$, or $Q_1 \cap Q_2 = \varnothing$, or $Q_1 \cap Q_2$ consists of separatrix connections and singularities*;
- *if Q_1, Q_2 are different quasiminimal sets, there are simple closed transversals C_1, C_2 such that $C_i \cap Q_i \neq \varnothing$, $i = 1, 2$, and $C_i \cap Q_j = \varnothing$ for $i \neq j$, and $C_1 \cap C_2 = \varnothing$.*

Proof. The first statement follows immediately from Corollary 3.9. Let us prove the second statement. Take a semileaf $l_1 \subset Q_1$ that is dense in Q_1, and consider a transversal segment Σ through a point $m \in \Sigma$. Since Q_1, Q_2 are different, l_1 does not belong to Q_2. Therefore, there exists an open interval $J \subset \Sigma$ such that $m \in J$ and $J \cap Q_2 = \varnothing$. As a consequence, there is a J-arc of l_1 that does not intersect Q_2. Following the proof of Theorem 3.4, one can construct from the J-arc the transversal C_1 such that $C_1 \cap Q_1 \neq \varnothing$ and $C_1 \cap Q_2 = \varnothing$. The similar method gives us a transversal C_2 such that $C_2 \cap Q_2 \neq \varnothing$ and $C_2 \cap Q_1 = \varnothing$. By construction, $C_1 \cap C_2 = \varnothing$. □

3.8. Geodesic laminations

A lamination whose leaves are geodesics is called a *geodesic lamination*. One can reformulate this definition in the traditional way: a geodesic lamination is a family of pairwise disjoint simple geodesics such that their union is a closed set[2]. Recall that a geodesic is said to be *simple* if it has no transversal self-intersections. When there is a nontransversal self-intersection (which is admitted by this definition of simplicity), a geodesic is a closed curve. Thus, a *simple geodesic* is either an infinite curve without self-intersections or a simple closed curve. Denote by $\mathscr{L}(M) = \mathscr{L}$ the set of geodesic laminations on M.

[2]Remark that the equivalence of these definitions is not obvious as it possibly seems at first glance.

The simplest geodesic lamination is any union of simple pairwise disjoint closed geodesics, Fig. 3.37, (a). Only such geodesic laminations exist on the Klein bottle K^2. For the torus T^2, one can add to such simplest laminations a linear irrational foliation. Later on, in this section, we'll consider geodesic laminations on closed hyperbolic surfaces M.

After simplest geodesic laminations, a few complicated example of a geodesic lamination on a hyperbolic surface, one gets by adding to a simplest geodesic lamination a finite collection of non-closed geodesics that spirally tend (in both directions) to closed geodesics, Fig. 3.37, (b). Note that the non-closed geodesics are isolated i. e., any point on such a geodesic has a neighborhood that intersects with the geodesic lamination only along a unique arc of the geodesic containing this point. A geodesic lamination is *trivial* if it consists of closed geodesics and isolated non-closed geodesics. Denote the set of trivial geodesic laminations by $\Lambda_{triv}(M) = \Lambda_{triv}$.

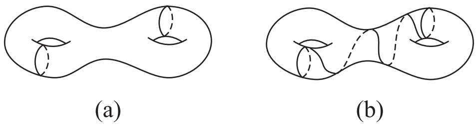


Figure 3.37. Simplest (a) and trivial (b) geodesic laminations.

To construct more complicated geodesic laminations, it is useful the following Lemma from the book [63], Lemma 3.2.

Lemma 3.33 *The topological closure of a nonempty union of simple pairwise disjoint geodesics is a geodesic lamination.*

This result allows us to construct geodesic laminations starting with a family of pairwise disjoint simple geodesics. In particular, this lemma implies that a finite family of geodesics pictured on Fig. 3.37, (b) forms really a (trivial) geodesic lamination.

Geodesic flow

There is a remarkable connection between behavior of geodesics and dynamics of geodesic flow. Recall that $T(M)$ means a tangent space of M, a smooth 4-dimensional manifold. Denote by $T_1(M)$ the space of unit tangent vectors, a smooth 3-dimensional manifold which is a tangent bundle $\xi\colon T_1(M) \to M$ over M with the fiber S^1. Take a vector $\vec{e}_0 \in T_1(M)$

with $\xi(\vec{e}_0) = p \in M$. Then there is a unique geodesic g through p with tangent vector $\vec{e}_0$ at p, Fig. 3.38. Let us endow g with the natural parametrization such that p corresponds the parameter 0. Then at the point with the parameter t the geodesic g has the tangent vector $\vec{e}_t$. Define the *geodesic flow* Geo^t on $T_1(M)$ putting $\mathrm{Geo}^t(\vec{e}_0) = \vec{e}_t$.

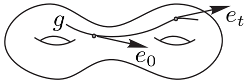


Figure 3.38. The geodesic flow Geo^t.

By the definition of geodesic flow, the projection $\xi\colon T_1(M) \to M$ maps every trajectory of Geo^t onto a geodesic endowed with some orientation, an *oriented geodesic*. Given any oriented geodesic g, one corresponds a unique trajectory of Geo^t, denoted by $\mathscr{T}(g)$. If g is a geodesic that is not endowed with an orientation, a *non-oriented geodesic*, then g corresponds two oriented geodesics denoted by g_+, g_- which are the same curve but endowed with two opposite orientations. In this case, $\mathscr{T}(g)$ is a union of two trajectories $\mathscr{T}(g_+)$, $\mathscr{T}(g_-)$ of geodesic flow. Thus, if g is an oriented geodesic, then $\mathscr{T}(g)$ is a unique trajectory, and the map

$$\xi|_{\mathscr{T}(g)}\colon \mathscr{T}(g) \to g \tag{3.12}$$

is one-to-one. If g is a non-oriented geodesic, then $\mathscr{T}(g)$ is a union of two trajectories $\mathscr{T}(g_+)$, $\mathscr{T}(g_-)$, and each of the maps

$$\xi|_{\mathscr{T}(g_+)}\colon \mathscr{T}(g_+) \to g_+, \quad \xi|_{\mathscr{T}(g_-)}\colon \mathscr{T}(g_-) \to g_- \tag{3.13}$$

is one-to-one.

Geodesic lamination and invariant sets of geodesic flow

Usually, geodesics of geodesic lamination are considered as curves with no orientations. For careful analysis of connection between geodesic laminations and invariant sets of geodesic flow, we need take into account orientations of geodesics. If a geodesic lamination $\mathscr{L}$ is orientable, it accepts two consistent orientations of its geodesics. So, $\mathscr{L}$ corresponds two oriented geodesic laminations $\mathscr{L}^+$ and $\mathscr{L}^-$ (to be precise, two geodesic laminations each geodesic of whose is endowed with the corresponding consistent orientation).

For an oriented geodesic lamination $\mathscr{L}$, $\mathscr{T}(\mathscr{L})$ is the union $\mathscr{T}(\mathscr{L}^+) \cup \cup \mathscr{T}(\mathscr{L}^-)$, where

$$\mathscr{T}(\mathscr{L}^+) = \bigcup_{g_+ \in \mathscr{L}^+} \mathscr{T}(g_+), \quad \mathscr{T}(\mathscr{L}^-) = \bigcup_{g_- \in \mathscr{L}^-} \mathscr{T}(g_-).$$

For a geodesic lamination $\mathscr{L}$ whose geodesics are not endowed with orientation (in particular, a non-oriented geodesic lamination),

$$\mathscr{T}(\mathscr{L}) = \bigcup_{g_+, g_- \in \mathscr{L}} \mathscr{T}(g_\pm).$$

Lemma 3.34 *Let $\mathscr{L}$ be a geodesic lamination* (*consisting of non-oriented geodesics*). *Then $\mathscr{T}(\mathscr{L})$ is an invariant closed set of* Geo^t. *Moreover, if $\mathscr{L}$ is an oriented geodesic lamination, the both $\mathscr{T}(\mathscr{L}^+)$ and $\mathscr{T}(\mathscr{L}^-)$ are invariant closed pairwise disjoint sets of* Geo^t.

Proof. Since $\mathscr{L}$ is a union of geodesics, $\mathscr{T}(\mathscr{L})$ is an invariant set of Geo^t. Take an accumulation point $\vec{e}$ of the set $\mathscr{T}(\mathscr{L})$. There is a sequence $\vec{e}_i \in \in \mathscr{T}(\mathscr{L})$ that tends to $\vec{e}$. Since $\operatorname{supp}\mathscr{L}$ is a closed set, $\xi(\vec{e}_i)$ tends to some point, say $p \in M$, of $\operatorname{supp}\mathscr{L}$. Therefore, there is the geodesic $g \in \mathscr{L}$ through p. It follows from [59] that $\vec{e}$ is a tangent vector to g. Hence, $\vec{e} \in \in \mathscr{T}(\mathscr{L})$ and $\mathscr{T}(\mathscr{L})$ is closed. Similar proofs of the closedness are for the sets $\mathscr{T}(\mathscr{L}^+)$ and $\mathscr{T}(\mathscr{L}^-)$.

If one assumes that $\mathscr{T}(\mathscr{L}^+)$ and $\mathscr{T}(\mathscr{L}^-)$ are intersected, then there is a geodesic $g \in \mathscr{L}$ such that $g_+, g_- \in \mathscr{L}^+$. This contradicts to the orientability of $\mathscr{L}^+$. □

Lemma 3.35 *Let N be an invariant closed set of* Geo^t. *Suppose that $\xi(N)$ is a union of pairwise disjoint geodesics. Then $\xi(N)$ is a geodesic lamination.*

Proof. To show that $\xi(N)$ is a geodesic lamination it is enough to prove that the set $\xi(N)$ is compact. Since $T_1(M)$ is a compact manifold, N is compact. Obviously that ξ is a continuous map. Hence, $\xi(N)$ is a compact set as an image of a compact set under a continuous map [53], thm. 7.6. □

Theorem 3.9 1) *Let $\mathscr{N}$ be a minimal set of* Geo^t *such that $\xi(\mathscr{N})$ is a union of pairwise disjoint geodesics* (*considered as curves with no orientations*). *Then $\xi(\mathscr{N})$ is a minimal geodesic lamination.*

2) *If* $\mathscr{L}$ *is a minimal oriented geodesic lamination, then the both* $\mathscr{T}(\mathscr{L}^+)$ *and* $\mathscr{T}(\mathscr{L}^-)$ *are minimal sets of* Geo^t.

3) *If* $\mathscr{L}$ *is a minimal non-oriented geodesic lamination, then* $\mathscr{T}(\mathscr{L})$ *is a minimal set of* Geo^t.

Proof. 1) By lemma 3.35, $\xi(\mathscr{N})$ is a geodesic lamination. Suppose $\xi(\mathscr{N})$ contains a proper nonempty sub-lamination $\mathscr{L}$. So, there is a geodesic $g^* \in$ $\in \xi(\mathscr{N})$ such that $g^* \notin \mathscr{L}$. Due to Lemma 3.34, $\mathscr{T}(\mathscr{L})$ is an invariant closed set of Geo^t. Obviously, $\mathscr{N} \cap \mathscr{T}(\mathscr{L}) \neq \varnothing$. Hence, $\mathscr{N} \cap \mathscr{T}(\mathscr{L})$ is a closed nonempty set of $\mathscr{N}$. Since $\mathscr{T}(g^*_\pm) \notin \mathscr{T}(\mathscr{L})$, $\mathscr{N} \cap \mathscr{T}(\mathscr{L})$ is a proper subset of $\mathscr{N}$. This contradicts the minimality of $\mathscr{N}$.

2) By Lemma 3.34, the both $\mathscr{T}(\mathscr{L}^+)$ and $\mathscr{T}(\mathscr{L}^-)$ are invariant closed disjoint sets of Geo^t. Suppose that $\mathscr{T}(\mathscr{L}^+)$ contains a proper nonempty closed invariant set $\mathscr{N}_*$ that is minimal set of Geo^t. By previous statement 1), $\xi(\mathscr{N}_*)$ is a minimal geodesic lamination. Taking in mind (3.12) and the orientation induced by trajectories of Geo^t on geodesics, we get that $\xi(\mathscr{N}_*) \subset \mathscr{L}^+$. Since $\mathscr{T}(\mathscr{L}^+) \neq \xi(\mathscr{N}_*)$, $\xi(\mathscr{N}_*) \neq \mathscr{L}^+$. This contradicts to the minimality of $\mathscr{L}$. Hence $\mathscr{T}(\mathscr{L}^+)$ is a minimal set of Geo^t. The similar proof holds for the minimality of $\mathscr{T}(\mathscr{L}^-)$.

3) Suppose $\mathscr{T}(\mathscr{L})$ contains a proper nonempty closed invariant set $\mathscr{N}$. Without loss of generality, one can assume that $\mathscr{N}$ is a minimal set. By previous statement 1), $\xi(\mathscr{N})$ is a minimal geodesic lamination whose geodesics are endowed with some orientations. Take any $g \in \xi(\mathscr{N})$, and assume for definiteness that $g = g_+$. It follows from Lemma 3.19 that any point on g_+ belongs to the limit set of the geodesic g_- (the same geodesic g but endowed with the opposite orientation) because g belongs to minimal non-orientable geodesic lamination. Hence, $g_- \in \xi(\mathscr{N})$. Therefore, $\xi(\mathscr{N})$ is a sub-lamination of $\mathscr{L}$. Due to the minimality of $\mathscr{L}$, $\mathscr{L} = \xi(\mathscr{N})$. It follows from (3.13) that $\mathscr{T}(\mathscr{L}) = \mathscr{N}$. Thus, $\mathscr{T}(\mathscr{L})$ is a minimal set of Geo^t. □

Nontrivial geodesic laminations

Before we saw that a trivial geodesic laminations consist of closed geodesics and isolated non-closed geodesics. So, it is natural to call a geodesic lamination *nontrivial* if it contains a non-closed geodesic that is non-isolated in the geodesic lamination. The example of a nontrivial geodesic lamination can be obtained as follows. Take a simple nontrivially recurrent geodesic (below, we'll see that such geodesics exist) g. Then, due to Lemma 3.33, the topological closure $\mathrm{clos}(g)$ is a nontrivial geodesic lamination, because g is non-isolated.

By definition, a nontrivial geodesic lamination can contain, in general, closed and, as well as, isolated geodesics.

Definition 3.15 *A nontrivial lamination is said to be* ***strongly nontrivial*** *if it consists of non-closed and non-isolated geodesics. A minimal strongly nontrivial geodesic lamination is called* ***weakly irrational****.*

As a consequence of Theorem 3.9, we get the following statement.

Corollary 3.11 *If $\mathscr{N}$ is a nontrivial minimal set of Geo^t such that $\xi(\mathscr{N})$ is a union of pairwise disjoint geodesics, then $\xi(\mathscr{N})$ is a weakly irrational lamination. Moreover, if $\mathscr{L}$ is an oriented weakly irrational geodesic lamination, then the both $\mathscr{T}(\mathscr{L}^+)$ and $\mathscr{T}(\mathscr{L}^-)$ are nontrivial minimal sets of Geo^t. If $\mathscr{L}$ is a non-oriented weakly irrational geodesic lamination, then $\mathscr{T}(\mathscr{L})$ is a nontrivial minimal set of Geo^t.*

Thus, there is a clear in sense correspondence between weakly irrational geodesic laminations and special nontrivial minimal sets of geodesic flow.

Denote by Λ the set of weakly irrational geodesic laminations. The following Corollary 4.7.2 from [63] gives an excellent criterium for a density of each geodesic in a geodesic lamination, i. e. for a minimality (we formulate this statement as a lemma).

Lemma 3.36 *Each geodesic in a geodesic lamination $\mathscr{L}$ is dense if and only if $\mathscr{L}$ is connected and has no isolated geodesics.*

In fact, Lemmas 3.33 and 3.36 imply the following assertion.

Lemma 3.37 *The topological closure of a nontrivially recurrent simple geodesic is a minimal nontrivial geodesic lamination.*

Proof. Let g be a nontrivially recurrent simple geodesic. By Lemma 3.33, its topological closure $\operatorname{clos} g$ is a geodesic lamination. As a topological closure of an arcwise connected set, the topological closure $\operatorname{clos} g$ is connected. Obviously, $\operatorname{clos} g$ has no isolated geodesics. By Lemma 3.36, each geodesic in the geodesic lamination $\operatorname{clos} g$ is dense. Hence, $\operatorname{clos} g$ is minimal and does not contain closed geodesics. □

The following Lemma 4.5 from [63] shows that a nontrivial geodesic lamination can not contain an infinite set of geodesics with the same asymptotic direction.

Lemma 3.38 *Let $\mathscr{L}$ be a geodesic lamination without closed geodesics on a closed orientable hyperbolic surface M, and let $\overline{\mathscr{L}}$ be its lift to Δ. Then there is no point on the circle at infinity S_∞ that is an ideal endpoint for an infinite set of geodesics of the covering lamination $\overline{\mathscr{L}}$.*

In fact, we obtain the following result as a corollary to Lemma 3.38.

Lemma 3.39 *Let $\mathscr{L}$ be a minimal nontrivial geodesic lamination on a closed orientable hyperbolic surface M, and let $\overline{\mathscr{L}}$ be its lift to Δ. Then any geodesic in $\overline{\mathscr{L}}$ has irrational ideal endpoints.*

Proof. Suppose the contrary. Then there exists a geodesic $\overline{g} \in \overline{\mathscr{L}}$ at least one ideal endpoint of which, say $\sigma \in S_\infty$, is rational. Hence, there exists a hyperbolic deck transformation γ with an axis $\overline{A}$ such that one ideal endpoint of $\overline{A}$ is σ.

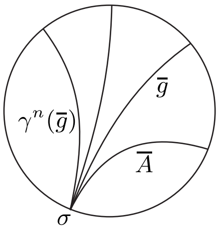


Figure 3.39. Geodesics $\overline{g}$ and $\overline{A}$.

Since $\overline{A}$ is projected to a closed geodesic on the surface and $\overline{g}$ is projected to a nonclosed geodesic, these geodesics do not coincide on Δ (Fig. 3.39). Therefore, the geodesics $\gamma^n(\overline{g})$, $n \in \mathbb{Z}$, are different but have the common endpoint σ. It is clear that $\gamma^n(\overline{g}) \in \overline{\mathscr{L}}$. Thus, σ is an ideal endpoint for an infinite set of geodesics of the covering lamination $\overline{\mathscr{L}}$. This result contradicts Lemma 3.38. □

Corollary 3.12 *The ideal endpoints of a geodesic that is projected to a nontrivially recurrent simple geodesic on a closed orientable hyperbolic surface M are irrational.*

After Lemma 3.36, one can describe a structure of strongly nontrivial geodesic laminations.

Lemma 3.40 *Let $\mathscr{L}$ be a strongly nontrivial geodesic lamination. Then*

- *every geodesic of $\mathscr{L}$ is nontrivially recurrent*;

- *$\mathscr{L}$ is a union of connected pairwise disjoint weakly irrational geodesic laminations*;
- *every geodesic of a weakly irrational geodesic lamination is dense in this lamination.*

Proof. Obviously, $\mathscr{L}$ is a union of pairwise disjoint connected components. Let $\mathscr{N}$ be a connected component of $\mathscr{L}$. Since a geodesic lamination has always a closed support, $\mathscr{N}$ has a closed support. Clearly, $\mathscr{N}$ is a union of pairwise disjoint simple geodesics. Therefore, $\mathscr{N}$ is a geodesic lamination. This lamination has no isolated geodesics because it belongs to a strongly nontrivial geodesic lamination. Due to Lemma 3.36, every geodesic of $\mathscr{N}$ is dense in $\mathscr{N}$. Hence, $\mathscr{N}$ is a minimal geodesic lamination. It follows from Lemma 3.27 that every geodesic of $\mathscr{N}$ is nontrivially recurrent. □

The following assertion says that a weakly irrational geodesic lamination looks like a quasiminimal set: the topological closure of a nontrivially recurrent leaf.

Lemma 3.41 *The topological closure of a nontrivially recurrent simple geodesic is a weakly irrational geodesic lamination, and vice versa, a weakly irrational geodesic lamination is a topological closure of every geodesic that belongs to.*

Proof. Let g be a nontrivially recurrent geodesic. By Lemma 3.33, its topological closure $\mathrm{clos}(g)$ is a geodesic lamination. As the closure of an arcwise connected set, $\mathrm{clos}(g)$ is connected. Obviously, $\mathrm{clos}(g)$ has no isolated geodesics because g is dense in $\mathrm{clos}(g)$. By Lemma 3.36, each geodesic in $\mathrm{clos}(g)$ is dense. Hence, $\mathrm{clos}(g)$ is minimal and does not contain closed geodesics. The last part of the statement follows from Lemma 3.40. □

By analogy with trajectories, let us introduce the concept of B-recurrentness for geodesics. Let g be a geodesic equipped with a natural parameter (= length). Then g is said to be *B-recurrent* or *recurrent in Birkhoff sense* if, given any $\varepsilon > 0$, there exists a number $T(\varepsilon) > 0$ such that any arc of g of length $T(\varepsilon)$ approximates the whole g with accuracy ε. Note that the definition does not depend on parametrization of g.

Clearly that a closed geodesic is B-recurrent. The following assertion is similar to the statement that a nonclosed B-recurrent trajectory is nontrivially recurrent (see Lemma 3.28).

Lemma 3.42 *A nonclosed B-recurrent simple geodesic is nontrivially recurrent* (*both in the positive and negative directions*).

Proof is similar to the proof of Lemma 3.28. Let g be a nonclosed B-recurrent geodesic. Fix some natural parametrization on g that induces the corresponding orientation. Take any positive ray $g^+ = \{g(t)\colon t \geqslant 0\}$ of g and arbitrary point $p \in g^+$. By definition of B-recurrentness, for any $\varepsilon > 0$, there is $T(\varepsilon) > 0$ such that any arc of g of the length $T(\varepsilon)$ approximates g with an accuracy ε. Given any $T_0 > 0$, take such arc $A \subset g^+$ with endpoints $g(t_1)$, $g(t_2)$, where $T_0 \leqslant t_1 \leqslant t_2 = t_1 + T(\varepsilon)$. Since A approximates g, there is a point $g(t) \in A$ with $t_1 \leqslant t \leqslant t_2$ ε-close to p. It follows that $p \in \omega(g^+)$. Hence, g^+ is nontrivially recurrent. By the similar way, one can prove that any negative ray of g is also nontrivially recurrent. □

Lemma 3.43 *Let g be a simple nonclosed B-recurrent geodesic. Then the both $\mathscr{T}(g_+)$ and $\mathscr{T}(g_-)$ are B-recurrent trajectories of* Geo^t. *Vice versa, if $\mathscr{T}(g_+)$ or $\mathscr{T}(g_-)$ is a B-recurrent trajectory of* Geo^t, *then g is a B-recurrent geodesic provided g is simple.*

Proof. According to Lemmas 3.41 and 3.42, each g_+ and g_- belongs to weakly irrational geodesic laminations (possibly, the same) provided g is a simple nonclosed B-recurrent geodesic. By Theorem 3.9 (see also Corollary 3.11), $\mathscr{T}(g_+)$ and $\mathscr{T}(g_-)$ belong to minimal sets of Geo^t. It follows from Theorem 3.8 that the both $\mathscr{T}(g_+)$ and $\mathscr{T}(g_-)$ are B-recurrent trajectories of Geo^t. The second part of the statement we leave to the Reader as exercise. □

The following result strengthens Lemma 3.41.

Lemma 3.44 *A weakly irrational geodesic lamination consists of nonclosed B-recurrent geodesics each being dense in it.*

Proof. Let $\mathscr{L}$ be a weakly irrational geodesic lamination and $g \in \mathscr{L}$ be an arbitrary geodesic. Since $\mathscr{L}$ is minimal, g is dense in $\mathscr{L}$. By Corollary 3.11, $\mathscr{T}(g_+)$ and $\mathscr{T}(g_-)$ belong to minimal sets of Geo^t. It follows from Lemma 3.43 and Theorem 3.8 that g is a B-recurrent geodesic. □

In general, a trajectory (or a leaf) that is nontrivially recurrent in one direction should not necessarily be the same in the opposite direction. For example, a separatrix of an irrational flow is nontrivially recurrent only in a unique direction. As well as a nontrivially recurrent trajectory is not necessarily B-recurrent (for example, any nontrivially recurrent trajectory of irrational flow with non-empty set of fixed points). The following lemma says that it is true for geodesics of weakly irrational geodesic laminations (it can be very roughly explained by the fact that a geodesic flow has no fixed points).

Lemma 3.45 *A simple geodesic that is nontrivially recurrent in the positive direction is nontrivially recurrent in the negative direction. A simple geodesic that is nontrivially recurrent* (*in both directions*) *is B-recurrent.*

Proof. Let g be a simple geodesic that is nontrivially recurrent in the positive direction. According to Lemma 3.33, the topological closure $\operatorname{clos} g$ is a geodesic lamination. It is easily seen that $\operatorname{clos} g$ is connected as the closure of an arcwise connected set.

Let us show that $\operatorname{clos} g$ has no isolated geodesics. Indeed, a geodesic that is different from g and belongs to $\operatorname{clos} g$ is a limit geodesic for g and, hence, is not isolated. The g itself is not isolated because it is nontrivially recurrent in the positive direction and, therefore, the positive ray of g is indefinitely close to any point of g.

By Lemma 3.36, each geodesic in $\operatorname{clos} g$ is dense. Hence, $\operatorname{clos} g$ does not contain closed geodesics, because a closed geodesic lies at a nonzero distance from any fixed point of the geodesic g and can not be dense in $\operatorname{clos} g$. Thus, $\operatorname{clos} g$ is a weakly irrational (i. e., minimal and strongly nontrivial) geodesic lamination. Now, the required assertions follow from Lemma 3.44. □

Geodesic garlands

An account of the general theory of laminations can be found in [165]. In particular, this paper contains a theorem on the decomposition of a measurable lamination of codimension one, see Theorem 3.2 [165]. Here, we formulate necessary results concerning geodesic laminations on hyperbolic surfaces. Mainly, the proofs can be found in [63]. Sometimes, we endow the statements with sketches of proofs.

Following [63], we will call a connected component of the set $M \setminus \mathscr{L}$ a *principal domain* for a geodesic lamination $\mathscr{L}$.

Lemma 3.46 *Let $\mathscr{L}$ be a geodesic lamination on a closed orientable hyperbolic surface M, and let D be a principal domain for $\mathscr{L}$. Then the closure of any lift $\overline{D}$ of the domain D to Δ is a contractible noncompact hyperbolic surface whose boundary is the union of geodesics.*

Corollary 3.13 *Let $\mathscr{L}$ be a geodesic lamination on a closed orientable hyperbolic surface M. If D is a principal domain for $\mathscr{L}$, then the homomorphism $\pi_1(D) \to \pi_1(M)$ induced by the embedding $D \subset M$ is a monomorphism.*

Let $\{\overline{g}_i\}$ be a family of pairwise disjoint geodesics on Δ. We will say that this family forms a *garland* if any two neighboring geodesics have a common ideal endpoint (in hyperbolic geometry, such geodesics are said to be parallel), Fig. 3.40. For a finite garland $\{\overline{g}_i\}$, a situation is possible in which the first and the last geodesics also have common ideal endpoints. In this case, a garland forms a geodesic polygon with ideal vertices on the absolute. Lemma 3.46 implies the following corollary.

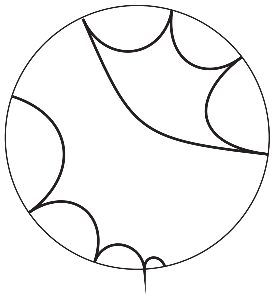


Figure 3.40. Finite garlands.

Corollary 3.14 *Suppose that the hypotheses of Lemma* 3.46 *hold and* $\overline{D}$ *is a lift of a principal domain of the geodesic lamination* $\mathscr{L}$. *Then, the boundary of the domain* $\overline{D}$ *is a union of pairwise disjoint garlands that consist of boundary geodesics.*

Lemma 3.47 *Let* $\mathscr{L}$ *be a geodesic lamination on a closed orientable hyperbolic surface* M. *Then* $\mathscr{L}$ *has only a finite set of principal domains, and each principal domain has only a finite set of boundary geodesics.*

Let $\{\overline{g}_i\}$ be an infinite garland, and let $\sigma_i \in S_\infty$ be a monotonic (in an obvious sense) sequence of ideal endpoints of the geodesics from the garland $\{\overline{g}_i\}$ that converges to $\sigma_* \in S_\infty$ (in the usual metric of the circle on S_∞). We will call the point σ_* an *accumulation point of the garland* $\{\overline{g}_i\}$. The compactness of S_∞ implies that any infinite garland has at least one accumulation point.

Lemma 3.48 *Let* $\mathscr{L}$ *be a nontrivial minimal geodesic lamination on a closed orientable hyperbolic surface* M, *and let the covering lamination* $\overline{\mathscr{L}}$ *have an infinite garland* $\{\overline{g}_i\}$. *Then, any accumulation point of the garland* $\{\overline{g}_i\}$ *is rational.*

Proof. Let $\sigma_i \in S_\infty$ be a monotonic sequence of ideal endpoints of the geodesics from the garland $\{\overline{g}_i\}$ that converges, as $i \to \infty$, to an accumulation point $\sigma_* \in S_\infty$. It follows from Lemma 3.47 that there are congruent geodesics, say $\overline{g}_{i_1}$ and $\overline{g}_{i_2}$, among $\{\overline{g}_i\}$. We can assume that $i_2 > i_1$. Denote by γ a deck transformation such that $\gamma(\overline{g}_{i_1}) = \overline{g}_{i_2}$.

According to Lemma 3.44, $\mathscr{L}$ consists of nonclosed Birkhoff recurrent geodesics. It follows from Lemma 5.6 and the definition of a garland that every point σ_i is reached by exactly two geodesics from the garland $\{\overline{g}_i\}$. Therefore, if two garlands from $\{\overline{g}_i\}$ are congruent (in particular, $\overline{g}_{i_1}$ and $\overline{g}_{i_2}$), then geodesics from the garland $\{\overline{g}_i\}$ that are parallel to them are also congruent. Hence, the sequence of geodesics $\overline{g}_i$ is invariant under γ. Hence, the sequence of ideal endpoints σ_i of these geodesics is also invariant under γ. The monotonicity of σ_i implies that $\gamma(\sigma_*) = \sigma_*$. Since $i_2 \neq i_1$, we have $\gamma \neq \mathrm{id}$. Therefore, σ_* is a rational point. □

Corollary 3.15 *Suppose that the hypotheses of Lemma* 3.48 *hold. Then,*

1) *the infinite garland $\{\overline{g}_i\}$ has exactly two different accumulation points, say $\sigma_{1*}, \sigma_{2*} \in S_\infty$;*
2) *both points are fixed with respect to one deck transformation, which we denote by γ_{12};*
3) *a geodesic $\overline{g}_{12}$ that connects the points σ_{1*} and σ_{2*} is projected to a simple closed geodesic on the surface that does not intersect $\mathscr{L}$.*

Proof. By virtue of Lemma 3.39, all geodesics from $\{\overline{g}_i\}$ have irrational ideal endpoints on the absolute. This and Lemma 3.48 imply that the accumulation points of the garland $\{\overline{g}_i\}$ do not belong to any garland of the covering lamination $\overline{\mathscr{L}}$. Therefore, the ideal endpoints of the geodesics from $\{\overline{g}_i\}$ form a bilateral sequence σ_i that tends to a certain point $\sigma_{1*} \in S_\infty$ as $i \to +\infty$, while as $i \to -\infty$, it tends to a certain point $\sigma_{2*} \in S_\infty$. According to Lemma 3.39, both points σ_{1*} and σ_{2*} are rational; i. e., there exists a deck transformation γ_j such that $\gamma_j(\sigma_{j*}) = \sigma_{j*}$ for $j = 1, 2$. It follows from the proof of Lemma 3.39 that the sequence σ_i is invariant under both transformations γ_1 and γ_2. Therefore, $\gamma_1 = \gamma_2 = \gamma_{12}$. Both points σ_{1*} and σ_{2*} are fixed with respect to γ_{12}. Since γ_{12} is a hyperbolic transformation, we have $\sigma_{1*} \neq \sigma_{2*}$. Hence, $\overline{g}_{12}$ is an axis of γ_{12} and is projected to a closed geodesic on the surface. Since the geodesics of the lamination $\mathscr{L}$ have no transversal self-intersections, $\overline{g}_{12}$ is projected to a simple closed geodesic. Obviously, the latter geodesic does not intersect $\mathscr{L}$. □

Note that Corollary 3.15 can also be derived from Lemma 4.4 in [63].

Bibliographic Notes and Panoramas

Chapter 3. The definition of a local lamination partly came from the theorem concerning the structure of a flow in a neighborhood of regular (non-fixed) point in the Theory of Differential Equations. This theorem states that in a certain neighborhood of regular point, the trajectories of a flow look like a family of parallel straight lines filling the whole neighborhood. Note that for a topological flow (i. e., for a C^0 flow), this assertion was proved only in the two-dimensional case in [218]. Certainly, this assertion for a smooth vector field is valid for any dimension.

(3.1). The notion of local lamination was introduced in [19, 20]. The notion of highly transitivity was introduced in [78]. The examples 3.7, 3.8 was constructed in [200].

(3.2). Here, we mainly follow to [26], Ch. 1, Sec. 2.

(3.3). Definition 3.5 is a straight generalization of the definition of a limit set for a semi-trajectory and trajectory of flow.

(3.4). Here, we mainly follow to [20].

(3.5). The notion of Σ-arc was introduces by Gutierrez [101].

(3.6). The proof of theorem 3.6 can be found in [7].

(3.7). Theorem 3.8 proved by G. Birkhoff [47].

The concept of a quasiminimal set was introduced by Hilmy [114] for a dynamical system as an ordinary dynamically indecomposable invariant set (see details in [114] or [172]). In the case of a dynamical system, our definition is equivalent to Hilmy's one for a compact manifold (see [168, Ch. 4]). In this case, a topological closure of a nontrivially recurrent semitrajectory contains a continuum nontrivially recurrent trajectories (due to Theorem 4.5 of Cherry [64]) each of whose is dense in this topological closure. Therefore, the topological closure of a nontrivially recurrent semitrajectory is a dynamically indecomposable set.

(3.8). Here, we mainly follow to [63].

CHAPTER 4

Poincaré–Bendixson Theory for Local Laminations

In the study of dynamical systems of physical or any other origin it is important to specify stationary (stabilized) motions. A mathematical model of such motions are limit sets, i. e. sets where all motions go as the time $t \to \pm\infty$. It is of prior interest therefore, to understand what kind of limit sets an individual motion may have. This investigation is traditionally called “Poincaré–Bendixson Theory” to honor two scientists who pioneered the study in the simplest case of flows on the plane and 2-sphere.

The central role in Poincaré–Bendixson Theory for surface flows (orientable foliations) plays the theorem on the absence of nontrivially recurrent semitrajectories for sphere and plane. We reformulate this result in a little general manner in Section 4.1.

On surfaces of higher genus, there are orientable foliations with nontrivially recurrent semileaves and leaves. As a consequence, the list of limit sets is extended by a quasiminimal set. For non-orientable foliations, nontrivially recurrent leaves exist even on a disk and sphere, see Examples 3.7, 3.9. To get the list of all limit sets we need deep results of T. Cherry, A. G. Maier and their generalizations on a structure of quasiminimal sets.

A quasiminimal set, in general, contains singularities. By the definition of quasiminimal set, at least one nontrivially recurrent positive (or negative) semileaf, say l, is dense in the quasiminimal set Q. Therefore, any singularity of Q belongs to $\omega(\alpha)$-limit set of l but l does not tend to this singularity (i. e., the $\omega(\alpha)$-limit set $\omega(\alpha)(l)$ does not coincide with any singularity). Such situation was studied by I. Bendixson [45] for an isolated singularity. Roughly speaking, he proved that in this case there are specific leaves, called separatrices, that tend to the singularity and form saddle sectors, see Section 4.2. In Section 4.3, one proves Cherry’s theorem on the existence of continual sets of nontrivially recurrent leaves in a quasiminimal set each being dense in this quasiminimal set. Although original Cherry's theorem concerns flows, it can be easily extended to local laminations. In Section 4.4, we present the series of remarkable Maier's theorems that clarify the structure of quasiminimal sets for surface local laminations. Taking in mind the methodological advantage, we begin with oriented foliations (flows) first historically considered by A. Maier.

4.1. Poincaré – Bendixson Theorems for Local Laminations

Theorem 4.1 *Let $\mathscr{F}$ be an orientable foliation on a surface* (*possibly, noncompact and non-orientable*) *M with the property that every simple two-sided closed curve separates M. Then $\mathscr{F}$ has no nontrivially recurrent semileaves.*

Proof. Suppose the contrary. Let l^+ be a nontrivially recurrent positive semileaf with the starting point $l^+(0)$. Due to Lemma 3.21 and Theorem 3.4, there exists a simple two-sided closed transversal, say C, for $\mathscr{F}$ that intersects l^+. Without loss of generality, one can assume that l^+ intersects C at a point after $l^+(0)$. By condition, C separates M into two parts M_1, M_2. To be definite, let l^+ enter in M_1. Then $l^+(0) \in M_2$. Since l^+ is nontrivially recurrent, l^+ must leave M_1. Leaving M_1 and entering in M_2, l^+ must intersect C with the opposite index of intersection to that when l^+ leaves M_2 and enters to M_1 because C is two-sided and separates M. This is impossible, since $\mathscr{F}$ is an orientable foliation, so all leaves intersect C with the same index of intersection. □

This theorem can be naturally extended to orientable local laminations as follows.

Theorem 4.2 *Let $\mathscr{D}$ be an orientable local lamination on an orientable surface* (*possibly, noncompact*) *M with the property that every simple closed curve separates M. Then $\mathscr{D}$ has no nontrivially recurrent semileaves.*

Proof. Suppose the contrary. Let l^+ be a nontrivially recurrent positive semileaf with the starting point $l^+(0)$. Because of l^+ is nontrivially recurrent, there exists a Σ-loop, say C, formed by a transversal segment Σ and some Σ-arc of l^+. Without loss of generality, one can assume Σ so small that l^+ intersects Σ with the same index of intersection. Since M is orientable, C is two-sided and separates M into two parts M_1, M_2. To be definite, let l^+ enter in M_1. Then $l^+(0) \in M_2$. Since l^+ is nontrivially recurrent, l^+ must leave M_1 intersecting Σ with the opposite index of intersection. This is impossible, since $\mathscr{D}$ is an orientable local lamination. □

Corollary 4.1 *Let $\mathscr{D}$ be an orientable local lamination on a surface M which is either the 2-sphere with without k, $k \geqslant 0$, disks or the projective plane with without m, $m \geqslant 0$, disks. Then $\mathscr{D}$ has no nontrivially recurrent semileaves.*

Proof. It remains to consider the case when M is the projective plane P^2 without m, $m \geqslant 0$, disks. Take a double cover for M that is the 2-sphere without $2m$ disks. By Lemma 3.10, there is the covering local lamination $\overline{\mathscr{D}}$ for $\mathscr{D}$. Since $\mathscr{D}$ is orientable, $\overline{\mathscr{D}}$ is also orientable. By Theorem 4.2, $\overline{\mathscr{D}}$ has no nontrivially recurrent semileaves. Due to Lemma 3.12, the local lamination $\mathscr{D}$ has no nontrivially recurrent semileaves as well. □

Often, Poincaré–Bendixson theorem is formulated as follows (again, we formulate in a little general manner).

Theorem 4.3 *Let $\mathscr{D}$ be an orientable local lamination on an orientable surface (possibly, noncompact) M with the property that every simple closed curve separates M. Suppose that positive semileaf l_0^+ belongs to the ω-limit set of positive semileaf l^+. Then if the ω-limit set $\omega(l_0^+)$ contains regular points, l_0^+ is a closed leaf.*

Proof. Take a regular point $m_0 \in \omega(l_0^+)$ and a transversal segment Σ through m_0. Suppose l_0^+ is not a closed leaf. Then there is a Σ-loop, say S_0, that divides M into two parts in each of whose there are points of l_0^+. By orientability, l^+ can intersect Σ in only one direction. This contradicts to the inclusion $l_0^+ \subset \omega(l^+)$. □

Note that the condition on the existence of regular points is essential: one can easily construct a loop of separatrix that belongs to ω-limit set of positive semileaf.

Corollary 4.2 *Let $\mathscr{D}$ be an orientable local lamination on an orientable surface (possibly, noncompact) M with the property that every simple closed curve separates M. Suppose that the ω-limit set of positive semileaf l^+ does not contain singularities. Then $\omega(l^+)$ is a closed leaf.*

Theorems 4.1, 4.3 allow to get the list of all possible limit sets for an individual semileaf of orientable foliation with finitely many singularities on a sphere, projective plane, plane domain, etc.

4.2. Bendixson theorems

We develop here results by Bendixson [45] for flows on a plane $\mathbb{R}^2$. We consider similar concepts and results for foliations, including non-oriented ones. Mainly, they are formulated and proved virtually without changes. Note that for greater generality, all the concepts can be introduced for so-called *Sing-extendible local laminations*: these are local laminations that can be extended, in a neighborhood of the set of singularities, to foliations with the same set of singularities. Since the considered concepts actually belong to the local theory, this restriction does not influence the proofs. Nevertheless, such a generalization proves usefulness, for example, when studying partial foliations [72].

Separatrix extensions

Let $l^+(m_0) = l^+ = \{m(t) \colon t \geqslant 0\}$ be a positive semileaf of leaf l of some foliation. Suppose that the ω-limit set $\omega(l^+)$ consists of exactly one isolated singularity, say s. Let us draw a transversal segment Σ through some point $m_0 \in l^+$. At m_0, l^+ defines naturally the normal orientation of Σ, and therefore, the right Σ_{R} and left Σ_{L} segments into which Σ is divided by m_0:

$$\Sigma_{\mathrm{R}} \cap \Sigma_{\mathrm{L}} = \varnothing, \quad \operatorname{clos} \Sigma_{\mathrm{R}} \cup \operatorname{clos} \Sigma_{\mathrm{L}} = m_0.$$

Let $U(s)$ be a neighborhood of s that does not contain m_0. Later on, by a neighborhood of a point we mean, for simplicity, a domain homeomorphic to an open disk with the boundary homeomorphic to a topological circle. Since $\omega(l^+) = s$, l^+ enters in $U(s)$ as the parameter increases. Hence, by Theorem 3.2 on continuous dependence of leaves on the initial conditions, every semileaf starting with Σ near m_0 and moving along l^+ enters in $U(s)$. The semileaf l^+ is called *extendible to the right* or *right extendible with respect to the neighborhood* $U(s)$ if all semileaves starting with Σ_{R} near m_0 and moving along l^+ enter $U(s)$ and after go out of $U(s)$, Fig 4.1 (a). The *left extendibility* is defined similarly (by replacing Σ_{R} with Σ_{L}). Sometimes, right extendible or left extendible semileaf, we simply call extendible.

Note that in the general case, the semileaf under consideration may belong to a nonorientable foliation. Therefore, the parametrization of leaves that lie close to l^+ and intersect Σ is not necessarily consistent with the parametrization of l^+. This may occur if, for example, l^+ is a separatrix of a thorn. However, if a foliation is orientable, then we will always assume that the parametrizations on close-lying semileaves are consistent.

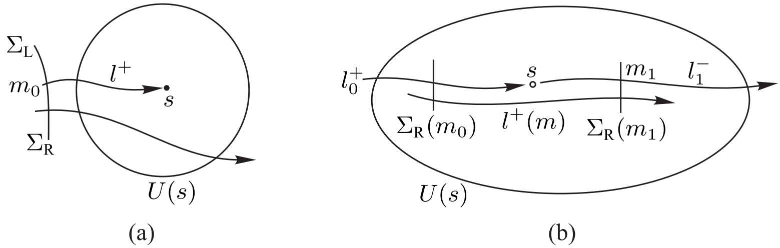


Figure 4.1. (a) Right extendible semileaf l^+; (b) Right Bendixson extension of the separatrix l_0^+.

It follows immediately from the definitions that a left or right extendible semileaf is a separatrix.

Lemma 4.1 *Let s be a singularity and $U(s)$ be a neighborhood of s such that the closure $\operatorname{clos} U(s) = \partial U(s) \cup U(s)$ does not contain singularities except s. Then there are finitely many semileaves that are extendible with respect to $U(s)$.*

Proof. Suppose that there are infinitely many semileaves $l(m_k)$, $k \in \mathbb{N}$, that are extendible with respect to $U(s)$, $m_k \notin U(s)$. Without loss of generality, we can assume that all $l(m_k)$ are positive semileaves with $\omega(l_k) = s$. Therefore, there is a last (with increasing parameter) point A_k where $l(m_k)$ intersects $\partial U(s)$. Thus, $l^+(A_k)$ belongs to $U(s)$, see Fig 4.2, (a). Let A be an accumulation point of A_k. Since $\partial U(s)$ does not contain singularities, A belongs to a leaf. By Theorem 3.2 on continuous dependence of leaves on the initial conditions, the semileaf $l^+(A)$ entering in $U(s)$ can not leave $U(s)$.

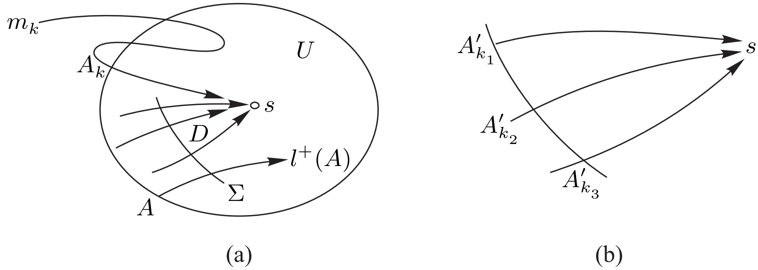


Figure 4.2. The semileaf $l(A'_{k_2})$ is not extendible with respect to $U(s)$.

Take a point $A' \in U(s)$ on $l^+(A)$, and draw a transversal segment Σ through A'. Again by Theorem 3.2, $l(m_k)$ must intersect Σ at point A'_k for all sufficiently large k. One can assume that $l(m_k)$ after A'_k goes to s with no intersecting Σ. Take any points A'_{k_1}, A'_{k_2}, A'_{k_3}, and suppose that A'_{k_2} is between A'_{k_1} and A'_{k_3} on Σ.

Then the semileaves $l(A'_{k_1})$, $l(A'_{k_3})$ that go to s together with the segment Σ and the point s form the closed curve C that bounds an open domain, say D (so-called, Bendixson sack), in $U(s)$. By conditions, there are no singularities in D. Therefore, $l(A'_{k_2})$ can not leave D, see Fig 4.2, (b). This contradicts the extendibility of $l(m_{k_2})$. $\square$

Now, we introduce the concept of the Bendixson extension with respect to a fixed neighborhood. For simplicity, we shall speak on the right Bendixson extension of an ω-separatrix in the positive direction. The left Bendixson extension and extensions in the negative direction (for an α-separatrix) are defined similarly.

Let l_0^+ and l_1^- be the ω- and α-separatrices respectively of the isolated singularity s, $\omega(l_0^+) = \alpha(l_1^-) = s$, and $U(s)$ be a neighborhood of s that does not contain other singularities. Then, l_1^- is called the *right Bendixson extension of l_0^+ with respect to $U(s)$* if

- l_1^- goes out of $U(s)$;
- for any $m_0 \in l_0^+ \cap U(s)$ and $m_1 \in l_1^- \cap U(s)$, there are segments without contacts with $\Sigma(m_0)$, $\Sigma(m_1) \subset U(s)$ that pass through the points m_0 and m_1, respectively, such that
  - any positive leaf $l^+(m)$ that passes through a point $m \in \Sigma_{\mathrm{R}}(m_0)$ intersects $\Sigma_{\mathrm{R}}(m_1)$ for the first time at a point $\widetilde{m}$ without going out of $U(s)$;
  - $\widetilde{m} \to m_1$ as $m \to m_0$, Fig. 4.1 (b).

Figures 4.3 (a), (b) show that Bendixson extensions depend on a neighborhood $U(s)$. Note that l_1^- could geometrically coincide with l_0^+ if, for example, s is a thorn.

If we fix a neighborhood of a singularity, then, according to the following lemma, the Bendixson extension always exists.

Lemma 4.2 *Suppose that a positive semileaf l^+ is right extendible with respect to a neighborhood $U(s)$ of a singularity s, $\omega(l^+) = s$. If $\operatorname{clos} U(s)$ contains no other singularities except s, then there exists a unique negative semileaf l_1^- with $\alpha(l_1^-) = s$ such that l_1^- is the right Bendixson extension of l^+ with respect to $U(s)$.*

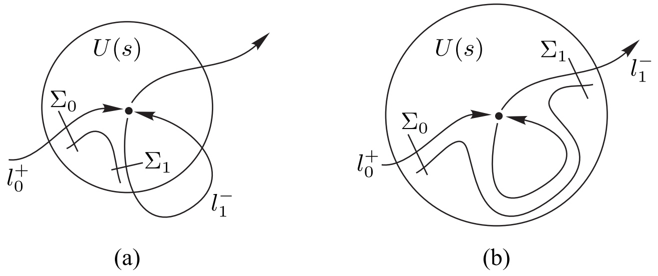


Figure 4.3. Right Bendixson extension of l_0^+ depends on neighborhoods.

Proof. Take a transversal (open) segment Σ through some point $m \in$ $\in l^+ \cap U(s)$. Assume that $m_i \to m$, $m_i \in \Sigma_{\mathrm{R}}$, and let A_i be the first point where $l^+(m_i)$ intersects $\partial U(s)$. Here we suppose that semileaves through m_i that satisfy the definition of extendibility are positive. Denote by A an accumulation point of points A_i, $i \in \mathbb{N}$, and by $l^-(A)$ the semileaf that enters in $U(s)$. Without loss of generality, one can assume that $l^-(A)$ is negative semileaf which enters in $U(s)$ when the parameter decreases. Without loss of generality, we can assume that the arc of every semileaf $l^+(m_i)$ between the points m_i, A_i intersects Σ only at m_i, $i \in \mathbb{N}$, see Fig. 4.4.

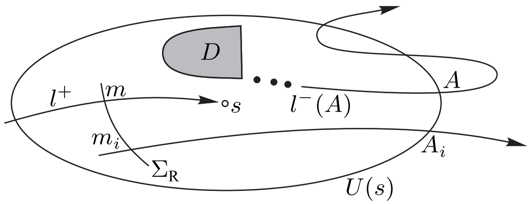


Figure 4.4. Semileaves $l^-(A)$.

Suppose that $l^-(A)$ leaves $\operatorname{clos} U(s)$ when the parameter decreases. Then there are two possibilities: 1) $l^-(A)$ does not intersect Σ; 2) $l^-(A)$ intersects Σ. In the case 1), by Theorem 3.2 on continuous dependence of leaves on the initial conditions, $l^-(A_i)$ must also leave $\operatorname{clos} U(s)$ with no intersecting Σ what is absurd. In the case 2), denote by m_* the first point of intersection $l^-(A)$ with Σ. Again by Theorem 3.2, one gets that $m_i \to m_*$, and hence $m =$ $= m_*$, which is impossible. Therefore, $l^-(A)$ does not leave $\operatorname{clos} U(s)$ when the parameter decreases. Hence, $\alpha(l^-(A)) \subset \operatorname{clos} U(s)$, and $\alpha(l^-(A)) \neq \varnothing$.

If $\alpha(l^-(A)) = s$, then there is nothing to prove, and $l^-(A) = l_1^-$ is the right Bendixson extension of the semileaf l^+ with respect to the neighborhood $U(s)$.

Suppose $\alpha(l^-(A)) \neq s$. Then $\alpha(l^-(A))$ consists of regular points. Certainly, $l^+ \notin \alpha(l^-(A))$, otherwise $l^-(A)$ leaves $\operatorname{clos} U(s)$. Hence, $\alpha(l^-(A)) \subset \subset \operatorname{clos} U(s) - l^+(m_*)$, where m_* is a point after whose l^+ does not leave $\operatorname{clos} U(s)$, i. e. $l^+(m_*) \subset \operatorname{clos} U(s)$. As a consequence, $\alpha(l^-(A))$ belongs to the simply connected set $\operatorname{clos} U(s) - l^+(m_*)$. Take a regular point $p \in \alpha(l^-(A))$. Without loss of generality, one can assume that $p \in U(s) - - l^+(m_*)$, because if not, we can take a neighborhood $U'(s) \supset \operatorname{clos} U(s)$ that does not contain any singularities except s. Let Σ_p be a transversal segment through p such that $\Sigma_p \subset \operatorname{clos} U(s) - l^+(m_*)$. Since p is a regular point, $l^-(A)$ intersects Σ_p infinitely many times. Consider any Σ_p-loop, say C, of $l^-(A)$. Then C bounds a disc D that belongs to $\operatorname{clos} U(s) - l^+(m_*)$. Obviously, $s \notin D$. On the other hand, D contains a singularity by Corollary 3.7. The contradiction proves lemma. □

The similar statement takes place for a left Bendixson extension. In the both cases, l_1^- is an α-separatrix, $\omega(l^+) = \alpha(l_1^-)$.

Lemma 4.3 *Any separatrix of an isolated singularity s is extendible with respect to a neighborhood $U(s)$ in whose there are no singularities except s.*

Proof. Let $l^+(m_0) = l^+$ be an ω-separatrix of s, $\omega(l^+) = s$. Let us take a transversal segment $\Sigma \subset U(s)$ through some point $m_0 \in l^+$. One can assume that $l^+(m_0) \subset U(s)$. The point m_0 divides naturally Σ into the right Σ_{R} and left Σ_{L} segments. For definiteness, suppose that l^+ is locally right-sided. Then by the definition of separatrix (see Section 3.7), there is a sequence $m_i \in \Sigma_{\mathrm{R}}$, $m_i \to m_0$, $i \in \mathbb{N}$, such that $l^+(m_i)$ leaves $U(s)$. We have to prove that given any point $\widetilde{m} \in \Sigma_{\mathrm{R}}$ sufficiently close to m_0, $l^+(\widetilde{m})$ leaves $U(s)$. If we suppose the contrary, one can construct a Bendixson sack, Fig. 4.2 (b), that prevents l^+ to be a locally right-sided separatrix. □

Let l^+ be a separatrix tending to the singularity s. Then l^+ is (right or left) extendible with respect to a neighborhood $U(s)$. By Lemma 4.2, l^+ has a Bendixson extension, say l^-, which is an α-separatrix. Actually, it is possible that if we take a smaller neighborhood $U'(s) \subset U(s)$, one can get a Bendixson extension different from l^-, see Figures 4.3 (a), (b). So it is natural to give the following definition. The separatrix l^- is called a (*right or left*) *Bendixson extension of* l^+ if l^- is a (right or left) Bendixson extension of l^+ with respect to any neighborhood $U'(s) \subset U(s)$.

In general, a separatrix does not always have a Bendixson extension. However, if a singularity has a finite number of separatrices, then a separatrix always has a Bendixson extension.

Lemma 4.4 *Let l^+ be an ω-separatrix that tends to an isolated singularity s, $\omega(l^+) = s$. If s has only a finite number of separatrices, l^+ has a unique right Bendixson extension l_1^- that is an α-separatrix of the singularity $s = \alpha(l_1^-)$.*

Proof. Take a sequence of neighborhoods $U_1 \supset \cdots \supset U_n \supset \cdots$ such that $\bigcap\limits_{n \geqslant 1} U_n = s$ and U_1 has no singularities except s. By Lemma 4.2, there is the right Bendixson extension l_n^- (which is a separatrix) of l^+ with respect to $U_n(s)$ for every $n \geqslant 1$. Since s has only a finite number of separatrices, l^+ has the same right Bendixson extension beginning with some $n \geqslant n_1$. □

Thus, if an isolated singularity s has a finitely many separatrices, every left or right sided separatrix l_1 of s has a uniquely defined left or right Bendixson extension l_2 respectively that is also a separatrix of s. Sometimes, l_1 and l_2 are called *adjacent*.

The next statement is a useful generalization of Lemma 3.32 that gives a sufficient condition of a semileaf to be a separatrix.

Lemma 4.5 *Let s be an isolated singularity and $U(s)$ be a neighborhood of s that has no other singularities. Then the following statements hold.*

1) *If there is a positive semileaf l^+ that tends to s, $\omega(l^+) = s$, and l^+ lies in the limit set of a semileaf l_*, then l^+ is an ω-separatrix of s. Moreover, the limit set of l_* contains s and the Bendixson extension (right or left that depends on which side l_* indefinitely approaches l^+) with respect to $U(s)$;*
2) *If s lies in the ω-limit set of a positive semileaf l_* but $s \neq \omega(l_*)$, then s has an ω-separatrix that lies in $\omega(l_*)$.*

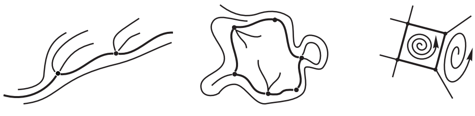
Proof. The first statement directly follows from Lemma 3.32 and the definition of Bendixson extension. To prove the second statement, take a neighborhood $U(s)$ with points of $\omega(l_*)$ outside of it and without singularities except s. Since $s \in \omega(l_*)$, there is a sequence $m_k \in l_* \cap U(s)$ that tends to s as $k \to \infty$. The semileaves $l^+(m_k)$ leave $U(s)$ because there are points of $\omega(l_*)$ outside of $U(s)$. Denote by A_k the first point where $l^+(m_k)$ intersects $\partial U(s)$, and let A^+ be an accumulation point of the sequence A_k, $k \in \mathbb{N}$. Similar to the proof of Lemma 4.2, one can prove that $l^-(A^+)$ is an α-separatrix of s. Now the second statement follows from the first one. □

This lemma holds if the positive semileaf l_* and its ω-limit set are replaced by a negative semileaf and its α-limit set respectively.

Generalized leaves

Let $\mathscr{F}$ be a foliation with isolated singularities each of whose has only finitely many separatrices. A generalized leaf is, roughly speaking, a sequence of consecutive adjacent separatrices. To be precise, a sequence of leaves $l_1, l_2, \ldots, l_k$ and singularities $s_1, \ldots, s_{k-1}$, $k \geqslant 2$, is called a *finite nonclosed generalized leaf* if one can introduce parametrizations on l_i, $1 \leqslant i \leqslant k$, such that

1) $l_1, l_2, \ldots, l_{k-1}$ are ω-separatrices, and $l_2, \ldots, l_k$ are α-separatrices (hence, $l_2, \ldots, l_{k-1}$ are separatrix connections), where $\omega(l_i) = s_i$ for any $1 \leqslant i \leqslant k-1$ and $\alpha(l_i) = s_{i-1}$ for any $2 \leqslant i \leqslant k$.
2) $l_1, l_2, \ldots, l_k$ are either left-sided or right-sided separatrices. Accordingly, l_i is a left (right) Bendixson extension of the leaf l_{i-1} for any $2 \leqslant i \leqslant k$;
3) the leaf l_1 is not an α-separatrix, and l_k is not an ω-separatrix, Fig. 4.5 (a).



(a) (b) (c)

Figure 4.5. Generalized leaves.

If we replace the last condition by the equality $l_1 = l_k$, then we obtain the definition of a *closed generalized leaf*, or a *closed generalized contour*, Fig. 4.5 (b).

Lemma 4.6 *Let L be a closed generalized leaf of a foliation $\mathscr{F}$ with finitely many singularities and separatrices. Suppose that the ω-limit set $\omega(l)$ of a positive semileaf l contains L. Then $\omega(l) = L$.*

Proof is quite similar to the proof of Lemma 3.25. We omit it. □

In the theory of dynamical systems on surfaces, a closed generalized contour is often called a *one-sided separatrix contour*. It is clear how one should modify the definition of a finite generalized leaf in order to obtain the definition of an *infinite generalized leaf*. It is easy to see that if a closed one-sided generalized leaf represents a non-null-homotopic curve on a surface, then its lift on a universal covering plane is an infinite generalized leaf.

Lemma 4.7 *Let Q be a quasiminimal set of a foliation $\mathscr{F}$ with finitely many singularities and separatrices. Then any separatrix connection $l \subset Q$ belongs to a finite generalized leaf $L \subset Q$ such that the first and the last semileaves of L are not separatrix connections.*

Proof. Suppose for definiteness that l is approached locally from the right side by some nontrivially recurrent leaf $l_* \subset Q$, i.e. l is right improper. Since Q contains a finitely many singularities and separatrices, right Bendixson extensions of l form a finite generalized leaf $L \subset Q$. If we assume that L is a closed curve, then by Lemma 4.6, L is a limit set of l_*, that is impossible since the limit set of l_* is Q. □

Note that a separatrix connection can belong to different generalized leaves one of them could be finite, see Fig. 4.5 (c), but this finite generalized leaf does not belong to a quasiminimal set.

Lemma 4.5, combined with the Theorem 3.2 on the continuous dependence on the initial conditions, implies the following statement.

Lemma 4.8 *Let $\mathscr{F}$ be a foliation with a finitely many singularities and separatrices on an orientable surface and l be a nontrivially recurrent leaf in the positive direction that is not so in the negative direction. Suppose that l is left improper. Then,*

1) *the α-limit set $\alpha(l)$ consists of exactly one singularity, say s_0;*
2) *l belongs to a generalized leaf that consists of a finite number of (one-dimensional) leaves $l_k, l_{k-1}, \ldots, l_0 \overset{\text{def}}{=} l$, $k \geqslant 1$, and singularities $s_{k-1}, \ldots, s_0$, where*
   a) *the leaves $l_k, l_{k-1}, \ldots, l_1$ are left-sided ω-separatrices, and the leaves $l_{k-1}, \ldots, l_0$ are left-sided α-separatrices (hence, $l_{k-1}, \ldots, l_1$ are separatrix connections);*
   b) *the leaf l_{i-1} is a left Bendixson extension of l_i for any $1 \leqslant i \leqslant k$;*
   c) *$\omega(l_i) = \alpha(l_{i-1}) = s_{i-1}$ for any $1 \leqslant i \leqslant k$;*
   d) *the α-limit set $\alpha(l_k)$ of the leaf l_k does not consist of a single singularity.*

Structure of isolated singularity

The following theorem is the first step to describe the topological structure of isolated fixed point. Remark that the both possibilities stated in this theorem are realized, Fig. 4.6.

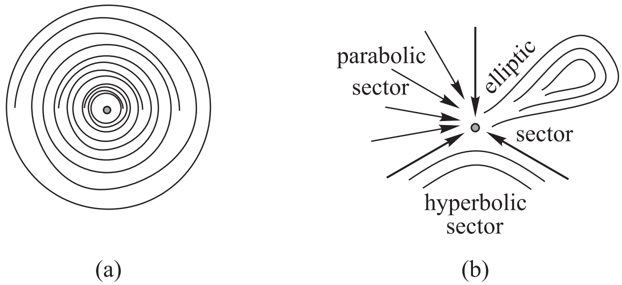


Figure 4.6. Structure of isolated singularity.

Theorem 4.4 *Let s_0 be an isolated singularity of a foliation $\mathscr{F}$. Then either any neighborhood of s_0 contains a closed leaf surrounding s_0, or there exists a semileaf tending to s_0.*

Proof. Let $U(s_0)$ be a neighborhood diffeomorphic to an open disk with no singularities in $\operatorname{clos} U(s_0)$ except s_0. First, we show that there is a semileaf lying entirely in $U(s_0)$. Take a sequence of points $m_k \to s_0$ as $k \to \infty$. Assume that the both semileaves $l^+(m_k)$, $l^-(m_k)$ leave $U(s_0)$. Such situation appeared in the proof of the second statement of Lemma 4.5 when s_0 belongs to limit set of a semileaf that leaves $U(s_0)$. Word for word, one can show that there is a separatrix of s_0 (see also Lemma 4.2).

Let $l^\cdot$ be a semileaf lying entirely in $U(s_0)$. For the definiteness, we take $l^\cdot$ to be a positive semileaf, $l^\cdot = l^+$. Then $\omega(l^+) \subset U(s_0)$. If $\omega(l^+) = s_0$ then there is nothing to prove. Let $m \in \omega(l^+)$ be a regular point. Then the leaf $l(m)$ belongs to $U(s_0)$. If $l(m)$ is closed, it must surround s_0, and the theorem is proved since $U(s_0)$ is an arbitrary neighborhood.

Suppose $l(m)$ is not a closed leaf, and the ω- or α-limit set of $l(m)$ does not coincide with s_0. Then there is a transversal segment $\Sigma \subset U(s_0)$ intersected by $l(m)$ at least two times. Hence, there is a Σ-loop of $l(m)$, say Σ_0, that divides $U(s_0)$ into two parts d_1, d_2. Assume $s_0 \in d_1$. Since $l(m) \subset \omega(l^+)$, l^+ must intersect Σ infinitely many times. As a consequence, there is a Σ-arc of l^+ belonging to d_2. The corresponding Σ-loop bounds a disk d that lays in d_2. By Corollary 3.7, d contains a singularity, which is impossible since s_0 is a unique singularity in $U(s_0)$. This contradiction shows that ω- or α-limit set of $l(m)$ coincides with s_0. □

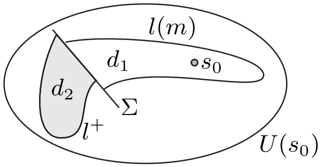


Figure 4.7. The domains d_1 and d_2.

The first possibility of Theorem 4.4 could be realized by the system of differential equations on $\mathbb{R}^2$:

$$\dot{x} = -y, \quad \dot{y} = x. \tag{4.1}$$

Here, any one-dimensional trajectory is periodic. The fixed point (0; 0) is called a *center* or *center-type* fixed point, Fig. 4.8 (a). One can construct an isolated singularity with arbitrarily small neighborhood containing periodic and non-periodic trajectories, Fig. 4.6 (a).

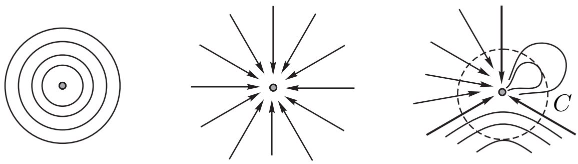


Figure 4.8. Center-type fixed point (a); attractive node (b); the curve C.

Now we represent the description by Bendixson [45] of the second possibility of Theorem 4.4. Let s_0 be an isolated singularity and $U(s_0)$ be a neighborhood diffeomorphic to an open disk with no singularities in $\operatorname{clos} U(s_0)$ except s_0. Take a leaf $l(m)$ through a current point $m \in U(s_0)$. There are three cases: 1) one semileaf of $l(m)$ tends to s_0 while another semileaf of $l(m)$ leaves $U(s_0)$; 2) the both semileaves of $l(m)$ go out of $U(s_0)$; 3) the both semileaves of $l(m)$ tend to s_0.

If case 1) holds for every $m \in U(s_0)$, s_0 is called a *node*, Fig. 4.8 (b). For oriented foliation, s_0 is *attractive* or *stable* provided all positive semileaves $l^+(m)$ tend to s_0, i.e. $\omega(l(m)) = s_0$, otherwise s_0 is *repelling* or *unstable*. An attractive node can be defined by the following system of differential equations:

$$\dot{x} = -x, \quad \dot{y} = -y. \tag{4.2}$$

In general, the leaves of the types 1), 2), and 3) form *parabolic* (*node*), *hyperbolic*, and *elliptic* sectors respectively, see Fig. 4.6 (b). Suppose that s_0 has a finitely many sectors, namely h hyperbolic sectors, e elliptic sectors, and n node sectors. Take a closed curve C being in general position with respect to leaves and surrounding s_0. One can choose C so that it has h external and e internal tangency points, Fig. 4.8 (c). By (3.10), we get

$$\operatorname{ind}(s_0) = \frac{1}{2}(2 - h + e). \tag{4.3}$$

If all sectors of s_0 are hyperbolic, s_0 is called a *saddle-type singularity* or simply *saddle*.

Lemma 4.9 *Suppose a foliation $\mathscr{F}$ has only isolated singularities each with finitely many separatrices. If $\mathscr{F}$ is transitive, every singularity is a saddle.*

Proof. By condition, $\mathscr{F}$ has a dense leaf. Lemma 4.5 implies that every singularity has separatrices that divide a small neighborhood of singularity into sectors. Because of transitivity, there are no elliptic and parabolic sectors. □

4.3. Cherry theorem

In this section, we present the Cherry theorem for local laminations (including foliations) on surfaces. In 1937, T. Cherry [64] proved that a quasiminimal set N of a flow (recall that a quasiminimal set is a closure of some nontrivially recurrent semitrajectory) contains nondenumerable (noncountable) set of nontrivially recurrent trajectories, each being dense in N. The idea was to construct a family of flow boxes which generates an "infinite tree", see Fig. 4.11. An infinite path in the tree corresponds to a point that belongs to a nontrivially recurrent trajectory which is everywhere dense in N. Since there is a nondenumerable set of paths in the tree, one gets the nondenumerable family of nontrivially recurrent trajectories. We follow this idea of Cherry.

Theorem 4.5 *Let l^+ be a nontrivially recurrent semileaf of a local lamination $\mathscr{D}$ on a surface M^2. Suppose that the intersection of the support* $\operatorname{supp}\mathscr{D}$ *with any compact transversal segment is a closed set. Then the quasiminimal set* $\operatorname{clos} l^+ \stackrel{\text{def}}{=} Q$ *contains a nondenumerable family of nontrivially recurrent leaves each of which is dense in Q.*

Proof. Due to Lemma 3.12, one can assume that M^2 is orientable. We also can assume that l^+ is endowed with the injective parametrization $t\colon l^+ \to \mathbb{R}^+$ with a starting point m_0, $t(m_0) = 0$.

Let P be a closed trivially foliated box. We say that P *has a parametric length at least* $T > 0$, if P contains the arc $\widehat{pq}$ of l^+ such that $p, q \in \partial P$ and $|t(p) - t(q)| \geqslant T$. Recall that P does not contain singularities, and the boundary ∂P of P consists of two transversal segments and two arcs of some leaves. Here, the points p, q belong to the different transversal segments that belong to ∂P.

Note that since l^+ is nontrivially recurrent, l^+ intersects any transversal segment Σ infinitely many times provided l^+ intersects Σ in an interior point of Σ. In particular, if Σ is an open segment and $l^+ \cap \Sigma \neq \varnothing$, then l^+ intersects Σ infinitely many times. It is now convenient to divide the proof into three steps.

Step 4.1 *Given any numbers $\varepsilon > 0$, $T > 0$ and transversal segments Σ, Σ_* such that $\Sigma \cap \Sigma_* = \varnothing$, $\Sigma \cap l^+ \neq \varnothing$, $\Sigma_* \cap l^+ \neq \varnothing$, and the length of $\Sigma_* \stackrel{\text{def}}{=} \Sigma_\varepsilon$ is less than ε, there are a compact segment $\Phi(\Sigma, \varepsilon, T) \subset \Sigma$ and two foliated boxes $P_1(\Sigma, \varepsilon, T) = P_1$, $P_2(\Sigma, \varepsilon, T) = P_2$ such that*

1) $\operatorname{int}\Phi(\Sigma, \varepsilon, T) \cap l^+ \neq \varnothing$;

2) *the foliated boxes P_1, P_2 have disjoint interiors,* $\operatorname{int} P_1 \cap \operatorname{int} P_2 = \varnothing$, *and $\Phi(\Sigma, \varepsilon, T)$ is a unique common part of its boundaries,* $\operatorname{clos} P_1 \cap \operatorname{clos} P_2 = \Phi(\Sigma, \varepsilon, T)$;

3) *the boundary of P_1 and P_2 contains two disjoint segments of Σ_ε;*

4) *the both P_1 and P_2 are of parametric length at least T. To be precise, given any compact subset $K \subset l^+$, there are arcs $\widehat{m_1 m_2} \subset P_1$, $\widehat{m_2 m_3} \subset P_2$ of l^+ such that*

$$t(m_1) < t(m_2) < t(m_3), \quad t(m_2) - t(m_1) \geqslant T, \quad t(m_3) - t(m_2) \geqslant T, \tag{4.4}$$

where $m_2 \in \Phi(\Sigma, \varepsilon, T)$, $m_1, m_3 \in \Sigma_\varepsilon$, and the arc $\widehat{m_1 m_3} \subset l^+$ has no intersection with K;

5) *the length of $\Phi(\Sigma, \varepsilon, T)$ is less than ε, Fig.* 4.9.

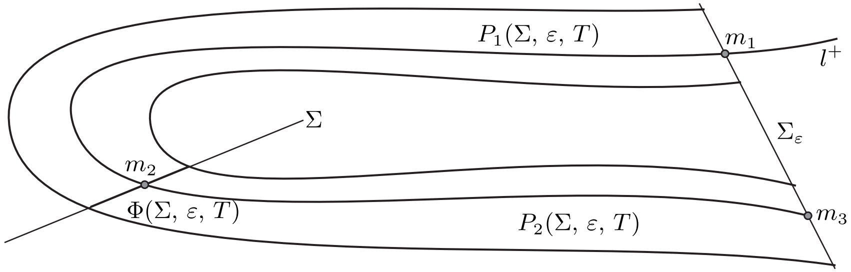


Figure 4.9. The segment $\Phi(\Sigma,\, \varepsilon,\, T)$.

Proof of step 4.1. It follows from $\Sigma_\varepsilon \cap l^+ \neq \varnothing$ and $\Sigma \cap l^+ \neq \varnothing$ that l^+ intersects the both Σ_ε and Σ infinitely many times at points with arbitrary large parameters. Therefore, there is the arc $\widehat{m_1 m_3} \subset l^+$ that satisfies (4.4). Note that one can take $m_1 = m_0$. By Theorem 3.1, there is the closed foliated box $\widetilde{P}_1$ containing $\widehat{m_1 m_2}$ whose transversal boundary consists of two segments $\Sigma_1 \subset \Sigma_\varepsilon$ and $\Sigma_2 \subset \Sigma$. Similarly, there is the foliated box $\widetilde{P}_2$ whose transversal boundary consists of two segments $\Sigma_2' \subset \Sigma$ and $\Sigma_3 \subset \Sigma_\varepsilon$. Since l^+ is nonclosed, $m_1 \neq m_3$. Therefore we can assume that $\Sigma_1 \cap \Sigma_3 = \varnothing$. Since $\widehat{m_1 m_2} \cap \widehat{m_2 m_3} = m_2$, there are segment

$$\Phi(\Sigma,\, \varepsilon,\, T) \subset \Sigma_2' \cap \Sigma_2, \quad m_2 \in \operatorname{int} \Phi(\Sigma,\, \varepsilon,\, T),$$

and foliated boxes

$$P_1(\Sigma,\, \varepsilon,\, T) \subset \widetilde{P}_1, \quad P_1(\Sigma,\, \varepsilon,\, T) \subset \widetilde{P}_1,$$

that satisfy items 2), 3), 4). Since $m_2 \in \operatorname{int} \Phi(\Sigma,\, \varepsilon,\, T)$, $\operatorname{int} \Phi(\Sigma,\, \varepsilon,\, T) \cap l^+ \neq \varnothing$ which gives item 1). Clearly, the length of $\Phi(\Sigma,\, \varepsilon,\, T)$ can be taken arbitrary small. $\diamondsuit$

The segment $\Phi(\Sigma,\, \varepsilon,\, T)$ satisfying the conditions of step 4.1 is said to be *fitting segment* for Σ with *neighbor foliated boxes* $P_1(\Sigma,\, \varepsilon,\, T)$, $P_2(\Sigma,\, \varepsilon,\, T)$. In sense, the second step is a double first one.

Step 4.2 *Suppose that the condition of step* 4.1 *holds. Then there are disjoint segments* $\Sigma_0,\, \Sigma_1 \subset \Sigma$ *and fitting segments* $\Phi(\Sigma_0,\, \varepsilon,\, T)$, $\Phi(\Sigma_1,\, \varepsilon,\, T)$ *for* Σ_0, Σ_1 *with neighbor foliated boxes* $P_1(\Sigma_0,\, \varepsilon,\, T)$, $P_2(\Sigma_0,\, \varepsilon,\, T)$ *and* $P_1(\Sigma_1,\, \varepsilon,\, T)$, $P_2(\Sigma_1,\, \varepsilon,\, T)$ *respectively such that*

1) $\operatorname{int} \Sigma_i \cap l^+ \neq \varnothing$, $i = 0,\, 1$;

2) $(P_1(\Sigma_0,\, \varepsilon,\, T) \cup P_2(\Sigma_0,\, \varepsilon,\, T)) \cap (P_1(\Sigma_0,\, \varepsilon,\, T) \cup P_2(\Sigma_0,\, \varepsilon,\, T)) = \varnothing$;

3) *all neighbor foliated boxes satisfy the conditions of item* 4) *of step* 4.1;
4) *the length of each* $\Phi(\Sigma_0, \varepsilon, T)$ *and* $\Phi(\Sigma_1, \varepsilon, T)$ *is less than* ε.

Proof of step 4.2. It obviously follows from the fact that the segment Σ is intersected by l^+ infinitely many times the existence of disjoint segments $\Sigma_0, \Sigma_1 \subset \Sigma$ with $\operatorname{int} \Sigma_i \cap l^+ \neq \varnothing$, $i = 0, 1$. Now given $\varepsilon > 0$, $T > 0$ and transversal segments Σ_0, Σ_*, there is a fitting segment $\Phi(\Sigma_0, \varepsilon, T)$ with neighbor foliated boxes $P_1(\Sigma_0, \varepsilon, T)$, $P_2(\Sigma_0, \varepsilon, T)$ by step 4.1. In particular, the length of $\Phi(\Sigma_0, \varepsilon, T)$ is less than ε. Taking all of them and the segment Σ_1 thinner, if necessary, one can assume that Σ_1 has no intersections with the both $P_1(\Sigma_0, \varepsilon, T)$ and $P_2(\Sigma_0, \varepsilon, T)$ keeping the property that Σ_1 is intersected by l^+ infinitely many times. Let us choose and fix some arc A of l^+ in $\Sigma_0, \varepsilon, T) \cup P_2(\Sigma_0, \varepsilon, T)$ with endpoints at the transversal segment Σ_*. By step 4.1, where we take $K = A$, there is a fitting segment $\Phi(\Sigma_1, \varepsilon, T)$ with neighbor foliated boxes $P_1(\Sigma_1, \varepsilon, T)$, $P_2(\Sigma_1, \varepsilon, T)$ that satisfy items 1), 3) and 4). Again, taking all fitting segments and corresponding neighbor foliated boxes thinner, if necessary, one gets that all neighbor foliated boxes satisfy item 2). $\diamondsuit$

Take decreasing sequence ε_j tending to 0 and increasing sequence T_j tending to $+\infty$ as $j \to \infty$. Then

$$\Sigma_{\varepsilon_1} \supset \Sigma_{\varepsilon_2} \supset \ldots \Sigma_{\varepsilon_j} \supset \ldots$$

are nested sequence of segments with the middle point m_0. Denote Σ by Σ_1. Applying step 4.1 for $\varepsilon = \varepsilon_1$ and $T = T_1$, we get the fitting segment $\Phi(\Sigma_1, \varepsilon_1, T_1) \subset \Sigma$ and corresponding foliated boxes $P_1(\Sigma_1, \varepsilon_1, T) = P_1$, $P_2(\Sigma_1, \varepsilon_1, T_1) = P_2$ (the union of these foliated boxes looks like the largest horseshoe in Fig. 4.10).

Due to step 4.2, there are segments $\Sigma_{10}, \Sigma_{11} \subset \Phi(\Sigma_1, \varepsilon_1, T_1)$ such that $\Sigma_{10} \cap \Sigma_{11} = \varnothing$ and $\operatorname{int} \Sigma_{1i} \cap l^+ \neq \varnothing$, $i = 0, 1$. Moreover, for each of segments Σ_{10}, Σ_{11} and numbers $\varepsilon = \varepsilon_2$, $T = T_2$, we get the fitting segments $\Phi(\Sigma_{1i}, \varepsilon_2, T_2) \subset \Sigma_{1i}$, $(i = 0, 1)$ and corresponding foliated boxes satisfying the statements of step 4.1. Again by step 4.2, one can choose in every segment $\Phi(\Sigma_{1i}, \varepsilon_2, T_2)$ two segments $\Sigma_{1i0}, \Sigma_{1i1} \subset \Phi(\Sigma_{1i}, \varepsilon_2, T_2)$ such that $\Sigma_{1i0} \cap \Sigma_{1i1} = \varnothing$ and $\operatorname{int} \Sigma_{1ik} \cap l^+ \neq \varnothing$, $k = 0, 1$, Fig. 4.10.

Continuing this procedure, one gets the family of closed segments Σ_1, Σ_{10}, Σ_{11}, $\ldots$, $\Sigma_{i_1 i_2 \ldots i_n}$, $\ldots$, where $i_1 = 1$, $i_n \in \{0; 1\}$ for $n \geqslant 2$, with the following properties:

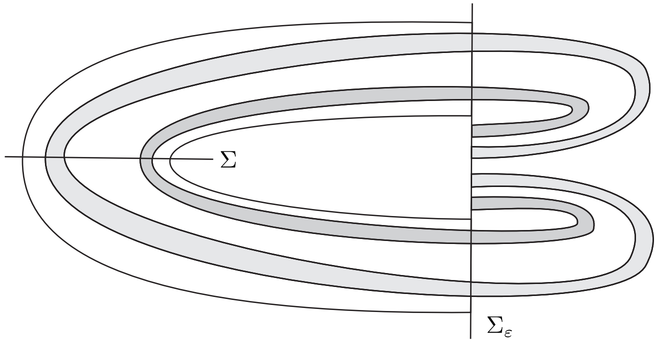


Figure 4.10. Construction of foliated boxes.

1) $\Sigma_{i_1 i_2 \ldots i_{n-1} i_n} \subset \Sigma_{i_1 i_2 \ldots i_{n-1}}$ for the both $i_n = 0$ and $i_n = 1$.
2) The length of $\Sigma_{i_1 i_2 \ldots i_n}$ is less than ε_n.
3) $\Sigma_{i_1 i_2 \ldots i_n} \cap \Sigma_{j_1 j_2 \ldots j_n} = \varnothing$, if there is at least one index k with $i_k \neq j_k$.
4) $\operatorname{int} \Sigma_{i_1 i_2 \ldots i_n} \cap l^+ \neq \varnothing$.
5) For any segment $\Sigma_{i_1 i_2 \ldots i_n}$, there are two foliated boxes $P_k(\Sigma_{i_1 i_2 \ldots i_n}, \varepsilon_{n+1}, T_{n+1})$, $k = 0, 1$, satisfying step 4.1, see Fig. 4.11.

$$\Sigma \supset \Phi(\Sigma_1, \varepsilon_1, T_1)$$

$$\Phi(\Sigma_1, \varepsilon_1, T_1) \supset \Sigma_{10}, \qquad \Phi(\Sigma_1, \varepsilon_1, T_1) \supset \Sigma_{11}$$

$$\Sigma_{10} \supset \Phi(\Sigma_{10}, \varepsilon_2, T_2), \qquad \Sigma_{11} \supset \Phi(\Sigma_{11}, \varepsilon_2, T_2)$$

$$\Phi(\Sigma_{10}, \varepsilon_2, T_2) \supset \Sigma_{100}, \; \Sigma_{101}, \qquad \Phi(\Sigma_{11}, \varepsilon_2, T_2) \supset \Sigma_{110}, \; \Sigma_{111}$$

$$\Sigma_{100} \supset \ldots \qquad \Sigma_{101} \supset \ldots \qquad \Sigma_{110} \supset \ldots \qquad \Sigma_{111} \supset \ldots$$

Figure 4.11. The family of segments $\Sigma_{i_1 i_2 \ldots i_n \ldots}$.

Since all segments $\Sigma_{i_1 i_2 \ldots i_n \ldots}$ are closed and $\varepsilon_n \to 0$, the intersection

$$\bigcap_{i_1 = 1,\, i_n \in \{0; 1\},\, n \geqslant 2} \Sigma_{i_1 i_2 \ldots i_n} \stackrel{\text{def}}{=} Q_{i_1 i_2 \ldots i_n \ldots}$$

is a point for every concrete sequence $i_2, \ldots, i_n, \ldots$. It follows from $\operatorname{int}\Sigma_{i_1 i_2 \ldots i_n} \cap l^+ \neq \varnothing$ that any point $Q_{i_1 i_2 \ldots i_n \ldots}$ belongs to $\operatorname{clos}(l^+)$. Since the intersection of the support $\operatorname{supp}\mathscr{D}$ with any compact transversal segment is a closed set, there is a (one-dimensional) leaf $l(Q_{i_1 i_2 \ldots i_n \ldots})$ through $Q_{i_1 i_2 \ldots i_n \ldots}$. Since the cardinality of set of sequences $i_1 = 1$, i_2, ..., i_n, ... is nondenumerable, the set $\mathscr{Q} = \bigcup\limits_{i_n \in \{0;\,1\}} Q_{i_1 i_2 \ldots i_n \ldots}$ is nondenumerable also.

Step 4.3 *Any leaf* $l(Q_{i_1 i_2 \ldots i_n \ldots})$ *is nontrivially recurrent and dense in* $\operatorname{clos}(l^+)$.

Proof of step 4.3. Take the semileaves $l^{\pm}(Q_{i_1 i_2 \ldots i_n \ldots})$, positive and negative, of the leaf $l(Q_{i_1 i_2 \ldots i_n \ldots})$. Since $Q_{i_1 i_2 \ldots i_n \ldots} \stackrel{\text{def}}{=} Q \in \operatorname{clos}(l^+)$, the both semileaves $l^{\pm}(Q)$ and its limit set belong to $\operatorname{clos}(l^+)$. It follows from $\varepsilon_j \to 0$ and $T_j \to \infty$ that m_0 belongs to the limit set of each of the semileaves $l^+(Q)$ and $l^-(Q)$. Hence, l^+ belongs to the limit set of each of the semileaves $l^+(Q)$ and $l^-(Q)$ as well. Therefore, $l^{\pm}(Q)$ are nontrivially recurrent and dense in $\operatorname{clos}(l^+)$. $\diamondsuit$

Since every leaf $l(Q_{i_1 i_2 \ldots i_n \ldots})$ intersects Σ at countable set of points and the set $\mathscr{Q}$ is noncountable, the set of disjoint leaves $l(Q_{i_1 i_2 \ldots i_n \ldots})$ is noncountable as well. This concludes the proof. $\square$

4.4. Maier theorems

Recall that in the late 1930s and early 1940s, Cherry [64] and Maier [142] obtained classical results concerning the structure of quasiminimal sets of flows. Although A. Maier considered smooth flows with finitely many fixed points, his theorems hold for topological flows with arbitrary set of fixed points (the conditions on smoothness and cardinality of fixed point set, one needs for Maier's descriptions of so-called essential trajectories and the topological type of domains on what M is divided by essential trajectories). Below, we'll show that Maier's theorems hold even for noncompact surfaces but of finite genus. We begin with the original Maier's theorems for flows (orientable foliations).

The main topological feature used in the proof of Poincaré–Bendixson theorem (see Theorems 4.1, 4.3) is that any simple closed curve divides a plane domain. For surfaces we need the following separation property. Let M_p^2 be a closed orientable surface of genus $p \geqslant 0$. It is well-known (see exm., [154]) that any family of simple closed curves $C_1, \ldots, C_n$ that have intersection only at some point $x_0 \in M_p^2$ bounds an open disk provided $n \geqslant 2p+1$. This means that there is a component of the complement $M_p^2 - \bigcup_{i=1}^{n} C_i$ to these curves that is homeomorphic to an open disk. The similar statement holds for surfaces with holes and punctures.

Lemma 4.10 *Let M be an orientable surface of finite genus and $x_0 \in M$ be an arbitrary point. Let $l_1, \ldots, l_n, \ldots$ be a sequence of simple* (*i. e. without self-intersections*) *closed curves with the only one common point x_0. Then there is a number n_0 which does not depend on a sequence l_i such that*

$$M - \left(l_1 \cup \ldots \cup l_{n_0}\right)$$

contains a component homeomorphic to a planar domain (*a sphere without some family of open disks, and closed disks, and points*).

Proof. The surface M is homeomorphic to a closed orientable surface M_0 after deleting some family of open disks, and closed disks, and points [197]. Every of these open disks, and closed disks, and points can be surrounded by a closed simple curve that bounds a planar domain. Therefore the statement reduces to the case of closed surface for which the result is true. □

Oriented local laminations

Following a historical canvas, we begin with the Maier's results for flows, which compose important part of theory of oriented local laminations. Consider a nontrivially $\omega(\alpha)$-recurrent trajectory l. It is obvious that the $\omega(\alpha)$-limit set of l contains regular points (any point of l) and l belongs to a limit set of some trajectory, for example $l \in \omega(\alpha)(l)$. Maier [141, 142] proved that these conditions are sufficient for a non-periodic trajectory to be nontrivially recurrent.

Theorem 4.6 *Let f^t be a flow on a surface M* (*possibly, noncompact or non-orientable*) *of finite genus. Suppose f^t has a positive semitrajectory l^+ which intersects some compact transversal segment infinitely many times. If l^+ belongs to the limit set of some semitrajectory l_1^+ then l^+ is a nontrivially ω-recurrent semitrajectory.*

Proof. Due to Lemma 3.12, it is enough to prove the theorem for orientable M because we can take an orientable two-sheeted covering surface with a covering flow.

Let Σ be a compact transversal segment intersected by l^+ infinitely many times. Then there is an accumulating point $m_* \in \Sigma$ of $l^+ \cap \Sigma$. So there exists a sequence of points $m_i \in l^+ \cap \Sigma$ such that $m_i \to m_*$ and points m_i correspond to increasing time parameters as $i \to \infty$. Taking a subsequence, if necessary, we can assume that all points m_i belong to the same component of $\Sigma - m_*$ and form monotone sequence on Σ.

Assume that the theorem is false. Then there is an interval $J \subset \Sigma$ such that the point m_1 is a unique point of intersection l^+ with J, $m_1 = J \cap l^+$. Denote by $\Sigma_0 \subset \Sigma$ the open segment between m_* and m_1. By construction, $m_i \in \Sigma_0$ for $i \geqslant 2$.

Since all points m_i, $i \in \mathbb{N}$, belong to the limit set of l_1^+, there are infinitely many J-arcs[1] of the semitrajectory l_1^+ that intersect Σ_0. Take any such J-arc $\widehat{p_1 q_1} \subset l_1^+$. Let j_1 be the maximal index such that $\widehat{p_1 q_1}$ intersects Σ_0 between points m_{j_1}, m_{j_1+1}, and denote $\widehat{p_1 q_1}$ by l_{j_1}. Clearly, we can assume that the time-parameter of q_1 is more than the time-parameter of p_1. The positive semitrajectory $l_1^+(q_1)$ contains in its limit set all points m_i with $i \geqslant j_1 + 1$. Therefore there is the J-arc of $l_1^+(q_1)$, denoted by $l_{j_2}^+$, intersecting Σ_0 between the points m_{j_2}, m_{j_2+1}, where $j_2 > j_1$, and $l_{j_2}^+$ does not intersect Σ_0 between the points m_*, m_{j_2+1}. Continuing this procedure, we get the sequence of points m_{j_k} and the sequence of pairwise disjoint J-arcs l_{j_k} of l_1^+ ($k \in \mathbb{N}$) such that

- the arc l_{j_k} intersects Σ between the points m_{j_k}, m_{j_k+1} and does not intersect Σ between the points m_{j_k+1}, m_*;
- $j_k + 1 \leqslant j_{k+1}$, $k \in \mathbb{N}$, Fig. 4.12.

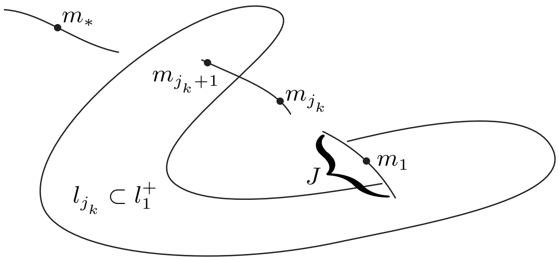


Figure 4.12. J-arcs l_{j_k}.

[1] Recall that the arc $\widehat{pq}$ of l_1^+ with endpoints p, $q \in l_1^+$ is a J-arc if $J \cap \widehat{pq} = p \cup q$.

By Lemma 4.10, there is the index n_0 such that some component of $M - \left(J \cup l_{j_1} \cup \ldots \cup l_{j_{n_0}}\right)$, say G, is a planar domain. Without loss of generality we can assume that $l_{j_{n_0}}$ belongs to the boundary of G. By properties quoted above, $l_{j_{n_0}}$ intersects Σ between the points $m_{j_{n_0}}$ and $m_{j_{n_0}+1}$, but other curves l_{j_k}, $1 \leqslant k < n_0$, do not intersect Σ between the points $m_{j_{n_0}}$, $m_{j_{n_0}+1}$. Let us prove that exactly one of the points $m_{j_{n_0}}$, $m_{j_{n_0}+1}$ belongs to G, while the another point belongs to $M - G$. Indeed, when we are moving along $l_{j_{n_0}}$, the domain G lies only from one side that is either left or right. Since $l_{j_{n_0}}$ intersects Σ_0 with the same index of intersection at every point, $m_{j_{n_0}}$ and $m_{j_{n_0}+1}$ lie in different sides from $l_{j_{n_0}}$. Taking in the mind that l_{j_k} for $1 \leqslant k < n_0$ do not intersect Σ between the points $m_{j_{n_0}}$, $m_{j_{n_0}+1}$, one gets the result.

To simplify matters, one replaces the index j_{n_0} by n_0. Let us consider the both possibilities.

1) $m_{n_0} \in G$, $m_{n_0+1} \notin G$. The positive semitrajectory $l^+(m_{n_0})$ has to leave G because $m_{n_0+1} \notin G$. However, the boundary of G consists of arcs of l_1^+ and J. Hence, $l^+(m_{n_0})$ must intersect J at a point different from m_1 because l^+ is (obviously) nonclosed. This contradicts to the equality $m_1 = J \cap l^+$, Fig. 4.13.

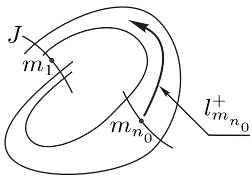


Figure 4.13. The semitrajectory $l^+(m_{n_0})$ must intersect J.

2) $m_{n_0} \notin G$, $m_{n_0+1} \in G$. In this case, the positive semitrajectory $l^+(m_{n_0})$ must enter to G because points m_i correspond to increasing time parameters as $i \to \infty$. Again, since the boundary of G consists of arcs of l_1^+ and J, $l^+(m_{n_0})$ has to intersect J at a point different from m_1. We get the same contradiction. $\square$

Corollary 4.3 *Let f^t be a flow on M of finite genus (possibly, noncompact or non-orientable). Suppose f^t has a positive semitrajectory l^+ which contains* *in its ω-limit set $\omega(l^+)$ a regular point. If l^+ belongs to the limit set of some semitrajectory l_1^+ then l^+ is a nontrivially ω-recurrent semitrajectory.*

Clearly that the similar statements hold for negative semitrajectories and its α-limit sets. Theorem 4.6 has an obvious generalization for local laminations.

Theorem 4.7 *Let $\mathscr{D}$ be an orientable local lamination on a surface M (possibly, noncompact or non-orientable) of finite genus. Suppose $\mathscr{D}$ has a positive semileaf l^+ which intersects some compact transversal segment infinitely many times. If l^+ belongs to the limit set of some semileaf l_1^+ then l^+ is a nontrivially ω-recurrent semileaf.*

Proof repeats, word for word, the proof of Theorem 4.6. The only place modified is the following: due to orientability, there is a subsegment $\Sigma_* \subset \Sigma$ containing the accumulation point m_* which is intersected by leaves of $\mathscr{D}$ with the same index of intersection. Next we continue with a sequence of points $m_i \in l^+ \cap \Sigma_*$. $\Box$

Mutual limiting of nontrivially recurrent trajectories

By a similar way, A. Maier proved the theorem which says that if a nontrivially recurrent semitrajectory belongs to the limit set of another nontrivially recurrent semitrajectory then vice versa, i. e. the second nontrivially recurrent semitrajectory belongs to the limit set of the first one. This theorem has an obvious generalization for local laminations.

Theorem 4.8 *Let l_1^+, l_2^+ be positive nontrivially recurrent semileaves of orientable local lamination $\mathscr{D}$ on a surface M of finite genus. If l_2^+ belongs to the limit set of l_1^+, then l_1^+ belongs to the limit set of l_2^+.*

Proof. Again, due to Lemma 3.12, it is enough to prove the theorem for orientable M. Take a transversal segment Σ intersected by l_2^+ at some interior point of Σ. Since l_2^+ is nontrivially ω-recurrent, Σ is intersected by l_2^+ infinitely many times. Then there is an accumulating point $m_* \in \Sigma$ of $l_2^+ \cap \Sigma$ such that there exists a sequence $m_i \in l_2^+ \cap \Sigma$, $i \in \mathbb{N}$, tending to m_* and corresponding to increasing time parameters as $i \to \infty$. Taking a subsequence, if necessary, we can assume that all points m_i belong to the same component of $\Sigma - m_*$ and form monotone sequence on Σ. Since $l_2^+ \subset \omega(l_1)$, l_1^+ intersects Σ infinitely many times as well. Take any point $m_0 \in l_1 \cap \operatorname{int} \Sigma$. Suppose the theorem is false. Then there is the arc $J \subset \Sigma$ around m_0 such that $l_2^+ \cap J = \varnothing$. Since l_1

is nontrivially ω-recurrent, l_1^+ intersects J infinitely many times. Hence, there are infinitely many J-arcs of l_1^+. Next, similarly to the proof of Theorem 4.6, we construct a special sequence of J-arcs of l_1^+ that separates the surface, and show that either l_2^+ intersects J or l_2^+ belongs to a planar domain, which is impossible. □

Non-orientable local laminations

Here we consider Maier theorems for non-orientable local laminations including foliations. First of all, we introduce a basic notation, a wide disposition, and prove Maier theorems for widely disposed local laminations.

Widely disposed local laminations

Suppose that a simple curve l intersects an arc Σ at two points a and b. Recall that the arc $\widehat{ab}$ of l between points a and b is called a Σ-*arc* if $\Sigma \cap \widehat{ab} = a \cup b$. Together with the subsegment of Σ between the points a and b, the Σ-arc $\widehat{ab}$ forms a simple closed curve that is called a Σ-*loop*. When Σ is a closed curve, a Σ-arc and Σ-loop are defined similarly (actually, there are two Σ-loops, one of them is chosen).

Definition 4.1 *Denote by $\mathscr{T}$ an arc or a closed curve that intersects a simple curve l transversally. Then l is said to be* ***widely disposed with respect to $\mathscr{T}$*** *if there are no $\mathscr{T}$-loops that bound a disk.*

On the torus T^2, the wide disposition with respect to a non-null-homotopic simple closed curve means the orientability of intersections with this curve. Indeed, cutting T^2 along a simple non-null-homotopic closed curve C, we obtain an annulus. Therefore, if a simple curve l^+ is widely disposed with respect to C, it intersects C orientably. It can easily be shown that the orientability of the intersection implies the wide disposition on any surface. We formulate this assertion as a lemma for references.

Lemma 4.11 *Let a simple curve l^+ orientably intersect a simple closed curve C. Then, l^+ is widely disposed with respect to C.*

Proof. If there exists a C-loop that bounds a disk, then, obviously, at the endpoints of the corresponding C-arc, the curve l^+ intersects C with different intersection indices. □

On a hyperbolic surface, one can easily construct a simple curve that is widely disposed with respect to a simple closed curve C and intersects C non-orientably. The following lemma follows immediately from definition 4.1 and Theorem 3.2 on the continuous dependence of leaves on the initial conditions on compact arcs.

Lemma 4.12 *Let $\mathscr{D}$ be a local lamination and let l^+ be a nontrivially recurrent semileaf of $\mathscr{D}$ that is widely disposed with respect to a transversal C. Then, any semileaf that lies in the closure of l^+ is widely disposed with respect to C.*

Before continuing, we give the statement showing that there are numerous local laminations with widely disposed leaves.

Lemma 4.13 *Let $\mathscr{F}$ be a foliation with finitely many singularities each of whose is of negative index. Then every leaf is widely disposed with respect to any transversal to $\mathscr{F}$.*

Proof. Suppose the contrary. Let l be a leaf that is not widely disposed with respect to some transversal Σ, i. e. there exists a Σ-loop, say C, of l bounding the disk D. By Corollary 3.7, D contains at least one singular point of positive index. This contradicts to the condition. □

Nontrivial recurrentness of leaves for local laminations

The heart of Maier theorems is the following criterion of a positive semileaf being nontrivially recurrent (clearly, the statement holds for negative semileaf).

Theorem 4.9 *Let $\mathscr{D}$ be a local lamination on a surface M (possibly, noncompact or non-orientable) of finite genus with finitely many boundary components (including ideal boundary). Suppose $\mathscr{D}$ has a positive semileaf l^+ which intersects some compact transversal segment Σ infinitely many times. If l^+ belongs to the limit set of some semileaf l_1^+ that is widely disposed with respect to Σ, then l^+ is a nontrivially ω-recurrent semileaf.*

Proof. We begin in a similar way with the proof of Theorem 4.6. Due to Lemma 3.12, it is enough to prove the theorem for orientable M because we can take an orientable two-sheeted covering surface with covering local lamination. We can assume that the both l^+ and l_1^+ are endowed with some injective parametrizations.

Since Σ is intersected by l^+ infinitely many times, there is an accumulating point $m_* \in \Sigma$ of $l^+ \cap \Sigma$, and there exists a sequence of points $m_i \in l^+ \cap \Sigma$ such that $m_i \to m_*$, and the points m_i correspond to increasing parameters as $i \to \infty$. One of the way to construct such sequence with additional properties is the following. Obviously that one of the components of $\Sigma - m_*$ is intersected by l^+ infinitely many times. Denote this component by Σ (if the both components of $\Sigma - m_*$ are intersected by l^+ infinitely many times, choose one of them). So, now Σ is a segment with m_* being an endpoint. Take some point $m_1 \in l^+ \cup \Sigma$. Without loss of generality, one can assume that m_1 is an initial point of l^+ corresponding to the parameter $t_1 = 0$. Denote by $\Sigma_0 \subset \Sigma$ the open segment between m_* and m_1. Since m_* is an accumulating point of $l^+(m_1) \cap \Sigma$, there is the first intersection $m_2 \in \Sigma_0$ of $l^+(m_1)$ with Σ_0, which corresponds to the parameter $t_2 > t_1$. Certainly, m_* is an accumulation point for the semileaf $l^+(m_2)$. Hence there is the (not necessary first) intersection $m_3 \in \Sigma_0$ of $l^+(m_2)$ with Σ_0, which corresponds to the parameter $t_3 > t_2$ such that m_3 lies between m_* and m_2 on Σ_0. Continuing, we get a sequence of points $m_i \in \Sigma$ that form monotone sequence on Σ_0. Taking m_{i+1} much more closer to m_* than m_i, one gets a sequence converging to m_*.

Assume that the theorem is false. Then there is an interval $J \subset \Sigma$ such that the point m_1 is a unique point of intersection l^+ with J, $m_1 = J \cap l^+$. By construction, $m_i \in \Sigma_0$ for $i \geqslant 2$.

Since all points m_i, $i \in \mathbb{N}$, belong to the limit set of l_1^+, there are infinitely many J-arcs of l_1^+ that intersect Σ_0. Take any such J-arc $\widehat{p_1 q_1} \subset l_1^+$. Let j_1 be the maximal index such that $\widehat{p_1 q_1}$ intersects Σ_0 between the points m_{j_1}, m_{j_1+1}, and denote $\widehat{p_1 q_1}$ by l_{j_1}. Clearly, we can assume that the parameter of q_1 is more than the parameter of p_1. The positive semileaf $l_1^+(q_1)$ contains in its limit set all points m_i with $i \geqslant j_1 + 1$. Therefore there is the J-arc of $l_1^+(q_1)$, denoted by $l_{j_2}^+$, intersecting Σ_0 between the points m_{j_2}, m_{j_2+1}, where $j_2 > j_1$, and $l_{j_2}^+$ does not intersect Σ_0 between the points m_*, m_{j_2+1}. Continuing this procedure, we get the sequence of points m_{j_k} and the sequence of pairwise disjoint J-arcs l_{j_k} of l_1^+ ($k \in \mathbb{N}$) such that

- the J-arc l_{j_k} intersects Σ between the points m_{j_k}, m_{j_k+1} and does not intersect Σ between m_{j_k+1}, m_*;
- $j_k + 1 \leqslant j_{k+1}$, $k \in \mathbb{N}$, Fig. 4.12.

Denote by L_{j_k} the J-loop formed by the J-arc l_{j_k} and corresponding subsegment of J. Recall that L_{j_k} is a closed simple curve.

First, we consider a surface M being a planar domain. By condition of the theorem, this means that M is a sphere after deleting finitely many disks (closed or open) and points. Every such a disk and point is called a hole. The main property of a planar domain is that any closed simple curve splits M into two planar domains.

Any L_{j_k} divides M into two planar domains, say M_k^+ and M_k^-. Suppose that for some $k \in \mathbb{N}$ the points m_{j_k}, m_{j_k+1} belong to the different domains of $M - L_{j_k}$, say $m_{j_k} \in M_k^-$ and $m_{j_k+1} \in M_k^+$. The semileaf $l^+(m_{j_k})$ must leave M_k^- because $m_{j_k+1} \in l^+(m_{j_k})$. Since the boundary of M_k^- consists of J and L_{j_k}, $l^+(m_{j_k})$ must intersect J at a point different from m_1 because l^+ is not closed. This contradicts to $m_1 = J \cap l^+$.

Thus, given any k, the points m_{j_k}, m_{j_k+1} belong to the same domain of $M - L_{j_k}$, say M_k^-. Denote by J_k the segment of Σ between the points m_{j_k}, m_{j_k+1}. Since M_k^+ is a planar domain, l_{j_k} intersects J_k at even number of points. Moreover, there is J_k-arc, say J_k', belonging to l_{j_k} such that the corresponding J_k-loop bounds some planar domain $M_k' \subset M_k^+$, Fig. 4.14. Because of l_1^+ is widely disposed with respect to Σ, the domain M_k' must contain nonzero number of handles and/or holes. Since the segments J_k are pairwise disjoint, M_k' are also pairwise disjoint. Hence, the set of handles and/or holes must be infinite what contradicts to the condition of theorem. This proves the theorem for M being a planar domain.

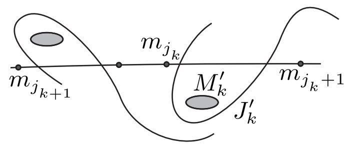


Figure 4.14. The domain $M_k' \subset M_k^+$.

Let M now not be a planar domain. If the family of the loops $L_{j_1}, \ldots, L_{j_k}$ for some $k \in \mathbb{N}$ divides M into several surfaces $M_1, M_2, \ldots$ such that the points m_{j_k}, m_{j_k+1} belong to different surfaces, we get the quite similar contradiction with $m_1 = J \cap l^+$ because the boundary of the surface M_ν containing m_{j_k} consists of J and $L_{j_1}, \ldots, L_{j_k}$.

Suppose that given any $k \in \mathbb{N}$, the family of the loops $L_{j_1}, \ldots, L_{j_k}$ does not separate the points m_{j_k}, m_{j_k+1}. By Lemma 4.10, there is n_0 such that some component, say G, of $M - \left(J \cup l_{j_1} \cup \ldots \cup l_{j_{n_0}}\right)$ is a planar domain. Without loss of generality we can assume that $l_{j_{n_0}}$ belongs to the boundary of G. By properties quoted above, $l_{j_{n_0}}$ intersects Σ between the points $m_{j_{n_0}}$ and $m_{j_{n_0}+1}$, but other J-arcs l_{j_k}, $1 \leqslant k < n_0$, do not intersect Σ between the points $m_{j_{n_0}}$, $m_{j_{n_0}+1}$. Due to our assumption, there are two possibilities: 1) all points m_{j_k}, $k \in \mathbb{N}$, belong to G; 2) all points m_{j_k}, $k \in \mathbb{N}$, belong to $M - G$.

Let us consider the both possibilities. In the first case, all J-arcs l_{j_n} beginning with $n \geqslant n_0 + 1$ belong to G because the boundary of G is formed by J and some arcs of semileaves (recall that every J-arc intersects J only at its endpoints). Thus this possibility reduces to the case $M = G$ being a planar domain considered above. In the second case, all J-arcs l_{j_n} beginning with $n \geqslant n_0 + 1$ belong to $M - G$ by the similar reason. If $M - G$ is a planar domain, we have actually the case 1). Suppose that $M - G$ is not a planar domain. To simplify matters, one replaces the index j_{n_0} by n_0. Note that since G is a planar domain and the points m_{n_0}, m_{n_0+1} do not belong to G, l_{n_0} intersects J_{n_0} at even number of points (where J_{n_0} is the segment of Σ between the points m_{n_0}, m_{n_0+1}). Furthermore, there is J_{n_0}-arc, say J'_{n_0}, belonging to l_{n_0} such that the corresponding J_{n_0}-loop bounds some planar domain $G' \subset G$. Because of l_1^+ is widely disposed with respect to Σ, the domain G' must contain nonzero number of handles and/or holes. Recall that all J-arcs l_{j_n} beginning with $n \geqslant n_0 + 1$ belong to $M - G$. Therefore, one can repeat the procedure for $M - G$ after renumbering $j_1 = n_0 + 1$ and so on. Since G contains nonzero number of handles and/or holes while the surface M has finitely many handles and holes, the above procedure must finish up with a finding a planar domain with infinitely many points m_{j_k} inside this planar domain. This gives us again the case 1). This proves the theorem. □

Mutual limiting of nontrivially recurrent semileaves

By a proof that is similar to one of Theorem 4.9 (see also the proof of Theorem 4.8), one can get the statement on the mutual limiting of nontrivially recurrent semileaves for local laminations.

Theorem 4.10 *Let $\mathscr{D}$ be a local lamination on a surface M of finite genus with finitely many boundary components* (*including ideal boundary*). *Let l_1, l_2 be nontrivially $\omega(\alpha)$-recurrent leaves of $\mathscr{D}$ such that l_2 intersects some transversal* *segment Σ infinitely many times. If l_1 is widely disposed with respect to Σ and $\omega(\alpha)(l_1) \supset l_2$, then $\omega(\alpha)(l_2) \supset l_1$.*

There are no convenient definition of wide disposition, in general, for a local lamination. Often, one says about wide disposition of some saturated (invariant) set with respect to some special family of transversals (here, by transversal, we mean a transversal segment or closed transversal).

Definition 4.2 *Let $\mathscr{D}$ be a local lamination and N be a saturated set of $\mathscr{D}$. Given transversal Σ, N is said to be* ***widely disposed with respect to*** Σ *if* 1) *at least one leaf of N intersects Σ*; 2) *every nontrivially recurrent leaf of N, if exists, intersects Σ*; 3) *each semileaf of N is widely disposed with respect to Σ.*

A saturated set N is called ***widely weak disposed****, if there is a transversal Σ such that N is widely disposed with respect to Σ.*

At last, N is called ***widely strong disposed*** *or simply* ***widely disposed****, if N is widely disposed with respect to any transversal that has a nonempty intersection with leaves of N.*

In particular, a *foliation is widely disposed* if each leaf is widely disposed with respect to any transversal of this foliation. Lemma 4.13 can be reformulated now as follows: **a foliation $\mathscr{F}$ is widely disposed provided $\mathscr{F}$ has finitely many singularities each of whose is of negative index**.

Definition 4.2 and Theorem 4.10 imply the following

Corollary 4.4 *Let $\mathscr{D}$ be a local lamination on a surface M of finite genus with finitely many boundary components* (*including ideal boundary*)*. Then any weak widely disposed quasiminimal set of $\mathscr{D}$ is a Maier one.*

For references, we resume Theorems 4.9 and 4.10 for widely disposed foliations as the following theorem.

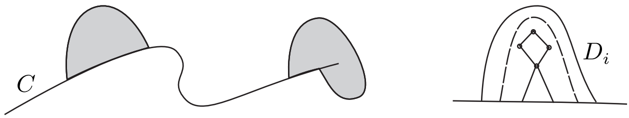
Theorem 4.11 *Let $\mathscr{F}$ be a widely disposed foliation on a surface M of finite genus with finitely many boundary components* (*including ideal boundary*)*. Then*

1) *if $\mathscr{F}$ has a positive semileaf l^+ that intersects some transversal infinitely many times* (*in particular, the ω-limit set $\omega(l^+)$ contains a regular point*) *and l^+ belongs to a limit set of a semileaf, then l^+ is nontrivially ω-recurrent*;

2) *if l_1, l_2 are nontrivially $\omega(\alpha)$-recurrent leaves, and $\omega(\alpha)(l_1) \supset l_2$, then $\omega(\alpha)(l_2) \supset l_1$.*

Maier theorems for foliations with finitely many singularities

Now we consider Maier theorems for foliations with finitely many singularities. Let $\mathscr{F}$ be a foliation on a surface M and C be a transversal segment or closed transversal for $\mathscr{F}$. Suppose that some leaf l intersects C at least at two points and is not widely disposed with respect to C. As a consequence, there are C-loops of l that bound disks. Denote by $\varepsilon(C, l)$ the set of disks bounded by C-loops of l, Fig. 4.15, (a). Note that $\varepsilon(C, l) = \varnothing$ provided l is widely disposed with respect to C. The lemma below means that roughly speaking one can mark a finite family of disks that are responsible for a foliation which is not widely disposed.



(a) (b)

Figure 4.15. The set $\varepsilon(C, l)$, (a), and disk D_i, (b).

Lemma 4.14 *Let $\mathscr{F}$ be a foliation with finitely many singularities on a surface M and C be a transversal segment or closed transversal for $\mathscr{F}$. If $\varepsilon(C, l) \neq \varnothing$, then there is a family $T(\mathscr{F})$ of pairwise disjoint disks $D_1, \ldots, D_k \in \varepsilon(C, l)$ such that*

1) *Given any $D \in \varepsilon(C, l)$, either there is* (*not necessary unique*) *$D_i \in T(\mathscr{F})$ such that $D_i \subset D$ or there is $D_i \in T(\mathscr{F})$ such that $D \subset D_i$ and D contains the same singularities with D_i.*
2) *Given any $D_i \in T(\mathscr{F})$, every $D \in \varepsilon(C, l)$ contains the same singularities with D_i* (*i.e. all singularities contained in D_i*) *provided $D \subset D_i$.*
3) *If l intersects C infinitely many times, then there is a point $m \in l \cap C$ such that $m \notin \bigcup_{i=1}^{k} D_i$.*
4) *If $M = S^2$, then $k \geqslant 4$.*

Proof. Take any $D_1' \in \varepsilon(C, l)$. There are two possibilities: 1) there is a disk $D_1'' \subset D_1'$ that does not contain all singularities from the disk D_1'; 2) any disk $D \in \varepsilon(C, l)$ that belongs to D_1' contains all singularities from the disk D_1', Fig. 4.15, (b). In case 1), we pass from D_1' to $D_1'' \stackrel{\text{def}}{=} D_1'$ and after one considers the possibilities 1), 2) above for new D_1'. Since there are finitely many singularities, this procedure is finished by the possibility 2). In case 2), we take D_1' to the family $T(\mathscr{F})$, $D_1' = D_1$.

If any disk $D \in \varepsilon(C, l)$ that is not in D_1 contains D_1, then the construction of the family $T(\mathscr{F})$ is finished. Suppose that there is $D_2' \in \varepsilon(C, l)$ that does not contain D_1 and is not in D_1. For D_2', we again consider the possibilities 1), 2) above and repeat the procedure that is ended by choosing $D_2 \in T(\mathscr{F})$. By construction, $D_1 \cap D_2 = \varnothing$. If any disk $D \in \varepsilon(C, l)$ that is not in $D_1 \cup D_2$ contains D_1 or D_2, the construction of $T(\mathscr{F})$ is finished. Otherwise, we continue in the similar way. Since there are finitely many singularities and any disk D_i contains at least one singularity, one gets the finite family $T(\mathscr{F})$ of disks satisfying the first two items.

Suppose that there is a disk $D_i \in T(\mathscr{F})$ such that inside D_i there exists a disk $D \in \varepsilon(C, l)$. Then one changes D_i by D in the family $T(\mathscr{F})$. Due to the properties of the disks $D_1, \ldots, D_k \in \varepsilon(C, l)$, the new family $T(\mathscr{F})$ satisfies the first two items of the lemma. Any point of the intersection of C and C-arc that belongs to the boundary of the old D_i is not in the union $\bigcup_{i=1}^{k} D_i$. This proves the item three. If we assume that given any $D_i \in T(\mathscr{F})$, there are no $D \in \varepsilon(C, l)$ inside D_i, we get the contradiction with the condition that l intersects C infinitely many times.

The proof of the last item, we leave to the reader like an exercise. □

Remark. The family $T(\mathscr{F})$ is in general not uniquely determined for fixed C and l. However, one can prove that the cardinality (number of disks) of this family does not depend on the concrete makeup of $T(\mathscr{F})$.

Theorem 4.12 *Let $\mathscr{F}$ be a foliation with finitely many singularities on a surface M of finite genus with finitely many boundary components* (*including ideal boundary*)*. Suppose $\mathscr{F}$ has a positive semileaf l^+ which contains in its ω-limit set a regular point. If l^+ belongs to the limit set of some semileaf l_1^+, then l^+ is nontrivially ω-recurrent.*

Proof. There are two ways to reduce the proof to Theorem 4.9. Take a regular point $m_0 \in \omega(l^+)$ and a transversal segment Σ through m_0. Obviously, l_1^+ intersects Σ infinitely many times. If $\varepsilon(\Sigma, l_1^+) = \varnothing$, the result follows from Theorem 4.9. Suppose $\varepsilon(\Sigma, l_1^+) \neq \varnothing$. By Lemma 4.14, there is the family $D_1, \ldots, D_k \in \varepsilon(\Sigma, l_1^+)$ of disks satisfying the conditions of this lemma.

Remove any singularity from each disk $D_1, \ldots, D_k$. Due to Lemma 4.14, we get the foliation $\mathscr{F}'$ on a surface M' such that the corresponding semileaves l^+ and l_1^+ (we keep the old notations) satisfy the conditions of Theorem 4.9, since $\varepsilon(\Sigma, l_1^+) = \varnothing$ for $\mathscr{F}'$. It follows that l^+ is a nontrivially ω-recurrent semileaf of the foliation $\mathscr{F}'$ and hence, of the foliation $\mathscr{F}$.

The second way is to choose carefully two points at each disk D_i, and one declares the set of these points a branched set. By Lemma 3.15, the foliation under the corresponding 2-sheeted covering map satisfies Theorem 4.9. We omit the details to the reader. □

Let us sketch the example showing that the condition on finitely many singularities is essential in Theorem 4.12. Take a curve L similar to an impulse type signal that approaches to the segment AB like the curve $y = \sin \frac{1}{x}$ approaches to the segment $[-1; +1]$ as $x \to 0$, Fig. 4.16. The curve l tends spirally to L. The both curves can be embedded in a foliation with infinitely many singularities, thorn and tripods, “half” of them approaches to A and other tends to B.

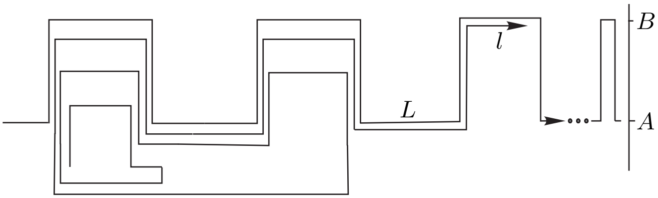


Figure 4.16. The curve L is in the limit set of the curve l.

The proof of the following theorem is similar to one of Theorem 4.12 (see the proof of Theorem 4.8). We omit it.

Theorem 4.13 *Let $\mathscr{F}$ be a foliation with finitely many singularities on a surface M of finite genus with finitely many boundary components* (*including ideal boundary*). *Let l_1, l_2 be nontrivially $\omega(\alpha)$-recurrent leaves of $\mathscr{F}$. If*

$$\omega(\alpha)(l_1) \supset l_2, \quad \textit{then} \quad \omega(\alpha)(l_2) \supset l_1.$$

Description of quasiminimal sets

Actually, as a consequence of Maier theorems, one gets the following description of leaves of quasiminimal sets for foliations with finitely many singularities and separatrices. We assume that every leaf is endowed with some orientation even for non-orientable foliations.

Theorem 4.14 *Let $\mathscr{F}$ be a foliation with finitely many singularities and separatrices on a surface M of finite genus with finitely many boundary components* (*including ideal boundary*). *Then any quasiminimal set Q of $\mathscr{F}$ is a Maier quasiminimal set* (*i. e. every nontrivially recurrent semileaf of Q is dense in Q*) *and contains only the following leaves and singularities*:

- *nontrivially recurrent leaves*;
- *$\omega(\alpha)$-separatrices that are nontrivially $\alpha(\omega)$-recurrent*;
- *separatrix connections*;
- *singularities each being ω- and α-limit set of at least one ω-separatrix and α-separatrix respectively.*

Moreover, any nontrivially recurrent semileaf of $\mathscr{F}$ belongs to a unique quasiminimal set.

Proof. Let l^+ be a positive semileaf of Q. By definition, $Q = \mathrm{clos}(l_0)$ for some nontrivially recurrent leaf l_0. There are two possibilities: 1) $\omega(l^+)$ contains only singularities; 2) $\omega(l^+)$ contains at least one regular point. In case 1), $\omega(l^+)$ is a unique singularity because an ω-limit set is connected. Lemma 3.32 implies that l^+ is ω-separatrix. In case 2), l^+ is nontrivially ω-recurrent by Theorem 4.12. The last item follows from Lemma 4.5. Theorem 4.13 implies that every nontrivially recurrent semileaf of Q is dense in Q. □

Bibliographic Notes and Panoramas

Chapter 4. At the beginning of the XX century, the Poincaré – Bendixson theory was developed for flows on compact flat domains and on the sphere. This theory provided the description of all possible limit sets of an individual semitrajectory for flows with a finite set of fixed points.

(4.1). The central result is the Poincaré – Bendixson theorem that says that there are no nontrivially recurrent semitrajectories and trajectories on a compact flat domain or sphere (this theorem is valid for flows with any sets of fixed points). Therefore, the list of limit sets is exhausted by a fixed point, a periodic trajectory, and a one-sided separatrix contour. [45, 192].

(4.2). The definitions and results for flows on a plane stated in this section were introduced and obtained by Bendixson [45]. This material is also expounded in the books [3, 26].

Note that an isolated singularity may have a countable set of separatrices. Moreover, in [3, pp. 557–558 of the Russian edition], Maier provided an example of an isolated singularity such that the intersection of the boundary of a certain neighborhood of this singularity with the separatrices of this singularity is everywhere dense (on the boundary). All concentric circles of greater radius with the center at the singularity also intersect the separatrices in an everywhere dense set. This singularity has a countable set of elliptic sectors whose diameters tend to zero. Outside the elliptic sectors, one has a domain in which the separatrices of one singularity are everywhere dense. However (according to theorems formulated later on), there is only a finite number of extendible semileaves with respect to a fixed neighborhood of a singularity.

(4.3). In 1937, T. Cherry [64] proved that if N is a quasiminimal set of a flow on arbitrary paracompact space, then N contains continuum of nontrivially recurrent trajectories, each being dense in N. Note that applying some results of General Topology Theory, the Cherry's proof can be simplified. Take a transversal segment Σ which is intersected by a nontrivially recurrent trajectory l lying in the quasiminimal set Ω. Then $\mathrm{clos}(l \cap \Omega) = \Omega_0$ is a perfect closed subset of Σ. Therefore Ω_0 consists of a continuum of points. Since any trajectory intersects Σ in at most countable set of points, there is a continuum of trajectories $l_\alpha \subset \Omega$ through Ω_0. By the Baire's second countability axiom, and since the intersection $l \cap \Sigma$ is dense in Ω_0, there is a continuum of the trajectories such that their ω- and α-limit sets contain the trajectory l.[2]

(4.4). In 1943, in remarkable series of theorems, A. Maier [142] clarified the structure of quasiminimal sets (see also [141], where the main results was announced) for flows on compact surfaces. Recall that Cherry proved the existence in a quasiminimal set of a flow (on a paracompact manifold of any dimension) a continuum of nontrivially recurrent trajectories each of which is everywhere dense in the quasiminimal set. For a two-dimensional manifold of finite genus, Maier obtained a more accurate result by proving that any nontrivial recurrent semitrajectory of a quasiminimal set is everywhere dense in this set. For this purpose, Maier derived a criterion for the nontrivially recurrence of a nonclosed semitrajectory and proved a remarkable theorem on the mutual limit property of two nontrivially recurrent semitrajectories. Note that the above-mentioned theorems by Cherry and Maier are valid for flows with any set of fixed points, although Maier formally required that the number

[2] We would like to thank prof. G. Hector who communicated us this proof. Prof. K. Yano indicated us a similar proof of Cherry's theorem for foliations of any codimension on a manifold M satisfying the second countability axiom.

of fixed points be finite, which he needed to prove other assertions. Note also that these theorems laid the foundation for the Poincaré – Bendixson theory for flows on closed orientable surfaces of genera greater than 1. Namely, it became possible to prove that the list of limit sets of an individual one-dimensional semitrajectory for flows with a finite number of fixed points is exhausted by a fixed point, a closed trajectory, a one-sided separatrix loop, and a quasiminimal set (see [172] for details).

In contrast to flows (orientable foliations), non-orientable foliations and laminations with nontrivially recurrent semileaves may exist even on a disk and, hence, on any surface. Such foliations may have a finite number of singularities (see Chapter 3), which means that the Maier theorems are not immediately extended to foliations and laminations. Nevertheless, introducing certain constraints, one can obtain analogues of these theorems.

Aranson and Zhuzhoma [38, 39] extended Maier theorems for non-orientable foliations with finitely many singularities. Note that the condition on the set of singularities is essential for this case.

CHAPTER 5

Introduction to Anosov–Weil Theory

Andre Weil and Dmitrii Anosov were the first who patently suggested to apply an universal covering space and circle at infinity to study surface dynamical systems using nonlocal asymptotic properties of trajectories and leaves of special local laminations arising in the dynamical systems. Anosov–Weil method was effectively applied by S. Kh. Aranson, V. Z. Grines, G. Levitt, N. Markley, E. Zhuzhoma and others to study important classes of surface dynamical systems and foliations. The generalization to infinite curves of Anosov–Weil approach sometimes now is called Anosov–Weil Theory that, roughly speaking, includes the following two parts:

- ***a study of nonlocal asymptotic properties of simple curves on a surface by lifting these curves to an universal covering, and making a "comparison" with lines of constant geodesic curvature*;**
- ***an application of nonlocal asymptotic properties for constructions of topological invariants for surface dynamical systems and foliations.***

Naturally, one regards surfaces endowed with a metric of constant curvature and with universal covering space homeomorphic to a disk. These objects and first basic notions are involved in Section 5.1. In Sections 5.2, 5.3, we consider statements by Weil and Anosov that are fundamental. In Sections 5.4, 5.5, we pay attention to special curves arising in surface dynamical systems. These sections are crucial for constructions of topological invariants of surface dynamical systems with chaotic behavior. The Reader who is interested in these aspects can go these sections strictly after Section 5.1. In Section 5.6, one considers the deviation of the cover curves that have asymptotic directions from co-asymptotic geodesics. The relation between smoothness of flows and asymptotic properties of their trajectories one regards in Section 5.7. At last, in Section 5.8, we study asymptotic properties of the image of a geodesic under a cover homeomorphism.

5.1. Introductory concepts and notions

It is well-known [223] that a complete simply connected Riemannian 2-manifold of constant ***nonpositive*** curvature is homeomorphic to an open disk. For an observer standing on the such disk, there is a ***horizon***. Considering the horizon as an ideal boundary of the disk, this horizon becomes a so-called circle at infinity or absolute. Each point of this horizon is naturally identified with an asymptotic direction. Actually, the asymptotic direction is a central notion of the Anosov – Weil Theory.

Universal covering and the circle at infinity

Here, we consider surfaces being complete Riemannian manifolds M^2 of constant nonpositive curvature. The universal covering space $\overline{M}^2$ for M^2 is isometric either to the Euclidean plane $\mathbb{R}^2$ (in the case of zero curvature and Euler characteristic $\chi(M^2) = 0$) or to the hyperbolic plane Δ (in the case of negative curvature and Euler characteristic $\chi(M^2) < 0$), [223], Corollary 2.4.10. Accordingly, M^2 is isometric either to $\mathbb{R}^2/\Gamma$ or to Δ/Γ, where Γ is a properly discontinuous group of isometries. Denote by $\pi\colon \overline{M}^2 \to M^2$ the universal covering map, which is a local isometry. Denote by $\overline{d}$ the metric on $\overline{M}^2$, Fig. 5.1.

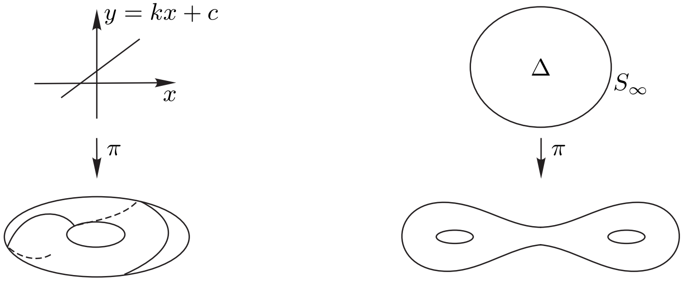


Figure 5.1. Universal coverings.

Given a curve $C \subset M^2$, a *lift* of C is an arcwise connected component of the pull-back $\pi^{-1}(C)$. Often the choice of this component is clear from the context.

For simplicity, we restrict ourself by closed surfaces. A closed surface of constant nonpositive curvature is either orientable surface M_p^2 of genus $p \geqslant 1$

or non-orientable surface N_p^2 of genus $p \geqslant 2$. On the other hand, such surfaces are divided into the two flat surfaces (the torus $T^2 = M_1^2$ and the Klein bottle $K^2 = N_2^2$) with zero curvature, and hyperbolic surfaces with negative curvature. Consider them separately.

Flat surfaces. The Euclidean plane $\mathbb{R}^2$ endowed with the standard quadratic form $ds^2 = dx^2 + dy^2$ is the simplest *flat* surface. As we said before, there exist only two closed flat surfaces: T^2 and K^2. For T^2, one can take Γ to be the group $\mathbb{Z}^2$ of integer translations S_{nk} of the form (1.2). For K^2, Γ consists of transformations (1.3).

It is convenient to use the unit disk D^2 with coordinates ξ, η as a universal covering space: $D^2 = \{(\xi, \eta) \colon \xi^2 + \eta^2 < 1\}$. One can check that the map

$$x = \frac{\xi}{\sqrt{1 - \xi^2 - \eta^2}}, \quad y = \frac{\eta}{\sqrt{1 - \xi^2 - \eta^2}} \tag{5.1}$$

is a homeomorphism denoted by $\tau \colon D^2 \to \mathbb{R}^2$. Then $\pi \circ \tau$ is also a universal covering map, see Fig. 5.2. The boundary $S_\infty = \partial D^2$ is called the *circle at infinity* or *absolute*.

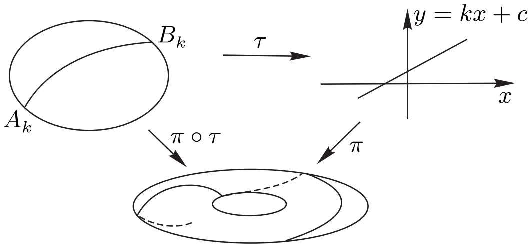


Figure 5.2. Coverings of the torus

Under τ^{-1}, the family of parallel lines $y = kx + c$, $c \in \mathbb{R}$, on $\mathbb{R}^2$ is mapped into a family of curves on D^2 with common diametrically opposite endpoints A_k, $B_k \in S_\infty$. The points A_k and B_k are called *rational* (*irrational*) if $k \in \mathbb{R}$ is rational (respectively, irrational). For the sake of generality, assume that ∞ is a rational "number".

Below, we will need a two-dimensional noncompact cylinder C^2, which we consider to be the quotient space $\mathbb{R}^2/\Gamma$, where $\Gamma = \mathbb{Z}^1$ consists of integer translations S_{n0}:

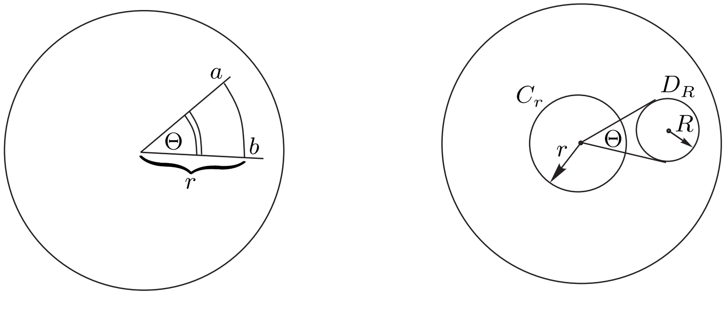
$$(x, y) \to (x + n, y), \quad n \in \mathbb{Z}. \tag{5.2}$$

Hyperbolic surfaces. $\Delta/\Gamma \cong M^2$ was introduced in Section 3.2. Here Δ is the Poincaré model of the hyperbolic plane. Sometimes we consider Δ as the unit disk on $\mathbb{R}^2$ with the topology and metric induced by $\mathbb{R}^2$. Denote by $\overline{d}_{\mathrm{E}}(\cdot,\cdot)$ (respectively, $\overline{d}_{\mathrm{NE}}(\cdot,\cdot)$) the Euclidean (respectively, non-Euclidean) metric on Δ. Let us quote some properties of Δ and Γ we need below.

If $\widehat{ab}$ is the arc of Euclidean circle C_r centered at the origin and of radius r, then its non-Euclidean length $l(\widehat{ab})$ equals

$$l(\widehat{ab}) = \Theta \sinh r \tag{5.3}$$

provided $\widehat{ab}$ is opposite angle Θ, see Fig. 5.3 (a).



(a) (b)

Figure 5.3. The arc $\widehat{ab}$ is opposite angle Θ (a); D_R is outside of C_r (b).

Let D_R be a circle of radius R lying outside of C_r, and Θ be formed by tangent rays to D_R. Then $\Theta \sinh r = l(\widehat{ab}) \leqslant 2R$. Hence, $\Theta \leqslant \frac{2R}{\sinh r}$. For $r \geqslant 2$, $\sinh r \geqslant 2\exp(\frac{r}{8})$. Thus,

$$\Theta \leqslant R \exp\left(-\frac{r}{8}\right) \quad \text{for } r \geqslant 2. \tag{5.4}$$

Recall that the elements of Γ are fractional linear transformations that take the circle at infinity S_∞ onto itself. A point $\sigma \in S_\infty$ is called *rational* if it is a fixed point of some $\gamma \in \Gamma$, $\gamma \neq id$. Otherwise, σ is *irrational*.

The following definition holds for both flat and hyperbolic surfaces. Two subsets of $\overline{M}^2 \cup S_\infty$ are said to be *congruent* if one of them can be mapped to the other by a transformation from Γ.

Limit set of a curve at infinity

Let $l^+ \subset M^2$ be a semi-infinite continuous curve endowed with an injective parametrization $t \to l^+(t)$, $t \in \mathbb{R}^+$, and $\overline{l}^+$ be a lift of l^+ under a universal covering map $\pi\colon \overline{M}^2 \to M^2$. The universal covering surface $\overline{M}^2$ can be thought as the open unit disk $D^2 \subset \mathbb{R}^2$ provided M^2 is a flat or hyperbolic surface. Let $\overline{d}_{\mathrm{E}}(\cdot,\cdot)$ be the metric on $S_\infty \cup D^2$ induced by the standard metric of $\mathbb{R}^2$. The parametrization of l^+ induces the parametrization of $t \to \overline{l}^+(t)$ such that $\pi(\overline{l}^+(t)) = l^+(t)$. A point $\sigma \in S_\infty$ is called the *remote limit point* of $\overline{l}^+$ if there is a sequence t_k, $\lim\limits_{k\to\infty} \to \infty$, such that $\overline{d}_{\mathrm{E}}(\sigma, \overline{l}^+(t_k)) \to 0$ as $k \to \infty$.

Definition 5.1 *The* ***limit set at infinity*** $\lim\limits_{\infty}(\overline{l}^+)$ *of* $\overline{l}^+$ *is the union of all remote limit points of* $\overline{l}^+$.

The union of ordinary limit set and the limit set at infinity is called the *complete limit set*,

$$\mathrm{Lim}(\overline{l}^+) = \lim(\overline{l}^+) \cup \lim_{\infty}(\overline{l}^+).$$

Lemma 5.1 *A complete limit set is connected, closed, and nonempty. The limit set at infinity is closed* (*if nonempty*).

Proof follows from Lemma 3.16 because the set $S_\infty \cup D^2$ is compact. □

The following example shows that the limit set at infinity may be disconnected even for trajectories of smooth flows.

Example 5.1 *A disconnected limit set at infinity.*

In a sufficiently small strip $0 < |y| \leqslant \varepsilon$ of the axis Ox, consider the flow that is defined by the system of differential equations:

$$\dot{x} = -\varphi(y)\left[\frac{1}{y^2}\sin\frac{1}{y} - \frac{1}{y^3}\cos\frac{1}{y}\right], \quad \dot{y} = \varphi(y),$$
$$\text{where} \quad \varphi(y) = -y\exp\left(-\frac{1}{y^2}\right).$$

All points on the line $y = 0$ are fixed points. Obviously, this flow can be extended to the plane $\mathbb{R}^2$ to a C^∞ flow $\overline{f}^t$ that covers some flow f^t on the torus. In the strip $0 < |y| \leqslant \varepsilon$, the integral curves of $\overline{f}^t$ are graphics of $x = \frac{1}{y}\sin\frac{1}{y} + \text{const}$; one can check that the limit set at infinity consists of two points, Fig. 5.5 (a).

Asymptotic directions and accessible points

There is a general introduction of the circle at infinity when a point at infinity (remote point) is considered as a family of parallel directed geodesics or geodesic rays, Fig. 5.4 (a), (b), see for example [69]. Any geodesic from this family is called a *representative* of the point at infinity. Our representation of S_∞ is in the agreement with the general one.

Figure 5.4. Parallel oriented geodesics on $\mathbb{R}^2$ (a) and Δ (b).

Now we quote some possible types of asymptotic behavior of semi-infinite curves. Let $l^+ = \{m(t)\colon t \geqslant 0\} \subset M^2$ be a semi-infinite simple curve and $\overline{l}^+ = \{\overline{m}(t)\colon t \geqslant 0\}$ be its lift to the universal covering $\overline{M}^2$ that is either $\mathbb{R}^2$ or Δ. One says that $\overline{l}^+$ *leaves any compact subset* of $\overline{M}^2$, or is *unbounded*, if

$$\limsup_{t\to+\infty} \overline{d}(\overline{a}_0, \overline{m}(t)) = +\infty, \tag{5.5}$$

where $\overline{a}_0 \in \overline{M}^2$ is an arbitrary point, Fig. 5.5 (a). It is clear that this definition does not depend on the choice of $\overline{a}_0$. A curve that belongs to some compact subset of $\overline{M}^2$ is called *bounded*.

We say that $\overline{l}^+$ *goes to infinity* if

$$\lim_{t\to+\infty} \overline{d}(a_0, \overline{m}(t)) = +\infty. \tag{5.6}$$

In general, (5.5) does not imply 5.6, see example 5.1. Obviously, the converse is true: a curve that goes to infinity sooner or later leaves any compact subset of $\overline{M}^2$ and never returns to it, Fig. 5.5 (b).

One of basic definitions of the Anosov – Weil Theory is the following one.

Definition 5.2 *Let $l^+ = \{m(t)\colon t \geqslant 0\} \subset M^2$ be a semi-infinite simple curve and $\overline{l}^+ = \{\overline{m}(t)\colon t \geqslant 0\} \subset \overline{M}^2$ be its lift. If $\overline{l}^+$ tends exactly to one point $\sigma \in S_\infty$, $\operatorname{Lim}(\overline{l}^+) = \lim_\infty(\overline{l}^+) = \sigma$, then we say that $\overline{l}^+$* ***has an asymptotic direction*** *σ.*

Roughly speaking, for an observer situated on $\overline{M}^2$, the curve $\overline{l}^+$ goes exactly to one point of the horizon, Fig. 5.5 (c). Actually, the asymptotic direction is the ω-limit set $\omega(\overline{l}^+)$ that coincides with the complete limit set, Fig. 5.5. $\sigma = \omega(\overline{l}^+)$ is called a *point accessible* (*or reached*) *by the curve* $\overline{l}^+$. One also says that $\omega(\overline{l}^+)$ is attained by $\overline{l}^+$. Clearly, if some lift has an asymptotic direction, then any lift also has an asymptotic direction. For a curve $\overline{l}^-$ that is semi-infinite in the negative direction, the asymptotic direction and its accessible point $\alpha(\overline{l}^-)$ are defined similarly.

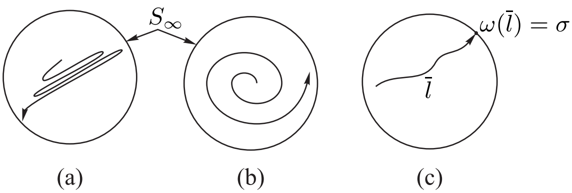


Figure 5.5. Unbounded curve (a); a curve goes to infinity (b); the point $\omega(\overline{l}^+)$ is reached by $\overline{l}^+$ (c).

An asymptotic direction is called *rational* (*irrational*) if the point $\sigma \in S_\infty$ is rational (respectively, irrational).

Co-asymptotic geodesics

Let $l = \{m(t) \in M^2\colon -\infty < t < +\infty\} \subset M^2$ be a simple infinite continuous curve. The point $m(0)$ divides l into two semi-infinite curves: a *positive* $l^+ = \{m(t) \in M^2\colon\ t \geqslant 0\}$ and a *negative* $l^- = \{m(t) \in M^2\colon\ t \leqslant 0\}$ ones. Let $\overline{l} = \{\overline{m}(t)\colon -\infty < t < +\infty\} \subset \overline{M}^2$ be a lift of l. Then $\overline{l}^+ = \{\overline{m}(t)\colon t \geqslant 0\}$ and $\overline{l}^- = \{\overline{m}(t)\colon\ t \leqslant 0\}$ are the lifts of l^+ and l^-, respectively. Suppose that $\overline{l}^+$ and $\overline{l}^-$ have asymptotic directions $\omega(\overline{l}) \in S_\infty$ and $\alpha(\overline{l}) \in S_\infty$, respectively, and $\alpha(\overline{l}) \neq \omega(\overline{l})$. Then, there exists a geodesic $\overline{g}(\overline{l})$ with the same ideal endpoints $\alpha(\overline{l})$ and $\omega(\overline{l})$ oriented from $\alpha(\overline{l})$ to $\omega(\overline{l})$. The geodesic $\overline{g}(\overline{l})$ is called *co-asymptotic* or *corresponding* to $\overline{l}$, Fig. 5.6. Thus, the geodesic that is co-asymptotic for $\overline{l}$ is co-asymptotic for $\overline{l}^+$ and $\overline{l}^-$ in the positive and negative directions, respectively. It is easy to see that the geodesic $\pi(\overline{g}(\overline{l})) \stackrel{\text{def}}{=} g(l)$ on M^2 does not depend on the choice of $\overline{l} \in \pi^{-1}(l)$ and is called *co-asymptotic* or *corresponding* to l.

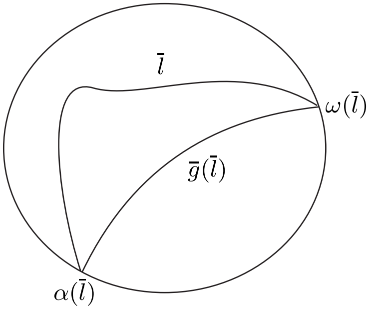


Figure 5.6. Co-asymptotic geodesic.

Similarly, one can introduce *co-asymptotic* or *corresponding geodesic to a simple closed curve* that is non-null-homotopic on the surface. This is actually a closed geodesic that is freely homotopic to the closed curve. For the hyperbolic surface, the co-asymptotic geodesic is unique, due to properties of Hyperbolic Geometry. For the torus, the co-asymptotic geodesic is not unique and must be specified. For the Klein bottle, a uniqueness of co-asymptotic geodesic depends on a homotopy class of simple closed curve. Anyway the following statement holds.

Lemma 5.2 *Let $C \subset M^2$ be a simple closed curve such that C does not bound neither a disk nor a punctured disk (in particular, C is a non-null-homotopic curve) on a surface M^2 of constant non-positive curvature. Given any lift $\overline{C} \subset \overline{M}^2$, there is a co-asymptotic geodesic $\overline{g}(\overline{C})$ that is projected to the closed geodesic $\pi(\overline{g}(\overline{C})) \subset M^2$ freely homotopy to C. If M^2 is hyperbolic, then $\overline{g}(\overline{C})$ is unique.*

Proof. Take points $m_0 \in C$ and $\overline{m}_0 \in \pi^{-1}(m_0)$. According to the Covering Homotopy Theorem, there exists an arc $\overline{C}_{01}$ with the starting point $\overline{m}_0$ that covers C. Since C is non-null-homotopic, $\overline{C}_{01}$ is nonclosed and homeomorphic to an interval. Since $\overline{C}_{01}$ covers a closed curve, $\overline{m}_0$ is congruent to the other endpoint of $\overline{C}_{01}$ denoted by $\overline{m}_1$. Hence there exists a $\gamma \in \Gamma$ such that $\gamma(\overline{m}_0) = \overline{m}_1$, where Γ is a deck group such that $M^2 = \Delta/\Gamma$. Since the limit set of Γ belongs to S_∞, the both $\gamma^n(\overline{m}_0)$ and $\gamma^{-n}(\overline{m}_0)$ tend to some points σ^+, $\sigma^- \in S_\infty$, respectively as $n \to +\infty$ in the Euclidean metric $\overline{d}_E$ on $\overline{M}^2 \cup S_\infty$. Hence,

$$\bigcup_{n \in \mathbb{Z}} \gamma^n(\overline{C}_{01}) = \overline{C}$$

is a nonclosed curve with ideal endpoints on S_∞. It is clear that $\overline{C}$ is a lift of C through $\overline{m}_0$.

If M^2 is flat, γ is an integer shift. Hence, $\sigma^+ \neq \sigma^-$, and there is a co-asymptotic to $\overline{C}$ geodesic, a straight line, that is invariant under γ. Therefore, $\pi(\overline{g}(\overline{C}))$ is a closed geodesic.

Consider the case of hyperbolic M^2. Note that γ is unique because Γ does not contain elliptic transformations. Let us show that $\sigma^+ \neq \sigma^-$. Suppose the contrary. Then, the union $\overline{C} \cup \sigma^+$ is a closed curve that is invariant under γ. According to Lemma 3.7, the transformation γ is parabolic, and the curve C bounds a punctured disk on M^2, which is impossible. Therefore, there is $\overline{g}(\overline{C})$. It follows that γ is a hyperbolic isometry, and $\overline{g}(\overline{C})$ an axis of γ. Hence, $\pi(\overline{g}(\overline{C}))$ is a closed geodesic.

Since the both $\overline{C}$ and $\overline{g}(\overline{C})$ are invariant under the same isometry γ, C is freely homotopy to $\pi(\overline{g}(\overline{C}))$. □

We say that a pair of points $a, b \in S_\infty$ is *separated* by a pair of points $c, d \in S_\infty$ (or the *pairs are separated*) if each arc $S_\infty - (a \cup b)$ contains exactly one point from the pair c, d, Fig. 5.7. Clearly that paths joining separated pairs must be intersected. This simple observation is used in the following lemma.

Lemma 5.3 *A geodesic that is co-asymptotic to a simple curve is simple* (*i. e. has no transversal self-intersections*).

Proof. First, consider the surfaces covered by $\mathbb{R}^2$: the cylinder, torus, and Klein bottle. Since the group of deck transformations for the cylinder and torus consists of integer translations (1.2) and (5.2), there are no geodesics with transversal self-intersections at all. The group of deck transformations (1.3) for the Klein bottle K^2 contains glide reflections, and geodesics with transversal self-intersections on K^2 exist. Such a geodesic has the lifts

$$y = kx, \quad y = -kx - k, \quad k \neq 0,$$

that are congruent under the deck transformation $P\colon (x, y) \to (x + 1, -y)$. Consider closed cones K_1 and K_2 that are bounded by the straight lines $y = (k - \varepsilon)x$, $y = (k + \varepsilon)x$ and contain the line $y = kx$ in their union $K_1 \cup K_2$; the parameter $\varepsilon > 0$ is defined below. Obviously, $P(K_1 \cup K_2)$ contains the line $y = -kx - k$. Since $k \neq 0$, there exists an arbitrarily small $\varepsilon > 0$ such that the intersection of $K_1 \cup K_2$ with $P(K_1 \cup K_2)$ is compact. Suppose that the line $y = kx$ is a co-asymptotic geodesic for some lift $\bar{l}$ of a simple curve $l = \pi(\bar{l}) \subset K^2$. Then, $P(\bar{l})$ is a lift of the same curve with the co-asymptotic geodesic $y = -kx - k$. It follows from the definition of the co-asymptotic direction that, except for a finite number of compact arcs, the curve $\bar{l}$ lies in $K_1 \cup K_2$ and the curve $P(\bar{l})$ lies in $P(K_1 \cup K_2)$. Hence, $\bar{l}$ intersects $P(\bar{l})$, which contradicts the simplicity of l.

Now, consider the surface M^2 covered by Δ. Suppose that the co-asymptotic geodesic $g(l)$ to simple infinite curve $l \subset M^2$ has transversal self-intersections. Then $g(l)$ has two transversally intersecting lifts $\overline{g}_1$ and $\overline{g}_2$ on Δ that are co-asymptotic geodesics for the lifts $\bar{l}_1$ and $\bar{l}_2$ of l, respectively, $\overline{g}_1 = \overline{g}(\bar{l}_1)$ and $\overline{g}_2 = \overline{g}(\bar{l}_2)$, see Fig. 5.7.

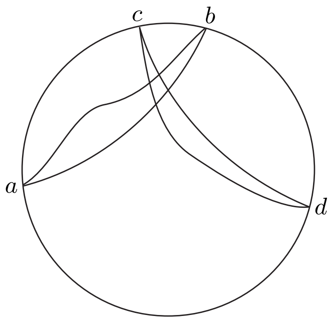


Figure 5.7. Co-asymptotic geodesics must intersect.

Since any geodesics on Δ intersect at most at one point, the ideal endpoints of $\overline{g}(\bar{l}_1)$ and $\overline{g}(\bar{l}_2)$ are separated on S_∞. The same for the ideal endpoints of $\bar{l}_1$ and $\bar{l}_2$. Therefore, $\bar{l}_1$ intersects $\bar{l}_2$, which contradicts the simplicity of l. □

Take two simple disjoint arcs $K_1, K_2 \subset \Delta$ such that each divides Δ, Fig. 5.8 (a). Denote by $D^+(K_1)$ the component of $\Delta - K_1$ that contains K_2, $K_2 \subset D^+(K_1)$. Denote the component of $\Delta - K_2$ that belongs to $D^+(K_1)$ by $D^+(K_2)$, $D^+(K_2) \subset D^+(K_1)$. Such notions are called *compatible*.

Let $\{\overline{C}_i\}_{i=1}^{\infty}$ be a sequence of pairwise disjoint simple arcs each of which divides Δ. This sequence is called *monotone*, if it is possible to introduce the compatible notion for every consecutive pair $\overline{C}_i$, $\overline{C}_{i+1}$, Fig. 5.8 (b). In this case, the sets $D^+(\overline{C}_i)$ form a nested sequence,

$$D^+(\overline{C}_1) \supset D^+(\overline{C}_2) \supset \cdots \supset D^+(\overline{C}_i) \supset \cdots .$$

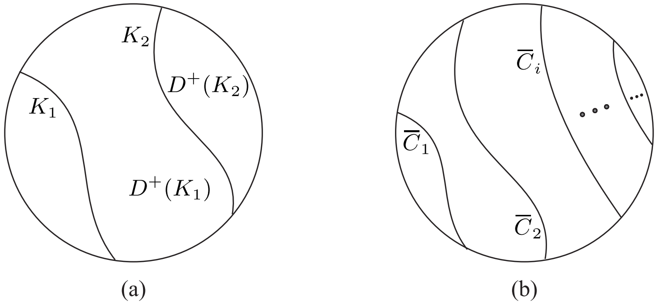


Figure 5.8. Compatible arcs K_1, K_2 (a); monotone sequence $\{\overline{C}_i\}_{i=1}^{\infty}$.

Note that if $C \subset M^2$ is a simple homotopy nontrivial closed curve on a hyperbolic surface M^2, the set $\pi^{-1}(C)$ consists of disjoint arcs each of which divides Δ.

Lemma 5.4 *Let $C \subset M^2$ be a simple homotopy nontrivial closed curve on a hyperbolic surface M^2, and $\{\overline{C}_i\}_{i=1}^{\infty} \in \pi^{-1}(C)$ be a monotone sequence, where $\overline{C}_j \neq \overline{C}_i$ if $j \neq i$. Then the topological limit of $\{\overline{C}_i\}_{i=1}^{\infty}$ is a unique point of S_∞.*

Proof. By lemmas 5.2, 5.3, C is freely homotopy to the co-asymptotic geodesic $g(C) = \pi(\overline{g}(\overline{C}))$ that is simple. Lifting a homotopy of M^2 which carries C to $g(C)$, we see that there is a uniform bound to the non-Euclidean distance d_{NE} between $\overline{C}_i$ and $\overline{g}(\overline{C}_i) \stackrel{\text{def}}{=} \overline{g}_i$ for all i. Hence, it is enough to prove the result for the sequence $\{\overline{g}_i\}_{i=1}^{\infty}$ which is obviously monotone.

Since $\overline{g}_i$ are arcs of Euclidean circles which are transversal to S_∞, one should prove that the sequences e_i^1, $e_i^2 \in S_\infty$ of the endpoints of $\overline{g}_i$ converge to the same point. Suppose the converse. The monotonicity of $\{\overline{g}_i\}_{i=1}^{\infty}$ implies that the both sequences e_i^1, e_i^2 are monotone on S_∞ and hence, have limits, say $E_1 = \lim_{i\to\infty} e_i^1$, $E_2 = \lim_{i\to\infty} e_i^2$, where $E_1 \neq E_2$. This means that the geodesics $\overline{g}_i$ converge to the geodesic $\overline{g}_{12}$ with the ideal endpoints E_1, E_2. Since $g(C) = \pi(\overline{g}_i)$ is a closed geodesic, any point of M^2 has a neighborhood U such that the intersection $U \cap g(C)$ is a unique arc of $g(C)$, if $U \cap g(C) \neq \varnothing$. Therefore, every point of $\overline{g}_{12}$ is a limit of a sequence of congruent points. This is impossible, since the limit set of the group of deck transformations (recall that this group is crystallographic) belongs to S_∞, see Section 3.2. □

Remark that Lemma 5.4 is not true for flat surfaces, the torus and Klein bottle. For these surfaces, the topological limit of $\{\overline{C}_i\}_{i=1}^{\infty}$ is a “half” of S_∞.

Description of points of the circle at infinity

Here, we present some ways for specifying points of S_∞. For the torus and Klein bottle, the universal covering $\overline{M}^2$ is $\mathbb{R}^2$. Every point $\sigma \in S_\infty$ corresponds to oriented parallel rays with the same angular coefficient (including ∞), say k. Rationality (irrationality) of k corresponds to rationality (irrationality) of σ. Any ray with $k \in \mathbb{Q}$ (respectively, $k \notin \mathbb{Q}$) projects to a closed (respectively, unclosed) geodesic, and vice versa, rational points of S_∞ are exactly ideal endpoints of lifts of closed geodesics.

Note that a family of parallel lines on $\mathbb{R}^2$ corresponds to two diametrically opposite points of S_∞. Hence, one angular coefficient corresponds to a pair of diametrically opposite points of S_∞. Thus, pairs of diametrically opposite points are parameterized by angular coefficients $k \in \mathbb{R} \cup \{\infty\}$. This specification of points of S_∞ is often quite sufficient for the case of flat surfaces.

Further, we consider the description of points of S_∞ for the hyperbolic (Lobachevsky) plane Δ, which is a universal covering for hyperbolic surfaces. Recall that for a fixed group Γ of deck transformations that acts on Δ, *rational points* $\mathscr{R}$ are exactly ideal endpoints of lifts of all closed geodesics. The remaining points $\mathscr{I} = S_\infty - \mathscr{R}$ are called *irrational points*. When it is necessary to stress that the above sets depend on the group Γ, we will denote them by $\mathscr{R}(\Gamma)$ and $\mathscr{I}(\Gamma)$, respectively.

Every point of the circle at infinity corresponds to a family of oriented collinear parallel geodesics. However, there does not exist a convenient generally accepted method for assigning a certain number to such a family of geodesics. There are different types of coding that depend on the choice of the generators of the fundamental group of a surface (see [131, 166, 173, 206]). To specify points of S_∞, we consider ideal endpoints of the geodesics that belong to lifts of special geodesic laminations. We'll see that the description of irrational points is much more rich than in the case of flat surfaces.

Let G be a geodesic lamination on a hyperbolic surface M^2. It is clear that the preimage $\pi^{-1}(G) \overset{\text{def}}{=} \overline{G}$ is a geodesic lamination on the universal covering Δ. If $\overline{G}$ has a geodesic with an ideal endpoint $\sigma \in S_\infty$, we say that σ is *accessible* (*or, reached, or attained*) *by the lamination* $\overline{G}$. Taking a certain liberty, we will also say that σ is accessible by the lamination G, although this lamination lies on the surface. Denote by $\overline{G}_\infty \subset S_\infty$ the set of points on S_∞ that are accessible by the lamination $\overline{G}$. Again, taking a certain liberty, we will use the notation G_∞. Sometimes, when the subscript is in use, we will denote the set of accessible points by $\overline{G}(\infty)$ or $G(\infty)$. Everywhere below in this section, we will consider geodesic laminations on a *closed orientable hyperbolic surface* M^2.

Recall that Λ is the set of weakly irrational (i. e., minimal strongly nontrivial) geodesic laminations. According to Lemma 3.44, each lamination in Λ consists of nonclosed B-recurrent (and, hence, nontrivially recurrent) geodesics. Moreover, each geodesic in such lamination is everywhere dense in it. Denote by Λ_{or} (respectively, Λ_{non}) the set of orientable (respectively, non-orientable) weakly irrational geodesic laminations on M^2,

$$\Lambda = \Lambda_{or} \cup \Lambda_{non}.$$

Denote by $\overline{\Lambda}_{or}$ (respectively, $\overline{\Lambda}_{non}$) the set of lifts of Λ_{or} (respectively, Λ_{non}) to Δ. Set

$$\overline{\Lambda} = \overline{\Lambda}_{or} \cup \overline{\Lambda}_{non}.$$

By Lemma 3.39, every geodesic of any lamination in $\overline{\Lambda}$ has irrational ideal endpoints on S_∞. To consider the family Λ in greater detail, we must consider the properties of nontrivially recurrent geodesics.

Lemma 5.5 *Let g_1 and g_2 be nontrivially recurrent simple geodesics on M^2, and let there exist their lifts $\overline{g}_1$ and $\overline{g}_2$ to Δ that have a common ideal endpoint $\sigma \in S_\infty$. If $g_1 \neq g_2$, then g_1 and g_2 do not intersect.*

Proof. Suppose the contrary. Without loss of generality, we can assume that $\overline{g}_1$ and $\overline{g}_2$ are oriented toward σ and induce an orientation on g_1 and g_2. It is well-known (see, for example, [46]) that geodesics on Δ with a common ideal endpoint exponentially approach each other and the angle between the tangent vectors to these geodesics at appropriate points tends to zero. Hence, g_1 and g_2 also exponentially approach each other, and the angle between the tangent vectors at appropriate points tends to zero. Since $g_1 \neq g_2$ by the assumption, g_1 and g_2 intersect transversally. Let us fix a certain intersection point m of these geodesics. Since g_2 is a nontrivially recurrent geodesic, it intersects g_1 infinitely many times near the point m at angles that are bounded away from zero by a positive constant (Fig. 5.9).

Since g_1 and g_2 exponentially approach each other and the angle between the tangent vectors to these geodesics at appropriate points tends to zero, g_1 must self-intersect, which contradicts the assumption. □

Lemma 5.6 *No point of S_∞ can be an ideal endpoint for three or more geodesics that are projected onto a surface as nontrivially recurrent simple geodesics.*

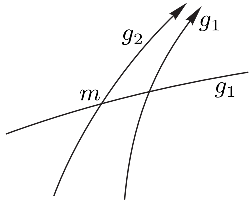


Figure 5.9. Geodesics g_1 and g_2 near the point m.

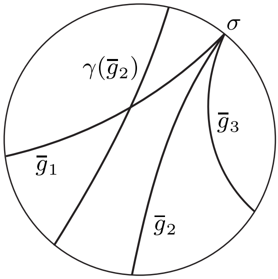


Figure 5.10. Geodesics $\overline{g}_1$, $\overline{g}_2$ and $\overline{g}_3$.

Proof. Suppose the contrary. Let $\sigma \in S_\infty$ be an ideal endpoint for geodesics $\overline{g}_1$, $\overline{g}_2$, and $\overline{g}_3$ that are projected to nontrivially recurrent simple geodesics. Then, one of them, say $\overline{g}_2$, lies in a domain bounded on Δ by the geodesics $\overline{g}_1$ and $\overline{g}_3$. Since $\overline{g}_2$ is a lift of a nontrivially recurrent geodesic and the point σ is irrational (see Corollary 3.12), there exists a geodesic $\gamma(\overline{g}_2)$, congruent to $\overline{g}_2$, that intersects either $\overline{g}_1$ or $\overline{g}_3$ (see Fig. 5.10). But this result contradicts Lemma 5.5. □

Recall that any directed geodesic $\overline{g}$ on Δ with an endpoint $\sigma \in S_\infty$ that is oriented toward σ is called a *geodesic representative* of the point σ. If $\overline{g}$ is projected to a nontrivially recurrent simple geodesic on the surface, then we call $\overline{g}$ a *nontrivially recurrent geodesic representative*.

According to Lemma 5.6, each point belonging to Λ_∞ has at most two nontrivially recurrent geodesic representatives. We will call $\sigma \in \Lambda_\infty$ a *point of the first* (respectively, *second*) *kind* if σ has one (respectively, two) nontrivially recurrent geodesic representative(s). Denote by $\Lambda_{1,\infty}(M^2)$ (respectively, $\Lambda_{2,\infty}(M^2)$) the set of points of the first (respectively, second) kind. Obviously,

$$\Lambda_\infty = \Lambda_{1,\infty}(M^2) \cup \Lambda_{2,\infty}(M^2).$$

We will also use the following notation for the subsets of Λ_∞:

$$\Lambda_{i,or}(\infty) \overset{\text{def}}{=} \Lambda_{or}(\infty) \cap \Lambda_{i,\infty}, \qquad i = 1,\, 2,$$

$$\Lambda_{i,non}(\infty) \overset{\text{def}}{=} \Lambda_{non}(\infty) \cap \Lambda_{i,\infty}, \qquad i = 1,\, 2.$$

There is a close relation between nontrivially recurrent geodesic representatives of the same point from $\Lambda_{2,\infty}(M^2)$.

Lemma 5.7 *Let g_1 and g_2 be nontrivial recurrent simple geodesics on M^2, and let there exist their lifts $\overline{g}_1$ and $\overline{g}_2$ to Δ that have a common ideal endpoint $\sigma \in \Lambda_{2,\infty}(M^2)$. Then, the geodesics g_1 and g_2 belong to the same minimal nontrivial geodesic lamination* $\operatorname{clos} g_1 = \operatorname{clos} g_2$.

Proof. If the point belongs to $\operatorname{clos} g_1$, then it also belongs to $\operatorname{clos} g_2$ because the geodesics g_1 and g_2 exponentially approach each other. Therefore, $\operatorname{clos} g_1 \subset$ $\subset \operatorname{clos} g_2$. The converse inclusion can BE proved in a similar way. $\square$

Corollary 5.1 *An arbitrary point $\sigma \in \Lambda_\infty$ defines a unique weakly irrational* (*minimal strongly nontrivial*) *geodesic lamination on the surface M^2 by the following rule*: *take an arbitrary nontrivially recurrent geodesic representative $\overline{g}$ of the point σ and set*

$$\sigma \mapsto \operatorname{clos} \pi(\overline{g}) \overset{\text{def}}{=} G(\sigma).$$

Proof. The geodesic $\overline{g}$ is projected to a nontrivially recurrent simple geodesic $\pi(\overline{g})$ on M^2. Therefore, $\operatorname{clos} \pi(\overline{g})$ is a minimal strongly nontrivial geodesic lamination. By Lemma 5.7, this lamination does not depend on the choice of the nontrivially recurrent geodesic representative of the point σ. $\square$

Thus, there is the correspondence

$$\overline{\Lambda}_\infty \implies \{\text{weakly irrational geodesic laminations}\}.$$

Corollary 5.1 can be reformulated as follows. We say that two geodesic laminations G_1 and G_2 *intersect at infinity* if there exist geodesics $g_1 \in G_1$ and $g_2 \in G_2$ such that some of their coverings have the same ideal endpoints.

Corollary 5.2 *If two weakly irrational geodesic laminations intersect at infinity, then they coincide.*

An important class of geodesic laminations is given by irreducible laminations.

Definition 5.3 *A geodesic lamination G is called irreducible if any closed geodesic on M^2 intersects G.*

Note that if M^2 is a closed orientable hyperbolic surface, this condition is equivalent to the fact that any component of the set $M^2 \setminus G$ is simply connected (see Corollary 3.13).

Definition 5.4 *A geodesic lamination G is called strongly irrational if G is weakly irrational and irreducible.*

Denote by $\Lambda^{irr} \subset \Lambda$ the set of strongly irrational (that is weakly irrational and irreducible) geodesic laminations. We'll say a geodesic lamination from Λ^{irr}.

Set

$$\Lambda_{or} \cap \Lambda^{irr} \overset{\text{def}}{=} \Lambda_{or}^{irr}, \quad \Lambda_{non} \cap \Lambda^{irr} \overset{\text{def}}{=} \Lambda_{non}^{irr}.$$

By Corollary 5.2, any point $\sigma \in \Lambda^{irr}(\infty)$ is reached by exactly one irrational geodesic lamination.

Recall that a nontrivially recurrent geodesic is called *internal* if it is improper on both sides. Otherwise, such a geodesic is said to be a *boundary* one. Taking a certain liberty, we can say that an internal nontrivially recurrent geodesic approaches itself from both sides, while a boundary nontrivially recurrent geodesic approaches itself from only one side.

Lemma 5.8 *Let g be a geodesic of a weakly irrational geodesic lamination G on M^2. Then,*

1) *g is an internal geodesic if and only if the ideal endpoints of an arbitrary lift $\overline{g}$ of g belong to the set $\Lambda_{1,\infty}(M^2)$;*
2) *g is a boundary geodesic if and only if the ideal endpoints of an arbitrary lift $\overline{g}$ of g belong to the set $\Lambda_{2,\infty}(M^2)$.*

Proof. It suffices to prove only one of these assertions, say the first. Let g be internal and suppose that $\overline{g}$ reaches a point σ from the set $\Lambda_{2,\infty}(M^2)$. Then, there exists a lift $\overline{g}_*$ of an another geodesic $g_* \in G$ with the ideal endpoint σ. Since g is internal, there exists a geodesic congruent to $\overline{g}$ that intersects $\overline{g}_*$. Hence, g intersects g_*, which is impossible because the geodesics in a lamination are pairwise disjoint.

Suppose that the lift $\overline{g}$ has ideal endpoints from the set $\Lambda_{1,\infty}(M^2)$ and let g be a boundary geodesic. Then, $\overline{g}$ belongs to the boundary of the lift $\overline{D}$

of a certain component of the set $M^2 - G$. By Corollary 3.14, the point σ is an ideal endpoint of a geodesic that is different from $\overline{g}$ and belongs to the boundary of the domain $\overline{D}$. This means that σ lies in the set $\Lambda_{2,\infty}(M^2)$. The contradiction obtained proves the lemma. □

The Fréchet distance and the F-equivalence

Let l be an infinite curve endowed with two parametrizations $m' : \mathbb{R} \to l$, $m'' : \mathbb{R} \to l$. These parametrizations are called *equivalent* if there exist continuous nondecreasing functions

$$t',\, t'' : \mathbb{R} \to \mathbb{R} \text{ such that } m' \circ t'(t) = m'' \circ t''(t) \text{ for any } t \in \mathbb{R}.$$

In other words, parametrizations m_1 and m_2 are equivalent if the points $m'(t)$ and $m''(t)$, where $t \in \mathbb{R}$, run through the same set in the same order. The class of equivalent parameterizations defines an *oriented infinite curve*. We denote this curve either by the same symbol l or (when this is important) by $[l]$.

Let $[l_1]$ and $[l_2]$ be two oriented infinite curves on a manifold M^2 equipped with a metric $d(\cdot,\cdot)$. In the following definition, M^2 may be a noncompact surface (in particular, a universal covering). The *Fréchet distance* between the curves $[l_1]$ and $[l_2]$ is defined as follows:

$$\rho_F([l_1],\, [l_2]) = \inf_{l_1 \in [l_1],\, l_2 \in [l_2]} \sup_t d(m_1(t),\, m_2(t)),$$

where $m_i : \mathbb{R} \to M^2$ is a parametrization of the curve l_i, $i = 1, 2$. The Fréchet distance between semi-infinite curves is defined similarly. This distance is not a metric in the sense of metric spaces because it may be infinite. One can easily verify that

$$\rho_F([l_1],\, [l_2]) = \rho_F([l_2],\, [l_1]), \quad \rho_F([l],\, [l]) = 0,$$
$$\rho_F([l_1],\, [l_3]) \leqslant \rho_F([l_1],\, [l_2]) + \rho_F([l_2],\, [l_3]).$$

Note that the first and the third relations make sense even for infinite Fréchet distances.

For semi-infinite nonsimple curves and for infinite simple curves, the distance $\rho_F([l_1],\, [l_2])$ may generally vanish, while $[l_1] \neq [l_2]$. Nevertheless, one can prove that if $[l_1]$ and $[l_2]$ are simple semi-infinite curves such that $\rho_F([l_1],\, [l_2]) = 0$, then $[l_1] = [l_2]$, see details in [19, 20]. Thus, the distance ρ_F is a regular metric for simple semi-infinite and infinite curves (with the only reservation that this distance may be infinite).

Definition 5.5 *We say that curves* $[l_1]$ *and* $[l_2]$ *on* M^2 *are* ***F-equivalent*** *or* ***Fréchet equivalent*** *if they lie at a finite Fréchet distance from each other,* $\rho_F([l_1], [l_2]) < \infty$.

The F-equivalence of curves $[l_1]$ and $[l_2]$ is equivalent to the fact that

$$\sup_{t \geqslant 0} \overline{d}(m_1(t), m_2(t)) < \infty \tag{5.7}$$

for certain parameterizations $m_1(t)$ and $m_2(t)$ of these curves. It is clear that the F-equivalence of curves is indeed an equivalence relation.

All main properties studied in this chapter are identical for F-equivalent curves. For example, if a certain lift $[\overline{l}]$ of the curve $[l]$ is a bounded (or unbounded) curve, then any lift of $[l]$ is also a bounded (respectively, unbounded) curve. If $[\overline{l}]'$ is F-equivalent to $[\overline{l}]$, then $[\overline{l}]'$ is bounded or unbounded simultaneously with $[\overline{l}]$.

5.2. The theorem and conjecture of Weil

Let us formulate here the theorem and conjecture of Weil.

Theorem 5.1 *Let* $l = \{m(t) \colon t \geqslant 0\}$ *be a semi-infinite (continuous) curve without self-intersections on the torus* T^2*, and let* $\overline{l} = \{\overline{m}(t) \colon t \geqslant 0\}$ *be its lift to* D^2*. If the curve* $\overline{l}$ *goes to infinity, it has an asymptotic direction.*

The conjecture is analogous to the theorem, but it refers to a hyperbolic surface.

Conjecture 5.1 *Let* $l = \{m(t) \colon t \geqslant 0\}$ *be a semi-infinite (continuous) curve without self-intersections on a closed hyperbolic surface* M*, and let* $\overline{l} = \{\overline{m}(t) \colon t \geqslant 0\}$ *be its lift to* Δ*. If the curve* $\overline{l}$ *goes to infinity, it has an asymptotic direction.*

To prove the theorem and conjecture, we'll need a condition for the nonsimplicity of an arc (or a curve); this condition will also be used in other sections. Since the Weil theorem and the Weil conjecture represent the same assertion but for surfaces of zero and negative curvature, respectively, we formulate the nonsimplicity condition as two theorems, for the torus and a hyperbolic surface, respectively. First, we consider the case of the torus, which is covered by the Euclidean plane $\mathbb{R}^2$ with Cartesian coordinates x, y.

Theorem 5.2 *Let* $A = \{m(t)\colon t_1 \leqslant t \leqslant t_2\}$ *be an arc on the torus, and let*

$$\overline{A} = \{\overline{m}(t) = (x(t),\, y(t))\colon t_1 \leqslant t \leqslant t_2\}$$

be its covering arc on the Euclidean plane. Suppose that $y(t_1) = y(t_2)$ *and* $y(t) \neq y(t_1)$ *for* $t_1 < t < t_2$. *If* $|x(t_2) - x(t_1)| \geqslant 1$, *then the arc* A *has self-intersections*; *i. e., there exist* $t_\alpha,\, t_\beta \in [t_1;\, t_2]$, $t_\alpha \neq t_\beta$, *such that* $m(t_\alpha) = m(t_\beta)$.

Proof. Assume, for definiteness, that

$$y(t) > y(t_1) \text{ for } t_1 < t < t_2 \text{ and } x_2 \overset{\text{def}}{=} x(t_2) > x(t_1) \overset{\text{def}}{=} x_1.$$

In other cases, the proof is analogous. Suppose the contrary. Then, $x(t_2) > x(t_1) + 1$. Denote by

$$I = \{(x,\, y)\colon x_1 \leqslant x \leqslant x_2,\; y = y(t_2)\}$$

the segment with endpoints $E_1 = (x_1,\, y(t_1))$ and $E_2 = (x_2,\, y(t_2))$, and by D the domain bounded by I and the arc $\overline{A}$, see Fig. 5.11. Recall that S_{nk} denotes the translation

$$(x,\, y) \to (x + n,\, y + k), \quad n,\, k \in \mathbb{Z}.$$

According to the assumption of the contrary, the point $S_{10}(x_1)$ lies on I between x_1 and x_2. Since $y(t) > y(t_1)$ for $t_1 < t < t_2$, the arc $S_{10}(\overline{A})$ enters the domain D near the point $S_{10}(E_1)$; i. e., the point $S_{10}(\overline{m}(t_1 + \varepsilon)) \in S_{10}(\overline{A})$ lies inside D for a sufficiently small $\varepsilon > 0$. Since the straight line $y(t) = y(t_1)$ intersects the domain D only along the segment I and all other points of D lie above the straight line $y(t) = y(t_1)$, $S_{10}(E_2) \notin D$. Hence, $S_{10}(\overline{A})$ intersects the boundary of D, i. e., either $\overline{A}$ or I. If $S_{10}(\overline{A}) \cap \overline{A} \neq \varnothing$, then the arc A is nonsimple and the theorem is proved. If $S_{10}(\overline{A}) \cap I \neq \varnothing$, then $S_{10}(\overline{m}(t)) \in I$ for a certain $t_1 + \varepsilon \leqslant t \leqslant t_2$. Then, $t = t_2$ by the hypothesis of the theorem. But the point $S_{10}(E_2)$ lies outside I. □

Remark. The analysis of the proof of Theorem 5.2 shows that of all properties of the torus, we have used only the congruence of the arcs $\overline{A}$ and $S_{10}(\overline{A})$. Therefore, Theorem 5.2 is also valid for a cylinder $C^2 \cong \mathbb{R}^2/\Gamma$, where the group $\Gamma = \mathbb{Z}^1$ consists of integer translations S_{n0}. The only closed geodesics on C^2 are the curves of the form $\pi(y = \text{const})$, each of which divides C^2 into two cylinders. The theorem implies that if the arc A intersects a certain closed geodesic on C^2 only at its endpoints and makes at least one "turn" along this geodesic, then A should have self-intersections.

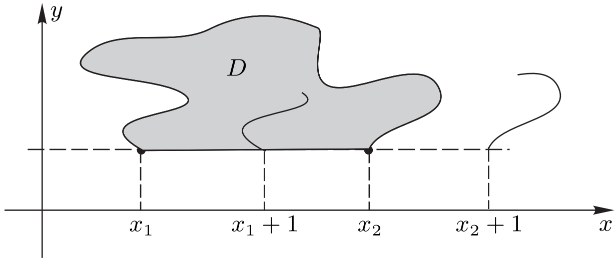


Figure 5.11. Domain D.

A similar theorem holds for an arc on a hyperbolic surface $\Delta/\Gamma \cong M^2$. First, we introduce a certain notation. Let $\overline{g}$ be a geodesic or an equidistant curve of a certain geodesic on Δ. Suppose that $\overline{g}$ is projected onto a closed curve $\pi(\overline{g}) = g$. Therefore, $\overline{g}$ is invariant under a certain transformation (and hence under all its iterations) of the group Γ. Denote by $\gamma(\overline{g}) \in \Gamma$ a nonidentity transformation that maps $\overline{g}$ into itself and translates the points on $\overline{g}$ by the least possible non-Euclidean distance. Since the group Γ is disconnected, such a transformation exists.

Theorem 5.3 *Let $A = \{m(t)\colon t_1 \leqslant t \leqslant t_2\}$ be an arc on the hyperbolic surface $\Delta/\Gamma \cong M^2$, and let $\overline{A} = \{\overline{m}(t)\colon t_1 \leqslant t \leqslant t_2\}$ be its covering arc on the hyperbolic plane Δ whose endpoints $\overline{m}(t_1)$ and $\overline{m}(t_2)$ lie on a geodesic $\overline{g}$ or on the equidistant curve of a certain geodesic (denoted by the same symbol $\overline{g}$). Suppose that $\overline{g}$ is projected onto a closed curve and intersects the arc $\overline{A}$ only at its endpoints $\overline{m}(t_1)$ and $\overline{m}(t_2)$. Let $\gamma(\overline{g}) \in \Gamma$ be a nonidentity transformation that maps $\overline{g}$ into itself and translates the points on $\overline{g}$ by the least possible non-Euclidean distance. If one of the points $\gamma(\overline{g})(\overline{m}(t_1))$ or $\gamma(\overline{g})(\overline{m}(t_2))$ lies on $\overline{g}$ between the points $\overline{m}(t_1)$ and $\overline{m}(t_2)$, then the arc A has self-intersections.*

The proof is analogous to the proof of Theorem 5.2, and we omit it. Note that this theorem holds if one takes a cylindrical neighborhood of the curve $\pi(\overline{g}) = g$ containing the arc A and appropriately maps this neighborhood onto the cylinder C^2. Then, one should apply the remark to Theorem 5.2.

Proof of the Weil theorem

First, we formulate the Weil theorem in a form that is more convenient for us.

Theorem 5.4 *Let* $l = \{m(t)\colon t \geqslant 0\}$ *be a semi-infinite continuous curve without self-intersections on the torus* $\mathbb{T}^2$*, and let* $\overline{l} = \{\overline{m}(t)\colon t \geqslant 0\}$ *be its lift to the Euclidean plane* $\mathbb{R}^2$*. If* $\overline{l}$ *goes to infinity, then there exists a limit*

$$\lim_{t\to+\infty} \frac{\vec{O\overline{m}}(t)}{|O\overline{m}(t)|},$$

where $|O\overline{m}(t)|$ *denotes the length of the radius vector* $\vec{O\overline{m}}(t)$ *of the point* $\overline{m}(t)$*.*

A key role in the proof of this result is Theorem 5.5 that is proved latter. Before, we need several technical lemmas.

Lemma 5.9 *Let* $l = \{m(t)\colon t \geqslant 0\}$ *be a semi-infinite continuous curve without self-intersections on the cylinder* $C^2 \cong \mathbb{R}^2/\Gamma$*, where the group* $\Gamma = \mathbb{Z}^1$ *consists of the integer translations* S_{n0}*, and let* $\overline{l} = \{\overline{m}(t) = (x(t),\, y(t))\colon t \geqslant 0\}$ *be a lift of this curve to the Euclidean plane* $\mathbb{R}^2$*. Suppose that there is an open unit interval* I *on the axis* Ox *that does not intersect* $\overline{l}$*. Then, the intersection of* $\overline{l}$ *with the axis* Ox *lies either to the right or to the left of the interval* I*.*

Proof. The idea of the proof is simple. If $\overline{l}$ intersects Ox both to the right and left of the interval I, then there should exist an arc that intersects Ox only at its endpoints lying on different sides of I. But this fact contradicts Theorem 5.2. Now, we give a formal proof under the assumption that $\overline{l}\cap Ox \neq \neq \varnothing$ (otherwise, there is nothing to prove).

Without loss of generality, we may assume that $\overline{m}(0) \in Ox$ since the position of the arc of the curve before the first intersection with Ox is not essential. Let us divide the values of the parameter $t \geqslant 0$ into two disjoint sets P_l and P_r as follows: t belongs to P_l (respectively, P_r) if

1) either the point $\overline{m} = \overline{m}(t) \in Ox$ lies to the left (respectively, to the right) of I,
2) or the point $\overline{m}$ lies on a compact arc

$$\{\overline{m}(t)\colon \alpha \leqslant t \leqslant \beta\} \stackrel{\text{def}}{=} A_{\alpha\beta}$$

of the curve $\overline{l}$ both endpoints of which $\overline{m}(\alpha)$, $\overline{m}(\beta) \in Ox$ lie to the left (respectively, to the right) of I,
3) or the point $\overline{m}$ lies on a semi-infinite subcurve

$$\{\overline{m}(t)\colon t \geqslant t_0\} \stackrel{\text{def}}{=} A_{t_0,+\infty}$$

whose endpoint $\overline{m}(t_0) \in Ox$ lies to the left (respectively, to the right) of I.

According to Theorem 5.2, the set of parameters is given by the union $P_l \cup P_r$. Let us show that P_l is closed. Let $t_n \in P_l$ and $t_n \to t_*$ as $n \to \infty$. Denote by x_0 and $x_0 + 1$ the endpoints of the interval I. If $\overline{m}(t_n) \in Ox$ for an infinite number of n, then, by virtue of the continuity of the curve, we have $x(t_*) \leqslant x_0$. Hence, $t_* \in P_l$. If an infinite set of points $\overline{m}(t_n)$ belongs to the same arc $A_{\alpha\beta}$, then $\overline{m}(t_*) \in A_{\alpha\beta}$ because $A_{\alpha\beta}$ is a closed subset of the plane. If $\overline{m}(t_n) \in A_{t_0,+\infty}$, then $\overline{m}(t_*) \in A_{t_0,+\infty}$. In both cases, $t_* \in P_l$. Finally, let the points $\overline{m}(t_n)$ be situated on an infinite number of arcs $A_{\alpha_n\beta_n}$ (we may leave out the semi-infinite curve $A_{t_0,+\infty}$ because it contains only a finite number of points $\overline{m}(t_n)$ in the remaining case). Passing to a certain subsequence and changing the numbering, we can assume that $\alpha_n \leqslant t_n \leqslant \beta_n$ and that the subsequence t_n monotonically converges to t_*. Then, α_n, $\beta_n \to t_*$. Since $y(\alpha_n) = y(\beta_n) = 0$ and $x(\alpha_n) \leqslant x_0$, we have $y(t_*) = 0$, $x(t_*) \leqslant x_0$. Therefore, $t_* \in P_l$.

The closedness of P_r is proved similarly. The connectedness of the set of parameters $[0; \infty)$ and the equality $[0; \infty) = P_l \cup P_r$ imply that either P_l or P_r is empty. $\square$

The following lemma constitutes the main part of the proof of Theorem 5.5.

Lemma 5.10 *Let $l = \{m(t)\colon t \geqslant 0\}$ be a semi-infinite continuous curve without self-intersections on the cylinder $C^2 \cong \mathbb{R}^2/\Gamma$, where the group $\Gamma = \mathbb{Z}^1$ consists of the integer translations S_{n0}, and let $\overline{l} = \{\overline{m}(t) = (x(t), y(t))\colon t \geqslant 0\}$ be a lift of this curve to the Euclidean plane $\mathbb{R}^2$. Suppose that $\overline{m}(0) = (0, 0)$ and $\overline{l}$ does not intersect the axis Ox for $x < 0$. Let $t = a$ (where $0 \leqslant a$, and the equality $a = 0$ is possible) be the last time when $\overline{l}$ intersects the unit interval $0 \leqslant x \leqslant 1$ on the axis Ox; in particular, $y(t)$ has a definite constant sign for $t > a$ sufficiently close to a. If this sign is positive (negative), then*

$$y(t) \geqslant \min_{0 \leqslant s \leqslant a} y(s) \textit{ for any } t \geqslant 0$$

$$(\textit{respectively, } y(t) \leqslant \max_{0 \leqslant s \leqslant a} y(s) \textit{ for any } t \geqslant 0).$$

Proof. We can assume that $y(t) > 0$ when $t > a$ is sufficiently close to a because we can reduce the case of the opposite sign to the present case by changing y to $-y$. If the curve $\overline{l}$ does not intersect the open unit interval $0 < x < 1$ on the axis Ox, then, by Lemma 5.9, the curve $\overline{l}$ does not intersect the axis Ox after $t = 0$, and the lemma is proved because $\min\limits_{0 \leqslant s \leqslant a} y(s) = 0$. Therefore, we will assume that $\overline{l}$ intersects the interval $0 < x < 1$ on the axis Ox. Suppose that the lemma does not hold. Then, there exists $t = c$ such that $y(c) < \min_{0 \leqslant s \leqslant a} y(s)$. Set

$$m \stackrel{\text{def}}{=} \min_{0 \leqslant s \leqslant c} y(s), \quad b = \min\{t \colon y(t) = m\}.$$

Then, $b > a$ and $y(t) > y(b)$ for $0 \leqslant t < b$ (see Fig. 5.12).

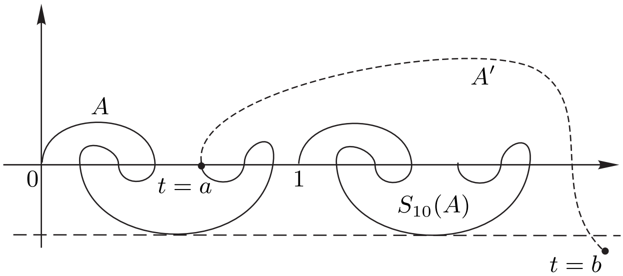


Figure 5.12. The arcs A, $S_{10}(A)$, and A'.

Denote by A the arc $\{\overline{m}(t) \colon 0 \leqslant t \leqslant b\}$. A further proof consists in obtaining a contradiction to the simplicity of the curve l. We will show that A should intersect the arc $S_{10}(A)$ by proving that $S_{10}(A)$ should intersect the sub-arc

$$A' \stackrel{\text{def}}{=} \{\overline{m}(t) \colon a \leqslant t \leqslant b\}.$$

First, we extend the arc A' to an infinite continuous curve C by adding two rays to A' in the following way. Set

$$\overline{w}(t) = \begin{cases} (x(a) + (t-a);\, 0) & \text{if } t \leqslant a, \\ \overline{m}(t) & \text{if } a \leqslant t \leqslant b, \\ (x(b);\, y(b) - (t-b)) & \text{if } b \leqslant t. \end{cases}$$

The infinite curve $C = \{\overline{w}(t) \colon -\infty < t < +\infty\}$ consists of the ray

$$(-\infty;\, x(a)) \times 0, \tag{5.8}$$

the arc A', and the ray

$$x(b) \times (-\infty;\, y(b)). \tag{5.9}$$

Since $y(t) > y(b)$ for $0 \leqslant t < b$ and A' does not intersect the axis Ox to the left of the point $x(a)$, C is a simple infinite curve on the Euclidean plane $\mathbb{R}^2$. Moreover, it is clear that C divides $\mathbb{R}^2$. Now, we will prove that the endpoints of the arc $S_{10}(A)$ lie in different components of the set $\mathbb{R}^2 - C$.

Indeed, the arc $S_{10}(A)$ does not intersect the ray (5.8) because $S_{10}(A)$ does not intersect Ox even to the left of the point $x = 1$ (recall that $\overline{l}$ does not intersect Ox for $x < 0$). Next, $S_{10}(A) \cap A' = \varnothing$ because l is a simple curve. Finally, $S_{10}(A)$ does not intersect the ray (5.9) because $y(t) > y(b)$ for $0 \leqslant t < b$. Therefore, $S_{10}(A) \cap C = \varnothing$.

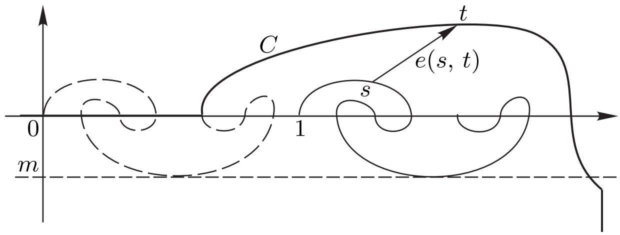


Figure 5.13. The curve C and the vector $e(s, t)$.

For $0 \leqslant s \leqslant b$ and $-\infty < t < +\infty$, we denote by $e(s, t)$ a unit vector directed from the point $S = S_{10}(\overline{m}(s))$ to the point $\overline{w}(t)$ (see Fig. 5.13). Formally,

$$e(s, t) = \frac{\overline{w}(t) - S_{10}(\overline{m}(s))}{|\overline{w}(t) - S_{10}(\overline{m}(s))|}.$$

Since $S_{10}(A) \cap C = \varnothing$, the vector $e(s, t)$ is well defined and continuously depends on (s, t). By continuity (see Fig. 5.13), we can set

$$e(s, -\infty) = (-1, 0), \quad e(s, +\infty) = (0, -1) \quad \text{for any } 0 \leqslant s \leqslant b.$$

Denote by $\varphi(s, t)$ the angle measured counterclockwise from the positive direction of the axis Ox toward $e(s, t)$. In fact, our definition specifies a family of "branches" (single-valued continuous functions) of a multivalued function that differ from each other by $2\pi k$. Choose $\varphi(s, t)$ by setting $\varphi(0, t) = \pi$ for $t \leqslant a$. By continuity, we can define $\varphi(s, t)$ at $t = -\infty$ by the equality $\varphi(s, -\infty) = \pi$ for any $0 \leqslant s \leqslant b$.

Since the curve C for $t \geqslant b$ is the ray (5.9), there exists a finite limit

$$\lim_{t \to +\infty} \varphi(s, t) \stackrel{\text{def}}{=} \varphi(s, +\infty),$$

$\varphi(s, +\infty) = \frac{3\pi}{2} + 2\pi k(s)$, where $k(s) \in \mathbb{Z}$. This and the continuity of $\varphi(s, t)$ imply that the function $\varphi(s, +\infty)$ is continuous in s and, hence, does not depend on s,

$$\varphi(s, +\infty) = \frac{3\pi}{2} + 2\pi k_0,$$

where the integer $k_0 \in \mathbb{Z}$ is fixed.

To determine k_0, we consider $\varphi(0, t)$. For $t \leqslant a$, we have $\varphi(0, t) = \pi$. When $t > a$ is sufficiently close to a, the y-component of the difference $\overline{w}(t) - S_{10}(\overline{m}(0))$ is positive. Therefore, $\varphi(0, t) < \pi$. However, for any $t > a$, we have

$$\overline{w}(t) \notin (-\infty; 1) \times 0,$$

so $\varphi(0, t) \notin \pi + 2\pi n$, $n \in \mathbb{Z}$. Hence, $-\pi < \varphi(0, t) < \pi$. The only number of the form $\frac{3\pi}{2} + 2\pi k$ that satisfies the last inequality, $-\frac{\pi}{2}$, is obtained when $k_0 = -1$.

Now, consider $\varphi(b, t)$. Since the points $\overline{m}(t)$ for $0 \leqslant t < b$ satisfy the inequality $y(t) > y(b)$, the y-component of the difference $\overline{w}(t) - S_{10}(\overline{m}(b))$ is positive. The equality $\varphi(b, -\infty) = \pi$ and the fact that the point $\overline{w}(t)$ runs over the ray (5.8) for $-\infty < t \leqslant 0$ imply that $\varphi(b, t) < \pi$ for $-\infty < t < b$. This and the equality

$$\overline{w}(b) - S_{10}(\overline{m}(b)) = \overline{m}(b) - S_{10}(\overline{m}(b)) = (-1, 0)$$

imply that $\varphi(b, b) = \pi$. Next, as t increases for $t \geqslant b$, the point $\overline{w}(t)$ descends along the ray (5.9) and

$$\overline{w}(t) - S_{10}(\overline{m}(b)) = (-1, b - t).$$

Hence, taking into account $\varphi(b, b) = \pi$, we obtain the equality $\varphi(b, +\infty) = \frac{3\pi}{2}$; i. e., $k_0 = 0$. However, the integer k_0 does not depend on s. The contradiction obtained proves the lemma. □

Corollary 5.3 *Let $l = \{m(t) : t \geqslant 0\}$ be a semi-infinite continuous curve without self-intersections on the cylinder $C^2 \cong \mathbb{R}^2/\Gamma$, where the group $\Gamma = \mathbb{Z}^1$ consists of the integer translations S_{n0}, and let $\overline{l} = \{\overline{m}(t) = (x(t), y(t)) : t \geqslant 0\}$ be a lift of this curve to the Euclidean plane $\mathbb{R}^2$. Suppose that $\overline{m}(0) = (0, 0)$ and $\overline{l}$ does not intersect the axis Ox for $x < 0$. Let there exist a finite parameter t after which $\overline{l}$ does not intersect the unit interval $0 \leqslant x \leqslant 1$ on the axis Ox. Then,*

$$\sup_{t \geqslant 0} y(t) < +\infty \quad \text{or} \quad \inf_{t \geqslant 0} y(t) > -\infty.$$

The following theorem, which is of independent interest, plays a key role in the proof of the Weil theorem.

Theorem 5.5 *Let* $l = \{m(t)\colon t \geqslant 0\}$ *be a semi-infinite continuous curve without self-intersections on the torus* T^2*, and let* $\overline{l} = \{\overline{m}(t) = (x(t), y(t))\colon t \geqslant 0\}$ *be a lift of this curve to the Euclidean plane* $\mathbb{R}^2$*. Suppose that* $\overline{l}$ *goes to infinity and intersects a certain ray* $\overline{r}_0$ *with a rational asymptotic direction for arbitrarily large values of the parameter. Then,* $\overline{l}$ *can not unboundedly deviate from this ray to both sides simultaneously.*

Proof. The ray $\overline{r}_0$ is projected to a simple closed geodesic on T^2 that can be mapped to a "meridian" or a "parallel" by a certain affine transformation of the torus. Therefore, without loss of generality, we may assume that $\overline{r}_0$ is the positive half-line of the axis Ox.

Let us show that $\overline{l}$ does not intersect the negative half-line of Ox at infinity. Suppose the contrary. Since $\overline{l}$ goes to infinity, $\overline{l}$ does not intersect the unit interval $I = \{(x, y)\colon 0 \leqslant x \leqslant 1,\ y = 0\}$ starting from a certain value of the parameter. It follows from the assumption that there exists an arc that intersects Ox only at its endpoints that lie on the opposite sides of I. This fact contradicts Lemma 5.9 because we may "shorten" $\overline{l}$ and assume that one of the endpoints of the arc is the starting point of $\overline{l}$.

The fact that $\overline{l}$ goes to infinity implies that there exists a parameter after which $\overline{l}$ does not intersect the negative half-line of Ox. It suffices to prove the theorem for any subcurve $\{\overline{m}(t)\colon t \geqslant t_0\}$; hence, we may assume that the starting point $\overline{m}(0)$ coincides with the origin and that $\overline{l}$ does not intersect Ox for $x < 0$.

Again, the fact that $\overline{l}$ goes to infinity implies that there exists an ultimate value of the parameter after which $\overline{l}$ does not intersect the interval I. Now, the required assertion follows from Corollary 5.3. □

Proof of Theorem 5.4. Without loss of generality, we may assume that $\overline{l}$ emanates from the origin of the Cartesian system of coordinates. A further proof is divided into two parts:

1) $\overline{l}$ does not intersect at infinity any ray that emanates from the origin and has a rational asymptotic direction;
2) there exists at least one ray with a rational asymptotic direction that is intersected by $\overline{l}$ at infinity.

Consider case (1). Let us draw two arbitrary rational rays $\overline{r}_1$ and $\overline{r}_2$ from the origin. Then, starting from a certain point, $\overline{l}$ lies completely within the angle $\overline{r}_1 O \overline{r}_2$ (one of the two complementary angles). Let us draw a rational ray $\overline{r}_3$ in $\overline{r}_1 O \overline{r}_2$. Then, starting from a certain point, $\overline{l}$ lies completely in one of the two angles obtained. Continuing this procedure, we construct a sequence of rays $\overline{r}_1, \ldots, \overline{r}_n, \ldots$ that emanate from the origin, have a rational asymptotic direction, and are such that the least angle (of the two complementary angles) between $\overline{r}_n$ and $\overline{r}_{n+1}$ tends to zero and contains the curve $\overline{l}$ starting from a certain moment. We may assume that the sequence of rays has a limit ray $\overline{r}$. Then, the curve $\overline{l}$ has an asymptotic direction that corresponds to the ray $\overline{r}$.

Consider case (2). It follows from Theorem 5.5 that $\overline{l}$ does not intersect any ray at infinity except for $\overline{r}_0$. Indeed, if this was not the case and $\overline{l}$ intersected some other ray $\overline{r}_1$ at infinity, then $\overline{l}$ should intersect at infinity any ray with a rational asymptotic direction that lies between the rays $\overline{r}_0$ and $\overline{r}_1$. The curve $\overline{l}$ should unboundedly deviate from this ray to both sides, which contradicts Theorem 5.5. Hence, the curve $\overline{l}$ has an asymptotic direction corresponding to the ray $\overline{r}_0$. This proves the Weil theorem.

Proof of the Weil conjecture

Curiously enough, the proof of the Weil conjecture proved to be simpler than the proof of the Weil theorem.

Proof of Conjecture 5.1. Suppose that $\overline{l}$ does not have an asymptotic direction. Since the curve $\overline{l}$ goes to infinity, its limit set at infinity contains at least two points and coincides with the complete limit set, $\mathrm{Lim}(\overline{l}) = \lim_{\infty}(\overline{l})$. By Lemma 5.1, the complete limit set of the lift of a semi-infinite curve is closed and connected. Therefore, $\mathrm{Lim}(\overline{l})$ contains a nontrivial interval, which we denote by $I \subset S_\infty$.

Since the group Γ of deck transformations is a Fuchsian group of the first kind, there exists a hyperbolic isometry $\gamma \in \Gamma$ such that the ideal endpoints of its axis $O(\gamma)$ belong to the interval I. Note that $O(\gamma)$ is projected to a closed geodesic on the surface.

Take a sufficiently long interval $\overline{A} \subset O(\gamma)$ such that one of its endpoints is mapped by γ into $\overline{A}$. The interval $\overline{A}$ divides the axis $O(\gamma)$ into two subintervals $\overline{A}_1$ and $\overline{A}_2$. Each of these subintervals has one ideal endpoint in I. Since the curve $\overline{l}$ goes to infinity, it does not intersect $\overline{A}$ starting from a certain moment. The fact that I belongs to the limit set of the curve $\overline{l}$ implies that there exists an arc $\overline{S}$ of the curve $\overline{l}$ that intersects $O(\gamma)$ only at the endpoints, such that one of the endpoints is in $\overline{A}_1$ and the other in $\overline{A}_2$, Fig. 5.14. But then, according to Theorem 5.3, the curve l has self-intersections because $\overline{S}$ and $\gamma(\overline{S})$ intersect. This contradicts the assumption. □

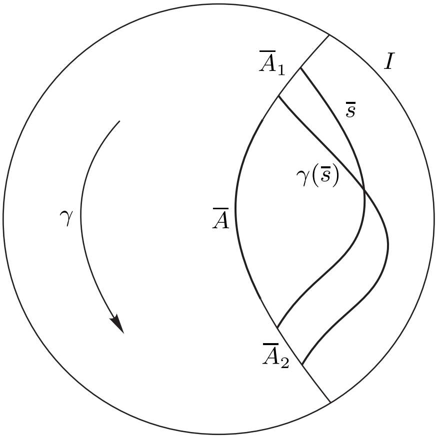


Figure 5.14. The arcs $\overline{S}$ and $\gamma(\overline{S})$ intersect.

Taking in mind that the group of deck transformations for the torus consists of integer translations of $\mathbb{R}^2$, and using Theorem 5.2, we can prove the following lemma.

Lemma 5.11 *Let $l = \{m(t) \colon t \geqslant 0\}$ and $l_1 = \{m_1(t) \colon t \geqslant 0\}$ be disjoint semi-infinite continuous curves each without self-intersections on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and let $\overline{l} = \{\overline{m}(t) \colon t \geqslant 0\}$ and $\overline{l}_1 = \{\overline{m}_1(t) \colon t \geqslant 0\}$ be its lifts to the Euclidean plane $\mathbb{R}^2$ respectively. If the both $\overline{l}$ and $\overline{l}_1$ go to infinity, then their asymptotic directions are either coincident or diametrically opposite.*

We leave the proof to the Reader (see details in [11]).

5.3. Anosov theorems on asymptotic directions and approximations of curves

In this section, we consider the following Anosov's Theorems 5.6, 5.7, and 5.8 on the existence of asymptotic directions of semitrajectories for flows on flat and hyperbolic surfaces.

Theorem 5.6 *If the set of fixed points of a topological flow f^t on a closed surface M of nonpositive Euler characteristic is finite, then any semitrajectory of the covering flow $\overline{f}^t$ on $\overline{M}$ is either bounded or has an asymptotic direction.*

To formulate Theorem 5.7, we need the following definition. A subset $F \subset M$ is called *contractible to a point* if there exists a continuous mapping $\varphi\colon F \times [0; 1] \to M$ such that $\varphi(m, 0) = m$ and $\varphi(m, 1) = m_0$ for any $m \in M$, where $m_0 \in M$ is a certain point.

Theorem 5.7 *If the set of fixed points of a topological flow f^t on a closed surface M of nonpositive Euler characteristic is contractible to a point, then any semitrajectory of the covering flow $\overline{f}^t$ on $\overline{M}$ is either bounded or has an asymptotic direction.*

Theorem 5.8 *If a flow f^t on a closed surface M of nonpositive Euler characteristic is analytic, then any semitrajectory of the covering flow $\overline{f}^t$ on $\overline{M}$ is either bounded or has an asymptotic direction.*

Theorem 5.8 does not follow from Theorem 5.7 because the set of fixed points of an analytic flow may contain, for instance, homotopically nontrivial closed curves and, hence, may not be contractible to a point. Naturally, Theorem 5.6 follows from Theorem 5.7. However, to demonstrate the idea of the proof in a simpler situation, we prove carefully only Theorem 5.6. Then we show what changes should be made in order to prove Theorem 5.8.

Presently, Theorem 5.7 gives the most general sufficient conditions for an unbounded semitrajectory of a topological flow to have an asymptotic direction.

Note that the problem of whether a closed curve (or a semi-infinite periodic curve) has an asymptotic direction is solved without difficulty, due to Lemma 5.2.

Theorem 5.9 *Let C be a closed curve* (*or a semi-infinite periodic curve*) *on a surface M. Then,*

1) *if C is null-homotopic, then any of its lifts $\overline{C}$ to $\overline{M}$ is a closed* (*and, hence, bounded*) *curve*;
2) *if C is non-null-homotopic, then any of its lifts $\overline{C}$ to $\overline{M}$ is a nonclosed infinite curve both of whose semi-infinite curves have a rational asymptotic direction.*

This assertion can be reformulated for periodic curve (recall that a semi-infinite curve l^+ is called periodic if there exist a parametrization $m\colon \mathbb{R}^+ \to M$ of l^+ and a number $T > 0$ such that $m(t) = m(t + nT)$ for any $n \in \mathbb{Z}$ and any $t \geqslant 0$) as follows.

Theorem 5.10 *Let l^+ be a semi-infinite periodic curve on a closed surface M of nonpositive Euler characteristic, and let $\overline{l}^+$ be its lift to $\overline{M}$. If l^+ is non-null-homotopic, then $\overline{l}^+$ goes to infinity and has a rational asymptotic direction.*

It is clear that l^+ can be represented as the image of the circle S^1 in M under a continuous mapping (i. e., l^+ is a closed curve in the topological sense), and we can speak of the null-homotopy or non-null-homotopy of such curve.

By Theorem 5.10, in all three Theorems 5.6, 5.7, and 5.8, it suffices to consider nonperiodic trajectories.

Recall that $U_\varepsilon(N)$ is the ε-neighborhood of a subset N of M or $\overline{M}$.

Proof of Theorem 5.6

Taking into account the Weil theorem and conjecture, it suffices to prove that under the hypotheses of Theorem 5.6, any unbounded semitrajectory of the covering flow goes to infinity. Suppose the contrary. Then, the covering flow $\overline{f}^t$ on $\overline{M}$ has a semitrajectory (for definiteness, we will assume that it is positive) $\overline{l}^+$ that is unbounded and returns to a certain compact domain $\overline{K} \subset \overline{M}$ at arbitrarily large values of time. According to Theorem 5.10, we can assume that $\overline{l}^+$ is not a lift of a periodic trajectory that is non-null-homotopic on M.

Let $\operatorname{Fix} f^t \stackrel{\text{def}}{=} F$ be the set of fixed points of f^t. Since the set F is finite, there exists $\varepsilon > 0$ such that the ε-neighborhood $U_\varepsilon(F)$ is a finite set of pairwise disjoint ε-disks. Therefore, $\pi^{-1}(U_\varepsilon(F))$ is a disjoint union of a countable set of ε-disks on $\overline{M}$. For a sufficiently small $\varepsilon > 0$, the disks from $\pi^{-1}(U_\varepsilon(F))$ do not cover $\overline{K}$. Moreover, since $\overline{l}^+$ is unbounded, the limit set of the semitrajectory l^+ can not consist of a single fixed point. Therefore, $\overline{l}^+$ has an infinite number of points with arbitrarily large values of time outside $\pi^{-1}(U_\varepsilon(F))$, and one may assume that these points lie in $\overline{K}$. Hence, the semitrajectory $\overline{l}^+$ has at least one limit point (denote it by $\overline{m}_*$) in $\overline{K}$ that is different from a fixed point. Let us draw a transversal segment $\overline{\Sigma}$ through $\overline{m}_*$, Fig. 5.15. Then, the semitrajectory $\overline{l}^+$ intersects $\overline{\Sigma}$ at a countable set of points. Denote by $\overline{m}(t_i) \in \overline{l}^+ \cap \overline{\Sigma}$ successive (in time) intersections of $\overline{l}^+$ with $\overline{\Sigma}$, $0 \leqslant t_1 < \ldots < t_i < \ldots$. The arc of the semitrajectory $\overline{l}^+$ between the points $\overline{m}(t_i)$ and $\overline{m}(t_{i+1})$ (the so-called $\overline{\Sigma}$-arc) and a part of the segment $\overline{\Sigma}$ between these points form a simple closed curve, which we denote by $\overline{C}(t_i)$. Since $\overline{M}$ is simply connected, $\overline{C}(t_i)$ bounds a domain $\overline{D}(t_i)$ on $\overline{M}$.

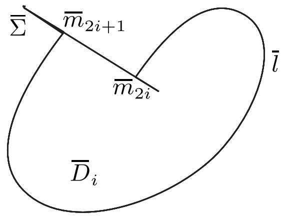


Figure 5.15. The transversal segment $\overline{\Sigma}$.

If we assume that $\overline{l}^+$ enters $\overline{D}(t_i)$ after the point $\overline{m}(t_{i+1})$, then $\overline{l}^+$ must remain in $\overline{D}(t_i)$, which contradicts the unboundedness of $\overline{l}^+$. Hence, the semitrajectory $\overline{l}^+$ goes out of $\overline{D}(t_i)$ after $\overline{m}(t_{i+1})$ and then it can not intersect $\overline{\Sigma}$ between the points $\overline{m}(t_i)$ and $\overline{m}(t_{i+1})$. This implies the monotonicity of the sequence of points $\overline{m}(t_i) \in \overline{l}^+ \cap \overline{\Sigma}$ and the inclusion $\overline{D}(t_i) \subset \overline{D}(t_{i+1})$. Making some changes in the notations, we find that there exists a sequence of points $\overline{m}_i \in \overline{l}^+ \cap \overline{\Sigma}$ that satisfy the following conditions:

1) the arc $\overline{l}_i$ of the semitrajectory $\overline{l}^+$ with the endpoints $\overline{m}_i$ and $\overline{m}_{i+1}$ and the segment $\overline{I}_i = [\overline{m}_i; \overline{m}_{i+1}] \subset \overline{\Sigma}$ form a simple closed curve that bounds an open domain $\overline{D}_i$ on $\overline{M}$;
2) $\overline{D}_0 \subset \ldots \subset \overline{D}_i \subset \overline{D}_{i+1} \subset \ldots$.

Let us prove that for a certain index i, the domain $\overline{D}_i$ contains congruent points strictly inside it. To this end, it suffices to find $\overline{D}_i$ whose area is greater than the area of the fundamental domain of the group Γ (by the area of a domain $\overline{D}_i$ that is bounded by a continuous non-self-intersecting curve, we mean its two-dimensional measure induced by the metric on $\overline{M}$; the area of the fundamental domain of the group Γ is equal to the area of the surface M). The idea of the proof of this assertion is simple: every point of the one-dimensional trajectory lies in a certain neighborhood of a constant field, and the arcs $\overline{l}_i$ can not form "tongues" of arbitrarily small "width". Therefore, the increasing sequence of domains $\overline{D}_i$ contains the required sufficiently large domain.

Since the group Γ of deck transformations of the covering $\pi\colon \overline{M} \to M$ is disconnected, there exists a number $\rho > 0$ such that any disk of radius ρ on $\overline{M}$ does not contain congruent points. Then, any set on $\overline{M}$ whose diameter is less than ρ is projected onto its own image on the surface M homeomorphically, and even isometrically.

It is well-known [218] that each nonsingular point (i. e., a point different from a fixed point) $z \in M$ of a topological flow has a neighborhood $U_{2\delta}(z) = \psi_z(S_{2\delta})$, where ψ_z is a rectifying homeomorphism of the square

$$S_{2\delta} = (-2\delta;\, 2\delta) \times (-2\delta;\, 2\delta) \subset \mathbb{R}^2, \quad \delta = \delta(z) > 0,$$

that maps segments of the form $y = \mathrm{const}$ to the components of the intersection of trajectories with $U_{2\delta}(z)$. The neighborhood $U(z)$ is a flow box with the structure of a constant field. One may assume that the diameters of all the neighborhoods $U_{2\delta}(z)$ are less than ρ. Slightly smaller neighborhoods $U_\delta(z) = \psi_z(S_\delta)$ form a covering of the compact set $M - U_\varepsilon(F)$. Therefore, there exists a finite subcovering $U_\delta(z_1), \ldots, U_\delta(z_k)$. Obviously, there exists $\mu > 0$ such that the areas of all the subdomains

$$\begin{aligned} V_i^+ &= \psi_{z_i}((-\delta;\, \delta) \times (\delta;\, 2\delta)), \\ V_i^- &= \psi_{z_i}((-\delta;\, \delta) \times (-2\delta;\, -\delta)) \subset U_{2\delta}(z_i), \quad i = 1, \ldots, k, \end{aligned}$$

are no less than μ.

Since $\operatorname{diam} U_{2\delta}(z_i) < \rho$ for any fixed $i = 1, \ldots, k$, $\pi^{-1}(U_{2\delta}(z_i))$ is a countable family $\{\overline{U}_{2\delta}(\overline{z}_{ij})\}_{j\geqslant 1}$ of pairwise disjoint sets $\overline{U}_{2\delta}(\overline{z}_{ij})$ each of which is a neighborhood with the product structure for the covering flow. The subfamily

$$\bigcup_{j\geqslant 1} \overline{U}_\delta(\overline{z}_{ij}) = \pi^{-1}(U_\delta(z_i))$$

forms a covering of the set $\overline{M} - \pi^{-1}(U_\varepsilon(F))$. Similarly, for any fixed $i = 1, \ldots, k$, the preimage $\pi^{-1}(V_i^{\pm})$ is also a countable set $\{\overline{V}_{ij}^{\pm}\}_{j\geqslant 1}$ of pairwise disjoint sets $\overline{V}_{ij}^{\pm}$ each of which is isometrically projected into $V_i^{\pm}$ and, hence, has the area at least μ.

Let us show that if a fixed neighborhood $\overline{U}_{2\delta}(\overline{z}_{ij})$ does not intersect the segment $\overline{\Sigma}$, then any arc $\overline{l}_n$ intersects $\overline{U}_{2\delta}(\overline{z}_{ij})$ along at most one component. Indeed, if a certain $\overline{l}_n$ intersects $\overline{U}_{2\delta}(\overline{z}_{ij})$ along more than one component, then $\overline{l}_n$ at least twice intersects a certain segment without contact that lies in $\overline{U}_{2\delta}(\overline{z}_{ij})$. This segment without contact and the corresponding sub-arc $\overline{l}_n' \subset \overline{l}_n$ form a closed curve, which we denote by $\overline{c}$. Note that $\overline{l}_n' \neq \overline{l}_n$ and, moreover, the endpoints of $\overline{l}_n'$ are internal points of $\overline{l}_n$. Exactly one endpoint of the arc $\overline{l}_n$ must lie inside $\overline{c}$ and, hence, $\overline{c}$ must contain the segment $\overline{\Sigma}$ because $\overline{\Sigma}$ intersects $\overline{l}_n$ only at the endpoints of $\overline{l}_n$. On the other hand, the other endpoint of the arc $\bar{l}_n$ lies outside $\bar{c}$; therefore, the segment $\overline{\Sigma}$ must lie outside $\bar{c}$. The contradiction obtained proves the required assertion.

Without loss of generality, we may assume that $\overline{\Sigma}$ and all the points $\overline{m}_i \in \in \bar{l}^+ \cap \overline{\Sigma}$ lie in the ρ-neighborhood of the point $\overline{m}_*$. Denote by ϱ_i the largest distance from the points of the arc $\bar{l}_i$ to $\overline{m}_*$. Since $\bar{l}^+$ is unbounded by the assumption, ϱ_i tends to infinity. We may assume that $\varrho_i > 4$. To simplify the expressions, we will assume below that $\rho = \frac{1}{4}$, changing, if necessary, the scale of the metric.

Let us fix an index i. The arc $\bar{l}_i$ leaves the circle $U_{\varrho_i}(\overline{m}_*)$ of radius ϱ_i centered at $\overline{m}_*$. This arc intersects at least $[\varrho_i] - 2 \overset{\text{def}}{=} s_i$ (the square brackets denote the integer part) concentric rings $A_1, \ldots, A_{s_i}$, where $A_j = U_{j+1}(\overline{m}_*) - - U_j(\overline{m}_*)$, $1 \leqslant j \leqslant s_i$ (Fig. 5.16). Therefore, on the medial line of the ring A_j (i.e., on the boundary of the circle $U_{j+1/2}(\overline{m}_*)$), there exists a point (denote it by $\overline{m}_{ij}$) that lies on the arc $\bar{l}_i$. Note that by the construction, the open disk $U_{2\rho}(\overline{m}_{ij})$ with the center at $\overline{m}_{ij}$ and with radius 2ρ lies in the ring A_j.

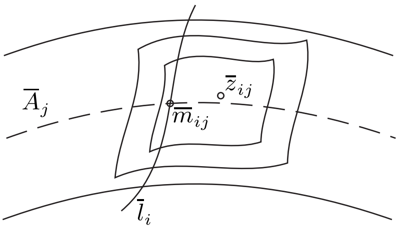


Figure 5.16. The rings A_j.

The point $\overline{m}_{ij}$ belongs to a certain neighborhood with the structure of a constant field of the form $\overline{U}_\delta(\overline{z}_{pq})$. Denote this neighborhood by $\overline{U}_\delta(\overline{z}_{ij})$. Recall that $\overline{U}_\delta(\overline{z}_{ij})$ lies inside a neighborhood with the structure of a constant field $\overline{U}_{2\delta}(\overline{z}_{ij})$. Since $\operatorname{diam} U_{2\delta}(z_i) < \rho = \frac{1}{4}$, $\overline{U}_{2\delta}(\overline{z}_{ij})$ lies in the ring A_j. Obviously, the neighborhoods $\overline{U}_{2\delta}(\overline{z}_{ij})$ (and certainly $\overline{U}_\delta(\overline{z}_{ij})$) are pairwise disjoint ($1 \geqslant j \geqslant s_i$, and the index i is fixed). Note that $\overline{\Sigma}$ and all the points $\overline{m}_i \in \bar{l}^+ \cap \overline{\Sigma}$ lie in $U_1(\overline{m}_*)$. Therefore, each neighborhood $\overline{U}_\delta(\overline{z}_{ij})$, $1 \geqslant j \geqslant s_i$, does not intersect $\overline{\Sigma}$; hence, as shown above, the arc $\bar{l}_i$ intersects each $\overline{U}_\delta(\overline{z}_{ij})$ along exactly one component. This component divides $\overline{U}_\delta(\overline{z}_{ij})$ into two parts; one part, say $\bar{d}_{ij}$, lies in the domain $\overline{D}_i$. However, we showed above that $\bar{l}_i$ intersects not only $\overline{U}_\delta(\overline{z}_{ij})$ but also $\overline{U}_{2\delta}(\overline{z}_{ij})$ along exactly one component. Hence, the domain $\overline{D}_i$ contains a domain of the form $\overline{V}^{\pm}_{ij}$ that adjoins $\overline{d}_{ij}$.

Thus, $\overline{D}_i$ contains $[\varrho_i] - 2$ disjoint subsets of the form $\overline{V}^{\pm}_{ij}$ of area greater than μ each. It follows from $\limsup\limits_{i\to\infty} \varrho_i = \infty$, that there exists an i such that the area of $\overline{D}_i$ is greater than the area of the surface M. Hence, the projection $\pi(\overline{D}_i)$ covers twice a certain point on M together with some neighborhood, so there are two congruent points strictly inside $\overline{D}_i$.

Fix this i and the domain $\overline{D}_i = \overline{D}$. There exists an isometry $\gamma \in \Gamma$, $\gamma \neq \mathrm{id}$, such that

$$\gamma(\overline{D}) \cap \overline{D} \neq \varnothing. \tag{5.10}$$

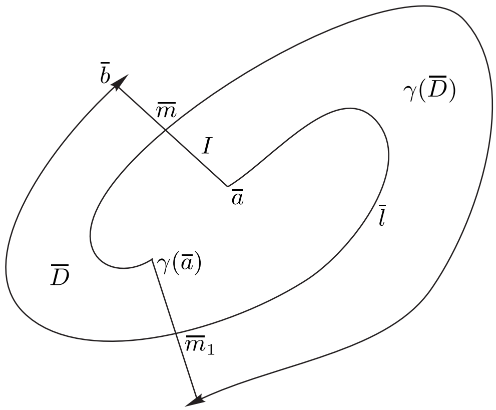


Figure 5.17

Let us show that this is impossible. We can take $\overline{I}_i = I$ so small that $\gamma(I) \cap \cap I = \varnothing$. For short, we denote $\bar{l}_i = \bar{l}$. Since the semitrajectory $\bar{l}^+$ is unbounded, the trajectories of the flow $\overline{f}^t$ should leave $\overline{D}$ through a segment without contact I as time increases. Let $\overline{a}$ be an endpoint of the segment I such that its negative semitrajectory lies in $\overline{D}$ (see Fig. 5.17). From 5.10, we obtain

$$\gamma(\bar{l} \cup I) \cap (\bar{l} \cup I) \neq \varnothing.$$

The arc $\bar{l}$ of the semitrajectory $\bar{l}^+$ that belongs to the boundary $\overline{D}$ can not have congruent points, because otherwise the semitrajectory $\overline{l}^+$ would be a covering for a periodic non-null-homotopic trajectory (and would go to infinity). Therefore, either $\overline{l} \cap \gamma(I) \neq \varnothing$, or $\gamma(\overline{l}) \cap I \neq \varnothing$, or both these inequalities hold. Replacing, if necessary, γ by γ^{-1}, we can assume that $\gamma(\overline{l}) \cap I \neq \varnothing$; this intersection consists of a single point, which we denote by $\overline{m}$.

After intersecting the boundary of the domain $\overline{D}$ at $\overline{m} \in I$, the arc $\gamma(l)$ can not intersect I once again and return to $\overline{D}$ because I is a segment without contact. Therefore, one endpoint of the arc $\gamma(\overline{l})$ lies inside $\overline{D}$ (obviously, this is the point $\gamma(\overline{a})$), while the other endpoint $\gamma(\overline{b})$ lies outside the domain $\overline{D}$. Hence, $\gamma(I) \cap \overline{l} \neq \varnothing$, and this intersection consists of a single point, which we denote by $\overline{m}_1$. Then, the negative semitrajectory of the point $\overline{m}_1$ contains the negative semitrajectory of the point $\overline{a}$. Therefore, the negative semitrajectory of the point $\gamma^{-1}(\overline{m}_1)$ contains the negative semitrajectory of the point $\gamma^{-1}(\overline{a})$.

The inclusion $\gamma(\overline{a}) \in \overline{D}$ and the fact that I is a segment without contact imply that the negative semitrajectory of the point $\gamma(\overline{a})$ lies in $\overline{D}$. On the other hand, since $\gamma^{-1}(\overline{m}_1) \in I$, the negative semitrajectory of the point $\gamma^{-1}(\overline{m}_1)$ lies in $\overline{D}$. Hence, the negative semitrajectory of the point $\gamma^{-1}(\overline{a})$ also lies in $\overline{D}$. Hence, the negative semitrajectory of the point $\gamma(\overline{a})$ lies in $\gamma^2(\overline{D})$. Since the negative semitrajectory of the point $\gamma(\overline{a})$ lies in $\overline{D}$, we have

$$\overline{D} \cap \gamma^2(\overline{D}) \neq \varnothing.$$

Repeating this line of reasoning with the element γ replaced by γ^2, we obtain $\overline{D} \cap \gamma^4(\overline{D}) \neq \varnothing$. Hence, $\overline{D} \cap \gamma^{2^n}(\overline{D}) \neq \varnothing$ for any $n \in \mathbb{N}$. However, since the domain $\overline{D}$ is bounded and the group Γ is disconnected, only a finite number of intersections $\overline{D} \cap \gamma^i(\overline{D})$ may be nonempty. The contradiction proves the theorem. □

Previous results on analytic flows

The proof of Theorem 5.8 is based on the description of the set $\mathrm{Fix}\, f^t$ of fixed points of an analytic flow f^t and on the analysis of the structure of f^t in the neighborhood of the set $\mathrm{Fix}\, f^t$. The results we need one formulates in the following lemma.

Lemma 5.12 *Let* $\mathrm{Fix}\, f^t$ *be the set of fixed points of an analytic flow* f^t *on* M. *Then,* $\mathrm{Fix}\, f^t$ *contains a finite number of isolated points and a finite number of isolated simple closed curves. The remaining subset of the set* $\mathrm{Fix}\, f^t$ *contains a finite number of points* S_1 *that divide this subset into a finite number of* *pairwise disjoint simple arcs with the endpoints in* S_1. *Outside* S_1, *all curves from* $\operatorname{Fix} f^t$ *are analytic.*

The proof is omitted. We note only that it is based on the Weierstrass preparation theorem and on the simple original fact that locally, the set $\operatorname{Fix} f^t$ is defined by the equations $f(x, y) = 0$ and $g(x, y) = 0$, where f and g are analytic functions in a certain domain of the plane $\mathbb{R}^2$.

Note that Theorem 5.8 gives a final solution to the problem of the existence of asymptotic directions for the covering semitrajectories of analytic flows.

The idea of the proof of Theorem 5.8 follows the idea of the proof of Theorem 5.7. Despite the fact that the set $\operatorname{Fix} f^t \overset{\text{def}}{=} F$ may be continual and even noncontractible, its local topological structure without a finite set of singular points S_1 is known (an open interval). By virtue of the analyticity, the behavior of the trajectories of the flow near $\operatorname{Fix} f^t - S_1$ is also obvious: all points that are sufficiently close to $\operatorname{Fix} f^t - S_1$ and that lie locally on the same side of the arc of the set $\operatorname{Fix} f^t - S_1$ move along the trajectories in the same direction (i. e., they do not behave like in Example 5.1). This fact allows us to prove the theorem by analogy with the proof of Theorem 5.7.

Proof of theorem 5.8

Let $\varepsilon_1 > 0$ be so small that the set $U_{2\varepsilon_1}(\pi^{-1}(S_1))$ consists of disjoint disks of radius $2\varepsilon_1$. The set $F - U_{\varepsilon_1}(S_1)$ is covered by a finite number of real analytic charts (V, ψ) such that

$$\psi(V) = (-2\delta; 2\delta) \times (-2\delta; 2\delta), \quad \psi(F \cap V) = (-2\delta; 2\delta) \times 0; \qquad (5.11)$$

moreover, we may assume that even reduced charts with $|r|, |s| < \delta$ cover $F - U_{\varepsilon_1}(S_1)$. Denote by (f, g) the components of the vector field that defines the flow f^t. These functions vanish for $s = 0$, so that

$$f = s^m f_1(r, s), \quad g = s^n g_1(r, s), \quad f_1(r, 0) \neq 0, \quad g_1(r, 0) \neq 0,$$

except for the case when $f \equiv 0$ or $g \equiv 0$. Let us refer a point $z \in F - U_{\varepsilon_1}(S_1)$ to the set S_2 if, for some of our charts that contain z, one of the following equations holds for the corresponding r:

$$f_1(r, 0) = 0 \text{ for } m < n \text{ or for } g \equiv 0,$$
$$g_1(r, 0) = 0 \text{ for } n < m \text{ or for } f \equiv 0,$$
$$f_1(r, 0) = 0 \text{ or } g_1(r, 0) = 0 \text{ for } m = n.$$

It is clear that S_2 is finite. Slightly reducing ε_1 if necessary, we may assume that $F - U_{\varepsilon_1}(S_1)$ is covered, as before, by the charts with $|r|,\ |s| < \delta$ and that the boundary of the disks that form $U_{\varepsilon_1}(S_1)$ does not have any points from S_2. Now, let $\varepsilon_2 > 0$ be so small that the set $\pi^{-1}(U)$ with

$$U = U_{\varepsilon_1}(S_1) \cup U_{\varepsilon_2}(S_2)$$

consists of disjoint disks of radius ε_1 with centers at the points of S_1 and of radius ε_2 with centers at the points of S_2.

Near the points of the set $F - U$, we have a sufficiently informative description of the flow. Namely, $F - U$ can be covered by charts $(V,\ \psi)$ such that (5.11) holds as before, the reduced charts with $|r|,\ |s| < \delta$ also cover $F - U$,

$$f = s^m f_1(r,\ s) \ \text{ or } \ f \equiv 0 \qquad \text{and} \qquad g = s^n g_1(r,\ s) \ \text{ or } \ g \equiv 0$$

in terms of the coordinates $(r,\ s)$, the functions f and g are not identically zero simultaneously, and f_1 and g_1 are different from zero everywhere in $(-2\delta;\ 2\delta)^2$. We will distinguish three types of charts.

1) $m < n$ or $g \equiv 0$, and m is even.
2) $m < n$ or $g \equiv 0$, and m is odd.
3) $m \geqslant n$ or $f \equiv 0$.

A point $z \in F$ that lies in a chart of type C can not be an ω-limit point for a semitrajectory $L = \pi(\overline{L})$. Indeed, within a chart of type C, the points move along the integral curves of the differential equation

$$\frac{dr}{ds} = s^{m-n} \frac{f_1(r,\ s)}{g_1(r,\ s)} \qquad \left(\text{or } \frac{dr}{ds} \equiv 0 \text{ if } f \equiv 0\right).$$

These curves intersect the axis r; however, while moving along such a curve γ with the phase velocity $(f,\ g)$, a point can not intersect the axis r but only indefinitely approaches the point $\gamma \cap (\mathbb{R} \times 0)$ as $t \to +\infty$ or $t \to -\infty$. Hence, either a point of the axial line $\psi^{-1}((-2\delta;\ 2\delta) \times 0)$ of our chart can not be an ω-limit point, or it is not only an ω-limit point but also a unique limit of the corresponding motion as $t \to \infty$. If L tended to z, then $\overline{L}$ would be bounded, in contrast to the assumption.

Thus, the set $\omega(L) \cap (F - U)$ lies in charts of types A or B. It is a closed set. Let us show that at the same time, it is open in $F - U$, so that the set $F - (\omega(L) \cup U)$ is also closed. Let $z \in \omega(L)$ and $z \in F - U$. Then, z lies in a certain chart (V, ψ) in which either $m < n$ or $g \equiv 0$. Near the axial line $\psi^{-1}((-2\delta; 2\delta) \times 0)$, or near the segment $(-2\delta; 2\delta)$ of the axis r in terms of the coordinates (r, s), the points move along the integral curves of the differential equation

$$\frac{ds}{dr} = s^{n-m}\frac{g_1(r, s)}{f_1(r, s)} = O(s^{n-m}) \quad \left(\text{or } \frac{ds}{dr} \equiv 0 \text{ if } g \equiv 0\right) \tag{5.12}$$

uniformly with respect to r on any segment $[a; b] \subset (-2\delta; 2\delta)$. Therefore, the connected arc of $L \cap V$ that is close to z is also close to all points of the axial line located in a certain neighborhood of the point z. Together with z, all these points should be ω-limit points for L.

Let us modify the charts of types A and B so that they look like rectifying charts. Namely, keeping the coordinate r fixed, we replace s by a new coordinate (denoted again by s), so that equation (5.12) takes the form $s = \text{const}$ in the new coordinates. The relevant construction is similar to the construction of current tubes, but it is based on the differential equation (5.12). Naturally, the charts may become smaller in this case; however, we may add new charts. In the modified chart A or B, we distinguish parts W and $V^{\pm}$ as in the proof of Theorem 5.6. Let

$$d_0 = \min\left\{\frac{1}{4}, \overline{d}(L, F - (\omega(L) \cup U))\right\}. \tag{5.13}$$

Let us cover $M - U$ by rectifying charts of types A, B, and C, all of which have diameter $< d_0$. Choose a finite system of charts such that, even when reduced to W, they still cover $M - U$. Let us lift this system of charts to $\overline{M}$ and schematically reproduce the proof of Theorem 5.6.

We construct "Bendixson bags" B_i. The curve $\overline{L}$ has an ω-limit point w that lies outside $\pi^{-1}(U)$. If $w \notin F$ or if $w \in F$ and w lies in a chart of type A, then we repeat verbatim the construction from the proof of Theorem 5.6 (it is essential that in a chart of type A, the points move in the same direction on both sides of the axial line). If $w \in F$ and w lies in a chart of type B, then the construction is slightly modified. Let

$$\psi(w) = (r, 0), \quad J \stackrel{\text{def}}{=} \psi^{-1}(r \times (-2\delta; 2\delta)).$$

At least one of the two parts into which w divides J contains an infinite set of points a_n such that $a_n \in \overline{L}$ and $a_n \to w$. Denote this part of J by I. Let us enumerate the points a_n in increasing time. By analogy with the proof of Theorem 5.6, take points $z_{ij} \in \overline{L}_i$ that are located sufficiently far from w, from each other, and from the points $\pi^{-1}(S_1)$. Consider the charts W_{ij} that contain the points z_{ij}, and the larger charts (V_{ij}, ψ_{ij}). There are no charts of type C among the latter charts. If V_{ij} is a rectifying chart or a chart of type A, we can repeat word for word the reasoning of the proof of Theorem 5.6, according to which either $V_{ij}^+ \subset B_i$ or $V_{ij}^- \subset B_i$. If V_{ij} is a chart of type B, then the line of reasoning should be different.

Let $\psi_{ij}(z_{ij}) = (r_{ij}, s_{ij})$ and let, for definiteness, $s_{ij} > 0$ (which can be achieved by changing the sign of s). Then, B_i contains either all points $\psi_{ij}^{-1}(r, s)$ such that $|s - s_{ij}|$ is sufficiently small and $s > s_{ij}$ or all points $\psi_{ij}^{-1}(r, s)$ such that $|s - s_{ij}|$ is sufficiently small and $s < s_{ij}$. In the first case, we find that B_i contains all points $\psi_{ij}^{-1}(r, s)$ with $s \geqslant s_{ij}$; moreover, since $z_{ij} \in W_{ij}$, we have $B_i \supset V_{ij}^+$. In the second case, we can only conclude that B_i contains all points $\psi_{ij}^{-1}(r, s)$ with $0 < s < s_{ij}$. Hence, the axial line

$$l_{ij}^0 \overset{\text{def}}{=} \psi_{ij}^{-1}((-2\delta; 2\delta) \times 0)$$

also lies in B_i because the boundary of B_i has no fixed points; in particular, it has no points of l_{ij}^0. However, on the other side of l_{ij}^0, the direction of motion is changed, so that $\overline{L}_i$ may again pass through V_{ij} very close to l_{ij}^0; but then the area of $B_i \cap V_{ij}$ may be very small, and we can not associate with z_{ij} a guaranteed increment of the area of B_i by at least μ.

Let us show that, in fact, $l_{ij}^0 \notin B_i$, so the second case is impossible. Take $v \in l_{ij}^0$. It follows from 5.12 and the inequality $\operatorname{diam} V_{ij} < d_0$ that $\pi(v) \in \omega(L)$. If $v \in \omega(\overline{L})$, then it is clear that $v \notin B_i$ because $\overline{L}_i$ can not again enter B_i. Assume that $v \notin \omega(\overline{L})$. There exist $t_n \to \infty$ such that $f^{t_n}\pi(a_1) \to \pi(v)$. Hence, there exist integer translations T_n^{-1} such that

$$\overline{d}(\overline{f}^{t_n}(a_1), T_n^{-1}(v)) \to 0, \quad \overline{d}(T_n\overline{f}^{t_n}(a_1), v) \to 0. \tag{5.14}$$

Note that since $v \notin \omega(\overline{L})$, we have $T_n \neq \mathrm{id}$. By virtue of (5.14), $T_n\overline{f}^{t_n}(a_1) \in B_i$ for sufficiently large n. Since the semitrajectories can not enter B_i, the part of the semitrajectory $T_n(\overline{L})$ preceding the point $T_n\overline{f}^{t_n}(a_1)$ lies in B_i. Take a sequence of integers $i_n \to \infty$ such that

$$a_{i_n+1} = \overline{f}^{\tau_n}(a_1) \quad \text{with } \tau_n < t_n.$$

Then, $T_n(\overline{L}_{i_n}) \subset B_i$. Next, the segment $T_n(I_{i_n})$ that "completes" $T_n(\overline{L}_{i_n})$ to the Bendixson bag $T_n(B_{i_n})$ also lies in B_i: the endpoints of $T_n(I_{i_n})$ lie in B_i, and this segment intersects neither $\overline{L}_i$ nor I_i. Hence, $T_n(B_{i_n}) \subset B_i$. But this is impossible: the area of $T_n(B_{i_n})$ is equal to the area of B_{i_n}, which is greater than the area of B_i for $i_n > i$ because $B_i \subset B_{i+1}$ and $B_{i+1} - B_i$ contains internal points. The contradiction obtained proves the theorem. □

Now, we formulate with no proving one of the fundamental results in the field of inquiry, the Anosov theorem [8] on the approximation, from the viewpoint of the Fréchet distance, of a semi-infinite continuous curve by a semi-trajectory of a smooth flow.

Theorem 5.11 *Let $l = \{m(t)\colon t \geqslant 0\}$ be a semi-infinite continuous curve without self-intersections on a surface M. Then, for any $r > 0$, there exists a C^∞ flow f^t on M such that one of its semitrajectories $T = \{f^t(m_0)\colon m_0 \in M,\ t \geqslant 0\}$ lies at the Fréchet distance $\leqslant r$ from l; i. e.,*

$$\rho_F([l],\ [T]) \leqslant r.$$

Recall that the inequality $\rho_F([l],\ [T]) \leqslant r$ means the following: there exists a homeomorphism $s\colon [0;\ +\infty) \to [0;\ +\infty)$ such that

$$\sup_{t \geqslant 0} d(m \circ s(t),\ f^t(m_0)) \leqslant r,$$

where $d(\cdot, \cdot)$ is a metric on the manifold M.

The main idea of the proof of Theorem 5.11 is to approximate the curve l by a C^∞-embedded curve l_∞ that is r-close to l in the sense of the Fréchet metric and is obtained by a successive construction of arcs of increasing length. Since l_∞ is smoothly embedded, it is embedded into M together with a certain strip. Next, we declare all boundary points of this strip fixed points and construct a C^∞ flow with a semitrajectory l_∞.

In order to obtain the approximation by a smooth embedded curve, we apply approximations by topologically embedded arcs followed at each step by smoothing that does not spoil the smoothing made at preceding steps. Note that the initial curve l may contain points of its own limit set or may even completely belong to its own limit set. Therefore, the construction of l_∞ must be accompanied by "extruding the tails" of intermediate semi-infinite curves from a certain neighborhood of their initial arcs. This operation is described below, when we construct the curve l_∞.

In 1995, Anosov [13] generalized Theorem 5.11 and obtained its metric (in the sense of measure theory) version.

Theorem 5.12 *Let $l = \{m(t)\colon t \geqslant 0\}$ be a semi-infinite continuous curve without self-intersections on a surface M and μ be a smooth measure on M with everywhere positive C^∞ density. Then, for any $r > 0$, there exists a C^∞ flow f^t that preserves the measure μ and is such that one of its semitrajectories $T = \{f^t(m_0)\colon m_0 \in M,\ t \geqslant 0\}$ lies at a Fréchet distance $\leqslant r$ from l; i. e.,*

$$\rho_F([l], [T]) \leqslant r.$$

5.4. Nonlocal asymptotic behavior of special curves

Here, we consider the question on the existing of an asymptotic direction for special curves (nontrivial recurrent semitrajectories, semitrajectories of analytic flows, semileaves of foliations, one-dimensional stable and unstable manifolds of the points of surface diffeomorphisms, etc.), which play an important role in the theory of dynamical systems and foliations. Often, the presence of asymptotic directions allows one to construct an effective topological invariant for a wide class of dynamical systems and foliations that have invariant sets with nontrivially recurrent behavior.

First of all, recall (see Theorem 5.9) that if C is a closed curve (or a semi-infinite periodic curve) on a surface M, then,

- if C is null-homotopic, then any of its lifts $\overline{C}$ to $\overline{M}$ is a closed (and, hence, bounded) curve;
- if C is non-null-homotopic, then any of its lifts $\overline{C}$ to $\overline{M}$ is a nonclosed infinite curve both of whose semi-infinite curves have a rational asymptotic direction.

A similar assertion is valid for semileaves of a local lamination that tend to a closed leaf.

Theorem 5.13 *Let l^+ be a semileaf of a local lamination $\mathscr{D}$ on M such that its limit set $\lim(l^+)$ consists of a closed leaf C. Then,*

1) *if C is null-homotopic, then any lift $\overline{l}^+$ of l^+ to $\overline{M}$ is a bounded curve*;
2) *if C is non-null-homotopic, then any lift $\overline{l}^+$ of l^+ to $\overline{M}$ has a rational asymptotic direction.*

Proof. It follows from the definition of a local lamination and the compactness of C that C is covered by a finite family of neighborhoods in each of which $\mathscr{D}$ has a structure of a linear local lamination. This and the hypothesis of the theorem imply that l^+ approaches C in a spiral-like fashion. Now, the required result follows from Theorem 5.9. $\square$

The question on the existing of asymptotic directions is harder to solve in the case of nonclosed semi-infinite curves, such as nontrivially recurrent semileaves of local laminations, which are of great interest from the standpoint of applications. One can construct a foliation (and, hence, a lamination) in a disk with a nontrivially recurrent semileaf (see Exm. 3.7 in Section 3.1). Obviously, such a semileaf and any of its lifts on a universal covering have no asymptotic direction. Therefore, to guarantee that a nontrivially recurrent semileaf (and, moreover, an arbitrary curve) has an asymptotic direction, we have to impose certain constraints on this semileaf.

We begin with sufficient conditions for so-called widely disposed simple semi-infinite continuous curve; these conditions is then applied to special curves. We'll consider the cases of flat and hyperbolic surfaces separately because, for hyperbolic surfaces, we prove certain additional results that will be needed subsequently.

Widely disposed curves on flat surfaces

Among flat surfaces, we restrict our consideration to the torus. The definition of the orientability of the intersection of curves is given in Section 3.4.

Theorem 5.14 *Let C be a simple closed curve on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and suppose that a simple semi-infinite curve l^+ orientably intersects C infinitely many times. Then, any lift of l^+ to the universal covering $\mathbb{R}^2$ has an asymptotic direction.*

Proof. Since l^+ orientably intersects C infinitely many times, the curve C does not divide the torus and, hence, is non-null-homotopic. This, combined with Theorem 5.9, implies that the preimage $\pi^{-1}(C)$ consists of pairwise disjoint nonclosed curves each of whose semi-infinite curves has a rational asymptotic direction. Moreover, since the group of deck transformations for the torus consists of integer translations of $\mathbb{R}^2$, these semi-infinite curves have diametrically opposite directions (see Lemma 5.11).

Let us show that a lift $\overline{l}^+$ of l^+ can intersect any fixed lift $\overline{C}$ at most once. Suppose the contrary. According to what was proved above, $\overline{C}$ is a nonclosed curve that has diametrically opposite rational asymptotic directions in the positive and negative directions. Therefore, $\overline{C}$ divides the universal covering $\mathbb{R}^2$ into two domains that are homeomorphic to an open disk. If $\overline{l}^+ \cap \overline{C}$ consists of more than one point, then there exists an arc of the curve $\overline{l}^+$ that, together with a certain arc of the curve $\overline{C}$, bounds a disk on $\mathbb{R}^2$. Therefore, the intersection $l^+ \cap C$ must contain points with different intersection indices; but this contradicts the conditions.

Since l^+ orientably intersects C infinitely many times, $\overline{l}^+$ intersects a sequence of pairwise disjoint lifts $\overline{C}_n \in \pi^{-1}(C)$ of the curve C. Due to the discontinuity of the group of deck transformations, any compact set on the plane is intersected only by a finite number of curves from $\pi^{-1}(C)$. Therefore, starting from a certain moment, $\overline{l}^+$ leaves any compact set of the plane and never returns to it. Hence, $\overline{l}^+$ goes to infinity. According to Theorem 5.4, $\overline{l}^+$ has an asymptotic direction. □

Theorem 5.15 *Let C be a simple closed curve on the torus $\mathbb{T}^2$, and suppose that a simple infinite curve l orientably intersects C infinitely many times in such a way that the positive and negative semi-infinite curves l^+ and l^- of l also intersect C infinitely many times. Then any lift $\overline{l}$ of l to the universal covering is an infinite curve whose positive and negative semi-infinite curves $\overline{l}^+$ and $\overline{l}^-$ have diametrically opposite asymptotic directions.*

Proof. It follows from Theorem 5.14 and the definition of the orientability of the intersection of an infinite curve with C that $\overline{l}^+$ and $\overline{l}^-$ have asymptotic directions. Lemma 5.11 implies that these directions are diametrically opposite. □

Denote by $\mathscr{T}$ an arc or a closed curve that intersects a semi-infinite curve l^+ transversally. Recall that the curve l^+ is said to be *widely disposed with respect to* $\mathscr{T}$ if there do not exist any $\mathscr{T}$-loops that bound a disk. On the torus, the concept of wide disposition with respect to a non-null-homotopic simple closed curve coincides with the concept of orientability of the intersection with this curve. Indeed, cutting the torus T^2 along a simple non-null-homotopic closed curve $C \subset T^2$, we obtain a ring. Therefore, if a semi-infinite curve l^+ is widely disposed with respect to C, it intersects C orientably. It can easily be shown that the orientability of the intersection implies the wide disposition on any surface. We formulate this assertion as a lemma for references.

Lemma 5.13 *Let a simple semi-infinite curve l^+ orientably intersect a simple closed curve C. Then l^+ is widely disposed with respect to C.*

Proof. If there exists a C-loop that bounds a disk, then, obviously, at the endpoints of the corresponding C-arc, the curve l^+ intersects C with different intersection indices. □

Widely disposed curves on hyperbolic surfaces

On a hyperbolic surface, one can easily construct an example of a semi-infinite curve that is widely disposed with respect to C and intersects the curve C non-orientably. Let us formulate a sufficient condition for the existence of an asymptotic direction of a widely disposed semi-infinite curve on a hyperbolic surface that may be noncompact and non-orientable.

Theorem 5.16 *Let C be a simple closed curve on a hyperbolic surface M^2, and suppose that a simple semi-infinite curve l^+ is widely disposed with respect to C and transversally intersects C infinitely many times. Then any lift $\overline{l}^+ \subset \Delta$ of l^+ has an asymptotic direction. Moreover, the point of S_∞ that is accessible by $\overline{l}^+$ is the topological limit of the lifts $\overline{C}_i$ of C that are successively intersected by $\overline{l}^+$ as the parameter increases.*

Proof. Since l^+ is widely disposed with respect to C, C is non-null-homotopic on M^2 and does not bound a disk with a puncture (or a hole) on M^2. Due to Lemma 5.2 (see Theorem 5.9 as well), the set $\pi^{-1}(C)$ consists of pairwise disjoint nonclosed curves such that the ideal endpoints of each of them lie on the circle at infinity S_∞. Thus, any lift $\overline{C} \in \pi^{-1}(C)$ of C has two different ideal endpoints that coincide with the endpoints of the co-asymptotic geodesic $\overline{g}(\overline{C})$.

Let us show that a lift $\overline{l}^+$ of l^+ may intersect any lift $\overline{C}$ at most once. Suppose the contrary. The curve $\overline{C}$ divides Δ into two domains that are homeomorphic to an open disk. If $\overline{l}^+ \cap \overline{C}$ consists of more than one point, then there exists a $\overline{C}$-loop of $\overline{l}^+$ that bounds a disk on Δ. This disk projects to a disk on M^2 that contradicts to a widely disposition of l^+ with respect to C.

Taking into account that l^+ intersects C infinitely many times, we get that $\overline{l}^+$ successively intersects pairwise disjoint lifts $\overline{C}_n \in \pi^{-1}(C)$ of C as the parameter indefinitely increases, Fig. 5.18. Note that applying the Weil

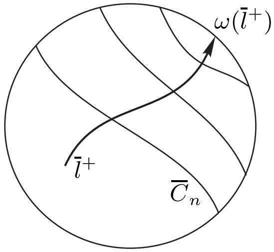


Figure 5.18. Asymptotic direction of a curve $\bar{l}^+$.

conjecture (see section 5.2), we already get that $\bar{l}^+$ has an asymptotic direction. However, we must prove a more informative result.

Since $\bar{l}^+$ intersects successively $\overline{C}_n$, the sequence $\{\overline{C}_n\}$ is monotone. By Lemma 5.4, the topological limit of the curves $\overline{C}_n \in \pi^{-1}(C)$ consists of a unique point of S_∞, which we denote by $\omega(\bar{l}^+)$. Clearly, that $\omega(\bar{l}^+)$ defines the asymptotic direction of $\bar{l}^+$. □

Theorem 5.17 *Let C be a simple closed curve that has a cylindrical tubular neighborhood on a hyperbolic surface M^2, and suppose that a simple semi-infinite curve l^+ orientably intersects C infinitely many times. Then any lift $\bar{l}^+$ of l^+ to the universal covering has an asymptotic direction. Moreover, the point on S_∞ that is accessible by the curve $\bar{l}^+$ is the topological limit of the lifts of C that are successively intersected by $\bar{l}^+$ as the parameter increases.*

Proof. By Lemma 5.13, the curve l^+ is widely disposed with respect to C. Then, the required assertion follows from Theorem 5.16. □

Now, we study the existence of asymptotic directions for semi-infinite curves belonging to a simple curve that is infinite in both directions.

Theorem 5.18 *Let C be a simple closed curve on a hyperbolic surface M^2, and suppose that a simple infinite curve l is widely disposed with respect to C; moreover, the positive and negative semi-infinite curves l^+ and l^- of l transversally intersect C infinitely many times. Then any lift $\bar{l}$ of the curve l on the universal covering is an infinite curve whose positive and negative semi-infinite curves $\bar{l}^+$ and $\bar{l}^-$ have asymptotic directions. Moreover,*

$$\omega(\bar{l}^+) = \omega(\bar{l}) \neq \alpha(\bar{l}) = \alpha(\bar{l}^-).$$

Proof. Choose a starting point $\overline{m}_0$ on $\bar{l}$, thus fixing the positive and negative semi-infinite curves

$$\bar{l}^+, \bar{l}^- \subset \bar{l}, \quad \bar{l}^+ \cup \bar{l}^- = \bar{l}, \quad \bar{l}^+ \cap \bar{l}^- = \overline{m}_0.$$

By Theorem 5.16, $\bar{l}^+$ and $\bar{l}^-$ have asymptotic directions. Since l is widely disposed with respect to C, $\bar{l}$ may intersect any fixed lift $\overline{C}$ at most once. Therefore, $\bar{l}^+$ and $\bar{l}^-$ intersect disjoint sequences $\overline{C}_n^+, \overline{C}_n^- \in \pi^{-1}(C)$ of the lifts of C. It follows from the proof of Theorem 5.16 that any two lifts of the curve C have pairwise different ideal endpoints on the absolute. This implies the inequality $\omega(\bar{l}) \neq \alpha(\bar{l})$. □

Theorem 5.19 *Let C be a simple closed curve on a hyperbolic surface M^2, and suppose that a simple infinite curve l orientably intersects C infinitely many times; moreover, the positive and negative semi-infinite curves l^+ and l^- of the curve l also intersect C infinitely many times. Then, any lift $\bar{l}$ of l on the universal covering is an infinite curve whose positive and negative semi-infinite curves $\bar{l}^+$ and $\bar{l}^-$ have asymptotic directions. Moreover,*

$$\omega(\bar{l}^+) = \omega(\bar{l}) \neq \alpha(\bar{l}) = \alpha(\bar{l}^-).$$

Proof. The proof follows from Lemma 5.13 and Theorem 5.18. □

Semileaves of orientable foliations

It is convenient to consider orientable foliations as flows using the corresponding terminology. The question of whether periodic trajectories and semi-trajectories that tend to a periodic trajectory have an asymptotic direction is actually solved in Theorems 5.9 and 5.13.

- If a periodic trajectory is null-homotopic (as a curve), then it has no asymptotic direction. If a periodic trajectory is non-null-homotopic, then its lift is an infinite curve both of whose semi-infinite curves have rational asymptotic directions.
- If a semitrajectory tends to a null-homotopic periodic trajectory, then it has no asymptotic direction. If a semitrajectory tends to a non-null-homotopic periodic trajectory, then it has a rational asymptotic direction.

A similar situation occurs for a semitrajectory that tends to a loop composed of separatrix connections and saddles. Obviously, a semitrajectory that tends to a single fixed point has no asymptotic direction. It remains to consider the question of whether semitrajectories that tend to trajectories whose limit set contains one-dimensional trajectories (or, which is the same, regular points) have an asymptotic direction. According to Corollary 4.3 for flows on compact surfaces, such semitrajectories tend to nontrivially recurrent trajectories. Therefore, it is natural to consider first the question of whether nontrivially recurrent semitrajectories have an asymptotic direction.

Nontrivially recurrent semitrajectories

Recall that a nontrivially recurrent semitrajectory is a nonclosed semitrajectory that belongs to its own limit set (i. e., a "self-limit"). Such semitrajectories may exist only on orientable surfaces of genus $g \geqslant 1$ and on non-orientable surfaces of genus $g \geqslant 3$ [21, 45, 142, 192]. The Euler characteristic of these surfaces is nonpositive, and their universal covering is homeomorphic to a disk. Therefore, we can speak of the nonlocal asymptotic behavior of nontrivially recurrent semitrajectories. The following theorem shows that a nontrivially recurrent semitrajectory of a flow with any set of fixed points has an asymptotic direction, and this asymptotic direction is irrational.

Theorem 5.20 *Let l be a nontrivially recurrent semitrajectory of a flow f^t on a closed surface M of nonpositive Euler characteristic, and let $\overline{l}$ be its lift to the universal covering $\overline{M}$ (i. e., $\overline{l}$ is a semitrajectory of the covering flow $\overline{f}^t$ on $\overline{M}$ that is projected onto l). Then $\overline{l}$ has an irrational asymptotic direction.*

Proof. By virtue of Lemma 3.12, the surface M can be assumed orientable. Since l is a nontrivially recurrent semitrajectory, there exists a closed transversal C that is intersected by l at a countable set of points (see Corollary 3.5). Since M is orientable, the transversal C has a tubular cylindrical neighborhood. As a semitrajectory of a flow, l intersects C orientably. This result, combined with Theorems 5.14 and 5.17, implies that $\overline{l}$ has an asymptotic direction, which we denote by $\omega(\overline{l})$.

Let us prove that the point $\omega(\overline{l})$ is irrational. For flows on the torus, it is well-known (see, for example, [26, 168, 180]) that if a flow has nontrivially recurrent trajectories, then the rotation number of this flow is irrational. Moreover, the rotation number is calculated with the use of an arbitrary semitrajectory that has an asymptotic direction, and is independent of the choice of the semitrajectory. This and the definition of the irrationality of the asymptotic direction for semi-infinite curves on the torus imply that the point $\omega(\bar{l})$ is irrational in the case of the torus (recall that the torus is the only closed flat surface that admits nontrivially recurrent semitrajectories).

Now, let us show that the point $\omega(\bar{l})$ is irrational in the case of a hyperbolic surface M. First, we assume that l belongs to a nontrivially recurrent trajectory, which we denote by the same symbol l. According to Theorem 5.19, the α-limit set $\alpha(\bar{l})$ of the trajectory $\bar{l}$ is a point that lies on the absolute and does not coincide with $\omega(\bar{l})$. Suppose that $\omega(\bar{l})$ is a rational point. Then, its stabilizer $\Gamma_{\omega(\bar{l})}$ in the group Γ is nonempty. According to Theorem 8.1.2 from [46], $\Gamma_{\omega(\bar{l})}$ is an infinite cyclic group generated by a certain element η. Since η is an orientation-preserving non-Euclidean translation of Δ and the trajectory $\bar{l}$ divides Δ, the trajectories $\eta(\bar{l})$ and $\eta^{-1}(\bar{l})$ bound a domain on Δ, say $\overline{D}$, that contains the trajectory $\bar{l}$. Since η is an element that generates the infinite cyclic group $\Gamma_{\omega(\bar{l})}$, $\overline{D}$ contains no trajectories $\eta^j(\bar{l})$ for $|j| \geqslant 2$, Fig. 5.19.

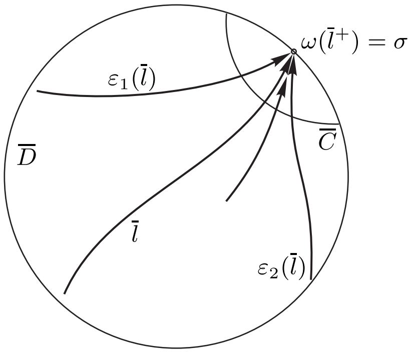


Figure 5.19

By Theorem 5.17, the point $\omega(\eta(\bar{l})) = \omega(\eta^{-1}(\bar{l})) = \omega(\bar{l})$ is the topological limit of the lifts of the transversal C that are successively intersected by the trajectory $\bar{l}$ as time increases. Therefore, there exists a lift $\overline{C}$ of the transversal C that intersects all three trajectories $\eta(\bar{l})$, $\eta^{-1}(\bar{l})$, and $\bar{l}$. It follows from geometrical considerations that the point of intersection $\bar{l} \cap \overline{C}$ lies on $\overline{C}$ between the points $\eta^{-1}(\bar{l}) \cap \overline{C}$ and $\eta(\bar{l}) \cap \overline{C}$. Since l is a nontrivially recurrent trajectory, it follows that for any point $\bar{z}_0 \in \bar{l}$ in the domain $\overline{D}$, there exists a trajectory $\gamma(\bar{l})$, $\gamma \in \Gamma$, that is congruent to $\bar{l}$ and is arbitrarily close to $\bar{z}_0$ and, in particular, to the point $\bar{l} \cap \overline{C}$. Set $\bar{l} \cap \overline{C} = \bar{z}_0$ and assume that $\gamma(\bar{l})$ intersects $\overline{C}$ between the points

$$\eta(\bar{l}) \cap \overline{C} = \bar{z}_1, \quad \eta^{-1}(\bar{l}) \cap \overline{C} = \bar{z}_2.$$

Denote the arc of $\overline{C}$ enclosed between the points $\bar{z}_1$ and $\bar{z}_2$ by $\overline{A}$. Together with the arc $\overline{A}$, the positive semitrajectories of the trajectories $\eta(\bar{l})$ and $\eta^{-1}(\bar{l})$ that emanate from the points $\bar{z}_1$ and $\bar{z}_2$, respectively, bound a domain (a curvilinear triangle) on Δ, which we denote by T. The trajectory $\gamma(\bar{l})$ enters T and can not leave T, because $\overline{C}$ is a transversal of the covering flow. Since, according to what was proved above, $\gamma(\bar{l})$ has an asymptotic direction, its ω-limit set must coincide with the point $\omega(\bar{l})$. Hence, $\gamma \in \Gamma_{\omega(\bar{l})}$. This contradicts the assumption that η generates the stabilizer $\Gamma_{\omega(\bar{l})}$.

If l is a nontrivially recurrent trajectory only in the positive direction but is not so in the negative direction (i. e., if it does not belong to a trajectory that is nontrivially recurrent in both directions), then we should apply the Cherry Theorem [64], according to which the topological closure $\operatorname{clos} l$ of the trajectory l contains a trajectory l_1 that is nontrivially recurrent in both directions and is such that $l \subset \operatorname{clos} l_1$. By analogy with the previous case, we construct for l a curvilinear triangle T that is bounded by the arc $\overline{A}$ and the trajectories $\eta(\bar{l})$ and $\eta^{-1}(\bar{l})$. By virtue of the inclusion $l \subset \operatorname{clos} l_1$, there exists a lift $\bar{l}_1$ that intersects $\overline{A}$ and enters T. The fact that $\bar{l}_1$ intersects $\overline{A}$ and enters the curvilinear triangle T implies that $\omega(\bar{l}) = \omega(\bar{l}_1)$. The irrationality, proved above, of the asymptotic direction for a nontrivially recurrent (in both directions) trajectory implies that $\omega(\bar{l}) = \omega(\bar{l}_1)$ is an irrational point. □

Corollary 5.4 *Let l be a nontrivially recurrent trajectory of a flow f^t on a closed surface M of nonpositive Euler characteristic, and let $\bar{l}$ be its lift to the universal covering $\overline{M}$. Then $\bar{l}$ has irrational asymptotic directions $\omega(\bar{l})$, $\alpha(\bar{l}) \in$ $\in S_\infty$; moreover, $\omega(\bar{l}) \neq \alpha(\bar{l})$.*

Corollary 5.5 *Let l be a generalized trajectory of an irrational flow f^t on a closed surface M of nonpositive Euler characteristic, and let $\bar{l}$ be its lift to the universal covering $\overline{M}$. Then, $\bar{l}$ has irrational asymptotic directions $\omega(\bar{l})$, $\alpha(\bar{l}) \in$ $\in S_\infty$; moreover, $\omega(\bar{l}) \neq \alpha(\bar{l})$.*

The following theorem is proved by analogy with the part of the proof of Theorem 5.20 concerning the existence of an asymptotic direction.

Theorem 5.21 *Let l be a positive (negative) semitrajectory of a flow f^t on a closed surface M of nonpositive Euler characteristic, and let $\bar{l}$ be its lift* *to the universal covering $\overline{M}$. If l contains a positive (negative) nontrivially recurrent semitrajectory in its ω-limit (respectively, α-limit) set, then $\bar{l}$ has an asymptotic direction.*

Proof. By Lemma 3.12, the surface M can be assumed orientable. It follows from the hypothesis that l infinitely many times intersects a certain closed transversal that is non-null-homotopic (and even not homologous to zero) on M. Further proof repeats, with some unessential modifications, the initial part of the proof of Theorem 5.20. □

Analytic and topological flows with finitely many of fixed points

It follows from Anosov's Theorems 5.6 and 5.8 that for an analytic flow or for a flow that has a finite number of fixed points, any semitrajectory of the covering flow on $\overline{M}$ either is bounded or has an asymptotic direction. Now, it is natural to consider the question concerning the relation between the "arithmetic" (i. e., the rationality or irrationality) of the asymptotic direction of a semitrajectory and its possible limit sets. Recall that an individual positive semitrajectory of a flow with a finite number of fixed points on an orientable compact surface may have an ω-limit set of exactly one of the following types (obviously, an analogous list can be written for a negative semitrajectory and its α-limit set):

- a single fixed point;
- a single periodic trajectory;
- a single one-sided loop formed by fixed points and separatrix connections (some of them may be separatrix loops);
- a single quasiminimal set.

Note that this assertion does not generally hold for flows with an infinite set of fixed points. However, Lemma 5.12 shows that for analytic flows (which may even have a continuum of fixed points) this assertion holds true. One should only keep in mind that a one-sided loop may contain arcs or closed curves that consist of fixed points. Now, using this result, we can obtain a list of limit sets for an individual semitrajectory that has an asymptotic direction both for analytic flows and topological flows with a finite number of fixed points.

Lemma 5.14 *Let f^t be an analytic flow or a topological flow with a finite number of fixed points on a closed orientable surface M of nonpositive constant curvature, and let l^+ be a positive semitrajectory of the flow f^t. Let $\bar{l}^+$*

be a lift of the semitrajectory l^+ to the universal covering. Then, $\bar{l}^+$ has an asymptotic direction if and only if the ω-limit set $\omega(l^+)$ of the semitrajectory l^+ falls under one of the following types:

1) $\omega(l^+)$ *is a periodic non-null-homotopic trajectory*;
2) $\omega(l^+)$ *is a homotopically nontrivially one-sided loop formed by* (*possibly, trivial*) *arcs that consist of fixed points and by separatrix connections* (*some of which may be separatrix loops*);
3) $\omega(l^+)$ *is a quasiminimal set.*

Proof. Since $\bar{l}^+$ has an asymptotic direction, $\omega(l^+)$ can not be a fixed point. This and the list of possible limit sets for an individual semitrajectory of a flow with a finite number of fixed points imply the required assertion for flows with a finite number of fixed points. It remains to prove the lemma for an analytic flow f^t. According to Lemma 5.12, the set $\operatorname{Fix} f^t$ is decomposed into three subsets (some of which may be empty), which we denote by A, B, and C, in the following fashion: (1) the subset A consists of a finite set of isolated points; (2) the subset B consists of a finite set of simple analytic isolated closed curves; (3) the subset C consists of a finite set of analytic pairwise disjoint simple arcs and a finite set N of the endpoints of these simple arcs.

Let K be a component of the set $M - \operatorname{Fix} f^t$ that contains l^+. According to Lemma 5.12, K is an open topological surface of finite genus with a finite number of punctures and holes that correspond to the points and curves from the above-mentioned subsets A, B, and C. Here, the punctures correspond to isolated fixed points of the flow f^t, i. e., to a certain subset $A_0 \subset A$, and the holes correspond to curves filled with fixed points that belong to subsets $B_0 \subset B$ and $C_0 \subset C$. Let us compactify K gluing up the punctures by the points of the set A_0 and adding the arcs from B_0 and C_0. As a result, we obtain a compact surface, which we denote by K_0. Let us declare that each point of the added set $A_0 \cup B_0 \cup C_0$ is an equilibrium state. Then, the initial flow induces a topological flow on K_0 whose isolated fixed points coincide with A_0 and the nonisolated points form the boundary of the surface K_0. Let us contract each boundary component to a point and declare it an equilibrium state. Then, we obtain a topological flow with a finite number of fixed points on the closed surface. Now, the required result follows from the list of limit points for the flows with a finite number of fixed points. □

Lemma 5.15 *Suppose that the hypotheses of Lemma* 5.14 *are fulfilled, and let l^+ be a nonperiodic trajectory. Then,*

1) $\overline{l}^+$ *has a rational asymptotic direction if and only if* $\omega(l^+)$ *is either a periodic non-null-homotopic trajectory or a homotopically nontrivially one-sided loop that consists of separatrix connections and* (*possibly, trivial*) *arcs formed by fixed points*;
2) $\overline{l}^+$ *has an irrational asymptotic direction if and only if* $\omega(l^+)$ *is a quasiminimal set.*

Proof. If $\omega(l^+)$ is either a periodic trajectory or a one-sided loop, then, obviously, $\overline{l}^+$ has a rational asymptotic direction (see Theorem 5.13). Therefore, by virtue of Lemma 5.14, if $\overline{l}^+$ has an irrational asymptotic direction, then $\omega(l^+)$ is a quasiminimal set. Let us prove the converse. If l^+ is a nontrivially recurrent semitrajectory, then the irrationality of the asymptotic direction of the semitrajectory $\overline{l}^+$ follows from Theorem 5.20. When l^+ is not a nontrivially recurrent semitrajectory, we note that nontrivially recurrent trajectories and semitrajectories are everywhere dense in the quasiminimal set $\omega(l^+)$. Lemma 5.12 and the results obtained by Gutierrez [103] concerning the structure of a quasiminimal set (see also [26, 172]) imply that starting from a certain moment, l^+ enters an open strip the part of whose boundary that is accessible on the inside consists of two nontrivially recurrent positive semitrajectories l_1^+ and l_2^+ that approach l^+ (this strip can be interpreted as a cell in the Denjoy flow on the torus). In this case, there exist lifts $\overline{l}^+$, $\overline{l}_1^+$, and $\overline{l}_2^+$ of the semitrajectories l^+, l_1^+, and l_2^+, respectively, that have the same asymptotic direction. Since the lifts $\overline{l}_1^+$ and $\overline{l}_2^+$ have an irrational asymptotic direction, $\overline{l}^+$ also has an irrational asymptotic direction. □

Semileaves of non-orientable foliations

Let M be a surface (possibly, with boundary ∂M). Recall that a *foliation* $\mathscr{F}$ on M with a set of singularities $\operatorname{Sing}(\mathscr{F})$ is a partition of $M - (\operatorname{Sing}(\mathscr{F}) \cup \partial M)$ into disjoint curves (called *leaves*) that are locally homeomorphic to a family of parallel straight lines. As for the set of singularities $\operatorname{Sing}(\mathscr{F})$, this set should be described separately for each class of foliations. Each point of the set $\operatorname{Sing}(\mathscr{F})$ is called a *singularity* of the foliation $\mathscr{F}$. The behavior of leaves near the boundary ∂M is also described separately, if necessary. It is usually assumed that either the leaves are transversal to ∂M or each component of the boundary ∂M is a leaf.

An important class of foliations on surfaces is given by foliations whose singularities are topological saddles (or saddle type singularities). Recall that the index of a topological saddle O with ν separatrices is calculated by the formula $\operatorname{ind} O = 1 - \frac{\nu}{2}$. Thus, among the topological saddles, only a thorn has a positive index equal to $\frac{1}{2}$. A fake saddle has the index 0, and all the other topological saddles have negative indices. In the general case, when an isolated singularity has elliptic, parabolic, and hyperbolic sectors, the index of singularity O is calculated by the formula

$$\operatorname{ind} O = 1 + \frac{e-h}{2},$$

where e is the number of elliptic sectors and h is the number of hyperbolic sectors of O. For the index of an arbitrary isolated singularity and the index of a simple closed curve in a general position with respect to a foliation, see Section 3.6.

Consider a class of foliations with isolated singularities of nonpositive index (i. e., a singularity may have elliptic sectors, but the number of these sectors must be less by two than the number of hyperbolic sectors).

Theorem 5.22 *Let $\mathscr{F}$ be a foliation on a closed surface M of negative Euler characteristic. Suppose that the singularities of the foliation $\mathscr{F}$ are isolated and have nonpositive indices. Then, any semileaf that does not tend to a singularity has an asymptotic direction.*

Proof. Passing, if necessary, to a two-sheeted covering, we may assume that the surface M is orientable. Let l be a semileaf that does not tend to a singularity. Without loss of generality, we may assume that l is a positive semileaf. Then, the $\omega(l)$-limit set of this semileaf contains a one-dimensional leaf l_0. If l_0 is a closed leaf, then it is non-null-homotopic on the surface. Indeed, otherwise, by virtue of Corollary 3.6, l_0 would bound a disk that contains singularities of positive index (since the index of l_0, as the index of a curve, is equal to $+1$). The fact that l_0 is non-null-homotopic and Theorem 5.13 imply that l has an asymptotic direction.

Consider the case when l_0 is not a closed leaf. If the ω-limit set of the leaf l_0 does not consist only of singularities, then, by virtue of Theorem 4.12, l_0 is a nontrivially recurrent semileaf. According to Theorem 3.4, there exists a closed transversal C that is intersected by the leaf l_0 infinitely many times. Since $l_0 \subset \omega(l)$, l also intersects C infinitely many times.

Let us show that l is widely disposed with respect to C. Suppose the contrary. Then, there exists a disk D bounded by an arc of the leaf l and a segment of the transversal C. The index of the curve that bounds D is positive. Hence, D must contain singularities of positive index, which contradicts the assumption (see Corollary 3.7). This proves that l is widely disposed with respect to C. By Theorem 5.16, l has an asymptotic direction.

It remains to consider the case when l_0 is not a closed leaf and the ω-limit set of the leaf l_0 consists only of singularities. Since the singularities are isolated, $\omega(l_0)$ consists of exactly one singularity. Then, the Bendixson extensions of the leaf l_0 form a closed one-sided loop K that coincides with $\omega(l)$. The index of this loop, as the index of a curve, is equal to $+1$. Therefore, K is non-null-homotopic. This and Theorem 5.13 imply that l has an asymptotic direction. □

For references, we formulate the following theorem, which is derived from Theorems 5.22 and 5.18 (we omit the proof).

Theorem 5.23 *Suppose that a foliation $\mathscr{F}$ with isolated singularities of negative indices is defined on a closed hyperbolic orientable surface M^2. Then, the semileaves of any leaf that is not a separatrix of any singularity have different asymptotic directions.*

On an orientable surface, consider foliations with singularities solely of saddle type with negative indices. For such foliations, there are the curves "generated" by separatrices and saddle type singularities. These additional curves are obtained as the union of separatrices with their one-sided Bendixson extensions (see Fig. 4.5), which are called *generalized leaves*. For flows, which can be interpreted as orientable foliations, generalized leaves are referred to as *generalized trajectories*. Note that a generalized trajectory of a covering flow $\overline{f}^t$ may consist of a countable family of separatrices and fixed points. In this case, the generalized trajectory covers a one-sided non-null-homotopic loop of the flow f^t.

Recall that a generalized leaf endowed with a parametrization (and, hence, with an orientation) represents an infinite curve. Fixing the starting point, we divide a generalized leaf into two generalized semileaves, the positive and negative ones. Note that a generalized curve on a surface is not generally a simple semi-infinite curve. However, this is so on the universal covering in the class of foliations under consideration.

Lemma 5.16 *Let $\mathscr{F}$ be a foliation on a closed orientable hyperbolic surface M. Suppose that the singularities of $\mathscr{F}$ are topological saddles of negative index. Then, any generalized semileaf of the covering foliation $\overline{\mathscr{F}}$ on Δ is a simple semi-infinite curve.*

Proof. Suppose the contrary. Since the leaves are pairwise disjoint, the covering generalized semileaf has a separatrix loop. This loop bounds, on the universal covering, a disk that must contain singularities of positive index. But this contradicts the assumption. □

The following lemma shows that the semileaves of a generalized leaf have asymptotic directions and that these directions are different.

Theorem 5.24 *Let $\mathscr{F}$ be a foliation on a closed orientable surface M of negative Euler characteristic. Suppose that the singularities of the foliation $\mathscr{F}$ are topological saddles of negative index. Then, any generalized semileaf of the covering foliation $\overline{\mathscr{F}}$ on Δ has an asymptotic direction. Moreover, any generalized leaf $\bar{l}$ of the covering foliation $\overline{\mathscr{F}}$ is a simple infinite curve whose positive and negative semi-infinite curves $\bar{l}^+$ and $\bar{l}^-$ have asymptotic directions such that*

$$\omega(\bar{l}^+) = \omega(\bar{l}) \neq \alpha(\bar{l}) = \alpha(\bar{l}^-).$$

The proof is omitted.

Nontrivially recurrent leaves and semileaves

The analysis of the proof of Theorem 5.20 shows that similar assertions are valid for local laminations on a hyperbolic surface.

Theorem 5.25 *Let C be a simple closed curve on a hyperbolic surface M^2, and suppose that all leaves of a local lamination $\mathscr{D}$ are widely disposed with respect to C. Suppose that a nontrivially recurrent leaf l of the local lamination $\mathscr{D}$ transversally intersects C infinitely many times. Then, the positive and negative semileaves of the covering leaf $\bar{l}$ have different irrational asymptotic directions on the universal covering.*

Proof. Since l is nontrivially recurrent and transversally intersects C, it intersects C infinitely many times both in the positive and negative directions. This and Theorem 5.18 imply that the positive and negative semileaves of the covering leaf $\bar{l}$ on the universal covering have different asymptotic directions.

Let us prove that these directions are irrational. We will only prove the irrationality of the point $\omega(\bar{l})$ because the irrationality of the point $\alpha(\bar{l})$ can be proved in a similar way. Note that $\omega(\bar{l}) \neq \alpha(\bar{l})$ by Theorem 5.18.

Suppose the contrary; i. e., let $\omega(\bar{l}) \overset{\text{def}}{=} \sigma$ be a rational point and, hence, the stabilizer Γ_σ be nontrivial. Let us show that there exist transformations $\varepsilon_1, \varepsilon_2 \in \Gamma_\sigma$ such that the leaves $\varepsilon_1(\bar{l})$ and $\varepsilon_2(\bar{l})$ bound an open domain $\overline{D} \subset \Delta$ that contains $\bar{l}$ and is such that $\gamma(\bar{l}) \notin \overline{D}$ for any nonidentity transformation $\gamma \in \Gamma_\sigma$.

Indeed, if $\Gamma_{\alpha(\bar{l})} \cap \Gamma_\sigma = \{\mathrm{id}\}$, then the required result follows from Corollary 3.2. If $\Gamma_{\alpha(\bar{l})} \cap \Gamma_\sigma \neq \{\mathrm{id}\}$, then the stabilizers $\Gamma_{\alpha(\bar{l})}$ and Γ_σ contain a common hyperbolic transformation. Hence, each stabilizer is of type (1), (2), or (3) from Lemma 3.8. Consider these types.

Suppose that Γ_σ is generated by a parabolic or a hyperbolic element η. Since η is an orientation-preserving transformation, $\bar{l}$ belongs to a domain bounded by the leaves $\eta(\bar{l})$ and $\eta^{-1}(\bar{l})$. Thus, $\varepsilon_1 = \eta$ and $\varepsilon_2 = \eta^{-1}$. Suppose that Γ_σ is generated by a glide reflection η. The leaf $\bar{l}$ divides Δ into two domains L and R. Without loss of generality, we can assume that $\eta(\bar{l}) \subset R$. Since η is an orientation-reversing transformation, $\eta^{-1}(\bar{l}) \subset R$. Since η^2 is an orientation-preserving transformation, $\bar{l}$ lies in a domain bounded by the leaves $\eta^2(\bar{l})$ and $\eta^{-2}(\bar{l})$. Thus, $\varepsilon_1 = \eta^2$ and $\varepsilon_2 = \eta^{-2}$.

Thus, there exist transformations $\varepsilon_1, \varepsilon_2 \in \Gamma_\sigma$ such that the leaves $\varepsilon_1(\bar{l})$ and $\varepsilon_2(\bar{l})$ bound an open domain $\overline{D} \subset \Delta$ that contains $\bar{l}$ and is such that $\gamma(\bar{l}) \notin$ $\notin \overline{D}$ for any nonidentity transformation $\gamma \in \Gamma_\sigma$.

By Theorem 5.16, there exists a lift $\overline{C}$ of the curve C that intersects the leaves $\varepsilon_1(\bar{l})$ and $\varepsilon_2(\bar{l})$. Set $\varepsilon_i(\bar{l}) \cap \overline{C} = x_i$, $i = 1, 2$. According to Lemma 3.20, there exists a leaf $\bar{l}_1$ that is congruent to $\bar{l}$ and enters the curvilinear triangle $x_1 \sigma x_2$ because l is a nontrivially recurrent leaf. By virtue of the wide disposition, $\bar{l}_1$ tends to the point σ. This and the congruence of the leaves $\bar{l}$ and $\bar{l}_1$ imply that there exists $\gamma \in \Gamma_\sigma$ such that $\gamma(\bar{l}) = \bar{l}_1$. On the other hand, $\gamma(l) \subset \Delta - \overline{D}$ for any nonidentity transformation $\gamma \in \Gamma_\sigma$. The contradiction obtained proves the theorem. □

Theorem 5.26 *Let C be a simple closed curve on a hyperbolic surface M^2, and suppose that all leaves of a local lamination $\mathcal{D}$ are widely disposed with respect to C. Suppose that a nontrivially recurrent semileaf l^+ of the local lamination $\mathcal{D}$ transversally intersects C infinitely many times and belongs to the limit set of a certain nontrivially recurrent leaf that also transversally intersects C. Then, l^+ has an irrational asymptotic direction.*

Proof. By Theorem 5.16, the lift $\bar{l}^+$ of the semileaf l^+ has an asymptotic direction, which we denote by σ. Let us prove that the point σ is irrational. Suppose the contrary and let us avail ourselves of the notations from the proof of Theorem 5.25. Then, there exist pairwise congruent semileaves $\bar{l}_1^+$, $\bar{l}_2^+$, and $\bar{l}_3^+ = \bar{l}^+$ that tend to σ and intersect the curve $\overline{C}$ at points x_1, x_2, and x_3, respectively. Without loss of generality, we can assume that the point x_2 lies between the points x_1 and x_3 on the curve $\overline{C}$; i. e., $x_2 \in (x_1; x_3) \subset \overline{C}$. By the hypothesis, there exists a nontrivially recurrent (in both directions) leaf l_0 such that $l^+ \subset \operatorname{clos} l_0$. Therefore, there exists a lift $\overline{l}_0$ of the leaf l_0 such that $\overline{l}_0$ intersects the interval $(x_1; x_3)$. By virtue of Lemma 4.12, the leaf l_0 is widely disposed. Hence, $\overline{l}_0$ can not intersect $\overline{C}$ twice and therefore tends to the point σ (in the "positive" direction). According to Theorem 5.25, σ is an irrational point. The contradiction obtained proves the theorem. □

Corollary 5.6 *Suppose that the hypotheses of Theorem* 5.26 *are fulfilled, and let l^+ be an internal semileaf. Then, any semileaf of the local lamination $\mathscr{D}$ that transversally intersects C infinitely many times can not have the same asymptotic direction as the semileaf l^+.*

Proof. The proof is carried out by contradiction, following the scheme of the proof of Theorem 5.26. Using the notations from the proof of that theorem, we find that the interval $(x_1; x_3)$ must intersect a lift of the semileaf l^+, because l^+ is internal. This lift tends to the point σ, which contradicts the irrationality of σ. □

Transversal local laminations

Let $\mathscr{D}_1$ and $\mathscr{D}_2$ be two one-dimensional local laminations on a surface M. They are *transversal* if any intersecting leaves $l_1 \in \mathscr{D}_1$ and $l_2 \in \mathscr{D}_2$ intersect transversally at any intersection point.

Local laminations $\mathscr{D}_1$ and $\mathscr{D}_2$ are called *widely disposed with respect to each other* if

- each leaf $l_1 \in \mathscr{D}_1$ intersects at least one leaf from $\mathscr{D}_2$ and vice versa;
- any leaf $l_1 \in \mathscr{D}_1$ is widely disposed with respect to the leaves of the local lamination $\mathscr{D}_2$ and vice versa.

By analogy with Theorems 5.25 and 5.26, let us formulate two sufficient conditions that guarantee the existence of an irrational asymptotic direction for a nontrivially recurrent semileaf and a nontrivially recurrent leaf, respectively, of a local lamination that has a transversal lamination widely disposed with respect to it.

Theorem 5.27 *Let $\mathscr{D}_1$ and $\mathscr{D}_2$ be transversal local laminations on a hyperbolic surface M that are widely disposed with respect to each other. Then, any nontrivially recurrent semileaf l^+ of the local lamination $\mathscr{D}_1$ has an asymptotic direction. If l^+ belongs to the limit set of a certain nontrivially recurrent leaf from $\mathscr{D}_1$, then l^+ has an irrational asymptotic direction.*

Proof. Let $l_1 \in \mathscr{D}_1$ be a nontrivially recurrent semileaf. By the definition of the wide disposition, there exists a leaf $l_2 \in \mathscr{D}_2$ that is transversally intersected by l_1 and with respect to which l_1 is widely disposed. Therefore, there exists an arc A of the leaf l_2 that is intersected by l_1 infinitely many times. Take an A-arc $\widehat{ab}$ of the semileaf l_1 with endpoints $a, b \in A \cap l_1$ and denote by $\overline{ab} \subset A$ the segment of the arc A between the points a and b. Since l_1 is a nontrivially recurrent semileaf of a local lamination, we can take $\widehat{ab}$ in such a way that the segment $\overline{ab}$ intersects l_1 at internal points.

Denote the A-loop $\overline{ab} \cup \widehat{ab}$ by C. After the A-arc $\widehat{ab}$, the semileaf l_1 intersects C infinitely many times. By the hypothesis, l_1 is widely disposed with respect to C because it intersects C only on the segment $\overline{ab}$, which belongs to l_2. Then, according to Theorem 5.16, l_1 has an asymptotic direction. By Theorem 5.26, if l^+ belongs to the limit set of a certain nontrivially recurrent leaf from $\mathscr{D}_1$, then l^+ has an irrational asymptotic direction. □

Theorem 5.28 *Let $\mathscr{D}_1$ and $\mathscr{D}_2$ be transversal local laminations on a hyperbolic surface M that are widely disposed with respect to each other, and let l be a nontrivially recurrent leaf of the local lamination $\mathscr{D}_1$. Then, the positive and negative semileaves of the covering leaf $\bar{l}$ on the universal covering have different irrational asymptotic directions.*

Proof. The initial part of the proof almost word for word repeats the initial part of the proof of Theorem 5.27. Further, instead of Theorem 5.26, we refer to Theorem 5.25. □

Invariant manifolds of the points of basic sets

The main facts of this subsection concerning the nonlocal asymptotic behavior of invariant manifolds were proved in [82, 186, 188]. Here, we show how they follow from our general results.

Let f be a C^∞ diffeomorphism that satisfies Smale's Axiom A [208] and is defined on a closed surface M (an *A-diffeomorphism*). Axiom A implies that the nonwandering set $NW(f)$ of the diffeomorphism f is hyperbolic and that periodic points are everywhere dense in $NW(f)$. According to the Spectral Decomposition Theorem [208], $NW(f)$ is represented as a finite union of pairwise disjoint closed invariant sets $\Omega_1, \ldots, \Omega_k$, called *basic sets*, each of which contains an orbit that is everywhere dense in this basic set. It is well-known [208] that through any point $x \in NW(f)$, there passes a *stable* *manifold* W_x^s that is the image of an injective immersion $J_x^q\colon \mathbb{R}^q \to M$, $W_x^s = J_x^q(\mathbb{R}^q)$ (for a surface, $q \in \{0, 1, 2\}$). A point $y \in M$ belongs to W_x^s if and only if $d(f^n(x), f^n(y)) \to 0$ as $n \to \infty$ (d is the metric on M). The *unstable manifold* W_x^u is defined as the stable manifold with respect to the diffeomorphism f^{-1}.

If f has a two-dimensional basic set, then this set coincides with the entire surface, and f is the Anosov diffeomorphism of the two-dimensional torus [208]. Therefore, below, we mainly focus on the case when the diffeomorphism f has one-dimensional and nontrivial zero-dimensional basic sets. In this case, stable and unstable manifolds of the points of these basic sets are infinite curves without self-intersections ($q = 1$).

Let Ω be a one-dimensional basic set. According to [185] and [222], Ω is either an attractor or a repeller and has a local structure of the direct product of a segment and a Cantor set. As for the nontrivial zero-dimensional basic set, due to its nontriviality, it is locally homeomorphic to the direct product of two Cantor sets [208]. The following assertion is crucial for further consideration.

Lemma 5.17 *Let Ω be either a one-dimensional or a nontrivial zero-dimensional basic set of an A-diffeomorphism $f\colon M \to M$ of a closed surface M. Then, the stable and unstable manifolds*

$$\{W^s(m)\colon m \in \Omega\} \overset{\text{def}}{=} \mathscr{W}^s(\Omega), \quad \{W^u(m)\colon m \in \Omega\} \overset{\text{def}}{=} \mathscr{W}^u(\Omega)$$

form continuous local laminations each of which consists of a continuum set of simple infinite curves. Moreover, if Ω is a one-dimensional attractor (repeller), then $\mathscr{W}^u(\Omega)$ (respectively, $\mathscr{W}^s(\Omega)$) forms a C^1 lamination. Each of the laminations $\mathscr{W}^u(\Omega)$ and $\mathscr{W}^s(\Omega)$ has a finite (possibly, zero) number of leaves that are nontrivially recurrent only in one direction. All the other leaves are nontrivially recurrent in both directions. The local laminations $\mathscr{W}^u(\Omega)$ and $\mathscr{W}^s(\Omega)$ are transversal.

Proof. The proof of the first part of the lemma follows from [5, 116, 118, 222] (see also [18, 124, 180, 198, 208]). The proof of the second part can be deduced from [6, 52, 82, 186, 188]. □

Since (stable and unstable) invariant manifolds are homeomorphic to a straight line, they can be endowed, together with the "intrinsic" orientation, with a transversal orientation in a way. For example, define, at a certain point of an invariant manifold, a normal (in the topological sense) vector and extend it by continuity over the whole invariant manifold. Then, the index of intersection is defined at the point of transversal intersection of invariant manifolds. Following [82], we call the basic set Ω *orientable* if, for any point $x \in \Omega$, the index of the intersection $W^s_x \cap W^u_x$ is the same at all points of the intersection. Otherwise, the set Ω is called *non-orientable*. Note that since the invariant manifolds W^s_x and W^u_x intersect transversally at the point $x \in W^s_x \cap W^u_x$ (hence, the intersection index at x is nonzero), the orientability of Ω implies the transversality of the intersection of these invariant manifolds at any point of the intersection $W^s_x \cap W^u_x$.

In [186], Plykin introduced the concept of wide disposition of a basic set, which is more general than the orientability. A basic set Ω is said to be *widely disposed* if there does not exist a null-homotopic loop formed by a pair of segments of stable and unstable manifolds of a certain point from Ω.

Theorem 5.29 *Let $f: M \to M$ be an A-diffeomorphism of a closed orientable surface M of nonpositive Euler characteristic. Let Ω be a one-dimensional widely disposed attractor (repeller) of the diffeomorphism f, and let $W^{u(s)}(x)$ be the unstable (respectively, stable) manifold of a point $x \in \Omega$. Then, the curves $W^{u(s)}(x) - x$ have different irrational asymptotic directions.*

Proof. It suffices to note that the wide disposition of the basic set Ω implies the wide disposition of the local laminations $\mathscr{W}^u(\Omega)$ and $\mathscr{W}^s(\Omega)$ with respect to each other. Now, the required assertion follows from Lemma 5.17 and Theorem 5.28. □

Theorem 5.29 does not generally hold for the stable (unstable) manifold $W^{s(u)}(x)$ of a point from the attractor (respectively, repeller) Ω because one of the curves $W^{s(u)}(x) - x$ may contain a periodic boundary point. Recall that a point $q \in \Omega$ is said to be *boundary* if one of the connected components of the set $W^{s(u)}(q) - q$ does not intersect Ω.

The proof of the theorems formulated below is analogous to the proof of Theorem 5.29, except that the reference to Theorem 5.28 should be replaced by a reference to Theorem 5.27.

Theorem 5.30 *Let $f: M \to M$ be an A-diffeomorphism of a closed surface M of nonpositive Euler characteristic, and let Ω be a one-dimensional widely disposed attractor (repeller) of the diffeomorphism f. Let $W^{s(u)}(x)$ be the stable (respectively, unstable) manifold of a point $x \in \Omega$ and L^σ be one of the connected components of the set $W^{s(u)}(x) - x$ that does not contain a periodic boundary point. Then, L^σ has an irrational asymptotic direction.*

Theorem 5.31 *Let $f\colon M \to M$ be an A-diffeomorphism of a closed surface M of nonpositive Euler characteristic, and let Ω be a zero-dimensional nontrivial widely disposed basic set of the diffeomorphism f. Let $W^{s(u)}(x)$ be the stable (unstable) manifold of a point $x \in \Omega$ and L^σ be one of the connected components of the set $W^{s(u)}(x) - x$ that does not contain a periodic boundary point. Then, L^σ has an irrational asymptotic direction.*

Theorems 5.29–5.31 will be proved in Chapter 8 (Section 8.1) using dynamical properties of basic sets.

5.5. Geodesic frameworks of local laminations

Applying medical terminology, we can say that the geodesic framework of a local lamination is its geodesic skeleton around which leaves that have asymptotic directions are grouped. The geodesic framework contains full information on the asymptotic directions of leaves of a given local lamination. The geodesic framework of a local lamination is defined only if this lamination has a leaf or a generalized leaf that has a co-asymptotic geodesic. Let us pass on to the formal definition.

Let $\mathscr{D}$ be a local lamination on M^2. Denote by $\mathscr{A}^{\pm}(\mathscr{D})$ the union of all leaves and generalized leaves of $\mathscr{D}$ that have co-asymptotic geodesics.

Definition 5.6 *The topological closure*

$$G(\mathscr{D}) \overset{\text{def}}{=} \operatorname{clos} \bigcup_{l \in \mathscr{A}^{\pm}(\mathscr{D})} g(l)$$

is called the ***geodesic framework of the local lamination*** *$\mathscr{D}$.*

Since a lamination and a foliation are local laminations, we have defined the concepts of geodesic framework $G(\mathscr{F})$ of a foliation $\mathscr{F}$ and geodesic framework $G(\mathscr{L})$ of a lamination $\mathscr{L}$. It follows immediately from the definition and Lemma 3.33 that a geodesic framework is a geodesic lamination.

The geodesic framework of an arbitrary invariant set of a local lamination with closed support is defined similarly as follows. Let $\mathscr{D}$ be a local lamination with closed support $\operatorname{supp} \mathscr{D}$, and let $\mathscr{N}$ be its invariant set. Denote by $\mathscr{A}'(\mathscr{N})$ the union of all leaves and generalized leaves from $\mathscr{N}$ that have co-asymptotic geodesics. The topological closure

$$G(\mathscr{N}) \overset{\text{def}}{=} \operatorname{clos} \bigcup_{l \in \mathscr{A}'(\mathscr{N})} g(l)$$

is called the *geodesic framework of the invariant set $\mathscr{N}$*.

Note that geodesic frameworks on flat surfaces, such as torus, Klein bottle, and cylinder, have a sufficiently clear structure because geodesic laminations on these surfaces are easily described. On the torus, a geodesic lamination either forms an irrational linear flow (hence, this lamination fills the whole torus) or is a family of pairwise homotopic closed geodesics. On Klein bottle and cylinder, geodesic laminations may consist only of pairwise homotopic closed geodesics. Therefore, below in this section, we will consider geodesic frameworks on closed orientable hyperbolic surfaces.

Geodesic frameworks of quasiminimal sets

Let l be a nontrivially recurrent leaf that is widely disposed with respect to a certain simple closed curve C. By Theorem 5.18, l has a co-asymptotic geodesic $g(l)$ on M^2.

Lemma 5.18 *Let l be a nontrivially recurrent leaf of a local lamination $\mathscr{D}$, and suppose that l is widely disposed with respect to a certain simple closed curve C and transversally intersects C. Then, the co-asymptotic geodesic $g(l)$ is a nonclosed B-recurrent geodesic. In particular, $g(l)$ is nontrivially recurrent.*

Proof. According to Lemma 3.45, it suffices to prove that the geodesic $g(l)$ is nontrivially recurrent only in one direction (either positive or negative). Consider a covering leaf $\overline{l}$ for l and a covering curve $\overline{C}$ for C that intersects $\overline{l}$ at a certain point $\overline{m}$. Since the leaf l is nontrivially recurrent, it intersects C infinitely many times. This and the wide disposition of l with respect to C imply that C is not homologous to zero (and, hence, is non-null-homotopic). Therefore, any lift of C is a curve with different ideal endpoints lying on the absolute.

By Theorem 5.17, the points $\alpha(\overline{l})$ and $\omega(\overline{l})$ are the topological limits of curves $\overline{C}_{-k}$, $\overline{C}_k \subset \pi^{-1}(C)$, respectively, as $k \to +\infty$ (Fig. 5.20).

It follows from the nontrivial recurrence of l that there exists a sequence of leaves $\overline{l}_n$ that are congruent to $\overline{l}$ and intersect $\overline{C}$ at an indefinitely close distance from the point $\overline{m}$; i. e.,

$$\gamma_n(\overline{l}) = \overline{l}_n \text{ for some } \gamma_n \in \Gamma, \quad \overline{l}_n \cap \overline{C} \to \overline{m} \text{ as } n \to \infty.$$

According to Theorem 3.2 on the continuous dependence of leaves on the initial conditions, for an arbitrary fixed positive integer k, the leaves $\overline{l}_n$ with sufficiently large n intersect both curves $\overline{C}_{-k}$ and $\overline{C}_k$. The ideal endpoints of these curves tend, as $k \to +\infty$, to the points $\alpha(\overline{l})$ and $\omega(\overline{l})$, respectively,

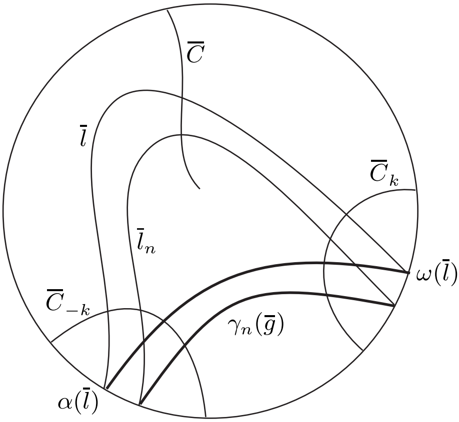


Figure 5.20. Congruent geodesics $\overline{g}_n = \gamma_n(\overline{g})$.

in the Euclidean metric on the absolute S_∞; hence, the ideal endpoints of the leaves $\overline{l}_n$ tend to the corresponding endpoints of the leaf $\overline{l}$. Formally,

$$\alpha(\overline{l}_n) \to \alpha(\overline{l}), \quad \omega(\overline{l}_n) \to \omega(\overline{l}) \quad \text{as } n \to \infty.$$

Hence,

$$\alpha[\overline{g}(\overline{l}_n)] \to \alpha[\overline{g}(\overline{l})], \quad \omega[\overline{g}(\overline{l}_n)] \to \omega[\overline{g}(\overline{l})] \quad \text{as } n \to \infty$$

in the Euclidean metric on the absolute. Then, any point of the geodesic $g(l) = \pi[\overline{g}(\overline{l})]$ is a limit point for a certain sequence of points lying on $g(l) = \pi[\overline{g}(\overline{l})] = \pi(\gamma_n[\overline{g}(\overline{l})])$ with indefinitely large (in absolute value) values of the parameter. This means that the geodesic $g(l)$ is nontrivially recurrent in one direction. □

Corollary 5.7 *Suppose that the hypotheses of Lemma* 5.18 *are fulfilled, and let l be an internal leaf. Then, the co-asymptotic geodesic $g(l)$ is also an internal nontrivially recurrent geodesic, and any of its lifts reaches points from the set $\Lambda_{1,\infty}(M)$ in either direction.*

Proof. Let $\overline{l}$ be a leaf that covers l. It follows from Lemma 5.18 that $\overline{l}$ reaches some points from the set $\Lambda_\infty(M)$ in either direction. Let us make use of the notations from the proof of Lemma 5.18. Since the leaf l is internal, there exists a sequence of leaves $\overline{l}_n$ that are congruent to $\overline{l}$ and intersect $\overline{C}$ at an indefinitely close distance from the point $\overline{m}$ on both sides of $\overline{m}$. Hence, the co-asymptotic geodesic $\overline{g}(\overline{l})$ is also internal. By Lemma 5.8, $\overline{g}(\overline{l})$ reaches (in either direction) some points from the set $\Lambda_{1,\infty}(M)$. □

Consider a family of sets $\overline{K}_n$ on the closed unit disk $\Delta \cup S_\infty$ of the Euclidean plane. Recall that a point $\overline{z}_* \in \Delta \cup S_\infty$ belongs to the *topological limit* of $\overline{K}_n$ if there exists a sequence of points $\overline{z}_n \in \overline{K}_n$ that tend to $\overline{z}_*$ in the Euclidean topology:

$$\overline{d}_{\mathrm{E}}(\overline{z}_n, \overline{z}_*) \to 0 \quad \text{as } n \to \infty.$$

Lemma 5.19 *Let l be a simple infinite curve with a co-asymptotic geodesic $g(l)$, and let $\overline{l}_k$, $k \geqslant 0$, be an arbitrary family of lifts of l to Δ. Then the topological limit of $\overline{l}_k$ does not contain a nontrivial interval of S_∞.*

Proof. Denote by $\overline{K}$ the topological limit of the family $\overline{l}_k$ on $\Delta \cup S_\infty$ and show that the intersection of $\overline{K}$ with S_∞ does not contain nontrivial intervals. Suppose the contrary. Then, there exists an arc $(\alpha, \beta) \subset \overline{K} \cap S_\infty$, $\alpha \neq \beta$. Since the surface $M = M_g^2$, $g \geqslant 2$, is orientable and closed, the group Γ of deck transformations is a Fuchsian group of the first kind [173]. Therefore, there exists a hyperbolic transformation $\gamma \in \Gamma$ with an attracting point on the arc (α, β) and a repelling point outside (α, β). Hence, the ideal endpoints of the curve $\gamma^s(\overline{l}_0)$ lie strictly inside (α, β) for a certain $s \in \mathbb{N}$. Denote them by α_s and β_s. Since l has a co-asymptotic geodesic, $\alpha_s \neq \beta_s$.

The arc (α_s, β_s) and the curve $\gamma^s(\overline{l}_0)$ bound a domain $\overline{D}_s$ in Δ. Since any point of the arc (α_s, β_s) lies in $\overline{K}$, the curves $\overline{l}_k$ intersect $\overline{D}_s$ for a sufficiently large k. Since there are points outside (α_s, β_s) that belong to the topological limit of the curves $\overline{l}_k$, $\overline{l}_k$ must intersect $\gamma^s(\overline{l}_0)$ (see Fig. 5.21), which is impossible because l is a simple curve. □

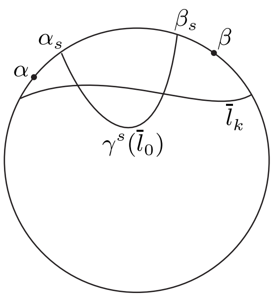


Figure 5.21. The arcs (α, β) and (α_s, β_s).

It follows from Theorem 4.5 that any quasiminimal set with closed support contains a continuum set of nontrivially recurrent leaves each of which is everywhere dense in the quasiminimal set. Recall that a quasiminimal set Q is called a Maier quasiminimal set if each semileaf from Q that does not tend exactly to one singularity is everywhere dense in Q. In particular, a leaf from Q that is different from a separatrix connection is everywhere dense in Q and is nontrivially recurrent in, at least, one direction. The following theorem describes a geodesic framework of Maier quasiminimal set.

Theorem 5.32 *Let Q be a Maier quasiminimal set of a local lamination $\mathscr{D}$ with closed support* $\operatorname{supp}\mathscr{D}$. *Suppose that every nontrivial recurrent leaf from Q is widely disposed with respect to a certain simple closed curve C and intersects C transversally. Let Q have a finitely many singularities and separatrices. Then*

1) *for any nontrivially recurrent leaf $l \in Q$, the geodesic framework $G(Q)$ of the quasiminimal set Q is equal to* $\operatorname{clos} g(l)$, *where $g(l)$ is a co-asymptotic geodesic of l*;
2) *the geodesic framework $G(Q)$ is a weakly irrational geodesic lamination that consists of nonclosed B-recurrent geodesics*;
3) *any geodesic from $G(Q)$ is the co-asymptotic geodesic of a certain leaf or a generalized leaf that belongs to Q.*

Proof. Theorem 5.18 implies that the co-asymptotic geodesic $g(l)$ exists. By Lemma 5.18, $g(l)$ is a nonclosed B-recurrent geodesic. In view of Lemma 3.41, in order to prove the first two assertions, it suffices to prove the equality $G(Q) = \operatorname{clos} g(l)$. The definition of the geodesic framework implies $\operatorname{clos} g(l) \subset G(Q)$. Let us prove the reverse inclusion. Take an arbitrary geodesic g_* from $G(Q)$. We must show that $g(l)$ intersects any neighborhood of an arbitrary point of the geodesic g_*. Again, from the definition of the geodesic framework we derive that a fixed neighborhood of a fixed point on the geodesic g_* is intersected by a geodesic g_1 from $G(Q)$ that is either

1) a co-asymptotic geodesic for a leaf, say l_1, that does not belong to any generalized leaf or
2) a co-asymptotic geodesic for a certain generalized leaf, say L_1.

In case (1), l_1 is a nontrivially recurrent leaf that is everywhere dense in Q because Q is a Maier quasiminimal set. Therefore, l belongs to the limit set of the leaf l_1. This result, combined with the fact that l intersects the curve C

infinitely many times, implies that l_1 also intersects C infinitely many times. Since l_1 belongs to the closure of the leaf l and l is widely disposed with respect to the curve C, l_1 is also widely disposed with respect to C.

Take one of the coverings $\overline{l}_1$ for the leaf l_1. By Theorem 5.18, $\overline{l}_1$ tends to different points of the absolute in different directions: $\alpha(\overline{l}_1), \omega(\overline{l}_1) \in S_\infty$, $\alpha(\overline{l}_1) \neq \omega(\overline{l}_1)$. According to Theorem 5.16, the points $\alpha(\overline{l}_1)$ and $\omega(\overline{l}_1)$ are the topological limits of the lifts of C that are successively intersected by $\overline{l}_1$ as the parameter decreases and increases, respectively. Next, by analogy with the proof of Lemma 5.18, we show that the geodesic $\overline{g}(\overline{l}_1)$ is the topological limit of geodesics that are congruent to $\overline{g}(\overline{l})$. This means that $g(l_1) \subset \operatorname{clos} g(l)$.

In case (2), by Lemma 4.7, the generalized leaf L_1 is finite; i. e., its first and last leaves are nontrivially recurrent in one direction. Each of them is everywhere dense in Q, intersects C infinitely many times, and is widely disposed with respect to C. Hence, by analogy with the previous case, we obtain the required inclusion $g(L_1) \subset \operatorname{clos} g(l)$. Thus, $G(Q) \subset \operatorname{clos} g(l)$.

Let us prove the last assertion of the theorem. Take an arbitrary geodesic $g_0 \subset \operatorname{clos} g(l)$. Let $\overline{g}_0$ be a lift of g_0. Since $g_0 \subset \operatorname{clos} g(l)$, there exist sequences of geodesics $\overline{g}_k \in \pi^{-1}(g(l))$, $k \geqslant 1$, and of points $\overline{m}_k \in \overline{g}_k$ that converge to a certain point of the geodesic $\overline{g}_0$ as $k \to \infty$. Since $\pi(\overline{g}_k) = g(l)$, there exists a sequence of lifts $\overline{l}_k$ of the leaf l such that $\overline{g}_k = \overline{g}(\overline{l}_k)$. Since $\operatorname{clos} g(l)$ is a geodesic lamination, we may assume that the geodesics $\overline{g}_0$ and $\overline{g}_k$, $k \geqslant 1$, are pairwise disjoint and that the ideal endpoints of the geodesics $\overline{g}_k$ tend, in the Euclidean metric on S_∞, to the corresponding ideal endpoints of the geodesic $\overline{g}_0$ as $k \to \infty$. Let us orient $\overline{g}_0$ and $\overline{g}_k$ so that

$$\omega(\overline{g}_k) \stackrel{\text{def}}{=} \omega_k \to \omega(\overline{g}_0) \stackrel{\text{def}}{=} \omega_0, \quad \alpha(\overline{g}_k) \stackrel{\text{def}}{=} \alpha_k \to \alpha(\overline{g}_0) \stackrel{\text{def}}{=} \alpha_0 \quad \text{as } k \to \infty.$$

Denote by $\overline{K}$ the topological limit of the leaves $\overline{l}_k$ on $\Delta \cup S_\infty$. As a topological limit of arcwise connected sets, $\overline{K}$ is a connected continuum containing the points ω_0 and α_0. Combined with Lemma 5.19, this implies that the intersection $\Delta \cap \overline{K}$ contains a continuum of points.

Since a quasiminimal set is closed, $\Delta \cap \overline{K}$ is projected into a part of the set Q, $\pi(\Delta \cap \overline{K}) \subset Q$. Since Q is a Maier quasiminimal set that contains only a finite number of singularities, there is a point in $\Delta \cap \overline{K}$, say $\overline{m}_0$, through which a (one-dimensional) leaf $\overline{l}_0$ passes (Fig. 5.22). It is clear that

$$\pi(\overline{l}_0) \stackrel{\text{def}}{=} l_0 \subset Q.$$

According to Lemma 4.7, if $\overline{l}_0$ tends to a certain singularity in one of the directions, then $\overline{l}_0$ belongs to a finite generalized leaf, which we denote by $\overline{L}_0$.

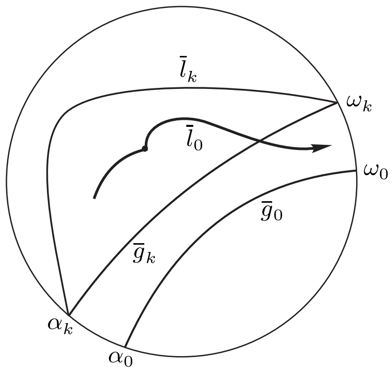


Figure 5.22

If $\bar{l}_0$ does not tend to a singularity, then l_0 is a nontrivially recurrent leaf. In this case, further reasoning is simpler than in the case when $\bar{l}_0$ belongs to $\overline{L}_0$. Therefore, we will consider only the case when $\bar{l}_0 \subset \overline{L}_0$.

The theorem on the continuous dependence on the initial conditions implies that $\pi(\overline{L}_0) \subset Q$. Since $\overline{L}_0$ is a finite generalized leaf, its "marginal" leaves $\bar{l}_b$ and $\bar{l}_f$ are ω- and α-separatrices, respectively (if l_0 is a nontrivial recurrent leaf, then we can set $\bar{l}_0 = \bar{l}_f = \bar{l}_b$ in further reasoning). It follows from the definition of the Maier quasiminimal set that $\pi(\bar{l}_b)$ and $\pi(\bar{l}_f)$ are everywhere dense in Q and are nontrivially recurrent in the negative and positive directions, respectively. Therefore, each of the curves $\pi(\bar{l}_b)$ and $\pi(\bar{l}_f)$ intersects the curve C and is widely disposed with respect to C. Hence, $\bar{l}_b$ and $\bar{l}_f$ have asymptotic directions $\alpha(\bar{l}_b)$, $\omega(\bar{l}_f) \in S_\infty$, respectively.

It remains to show that $\alpha(\bar{l}_b) = \alpha_0$ and $\omega(\bar{l}_f) = \omega_0$ because these equalities imply the required equality $\bar{g}(\overline{L}_0) = \bar{g}_0$. We will only prove that $\omega(\bar{l}_f) = \omega_0$ because the equality $\alpha(\bar{l}_b) = \alpha_0$ is proved in a similar way. Suppose the contrary; i. e., let $\omega(\bar{l}_f) \neq \omega_0$. According to Theorem 5.16, the point $\omega(\bar{l}_f) \in S_\infty$ is the topological limit of the lifts of the curve C that are successively intersected by the leaf $\bar{l}_f$ as the parameter increases. Therefore, there exists a lift $\overline{C}_*$ of the curve C that intersects the leaf $\bar{l}_f$ and whose ideal endpoints c_1^*, $c_2^* \in S_\infty$ separate the points $\omega(\bar{l}_f)$ and ω_0 on S_∞. In other words, if we denote by J_f and J_0 the arcs into which the absolute S_∞ is divided by the points c_1^* and c_2^*, then one of these arcs, say J_f, contains $\omega(\bar{l}_f)$, and the other arc, J_0, contains the point ω_0. Without loss of generality, we may assume that ω_0 lies strictly inside the arc J_0. Since the topological limit of the sequence $\overline{l}_k$ contains the point $\overline{m}_0$, the theorem on the continuous dependence on the initial conditions implies that starting from a certain number, all $\overline{l}_k$ intersect $\overline{C}_*$. The wide disposition of the leaf l with respect to C implies that $\overline{l}_k$ intersects $\overline{C}_*$ only once. Hence,

$$\omega(\overline{l}_k) = \omega_k \in J_f.$$

On the other hand, $\omega_k \to \omega_0 \in J_0$ as $k \to \infty$. The contradiction obtained proves the theorem. □

Corollary 5.8 *Let Q be a Maier quasiminimal set. Then,* $\operatorname{clos} g(l)$ *does not depend on the choice of a nontrivially recurrent leaf l from Q, and $G(Q) = \operatorname{clos} g(l)$.*

Geodesic frameworks of special foliations

Consider foliations with isolated singularities of negative index, in particular, foliations with singularities solely of the saddle type of negative index. According to Theorem 5.22, any semileaf of such foliations that does not tend to a singularity has an asymptotic direction. It follows from Theorem 5.23 that any leaf of such foliations that is not a separatrix has a co-asymptotic geodesic. According to Theorem 5.24, any generalized leaf of such foliations also has a co-asymptotic geodesic. The proof of the following lemma is analogous to the proof of Lemma 5.18 (we omit the proof).

Lemma 5.20 *Let $\mathscr{F}$ be a foliation with isolated singularities of negative index on a closed orientable hyperbolic surface M^2. Suppose that a generalized leaf l of the foliation $\mathscr{F}$ contains a nontrivially recurrent leaf. Then, the co-asymptotic geodesic $g(l)$ is a nonclosed B-recurrent geodesic. In particular, $g(l)$ is nontrivially recurrent.*

The so-called irrational foliations and flows play an especially important role among foliations with singularities of saddle type. Recall that a foliation on a surface is called *irrational* if any of its (one-dimensional) leaves is everywhere dense on the surface and all its singularities are saddles of nonzero index. As a consequence, an irrational foliation does not have any closed leaves or separatrix connections. Note that an irrational foliation may have saddle singularities of positive index (thorns). A generalization of the irrational foliation is an arational foliation introduced by Rosenberg [200]. A foliation is called *arational* if it has no closed leaves or separatrix connections. For a surface with a boundary, it is additionally required that a foliation should be transversal to the boundary and that there should not exist any compact leaves that connect the points on the boundary components.

Let $\mathscr{F}$ be a foliation (not necessarily arational) with singularities of nonpositive index on a closed orientable surface. The limit set of a semileaf that does not tend exactly to one singularity is either a closed non-null-homotopic leaf, a non-null-homotopic loop, or a quasiminimal set. Hence, the limit set of any semileaf of an arational foliation that is not a separatrix of a singularity is a quasiminimal set. According to Theorem 4.14, this quasiminimal set does not contain proper quasiminimal subsets (since we speak about an index of singularities, all of them are isolated and, hence, the number of them is finite). Thus, the geodesic framework of an arational foliation with singularities of nonpositive index is nonempty and is obtained as the union of the geodesic frameworks of quasiminimal sets. It can be shown that an arational foliation with singularities of nonpositive index has exactly one quasiminimal set. Therefore, *the geodesic framework of an arational foliation with singularities of nonpositive index is an strongly irrational geodesic lamination that consists of nonclosed B-recurrent geodesics*. For a more detailed description of the geodesic framework, we will need a few lemmas, which are of independent value.

Lemma 5.21 *Let $\mathscr{L}$ be a weakly irrational geodesic lamination on a closed orientable surface M^2. Then, $\mathscr{L}$ is irreducible* (*i. e., strongly irrational*) *if and only if each component of the set $M^2 - \mathscr{L}$ is a simply connected domain such that any of its lifts to the universal covering represents the interior of a geodesic polygon with a finite number of sides and with vertices lying on the absolute.*

Proof. Let $\mathscr{L}$ be irreducible, and let K be a connected component of the set $M^2 - \mathscr{L}$. According to Lemma 3.46, the lift $\overline{K}$ of this component to Δ is a contractible noncompact hyperbolic surface with a geodesic boundary. This result, combined with Corollary 3.14, implies that the boundary $\partial\overline{K}$ consists of garlands (see Section 3.8). If $\partial\overline{K}$ contains an infinite garland, then, by Corollary 3.15, there exists a closed geodesic on the surface that does not intersect $\mathscr{L}$. This fact contradicts the irreducibility of $\mathscr{L}$. Therefore, $\partial\overline{K}$ consists of finite garlands. This and Lemma 5.7 imply that $\partial\overline{K}$ consists of exactly one finite closed garland; i. e., $\overline{K}$ is the interior of a geodesic polygon with a finite number of sides and with vertices lying on the absolute. It is clear that the component K is simply connected.

The converse follows from Corollary 3.13. □

Lemma 5.22 *Let $\mathscr{F}$ be an arational foliation with singularities that are saddles of negative index on a closed orientable hyperbolic surface M^2, and let l^+ be a nontrivially recurrent semileaf of the foliation $\mathscr{F}$. Suppose that a lift $\overline{l}^+$ of l^+ reaches a point $\sigma \in \Lambda_\infty(M^2)$ and is oriented toward the point σ. Then,*

1) *if $\sigma \in \Lambda_{1,\infty}(M^2)$, then l^+ belongs to a nontrivially recurrent* (*in both directions*) *leaf and the co-asymptotic geodesic of this leaf is an internal nontrivially recurrent geodesic*;
2) *if $\sigma \in \Lambda_{2,\infty}(M^2)$, then either*
   a) *l^+ is an internal nontrivially recurrent semileaf that is an α-separatrix of a certain singularity, and the left and right Bendixson extensions in the negative direction of the leaf that contains l^+ have different asymptotic directions*; *or*
   b) *l^+ is a boundary nontrivially recurrent semileaf.*

Proof. Let $\sigma \in \Lambda_{1,\infty}(M^2)$, and suppose that l^+ is an α-separatrix of a certain singularity, say O. Then, $\overline{l}^+$ is an α-separatrix of the saddle $\overline{O}$ that covers O. Denote by $\overline{l}_1$ and $\overline{l}_2$, respectively, the left and right Bendixson extensions of the α-separatrix l^+ in the negative direction. Since the singularities have a negative index, $\overline{l}_1 \neq \overline{l}_2$. The leaves $\overline{l}_1$ and $\overline{l}_2$ are ω-separatrices of the saddle $\overline{O}$. In view of the arationality of the foliation $\mathscr{F}$, the α-limit sets of these leaves do not consist of a single singularity. Therefore, $\overline{l}_1$ and $\overline{l}_2$ have asymptotic directions represented by the points $\alpha(\overline{l}_1)$, $\alpha(\overline{l}_2) \in S_\infty$, respectively. Let us show that $\alpha(\overline{l}_1) \neq \alpha(\overline{l}_2)$. Suppose the contrary; i. e., let $\alpha(\overline{l}_1) = \alpha(\overline{l}_2) \stackrel{\text{def}}{=} \alpha$. Since $\mathscr{F}$ is arational, there exists a closed transversal C that intersects the leaves $\pi(\overline{l}_1)$ and $\pi(\overline{l}_2)$. By the assumption, there exists a lift $\overline{C}$ of the transversal C that intersects both leaves $\overline{l}_1$ and $\overline{l}_2$. Then, the saddle $\overline{O}$, the segment of the transversal $\overline{C}$ between the points of intersection of $\overline{C}$ with $\overline{l}_1$ and $\overline{l}_2$, and the corresponding semileaves of the leaves $\overline{l}_1$ and $\overline{l}_2$ form a closed curve, which we denote by $\overline{T}$. Since $\overline{C}$ is a transversal, the index of the curve $\overline{T}$ is equal to $\frac{1}{2}$. Therefore, there must exist singularities of positive index in $\overline{T}$, which is impossible. This contradiction shows that $\alpha(\overline{l}_1) \neq \alpha(\overline{l}_2)$.

Denote by $\overline{g}_1$ and $\overline{g}_2$ the geodesics that connect the point σ with the points $\alpha(\overline{l}_1)$ and $\alpha(\overline{l}_2)$, respectively. According to Lemma 5.20, $\overline{g}_1$ and $\overline{g}_2$ are projected to simple nontrivially recurrent geodesics on M^2. Hence, $\sigma \in$ $\in \Lambda_{2,\infty}(M^2)$, which contradicts the assumption. The contradiction obtained shows that l^+ belongs to a leaf, say l, that is not an α-separatrix. This and the arationality of the foliation $\mathscr{F}$ imply that l is a nontrivially recurrent leaf.

By Lemma 5.18, the co-asymptotic geodesic $g(l)$ is nontrivially recurrent. Lemma 5.8 and the inclusion $\sigma \in \Lambda_{1,\infty}(M^2)$ imply that $g(l)$ is internal.

Let, now, $\sigma \in \Lambda_{2,\infty}(M^2)$ and l^+ be an internal nontrivially recurrent semileaf. Let us show that l^+ is an α-separatrix of a certain singularity. Suppose the contrary. Then, l^+ belongs to a leaf, which we denote by l, that is nontrivially recurrent in view of the arationality of the foliation $\mathscr{F}$. By the theorem on the continuous dependence of leaves on the initial conditions, l is an internal leaf. By Lemma 5.8 and Corollary 5.7, the lift $\overline{l}$ of l and, hence, $\overline{l}^+$ reach a point from the set $\Lambda_{1,\infty}(M^2)$. The contradiction obtained proves the lemma. □

The following theorem provides a more detailed description for the geodesic framework of an arational (in particular, an irrational) foliation. Recall that if G is a geodesic lamination on a hyperbolic surface M^2 and $\overline{G}$ is its covering lamination, then $\overline{G}_\infty$ denotes the set of ideal endpoints of geodesics from $\overline{G}$.

Theorem 5.33 *Let $\mathscr{F}$ be an arational foliation on a closed orientable hyperbolic surface M^2 with singularities that are saddles of negative index. Then,*

1) *the geodesic framework $G(\mathscr{F})$ of $\mathscr{F}$ is a strongly irrational geodesic lamination that consists of nonclosed B-recurrent geodesics*;
2) *any geodesic from $G(\mathscr{F})$ is a co-asymptotic geodesic for a certain leaf or generalized leaf of $\mathscr{F}$. Moreover,*
   a) *any point $\sigma \in \Lambda_{1,\infty}(M^2) \cap \overline{G}(\mathscr{F})_\infty$ is reached by a leaf projected to a nontrivially recurrent leaf on M^2 whose co-asymptotic geodesic is internal*;
   b) *any point $\sigma \in \Lambda_{2,\infty}(M^2) \cap \overline{G}(\mathscr{F})_\infty$ is reached by a leaf $\overline{l}$ that is either a boundary or an internal leaf. In the latter case, $\overline{l}$ is a separatrix of a singularity, and the left and right Bendixson extensions of $\overline{l}$ in the negative direction* (*here, the direction toward the point σ is assumed positive*) *have different asymptotic directions α_1 and α_2. Two geodesics that connect σ with the points α_1 and α_2 are sides of a geodesic polygon with a finite number of sides that belong to $\overline{G}(\mathscr{F})$, and these geodesics are projected to boundary geodesics on M^2*;
3) *each component of the set $M^2 - G(\mathscr{F})$ is a simply connected domain any of whose lifts to the universal covering represents the interior of*

a geodesic polygon with a finite number of sides and with vertices lying on the absolute. In this case, the vertices are reached by the separatrices of the saddles of the covering foliation.

Proof. We must prove only assertions (2) and (3). Assertion (2) follows from Theorem 5.32 and Lemma 5.22. The first part of assertion (3), except for the last statement, follows from Lemma 5.21. The last statement follows from the arationality of the foliation. □

Note that in the general case, in the last assertion of Theorem 5.33, one can not claim that the vertices of the geodesic polygon are reached by separatrices of the same saddle. However, this statement can be claimed for a strongly irrational foliation.

Theorem 5.34 *Let $\mathscr{F}$ be an irrational foliation on a closed orientable hyperbolic surface M^2 with singularities that are saddles of negative index. Then*

1) *the geodesic framework $G(\mathscr{F})$ of $\mathscr{F}$ is a strongly irrational geodesic lamination that consists of nonclosed B-recurrent geodesics*;
2) *any geodesic from $G(\mathscr{F})$ is a co-asymptotic geodesic for a certain leaf or generalized leaf of $\mathscr{F}$. Moreover,*
   a) *any point $\sigma \in \Lambda_{1,\infty}(M^2) \cap \overline{G}(\mathscr{F})_\infty$ is reached by a leaf projected to an internal nontrivially recurrent leaf on M^2 whose co-asymptotic geodesic is also internal*;
   b) *any point $\sigma \in \Lambda_{2,\infty}(M^2) \cap \overline{G}(\mathscr{F})_\infty$ is reached by a leaf $\bar{l}$ that is an α-separatrix of a singularity, and the left and right Bendixson extensions of the leaf $\bar{l}$ in the negative direction*[1] *have different asymptotic directions α_1 and α_2. Two geodesics that connect σ with the points α_1 and α_2 are sides of a geodesic polygon with a finite number of sides that belong to $\overline{G}(\mathscr{F})$, and these geodesics are projected to boundary geodesics on M^2*;
3) *each component of the set $M^2 - G(\mathscr{F})$ is a simply connected domain any of whose lifts to the universal covering is the interior of a geodesic polygon with a finite number of sides and with vertices lying on the absolute. In this case, the sides of the geodesic polygon belong to $\overline{G}(\mathscr{F})$, and each vertex is reached by exactly one separatrix of a certain saddle of the covering foliation. Conversely, each saddle of the covering foliation corresponds to a unique geodesic polygon formed by geodesics from $\overline{G}(\mathscr{F})$,*

[1] Without loss of generality, we may assume that the leaf $\bar{l}$ is oriented toward the point σ.

such that the separatrices of the saddle reach all vertices of the polygon and the number of separatrices is equal to the number of vertices.

Proof. Note that all leaves of the irrational foliation are internal. Therefore, we must only prove the assertion that the vertices of the geodesic polygon are reached by the separatrices of the same saddle. Suppose the contrary. Then, there exists a simply connected domain that is bounded by generalized leaves of several (at least two) different saddles. In view of the simple connectedness, this domain can not contain nontrivially recurrent leaves, which contradicts the irrationality of the foliation. □

5.6. Deviations of curves from co-asymptotic geodesics

In this section, we focus our attention on the deviation of the lifts of simple curves that have asymptotic directions (so, they have co-asymptotic geodesics) from co-asymptotic geodesics on a universal covering. First, we consider examples of curves with unbounded deviation; here, the cases of irrational and rational asymptotic directions are considered separately because there is a fundamental difference between the relevant constructions.

We begin with examples of curves with unbounded deviation. Recall that according to Theorem 5.11, for any simple semi-infinite curve, there exists a semitrajectory of a smooth flow with the same asymptotic direction and with the same property of (bounded or unbounded) deviation. Therefore, all the curves with unbounded deviation can be represented in the form of semitrajectories and trajectories of smooth flows.

Here, we give relevant examples of curves with an irrational asymptotic direction. In this case, examples for orientable flat and orientable hyperbolic surfaces are constructed in essentially different ways. Concerning rational directions, we represent an example of a curve with unbounded deviation and with a rational asymptotic direction on the torus. This example is basic in the sense that one can easily derive similar examples for hyperbolic surfaces from it.

Following these examples, it is natural to consider the question concerning the conditions under which a deviation of special curves is bounded. We prove the property of bounded deviation for semitrajectories of flows with a finite number of fixed points, for semileaves of foliations with a finite number of saddle-type singularities of negative index, for semitrajectories and semileaves of a wide class of flows and foliations on the torus, and for nontrivially recurrent semitrajectories on the torus without constraints on the set of fixed points, as well as for leaves of certain special laminations that are encountered in the theory of smooth hyperbolic dynamical systems.

At last, we consider the property of uniformly bounded deviation from geodesic frameworks for the leaves and generalized leaves of certain local laminations.

Examples of unbounded deviation

One can prove that there are no simple curves with an irrational asymptotic direction on the Klein bottle [11]. Thus, among closed flat surfaces, the torus is unique surface on which there exist simple semi-infinite curves with an irrational asymptotic direction.

Recall that by virtue of Theorem 5.5, a covering curve for a simple semi-infinite curve that has a rational asymptotic direction can not unboundedly deviate from a co-asymptotic geodesic on both sides simultaneously. For an irrational asymptotic direction, an unbounded deviation on both sides is possible.

Example 5.2 *Curve with irrational asymptotic direction on the torus.*

The construction is based on the properties of the approximation of an irrational number by rational numbers. We will construct this curve on the universal covering, the Euclidean plane $\mathbb{R}^2$, by deforming the ray l: $y = \alpha x$, $x \geqslant 0$, where $\alpha \in \mathbb{R} - \mathbb{Q}$ is an irrational number. Let $\frac{p_i}{q_i}$ be the convergents of the continued fraction for α. The convergents of even (odd) order form an increasing (respectively, decreasing) sequence that converges to α:

$$\frac{p_0}{q_0} < \frac{p_2}{q_2} < \ldots < \frac{p_{2n}}{q_{2n}} < \ldots \to \alpha \leftarrow \ldots < \frac{p_{2n-1}}{q_{2n-1}} < \ldots < \frac{p_1}{q_1}.$$

Denote by S_i a strip bounded by the straight lines

$$P_i\colon y = \frac{p_i}{q_i}x + \frac{(-1)^i}{q_i}, \quad P_{i,0}\colon y = \frac{p_i}{q_i}x.$$

The ray l enters S_i intersecting $P_{i,0}$ at the origin and leaves S_i intersecting P_i at a point m_i with the abscissa $1/|q_i\alpha - p_i|$ (Fig. 5.23). Denote by l_i the segment of the ray l between the origin and the point m_i. The idea of the construction is to "stretch out" the segment l_i near the point m_i along the curve P_i (Fig. 5.24).

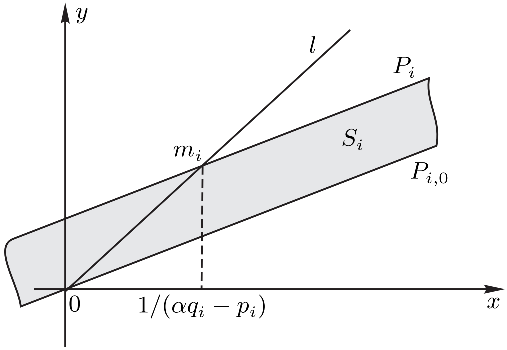


Figure 5.23. i is even.

Let us fix an index i and set $v = q_i y - p_i x$. For a fixed value of v, the equation $v = q_i y - p_i x$ defines a straight line, which we denote by P_v. When the parameter v varies from 0 to $(-1)^i$, the straight line P_v "sweeps out" the strip S_i from the line $P_i = P_0$ to the line $P_{i,0}$.

Take a nonnegative continuous function (its specific form is defined below) φ of a single variable with period 1 and consider the mapping

$$\overline{F}\colon (x,\, y) \mapsto (x + \varphi(v)q_i,\, y + \varphi(v)p_i) = (x,\, y) + \varphi(q_i y - p_i x)(q_i,\, p_i).$$

It is clear that $\overline{F}$ translates each straight line P_v by the vector $\varphi(v)(q_i,\, p_i)$. Therefore, $\overline{F}$ is a homeomorphism of the plane $\mathbb{R}^2$. Since any point $(x,\, y) \in \mathbb{R}^2$ is shifted by the vector $\varphi(q_i y - p_i x)(q_i,\, p_i)$ under $\overline{F}$ and φ has period 1, $\overline{F}$ commutes with integer translations:

$$\overline{F} \circ S_{nk} = S_{nk} \circ \overline{F} \quad \text{for any } (n,\, k) \in \mathbb{Z}^2. \tag{5.15}$$

This implies that $\overline{F}$ is a covering homeomorphism for a certain homeomorphism of the torus.

Denote the coordinates of the point $z \in \mathbb{R}^2$ by $x(z)$ and $y(z)$. Since the function φ is nonnegative, we have

$$x(\overline{F}(z)) \geqslant x(z), \quad y(\overline{F}(z)) \geqslant y(z). \tag{5.16}$$

To define the function φ more exactly, we need the function

$$\psi(x) = 1 - 2\left|x - \frac{1}{2}\right| \text{ for } x \in [0;\, 1] \quad \text{and} \quad \psi(x) = 0 \text{ for } x \notin [0;\, 1].$$

For a certain fixed $0 < \varepsilon_i < 1$, we set

$$\varphi_{\varepsilon_i}(z) = \sum_{k=-\infty}^{\infty} \psi\left(\frac{z-k}{\varepsilon_i} + \frac{(-1)^i + 1}{2}\right),$$

so that

$$0 \leqslant \varphi_{\varepsilon_i} \leqslant 1, \quad \max \varphi_{\varepsilon_i} = 1,$$

$$\varphi_{\varepsilon_i} = 0 \quad \text{outside the segments } [n - \varepsilon_i;\, n] \text{ for even } i,$$

$$\varphi_{\varepsilon_i} = 0 \quad \text{outside the segments } [n;\, n + \varepsilon_i] \text{ for odd } i.$$

Denote by θ_i the minimal angle between l and the straight line $P_{i,0}$. Set

$$\overline{F}_i(x,\, y) = (x,\, y) + c_i \varphi_{\varepsilon_i}(q_i y - p_i x)(q_i,\, p_i), \tag{5.17}$$

where c_i are some positive constants. In the strip S_i, only the points with $|v| \in \in (1 - \varepsilon_i,\, 1)$ are shifted under the action of $\overline{F}_i(x,\, y)$. The maximal shift is equal to $c_i\sqrt{p_i^2 + q_i^2}$ and is attained at the points with $|v| = 1 - \frac{\varepsilon_i}{2}$. Denote by z_i the image of the point on l_i with maximal shift.

Choose c_i so large that the distance d_i from z_i to the ray l indefinitely increases as $i \to \infty$, i. e.,

$$d_i = c_i\sqrt{p_i^2 + q_i^2}\,\sin\theta_i \to \infty \quad \text{as } i \to \infty, \quad \text{and that} \quad d_{i+1} > 3d_i. \tag{5.18}$$

It follows from the theory of continued fractions that

$$|p_1 - \alpha q_1| > \ldots > |p_i - \alpha q_i| > \ldots, \quad \lim_{i\to\infty}(p_i - \alpha q_i) = 0; \tag{5.19}$$

therefore,

$$l_1 \subset l_2 \subset \ldots \subset l_i \subset \ldots.$$

The numbers $\varepsilon_i > 0$ are chosen so that in the strip S_i, $\overline{F}_i(x,\, y)$ does not shift the points of the segment l_{i-1} near m_{i-1} and, moreover, the points of the segment l_j with $j < i - 1$.

Set $\overline{L}_n = \overline{F}_1 \circ \ldots \circ \overline{F}_n(l_n)$. As the image of a segment under a homeomorphism, $\overline{L}_n$ is a compact curve without self-intersections. Since

$$\overline{F}_1 \circ \ldots \circ \overline{F}_n \circ \overline{F}_{n+1} \circ \ldots \circ \overline{F}_{n+k}\big|_{l_n} = \overline{F}_1 \circ \ldots \circ \overline{F}_n\big|_{l_n}, \tag{5.20}$$

we have $\overline{L}_n \subset \overline{L}_{n+k}$ for any $k \in \mathbb{N}$. Hence, the union $\overline{L} \overset{\text{def}}{=} \bigcup\limits_n \overline{L}_n$ is also a curve without self-intersections.

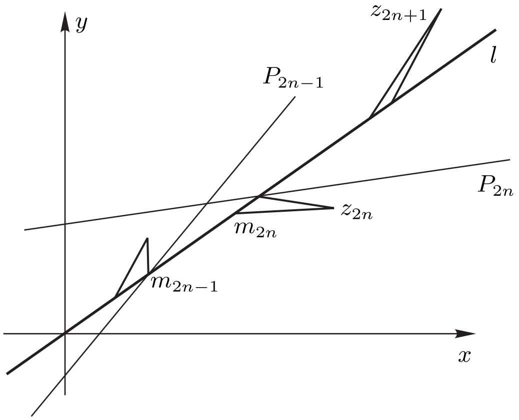


Figure 5.24

It follows from (5.15) that

$$\overline{F}_1 \circ \ldots \circ \overline{F}_n \circ S_{jk}(\bar{l}_n) = S_{jk} \circ \overline{F}_1 \circ \ldots \circ \overline{F}_n(\bar{l}_n) \text{ for any } (j,\, k) \in \mathbb{Z}^2.$$

Since $\overline{F}_1 \circ \ldots \circ \overline{F}_n$ is a homeomorphism and $S_{jk}(l_n) \cap l_n = \varnothing$ for $(j,\, k) \neq$ $\neq (0,\, 0)$, we have

$$S_{jk}(\overline{L}_n) \cap \overline{L}_n = \varnothing \quad \text{for } (j,\, k) \neq (0,\, 0),\; n \in \mathbb{N}.$$

Hence, we find that $S_{jk}(\overline{L}) \cap \overline{L} = \varnothing$ for any $(j,\, k) \neq (0,\, 0)$; i. e., $L \stackrel{\text{def}}{=} \pi(\overline{L})$ is a simple semi-infinite curve. It follows from (5.18) that $\overline{L}$ unboundedly deviates from the ray l.

The original ray l admits a natural parametrization if we assign a parameter $t = x$ to each point $(x,\, \alpha x) \in l$. Let us parameterize the arc $\overline{L}_i$ so that its endpoint m_i has a parameter $x(m_i)$; i. e., $t(m_i) = x(m_i)$. According to (5.20), the curve $\overline{L}$ admits a consistent parametrization $z\colon [0;\, +\infty) \to \overline{L}$ if we set

$$z(t) = \overline{F}_1 \circ \ldots \circ \overline{F}_n(t,\, \alpha t) \quad \text{for } t \in [0;\, x(m_n)]. \tag{5.21}$$

The curve $\overline{L}$ goes to infinity because, in view of (5.16),

$$x(z(t)) = x\left(\overline{F}_1 \circ \ldots \circ \overline{F}_n(t,\, \alpha t)\right) \geqslant x(t,\, \alpha t) = t.$$

One can show that $\overline{L}$ has an asymptotic direction defined by the ray l. Thus, $L \stackrel{\text{def}}{=} \pi(\overline{L})$ is the required curve on the torus (see details in [9–11, 20]). $\diamondsuit$

Example 5.3 *Curve with an irrational asymptotic direction on a hyperbolic surface.*

In the case of hyperbolic surfaces, we will construct a foliation that has a nontrivially recurrent leaf with unbounded deviation from the co-asymptotic geodesic. It will be clear from the construction that a deviation can be made unbounded either only on one side or on both sides. A similar example can be constructed on any closed non-orientable surface of genus $p \geqslant 5$ (one needs at least two handles).

On a closed orientable surface $M^2_{g_1}$ of genus $g_1 \geqslant 2$, consider an irrational foliation $\mathscr{F}_1$ that has a topological saddle s_1 with $k \geqslant 3$ separatrices (hence, the index of the saddle is equal to $\operatorname{ind} s_1 = 1 - \frac{k}{2}$). Since all saddles of the foliation $\mathscr{F}_1$ have a negative index, $\mathscr{F}_1$ is a widely disposed foliation with respect to any closed transversal and a transversal segment. On another surface $M^2_{g_2}$ of genus $g_2 \geqslant 2$, take a Denjoy-type foliation $\mathscr{F}_2$ with a minimal set $\Omega(\mathscr{F}_2)$ such that the set $M^2_{g_2} - \Omega(\mathscr{F}_2)$ has a component S_2 of index $\operatorname{ind} s_1$ (Fig. 5.25). Let us place the saddle s_1 inside a disk D_1 whose boundary ∂D_1 is transversal to the foliation $\mathscr{F}_1$ everywhere except for points $a_1, \ldots, a_k \in \partial D_1$ that are arranged in the order corresponding to the positive (counterclockwise) orientation of ∂D_1. Without loss of generality, we may assume that the leaves passing through the points $a_1, \ldots, a_k$ are pairwise different and are not separatrices of any saddles of the foliation $\mathscr{F}_1$. Between the points a_i and a_{i+1}, $i = 1, \ldots, k$ (where $a_{k+1} = a_1$), on ∂D_1, there is a unique point of intersection of a separatrix of the saddle s_1 with ∂D_1, which we denote by c_i, such that the arc $(s_1;\, c_i)$ of the separatrix does not intersect ∂D_1 (we assume that $c_{k+1} = c_1$). Then, the foliation $\mathscr{F}_1$, considered in D_1, induces the fiber mapping (or, what is the same, the first-return map)

$$\phi_1 \colon \partial D_1 - \bigcup_{i=1}^{k} c_i \to \partial D_1 - \bigcup_{i=1}^{k} c_i,$$
$$\phi_1|_{(c_i;\, a_i]} : (c_i;\, a_i] \to [a_i;\, c_{i+1}),$$
$$\phi_1|_{[a_i;\, c_{i+1})} : [a_i;\, c_{i+1}) \to (c_i;\, a_i], \quad i = 1, \ldots, k.$$

By the construction, $\phi_1^2 = \mathrm{id}$ (see Fig. 5.25).

In the component S_2, take an open disk $D_2 \subset S_2$ whose boundary ∂D_2 intersects $\Omega(\mathscr{F}_2)$ only at points $b_1, \ldots, b_k \in \partial D_2$ that are arranged in the order corresponding to the negative (clockwise) orientation of ∂D_2. In addition, let

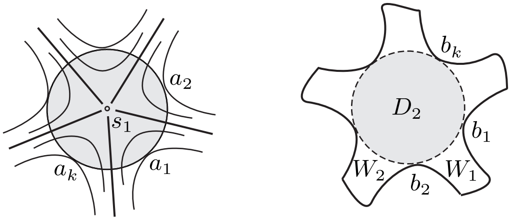


Figure 5.25. The saddle s_1 and the component S_2 of index $\operatorname{ind} s_1$.

us require that the disk D_2 divides S_2 into k domains W_i, $i = 1, \dots, k$, that are homeomorphic to an open strip (see Fig. 5.25). Since the index of the component S_2 is $\operatorname{ind} s_1 = 1 - \frac{k}{2}$, this can be done.

Let us declare that the points a_i and c_i, $i = 1, \dots, k$, are the singularities of the foliation $\mathscr{F}_1$, and denote the foliation obtained by $\mathscr{F}_1'$. Note that since the leaves passing through the points $a_1, \dots, a_k$ are pairwise different and are not separatrices, any one-dimensional leaf of the foliation $\mathscr{F}_1'$ different from the leaves of the form $(s_1; c_i)$ is everywhere dense on $M^2_{g_1}$.

Let us modify the foliation $\mathscr{F}_2$ by placing a Reeb foliation in each strip W_i, $i = 1, \dots, k$, as shown in Fig. 5.26, and declaring each point of the set $\Omega(\mathscr{F}_2)$ a singularity. The points $d_i \in (b_i; b_{i+1}) \subset \partial D_2$, $i = 1, \dots, k$, are chosen arbitrarily, where $b_{k+1} = b_1$. The foliation is extended arbitrarily into the interior of D_2. In addition, declare the points b_i and d_i, $i = 1, \dots, k$, singularities and denote the foliation obtained by $\mathscr{F}_2'$. Below, we specify the foliation $\mathscr{F}_2'$ on every strip W_i. It follows from the construction that the restriction of $\mathscr{F}_2'$ to W_i induces a fiber homeomorphism

$$\phi_2 \colon \partial D_2 - \bigcup_{i=1}^{k} b_i \to \partial D_2 - \bigcup_{i=1}^{k} b_i,$$
$$\phi_2|_{(b_i;\, d_i]} \colon (b_i; d_i] \to [d_i; b_{i+1}),$$
$$\phi_2|_{[d_i;\, b_{i+1})} \colon [d_i; b_{i+1}) \to (b_i; d_i], \quad i = 1, \dots, k.$$

Obviously, $\phi_2^2 = \mathrm{id}$.

Let $\Theta \colon \partial D_1 \to \partial D_2$ be an orientation-reversing homeomorphism such that

- $\Theta \colon [c_i; c_{i+1}] \to [b_i; b_{i+1}]$ for $i = 1, \dots, k$;
- $\Theta(a_i) = d_i$ for $i = 1, \dots, k$, where $d_{k+1} = d_1$ (see Fig. 5.27).

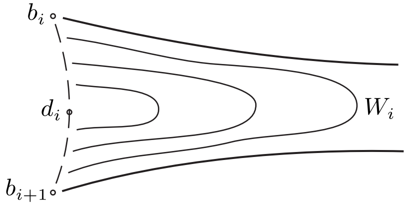


Figure 5.26. A Reeb foliation in the strip W_i.

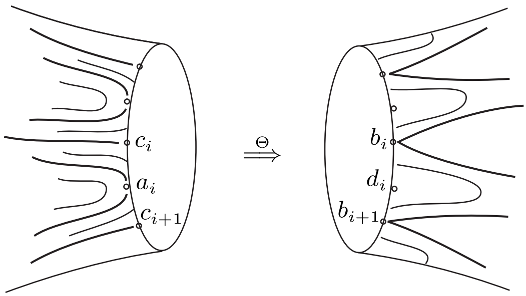


Figure 5.27. The homeomorphism $\Theta\colon \partial D_1 \to \partial D_2$.

Now, let us specify the foliation $\mathscr{F}_2'$ on each strip W_i. Since

$$\Theta\left(\partial D_1 - \bigcup_{i=1}^{k} c_i\right) = \partial D_2 - \bigcup_{i=1}^{k} b_i,$$

the following mapping is defined:

$$\Theta \circ \phi_1 \circ \Theta^{-1}\Big|_{\partial D_2 - \bigcup\limits_{i=1}^{k} b_i}, \quad \text{where} \quad \Theta \circ \phi_1 \circ \Theta^{-1}(d_i) = d_i, \quad i = 1, \dots, k.$$

It can easily be verified that this mapping is an involution. Finally, we impose the last restriction on $\mathscr{F}_2'$ by requiring that, for each strip W_i, the following relation holds:

$$\phi_2 = \Theta \circ \phi_1 \circ \Theta^{-1}. \tag{5.22}$$

Let us glue together two surfaces $M^2_{g_1} - \operatorname{Int} D_1$ and $M^2_{g_2} - \operatorname{Int} D_2$ by $\Theta\colon \partial D_1 \to \partial D_2$. As a result, we obtain a closed surface $M^2_{g_1+g_2}$ of genus $g_1 + g_2 \geqslant 4$, which is the connected sum $M^2_{g_1} \sharp M^2_{g_2}$ of the surfaces $M^2_{g_1}$

and $M^2_{g_2}$. The foliations $\mathscr{F}'_1$ and $\mathscr{F}'_2$ form a foliation on $M^2_{g_1+g_2}$, which we denote by $\mathscr{F}$. There are natural embeddings

$$i_1\colon M^2_{g_1} - \operatorname{Int} D_1 \subset M^2_{g_1+g_2}, \quad i_2\colon M^2_{g_2} - \operatorname{Int} D_2 \subset M^2_{g_1+g_2},$$

by which objects (for example, the leaves of foliations) on the surfaces $M^2_{g_1}$ and $M^2_{g_2}$ are identified with analogous objects on $M^2_{g_1+g_2}$. For simplicity, we will often omit the symbols i_1 and i_2 when it is clear from the context which objects are meant.

Take a leaf l_1 of the foliation $\mathscr{F}'_1$ that is not a separatrix of the singularity s_1 or a_i, $i = 1, \ldots, k$. Then, l_1 is a nontrivially recurrent leaf that is everywhere dense on $M^2_{g_1}$. By virtue of the equality (5.22), the arcs $l_1 - \operatorname{Int} D_1$ of this leaf are continued by the arcs of the foliation $\mathscr{F}'_2|_{D_2}$, and we obtain a leaf l of the foliation $\mathscr{F}$ (actually, each arc from $l_1 \cap D_1$ is replaced by the corresponding arc of the foliation $\mathscr{F}'_2|_{W_i}$, and the endpoints of these arcs are identified by Θ). It follows from (5.22) and the density of l_1 on $M^2_{g_1}$ that the leaf l is everywhere dense on $M^2_{g_1} - \operatorname{Int} D_1 \subset M^2_{g_1+g_2}$. In particular, l is a nontrivially recurrent leaf.

Let us show that l possesses the property of unbounded deviation. Since $g_1 + g_2 \geqslant 4$, one can assume that $M^2_{g_1+g_2}$ is a hyperbolic surface. One can also assume that the curve

$$\Theta(\partial D_1) = \partial D_2 \stackrel{\text{def}}{=} g_0 \subset M^2_{g_1+g_2}$$

is a geodesic (this can be done with the use of an isotopy of the foliation $\mathscr{F}$). Fix the open strip $W_1 \stackrel{\text{def}}{=} W$. The internally accessible boundary of this strip consists of the separatrices $\mathscr{S}_1, \mathscr{S}_2 \subset M^2_{g_2} - \operatorname{Int} D_2$ of the singularities b_1 and b_2, respectively, and the segment

$$\Theta([c_1; c_2]) = [b_1; b_2] \stackrel{\text{def}}{=} \Sigma \subset g_0.$$

Take a lift $\overline{W}$ of the strip W; it is bounded by the corresponding lifts $\overline{\mathscr{S}}_1$, $\overline{\mathscr{S}}_2$, and $\overline{\Sigma} \subset \overline{g}_0$. Since $\mathscr{F}_2$ is a Denjoy-type foliation, the "width" of the strip W tends uniformly to zero. Hence, $\overline{\mathscr{S}}_1$ and $\overline{\mathscr{S}}_2$ have the same asymptotic direction, say $\sigma \in S_\infty$.

The ideal endpoints σ_0^+, $\sigma_0^- \in S_\infty$ of the geodesic $\overline{g}_0$ divide the absolute into two arcs. Denote by I_σ the arc that contains the point σ. Note that since $W \cap g_0 = \varnothing$, we have $\overline{W} \cap \overline{g}_0 = \varnothing$. Therefore, $\overline{W}$ lies in one of the half-planes into which $\overline{g}_0$ divides Δ. Denote by Δ^- the half-plane that does not contain the strip $\overline{W}$. Take the lift $\overline{l}$ of the leaf l that intersects the segment $\overline{\Sigma}$. It follows from the construction that there exists an arc $A \subset \overline{l} \cap \overline{W}$ with endpoints e_1 and e_2 on $\overline{\Sigma}$ (see Fig. 5.28). Let us show that $\overline{l} - A \subset \Delta^-$. In other words, the semileaves $\overline{l}^-(e_1) \subset \overline{l}$ and $\overline{l}^+(e_2) \subset \overline{l}$ that do not contain the arc A do not intersect the geodesic $\overline{g}_0$.[2] Suppose the contrary. Assume, for definiteness, that $\overline{l}^+(e_2) \cap \overline{g}_0 \neq \varnothing$. Then, there exists a $\overline{g}_0$-arc A_{23} of the semileaf $\overline{l}^+(e_2)$ with endpoints e_2 and e_3 belonging to $\overline{l}^+(e_2) \cap \overline{g}_0$. The closed curve C formed by the $\overline{g}_0$-arc A_{23} and the segment $[e_2;\, e_3]$ of the geodesic $\overline{g}_0$ is null-homotopic. Therefore, $\pi(C)$ is also a g_0-arc of the leaf l that is null-homotopic on the surface $M^2_{g_1+g_2}$. It follows from the definition of the g_0-arc that $\pi(C) \subset M^2_{g_1}$. Hence, the foliation $\mathscr{F}_1$ is not widely disposed. On the other hand, as an irrational foliation with saddles of nonpositive index, $\mathscr{F}_1$ is a widely disposed foliation. The contradiction obtained shows that $\overline{l}$ intersects $\overline{g}_0$ only at the endpoints e_2 and e_3 of the arc A.

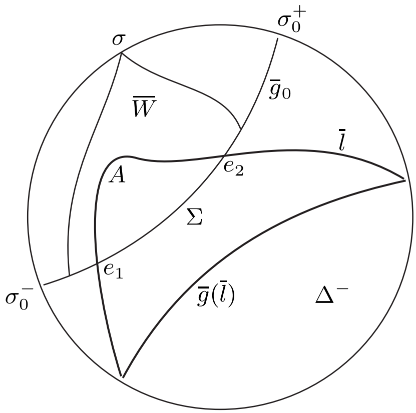


Figure 5.28. The arc $A \subset \overline{l} \cap \overline{W}$.

The fact that $\overline{l} - A \subset \Delta^-$ implies that the corresponding geodesic $\overline{g}(\overline{l})$ lies in Δ^-. Therefore, for any point $\overline{z} \in A$, we have

$$\overline{d}_{\mathrm{NE}}(\overline{z},\, \overline{g}_0) \leqslant \overline{d}_{\mathrm{NE}}(\overline{z},\, \overline{g}(\overline{l})).$$

[2]We assume that the leaf $\overline{l}$ is oriented from e_1 to e_2.

In particular, this inequality holds for the point $\overline{z}_{\max}(A) \in A$ that corresponds to the maximal deviation of the arc A from the geodesic $\overline{g}_0$,

$$\overline{d}_{\mathrm{NE}}(\overline{z}_{\max}(A), \overline{g}_0) \leqslant \overline{d}_{\mathrm{NE}}(\overline{z}_{\max}(A), \overline{g}(\overline{l})), \quad \overline{z}_{\max}(A) \in A.$$

It follows from the construction and the fact that the leaf l is everywhere dense on the surface that there exists a sequence of leaves $\overline{l}_k$ that are congruent to the leaf $\overline{l}$ and intersect the strip $\overline{W}$ along arcs $A_k = \overline{W} \cap \overline{l}_k$ such that A_k indefinitely approach the point σ in the Euclidean metric. In other words,

$$\overline{d}_{\mathrm{NE}}(\overline{z}_{\max}(A_k), \overline{g}_0) \to \infty \quad \text{as } k \to \infty.$$

Hence,

$$\overline{d}_{\mathrm{NE}}(\overline{z}_{\max}(A_k), \overline{g}(\overline{l}_k)) \to \infty \quad \text{as } k \to \infty.$$

Therefore, there exists a sequence of points $\overline{m}_k \in \overline{l}$ such that

$$\overline{d}_{\mathrm{NE}}(\overline{m}_k, \overline{g}(\overline{l})) \to \infty \quad \text{as } k \to \infty.$$

Since the leaf $\overline{l}$ tends, in the Euclidean metric, to one point of the absolute in either (the positive or negative) direction, there is a subsequence of points among $\overline{m}_k \in \overline{l}$ whose parameters tend to infinity, while the points themselves tend to the absolute in the Euclidean metric. This means that the leaf $\overline{l}$ possesses the property of unbounded deviation. $\diamondsuit$

Example 5.4 *Curve with a rational asymptotic direction.*

The construction of a curve with a rational direction and with the property of unbounded deviation is a more complicated and cumbersome problem as compared to the construction of a curve with an irrational asymptotic direction. Therefore, we restrict ourselves only by a description of the idea of the construction (see details in [9, 20]).

Figures 5.29 and 5.30 show the first steps in the construction of a required simple semi-infinite curve $L = \{z(t) \colon t \geqslant 0\}$ on the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ that has a covering $\overline{L}$ on $\mathbb{R}^2$. The curve $\overline{L}$ has an asymptotic direction defined by the positive half-axis Ox, $\frac{y(t)}{x(t)} \to 0$ as $x(t) \to \infty$, and unboundedly deviates from Ox, $\sup y(t) = \infty$. The construction is carried out on $\mathbb{R}^2$. The curve $\overline{L}$ is shown in the figures by a heavy line; it starts at $(0, 0)$. Integer translations of some of its arcs are shown by thin lines.

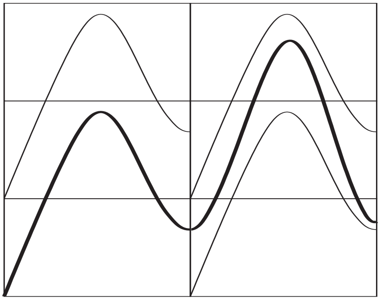


Figure 5.29

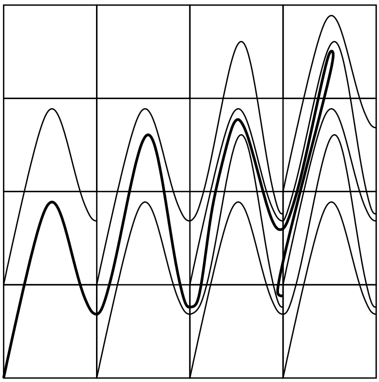


Figure 5.30

One can construct the required curves on the torus and the Klein bottle in such a way that they are not everywhere dense. Using these examples, one can construct similar curves with the property of unbounded deviation on any closed surface of negative Euler characteristic, by gluing a necessary number of handles and Möbius bands onto "free" parts of the torus or the Klein bottle.

Example 5.5 (*Markley – Vanderschoot*).

Here, we give a sketch of the example from [152] with a semitrajectory that has a rational asymptotic direction and possesses the property of unbounded deviation. Although the existence of such a flow follows from the previous examples and Theorem 5.11, it has other interesting (in our view) properties. Let us represent a closed ring A as a quotient space $\overline{A}/\Gamma$, where $\overline{A} = \{(x, y)\colon -1 \leqslant y \leqslant +1\}$ and the group Γ consists of integer translations $(x, y) \mapsto (x+n, y)$. Define on A a vector field $\vec{X}$ with a phase portrait whose lift is shown in Fig. 5.31. Denote by $\pi\colon \overline{A} \to A$ the natural projection. The boundary component $l_{+1} = \pi(y = +1)$ contains a finite number of fixed points $s_1, \dots, s_k$, and the other boundary component $l_{-1} = \pi(y = -1)$ is a periodic trajectory of the field $\vec{X}$. The neighboring points $s_1, \dots, s_k$ are connected by separatrix connections $L_1, \dots, L_k$ (see Fig. 5.31). These separatrix connections and the points $s_1, \dots, s_k$ form a closed curve, which we denote by L_*. In the domain between l_{-1} and L_*, any trajectory, say l, winds around L_* in the positive direction and around l_{-1} in the negative direction; i. e., $\omega(l) = L_*$, $\alpha(l) = l_{-1}$.

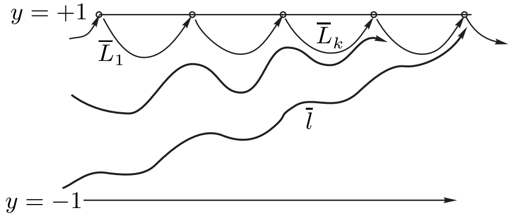


Figure 5.31. The phase portrait of the covering vector field for $\vec{X}$.

Take a diffeomorphism of the half-open ring $h\colon A - l_{+1} \to A - l_{+1}$ whose covering diffeomorphism has the form $\overline{h}(x, y) = (x + \lambda(y), y)$, where $\lambda\colon [-1; +1) \to \mathbb{R}^+$ is a strictly increasing C^∞ function such that $\lambda(y) \to +\infty$ as $y \to +1$. The field $\vec{X}$ is mapped by h into a smooth vector field $h_*(\vec{X})$ on $A - l_{+1}$. Take a smooth function $\rho\colon A \to \mathbb{R}^+$ that is strictly positive on $A - l_{+1}$ and, together with its derivatives, tends uniformly to zero as the argument tends to l_{+1}. If ρ, together with its derivatives, tends to zero sufficiently rapidly, then the field $\rho h_*(\vec{X})$ can be extended to a smooth vector field (which we denote by the same symbol) over the whole ring A, such that the curve l_{+1} consists of fixed points. The phase portrait of the covering vector field for $\rho h_*(\vec{X})$ is shown in Fig. 5.32. One can easily see that the trajectory $l_* \overset{\text{def}}{=} h(l)$ possesses the following properties:

1) the ω-limit set $\omega(l_*)$ of the trajectory l_* consists of one-dimensional trajectories $h(L_1), \dots, h(L_k)$ and the curve l_{+1};

2) $\alpha(l_*) = l_{-1}$;
3) the domain bounded by the trajectories $\bar{l}_*$ and $y = -1$ of the covering flow is simply connected.

Note that each of the trajectories $h(L_i)$, $1 \leqslant i \leqslant k$, bounds a domain D_i in the ring whose internally accessible boundary consists of $h(L_i)$. Let us remove two disks from a certain domain D_i and modify $\rho h_*(\vec{X})$ so that the boundary of one of the disks consists of fixed points and the boundary of the other disk is a periodic trajectory. Then, we remove these disks and glue their boundaries with l_{-1} and l_{+1} in a natural way. We obtain a pretzel M_2^2 on which $\rho h_*(\vec{X})$ induces a flow f^t with the following properties.

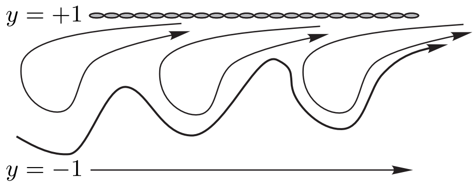


Figure 5.32. The phase portrait of the covering vector field for $\rho h_*(\vec{X})$.

1. The flow f^t has a periodic trajectory l_0 and a simple closed curve l^{fix} that consists of fixed points; l_0 and l^{fix} are non-null-homotopic on M_2^2 and are not homotopic to each other.

2. There are one-dimensional trajectories l, l_1, ..., l_k such that

- $\alpha(l_i) = \omega(l_i) = l^{\text{fix}}$ for $i = 1, \ldots, k$;
- $\omega(l) = l^{\text{fix}} \cup \bigcup\limits_{i=1}^{k} l_i$;
- $\alpha(l) = l_0$.

3. The trajectory l winds around l_0 in a spiral-like fashion in the negative direction, and the asymptotic directions of l and l_0 coincide.

4. The trajectory l does not wind around l_0 in a spiral-like fashion in the positive direction; however, l and l_0 have the same (rational) asymptotic direction.

5. The trajectory l possesses the property of unbounded deviation.

Note that the ω-limit set of the trajectory l is locally disconnected.

Bounded deviation of special curves

First, we consider the property of bounded deviation for semitrajectories of topological flows on the torus — an orientable closed surface of zero curvature. In this case, the sufficient conditions for the boundedness of deviation obtained thus far are of more general character than those in the case of semitrajectories of topological flows on hyperbolic surfaces. Next, we consider the property of bounded deviation for semitrajectories of analytic flows. Finally, we consider semileaves of foliations with saddle singularities and invariant manifolds of points of nontrivial basic sets, respectively.

Semitrajectories of topological flows on the torus

Theorem 5.35 *Let f^t be a topological flow on the torus $\mathbb{T}^2$ and $\overline{l}$ be a lift to $\mathbb{R}^2$ of a semitrajectory $l = \pi(\overline{l})$ that has an asymptotic direction. Suppose that one of the following conditions is fulfilled*:

- *the set of fixed points of the flow f^t is contractible to a point*;
- *l is a nontrivially recurrent semitrajectory.*

Then, $\overline{l}$ possesses the property of bounded deviation.

Scheme of the proof. If l is a closed trajectory, then it is non-null-homotopic since it has an asymptotic direction, and the result is obvious. It remains to consider the case of a nonclosed l. One of the key steps in the proof of both assertions is to prove the existence of a simple non-null-homotopic closed transversal (in the topological sense) that is intersected by the semitrajectory l. For a nontrivially recurrent semitrajectory, this fact is valid without additional assumptions concerning the set of fixed points in view of Corollary 3.5. When the semitrajectory l is not nontrivially recurrent, this fact follows from the existence of an open transversal segment that is intersected by l at least twice. The proof of this result requires delicate arguments involving assumptions concerning the set of fixed points.

After proving the existence of a simple non-null-homotopic closed transversal, we may assume, without loss of generality, that this transversal is the zero meridian of the torus, $C_0 = \pi(x = 0)$. If l intersects C_0 a finite number of times, then, starting from a certain moment, the covering semitrajectory $\overline{l}$ goes to infinity, while remaining in a certain strip $k \leqslant x \leqslant k + 1$. Hence, $\overline{l}$ boundedly deviates from the axis Oy, which determines the asymptotic direction of $\overline{l}$. If l intersects C_0 infinitely many times, then $\overline{l}$ has an asymptotic direction defined by a straight line $y = ax$. First, we prove that the points of intersection of $\bar{l}$ with the straight lines $x = n$ boundedly deviate from the points of intersection of the straight line $y = ax$ with the same lines $x = n$ and then, using arcs congruent to arcs of the semitrajectory $\bar{l}$, we show that the deviation of the whole $\bar{l}$ from the straight line $y = ax$ is bounded. (In a far-fetched interpretation, we can regard f^t as a flow obtained by blowing up a suspension over a homeomorphism of the circle C_0, i. e., as a flow that is semiconjugate to a suspension over C_0. The latter is topologically equivalent to a linear flow via a homeomorphism homotopic to the identity.) □

Let us give without proof an elegant result by Markley that was proved in [150].

Theorem 5.36 *Let f^t be a topological flow on the torus $\mathbb{T}^2$ and $\bar{l}$ be a lift to $\mathbb{R}^2$ of a positive semitrajectory $l = \pi(\bar{l})$ that has an asymptotic direction. Suppose that the ω-limit set of l contains a regular point* (*that is not a fixed point*)*. Then, $\bar{l}$ possesses the property of bounded deviation.*

Semitrajectories of flows on hyperbolic surfaces

Theorem 5.37 *Let f^t be a topological flow with a finite set of fixed points on a closed hyperbolic surface M. Let $\bar{l}^+$ be a positive semitrajectory of a covering flow $\bar{f}^t$ on $\overline{M} = \Delta$ that has an asymptotic direction. Then, $\bar{l}^+$ possesses the property of bounded deviation.*

Proof. Without loss of generality, we can assume that the surface M is orientable. Let $l^+ = \pi(\bar{l}^+)$, and denote by l a trajectory that contains l^+, $l^+ \subset l$. If l is a periodic trajectory, then the assertion is obvious. Therefore, we will assume in what follows that l is a nonclosed trajectory. It is well-known (see, for example, [3,26,172]) that the ω-limit set of a nonclosed semitrajectory of a flow with a finite number of fixed points on a compact surface falls under one of the following types:

- a fixed point;
- a periodic trajectory;
- a one-sided loop consisting of fixed points and separatrix connections;
- a quasiminimal set.

Since $\bar{l}^+$ has an asymptotic direction and, hence, goes to infinity, $\omega(l^+)$ can not be a fixed point. Other limit sets are realized under the stipulation that a periodic trajectory and a one-sided loop are non-null-homotopic. It is obvious that a periodic trajectory and a one-sided loop define a homotopy class that contains a simple closed non-null-homotopic curve. Therefore, this homotopy class contains a simple closed geodesic that is a co-asymptotic geodesic for the periodic trajectory or for the one-sided loop. If $\omega(l^+)$ is a non-null-homotopic periodic trajectory l_0, then l^+ approaches l_0 in a spiral-like way. Hence, $\overline{l}^+$ possesses the property of bounded deviation because l_0 possesses this property and l^+ and l_0 have the same co-asymptotic geodesic. Similar reasoning applies to the case when $\omega(l^+)$ is a non-null-homotopic closed loop.

Therefore, it remains to consider the case when $\omega(l^+)$ is a quasiminimal set. There are three possibilities:

1) l^+ belongs to the nontrivially recurrent trajectory l;
2) l^+ is a nontrivially recurrent semitrajectory (in the positive direction) but belongs to the trajectory l, which is not nontrivially recurrent in the negative direction;
3) l^+ is not a nontrivially recurrent semitrajectory.

The first possibility is essential, and the major part of the proof is devoted to this case. The last two possibilities are reduced to the first one.

Proof in the case when l^+ belongs to the nontrivially recurrent trajectory l. Suppose that the theorem does not hold in this case. Let $\overline{l}$ be one of the lifts of l to Δ that contains the lift $\overline{l}^+$. Then, the limit set of the trajectory $\overline{l}$ consists of two different points $\sigma^+ = \omega(\overline{l}) = \omega(\overline{l}^+) \in S_\infty$ and $\sigma^- = \alpha(\overline{l}) \in S_\infty$. Let $\overline{g}(\overline{l})$ be a co-asymptotic geodesic for the trajectory $\overline{l}$.

Denote by $\overline{m}(t) \in \overline{l}$ a current point on the trajectory $\overline{l}$ that corresponds to time t. Assume that $\overline{m}(t) \to \sigma^{+(-)}$ as $t \to +\infty\,(-\infty)$. Let $[\overline{m}(t); \overline{m}_0(t)]$ be a perpendicular dropped from the point $\overline{m}(t) \in \overline{l}$ to the geodesic $\overline{g}(\overline{l})$, where $\overline{m}_0(t) \in \overline{g}(\overline{l})$. Denote by $\overline{d}(t)$ the length of the perpendicular $[\overline{m}(t); \overline{m}_0(t)]$ (Fig. 5.33). Since l possesses the property of unbounded deviation, there exists a sequence t_n such that $\overline{d}(t_n) \to +\infty$ as $t_n \to +\infty$. Passing, if necessary, to a subsequence, we may assume that the sequence $\pi(\overline{m}_0(t_n)) \in M$ converges to a certain point $m_0 \in M$.

Take an arbitrary point $\overline{m}_0 \in \pi^{-1}(m_0)$. Since the sequence $\pi(\overline{m}_0(t_n))$ converges to m_0, there exists a sequence of deck transformations $\gamma_k \in \Gamma$ such that

$$\gamma_k(\overline{m}_0(t_k)) \stackrel{\text{def}}{=} \overline{m}_k \to \overline{m}_0 \quad \text{as } t_k \to +\infty.$$

By Lemma 5.18, $g(l) = \pi(\overline{g}(\overline{l}))$ is a nonclosed B-recurrent geodesic. Denote by $G(Q)$ the geodesic framework of the quasiminimal set $\operatorname{clos} g(l) =$

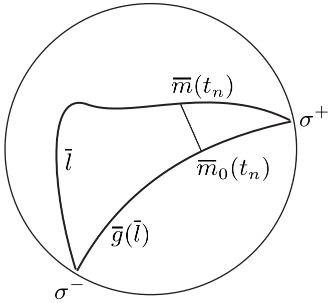


Figure 5.33. The sequence $\overline{m}(t_n)$.

$= Q$. According to Lemma 3.41, $G(Q)$ is a minimal geodesic lamination consisting of nonclosed geodesics that are everywhere dense in $G(Q)$.

Since the support of a geodesic lamination is a closed set, the point m_0 belongs to $G(Q)$ and a geodesic $g_0 \in G(Q)$ passes through this point. Hence, a covering geodesic $\overline{g}_0 \in \overline{G}(\overline{Q})$ of g_0 passes through the point $\overline{m}_0$. Set $\gamma_k(\overline{g}(\bar{l})) \overset{\text{def}}{=} \overline{g}_k$. All geodesics $\overline{g}_k$ are covering geodesics for $g(l)$, and we equip them with an orientation induced by the orientation of the geodesic $g(l)$. Denote by σ_k^+ and σ_k^- the ideal endpoints of the geodesic $\overline{g}_k$. Since the geodesics in the lamination are pairwise disjoint and $\overline{m}_k \to \overline{m}_0$, the points σ_k^+ and σ_k^- approach the corresponding ideal endpoints σ_0^+ and σ_0^- of the geodesic $\overline{g}_0$ in the Euclidean metric on S_∞. Since the geodesic $g_0 \in G(Q)$ is nonclosed, we may assume without loss of generality that the geodesics $\overline{g}_k$ are pairwise disjoint, the points $\overline{m}_k$ are also pairwise different, and none of these points coincides with the point $\overline{m}_0$.

Denote by Δ^+ and Δ^- the open half-planes into which $\overline{g}_0$ divides the plane Δ. Assume, for definiteness, that $\overline{m}_k \in \Delta^-$. Denote by $\overline{g}_\perp$ the geodesic that passes through the point $\overline{m}_0$ and is perpendicular to $\overline{g}_0$, and denote by $\overline{p}_+, \overline{p}_- \in S_\infty$ the endpoints of the geodesic $\overline{g}_\perp$. The geodesic segment $\gamma_k([\overline{m}(t_k); \overline{m}_0(t_k)]) \overset{\text{def}}{=} \overline{p}_k$ is a perpendicular to the geodesic $\overline{g}_k$ erected at the point $\overline{m}_k$. Since $\overline{m}_k \to \overline{m}_0$ and $\overline{d}(t_k) \to +\infty$, the endpoint $\gamma_k(\overline{m}(t_k)) \overset{\text{def}}{=} \overline{m}_k^+$ of the perpendicular $\overline{p}_k$ must lie in Δ^+ for sufficiently large k (Fig. 5.34). Therefore, the geodesic ray $[\overline{m}_0; \overline{p}_+] \overset{\text{def}}{=} \overline{g}_\perp^+$ of the geodesic $\overline{g}_\perp$ lies in the half-plane Δ^+.

Denote by $\overline{K}$ the topological limit of the augmented trajectories $\bar{l}_n \cup \sigma_n^+ \cup \cup \sigma_n^-$ on the augmented Lobachevsky plane $\Delta \cup S_\infty$. By the definition of the

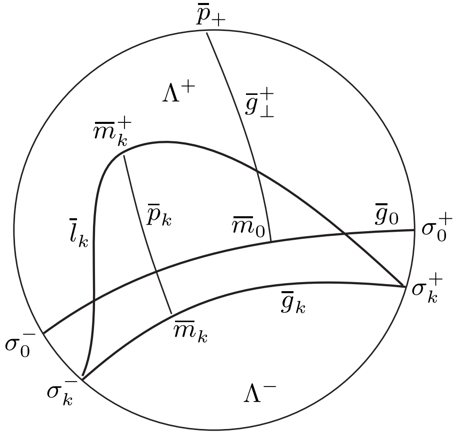


Figure 5.34. The perpendicular $\overline{g}_\perp^+$.

topological limit, a point $\overline{x}$ belongs to the set $\overline{K}$ if there exists a sequence of points $\overline{x}_n \in \overline{l}_n \cup \sigma_n^+ \cup \sigma_n^-$ that converges to $\overline{x}$ as $n \to \infty$ in the Euclidean metric on $\Delta \cup S_\infty$. It is obvious that σ_0^-, σ_0^+, $\overline{p}_+ \in \overline{K}$. The set $\overline{K}$ is connected as the topological limit of connected arcs. Since each trajectory $\overline{l}_n$ divides the Lobachevsky plane, $\overline{K}$ is nowhere dense and is closed in $\Delta \cup S_\infty$. By virtue of the theorem on the continuous dependence of trajectories on the initial conditions, the set $\overline{K} \cap \Delta \overset{\text{def}}{=} \overline{K}_\Delta$ is invariant; i. e., it consists of trajectories of the covering flow.

According to Lemma 5.19, the intersection of $\overline{K}$ with S_∞ does not contain nontrivial intervals. Hence, $\overline{K}_\Delta \neq \varnothing$. Note that the set $\pi(\overline{K}_\Delta)$ lies in the limit set of the trajectory l. It is well-known (see, for example, [26, Ch. 2, p. 54]) that the ω-limit set of a nontrivially recurrent trajectory is equal to its α-limit set, and both of them coincide with the topological closure of this trajectory:

$$\pi(\overline{K}_\Delta) \subset \operatorname{clos} l = \omega(l) = \alpha(l) = Q.$$

Let us show that $\pi(\overline{K}_\Delta)$ contains a trajectory that is nontrivially recurrent in the positive direction. The points σ_0^+ and σ_0^- divide the absolute into two intervals. Denote by $I_+ \subset S_\infty$ the interval that contains the point $\overline{p}_+$. Since $\overline{p}_+ \in \overline{K}$ and $\overline{K} \cap S_\infty$ does not contain nontrivial intervals, there exists a connected invariant subset $K_* \subset \overline{K}_\Delta$ whose closure in the Euclidean metric on $\Delta \cup S_\infty$ contains a point $k_1 \in I_+$. In particular, $k_1 \notin \{\sigma_0^+, \sigma_0^-\}$. Since the number of fixed points is finite, K_* contains at least one one-dimensional trajectory, say $\bar{l}_0$. In this case, $l_0 \subset \omega(l)$, and l indefinitely approaches the trajectory $\pi(\bar{l}_0) \stackrel{\text{def}}{=} l_0$ either from the right or from the left. It suffices to show that K_* contains a one-dimensional trajectory that is projected onto a nontrivial ω-recurrent trajectory. If $\omega(l_0)$ contains one-dimensional trajectories, we apply the Maier theorem [142, Theorem 1] (see also Theorem 2.2 in [26] or Theorem 2.4.4 in [172]), which states that if a nonclosed trajectory lies in the limit set of a certain trajectory and contains one-dimensional trajectories in its ω-limit set, then this trajectory is nontrivially recurrent in the positive direction. By virtue of Theorem 1 from [142], l_0 is a nontrivially ω-recurrent trajectory, and there is nothing to prove. Suppose that the ω-limit set of the trajectory l_0 does not contain one-dimensional trajectories. Then, l_0 tends to a fixed point m_0 in the positive direction, and $\omega(l_0) = m_0$. Denote by U_0 a neighborhood of the point m_0 that contains exactly one fixed point m_0. Since $l_0 \subset \omega(l)$ and l is a nontrivially recurrent trajectory, l enters and leaves the neighborhood U_0. Assume, for definiteness, that l indefinitely approaches l_0 from the left. Then, l_0 has a left Bendixson extension in the positive direction (which we denote by l_1) with respect to the neighborhood U_0. In this case, $\alpha(l_1) = m_0$, $l_1 \subset \omega(l)$, and l_1 leaves U_0. It is easily seen that l_1 belongs to $\pi(\overline{K}_\Delta)$, $l_1 \subset \pi(\overline{K}_\Delta)$. The theorem on the continuous dependence on the initial conditions implies that l indefinitely approaches l_1 from the left. If the ω-limit set of the trajectory l_1 does not contain one-dimensional trajectories, then l_1 tends to a fixed point m_1 in the positive direction, and $\omega(l_1) = m_1$. Note that m_1 may coincide with m_0. There is a neighborhood U_1 of the point m_1 that contains exactly one fixed point m_1 (if $m_1 = m_0$, then $U_1 = U_0$). Then, l_1 has a left Bendixson extension in the positive direction (which we denote by l_2) with respect to the neighborhood U_1. Continuing this process, we obtain sequences of one-dimensional trajectories $l_1, l_2, \dots, l_k, \dots$ and fixed points $m_0, m_1, \dots, m_k, \dots$ such that

- l_k is a left Bendixson extension of the trajectory l_{k-1} in the positive direction for any $k = 1, \dots$ with respect to the neighborhood U_{k-1}, which contains exactly one fixed point m_{k-1};
- $\alpha(l_k) = m_{k-1}$ for $k \geqslant 1$;
- $l_k \subset \omega(l)$ for $k \geqslant 1$;
- l_k^+ leaves the neighborhood U_k at least once.

Since the asymptotic direction of l is irrational, the sequence l_k can not be periodic.

Let us show that the sequence l_k, $k \geqslant 1$, is finite. Suppose the contrary. Consider a trajectory l_k for which the sets $\alpha(l_k)$ and $\omega(l_k)$ coincide. For each such l_k, denote by $l_k \cup [\operatorname{Fix} f^t]$ the loop formed by l_k and $\alpha(l_k) = \omega(l_k)$. First, we show that there is a finite number of non-null-homotopic loops of the form $l_k \cup [\operatorname{Fix} f^t]$. Indeed, since all l_k lie in the limit set of the nontrivially recurrent trajectory l, it follows from the Poincaré–Bendixson theorem that, for each loop $l_k \cup [\operatorname{Fix} f^t]$, there exists at most one loop that is homotopic to it and is of the same form. This and the finiteness of the genus of the surface M imply that there is a finite number of non-null-homotopic loops of the form $l_k \cup [\operatorname{Fix} f^t]$.

Now, consider null-homotopic loops $l_k \cup [\operatorname{Fix} f^t]$. Denote by $N(\operatorname{Fix} f^t)$ the union of all neighborhoods U_k. Note that $N(\operatorname{Fix} f^t)$ is a finite union of pairwise disjoint open domains each of which contains exactly one fixed point of the flow. According to the Poincaré–Bendixson theorem, the null-homotopic loops $l_k \cup [\operatorname{Fix} f^t]$ bound disjoint disks. If there exists an infinite number of null-homotopic loops that bound disjoint disks, then the points of the trajectories l_k that enter and leave the neighborhood $N(\operatorname{Fix} f^t)$ are accumulated at a certain point on the boundary $\partial N(\operatorname{Fix} f^t)$, and this point must be a fixed point, which is impossible.

Now, consider a trajectory l_k for which the sets $\alpha(l_k)$ and $\omega(l_k)$ are different fixed points. Suppose that l_j is a trajectory for which either the ω- or the α-limit set coincides with $\alpha(l_k)$ and either the α- or the ω-limit set coincides with $\omega(l_k)$, respectively. Then, the union $\alpha(l_k) \cup \omega(l_k) \cup l_k \cup l_j$ forms a closed curve. Similar to the previous case, we can show that there may be only a finite number of such closed curves. Since the set $\operatorname{Fix} f^t$ is finite, this proves that the sequence l_k, $k \geqslant 1$, is finite.

Since the sequence l_k, $k \geqslant 1$, is finite, it ends with a trajectory (which we denote by l_*) that does not tend to a fixed point in the positive direction. Hence, the ω-limit set of l_* contains at least one one-dimensional trajectory. It follows from [142, Theorem 1] that l_* is nontrivially recurrent in the positive direction. Hence, there exists a lift $\overline{l}_*$ of a semitrajectory of l_* that belongs to K_*.

As a lift of a nontrivially recurrent semitrajectory, the positive semitrajectory of the trajectory $\overline{l}_*$ has an asymptotic direction. Since l indefinitely approaches l_* from the left, this asymptotic direction is defined by the point k_1.

By Corollary 3.5, there exists a simple closed transversal C that intersects l_*, $C \cap l_* \neq \varnothing$. In view of the nontrivial recurrence of l_*, C is non-null-homotopic, and l_* intersects C countably many times. Therefore, $\overline{l}_*$ intersects a countable family of lifts $\overline{C}_1, \overline{C}_2, \ldots$ of the curve C. It follows from Theorem 5.17 that the ideal endpoints of the curves $\overline{C}_i$ converge to the point k_1 in the Euclidean metric on S_∞. Therefore, there exists a curve $\overline{C} \in \pi^{-1}(C)$ that intersects $\overline{l}_*$ and whose ideal endpoints belong to the interval $I_+ \subset S_\infty$. In particular, the ideal endpoints of $\overline{C}$ do not separate the points σ_0^+ and σ_0^- on the absolute (Fig. 5.35). Hence, for sufficiently large k, the trajectory $\gamma_k(\overline{l}) = \overline{l}_k$ intersects $\overline{C}$ at least twice, which is impossible because $\overline{C}$ is a transversal of the covering flow $\overline{f}^t$ and no semitrajectory can intersect the transversal $\overline{C}$ at more than two points. This contradiction proves the theorem in the case when l^+ belongs to a trajectory l that is nontrivially recurrent in both directions.

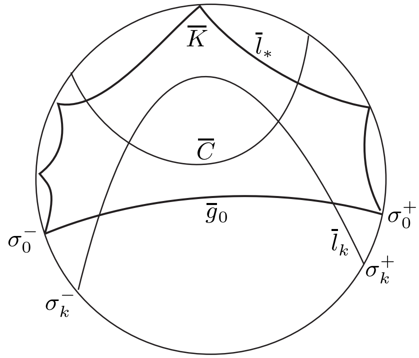


Figure 5.35. The semitrajectory $\overline{l}_*$ and the transversal $\overline{C}$.

Proof in the case when l^+ is a nontrivially recurrent positive semitrajectory that belongs to a trajectory l that is not nontrivially recurrent in the negative direction. In this case, the α-limit set of the trajectory l is a fixed point, say m_{-1}. Since l^+ is a nontrivially recurrent semitrajectory, it follows from the theorem on the continuous dependence of trajectories on the initial conditions that l^+ indefinitely approaches any point of l either from the right or from the left. Assume, for definiteness, that it approaches points of l from the left. Then, l has a left Bendixson extension in the negative direction, which we denote by l_{-1}. Let us show that l, m_{-1}, and l_{-1} belong to a generalized trajectory that consists of a finite number of trajectories. Note that l_{-1} belongs to the ω-limit set of the semitrajectory l^+ by virtue of the theorem on the continuous dependence of trajectories on the initial conditions. Hence, if the α-limit set of the semitrajectory l_{-1} contains one-dimensional trajectories, then l_{-1} is nontrivially recurrent in the negative direction [142, Theorem 1] and $l \cup m_{-1} \cup l_{-1}$ forms a generalized trajectory. Suppose that the α-limit set of the trajectory l_{-1} is a fixed point, say m_{-2}. Then, l_{-1} has a left Bendixson extension in the negative direction, which we denote by l_{-2}. Continuing this process, we obtain a sequence l_{-1}, l_{-2}, ... of one-dimensional trajectories each of which, starting from the second one, is a Bendixson extension of the preceding trajectory in the negative direction. By analogy with the case considered above, one can show that the sequence obtained is finite and ends with a trajectory that is an ω-separatrix of a certain fixed point and is nontrivially recurrent in the negative direction.

Thus, the Bendixson extensions of the trajectory l in the negative direction constitute a finite generalized trajectory L. A further proof is analogous to the case, considered above, when l is a nontrivially recurrent trajectory. Now, instead of l, we consider a generalized trajectory L that is obtained by adding a finite number of separatrix connections and fixed points to l. Since L starts and ends with nontrivially recurrent semitrajectories, the addition of a compact invariant arc allows us to repeat the proof with unessential changes.

Proof in the case when l^+ is not a nontrivially recurrent semitrajectory. In this case, the ω-limit set of the semitrajectory l^+ is a quasiminimal set in which nontrivially recurrent trajectories and semitrajectories are everywhere dense. It follows from the results of Gardiner [78] and Gutierrez [103] concerning the structure of a quasiminimal set (see also [26, 172]) that starting from a certain moment, l^+ enters an open strip whose boundary, accessible from the inside, consists of two nontrivially recurrent positive semitrajectories l_1^+ and l_2^+ that approach l^+ (this strip can be considered as a cell in a Denjoy flow on the torus) (Fig. 5.36). In this case, all three semitrajectories l^+, l_1^+, and l_2^+ have the same asymptotic direction. Hence, there exist lifts of these semitrajectories that tend to the same point of the absolute and have a common co-asymptotic geodesic. The fact that current points on the semitrajectories l^+, l_1^+, and l_2^+ lie at a rather small distance from each other implies that the corresponding

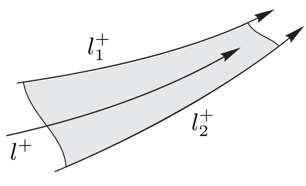


Figure 5.36. The semitrajectories l^+, l_1^+, and l_2^+.

coverings for l^+, l_1^+, and l_2^+ lie at a finite Fréchet distance from each other. Therefore, the required result follows from the already proven case for a non-trivially recurrent semitrajectory. □

Semitrajectories of analytic flows

Theorem 5.38 *Let f^t be an analytic flow on a closed orientable surface of constant nonpositive curvature, and let $\bar{l}$ be a semitrajectory of the covering flow that has an asymptotic direction. Then, $\bar{l}$ possesses the property of bounded deviation.*

Scheme of the proof. For the torus, the proof follows the scheme of the proof of Theorem 5.35. But now the existence of a closed transversal that intersects $l = \pi(\bar{l})$ is proved with the use of Lemma 5.12, which describes the structure of the set of fixed points for an analytic flow.

For a hyperbolic surface, the proof follows the scheme of the proof of Theorem 5.37. In fact, there are only two instances where the analyticity of the flow is taken into account. Henceforth, we use the notations from the proof of Theorem 5.37. The first instance relates to the assertion that the set K_* contains at least one one-dimensional trajectory. Now, this assertion is obtained with the use of Lemma 5.12. The second instance relates to the assertion that the sequence l_k, $k \geqslant 1$, is finite. Now, to prove this assertion, one should consider the connected components of the set of fixed points $\operatorname{Fix} f^t$ of the flow f^t. While in Theorem 5.37 we took into account the finiteness of the set of fixed points, now we use the finiteness of the number of connected components of the set $\operatorname{Fix} f^t$, which follows from Lemma 5.12. □

Semileaves of foliations

Theorem 5.39 *Let $\mathscr{F}$ be a foliation with a finite number of singularities on a closed surface M of nonpositive Euler characteristic. Suppose that all the singularities of $\mathscr{F}$ are saddles of negative index. Let $\bar{l}$ be a semileaf of the covering foliation $\overline{\mathscr{F}}$ on $\overline{M}$ that has an asymptotic direction. Then, $\bar{l}$ possesses the property of bounded deviation.*

The proof of this theorem is analogous to the proof of Theorem 5.37 in view of the fact that since the index of all singularities is negative, any semileaf of the covering foliation can not intersect twice a transversal whose endpoints lie on the absolute. See [34] for more details.

Note that Theorem 5.39 is generally incorrect for arbitrary foliations with a countable set of singularities. For instance, the curve from Anosov's example 5.4, which has an asymptotic direction and unboundedly deviates from the co-asymptotic geodesic, can be realized as a semileaf of a certain foliation with a countable set of singularities part of which are needles (i. e., have a positive index which is equal to $\frac{1}{2}$).

Remote limit points

Let l^+ be a positive semitrajectory of a flow f^t on a surface M of constant nonpositive curvature. Denote a current point on l^+ that corresponds to a parameter (say, time) t by $l^+(t)$. Suppose that a lift $\overline{l}^+$ of the semitrajectory l^+ has an asymptotic direction, and let $\overline{g}(\overline{l}^+)$ be a geodesic that is co-asymptotic for $\overline{l}^+$ in the positive direction. If l^+ has the property of unbounded deviation, then there exists a sequence of points $l^+(t_k)$ such that $t_k \to +\infty$ as $k \to \infty$ and the distance between the lifts $\overline{l}^+(t_k) \in \overline{l}^+$ of these points and the geodesic $\overline{g}(\overline{l}^+)$ indefinitely increases; i. e., $\overline{d}(\overline{l}^+(t_k), \overline{g}(\overline{l}^+)) \to \infty$, where $\overline{d}$ is a metric on the universal covering $\overline{M}$.

Following [153], we distinguish the subsets of the so-called remote and bounded limit points in the ω-limit set $\omega(l^+)$ of the semitrajectory l^+ in the following way. A point $m \in \omega(l^+)$ is a *remote limit point* if there exists a sequence of points $l^+(t_k)$ and their lifts $\overline{l}^+(t_k) \in \overline{l}^+$ such that

$$l^+(t_k) \to m, \quad \overline{d}(\overline{l}^+(t_k), \overline{g}(\overline{l}^+)) \to \infty, \quad t_k \to +\infty$$

as $k \to \infty$. Similarly, a point $m \in \omega(l^+)$ is a *bounded limit point* if there exists a sequence of points $l^+(t_k)$ and their lifts $\overline{l}^+(t_k) \in \overline{l}^+$ such that

$$l^+(t_k) \to m, \quad \overline{d}(\overline{l}^+(t_k), \overline{g}(\overline{l}^+)) \text{ is bounded}, \quad t_k \to +\infty$$

as $k \to \infty$. Denote the set of remote (respectively, bounded) limit points by $\omega_{\mathrm{R}}(l^+)$ (respectively, $\omega_{\mathrm{B}}(l^+)$). Note that the sets $\omega_{\mathrm{R}}(l^+)$ and $\omega_{\mathrm{B}}(l^+)$ may intersect.

The following theorem, which is proved in [153], states a fact that is common for all examples of flows with semitrajectories that have rational asymptotic direction and possess the property of unbounded deviation.

Theorem 5.40 *Let l^+ be a positive semitrajectory of a topological flow f^t on a closed orientable hyperbolic surface M. Suppose that l^+ has a rational* *asymptotic direction and possesses the property of bounded deviation. Then, each point of the remote limit set* $\omega_{\mathrm{R}}(l^+)$ *is a fixed point of* f^t.

Invariant manifolds of points of basic sets

Now, we pass on to the curves that play an important role in studying diffeomorphisms with hyperbolic nonwandering sets, namely, to one-dimensional stable or unstable manifolds of points that belong to hyperbolic nonwandering sets of surface diffeomorphisms. Using Theorem 5.39, we can prove the following theorem (which was first proved by other method in [87]).

Theorem 5.41 *Let* $f\colon M \to M$ *be an A-diffeomorphism of a closed surface* M *of nonpositive Euler characteristic. Let* Ω *be a one-dimensional widely disposed attractor (repeller) of the diffeomorphism* f*, and let* $l_x^{u(s)}$ *be the unstable (respectively, stable) manifold of a point* $x \in \Omega$*. Then both curves* $l_x^{u(s)} - x$ *have asymptotic direction and possess the property of bounded deviation.*

Scheme of the proof. Indeed, in view of the product structure in the neighborhood of any hyperbolic point, we can construct a foliation without singularities in the neighborhood of the attractor (respectively, repeller) such that the unstable (respectively, stable) manifolds of points $x \in \Omega$ are leaves of this foliation. The fact that Ω is widely disposed implies that this foliation can be extended over the whole surface so that the new foliation obtained has a finite number of saddle singularities, all being of negative index. Now, the required assertion follows from Theorem 5.39. □

The analysis of the aforementioned example of Robinson and Williams in [199] shows that for stable (respectively, unstable) manifolds of points of a one-dimensional attractor (respectively, repeller), Theorem 5.41 is generally incorrect. The above arguments do not work because the theorem on the product structure can not be applied to all points of stable (respectively, unstable) manifolds of points of a one-dimensional attractor (respectively, repeller). However, if we require that the diffeomorphism $f\colon M \to M$ is structurally stable, then we obtain the following result [86, 87]:

Theorem 5.42 *Let* $f\colon M \to M$ *be a structurally stable A-diffeomorphism of a closed orientable hyperbolic surface* M *and let* Ω *be a one-dimensional widely disposed attractor (respectively, repeller) of the diffeomorphism* f*. Let* $l_x^{s(u)}$ *be the stable (respectively, unstable) manifold of a point* $x \in \Omega$

and L^σ *be one of the connected components of the set* $l_x^{s(u)} - x$ *that does not contain a periodic boundary point. Then* L^σ *has an asymptotic direction and possesses the property of bounded deviation.*

The proof of this theorem employs the dynamical properties of structurally stable diffeomorphisms. Here, a key role is played by the fact that for structurally stable diffeomorphisms, stable and unstable manifolds of points from basic sets can not be tangent to each other but must intersect transversally.

On the torus (a closed orientable surface $M = M_p^2$ of genus $p = 1$), Theorem 5.42 is valid without the requirement that the diffeomorphism should be structurally stable [84, 87].

Theorem 5.43 *Let* $f : M \to M$ *be an A-diffeomorphism of the torus, and let* Ω *be a one-dimensional widely disposed attractor (repeller) of the diffeomorphism* f. *Let* $l_x^{s(u)}$ *be the stable (respectively, unstable) manifold of a point* $x \in \Omega$ *and* L^σ *be one of the connected components of the set* $l_x^{s(u)} - x$ *that does not contain a periodic boundary point. Then* L^σ *has an asymptotic direction and possesses the property of bounded deviation.*

A key moment in the proof is the fact that f is semiconjugate to an algebraic automorphism of the torus via a continuous mapping homotopic to the identity.

Uniformity of deviations of curves from geodesics

After establishing that certain special curves possess the property of bounded deviation under quite general assumptions, it is natural to investigate the uniformity of the bounded deviation. We will consider the same curves (but in the aggregate) that we dealt with in the previous subsection (semitrajectories of flows, semileaves of foliations, and stable or unstable manifolds of points from nonwandering hyperbolic sets).

For generalized leaves of foliations, the concept of bounded deviation is introduced by analogy with ordinary leaves. If the singularities of a foliation are topological saddles of negative index, then Theorems 5.37 and 5.39 and the fact that a generalized leaf l on a surface consists of a finite number of ordinary leaves imply that $d_l = \overline{d}_{\overline{l}} < \infty$, where $\overline{d}_{\overline{l}}$ is the maximal deviation of the covering generalized leaf $\overline{l}$ from the corresponding geodesic $\overline{g}(\overline{l})$.

The following two theorems show that the deviation of trajectories and generalized trajectories, as well as of leaves and generalized leaves, from a geodesic framework is uniformly bounded.

Theorem 5.44 *Let f^t be an analytic flow on a closed surface M of negative Euler characteristic. Suppose that the fixed points of the flow f^t are topological saddles of negative index. Then*

$$\sup \{\overline{d}_{\overline{L}}\} < \infty,$$

where the supremum is taken over all $\overline{L}$ that are generalized trajectories or ordinary trajectories of the covering flow $\overline{f}^t$.

Theorem 5.45 *Let $\mathscr{F}$ be a foliation on a closed surface M of negative Euler characteristic. Suppose that the singularities of the foliation $\mathscr{F}$ are topological saddles of negative index. Then*

$$\sup \{\overline{d}_{\overline{L}}\} < \infty,$$

where the supremum is taken over all $\overline{L}$ that are generalized leaves or ordinary leaves of the covering foliation $\overline{\mathscr{F}}$.

The proofs of both theorems can be performed according to the scheme of the proof of Theorem 5.37.

Thus, the deviation of trajectories and generalized trajectories (leaves and generalized leaves) of a flow (respectively, foliation) with saddle singularities from their respective geodesic frameworks is finite. Figuratively speaking, if a geodesic framework is considered as a skeleton of a flow (foliation), then, by virtue of Theorems 5.44 and 5.45, flows (foliations) with saddle singularities are "slender and neat".

It is natural to call the quantity $\sup\{\overline{d}_{\overline{L}}\}$ appearing in Theorems 5.44 and 5.45 a *deviation of a flow* (*respectively, foliation*) *from its geodesic framework*. Since a mapping that covers a homeomorphism homotopic to the identity shifts the points by a quantity bounded from above, the finiteness of the deviation of a flow (respectively, foliation) from its geodesic framework is a topological invariant with respect to homeomorphisms that are homotopic to the identity.

Now, consider the deviation of a one-dimensional widely disposed basic set from its geodesic framework. If Ω is a one-dimensional widely disposed attractor (repeller) of an A-diffeomorphism f, then the family of unstable (stable) manifolds of all of its points $x \in \Omega$ forms a lamination (denoted by the same character Ω) for which there exists a geodesic framework $G(\Omega)$.

By virtue of Theorem 5.41, each unstable (stable) manifold $l_x^{u(s)}$ of a point $x \in \Omega$ possesses the property of bounded deviation. Denote by $d_{\max}\left(l_x^{u(s)}\right)$ the maximal deviation of the covering $\overline{l}_x^{u(s)}$ for $l_x^{u(s)}$ from the corresponding geodesic $\overline{g}\left(\overline{l}_x^{u(s)}\right)$. It is obvious that this quantity does not depend on the choice of the covering for $l_x^{u(s)}$. Theorem 5.45 implies the following theorem, which was first proved in [34].

Theorem 5.46 *Let $f\colon M \to M$ be an A-diffeomorphism of a closed orientable hyperbolic surface M, and let Ω be a one-dimensional widely disposed attractor (repeller) of f. Then*

$$\sup\left\{d_{\max}(l_x^{u(s)})\right\} < \infty,$$

where the supremum is taken over all unstable (respectively, stable) manifolds $l_x^{u(s)}$ of the attractor (repeller) Ω.

Thus, the deviation of a one-dimensional widely disposed basic set from its geodesic framework is finite.

By analogy with the case of flows and foliations with saddle singularities of negative index, the finiteness of the deviation of a one-dimensional widely disposed basic set from its geodesic framework is an invariant of the topological conjugacy of A-diffeomorphisms with respect to homeomorphisms that are homotopic to the identity.

5.7. How smoothness depends on asymptotic directions

Recall that for analytic flows on closed surfaces of nonpositive Euler characteristic, Anosov [7] proved in 1987 that if a covering semitrajectory does not lie in a bounded part of a universal covering, then it goes to infinity in a certain asymptotic direction, see Theorem 5.8. Here, we consider the question concerning possible asymptotic directions of the semitrajectories of analytic flows. For the torus, this question is trivial: all asymptotic directions are realized by semitrajectories of analytic (and even linear) flows. Therefore, in this section, we consider analytic flows on a closed orientable hyperbolic surface M.

Denote by A_{fl}, A_∞, $A_{\text{an}} \subset S_\infty$ the sets of points that are accessible by semitrajectories of topological, C^∞-smooth, and analytic flows on M, respectively. According to Theorem 5.11, $A_{fl} = A_\infty$. Recall that Λ_{triv} is the family of trivial geodesic laminations on M.

The main result of this section is the following theorem, which provides a partial description of the set of asymptotic directions for the semitrajectories of analytic flows on a closed orientable hyperbolic surface.

Theorem 5.47 *Let M be a closed orientable hyperbolic surface. Then*

1) $\Lambda_{triv}(\infty) \subset A_{\mathrm{an}} \subset \Lambda_{triv}(\infty) \cup \Lambda_{ori}(\infty)$;
2) *The set $\Lambda_{nr}(\infty)$ is an everywhere dense continuum that has zero Lebesgue measure on S_∞. No point from $\Lambda_{nr}(\infty)$ can be reached by a lift of a semitrajectory of any analytic flow on M. At the same time, $\Lambda_{nr}(\infty)$ belongs to the set of points that are accessible by the lifts of the semitrajectories of C^∞-smooth flows on M;*
3) *Any C^∞ flow f^t that reaches a point from $\Lambda_{nr}(\infty)$ contains a continuum of fixed points. Moreover, if f^t reaches a point from $\Lambda_{nr}^{irr}(\infty)$, then f^t has neither nontrivially recurrent semitrajectories nor closed non-null-homotopic orbits without contacts.*

Assertion 2 implies that on a closed orientable hyperbolic surface, there exists a sufficiently “large” set of asymptotic directions that are realized by semitrajectories of C^∞-smooth flows but can not be realized by semitrajectories of any analytic flow.

We divide the proof of the theorem into several lemmas. The proof of assertion 1 follows from Lemmas 5.23–5.25 and 5.29; the proof of assertion 2 follows from Lemmas 5.29 and 5.30; and the proof of assertion 3 follows from Lemmas 5.26 and 5.29.

Lemma 5.23 $\Lambda_{triv}(\infty) \subset A_{\mathrm{an}}$; *i. e., for any point $\sigma \in \Lambda_{triv}(\infty)$, there exists an analytic flow f^t on M that reaches σ.*

Proof. Since $\sigma \in \Lambda_{triv}(\infty)$, there exists an axis A_σ of a certain hyperbolic isometry with the ideal endpoint σ. Moreover, A_σ is projected onto a simple closed geodesic $g = \pi(A_\sigma)$. It suffices to construct an analytic flow with a periodic trajectory that is freely homotopic to g.

The surface M can be defined by an analytic equation $F(x, y, z) = 0$ in the Euclidean space $\mathbb{R}^3$ (see Section 3.2). Since g is a simple curve, there exists an analytic function $G(x, y, z)$ such that the system of equations

$$\begin{cases} F(x, y, z) = 0, \\ G(x, y, z) = 0 \end{cases}$$

defines a simple closed curve l that is freely homotopic to g on M. Consider the analytic system

$$\begin{cases} \dfrac{dx}{dt} = F'_y G'_z - F'_z G'_y \stackrel{\text{def}}{=} X(x,\, y,\, z), \\ \dfrac{dy}{dt} = F'_z G'_x - F'_x G'_z \stackrel{\text{def}}{=} Y(x,\, y,\, z), \\ \dfrac{dz}{dt} = F'_x G'_y - F'_y G'_x \stackrel{\text{def}}{=} Z(x,\, y,\, z), \end{cases}$$

which is induced by the vector field $\vec{v} = (X,\, Y,\, Z)$ in $\mathbb{R}^3$. It is easily seen that l is a periodic trajectory of the field $\vec{v}$. Therefore, the projection of the field $\vec{v}$ defined by (1.10) yields an analytic vector field on M with the periodic trajectory l. □

Lemma 5.24 $A_{\text{an}} \cap \mathscr{R} = \Lambda_{triv}(\infty)$ (*here, $\mathscr{R}$ is the set of rational points on the absolute*).

Proof. Let $\xi \in S_\infty$ be a rational point, $\xi \in \mathscr{R}$. Then, there exists a hyperbolic isometry $\gamma_\xi \in \Gamma$ for which ξ is a fixed point. Without loss of generality, we may assume that ξ is a repelling point. There exists an axis A_ξ of the isometry γ_ξ with the ideal endpoint ξ. Denote by η the ideal endpoint of A_ξ different from ξ. The geodesic $\pi(A_\xi)$ is a closed geodesic on M.

Suppose that ξ is reached by a certain analytic flow f^t. We must show that the geodesic $\pi(A_\xi)$ is simple. Suppose the contrary. Then, there exists a $\gamma \in \Gamma$ such that $A_\xi \cap \gamma(A_\xi) \neq \varnothing$ and the pair of points $(\xi,\, \eta)$ is separated by the pair of points $(\gamma(\xi),\, \gamma(\eta))$ on S_∞. There exists a positive semitrajectory l^+ of the covering flow $\overline{f}^t$ that has an asymptotic direction defined by ξ. By virtue of Theorem 5.38, l^+ possesses the property of bounded deviation. Therefore, there exists a positive number E such that $\overline{l}^+$ lies in the non-Euclidean E-neighborhood (which we denote by $\aleph$) of the axis A_ξ, Fig. 5.37. Since γ_ξ is an isometry, $\gamma_\xi^n(\overline{l}^+)$ belongs to $\aleph$ for any integer n.

There exist Euclidean neighborhoods U_1 and U_2 of the points $\gamma(\xi)$ and $\gamma(\eta)$, respectively, on $\Delta \cup S_\infty$ such that $U_1 \cap \aleph = \varnothing$ and $U_2 \cap \aleph = \varnothing$ because $\gamma(\xi) \notin \aleph$ and $\gamma(\eta) \notin \aleph$. It is well-known that γ is continuously extended to $\Delta \cup S_\infty$. Therefore, there exist Euclidean neighborhoods U_ξ and U_η of the points ξ and η, respectively, that are mapped by γ into U_1 and U_2. Since η is an attracting point for γ_ξ, we have $\gamma_\xi^{n_0}(m_0) \in U_\eta$ for a certain $n_0 \in \mathbb{N}$,

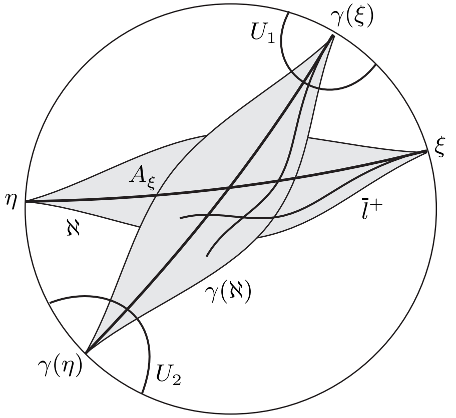


Figure 5.37. The semitrajectory $\gamma_\xi^n(\bar{l}^+)$ intersects the semitrajectory $\gamma(\gamma_\xi^{n_0}(\bar{l}^+))$.

where m_0 is the starting point of $\bar{l}^+$. Hence, $\gamma\left(\gamma_\xi^{n_0}(m_0)\right) \in U_2$. Recall that $\gamma(\xi) \in U_1$. Therefore, $\gamma_\xi^{n_0}(\bar{l}^+)$ must intersect $\gamma\left(\gamma_\xi^{n_0}(\bar{l}^+)\right)$ since the semitrajectory $\gamma_\xi^{n_0}(\bar{l}^+)$ lies in the non-Euclidean E-neighborhood $\aleph$ of the axis A_ξ. However, this is impossible. □

Lemma 5.25 *Suppose that an analytic flow f^t on M has a positive semitrajectory l^+ whose lift $\bar{l}^+$ has an asymptotic direction defined by an irrational point σ. Then, $\sigma \in \Lambda_\infty$.*

Proof. By Lemma 5.15, $\omega(l^+)$ is a quasiminimal set. According to [142], $\omega(l^+)$ is a Maier quasiminimal set (see Definition 3.14). By Theorem 5.32, the geodesic framework $G(\omega(l^+))$ of the set $\omega(l^+)$ belongs to Λ. Since the support of a geodesic lamination is a closed set, we have $\sigma \in \Lambda_\infty$. □

Recall that a lamination is *orientable* if each leaf can be equipped with an orientation so that, for any point of an arbitrary leaf, there exists a transversal segment that passes through this point and is intersected by the leaves in the same direction (of course, the transversal segment must be equipped with the normal orientation, and the intersection in the same direction means that the index of intersection of the leaves with this segment is the same). Therefore, the *non-orientability* of a lamination implies that, irrespectively of the orientation of the leaves, there always exists a point (which lies in a certain leaf since the support of the lamination is closed) such that an arbitrary transversal segment passing through this point is intersected by certain leaves in opposite directions. The following lemma follows easily from Lemma 3.19. For the reader's convenience, we give a short proof.

Lemma 5.26 *Let $\mathscr{L}$ be a non-orientable minimal nontrivial geodesic lamination and Σ be a geodesic segment that transversally intersects $\mathscr{L}$ at its internal points,* $\operatorname{int}\Sigma \cap \operatorname{supp}\mathscr{L} \neq \varnothing$. *Then, any directed geodesic $g \in \mathscr{L}$ intersects Σ in opposite directions.*

Proof. Since any geodesic from $\mathscr{L}$ is everywhere dense in $\mathscr{L}$, it suffices to show the existence of a geodesic that intersects Σ in opposite directions. Suppose that all geodesics intersect Σ in the same direction. Introduce the orientation on each geodesic assuming that the positive direction corresponds to the positive transversal orientation on Σ. By the assumption, this introducing is correct and specifies the orientation on each geodesic from $\mathscr{L}$ because all geodesics intersect Σ. Since the lamination $\mathscr{L}$ is non-orientable, there exists a geodesic $g_* \in \mathscr{L}$ such that any transversal segment Σ_0 intersecting g_* is intersected by certain geodesics from $\mathscr{L}$ in opposite directions. The geodesic g_* intersects Σ. The theorem on the continuous dependence of leaves on the initial conditions implies that Σ should also be intersected by certain geodesics from $\mathscr{L}$ in opposite directions. We get a contradiction. $\square$

Lemma 5.27 $\Lambda_{nr}(\infty) \subset A_\infty$.

Proof. For an arbitrary point $\sigma \in \Lambda_{nr}(\infty)$, there exists a lift $\overline{g}$ of a geodesic g with the ideal endpoint σ. The geodesic g belongs to a certain non-orientable geodesic lamination and, hence, has no self-intersections. According to Theorem 5.11, there exists a C^∞ flow f^t that has a trajectory whose lift $\overline{l}$ lies at a finite (and even arbitrarily small) Fréchet distance from $\overline{g}$. Therefore, $\overline{l}$ has an asymptotic direction defined by the point σ. Hence, f^t reaches σ and $\sigma \in A_\infty$. $\square$

Lemma 5.28 *Suppose that a flow f^t on M has a positive semitrajectory l^+ whose lift $\overline{l}^+$ has an asymptotic direction defined by a point from $\Lambda_{nr}(\infty)$. Then there does not exist a simple closed transversal that is intersected by the semitrajectory l^+ infinitely many times.*

Proof. Suppose the contrary. Let T be a simple closed transversal that is intersected by l^+ infinitely many times. Hence, T is non-null-homotopic (and is not even homologous to zero), and there exists a simple closed geodesic g_0 that is freely homotopic to T, see Fig. 5.38. Then, for any lift $\overline{T} \in \pi^{-1}(T)$ of the transversal T, there exists a lift $\overline{g}_0 \in \pi^{-1}(g_0)$ of the geodesic g_0, which we denote by $\overline{g}_0(\overline{T})$, that has the same ideal endpoints. Since $\overline{l}^+$ tends to σ and l^+ intersects T infinitely many times, σ is the topological limit of the lifts of T. Therefore, σ is also the topological limit of the lifts of the geodesic g_0. Thus, $\overline{g}$ must intersect an infinite number of lifts of the geodesic g_0 and, hence, g must intersect g_0 infinitely many times.

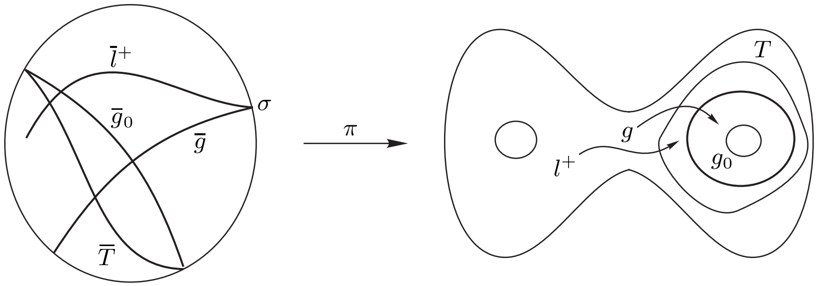


Figure 5.38. The geodesic g_0 that is freely homotopic to the transversal T, and their lifts.

Fix a natural parametrization $\theta\colon \mathbb{R} \to g$. According to Lemma 5.26, there exists an increasing sequence of parameters $t_i \in \mathbb{R}$ such that $t_i \to \infty$ as $i \to \infty$, $\theta(t_i) \in g_0$, and the index of the intersection $g \cap g_0$ at the point $\theta(t_i)$ equals $(-1)^i$, $i \in \mathbb{N}$. Hence, there exists a sequence $\overline{g}_i$ of the lifts of the geodesic g_0 that satisfies the following conditions:

1) $\overline{g}$ intersects $\overline{g}_i$ exactly at one point, say $\overline{m}_i$;
2) the topological limit of the geodesics $\overline{g}_i$ is equal to σ;
3) the index of the intersection $\overline{g} \cap \overline{g}_i$ at $\overline{m}_i$ is equal to $(-1)^i$ (see Fig. 5.39).

Since g_0 is freely homotopic to T, each $\overline{g}_i$ is co asymptotic for a certain lift $\overline{T}_i$ of the transversal T; i.e., $\overline{g}_i$ and $\overline{T}_i$ have the same ideal endpoints on S_∞. By virtue of condition 2, $\overline{l}^+$ intersects all $\overline{T}_i$ starting from a certain index $i \geqslant i_0$. According to condition 3, the index of the intersection $\overline{l}^+ \cap \overline{T}_i$ is equal to $(-1)^i$, $i \geqslant i_0$. Hence, l^+ must intersect T in opposite directions, which is impossible. The contradiction obtained completes the proof. □

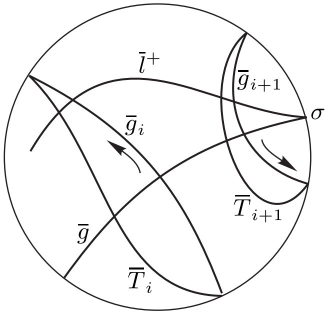


Figure 5.39. The index of the intersection $\overline{g} \cap \overline{g}_i$ equals to $(-1)^i$.

Lemma 5.29 *Suppose that a flow f^t on M has a positive semitrajectory l^+ whose lift $\overline{l}^+$ has an asymptotic direction defined by a point $\sigma = \omega(\overline{l}^+)$ from $\Lambda_{nr}(\infty)$. Then f^t is nonanalytic and has a continuum of fixed points. Moreover, if $\omega(\overline{l}^+)$ belongs to $\Lambda_{nr}^{irr}(\infty)$, then f^t has neither nontrivially recurrent semitrajectories nor non-null-homotopic closed transversals.*

Proof. Since $\sigma \in \Lambda_{nr}(\infty)$, there exists a non-orientable minimal nontrivial geodesic lamination $\mathscr{L}$ that reaches σ; i. e., there exists a geodesic $\overline{g} \in \overline{\mathscr{L}} = \pi^{-1}(\mathscr{L})$ that tends to σ in the positive direction. Suppose that the flow f^t is analytic. Since the point σ is irrational, $\omega(l^+)$ is a quasiminimal set by virtue of Lemma 5.15. According to Corollary 3.5, there exists a simple closed transversal T that intersects l^+ infinitely many times, which is impossible because of Lemma 5.28. The contradiction obtained shows that the flow f^t is nonanalytic.

Let us show that the flow f^t has a continuum of fixed points. Suppose the contrary. Since l^+ has an asymptotic direction, the ω-limit set $\omega(l^+)$ of the semitrajectory l^+ can not consist of a single point and contains at least two points. The connectedness of the ω-limit set of any semitrajectory and our assumption that the set of fixed points is not a continuum imply that $\omega(l^+)$ contains a one-dimensional trajectory, say l_1. Let S_1 be a transversal segment that passes through a certain point $a \in l_1$, $a \in \operatorname{int} S_1$. Then l^+ intersects S_1 infinitely many times because $a \in \omega(l^+)$. By virtue of Theorem 3.3, there exists a simple closed transversal T that intersects l^+. Since $\overline{l}^+$ has an asymptotic direction, T is non-null-homotopic, which is impossible in view of Lemma 5.28.

Now, consider the case when the point $\sigma = \omega(\overline{l}^+)$ is accessible by an irreducible non-orientable geodesic lamination $\mathscr{L}$. We will use the previous notations. Let us prove that f^t has neither nontrivially recurrent semitrajectories nor closed non-null-homotopic transversals. Since the existence of nontrivially recurrent semitrajectories implies the existence of closed non-null-homotopic transversals, it suffices to show that the flow f^t does not have any closed non-null-homotopic transversal. Suppose the contrary. Let T_0 be a homotopically nontrivial closed transversal. Then, there exists a closed geodesic g_0 that is freely homotopic to T_0. It follows from the irreducibility of the geodesic lamination that the geodesic g intersects g_0 infinitely many times. Since there exists a lift of the geodesic g that tends to σ, there exists a sequence of lifts of the geodesic g_0 whose topological limit is σ. Hence, $\overline{l}^+$ must intersect an infinite number of lifts of the transversal T_0. Therefore, l^+ intersects T_0 infinitely many times, which contradicts Lemma 5.28. The contradiction obtained completes the proof. □

To complete the proof of Theorem 5.47, we must show that $\Lambda_{nr}(\infty)$ has zero Lebesgue measure. Following [111], we call a geodesic $\overline{g} \subset \Delta$ *transitive* if, for any intervals $U_1, U_2 \subset S_\infty$, there exists a deck transformation $\gamma \in \Gamma$ such that one ideal endpoint of the geodesic $\gamma(\overline{g})$ lies in U_1 and the other in U_2. Denote by $TR(\Gamma) \subset S_\infty$ the set of points $a \in S_\infty$ with the following property: any geodesic with an ideal endpoint a that is directed toward a is transitive. Myrberg [167] proved that the Lebesgue measure of the set $TR(\Gamma)$ is equal to the Lebesgue measure of the absolute S_∞. In particular, $S_\infty - TR(\Gamma)$ has zero Lebesgue measure. Therefore, it suffices to prove the following lemma.

Lemma 5.30 *The set $\Lambda_{nr}(\infty)$ is everywhere dense on the absolute, and $\Lambda_{nr}(\infty) \subset S_\infty - TR(\Gamma)$. In particular, $\Lambda_{nr}(\infty)$ has zero Lebesgue measure.*

Proof. Since the set $\Lambda_{nr}(\infty)$ is invariant under the action of the group of deck transformations and, for a closed orientable hyperbolic surface, this group is a Fuchsian group of the first kind, $\Lambda_{nr}(\infty)$ is everywhere dense on the absolute (the orbit of any point under a Fuchsian group of the first kind is everywhere dense on the absolute).

Let us prove the inclusion $\Lambda_{nr}(\infty) \subset S_\infty - TR(\Gamma)$. Suppose that there exists a point $\sigma \in \Lambda_{nr}(\infty) \cap TR(\Gamma)$. Then, σ is an ideal endpoint of a lift $\overline{g}$ of a certain geodesic $g \in \Lambda_{nr}(\infty)$ without self-intersections. On the other hand, $\overline{g}$ is transitive by the definition of the set $TR(\Gamma)$. Therefore, g has self-intersections. We have arrived at a contradiction. □

5.8. The image of geodesic under a cover homeomorphism

Here, we prove a result that plays an important role in the Nielsen–Thurston theory on the homotopy classification of surface homeomorphisms. We formulate it in the spirit of our book as a proposition on the existence of asymptotic direction.

Theorem 5.48 *Let $h\colon M^2 \to M^2$ be a homeomorphism of a closed hyperbolic surface M^2, and $\overline{h}\colon \Delta \to \Delta$ be its lift to Δ. Then given any half-infinite geodesic $\overline{g} \subset \Delta$ (geodesic ray) the curve $\overline{h}(\overline{g})$ has an asymptotic direction.*

Proof. First, let's prove that the curve $\overline{h}(\overline{g})$ goes to infinity. Suppose the contrary. Then there is a sequence of points $A_i \in \overline{g}$ such that $d_{NE}(O,\, A_i) \to \infty$ and $\overline{h}(A_i) \to A_*$ as $i \to \infty$ for some point $A_* \in \Delta$. By continuity of $\overline{h}^{-1}$,

$$A_i = \overline{h}^{-1}(\overline{h}(A_i)) \to \overline{h}^{-1}(A_*) \in \Delta$$

as $i \to \infty$. This contradiction implies that $\overline{h}(\overline{g})$ goes to infinity and thus, $\operatorname{Lim} \overline{h}(\overline{g}) \subset S_\infty$. We have to prove that the complete limit set $\operatorname{Lim} \overline{h}(\overline{g})$ consists of a unique point.

Note that h is uniformly continuous mapping because M is compact. Hence, $\overline{h}$ is also uniformly continuous. Two geodesic rays (half-infinite geodesics) $\overline{g}_1$ and $\overline{g}_2$ with common ideal endpoint at the circle at infinity approach exponentially. As a consequence, if $\overline{h}(\overline{g}_1)$ has an asymptotic direction, then $\overline{h}(\overline{g}_2)$ also has an asymptotic direction, and vice versa. Therefore without loss of generality we can assume that $\overline{g}$ starts at the origin $O \in \Delta$. Moreover, we can assume that $\overline{h}(O) = O$ and hence, $\overline{h}^{-1}(O) = O$ (if it is not true we can consider a homeomorphism which is isotopic to h). To simplify matters, denote $d_{NE} = d$.

Since $\overline{h}$ and $\overline{h}^{-1}$ are lifts of continuous maps of closed surface, they are uniformly continuous with respect to d. Hence, there is $k \in \mathbb{N}$ such that

$$d(p,\, q) \leqslant \frac{1}{k} \Longrightarrow d(\overline{h}p,\, \overline{h}q) \leqslant 1, \tag{5.23}$$

$$d(P,\, Q) \leqslant \frac{1}{k} \Longrightarrow d(\overline{h}^{-1}P,\, \overline{h}^{-1}Q) \leqslant 1 \tag{5.24}$$

for any points p, $q \in \Delta$. If one denotes $P = \overline{h}p$ and $Q = \overline{h}q$, (5.24) is written as follows

$$d(\overline{h}p, \overline{h}q) \leqslant \frac{1}{k} \Longrightarrow d(p, q) \leqslant 1. \tag{5.25}$$

Take points $A, B \in \Delta$ such that $d(A, B) = \frac{m}{k} \in \mathbb{Q}$. By subdividing the geodesic joining A and B into m subintervals each of which equals to $\frac{1}{k}$, and applying the triangle inequality, it follows from (5.23) and (5.25) that

$$d(p, q) \leqslant \frac{m}{k} \Rightarrow d(\overline{h}p, \overline{h}q) \leqslant m, \quad d(\overline{h}p, \overline{h}q) \leqslant \frac{m}{k} \Rightarrow d(p, q) \leqslant m \tag{5.26}$$

for any $m \in \mathbb{N}$.

Denote by $A_t \in \overline{g}$ the point with $d(O, A_t) = t \in \mathbb{R}_+$. Given any $n \in \mathbb{N}$ and $\alpha \geqslant 0$, it follows from (5.26) that

$$d(O, \overline{h}(A_{n+\alpha})) > \frac{n-1}{k}. \tag{5.27}$$

Indeed, if $d(O, \overline{h}(A_{n+\alpha})) \leqslant \frac{n-1}{k}$, then

$$d(O, \overline{h}(A_{n+\alpha})) = d(\overline{h}(O), \overline{h}(A_{n+\alpha})) \leqslant \frac{n-1}{k} \Rightarrow d(O, A_{n+\alpha}) \leqslant n-1$$

that contradicts to $d(O, A_{n+\alpha}) = n + \alpha \geqslant n$.

Let $A_n A_{n+1} \subset \overline{g}$ be the geodesic segment with the endpoints A_n, A_{n+1}. By (5.26), the length of the arc $\overline{h}(A_n A_{n+1})$ is not more than k. Hence, $\overline{h}(A_n A_{n+1})$ belongs to the (non-Euclidean) disk $D_{n,k}$ with the center at $B_n = \overline{h}(A_n)$ and radius k.

Denote by L_t the geodesic ray that starts at O and passes through $\overline{h}A_t = B_t$ and by Θ_t the angle between L_t and L_{t+1}. Let $\operatorname{Var}\Theta_t$ be the variation of Θ_ν when the current point runs from B_t to B_{t+1}, $\operatorname{Var}\Theta_t = \max\limits_{0\leqslant\alpha\leqslant 1} |\Theta_t - \Theta_{t+\alpha}|$. Denote by $\operatorname{TVar}\Theta_t$ the total variation of Θ_ν when the current point B_ν runs all points for $\nu \geqslant t$. To prove the theorem, one needs to prove the existence of $\lim\limits_{t\to\infty} L_t$. This rules out "spiral" or "big oscillation" behavior of $\overline{h}(\overline{g})$ when it approaches S_∞. To do this we estimate $\operatorname{TVar}\Theta_t$ as follows.

Take and fix $n \geqslant 2k^2 + 2$. Hence, $\frac{n-1}{k} - k > \frac{n}{2k}$. As a consequence, $D_{n,k}$ is outside of the disk D_r with the center at O and radius $r = \frac{n}{2k}$, Fig. 5.40.

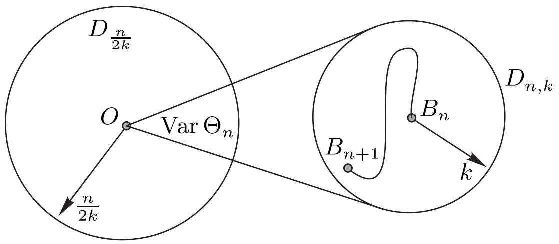


Figure 5.40

Applying (5.4) when $r = \frac{n}{2k}$ and $R = k$, one gets $\operatorname{Var}\Theta_n \leqslant k \exp\left(-\frac{n}{16k}\right)$. Note that if $n \geqslant 2k^2 + 2$, then $n \geqslant 4k$. Therefore, $r = \frac{n}{2k} \geqslant 2$, and we can use (5.4). Hence, the total variation equals to

$$\operatorname{TVar}\Theta_n \leqslant \sum_{i=0}^{\infty} k \exp\left(-\frac{n+i}{16k}\right) = \\ = k \exp\left(-\frac{n}{16k}\right)\left(\sum_{i=0}^{\infty} \exp\left(-\frac{i}{16k}\right)\right) = C_k \exp\left(-\frac{n}{16k}\right)$$

where

$$C_k = \frac{k}{1 - \exp\left(-\frac{1}{16k}\right)}$$

is a function of k only. Since

$$\lim_{n\to\infty} \exp\left(-\frac{n}{16k}\right) = 0,$$

$\lim\limits_{t\to\infty} L_t$ exists. This follows that $\overline{h}(\overline{g})$ has an asymptotic direction. □

Remark. If the half-infinite geodesic $\overline{g}$ projects onto a closed geodesic on M, the theorem 5.48 follows from Lemma 5.2.

Bibliographic Notes and Panoramas

Chapter 5. Some aspects of the Anosov – Weil Theory are considered in the surveys [10, 12, 15, 19, 97, 172]. The most complete panorama can be found in [20]. Multidimensional aspects are considered in [69].

(5.1). The notion *co-asymptotic geodesic* was suggested by A. M. Stepin.

(5.2). Weil [215] applied a covering flow to get a geometrical interpretation of Poincaré rotation number for a fixed-points-free flow on a torus. Let us briefly explain Weil's approach. Denote by $C(p, q)$ a trajectory passing through a point with the integer coordinates $(p, q) \in \mathbb{R}^2$. Let us split the set of $\mathbb{Q}$ onto two classes L, R as follows:

$$\frac{p}{q} \in \begin{cases} R \text{ if } C(p, q) \text{ over } C(0, 0) \\ L \text{ if } C(p, q) \text{ under } C(0, 0) \end{cases}$$

where $p \geqslant 1$. If $\frac{p}{q} \in R$ and $\frac{r}{s} > \frac{p}{q}$ then $\frac{r}{s} \in R$. Indeed, since $sp > rq$ then the curve $C(rp, sp)$ lies "over" the curve $C(rp, rq)$. Therefore $sp/rp = = s/r \in R$. (The same argument is applicable to the set L.) Since each rational number belongs to either of the classes R, L, the closure of the above classes defines a section of the set of real numbers. Every such a section specifies a real number, α, which is equal to the Poincaré rotation number. It can be seen that α is a slope of the straight line defining the asymptotic direction.

On account of the above terminology Weil proved that the covering trajectories must have an asymptotic direction. Weil's method of the study of asymptotic directions is more geometric then the Poincaré's method which consists in the study of first return mappings on the global cross-sections to the flows. What is more important Weil apparently inferred that his method works not exclusively for the torus flows but also for the higher genus flows and is applicable as well to the arbitrary families of curves not necessarily given by the differential equations. This led him to the two conjectures (quoted below) on a nonlocal asymptotic behavior of a lift for any curve without topologically transversal self-intersections. We quote the original of his talk at the First international topological conference in Moscow [216]:

"Dans la présente communication, l'auteur discute deux méthodes pouvant servir à l'etude de la question et d'autres analogues. La première, qui a déjà été développée dans un article du [215]*, consiste à considérer dans le plan* (x, y) *en même temps que la courbe C de la famille, toutes les courbes* $C_{p,q}$ *qui s'en déduisent par une translation* (p, q)*, p et q étant des entiers*: *la position relative de ces courbes par rapport à C permet, non seulement de déterminer le nombre de rotation, mais encore la transformation qui ramène la famille étudiée à une forme canonique. La méthode s'applique dans le cas de Poincaré, et plus généralement chaque fois que la famille ne présante pas de*

'col à l'infini' (*au sens de Niemytzky*).[3] *D'ailleurs cette dernière circonstance ne peut vraisemblablement pas se présenter si la famille ne contient pas de courbe fermée.*[4] *À cette méthode se relie encore le théorème suivant, d'ailleurs obtenu par une voie quelque peu différente*:

Soit, sur le tore, une courbe de Jordan, image continue de la demi-droite $0 \leqslant t < +\infty$; *on suppose que cette courbe soit sans point double*; *alors, si l'image de la courbe dans le plan* (x, y), *surface de recouvrement universelle du tore, tend vers l'infini avec* t, *elle y tend avec une direction asymptotique bien déterminée, c'est-à-dire que la rapport* $\frac{x(t)}{y(t)}$ *tend vers une limite quand* t *tend vers* $+\infty$.

Une généralisation très intéressante du probllème étudié, qui paraît susceptible d'être abordée par la même méthode, est l'étude, sur une surface close de genre p, *des solutions d'une équation différentielle du premier ordre n'ayant d'autres points singuliers que de cols, ou en termes topologiques, d'une famille de courbes dont tous les points singuliers sont d'indice négatif.*[5] *Un premier résultat est suivant*:

Sur le cercle hyperbolique, surface de recouvrement universelle de la surface étudiée, toute courbe de la famille tend, dans chaque direction, vers un point à l'infinie bien détetminé. . . .

English version is the following: *In the present talk the author discusses two methods which can help to study the question, and other analogies. The first method, which was already developed in the paper* [215], *consists in considering in the plane* (x, y) *along with the curve* C *of the family, all the curves* $C_{p,q}$ *that are obtained by translations* (p, q) *where* p, q *are integers*: *the relative position of these curves with regard to* C *allows not only to determine the rotation number, but also the transformation which brings the family to the canonical form. The method is applicable to the Poincaré's case, and more generally each time when the family does not have 'saddle at infinity'* (*in the sense of Nemytsky*). *Anyway, the latter possibility evidently cannot be realized unless the family contains a closed curve. Connected with this method, there is a theorem which can be obtained also in other way*:

Let a Jordan curve which is an image of a ray $0 \leqslant t < +\infty$ *on torus be given*; *one assumes that this curve has no double points*; *if the image of the*

[3]It means that the covering flow in the plane does not admit a strip in which the integral curves are homeomorphic to the family of parabolas $y = x^2 + C$. (Remark is our's.)

[4]Magnier obtained the proof of this result. (Remark of A. Weil.)

[5]Magnier obtained very interesting results on this question as well, they will be published shortly. (Remark of A. Weil.)

curve on the plane (x, y) which is the universal covering of torus, tends to infinity along with t, it tends there in a well-defined asymptotic direction, that is the quotient $\frac{x(t)}{y(t)}$ tends to a limit when t tends to $+\infty$.

A very interesting generalization of the studied problem which is likely solvable by the same method, is the study on a closed surface of genus p of the differential equations of the first order having only singular points of the saddle type, or in topological terms, of families of curves with singularities of the negative index. The first result is the following:

In the hyperbolic circle which is the universal covering of the studied surface all curves of the family tend in each direction to a well-defined point at the infinity. ...

Neither Weil nor Magnier have ever published the proof of their statements.

The Weil theorem was proved independently by Markley [147, 148] and Pupko [195]. However, Pupko's proof has essential errors, which actually lead to an incorrect Theorem 2 in [195] on bounded deviation. The proof of Markley [148] also contains flaws, but they are not essential. Markley communicated to us that these flaws were due to the necessity of restricting the volume of his paper. These flaws are mainly related to the proof of Theorem 5.5 on the boundedness (at least from one side) of the deviation of a curve from the corresponding straight line on the torus. For the cylinder, a similar theorem was proved in [8, Theorem 6]; hence, the corresponding assertion is valid for any closed surface of nonpositive Euler characteristic because a cylinder can be "pasted" in any such surface. Our proof follows [8].

The Weil conjecture was proved independently by Markley in 1966 [147] and Pupko in 1967 [195]. The arguments of these mathematicians are strikingly similar, although they considered the Weil conjecture in different hemispheres at about the same time.

(5.3). In 1966, at the symposium on general topology held in Tiraspol (Moldova, former part of Soviet Union), Anosov communicated the theorem that he proved stating that unbounded coverings for semitrajectories of smooth flows with a finite number of fixed points on a closed surface of nonpositive Euler characteristic have an asymptotic direction. Later, Anosov generalized this theorem in three directions [7]:

- He weakened the requirement for the smoothness class down to C^0 (topological flows).

- He weakened the requirement for the finiteness of the set of fixed points to its contractibility to a point (in particular, any zero-dimensional set of fixed points).
- He removed all the constraints imposed on the set of fixed points at the expense of increasing the smoothness (analytic flows).

At the middle of 1960s, D. Anosov proved the existence of asymptotic direction for a lift of semitrajectory of smooth flow with finitely many fixed points provided the lift is an unbounded curve. At this time, he suggested to study dynamical properties of surface flows by limit sets "at infinity" and compared trajectories with geodesics. D. Anosov put forth the concept that the topological key to Nonlocal Theory of Dynamical Systems and Foliations on any surface of nonpositive curvature is a study asymptotic behaviors of lifts of special infinite curves on an universal covering plane with a using of circle at infinity. Especially this approaching turned up effective for dynamical systems with nontrivially recurrent motions and nontrivially recurrent invariant manifolds (the most known of such dynamical systems are transitive flows, pseudo-Anosov homeomorphisms, Anosov and DA diffeomorphisms), and foliations with nontrivially recurrent leaves.

(5.4). Theorem 5.16 was proved in [27] for l^+ being a nontrivially recurrent semi-trajectory of a surface flow. Theorem 5.20 was proved in [27].

(5.5). Actually, the notation of geodesic framework for invariant sets with no name was introduced by Aranson and Grines [28], and for foliations by Levitt [133, 134].

(5.6). One of the conjectures of Anosov concerned a deviation of a curve from the co-asymptotic geodesic. In 1967, V. Pupko [195] stated the restricted deviation property for the curve without self-intersections but her proof was unclear. About 1972, Aranson and Grines came to Moscow to present their results on the classification of transitive flows on hyperbolic surfaces. The essential part of this presentation was the proof of the existence of an asymptotical direction for a nontrivially recurrent trajectory. Anosov asked Grines about the deviation property, and it looked like he doubted Pupko's statement. Soon, Grines realized that in the example by C. Robinson and F. Williams [199] there are curves with an unbounded deviation. Aranson and Grines constructed a counter-example (described by Anosov in [7]) to Pupko's statement even if a curve is a semi-trajectory of flow on closed orientable surface of genus $g = 2$.

Examples of curves with rational asymptotic direction and with the property of unbounded deviation on the torus and the Klein bottle were first constructed in [9, 11]. Recall some results. It follows from the results obtained by Weil [215] and Denjoy [67] that semitrajectories of flows without fixed points on the torus possess the property of bounded deviation. For flows on a closed surface of negative curvature, Markley [147] proved the property of bounded deviation for a B-recurrent semitrajectory.

Here, we present without proof a result obtained in [11] that concerns the bounded deviation of arbitrary semi-infinite curves on the Klein bottle. Recall that, on the Klein bottle K^2, which is represented as the quotient space $\mathbb{R}^2/\Gamma$, where Γ is generated by transformations (1.3), only four asymptotic directions are realized by simple semi-infinite curves. More precisely, if l^+ is a simple semi-infinite curve on the Klein bottle such that its lift $\overline{l}^+$ to the universal covering $\mathbb{R}^2$ has an asymptotic direction, then this asymptotic direction coincides with the asymptotic direction of one of the half-lines of either the abscissa or the ordinate axis. The following theorem shows that if the asymptotic direction coincides with one of the half-lines of the abscissa, then the deviation of $\overline{l}^+$ from the abscissa axis is bounded.

Theorem 5.49 *Let l^+ be a simple semi-infinite curve on the Klein bottle K^2 such that its lift $\overline{l}^+$ to the universal covering $\mathbb{R}^2$ has an asymptotic direction defined by the positive half-line of the abscissa. Then, $\overline{l}^+$ has the property of bounded deviation.*

As regards the asymptotic direction defined by one of the half-lines of the ordinate axis, one can guarantee (by analogy with the case of a surface for a rational asymptotic direction) that the curve boundedly deviates only to one side from the ordinate axis. Modifying the example of a curve on the torus from Section 5.6, we can construct a curve on the Klein bottle whose lift unboundedly deviates from the ordinate axis (see [11] for details).

Theorem 5.35 was proved in [13]. Theorem 5.37 was proved in [34].

For an infinite set of fixed points, the following theorem holds, which was proved in [153].

Theorem 5.50 *Let l^+ be a positive semitrajectory of a topological flow f^t with a locally connected and nowhere dense set of fixed points $\operatorname{Fix} f^t$ on a closed hyperbolic surface M. Suppose that the ω-limit set of the semitrajectory l^+ contains a point that is not a fixed point, $\omega(l^+) \not\subset \operatorname{Fix} f^t$. If a lift $\overline{l}^+$ of the semitrajectory l^+ to $\overline{M} = \Delta$ is an unbounded curve, then $\overline{l}^+$ has a ra-*

tional asymptotic direction and possesses the property of bounded deviation. Moreover, $\omega(l^+)$ *contains a non-null-homotopic closed invariant curve.*

For the torus, Theorem 5.38 was proved in [13], while for a hyperbolic surface, it was proved in [42]. Theorems 5.44, 5.45 were proved in [34].

(5.7). Theorem 5.47 was proved in [41].

(5.8). The proof of Theorem 5.48 follows to [63].

CHAPTER 6

Classification of Surface Foliations, Webs, and Homeomorphisms

Recall that two foliations $\mathscr{F}_1$, $\mathscr{F}_2$ on a surface M are *topologically equivalent* if there exists a homeomorphism $h\colon M \to M$ such that $h(\mathrm{Sing}(\mathscr{F}_1)) = \mathrm{Sing}(\mathscr{F}_2)$ and h sends every leaf of $\mathscr{F}_1$ onto a leaf of $\mathscr{F}_2$. One says that h *maps the foliation* $\mathscr{F}_1$ *onto the foliation* $\mathscr{F}_2$. One of the important goals of Theory of Foliations is the classification of foliations up to the topological equivalence. A some characteristic (possibly, of a geometric, algebraic, etc., nature) which is common for topologically equivalent foliations is called a (topological) *invariant*.

It is impossible to classify all surface foliations. But if we restrict ourselves to special classes, this problem could be manageable. In general, the classification assumes the following (independent) steps.

1) Find a constructive topological invariant which takes the same values for topologically equivalent foliations.
2) Describe all topological invariants which are admissible, i. e. may be realized in the chosen class of foliations.
3) Find a standard representative in each equivalence class, i. e. given any admissible invariant, one constructs a foliation whose invariant is the admissible one.

An invariant is called *complete* if it takes the same value if and only if two foliations are topologically equivalent. The "if" part only gives a *relative* invariant.

A classical example of constructing an effective topological invariant is given by the Poincaré rotation number for fixed-point-free flows on the two-dimensional torus $\mathbb{T}^2$. This rotation number determines the "asymptotics of the rotation" of trajectories along the meridians and parallels of $\mathbb{T}^2$ (note that historically, this invariant was introduced for the first-return maps on global sections as the limit of "averaged iterations", see Section 2.2). If we consider a lift of an arbitrary semitrajectory to $\mathbb{R}^2$, which is the universal covering for $\mathbb{T}^2$, it turns out that this lift has an asymptote on $\mathbb{R}^2$. The angular coefficient of the asymptote is equal to the Poincaré rotation number of the flow. It is well-known that for minimal flows on $\mathbb{T}^2$, the rotation number is a complete topological invariant up to the recalculation by an integer unimodular matrix.

The Web Theory is a classical area of Geometry and is mainly devoted to solving local problems. However, 2-webs also naturally appear in the Theory of Dynamical Systems on surfaces as pairs of stable and unstable foliations of Smale horseshoes, Anosov diffeomorphisms, pseudo-Anosov homeomorphisms, and diffeomorphisms with Plykin attractors. The topological equivalence of these webs is clearly a necessary condition for the classification up to conjugacy of these diffeomorphisms and homeomorphisms.

In Sections 6.1, 6.7, we expose elements of the Nielsen – Thurston Theory. In Section 6.2, we formulate with no proofs the classical results on the classification of fixed-point-free torus flows. The results of Nielsen – Thurston Theory help to classify irrational foliations and 2-webs on hyperbolic surfaces in Sections 6.3, 6.4 respectively. In Section 6.5, we consider classification of nontrivial minimal sets. In Section 6.6, one gets the classification of surface AP-homeomorphisms that includes pseudo-Anosov homeomorphisms.

6.1. Elements of the Nielsen – Thurston Theory

In this section, we consider M to be closed orientable hyperbolic surface. Jakob Nielsen [173] developed a theory of orientation preserving self-homeomorphisms $M \to M$ based on an elaborate analysis of their fixed points. William Thurston [214], by entirely different methods, realized that some Nielsen's classes of homeomorphisms contain homeomorphisms that preserve a pair of transverse measured foliations. Here, we give first results of the Nielsen – Thurston Theory.

Automorphisms of deck group

Let $h\colon M \to M$ be an orientation preserving homeomorphism of closed orientable hyperbolic surface $M = \Delta/\Gamma$, and $\overline{h}\colon \Delta \to \Delta$ be a lift of h. Recall that the group Γ of deck transformations consists of hyperbolic isometries $\gamma \in \Gamma$ such that

$$\pi(\overline{z}) = \pi(\gamma(\overline{z})) \text{ for any } \overline{z} \in \Delta, \tag{6.1}$$

where the natural projection $\pi\colon \Delta \to \Delta/\Gamma = M$ is a universal covering projection. Equality (6.1) is crucial for γ to be a deck transformation.

Lemma 6.1 *Let $\beta\colon \Delta \to \Delta$ be a homeomorphism such that $\pi(\overline{z}) = \pi(\beta(\overline{z}))$ holds for any point $\overline{z} \in \Delta$. Then $\beta \in \Gamma$.*

Proof. Given any $\overline{z} \in \Delta$, there is $\gamma_{\overline{z}} \in \Gamma$ that depends on $\overline{z}$ such that $\gamma_{\overline{z}}[\beta(\overline{z})] = \overline{z}$. Since every point of the set $\Gamma(\overline{z})$ is uniformly isolated (i. e. the hyperbolic distance between any different points of $\Gamma(\overline{z})$ is more than a positive constant), $\gamma_{\overline{z}}$ depends continuously on $\overline{z}$. Because of the group Γ is properly discontinuous, the equality $\gamma_{\overline{z}}[\beta(\overline{z})] = \overline{z}$ holds for every $\overline{z} \in \Delta$. Therefore, $\gamma_{\overline{z}} = \gamma$ and $\beta = \gamma^{-1} \in \Gamma$. □

The next lemma shows that a lift $\overline{h}$ of h induces an automorphism of the group Γ.

Lemma 6.2 *The mapping $\overline{h}_*\colon \gamma \to \overline{h} \circ \gamma \circ \overline{h}^{-1}$ is an automorphism of the group Γ.*

Proof. One can check directly that $\overline{h}_*(\gamma_1 \circ \gamma_2) = \overline{h}_*(\gamma_1) \circ \overline{h}_*(\gamma_2)$ and $\overline{h}_*(id) = id$. One remains to prove that $\overline{h} \circ \gamma \circ \overline{h}^{-1} \in \Gamma$. Since $\overline{h}$ and $\overline{h}^{-1}$ are cover transformations, $\pi \circ \overline{h} = h \circ \pi$ and $\pi \circ \overline{h}^{-1} = h^{-1} \circ \pi$. Then

$$\pi(\overline{h} \circ \gamma \circ \overline{h}^{-1}) = h \circ \pi(\gamma \circ \overline{h}^{-1}) = h \circ \pi(\overline{h}^{-1}) = h \circ h^{-1} \circ \pi = \pi.$$

By Lemma 6.1, $\overline{h} \circ \gamma \circ \overline{h}^{-1} \in \Gamma$. □

Note that if we choose a base point $z \in M$ and $\overline{z} \in \pi^{-1}(z)$, then Γ is naturally isomorphic to the fundamental group $\pi_1(M, z)$: given any $\gamma \in \Gamma$, one corresponds the element of $\pi_1(M, z)$ containing the closed curve $\pi(\widehat{\overline{z}\gamma(\overline{z})}) \subset$ $\subset M$, where $\widehat{\overline{z}\gamma(\overline{z})}$ is any path connecting $\overline{z}$ and $\gamma(\overline{z})$.

Recall that each non-identity transformation $\gamma \in \Gamma$ has the axis, a unique geodesic $\overline{g}(\gamma) \subset \Delta$ that is invariant under γ. Its projection, $\pi(\overline{g}(\gamma)) \overset{\text{def}}{=} g_\gamma$, is a closed geodesic on M. Every nontrivial element of $\pi_1(M)$ contains a closed geodesic (as a representative of free homotopy class of closed freely homotopic curves) which is the projection of the axis of some $\gamma \in \Gamma$, $\gamma \neq id$. The automorphism $\overline{h}_*$ can be interpreted by using closed geodesics as follows.

Lemma 6.3 *$\overline{h}_*(\gamma) = \gamma'$ if and only if the curve $h(g_\gamma)$ is freely homotopic to the geodesic $g_{\gamma'}$.*

Proof. Let $\overline{h}_*(\gamma) = \gamma' = \overline{h} \circ \gamma \circ \overline{h}^{-1}$. Then

$$\gamma' \circ \overline{h}(\overline{g}(\gamma)) = \overline{h}\gamma(\overline{g}(\gamma)) = \overline{h}(\overline{g}(\gamma)).$$

Hence, $\overline{h}$ takes $\overline{g}(\gamma)$ to the curve $\overline{h}(\overline{g}(\gamma))$ invariant under γ'. By Theorem 5.48, the ideal endpoints of $\overline{h}(\overline{g}(\gamma))$ belong to S_∞. Therefore, $\overline{h}(\overline{g}(\gamma))$ and $\overline{g}(\gamma')$ have the same ideal points because they are unique points invariant under γ'. So, $\overline{g}(\gamma')$ is a co-asymptotic geodesic for $\overline{h}(\overline{g}(\gamma))$. It follows that $h(g_\gamma)$ is freely homotopic to the geodesic $g_{\gamma'}$.

Suppose that the curve $h(g_\gamma)$ is freely homotopic to $g_{\gamma'}$. Hence, $\overline{g}_{\gamma'}$ is a co-asymptotic geodesic for $\overline{h}(\overline{g}_\gamma)$. Therefore, there is a lift of $h(g_\gamma)$ that coincides with the curve $\overline{h}(\overline{g}(\gamma))$. This means that $\overline{h}_*(\gamma) = \gamma'$. □

Obviously that $h\colon M \to M$ has many lifts, for example $\gamma \circ \overline{h}$ for every $\gamma \in \Gamma$. Show that these types are all possible lifts.

Lemma 6.4 *Let $\overline{h}_1, \overline{h}_2 \colon \Delta \to \Delta$ be lifts of $h\colon M \to M$. Then there is $\gamma_{12} \in \Gamma$ such that $\overline{h}_2 = \gamma_{12} \circ \overline{h}_1$.*

Proof. Since $\overline{h}_1$, $\overline{h}_2$ are lifts of the same map, given any point $\overline{z} \in \Delta$, there is $\gamma_{12} \in \Gamma$ such that $\overline{h}_2(\overline{z}) = \gamma_{12} \circ \overline{h}_1(\overline{z})$. Because of the group Γ is discrete, γ_{12} does not depend on $\overline{z}$. □

As a consequence, we get the relation between automorphisms of the group Γ induced by different lifts of the same surface homeomorphism.

Corollary 6.1 *Let $\overline{h}_1, \overline{h}_2 \colon \Delta \to \Delta$ be lifts of $h\colon M \to M$ i. e., $\overline{h}_2 = \gamma_{12} \circ \overline{h}_1$ for some $\gamma_{12} \in \Gamma$. Then $\overline{h}_{2*} = \gamma_{12} \circ \overline{h}_{1*} \circ \gamma_{12}^{-1}$.*

Proof. Indeed, it follows from Lemmas 6.2 and 6.4 that

$$\overline{h}_{2*}(\gamma) = \overline{h}_2 \circ \gamma \circ \overline{h}_2^{-1} = \gamma_{12} \circ \overline{h}_1 \circ \gamma \circ \overline{h}_1^{-1} \circ \gamma_{12}^{-1} = \gamma_{12} \circ \overline{h}_{1*} \circ \gamma_{12}^{-1}. \quad \square$$

Extension of cover map to the circle at infinity

Lemma 6.5 *Given any point $P \in S_\infty$, there is a simple closed geodesic $\alpha \subset \subset M$ and lifts $\overline{\alpha}_i \subset \Delta$ such that $P = \bigcap_{j=1}^{\infty} N_j$, where N_j is the component of $\Delta - \overline{\alpha}_j$ whose closure in $\Delta \cup S_\infty$ contains P.*

Proof. Let $\overline{g}$ be the geodesic ray (half-infinite geodesic) connecting the origin $O \in \Delta$ and P. There is a collection of (non-disjoint) simple closed geodesics which partition M to a disk, Fig. 6.1 (a). Since the half-geodesic $g \stackrel{\text{def}}{=}$ $\stackrel{\text{def}}{=} \pi(\overline{g})$ can not be contained in a disk, g intersects some closed geodesic $\alpha_0 \subset$ $\subset M$ in a countable set of points. Note that g can a priori coincide with some of these closed geodesics. Each intersection of α_0 with g determines a lift $\overline{\alpha}_i$ of α_0 which intersects $\overline{g}$. By ordering $\overline{\alpha}_i$ appropriately we may assume that

$$N_1 \supset N_2 \supset \cdots \supset N_i \supset \cdots \supset P.$$

It follows that the geodesics $\overline{\alpha}_i$ form a monotone sequence, Fig. 6.1 (a). By Lemma 5.4, $P = \bigcap_{i \geqslant 1} N_i$ as desired. □

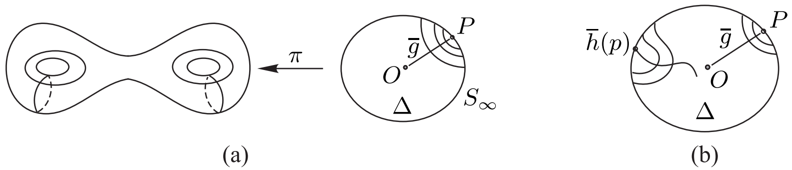


Figure 6.1. The geodesic ray $\overline{g}$ (a); monotone sequence $\{\overline{C}_i\}_{i=1}^{\infty}$ (b).

The following result is fundamental in the Nielsen–Thurston Theory.

Theorem 6.1 *Let $h\colon M \to M$ be an orientation preserving homeomorphism of a closed orientable hyperbolic surface M, and let $\overline{h}\colon \Delta \to \Delta$ be a lift of h to the universal covering surface Δ. Then $\overline{h}$ extends to a unique homeomorphism $\Delta \cup S_\infty \to \Delta \cup S_\infty$.*[1]

Proof. Take any $P \in S_\infty$. Let $\overline{g}$ be the geodesic ray connecting the origin $O \in \Delta$ and P. It follows from Theorem 5.48 that the curve $\overline{h}(\overline{g})$ has an asymptotic direction, say $Q \in S_\infty$. Put by definition, $\overline{h}(P) = Q$.

Since $\overline{h}\big|_\Delta : \Delta \to \Delta$ is a lift of a uniformly continuous homeomorphism, $\overline{h}\big|_\Delta$ is uniformly continuous one itself. Hence, Q does not depend on a representative of P: if a geodesic ray $\overline{g}_1$ has the endpoint P, then the curve $\overline{h}(\overline{g}_1)$ has the same asymptotic direction Q with $\overline{h}(\overline{g})$ because of $\overline{g}$ and $\overline{g}_1$ approach exponentially when current points tend to P.

Later on, the extension of $\overline{h}$ to S_∞, one denotes also by $\overline{h}$. To avoid a confusion, we sometimes will write $\overline{h}\big|_{S_\infty} : S_\infty \to S_\infty$ or $\overline{h}\big|_\Delta : \Delta \to \Delta$.

[1]Here, the disk $\Delta \cup S_\infty$ is endowed with the topology of a unit disk on the Euclidean plane.

Recall that $\mathscr{R}$ denotes the set of rational points of the circle at infinity S_∞. Let us show that $\overline{h}\big|_{S_\infty}$ is a one-to-one map $\mathscr{R} \to \mathscr{R}$. Given any point $\sigma \in \mathscr{R}$, there is a lift $\overline{g}_0$ of some closed geodesic $g_0 \subset M$ such that σ is reached by $\overline{g}_0$. Obviously, $\overline{h}(\overline{g}_0)$ is a lift of the closed curve $h(g_0)$. Since g_0 is not homotopic to zero, $h(g_0)$ is also not homotopic to zero. Hence, there is the closed geodesic $g_* \subset M$ that is freely homotopic to $h(g_0)$. By Lemma 5.2, g_* is the co-asymptotic geodesic for the curve $h(g_0)$. This implies that there is the lift $\overline{g}_*$ of g_* that is co-asymptotic to $\overline{h}(\overline{g}_0)$. It follows from the definition of $\overline{h}\big|_{S_\infty}$ that $\overline{h}\big|_{S_\infty}$ takes the ideal endpoint σ of $\overline{g}_0$ to the ideal endpoint of $\overline{g}_*$. Thus, $\overline{h}\big|_{S_\infty}(\mathscr{R}) \subset \mathscr{R}$. Since every rational point is reached by a unique lift of closed geodesic, the restriction of $\overline{h}\big|_{S_\infty}$ on the set $\mathscr{R}$ is one-to-one.

Note that any extension of $\overline{h}\big|_\Delta$ to S_∞ have to take the ideal endpoints of $\overline{g}_0$ to the ideal endpoints of $\overline{g}_*$ because of uniqueness of co-asymptotic geodesic for a closed homotopy nontrivial curve. This means that any two extensions of $\overline{h}\big|_\Delta$ to S_∞ are coincident on the set $\mathscr{R}$ which is dense in S_∞.

Now we have to prove that the extension $\overline{h}\colon \Delta \cup S_\infty \to \Delta \cup S_\infty$ is continuous. Let $U_1 \supset U_2 \supset \cdots \supset U_i \supset \cdots \supset P$ be a neighborhood system in $\Delta \cup S_\infty$ for P. By lemma 6.5, there is a simple closed geodesic $\alpha \subset M$ and lifts $\overline{\alpha}_i \subset \Delta$ such that $P = \bigcap\limits_{j=1}^{\infty} N_j$, where N_j is the component of $\Delta - \overline{\alpha}_j$ whose closure in $\Delta \cup S_\infty$ contains P. After passing to subsequences, one can assume that $U_i \supset N_i \supset U_{i+1}$. Therefore, in order to prove a continuity of $\overline{h}$, it is enough to show that $\overline{h}(N_i)$ form a neighborhood system for $\overline{h}(P)$ such that $\overline{h}(P) = \bigcap\limits_{i\geqslant 1} \overline{h}(N_i)$. Let $\overline{g}$ be the geodesic ray connecting the origin $O \in \Delta$ and P. Since $P = \bigcap\limits_{j=1}^{\infty} N_j$, the sequence $\overline{\alpha}_i$ is monotone, and $\overline{g}$ intersects the curves $\overline{\alpha}_i$ consecutively. Hence, the sequence $\overline{h}(\overline{\alpha}_i)$ is also monotone, and $\overline{h}(\overline{g})$ intersects the curves $\overline{h}(\overline{\alpha}_i)$ consecutively, Fig. 6.1 (b). Obviously, $h(\alpha)$ is a simple closed homotopy nontrivial curve, and $\overline{h}(\overline{\alpha}_i)$ are lifts of $h(\alpha)$. By Lemma 5.4, the topological limit of $\overline{h}(\overline{\alpha}_i)$ is a unique point of S_∞. Since $\overline{h}(\overline{g})$ intersects $\overline{h}(\overline{\alpha}_i)$ consecutively, Q is the topological limit of $\overline{h}(\overline{\alpha}_i)$. As a consequence, $Q = \overline{h}(P) = \bigcap\limits_{i\geqslant 1} \overline{h}(N_i)$.

Since h preserves orientation, $\overline{h}\big|_\Delta : \Delta \to \Delta$ also preserves orientation. Hence, $\overline{h}\big|_{S_\infty}$ is orientation preserving map of S_∞. We saw above that the restriction of $\overline{h}|_{\mathscr{R}}$ on the set $\mathscr{R}$ is one-to-one and $\mathscr{R}$ is dense in S_∞. It follows from the continuity of $\overline{h}|_{S_\infty}$ that the map $\overline{h}|_{S_\infty} : S_\infty \to S_\infty$ is one-to-one. Hence, $\overline{h}\colon \Delta \cup S_\infty \to \Delta \cup S_\infty$ is one-to-one and continuous. Again, the continuity of $\overline{h}|_{S_\infty}$ implies that the extension $\overline{h}|_{\Delta\cup S_\infty}$ is a unique one of the cover transformation $\overline{h}|_{\Delta}$.

By the similar way, one can prove that the inverse $\overline{h}^{-1}\Big|_{\Delta}$ has a unique extension to $\Delta \cup S_\infty$ that is one-to-one and continuous. This extension coincides with $\overline{h}^{-1}\Big|_{\mathscr{R}}$. Since $\mathscr{R}$ is dense in S_∞, the extension of $\overline{h}^{-1}\Big|_{\Delta}$ equals to $\overline{h}^{-1}\Big|_{\Delta\cup S_\infty}$. $\square$

Fixed points of extended homeomorphisms

The set of fixed points of a map f is denoted by $\mathrm{Fix}(f)$.

Lemma 6.6 *Let $h\colon M \to M$ be an orientation preserving homeomorphism of a closed orientable hyperbolic surface $M = \Delta/\Gamma$ and $\overline{h}$ be a lift of h extending to $\Delta \cup S_\infty$. Suppose $\overline{h}$ has a fixed point $\gamma^+ \in S_\infty$ that is a sink fixed point of some $\gamma \in \Gamma$, $\gamma^+ \in \mathrm{Fix}(\overline{h})$. Then $\overline{h} \circ \gamma = \gamma \circ \overline{h}$. Moreover, if $\overline{h}$ has a fixed point different from the fixed points γ^+, γ^- of γ, then these points γ^+, γ^- are non-isolated in* $\mathrm{Fix}(\overline{h})$.

Proof. One can assume that γ is indivisible. By Lemma 6.2, $\overline{h}\circ\gamma\circ\overline{h}^{-1} \in \Gamma$. For each point $\overline{z} \in \Delta$,

$$\lim_{n\to\infty} (\overline{h} \circ \gamma \circ \overline{h}^{-1})^n(\overline{z}) = \lim_{n\to\infty} (\overline{h} \circ \gamma^n \circ \overline{h}^{-1})(\overline{z}) = \\ = \overline{h} \lim_{n\to\infty} \gamma^n \circ \overline{h}^{-1}(\overline{z}) = \overline{h}(\gamma^+) = \gamma^+.$$

Hence, γ^+ is a sink fixed point of $\overline{h}\circ\gamma\circ\overline{h}^{-1}$. By Lemma 3.5, γ and $\overline{h}\circ\gamma\circ\overline{h}^{-1}$ have the common repelling fixed point γ^-, and the axes of γ and $\overline{h} \circ \gamma \circ \overline{h}^{-1}$ are equal. The discreteness of Γ implies that $\overline{h} \circ \gamma \circ \overline{h}^{-1} = \gamma^k$ for some $k \in \mathbb{N}$. Thus $\gamma = \overline{h}^{-1} \circ \gamma^k \circ \overline{h} = (\overline{h}^{-1} \circ \gamma \circ \overline{h})^k$. By the indivisibility of γ, $k = 1$. So, $\overline{h} \circ \gamma = \gamma \circ \overline{h}$.

Suppose that there is a point $P \in \mathrm{Fix}(\overline{h}) - \{\gamma^+, \gamma^-\}$. Since $\overline{h}$ and γ commute, $\gamma^n(P)$ is a fixed point of $\overline{h}$ for any $n \in \mathbb{Z}$. Then $\lim_{n\to\pm\infty} \gamma^n(P) = \gamma^\pm$. This means that γ^+ and γ^- are non-isolated in $\mathrm{Fix}(\overline{h})$. $\square$

Corollary 6.2 *Let the condition of Lemma* 6.6 *holds. Suppose that* $\overline{h}\big|_{S_\infty} = id$. *Then* $\overline{h}$ *commutes with every deck transformation.*

Corollary 6.3 *Let the condition of Lemma* 6.6 *holds. Suppose that* $\overline{h}\big|_{S_\infty} = id$. *If* $\overline{z} \in \mathrm{Fix}(\overline{h}) \cap \Delta$ *then* $\gamma(\overline{z}) \in \mathrm{Fix}(\overline{h})$ *for any* $\gamma \in \Gamma$.

Lemma 6.7 *Let* $h_1, h_2 \colon M \to M$ *be orientation preserving homeomorphisms of a closed orientable hyperbolic surface* M. *If* h_1 *is homotopic to* h_2, *then given any lift* $\overline{h}_1$, *there is a lift* $\overline{h}_2$ *of* h_2 *such that* $\overline{h}_1\big|_{S_\infty} = \overline{h}_2\big|_{S_\infty}$. *Vice versa, if* h_1 *and* h_2 *have lifts* $\overline{h}_1$, $\overline{h}_2$ *respectively such that* $\overline{h}_1\big|_{S_\infty} = \overline{h}_2\big|_{S_\infty}$, *then* h_1 *is homotopic to* h_2.

Proof. Let $H \colon M \times [0;\,1] \to M$ be a homotopy between h_1 and h_2. Let $\overline{H} \colon \Delta \times [0;\,1] \to \Delta$ be the lift of H such that $\overline{H}_1 \colon \Delta \times \{0\} \to \Delta$ equals to $\overline{h}_1$. Since H is uniformly continuous with respect to the hyperbolic metric, it follows that non-Euclidean lengths of the arcs $\overline{H}(z \times [0;\,1])$ are bounded for any $z \in \Delta$. Therefore the Euclidean distance between $\overline{h}_1(z)$ and $\overline{h}_2(z)$ tends to zero as z tends to S_∞. Hence, $\overline{h}_1\big|_{S_\infty} = \overline{h}_2\big|_{S_\infty}$.

Suppose $\overline{h}_1\big|_{S_\infty} = \overline{h}_2\big|_{S_\infty}$ for some lifts $\overline{h}_1$, $\overline{h}_2$ of h_1 and h_2 respectively. Choose base points $z_0 \in M$ and $\overline{z}_0 \in \pi^{-1}(z_0)$. After an isotopy of one of the homeomorphisms, say h_1, and the corresponding isotopy of $\overline{h}_1$, one can assume that $\overline{h} \stackrel{\text{def}}{=} \overline{h}_2^{-1} \circ \overline{h}_1$ fixes $\overline{z}_0$, $\overline{h}(\overline{z}_0) = \overline{z}_0$. By Corollaries 6.2 and 6.3, $\gamma(\overline{z}_0) \in \mathrm{Fix}(\overline{h})$ for any $\gamma \in \Gamma$. Then

$$\overline{h}_*(\gamma)(\overline{z}_0) = \overline{h} \circ \gamma \circ \overline{h}^{-1}(\overline{z}_0) = \gamma(\overline{z}_0)$$

for any $\gamma \in \Gamma$. Hence, $\overline{h}_*(\gamma) = \gamma$. It follows from Lemmas 6.2, 6.3 that $h = h_2^{-1} \circ h_1$ induces the identity map of $\pi_1(M)$. This means that $h_2^{-1} \circ h_1$ is homotopic to the identity. $\square$

6.2. Irrational foliations on 2-torus

Let $\mathscr{F}$ be a strongly irrational foliation on the torus $\mathbb{T}^2$. By Theorem 3.7,

$$\sum_{i=1}^{k} \mathrm{ind}\, s_i = \chi(\mathbb{T}^2) = 0,$$

where $s_1, \ldots, s_k$ are singularities of $\mathscr{F}$. Since any singularity of strongly irrational foliation has negative index, $\mathscr{F}$ has no singularities at all. We know that $\mathscr{F}$ has a homotopy nontrivial closed transversal C. Cutting $\mathbb{T}^2$ along C, one gets the cylinder A with the boundary components C_1, C_2. The foliation $\mathscr{F}$ becomes the foliation $\mathscr{F}_1$ on A which is transversal to the boundary $\partial A = C_1 \cup C_2$. It follows from Corollary 3.6 that every leaf of $\mathscr{F}_1$ starting from C_1 has to go to C_2. Hence, the foliation $\mathscr{F}$ is orientable and can be considered as a flow that is a suspension over Poincaré forward mapping $C \to C$.

As a result, we can study flows instead of strongly irrational foliations on $\mathbb{T}^2$. First of all, one introduces the classical invariant (not complete, in general) called the Poincaré rotation number of a flow. This invariant is defined only for a flow which has a semitrajectory with an asymptotic direction.

Poincaré rotation number for torus flows

Let f^t be a flow on the torus $\mathbb{T}^2$ and l^+ be a positive semitrajectory of f^t having asymptotic direction. Let $\overline{l}^+$ be a lift of l^+ on $\mathbb{R}^2$. The co-asymptotic geodesic of $\overline{l}^+$ is the straight line denoted by $L(\overline{l}^+)$. Since the torus $\mathbb{T}^2$ is a closed surface of genus $g = 1$ and any straight line divides the plane $\mathbb{R}^2$ into two half-planes, the line $L(\overline{l}^+)$ is the co-asymptotic geodesic for any semitrajectory (positive or negative) of f^t having asymptotic direction. Note that lifts of another semitrajectories can be diametrically opposite. Thus, $L(\overline{l}^+) = L(f^t)$ depends on the flow f^t but not on the choosing of a semitrajectory and their lifts.

The straight line $L(\overline{l}^+)$ is defined by the equation either $y = \mu x$ or $x = 0$. The last case means that $\mu = \infty$. In the both cases, the value μ is called a *Poincaré rotation number* of f^t (in short, *rotation number*) and is denoted by $\mathrm{rot}(f^t)$.

Remark. Along with a covering flow $\overline{f}^t$ on $\mathbb{R}^2$ we consider a covering flow $\widetilde{f}^t$ on the unit disk D^2 that is a universal covering space with respect to the covering map $\pi \circ \tau$, see Section 5.1. A lift $\widetilde{l}^+$ of l^+ has the asymptotic direction represented by a unique point

$$\sigma(\widetilde{l}^+) = (\mu_1, \mu_2) \in \partial D^2 = S_\infty,$$

$$\text{where} \quad \mu = \mathrm{rot}(f^t) = \frac{\mu_2}{\mu_1}, \quad \mu_1 = \frac{1}{\sqrt{1+\mu^2}}, \quad \mu_2 = \frac{\mu}{\sqrt{1+\mu^2}}.$$

We shall see that the rotation number up to recalculation under unimodular integer matrix is the topological invariant. To show this we need some information on the structure of mappings of torus and its covers. We formulate the statement for any dimension, and apply the following natural identification

$$\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n, \quad \mathbb{Z}^n = \pi_1(\mathbb{T}^n) = H_1(\mathbb{T}^n, \mathbb{Z}), \quad \mathbb{R}^n = H_1(\mathbb{T}^n, \mathbb{R}).$$

Lemma 6.8 *Let $h\colon \mathbb{T}^n \to \mathbb{T}^n$ be a continuous map, and $\overline{h}\colon \mathbb{R}^n \to \mathbb{R}^n$ be a covering mapping for h. Let $h_*\colon H_1(\mathbb{T}^n, \mathbb{R}) \to H_1(\mathbb{T}^n, \mathbb{R})$ be the map induced by h considered as the map of $\mathbb{R}^n = H_1(\mathbb{T}^n, \mathbb{R})$. Then there is the $n \times n$ integer matrix $A = (a_{ij})$ such that*

- *the map $\overline{h}_0\colon \mathbb{R}^n \to \mathbb{R}^n$ defined by $\vec{x} \mapsto A\vec{x}$ is a covering mapping for some linear map $h_0\colon \mathbb{T}^n \to \mathbb{T}^n$ that is homotopic to h;*
- $(h_0)_* = h_*$;
- *any covering mapping $\overline{h} = (\overline{h}_1, \ldots, \overline{h}_n)\colon \mathbb{R}^n \to \mathbb{R}^n$ has the form $\overline{h}(\vec{x}) = A\vec{x} + \vec{\psi}(\vec{x})$, $\vec{x} = (x_1, \ldots, x_n)$, $\vec{\psi}(\vec{x}) = (\psi_1(\vec{x}), \ldots, \psi_n(\vec{x}))$, i. e.,*

$$\begin{aligned}
\overline{h}_1(x_1, \ldots, x_n) &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + \psi_1(x_1, \ldots, x_n),\\
\overline{h}_2(x_1, \ldots, x_n) &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + \psi_2(x_1, \ldots, x_n),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
\overline{h}_n(x_1, \ldots, x_n) &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + \psi_n(x_1, \ldots, x_n),
\end{aligned}$$

where $\psi_1(x_1, \ldots, x_n)$, …, $\psi_n(x_1, \ldots, x_n)$ are continuous periodic functions with period 1 in each argument. If h is a homeomorphism, then $\det A = \pm 1$.

Proof. Since $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, h_* is defined by some integer matrix $A = (a_{ij})$. Given any point $\overline{x} \in \mathbb{R}^n$ with integer coordinates, $A\overline{x}$ is a point with integer coordinates as well. Therefore, the map $\overline{h}_0\colon \mathbb{R}^n \to \mathbb{R}^n$ defined by $\vec{x} \mapsto A\vec{x}$ is a covering mapping for some linear map

$$h_0\colon \mathbb{R}^n/\mathbb{Z}^n = \mathbb{T}^n \to \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n.$$

Since h_* is induced by h, $(h_0)_* = h_*$. As a consequence, h_0 is homotopic to h because of $\mathbb{T}^n$ is a $(\pi, 1)$-space (see [209], Ch. 8).

Let $\overline{h}$ be a covering mapping for h. There is a shift $\gamma_0\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $\gamma_0 \circ \overline{h}(0) = 0$. Since γ_0 maps congruent points to congruent points, $\gamma_0 \circ \overline{h}$ is a covering mapping for the transformation that is homotopic to h, and as a consequence, to h_0. It follows that the mapping $\gamma_0 \circ \overline{h} - A$ keeps the integer lattice $\mathbb{Z}^n$, and is defined by a periodic vector-function $\vec{\psi}'$. Then the desired vector-function is $\vec{\psi} = \vec{\psi} - \gamma_0$.

If h is a homeomorphism, then h_* is an isomorphism. Hence, $\det A = \pm 1$. □

Note that the last item of Lemma 6.8 is similar to Lemma 2.3.

Proposition 6.1 *Let f^t, g^t be flows on $\mathbb{T}^2$ and $\varphi\colon \mathbb{T}^2 \to \mathbb{T}^2$ be a homeomorphism that takes a semitrajectory l_f^+ of f^t into a semitrajectory l_g^+ of g^t. Suppose that l_f^+ has an asymptotic direction (thus, $\mathrm{rot}(f^t)$ exists). Then $\varphi(l_f^+) = l_g^+$ has an asymptotic direction (thus, $\mathrm{rot}(g^t)$ exists), and*

$$\mathrm{rot}(g^t) = \frac{c + d\,\mathrm{rot}(f^t)}{a + b\,\mathrm{rot}(f^t)}$$

for some integer unimodular matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

Proof. Let $\overline{f}^t$, $\overline{g}^t$ be covering flows for f^t, g^t respectively, and $\overline{\varphi}\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a covering mapping for φ such that $\overline{\varphi}$ takes the lift $\overline{l}_f^+$ of l_f^+ into the lift $\overline{l}_g^+$ of l_g^+. The semitrajectory $\overline{l}_f^+$ is described by parametric equations $x = x(t)$, $y = y(t)$. Since l_f^+ has an asymptotic direction, $x^2(t) + y^2(t) \to \infty$ as $t \to \infty$. Moreover, $\lim\limits_{t\to\infty} \frac{y(t)}{x(t)} = \mathrm{rot}(f^t)$. Here, we assume that $\lim\limits_{t\to\infty} x(t) = \infty$ (if $\lim\limits_{t\to\infty} y(t) = \infty$, the proof is similar). By Lemma 6.8, the mapping $\overline{\varphi}$ has the form

$$\overline{x} = ax + by + \psi_1, \quad \overline{y} = cx + dy + \psi_2$$

for some integer unimodular matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and continuous periodic functions ψ_1, ψ_2 with period 1 in each argument. Calculations show that $\overline{x}^2(t) + \overline{y}^2(t) \to \infty$ as $t \to \infty$. By Theorem 5.4, the semitrajectory l_g^+ has an asymptotic direction. Then

$$\mathrm{rot}(g^t) = \lim_{t\to\infty} \frac{\overline{y}(t)}{\overline{x}(t)} = \lim_{t\to\infty} \frac{cx + dy + \psi_2}{ax + by + \psi_1} = \lim_{t\to\infty} \frac{c + \frac{y}{x} d + \frac{\psi_2}{x}}{a + \frac{y}{x} b + \frac{\psi_1}{x}} = \frac{c + d \cdot \mathrm{rot}(f^t)}{a + b \cdot \mathrm{rot}(f^t)}. \quad \square$$

Proposition 6.1 means that a Poincaré rotation number is an invariant of the topological equivalence up to the recalculation by an integer unimodular matrix. As a consequence, one gets the following statement. However, we give the slightly different proof to get the more convenient formulas.

Theorem 6.2 *Let f_1^t and f_2^t be flows on $\mathbb{T}^2$ such that the both f_1^t and f_2^t have nontrivially recurrent trajectories. If f_1^t and f_2^t are topologically equivalent, then there is the integer unimodular matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ such that*

$$\operatorname{rot}(f_2^t) = \frac{-c + a \cdot \operatorname{rot}(f_1^t)}{d - b \cdot \operatorname{rot}(f_1^t)}, \quad d - b \cdot \operatorname{rot}(f_1^t) \neq 0. \tag{6.2}$$

Proof. Let $h\colon \mathbb{T}^2 \to \mathbb{T}^2$ be the homeomorphism that maps the trajectories of f_2^t into the trajectories of f_1^t, and $\overline{h}\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a covering mapping for h. By Lemma 6.8,

$$x = a\overline{x} + b\overline{y} + \psi_1, \quad y = c\overline{x} + d\overline{y} + \psi_2$$

for some integer unimodular matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and continuous periodic functions ψ_1, ψ_2 with period 1 in each argument. Denote $\lambda_1 = \operatorname{rot}(f_1^t)$, $\lambda_2 = \operatorname{rot}(f_2^t)$. The equality $y = \lambda_1 x$ becomes

$$\overline{y}(d - \lambda_1 b) = \overline{x}(-c + \lambda_1 a) + \lambda_1 \psi_1 - \psi_2.$$

Hence,

$$\lambda_2 = \lim_{\overline{x}\to\infty} \frac{\overline{y}}{\overline{x}} = \frac{-c + a\lambda_1}{d - b\lambda_1}. \quad \square$$

Classification of irrational flows on torus

Let f^t be a flow on a torus $\mathbb{T}^2$. Suppose that f^t has a nontrivially recurrent trajectory l. Let $\pi\colon \mathbb{R}^2 \to \mathbb{T}^2$ be the covering projection, and $\overline{l}$ be a lift of l. By Theorem 5.20, $\overline{l}$ has an asymptotic direction with the co-asymptotic geodesic which is a straight line $y = kx$. The existence of nontrivially recurrent trajectory implies the nonexistence of periodic trajectories that are non-homotopic to zero. Therefore, the rotation number $k = \operatorname{rot}(f^t)$ is irrational.

Now, let f^t be an irrational flow on a torus $\mathbb{T}^2$. Recall that all trajectories of f^t are nontrivially recurrent and f^t has no fixed points.

Theorem 6.3 *Let f^t be an irrational flow on $\mathbb{T}^2$. Then f^t is topologically equivalent to a linear flow of the form*

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = \mu. \tag{6.3}$$

Proof. According to Theorem 3.4 (see Corollary 3.5 also) and Lemma 3.22, there is a simple closed transversal that is an embedded circle $S^1 \subset \mathbb{T}^2$

which is non-homotopic to zero. The flow f^t induces the Poincaré forward mapping $f\colon S^1 \to S^1$. There is a homeomorphism $\mathbb{T}^2 \to \mathbb{T}^2$ taking the transversal S^1 to the meridian $l_0 = \pi(x = 0)$ of $\mathbb{T}^2$. Therefore, without loss of generality, one can assume that S^1 is l_0, and f^t is a suspension over f, $f^t = \mathrm{sus}^t(f)$. By Lemma 1.2, f is transitive, since f^t is transitive. According to Theorem 2.6, f conjugates to the rigid rotation $R_\mu\colon S^1 \to S^1$ via an orientation preserving homeomorphism. Clearly, the suspension $\mathrm{sus}^t(R_\mu)$ over R_μ is the linear flow of the form (6.3). The conjugacy mapping between f and R_μ can be extended to a homeomorphism $\mathbb{T}^2 \to \mathbb{T}^2$ which takes $\mathrm{sus}^t(f)$ to $\mathrm{sus}^t(R_\mu)$. Thus, we get the result. □

Remark that one can construct homeomorphism taking f^t to the linear flow to be homotopic to the identity (see [26, 66]).

As a consequence of Theorems 6.2, 6.3, we get the following classical result.

Theorem 6.4 *Let f_1^t and f_2^t be irrational flows on $\mathbb{T}^2$. Then f_1^t and f_2^t are topologically equivalent if and only if there is the integer unimodular matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ such that* (6.2) *holds.*

The linear flow (6.3) actually represents the geodesic framework of the original irrational flow because every trajectory of (6.3) is a geodesic on $\mathbb{T}^2$. Thus, we can reformulate the classification results for irrational flows on $\mathbb{T}^2$ in the spirit of the book as follows.

Theorem 6.5 *Any irrational flow on the torus $\mathbb{T}^2$ is topologically equivalent to its own geodesic framework via a homeomorphism homotopic to the identity; this geodesic framework is a linear irrational flow on $\mathbb{T}^2$. If two irrational flows on $\mathbb{T}^2$ are topologically equivalent via a homeomorphism $\mathbb{T}^2 \to \mathbb{T}^2$ homotopic to the identity, then their geodesic frameworks coincide.*

Two irrational flows on $\mathbb{T}^2$ are topologically equivalent if and only if there is a linear diffeomorphism $A\colon \mathbb{T}^2 \to \mathbb{T}^2$ defined by unimodular integer matrix such that A takes the geodesic framework of one flow to the geodesic framework of another flow.

6.3. Irrational foliations on hyperbolic surfaces

Recall that a foliation (in particular, an orientable foliation, a flow) on a surface M is called *irrational* if any (one-dimensional) leaf is everywhere dense on M and all singularities are saddles of nonzero index. The latter condition is sometimes formulated as the absence of fake saddles. The condition that each leaf is dense on M is equivalent to the transitivity and to the absence of separatrix connections (including separatrix loops). Recall that only the requirement that each leaf is dense (fake saddles may exist) means that the foliation is *highly transitive* [78]. Thus, an irrational foliation can be defined as a highly transitive foliation without fake saddles and with a finite number of singularities. A strongly irrational foliation is irrational one with no thorns (all singularities have negative index). Obviously, an irrational foliation contains nontrivially recurrent leaves. It follows from Theorem 3.4 that each such foliation has a closed simple transversal.

Lemma 6.9 *Let $\mathscr{F}$ be an irrational foliation and C be a closed transversal of $\mathscr{F}$. Then C intersects every leaf of $\mathscr{F}$. If $\mathscr{F}$ is strongly irrational, then C is a homotopy nontrivial curve.*

Proof. Since every leaf of $\mathscr{F}$ is dense, C intersects every leaf of $\mathscr{F}$. Suppose that $\mathscr{F}$ is strongly irrational and C is homotopy trivial. Then C bounds a disk $D \subset M$. It follows from Lemma 3.22 that $\mathrm{ind}(C, \mathscr{F}) = 1$. By Theorem 3.6, there are singularities inside of D with positive index. This contradicts to the condition that a strongly irrational foliation has only singularities with negative index. □

This lemma induces the following definition. A (simple) closed transversal C of a foliation $\mathscr{F}$ is a *global section* if C intersects all (one-dimensional) leaves of $\mathscr{F}$. Thus, a global section of strongly irrational foliation defines a nontrivial element of the fundamental group $\pi_1(M)$ that contains a simple curve. In other words, the global section is freely homotopic to a simple closed curve that represents a nontrivial element of $\pi_1(M)$. It is natural to pay attention to the question on existing of global sections in a given non-zero freely homotopy class that contains simple curves.

On existing of global sections

Let $\mathscr{F}$ be a strongly irrational foliation on a closed orientable surface M. Recall (see Theorem 5.34) that the geodesic framework $G(\mathscr{F})$ of $\mathscr{F}$ is a strongly irrational geodesic lamination such that each component of $M - G(\mathscr{F})$ is an open geodesic polygon with a finite number of sides and ideal vertices. Here, each ideal vertex corresponds to an asymptotical direction of pair neighbor sides of the polygon. Note that if C is a global section of $\mathscr{F}$ then C is freely homotopic to the co-asymptotic geodesic $g(C)$ which is a simple closed geodesic. Since all geodesics of $G(\mathscr{F})$ are non-closed, $g(C)$ intersects any geodesic of $G(\mathscr{F})$ transversally.

Lemma 6.10 *Let $\mathscr{F}$ be a strongly irrational foliation on a closed orientable hyperbolic surface M and C be a global section of $\mathscr{F}$. Then given any geodesic polygon P of $M - G(\mathscr{F})$, $g(C)$ can consequently intersect only neighbor sides of P.*

Proof. Suppose the contrary. Then there are a lift $\overline{g}$ of $g(C)$ and a lift $\overline{P}$ of P such that $\overline{g}$ intersects consequently two sides of $\overline{P}$ that are not neighbor. Hence, each of the arcs I^-, I^+ of S_∞ into which S_∞ is divided by the endpoints $\overline{g}^-$ and $\overline{g}^+$ of $\overline{g}$ contains at least two vertices of the polygon $\overline{P}$, see Fig. 6.2, (a).

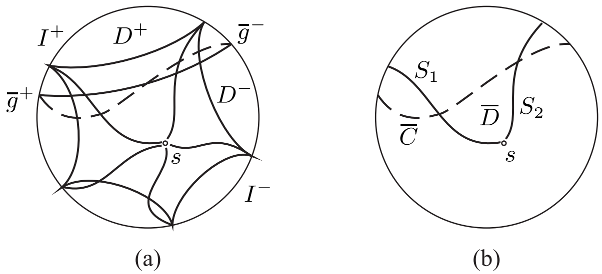


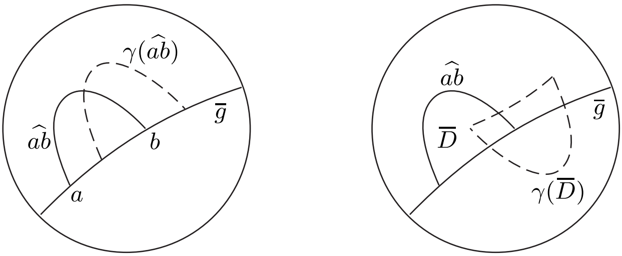
Figure 6.2

Let $\overline{C}$ be a lift of C that is co-asymptotic to $\overline{g}$. Denote by D^- and D^+ the domains of Δ into which Δ is divided by the curve $\overline{C}$. Without loss of generality, we can assume that the ideal boundary of D^+ is the arc I^+. Respectively, the ideal boundary of D^- is I^-. By Theorem 5.34, the polygon $\overline{P}$ corresponds to a saddle s of the covering foliation $\overline{\mathscr{F}}$ such that given any vertex of $\overline{P}$, there is a unique separatrix of s that reaches the vertex. Suppose, for definiteness, that s lies in D^-. Take two neighbor separatrices S_1, S_2 that reach points in I^+. Since $s \in D^-$, the both S_1 and S_2 intersect $\overline{C}$. Let $\overline{D}$ be a domain bounded by S_1, S_2, and $\overline{C}$, Fig. 6.2, (b). Due to Lemma 3.22 and Theorem 3.6, the sum of indices of singularities lying in $\overline{D}$ equals to $\frac{1}{2}$. This is impossible because of all singularities have negative indices. □

Now we are going to prove that the condition of Lemma 6.10 on a simple closed geodesic g can consequently intersect only neighbor sides of polygons from $M - G(\mathscr{F})$ is sufficient for the existence of global section which is freely homotopic to g. First, we prove two technical lemmas.

Lemma 6.11 *Let $\mathscr{F}$ be a foliation on a closed orientable hyperbolic surface M, $\overline{l}$ be a lift of the covering foliation $\overline{\mathscr{F}}$, and $\overline{g}$ be a lift of closed simple curve $g \subset M$. Suppose $\overline{l} \cap \overline{g} \neq \varnothing$. Then given any $\overline{g}$-arc $\widehat{ab}$ of $\overline{l}$ with endpoints $a, b \in \overline{g}$, the segment $[a, b] \subset \overline{g}$ has no congruent points provided l is non-closed.*

Proof. Suppose the contrary. Then there is an element $\gamma \in \Gamma$, where $M \simeq \simeq \Delta/\Gamma$ and Γ is a group of deck transformations, such that $\gamma(a) \in [a, b]$, $\gamma \neq id$. Because of g is a simple curve, $\gamma(\overline{g}) = \overline{g}$. Since γ is a orientation preserving hyperbolic transformation, $\gamma(b) \notin [a, b]$ and $\gamma(\widehat{ab})$ must intersect $\widehat{ab}$, Fig. 6.3, (a). This contradicts to the leaf l being a curve with no self-intersections. □



(a) (b)

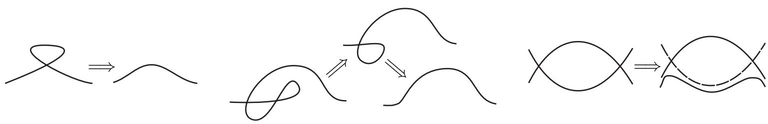
Figure 6.3

Lemma 6.12 *Let the condition of Lemma* 6.11 *holds. Suppose that the $\overline{g}$-arc $\widehat{ab}$ of $\overline{l}$ has no intersections with any curve congruent to $\overline{g}$ except $\overline{g}$. Then there are no congruent points in the closed disk bounded by the $\overline{g}$-loops $\widehat{ab} \cup [a, b]$.*

Proof. Denote by $\overline{D}$ the closed disk bounded by the $\overline{g}$-loops $\widehat{ab} \cup [a, b]$. Suppose the contrary. Then there is an element $\gamma \in \Gamma$ such that $\gamma(\overline{D}) \cap \overline{D} \neq \neq \varnothing$. By Lemma 6.11, $\gamma([a, b]) \cap [a, b] = \varnothing$. Moreover, $\gamma(\overline{g}) \cap \overline{g} = \varnothing$. Hence, $\gamma([a, b]) \cap \widehat{ab} \neq \varnothing$, Fig. 6.3, (b). Since $\gamma([a, b]) \subset \gamma(\overline{g})$, we get the contradiction with the condition that $\overline{l}$ has no intersections with any curve congruent to $\overline{g}$ except $\overline{g}$. □

Now we represent the procedure, called a *simplifying procedure*, that we'll often apply proving Lemma 6.13. Note that in Lemmas 6.11, 6.12, l could be just infinite simple curve, not necessaryly a non-closed leaf of foliation. This remark we take in mind describing the simplifying procedure.

Let C be a closed curve freely homotopic to a homotopy nontrivial simple closed curve, say C_0. Suppose that C has only transversal self-intersections. First, one deletes "small" loops formed by arcs of C as follows. It is convenient to use here a lift $\overline{C}$ of C on $\Delta/\Gamma \simeq M$. Then loops formed by arcs of $\overline{C}$ bound disks that can be easily deleted as shown in Fig. 6.4, (a), (b). Choose a loop, say L, that does not intersect other part of $\overline{C}$, and bounds a minimal (where the order is defined by an inclusion) disk, say D. By Lemmas 6.11, 6.12, D has no congruent points. Therefore, after that we delete D (see Fig. 6.4, (a)), one gets a lift of a closed curve freely homotopic to C_0. The consecutive deleting of "small" loops (see Fig. 6.4, (a)) gives us a curve $\overline{C}_1$ without self-intersections such that $\pi(\overline{C}_1)$ is a closed curve freely homotopic to C_0. A priori, $\overline{C}_1$ can intersect curves from $\Gamma(\overline{C})$, a set of curves congruent to $\overline{C}_1$. Take $\overline{C}_2 \in \Gamma(\overline{C})$ such that $\overline{C}_1 \cap \overline{C}_2 \neq \varnothing$.[2] The endpoints of curves $\overline{C}_1, \overline{C}_2 \in \Gamma(\overline{C})$ are not separated on the circle at infinity S_∞ because of they coincide with the endpoints of corresponding lifts of the same closed simple curve C_0. Therefore, the points of the intersection $\overline{C}_1 \cap \overline{C}_2$ divide each curve $\overline{C}_1, \overline{C}_2$ into arcs that bound finite disks. Consecutive deleting of minimal disks (see Fig. 6.4, (c)) gives a curve $\overline{C}_*$ such that the set $\Gamma(\overline{C}_*)$ consists of pairwise disjoint curves and the curve $\pi(\overline{C}_*)$ is a closed curve freely homotopic to C_0. Therefore, $\pi(\overline{C}_*)$ is simple. We say that the curve $\overline{C}_*$ is obtained from $\overline{C}$ by the simplifying procedure. Note that this procedure is not uniquely defined by $\overline{C}$.



(a) (b) (c)

Figure 6.4. A simplifying procedure.

The following lemma is in sense central for a classification of irrational foliations on hyperbolic surfaces. It says that to find a global section one needs

[2]Recall that given $N \subset \Delta$, $\Gamma(N)$ means the family of all sets $\gamma(N)$, $\gamma \in \Gamma$.

to look how a simple closed geodesic intersects polygons formed by a geodesic framework.

Lemma 6.13 *Let $\mathscr{F}$ be a strongly irrational foliation on a closed orientable hyperbolic surface $\Delta/\Gamma \simeq M$, and g_0 be a simple closed geodesic. In the case of orientable foliation $\mathscr{F}$, one assumes that g_0 orientably intersects all geodesics from the geodesic framework $G(\mathscr{F})$.*[3] *Suppose that given any geodesic polygon P of $M - G(\mathscr{F})$, g_0 consequently intersects only neighbor sides of P. Then there exists a global section C of $\mathscr{F}$ that is freely homotopic to g_0.*

Proof. Obviously, it suffices to prove the lemma for a foliation that is topologically equivalent to $\mathscr{F}$ via a homeomorphism homotopic to the identity. It follows from [103] that without loss of generality we can assume that $\mathscr{F}$ is a C^1 foliation. Slightly moving g_0, we may also assume that g_0 does not pass through the singularities, and is in a general position with respect to the leaves of $\mathscr{F}$, and any leaf is tangent to g_0 at most at one point.

Take a lift $\overline{g}_0$ of g_0 and a lift $\overline{l}$ of nontrivially recurrent leaf l such that the co-asymptotic geodesic $\overline{g}(\overline{l})$ intersects $\overline{g}_0$. Since g_0 intersects $G(\mathscr{F})$, such lifts exist. The intersection $\overline{g}_0 \cap \overline{g}(\overline{l})$ consists of a unique point. Therefore, $\overline{l}$ intersects $\overline{g}_0$. Without loss of generality, one can assume that the leaf l has no tangencies with g_0. Thus, $\overline{l}$ has no tangencies with $\overline{g}_0$. Let us endow $\overline{l}$ with some orientation. Recall that the $\omega(\alpha)$-limit set of $\overline{l}$ is a unique point $\omega(\alpha)(\overline{l})$ that belongs to the circle at infinity S_∞. Because of the endpoints $\omega(\overline{l})$, $\alpha(\overline{l})$ of $\overline{l}$ separate the endpoints g_0^-, g_0^+ of $\overline{g}_0$ on S_∞, the intersection $\overline{l} \cap \overline{g}_0$ consists of a finitely many (odd) points, Fig. 6.5, (a).

Given curves K_1, K_2, denote by $\sharp(K_1, K_2)$ the number of points of intersection $K_1 \cap K_2$. Suppose that $\sharp(\overline{l}, \overline{g}_0) \geqslant 3$. First, we construct a curve $\overline{g}$ whose endpoints are the same with $\overline{g}_0$ such that $\sharp(\overline{l}, \overline{g}) = 1$ and $\pi(\overline{g})$ is a closed simple curve freely homotopy to g_0. Denote by B and E the first and last points respectively of the intersection of the curve $\overline{l}$ with $\overline{g}_0$. By assumption, the open arc $A = (B, E) \subset \overline{l}$ intersects $\overline{g}_0$.

Let $\Gamma(\overline{g}_0) = \pi^{-1}(g_0)$ be the set of all geodesics congruent to $\overline{g}_0$. Given any geodesic $J \in \Gamma(\overline{g}_0)$ that intersects A, a J-loop of $\overline{l}$ bounds a disk. The set of such disks is finite and is endowed naturally with the order relation by an inclusion. Take the smallest disk, say D, formed by the A-arc $\widehat{pq} \subset A$

[3] It is assumed that g_0 is equipped with an orientation and all geodesics from $G(\mathscr{F})$ are oriented consistently with the oriented leaves of $\mathscr{F}$.

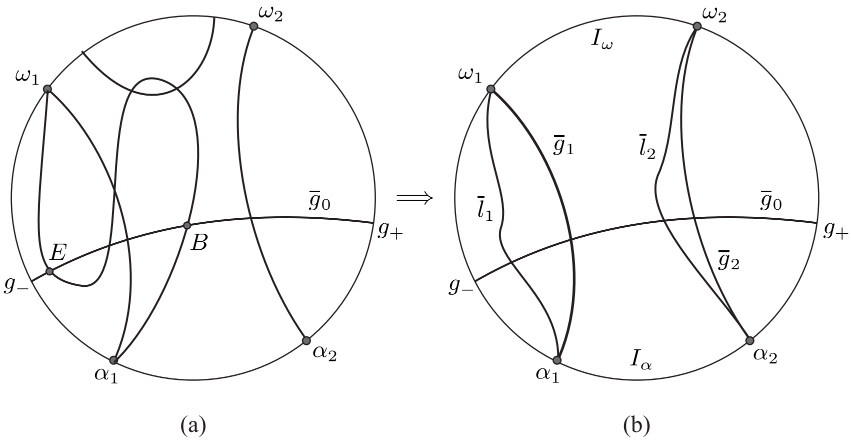


Figure 6.5. The co-asymptotic geodesics $\overline{g}_1 = \overline{g}_1(\overline{l}_1)$ and $\overline{g}_2 = \overline{g}_2(\overline{l}_2)$.

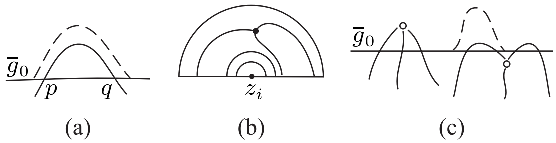


Figure 6.6. Reconstruction of arcs of the curve $\overline{g}_0$.

and the segment $[p, q] \subset G$, where $G \in \Gamma(\overline{g}_0)$. Replace G by a new curve, denoted by G', changing some neighborhood of $[p, q]$ by an arc, say $[p', q']$, as shown in Fig. 6.6, (a). Recall that the A-arc $\widehat{pq}$ is an arc of $\overline{l}$ that intersects transversally every curve from $\Gamma(\overline{g}_0)$. Therefore, $[p', q']$ can be constructed arbitrary close to $[p, q]$ with no more than one point of tangency. Moreover, one can assume that $\Gamma([p', q'])$ does not intersect $\overline{l}$ outside of A. Since the arc A has no congruent points, it is possible to get G' such that the intersection A with $\Gamma(G')$ has two points less and $\Gamma(G')$ consists of pairwise disjoint curves. Repeating this procedure with G' if necessary, one gets in a finitely many steps the curve $\overline{g}$ such that $A \cap \Gamma(\overline{g}) = A \cap \overline{g}$ is a unique point, and $\Gamma(\overline{g})$ consists of pairwise disjoint curves, and $\overline{l}$ does not intersect $\overline{g}$ outside of A.

By Lemma 6.12, every smallest disk, say D, has no congruent points. Hence, $\pi(G')$ is a closed curve that is freely homotopic to g_0. As a result, $\pi(\overline{g})$ is a closed curve freely homotopy to g_0. Moreover, since the curves in $\Gamma(\overline{g})$ are pairwise disjoint, $\pi(\overline{g})$ is simple.

The homotopy $g_0 \to \pi(\overline{g}) = g$ can be embedded into an isotopy ψ_t, $t \in [0; 1]$, of the surface M, where $\psi_0 = id$. Acting by the homeomorphism ψ_1^{-1} that is homotopic to the identity on $\mathscr{F}$, we get the foliation $\mathscr{F}_1$ such that there is a lift $\overline{g}_0$ of g_0 and a lift $\overline{l}_1$ of some nontrivially recurrent leaf l_1 of $\mathscr{F}_1$ such that the co-asymptotic geodesic $\overline{g}(\overline{l}_1)$ intersects $\overline{g}_0$ and $\overline{l}_1$ intersects $\overline{g}_0$ at a unique point. Moreover, g_0 does not pass through the singularities, and is in a general position with respect to the leaves of $\mathscr{F}_1$, and any leaf is tangent to g_0 at most at one point. For simplicity, denote $\mathscr{F}_1$ again by $\mathscr{F}$. The procedure of elimination of intersection points $A \cap \Gamma(\overline{g}_0)$ described above will be applied below for other compact arcs of leaves.

Denote by $\overline{a}_2 \in \overline{g}_0$ the point congruent to $\overline{a}_1 = \overline{g}_0 \cap \overline{l}_1$ such that there are no congruent points inside the segment $[\overline{a}_1; \overline{a}_2] \subset \overline{g}_0$. Let $\overline{g}_i = \overline{g}(\overline{l}_i)$ be the co-asymptotic geodesics for $\overline{l}_i$, $i = 1, 2$. By theorem 5.25, the points

$$\omega(\overline{l}_i) = \omega(\overline{g}_i) \overset{\mathrm{def}}{=} \omega_i, \quad \alpha(\overline{l}_i) = \alpha(\overline{g}_i) \overset{\mathrm{def}}{=} \alpha_i$$

are irrational and are separated by the ideal endpoints $g_-, g_+ \in S_\infty$ of $\overline{g}_0$ on S_∞ for each $i = 1, 2$, Fig. 6.5, (b).

Denote by I_ω the arc of S_∞ that is bounded by the points ω_1 and ω_2 and does not contain the points α_1 and α_2. Similarly, denote by I_α the arc of S_∞ that is bounded by α_1 and α_2 and does not contain ω_1 and ω_2. Obviously, I_ω and I_α are separated by points g_-, g_+. Denote by $\mathscr{D}^+$ the domain in Δ that is bounded by the segment $[\overline{a}_1; \overline{a}_2]$ and the positive semileaves $\overline{l}_1^+(\overline{a}_1)$ and $\overline{l}_2^+(\overline{a}_2)$. Similarly, $\mathscr{D}^-$ is bounded by $[\overline{a}_1; \overline{a}_2]$ and $\overline{l}_1^-(\overline{a}_1)$ and $\overline{l}_2^-(\overline{a}_2)$. Note that I_ω and I_α are included in the ideal boundary of $\mathscr{D}^+$ and $\mathscr{D}^-$ respectively. Set $\mathscr{D} = \mathscr{D}^+ \cup \mathscr{D}^-$.

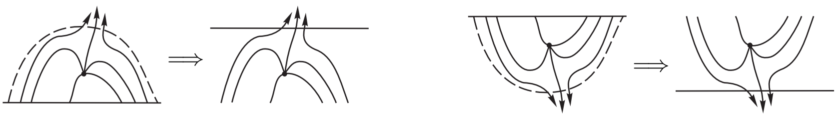
Consider the saddles $\mathscr{S}_\alpha$ of $\overline{\mathscr{F}}$ in $\mathscr{D}$ whose separatrices (not necessarily all) intersect $\overline{g}_0$ and all separatrices reach points in the interval I_α. Actually, one can require that separatrices of $\mathscr{S}_\alpha$ intersect just the segment $[\overline{a}_1; \overline{a}_2] \overset{\mathrm{def}}{=} \Sigma$ because of they can not intersect $\overline{g}_0$ outside of Σ. Let us show that there are finitely many of such saddles $\mathscr{S}_\alpha$. Suppose the contrary. Take a saddle $s \in \in \mathscr{S}_\alpha$ and its separatrices S_1, S_2. Two arcs of S_1, S_2 between first points of intersection x_1, x_2 respectively with $\overline{g}_0$ and s form the Σ-arc. Since the corresponding Σ-loop bounds a disk, the segment $[x_1, x_2]$ must have a tangency point, otherwise $\mathscr{F}$ has a singularity with positive index (see Lemma 3.22 and Theorem 3.6). The contrary assumption implies that there are infinitely many tangency points in Σ because of each saddle has at least three separatrices,

Fig. 6.6, (b). This contradicts the existence of only finitely many tangency points in Σ.

Σ-loops of separatrices bound disks. Take a maximal disk in $\mathscr{D}^+$ and replace the corresponding segment of $\overline{g}_0$ so that the new curve does not intersect saddles $\mathscr{S}_\alpha$ and their separatrices (in fact, $\overline{g}_0$ "runs around" such saddles and their separatrices "from above"), Fig. 6.6, (c). After the simplifying procedure, we make a similar replacement for saddles $\mathscr{S}_\omega$ in $\mathscr{D}$ all of whose separatrices reach points in the interval I_ω. As a result, one gets $\overline{\mathscr{F}}$ with $\mathscr{S}_\alpha = \varnothing$ and $\mathscr{S}_\omega = \varnothing$.

By the condition, $\overline{g}_0$ consequently intersects only neighbor sides of any geodesic polygon $\overline{P}$ of $\Delta \setminus G(\overline{\mathscr{F}})$. Therefore, the saddles from $\mathscr{D}$ whose separatrices intersect $\overline{g}_0$ split into two classes: $S_{1\omega}$, the saddles exactly one separatrix of which reaches a point in I_ω and all the other separatrices reach points in the interval I_α, and $S_{1\alpha}$, the saddles exactly one separatrix of which reaches a point in I_α and all the other separatrices reach points in the interval I_ω. Similarly $\mathscr{S}_\alpha$ and $\mathscr{S}_\omega$, one can prove that there are only finitely many saddles in $S_{1\alpha}$ and $S_{1\omega}$.

Take a saddle $s \in S_{1\omega}$ belonging to $\mathscr{D}^+$, and denote by s_u a unique separatrix that reaches a point at I_ω. Let s_1, s_2 be adjacent separatrices for s_u. By definition of the class $S_{1\omega}$, the both s_1 and s_2 intersect $\overline{g}_0$. Denote by s_{i*} the first point of the intersection s_i with Σ, $i = 1, 2$. For the compact arcs $\widehat{s_{i*}s}$ of s_i $(i = 1, 2)$, one makes the elimination and simplifying procedures so that $\widehat{s_{i*}s} \cap \Sigma = s_{i*}$. The arcs $\widehat{s_{i*}s}$ of s_i $(i = 1, 2)$ and s form a Σ-arc $\lambda = \widehat{s_{1*}s} \cup s \cup \widehat{s_{2*}s}$. Together with the segment $[s_{1*}, s_{2*}] \subset \overline{g}_0$, λ forms a closed simple curve that bounds a disk, say D. Suppose D is a maximal disk, where the order is defined by an inclusion. It is possible to replace $[s_{1*}, s_{2*}]$ by a transversal arc ν that is arbitrary close to λ and passes higher than λ. One gets the curve $(\Sigma \setminus \lambda) \cup \nu$ that has less tangency points than Σ. After a simplifying procedure and corresponding isotopy, $(\Sigma \setminus \lambda) \cup \nu$ becomes Σ. Continuing step by step, one gets $S_{1\omega} = \varnothing$. Similarly, $S_{1\alpha} = \varnothing$.



(a) (b)

Figure 6.7. Downing $S_{1\omega}$, (a); Elevation $S_{1\alpha}$, (b).

Our task is to deform $\overline{g}_0$ so as to eliminate all tangency points. We will describe the deformation of $\overline{g}_0$ only inside the segment $\Sigma \subset \overline{g}_0$. At each step, all congruent to Σ arcs are deformed simultaneously (one may assume that the deformation descends onto the surface M and then is lifted to the universal covering Δ). Without loss of generality, we may assume that all tangencies are one-sided; i. e., a leaf that is tangent to Σ lies locally either in $\mathscr{D}^+$ or in $\mathscr{D}^-$ near the tangency point. Accordingly, we will speak of a tangency from the domain $\mathscr{D}^+$ or $\mathscr{D}^-$. Here, three cases are possible: either all tangencies are from $\mathscr{D}^+$, all tangencies are from $\mathscr{D}^-$, or there are tangencies from both domains. The first two cases are symmetric. Therefore, it suffices to consider the last two cases, when there are tangencies from $\mathscr{D}^-$.

Let $\overline{z}_1, \ldots, \overline{z}_q$ be points of tangency from $\mathscr{D}^-$ on Σ. In $\mathscr{D}^+$, in a sufficiently small neighborhood of each point $\overline{z}_i$, $i = 1, \ldots, q$, the leaves form the simplest foliation on a half-disk, Fig. 6.6, (b). Therefore, for each point $\overline{z}_i$, we can construct a family of closed disks

$$D_\mu(i),\ \mu \in [0; \infty),\ \text{where } D_0(i) = z_i \text{ and } D_{\mu_1}(i) \subset D_{\mu_2}(i) \text{ for } \mu_1 < \mu_2,$$

which are embedded into each other and bounded by segments

$$[\alpha_\mu(i); \beta_\mu(i)] \subset \overline{g}_0 \text{ and arcs } {}^{\smile}[\alpha_\mu(i); \beta_\mu(i)]$$

of the leaves in $\mathscr{D}^+$. If the disks $D_\mu(i)$ are defined for $\mu < \mu_0$ and the limit arc ${}^{\smile}[\alpha_{\mu_0}(i); \beta_{\mu_0}(i)]$ is not tangent to Σ at the endpoints $\alpha_{\mu_0}(i)$ and $\beta_{\mu_0}(i)$ (see Fig. 6.6, (b) and Fig. 6.8), then there exist a segment $[\alpha; \beta] \supset \supset [\alpha_{\mu_0}(i); \beta_{\mu_0}(i)]$ and a leaf arc ${}^{\smile}[\alpha; \beta]$ that bound a disk $D' \subset \mathscr{D}^+$ containing the union $\bigcup_{0 \leqslant \mu < \mu_0} D_\mu(i)$. Setting $D' = D_{\mu_0}(i)$, we continue the process of constructing the disks $D_\mu(i)$. Note that the points $\alpha_\mu(i)$ and $\beta_\mu(i)$ can not both tend to $\overline{a}_1$ and $\overline{a}_2$ respectively as $\mu \to \infty$ since $\mathscr{F}$ is an irrational foliation with no Reeb components. Thus, the process of constructing the

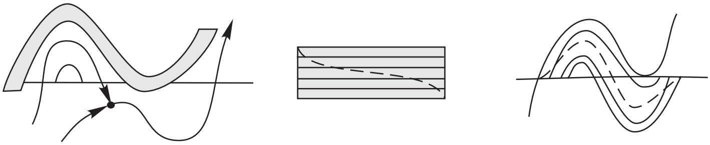


Figure 6.8. Elimination of tangencies.

disks $D_\mu(i)$ formally ends when a tangency arises at one (and only one) of the endpoints $\alpha_\mu(i)$ or $\beta_\mu(i)$. If for some value of μ_0 a tangency arises at one of the endpoints $\alpha_\mu(i)$ or $\beta_\mu(i)$, then the segment $[\alpha_\mu(i); \beta_\mu(i)]$ is replaced by an arc, as is shown in Fig. 6.8. In this case, one tangency point vanishes. The self-intersections that may arise are eliminated by a simplifying procedure. Continuing by this way, we eliminate all tangencies from $\mathscr{D}^-$. Similarly, one eliminates all tangencies from $\mathscr{D}^+$ in Σ. The projection of the segment Σ onto the surface yields the required simple closed transversal. □

Proof of the main theorems

The following four theorems give the complete classification of strongly irrational foliations on a closed orientable hyperbolic surface. These theorems correspond to three steps of the topological classification. The first and the second theorems correspond to a finding a constructive topological invariant which takes the same values for topologically equivalent foliations. The third theorem corresponds to a description of all topological invariants which are admissible, i. e. may be realized in the chosen class of foliations. At last, the firth theorem shows that given any admissible invariant, one constructs an irrational foliation whose invariant is the admissible one.

Theorem 6.6 *Let $\mathscr{F}_1$, $\mathscr{F}_2$ be strongly irrational foliations on a closed orientable hyperbolic surface M. Then $\mathscr{F}_1$, $\mathscr{F}_2$ are topologically equivalent if and only if the orbits of their geodesic frameworks coincide.*

Theorem 6.7 *Let $\mathscr{F}_1$, $\mathscr{F}_2$ be strongly irrational foliations on a closed orientable hyperbolic surface M. Then $\mathscr{F}_1$, $\mathscr{F}_2$ are topologically equivalent via a homeomorphism $M \to M$ homotopic to identity if and only if their geodesic frameworks coincide, $G(\mathscr{F}_1) = G(\mathscr{F}_2)$.*

We see that the orbit of a geodesic framework is a complete topological invariant for strongly irrational foliations. Thus, a geodesic framework is an analog of Poincaré rotation number for the class of strongly irrational foliations (flows) on a torus. The next theorem says that a geodesic framework of strongly irrational foliation is a strongly irrational geodesic lamination. This is completely similar to an irrationality of Poincaré rotation number.

Theorem 6.8 *Let $\mathscr{F}$ be a strongly irrational foliation on a closed orientable hyperbolic surface M. Then its geodesic framework $G(\mathscr{F})$ is strongly irrational, $G(\mathscr{F}) \in \Lambda^{irr}$.*

Theorem 6.9 *Given any strongly irrational geodesic lamination G on a closed orientable hyperbolic surface M, there is a strongly irrational foliation $\mathscr{F}$ on M such that $G(\mathscr{F}) = G$.*

Theorem 6.6 follows from Theorems 6.1 and 6.7. For simplicity, we give the proof of Theorem 6.7 for oriented irrational foliations and flows. Note that an irrational flow is automatically a strongly irrational one because of such flow can not have thorns.

Theorem 6.10 *Let f_1^t and f_2^t be irrational flows on a closed orientable hyperbolic surface M. Then f_1^t and f_2^t are topologically equivalent via a homeomorphism $M \to M$ homotopic to the identity if and only if their geodesic frameworks coincide.*

One of the key steps is the proof of the existence of freely homotopic global sections C_1 and C_2 for the flows f_1^t and f_2^t respectively provided that f_1^t and f_2^t have identical geodesic frameworks (the orientation on the geodesics can be neglected).

Lemma 6.14 *Let f_1^t and f_2^t be irrational flows on a closed orientable hyperbolic surface M that have identical* (*without regard to the orientation on the geodesics*) *geodesic frameworks. Then there exist freely homotopic* (*to each other*) *global sections C_1 and C_2 of the flows f_1^t and f_2^t, respectively.*

Proof follows from Lemmas 6.10 and 6.13. □

The crucial step in the proof of Theorem 6.10 is the construction, for each flow f_i^t, of a fundamental domain $\overline{\Phi}_i$ on the universal covering Δ.

On figure 6.9 is represented the construction of such fundamental domain Φ on $\overline{M}$ ($\pi(\Phi) = M$) for irrational flow f^t on closed orientable surface M of genus $p = 2$ containing two saddles O_1, O_2. On figure 6.9 by C is denoted a section for trajectories of the flow f^t. The numbers 1, 2, 3, 4 denote the separatrices of the saddle O_1 and the numbers 5, 6, 7, 8 — the separatrices of the saddle O_2. For simplicity of notations, the same numbers and letters on $\overline{M}$ denote the sections, saddles and their separatrices of the covering flow $\overline{f}^t$. Moreover, congruent points and curves on $\overline{M}$ are denoted by the same letters. Fundamental domain Φ is shaded.

Construction of a special fundamental domain. Let f^t be an irrational flow on a closed orientable hyperbolic surface M and C be a global section of f^t. Take a lift $\overline{C}$ of C and a point $\overline{z}_0 \in \overline{C}$ through which a lift $\overline{l}_0$ of a nontrivially

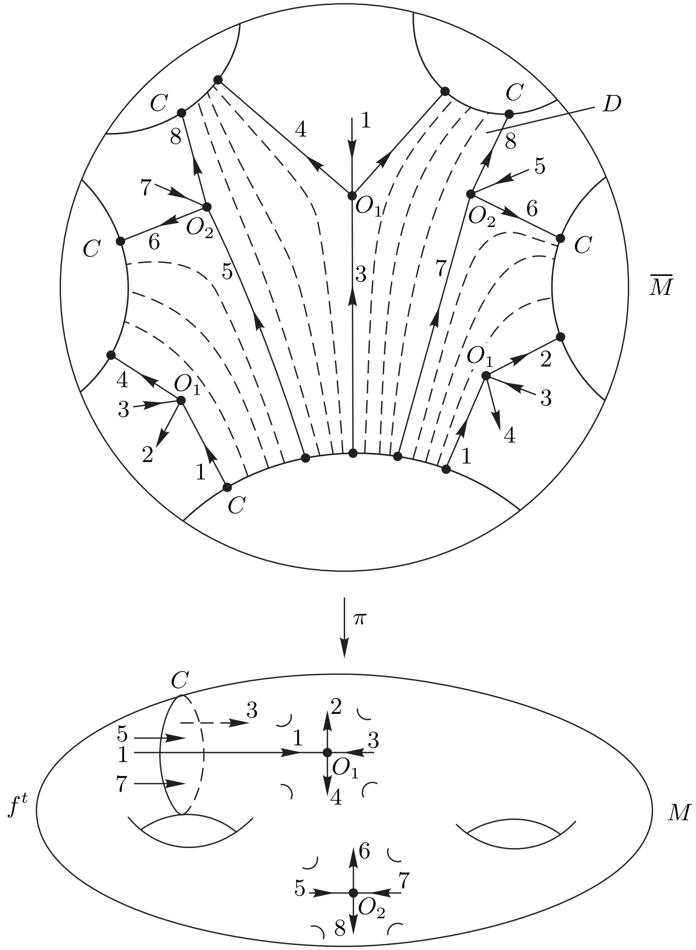


Figure 6.9. A fundamental domain. $\pi(\Phi) = M$

recurrent trajectory l_0 passes. Take a point $\overline{z}_{n+1} \in \overline{C}$ (the number n will be defined later) that is congruent to $\overline{z}_0$ such that there are no congruent points inside the interval $(\overline{z}_0; \overline{z}_{n+1}) \subset \overline{C}$. Obviously, a trajectory $\overline{l}_{n+1}$ which passed through $\overline{z}_{n+1}$ is congruent to $\overline{l}_0$. Denote by $\overline{C}_0$ and $\overline{C}_n$ the first, after $\overline{C}$, two lifts of C that are intersected by $\overline{l}_0$ and $\overline{l}_{n+1}$ respectively, as time indefinitely increases. Since the flow f^t is irrational, such $\overline{C}_0$ and $\overline{C}_n$ exist. Since M is not the torus, $\overline{C}_0 \neq \overline{C}_n$. Hence, there exists a point $\overline{z}_1 \in (\overline{z}_0; \overline{z}_{n+1})$ through which an ω-separatrix $\overline{l}_1$ of some saddle $\overline{O}_1$ passes such that the following conditions hold:

1) $\overline{l}_1$ does not intersect any curve congruent to $\overline{C}$ after $\overline{z}_1$;
2) the left Bendixson extension $\overline{l}_{1,l}$ of the ω-separatrix $\overline{l}_1$ in the positive direction intersects $\overline{C}_0$;
3) all positive semitrajectories that emanate from the interval $(\overline{z}_0; \overline{z}_1)$ intersect $\overline{C}_0$;
4) the right Bendixson extension $\overline{l}_{1,r}$ of the ω-separatrix $\overline{l}_1$ in the positive direction intersects a certain curve, say $\overline{C}_1$, that is congruent to $\overline{C}$.

It is easily seen that the point $\overline{z}_1$ with the indicated properties is unique. If $\overline{C}_1 \neq \overline{C}_n$, then there exists a point $\overline{z}_2 \in (\overline{z}_1; \overline{z}_{n+1})$ through which an ω-separatrix $\overline{l}_2$ of some saddle $\overline{O}_2$ with similar properties passes. The flow f^t has a finite number of saddles and separatrices; hence, continuing this process, we obtain the last point $\overline{z}_n \in (\overline{z}_{n-1}; \overline{z}_{n+1})$ through which an ω-separatrix $\overline{l}_n$ of some saddle $\overline{O}_n$ passes such that the right Bendixson extension $\overline{l}_{n,r}$ of the ω-separatrix $\overline{l}_n$ in the positive direction intersects the curve $\overline{C}_n$. Note that the points $\overline{z}_1, \ldots, \overline{z}_n$ and the separatrices passing through them could be obtained as follows. Consider all the saddles and their ω-separatrices on the surface M. The first intersections of these ω-separatrices with C under indefinite decrease of time are the projections of the points $\overline{z}_1, \ldots, \overline{z}_n$ to the surface.

Let $\overline{I}_0 \subset \overline{C}$ be the closed segment with the endpoints $\overline{z}_0$ and $\overline{z}_{n+1}$. Denote by $\overline{\Phi}(\overline{I}_0, \overline{C})$ the closure of the domain bounded by the curves $\overline{C}$, $\overline{C}_0$, $\overline{C}_1, \ldots, \overline{C}_n$, the trajectories $\overline{l}_0$, $\overline{l}_{1,l}$, $\overline{l}_{1,r}$, …, $\overline{l}_{n,l}$, $\overline{l}_{n,r}$, $\overline{l}_{n+1}$, and the saddles $\overline{O}_1, \ldots, \overline{O}_n$. Let us show that $\overline{\Phi}(\overline{I}_0, \overline{C})$ is a fundamental domain. First, we prove that there are no congruent points inside $\overline{\Phi}(\overline{I}_0, \overline{C})$. Suppose the contrary. Let points $\overline{m}_1, \overline{m}_2 \in \operatorname{Int} \overline{\Phi}(\overline{I}_0, \overline{C})$ be congruent. Then, the negative semitrajectories $\overline{l}^-(\overline{m}_1)$ and $\overline{l}^-(\overline{m}_2)$ that pass through these points are congruent and intersect the segment $\overline{I}_0$ at internal points, which must also be congruent. However, this contradicts the fact that the interior of the segment $\overline{I}_0$ does not contain congruent points by construction.

Consider the boundary $\partial\overline{\Phi}(\overline{I}_0, \overline{C})$ of the set $\overline{\Phi}(\overline{I}_0, \overline{C})$. Take the part of the α-separatrix $\overline{l}_{i,l}$, $i \in \{1, \ldots, n\}$, of the saddle $\overline{O}_i$ that is included in $\partial\overline{\Phi}(\overline{I}_0, \overline{C})$. This part of the boundary coincides with the negative semitrajectory $\overline{l}^-(\overline{z}_{i,l})$, where $\overline{z}_{i,l} = \overline{l}_{i,l} \cap \overline{C}_{i-1}$. Let $\overline{l}_*$ be an ω-separatrix of the saddle $\overline{O}_i$ such that $\overline{l}_{i,l}$ is the right Bendixson extension of the ω-separatrix $\overline{l}_*$ in the positive direction (Fig. 6.10). In view of the irrationality of the flow f^t, the separatrix $\overline{l}_*$ necessarily intersects a certain lift of the curve C as time indefinitely decreases. Since the segment $\overline{I}_0$ is projected onto the entire curve C, there exists an ω-separatrix $\overline{l}_j$ for a certain index $j \in \{1, \dots, n\}$, $j \neq i$, such that $\overline{l}_j$ is congruent to $\overline{l}_*$ (see Fig. 6.10). The orientability of the surface M implies that the group of deck transformations consists of orientation-preserving hyperbolic isometries that preserve the circular order of points. Hence, the saddle $\overline{O}_i$ is congruent to $\overline{O}_j$, and $\overline{l}^-(\overline{z}_{i,l})$ is congruent to $\overline{l}^-(\overline{z}_{j,r})$, where $\overline{z}_{j,r} = \overline{l}_{j,r} \cap \overline{C}_j$. Since there are no congruent points inside $\overline{I}_0$, $\overline{l}^-(\overline{z}_{j,r})$ is a unique α-separatrix that is included in the boundary $\partial\overline{\Phi}(\overline{I}_0, \overline{C})$ and is congruent to $\overline{l}^-(\overline{z}_{i,l})$.

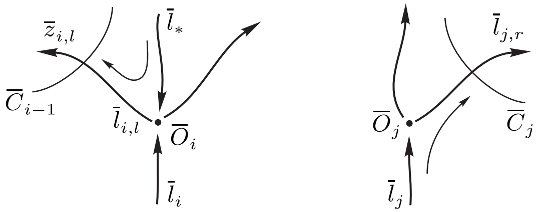


Figure 6.10. The separatrix $\overline{l}_{i,l}$ that is congruent to the separatrix $\overline{l}_{j,r}$.

The following fact is proved similarly: for each α-separatrix $\overline{l}_{i,r}$, $i \in \{1, \dots, n\}$, of the saddle $\overline{O}_i$, there exists a unique α-separatrix $\overline{l}_{k,l}$ of the saddle $\overline{O}_k$ with some index $k \in \{1, \dots, n\}$, $k \neq i$, that is congruent to the α-separatrix $\overline{l}_{i,r}$. Therefore, the parts of these separatrices that are included in the boundary $\partial\overline{\Phi}(\overline{I}_0, \overline{C})$ are congruent.

Denote the congruence by $\cong$. Set $\overline{z}_{0,r} = \overline{l}_0 \cap \overline{C}_0$ and $\overline{z}_{n+1,l} = \overline{l}_{n+1} \cap \overline{C}_n$ and consider the arcs $\overline{C}_{i,i+1} = [\overline{z}_{i,r}; \overline{z}_{i+1,l}] \subset \overline{C}_i$, $i = 0, \dots, n$. Let us show that they satisfy the following conditions:

1) the union of open arcs $\bigcup_{i=0}^{n} (\overline{z}_{i,r}; \overline{z}_{i+1,l})$ does not contain pairs of congruent points;
2) for the endpoints $\overline{z}_{i,r}$ and $\overline{z}_{i+1,l}$, $i \in \{0, \dots, n\}$, of the arc $\overline{C}_{i,i+1}$, there exists exactly one pair of points $\overline{z}_{j,l}$ and $\overline{z}_{q,r}$, $j \in \{1, \dots, n+1\}$, $q \in \{0, \dots, n\}$, that are endpoints of the arcs $\overline{C}_{j-1,j}$ and $\overline{C}_{q,q+1}$,

respectively, and are such that

$$\overline{z}_{i,r} \cong \overline{z}_{j,l}, \quad \overline{z}_{i+1,l} \cong \overline{z}_{q,r}, \quad \text{where } i+1 \neq j,\ i \neq q,\ j \neq q+1;$$

3) the union of closed arcs $\bigcup\limits_{i=0}^{n} \overline{C}_{i,i+1}$ is projected under the covering $\pi\colon \Delta \to \to M$ onto C; this projection is one-to-one except for the endpoints of the arcs $\overline{C}_{i,i+1}$.

The proof of condition 1 is analogous to the proof of the fact that there are no congruent points inside $\overline{\Phi}(\overline{I}_0, \overline{C})$. Let us prove condition 2. For the α-separatrix $\overline{l}_{i,r}$, there exists a unique congruent α-separatrix $\overline{l}_{j,l}$ some part of which is included in the boundary $\partial\overline{\Phi}(\overline{I}_0, \overline{C})$. Therefore, $\overline{z}_{i,r} \cong \overline{z}_{j,l}$. Let us show that $i+1 \neq j$. Suppose the contrary. Then, the endpoints of the arc $\overline{C}_{i,i+1}$ are congruent, and hence, the arc is projected onto the curve C. Since f^t has at least three different α-separatrices, there exists a negative semitrajectory of $\overline{f}^t$ that passes through the interior of the arc $\overline{C}_{i,i+1}$ and is an α-separatrix of some saddle $\overline{O} \in \overline{\Phi}(\overline{I}_0, \overline{C})$, which is impossible. The existence of a point $\overline{z}_{q,r} \cong \overline{z}_{i+1,l}$ with $i \neq q$ can be proved similarly. It remains to show that $j \neq q+1$. Suppose the contrary. Then $\pi(\overline{C}_{i,i+1} \cup \overline{C}_{j,j+1}) = C$. Again, taking into account that f^t has at least three different α-separatrices, we derive the existence of a negative semitrajectory of the covering flow that passes through the interior of one of the arcs $\overline{C}_{i,i+1}$ or $\overline{C}_{j,j+1}$ and is an α-separatrix of some saddle $\overline{O} \in \overline{\Phi}(\overline{I}_0, \overline{C})$, which is impossible.

Let us prove condition 3. Compose a union

$$[\overline{z}_{0,r}; \overline{z}_{1,l}] \cup \ldots \cup [\overline{z}_{i,r}; \overline{z}_{i+1,l}] \cup [\overline{z}_{q,r}; \overline{z}_{q+1,l}] \cup \ldots \cup [\overline{z}_{n,r}; \overline{z}_{n+1,l}],$$

where the adjacent intervals have congruent endpoints, $\overline{z}_{i+1,l} \cong \overline{z}_{q,r}$. Taking into account the already proven conditions 1 and 2, it suffices to show that the union composed contains all the arcs $\overline{C}_{i,i+1}$. Suppose the contrary. Then, by virtue of condition 2, the union $\bigcup\limits_{i=0}^{n} \overline{C}_{i,i+1}$ can be decomposed into two unions each of which is projected onto the curve C. This contradicts condition 1.

Thus, the boundary $\partial\overline{\Phi}(\overline{I}_0, \overline{C})$ is decomposed into a finite number of pairs of congruent arcs. Since C is a global section, it follows that for any point of the universal covering, there exists a congruent point in $\overline{\Phi}(\overline{I}_0, \overline{C})$. This completes the proof of the fact that $\overline{\Phi}(\overline{I}_0, \overline{C})$ is a fundamental domain.

Below, we need to indicate a flow in the notation of a fundamental domain. Denote the fundamental domain constructed above by $\overline{\Phi}(\overline{I}_0, \overline{C}, \overline{f}^t)$. Denote by $\overline{\Phi}([\overline{z}_i; \overline{z}_{i+1}], \overline{f}^t)$ the closed set bounded by the arcs $[\overline{z}_i; \overline{z}_{i+1}] \subset \overline{C}$ and $[\overline{z}_{i,r}; \overline{z}_{i+1,l}] \subset \overline{C}_{i,i+1}$, the ω-separatrices $\overline{l}_i$ and $\overline{l}_{i+1}$, the α-separatrices $\overline{l}_{i,r}$ and $\overline{l}_{i+1,l}$, and the saddles $\overline{O}_i$ and $\overline{O}_{i+1}$. Then,

$$\overline{\Phi}(\overline{I}_0, \overline{C}, \overline{f}^t) = \bigcup_{i=0}^{n} \overline{\Phi}([\overline{z}_i; \overline{z}_{i+1}], \overline{f}^t).$$

Proof of Theorem 6.10. By Lemma 6.7, a homeomorphism of the surface M^2 that is homotopic to the identity has a lift that is extended to the identity homeomorphism of S_∞. Therefore, if the flows f_1^t and f_2^t are orbitally topologically equivalent via a homeomorphism homotopic to the identity, then their geodesic frameworks are equal with regard to the orientation on the geodesics. In the case of topological equivalence, the geodesic frameworks are equal without regard to the orientation on the geodesics.

The main part of the proof of Theorem 6.10 consists of the proof of the converse proposition (the sufficiency). Let $G(f_1^t) = G(f_2^t)$. Reversing, if necessary, time on the trajectories of one of the flows, we may assume that the geodesics in $G(f_1^t)$ and $G(f_2^t)$ are oriented identically. Let us show that f_1^t and f_2^t are orbitally topologically equivalent via a homeomorphism homotopic to the identity. According to Theorem 6.14, the flows f_1^t and f_2^t have global sections C_1 and C_2, respectively, that are freely homotopic to each other. Then, for each lift $\overline{c}_1$ of the section C_1, there exists a unique lift $\overline{c}_2$ of the section C_2 such that $\overline{c}_1$ and $\overline{c}_2$ have identical ideal endpoints on the absolute. Such lifts $\overline{c}_1$ and $\overline{c}_2$ are said to be corresponding. Let us construct a mapping $\tau_{\overline{c}_1} : \overline{c}_1 \to \overline{c}_2$ as follows. Take a point $\overline{m}_1 \in \overline{c}_1$. There is a trajectory or a generalized trajectory of the flow $\overline{f}_1^t$ that passes through $\overline{m}_1$; denote it by $\overline{l}_1$. According to Corollaries 5.4 and 5.5, $\overline{l}_1$ reaches two different points σ^+ and σ^- on the absolute that are ideal endpoints of the co-asymptotic geodesic $\overline{g}(\overline{l}_1)$. Since $\overline{g}(\overline{l}_1) \in G(f_1^t) = G(f_2^t)$ and by virtue of Theorem 5.34, $\overline{g}(\overline{l}_1)$ is a co-asymptotic geodesic of a trajectory or a generalized trajectory $\overline{l}_2$ of the flow $\overline{f}_2^t$. Note that $\overline{l}_1$ and $\overline{l}_2$ are either trajectories or generalized trajectories simultaneously. Since the points σ^+ and σ^- are irrational and the ideal endpoints α and β of the curves $\overline{c}_1$ and $\overline{c}_2$ are rational, all the points are pairwise different. Since $\overline{c}_1$ is a section of the flow $\overline{f}_1^t$, the pair (σ^+, σ^-) is separated on the absolute by the pair of points (α, β). Therefore, $\overline{l}_2$ intersects $\overline{c}_2$. In view of the irrationality of the flow f_2^t, the intersection $\bar{l}_2 \cap \bar{c}_2$ consists of exactly one point, which we denote by $\overline{m}_2$. Set $\tau_{\bar{c}_1}(\overline{m}_1) = \overline{m}_2$.

Define a mapping $\tau\colon \pi^{-1}(C_1) \to \pi^{-1}(C_2)$ as the mapping that coincides with $\tau_{\bar{c}_1}$ on any lift $\bar{c}_1 \in \pi^{-1}(C_1)$. It is easy to verify that τ satisfies the relation

$$\tau \circ \gamma = \gamma \circ \tau \tag{6.4}$$

for any $\gamma \in \Gamma$.

Take the corresponding lifts $\overline{C}_1 \in \pi^{-1}(C_1)$ and $\overline{C}_2 \in \pi^{-1}(C_2)$ of the curves C_1 and C_2, respectively, and construct a fundamental domain $\overline{\Phi}(\overline{I}_1, \overline{C}_1, \overline{f}_1^t)$ for some segment $\overline{I}_1 \subset \overline{C}_1$. By virtue of (6.4), there are no congruent points inside the segment $\tau(\overline{I}_1) \subset \overline{C}_2$. Therefore, there exists a fundamental domain $\overline{\Phi}(\tau(\overline{I}_1), \overline{C}_2, \overline{f}_2^t)$. Recall that $\overline{\Phi}(\overline{I}_1, \overline{C}_1, \overline{f}_1^t)$ is a union of sets of the form $\overline{\Phi}([\overline{z}_i^{(1)}; \overline{z}_{i+1}^{(1)}], \overline{f}_1^t)$. According to (6.4), $\overline{\Phi}(\tau(\overline{I}_1), \overline{C}_2, \overline{f}_2^t)$ is a union of sets of the form

$$\overline{\Phi}(\tau([\overline{z}_i^{(1)}; \overline{z}_{i+1}^{(1)}]), \overline{f}_2^t) = \overline{\Phi}([\overline{z}_i^{(2)}; \overline{z}_{i+1}^{(2)}], \overline{f}_2^t).$$

The interior of each of the sets

$$\overline{\Phi}([\overline{z}_i^{(1)}; \overline{z}_{i+1}^{(1)}], \overline{f}_1^t), \quad \overline{\Phi}([\overline{z}_i^{(2)}; \overline{z}_{i+1}^{(2)}], \overline{f}_2^t)$$

is an open foliated box whose boundary consists of two segments without contact, four arcs of separatrices, and two saddles. Therefore, for each $i = 0, \ldots, n$, one can construct a homeomorphism

$$\overline{\psi}_i\colon \overline{\Phi}([\overline{z}_i^{(1)}; \overline{z}_{i+1}^{(1)}], \overline{f}_1^t) \to \overline{\Phi}([\overline{z}_i^{(2)}; \overline{z}_{i+1}^{(2)}], \overline{f}_2^t)$$

such that $\overline{\psi}_i$ are consistent at common boundary points and map arcs of trajectories into arcs of trajectories. Then, $\overline{\psi}_i$ define a homeomorphism

$$\overline{\psi}\colon \overline{\Phi}(\overline{I}_1, \overline{C}_1, \overline{f}_1^t) \to \overline{\Phi}(\tau(\overline{I}_1), \overline{C}_2, \overline{f}_2^t).$$

Moreover, fixing up, if necessary, $\overline{\psi}_i$ on the segments without contact, we can assume that $\overline{\psi}$ maps congruent points to congruent points. Hence, $\overline{\psi}$ is a covering for a homeomorphism $\psi\colon M \to M$ that maps the trajectories of the flow f_1^t into the trajectories of the flow f_2^t.

Denote the lift of the homeomorphism ψ that coincides with $\overline{\psi}$ on $\overline{\Phi}(\overline{I}_1, \overline{C}_1, \overline{f}_1^t)$ by the same letter $\overline{\psi}$. Since the fundamental domains used when constructing $\overline{\psi}$ are "suspended" on the trajectories and curves with identical ideal endpoints, we can show that the extension of the homeomorphism $\overline{\psi}$ to the absolute is identical. Therefore, ψ is homotopic to the identity and realizes an orbital topological equivalence of the flows f_1^t and f_2^t.

This completes the first part of the proof of Theorem 6.10 on the necessary and sufficient conditions for the (orbital) topological equivalence of flows. Let us prove the second part (the realization). Let f^t be an irrational flow on M. According to Theorem 5.33, the geodesic framework $G(f^t)$ is a nontrivial minimal and irreducible geodesic lamination. Since the trajectories form an orientable foliation, $G(f^t)$ is an orientable lamination.

Let G be an orientable nontrivial minimal and irreducible geodesic lamination. According to Lemma 5.21, each component of the set $M - G$ is a simply connected domain any of whose lifts to the universal covering is the interior of a geodesic polygon with a finite number of sides and with vertices lying on the absolute. Therefore, it is easy to construct, in each such component, an orientable foliation with one saddle so that, together with the lamination G, we obtain an orientable arational foliation on the entire surface. The application of the operation of blowing down yields a highly transitive foliation, which, due to the orientability, is embedded into the required irrational flow. $\square$

Proof of Theorem 6.8. By Theorem 5.33, the geodesic framework $G(\mathscr{F})$ is a strongly irrational (nontrivial minimal and irreducible) geodesic lamination. In the case of $\mathscr{F}$ being an orientable foliation, $G(\mathscr{F})$ is an orientable lamination. $\square$

Proof of Theorem 6.9. Let G be a strongly irrational geodesic lamination. By Lemma 5.21, each component of $M \setminus G$ is a simply connected domain whose lift to Δ is a geodesic polygon with a finite number of sides and with vertices lying on S_∞. Therefore, it is easy to construct in such component a foliation with one saddle so that, together with the lamination G, we obtain an arational foliation on M. The application of the operation of blowing down yields a strongly irrational foliation. In the case of $\mathscr{F}$ being an orientable foliation, $G(\mathscr{F})$ is an orientable lamination. Therefore, $G(\mathscr{F})$ can be extended to an orientable arational foliation, a flow. $\square$

6.4. Classification of irrational 2-webs

A 2-*web* on a surface is a pair of foliations such that they have a common singular set and are topologically transversal at all non-singular points. Suppose that two foliations $\mathscr{F}_1$ and $\mathscr{F}_2$ on a surface M form the 2-web denoted by $(\mathscr{F}_1, \mathscr{F}_2)$. The set of singularities of the foliation $\mathscr{F}_i$ (for any i) is called the *set of singularities* of $(\mathscr{F}_1, \mathscr{F}_2)$ denoted by $\mathrm{Sing}(\mathscr{F}_1, \mathscr{F}_2)$. A 2-web is *irrational or strongly irrational* if it consists of a pair of irrational or strongly irrational foliations respectively.

2-webs $(\mathscr{F}_1, \mathscr{F}_2)$ and $(\mathscr{F}'_1, \mathscr{F}'_2)$ are *topologically equivalent* if there is a homeomorphism $\varphi\colon M \to M$ that maps the foliations $\mathscr{F}_i$ $(i = 1, 2)$ to the corresponding foliations $\mathscr{F}'_i$ and $\varphi\,(\mathrm{Sing}(\mathscr{F}_1, \mathscr{F}_2)) = \mathrm{Sing}(\mathscr{F}'_1, \mathscr{F}'_2)$. For simplicity, we restrict ourselves by strongly irrational 2-webs.

Strongly irrational 2-webs on $\mathbb{T}^2$

On the torus $\mathbb{T}^2$, a strongly irrational 2-web consists of a pair of transversal irrational foliations without singularities.

Theorem 6.11 *Let $(\mathscr{F}_1, \mathscr{F}_2)$ be a strongly irrational 2-web on $\mathbb{T}^2$. Then $(\mathscr{F}_1, \mathscr{F}_2)$ is topologically equivalent via a homeomorphism homotopic to the identity to its own geodesic framework, which is a pair of linear transversal irrational foliations. Two strongly irrational 2-webs on $\mathbb{T}^2$ are topologically equivalent via a homeomorphism $\mathbb{T}^2 \to \mathbb{T}^2$ homotopic to the identity if and only if their geodesic frameworks coincide.*

Proof. Obviously, it suffices to prove the first part of the assertion on the topological equivalence of a 2-web to its own geodesic framework. By Theorem 6.5, for each $i = 1, 2$, there exists a homeomorphism $\phi_i\colon \mathbb{T}^2 \to \mathbb{T}^2$ homotopic to the identity that maps the foliation $\mathscr{F}_i$ into the linear foliation $G(\mathscr{F}_i)$, which is the geodesic framework of the foliation $\mathscr{F}_i$. There exists a lift $\overline{\phi}_i\colon \mathbb{R}^2 \to \mathbb{R}^2$ of ϕ_i that maps the covering foliation $\overline{\mathscr{F}}_i$ into the linear foliation $\overline{G}(\mathscr{F}_i)$ that is a lift of $G(\mathscr{F}_i)$. Since $\mathscr{F}_1$ and $\mathscr{F}_2$ are transversal, $\overline{G}(\mathscr{F}_1)$ and $\overline{G}(\mathscr{F}_2)$ are also transversal. It follows from the linearity of these foliations that any leaf of $\overline{G}(\mathscr{F}_1)$ intersects each leaf of $\overline{G}(\mathscr{F}_2)$ exactly at one point. Based on this result, we construct a covering conjugacy $\overline{\theta}$ as follows. Take a point $\overline{m} \in \mathbb{R}^2$. Leaves $\overline{l}_1$ and $\overline{l}_2$ of the foliations $\overline{\mathscr{F}}_1$ and $\overline{\mathscr{F}}_2$ respectively pass through this point. Define $\overline{\theta}$ assuming that $\overline{m}$ is mapped by $\overline{\theta}$ to the intersection point of the leaves $\overline{\phi}_1(\overline{l}_1)$ and $\overline{\phi}_2(\overline{l}_2)$. Since the pairs $(\overline{\mathscr{F}}_1, \overline{\mathscr{F}}_2)$ and $(\overline{G}(\mathscr{F}_1), \overline{G}(\mathscr{F}_2))$ form 2-webs, $\overline{\theta}$ is a homeomorphism $\mathbb{R}^2 \to \mathbb{R}^2$. It follows immediately from the definition that $\overline{\theta}$ maps $\overline{\mathscr{F}}_i$ into $\overline{G}(\mathscr{F}_i)$, $i = 1, 2$. It remains to verify that $\overline{\theta}$ is a covering for a homeomorphism of the torus that is homotopic to the identity.

Let S_{nk} be an integer translation of the plane $\mathbb{R}^2$. It is required to prove the equality

$$\overline{\theta} \circ S_{nk} = S_{nk} \circ \overline{\theta}, \quad (n, k) \in \mathbb{Z}^2.$$

Since $\overline{\phi}_i$ $(i = 1, 2)$ are coverings for mappings that are homotopic to the identity, $\overline{\phi}_i \circ S_{nk} = S_{nk} \circ \overline{\phi}_i$, $i = 1, 2$. If a leaf $\bar{l}_i(\overline{m})$ passes through the point $\overline{m}$, then the leaf $S_{nk}(\bar{l}_i(\overline{m}))$ passes through the point $S_{nk}(\overline{m})$. Therefore,

$$\begin{aligned}\overline{\theta}(S_{nk}(\overline{m})) = \overline{\phi}_1 \circ S_{nk}(\bar{l}_1(\overline{m})) \cap \overline{\phi}_2 \circ S_{nk}(\bar{l}_2(\overline{m})) = \\ = S_{nk} \circ \overline{\phi}_1(\bar{l}_1(\overline{m})) \cap S_{nk} \circ \overline{\phi}_2(\bar{l}_2(\overline{m})) = S_{nk}(\overline{\theta}(\overline{m})).\end{aligned}$$

The mapping $\overline{\theta}$ can be defined in a more explicit form by using the rotation numbers μ_1 and μ_2 of the foliations $\mathscr{F}_1$ and $\mathscr{F}_2$ respectively. Let

$$\overline{\phi}_i = \left(\phi_i^{(1)}, \phi_i^{(2)}\right) : \mathbb{R}^2 \to \mathbb{R}^2, \quad i = 1, 2.$$

Then, $\overline{\theta} = (\theta_1, \theta_2) \colon \mathbb{R}^2 \to \mathbb{R}^2$ is defined as follows:

$$\begin{aligned}\theta_1(\overline{m}) &= \frac{1}{\mu_2 - \mu_1}\left[\phi_1^{(2)}(\overline{m}) - \mu_1\phi_1^{(1)}(\overline{m})\right] - \frac{1}{\mu_2 - \mu_1}\left[\phi_2^{(2)}(\overline{m}) - \mu_2\phi_2^{(1)}(\overline{m})\right], \\ \theta_2(\overline{m}) &= \frac{\mu_2}{\mu_2 - \mu_1}\left[\phi_1^{(2)}(\overline{m}) - \mu_1\phi_1^{(1)}(\overline{m})\right] - \frac{\mu_1}{\mu_2 - \mu_1}\left[\phi_2^{(2)}(\overline{m}) - \mu_2\phi_2^{(1)}(\overline{m})\right].\end{aligned}$$

Note that $\mu_1 \neq \mu_2$ because the linear foliations $\overline{G}(\mathscr{F}_1)$ and $\overline{G}(\mathscr{F}_2)$ are transversal. Since $\overline{G}(\mathscr{F}_i)$ is a family of straight lines $y - \mu_i x = k$, we can easily verify that $\overline{\theta}$ thus defined satisfies the required conditions. □

Strongly irrational 2-webs on hyperbolic surfaces

Let us pass on to strongly irrational 2-webs on a closed orientable hyperbolic surface. According to Theorem 5.33, the geodesic framework of a strongly irrational foliation on a closed orientable hyperbolic surface M_p^2, $p \geqslant 2$, is a strongly irrational geodesic lamination. If foliations $\mathscr{F}_1$ and $\mathscr{F}_2$ form a strongly irrational 2-web, then their geodesic frameworks must satisfy the following *consistency* conditions:

- The sets $M_p^2 \setminus \operatorname{supp} G(\mathscr{F}_1)$ and $M_p^2 \setminus \operatorname{supp} G(\mathscr{F}_2)$ have the same number of simply connected components, which is equal to the number of singularities of the foliations $\mathscr{F}_1$ and $\mathscr{F}_2$ (which is the same for these foliations).

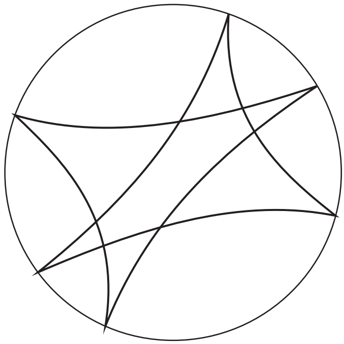


Figure 6.11. The polygons $\overline{P}_1$ and $\overline{P}_2$.

- For each simply connected component P_1 of the set $M_p^2 \setminus G(\mathscr{F}_1)$, there exists a simply connected component P_2 of the set $M_p^2 \setminus \operatorname{supp} G(\mathscr{F}_2)$ such that there exist lifts $\overline{P}_1$ and $\overline{P}_2$ of these components that are polygons with alternating ideal vertices on the absolute, Fig. 6.11.

Theorem 6.12 *Let $(\mathscr{F}_1, \mathscr{F}_2)$ and $(\mathscr{F}_1', \mathscr{F}_2')$ be strongly irrational 2-webs on a closed orientable hyperbolic surface M_p^2 of genus $p \geqslant 2$. Then $(\mathscr{F}_1, \mathscr{F}_2)$ and $(\mathscr{F}_1', \mathscr{F}_2')$ are topologically equivalent via a homeomorphism $M_p^2 \to M_p^2$ that is homotopic to the identity if and only if their geodesic frameworks coincide. For any pair of strongly irrational geodesic laminations that satisfy the consistency conditions, there exists a strongly irrational 2-web whose geodesic framework is equal to this pair of geodesic laminations.*

Proof. By Lemma 6.7, a homeomorphism of a surface that is homotopic to the identity has a lift that is extended to the identity homeomorphism of S_∞. Therefore, if the webs are topologically equivalent via a homeomorphism homotopic to the identity, then their geodesic frameworks coincide.

Suppose that the geodesic frameworks $(G(\mathscr{F}_1), G(\mathscr{F}_2))$ and $(G(\mathscr{F}_1'), G(\mathscr{F}_2'))$ coincide; i.e., let $G(\mathscr{F}_1) = G(\mathscr{F}_1')$, $G(\mathscr{F}_2) = G(\mathscr{F}_2')$. Consider the lifts $(\overline{\mathscr{F}}_1, \overline{\mathscr{F}}_2)$ and $(\overline{\mathscr{F}}'_1, \overline{\mathscr{F}}'_2)$ of the 2-webs $(\mathscr{F}_1, \mathscr{F}_2)$ and $(\mathscr{F}_1', \mathscr{F}_2')$ respectively. Let $\overline{m} \in \Delta$ be a point that is not a singularity of $(\overline{\mathscr{F}}_1, \overline{\mathscr{F}}_2)$. Denote by $\overline{l}_1$ and $\overline{l}_2$ the semileaves of the foliations $\overline{\mathscr{F}}_1$ and $\overline{\mathscr{F}}_2$ respectively that pass through $\overline{m}$. According to Theorem 5.34, their asymptotic directions are defined by some points σ_1 and σ_2 of S_∞ respectively. Since the foliations $\overline{\mathscr{F}}_1$ and $\overline{\mathscr{F}}_2$ are transversal outside the set of

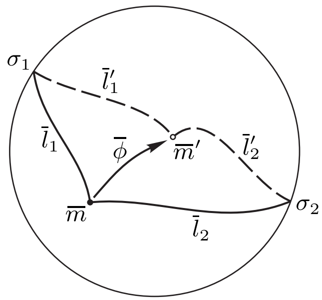


Figure 6.12. The mapping $\overline{\phi}$.

singularities, $\sigma_1 \neq \sigma_2$. By hypothesis, the points σ_1 and σ_2 are reached by the geodesic frameworks of $\overline{\mathscr{F}}'_1$ and $\overline{\mathscr{F}}'_2$ respectively. Therefore, by Theorem 5.34, there exist semileaves $\overline{l}'_1$ and $\overline{l}'_2$ of these foliations that reach the points σ_1 and σ_2 respectively. Note that according to Theorem 5.34, if $\overline{l}_i$ does not belong to a separatrix of a singularity, then $\overline{l}'_i$ does not belong to a separatrix of any singularity; and conversely, if $\overline{l}_i$ belongs to a separatrix of a singularity, then $\overline{l}'_i$ also belongs to a separatrix of a certain singularity, $i = 1, 2$. Since the co-asymptotic geodesics of the corresponding leaves or semileaves that contain $\overline{l}_i$ and $\overline{l}'_i$ coincide and the geodesic frameworks of the foliations $\overline{\mathscr{F}}'_1$ and $\overline{\mathscr{F}}'_2$ are transversal, the semileaves $\overline{l}'_1$ and $\overline{l}'_2$ intersect at some point, which we denote by $\overline{m}'$, Fig. 6.12. Since $\overline{\mathscr{F}}'_1$ and $\overline{\mathscr{F}}'_2$ form a 2-web, the point $\overline{m}'$ is unique. Denote the mapping $\overline{m} \to \overline{m}'$ by $\overline{\phi}$. We see that $\overline{\phi}$ is a homeomorphism that covers a certain homeomorphism $\phi\colon M^2_p \to M^2_p$. Moreover, $\overline{\phi}$ is extended to all singularities of the 2-web $(\overline{\mathscr{F}}_1, \overline{\mathscr{F}}_2)$ and realizes a topological equivalence of $(\overline{\mathscr{F}}_1, \overline{\mathscr{F}}_2)$ and $(\overline{\mathscr{F}}'_1, \overline{\mathscr{F}}'_2)$. By the construction, $\overline{\phi}$ is extended to the absolute as the identity mapping. By virtue of Theorem 5.34, ϕ is homotopic to the identity. It is clear that ϕ realizes a topological equivalence of the 2-webs $(\mathscr{F}_1, \mathscr{F}_2)$ and $(\mathscr{F}'_1, \mathscr{F}'_2)$. $\square$

6.5. Classification of Denjoy flows and nontrivial minimal sets

Recall that a minimal set of a flow is a nonempty closed set that is invariant (i. e., consists of trajectories of the flow) and does not contain proper subsets with the above-described properties. A similar definition applies to foliations, provided that the invariant means a union of leaves and singularities. The trivial minimal sets of flows include fixed points, periodic trajectories, and the minimal set that coincides with a closed surface, which is the torus in this case. The situation for foliations is analogous. A nontrivial minimal set is nowhere dense (see Lemma 3.26) and is locally homeomorphic to the product of a segment and the Cantor set. By Theorem 3.8, a nontrivial minimal set consists of nonclosed trajectories that are recurrent in the Birkhoff sense, in short B-recurrent. Moreover, every B-recurrent trajectory is everywhere dense in the minimal set.

A fixed-point-free flow f^t on the torus $\mathbb{T}^2$ is called a *Denjoy flow* if f^t has a nontrivial minimal set. This nontrivial set denoted by N is a unique minimal set of f^t. Therefore, the classification of Denjoy flows and their nontrivial minimal sets is intimately connected.

Nontrivial minimal sets on $\mathbb{T}^2$

On the torus $\mathbb{T}^2$, a co-asymptotic geodesic of nontrivially recurrent trajectory is a nontrivially recurrent geodesic that is dense on $\mathbb{T}^2$. Thus, the geodesic framework of a nontrivial minimal set is a linear irrational foliation.

Proposition 6.2 *Let N be a nontrivial minimal set of a flow f^t on $\mathbb{T}^2$ and $G(N)$ be the geodesic framework of N. Then there exists a continuous mapping $h\colon \mathbb{T}^2 \to \mathbb{T}^2$* (a blowing-down operation) *that is homotopic to the identity and possesses the following properties*:

- $h(N) = \mathbb{T}^2$;
- *each trajectory from N is homeomorphically mapped by h onto a geodesic of $G(N)$;*
- *let w be a component of the set $\mathbb{T}^2 \setminus N$. Then*
  - *$h(w)$ is a geodesic of $G(N)$;*
  - *w is simply connected*;
  - *the accessible from interior w boundary $\delta(w)$ of w consists of two trajectories l_1^w, $l_2^w \subset N$, and $h(w \cup l_1^w \cup l_2^w) = h\left(w \cup \delta(w)\right)$ is a geodesic of $G(N)$;*
  - *the restriction*

$$h|_{N \setminus \bigcup\limits_{w \in \mathbb{T}^2 \setminus N} \delta(w)} : N \setminus \bigcup_{w \in \mathbb{T}^2 \setminus N} \delta(w) \to \mathbb{T}^2$$

is a homeomorphism on its image, where $\bigcup_{w \in \mathbb{T}^2 \setminus N} \delta(w)$ *is the union over all components of the set* $\mathbb{T}^2 \setminus N$*.*

Proof. The set N consists of nontrivially recurrent trajectories. By Corollary 3.5, there is a closed simple transversal C intersecting N. Since every trajectory of N is dense in N, all trajectories transverse C. Hence, f^t induces the Poincaré forward mapping $N \cap C \to N \cap C$ that is a homeomorphism. We can consider C as a circle endowed with an orientation. Since $\mathbb{T}^2$ is an orientable surface, the homeomorphism $N \cap C \to N \cap C$ is monotone, and is extended to a homeomorphism $C \to C$ denoted by φ. The set $N \cap C$ is a minimal set of φ with no periodic points because of N is a nontrivial minimal set of f^t. Moreover, $C \neq N \cap C$ implies that $N \cap C$ is a Cantor set, and φ is a Denjoy homeomorphism. Therefore, the rotation number $\mathrm{rot}(\varphi) = \alpha$ of φ is irrational. It follows from Corollary 2.7 that there is the semi-conjugacy map $h_0 \colon C \to C$ between φ and the rotation R_α, $h_0 \circ \varphi = R_\alpha \circ h_0$.

Let $\mathrm{sus}^t(\varphi)$ be the suspension over φ and $\mathrm{sus}^t(R_\alpha)$ be the suspension over R_α, see Section 1.2. By construction, $\mathrm{sus}^t(\varphi)$ is a fixed-point-free Denjoy flow on $\mathbb{T}^2$, and $\mathrm{sus}^t(R_\alpha)$ is a fixed-point-free linear flow on $\mathbb{T}^2$. Note that $\mathrm{sus}^t(\varphi)$ has the minimal set N. It follows from the construction of suspension that h_0 can be naturally extended to a continuous mapping $h \colon \mathbb{T}^2 \to \mathbb{T}^2$. One can see that the items of Lemma 2.15 imply the corresponding items for the mapping h we required. □

Denote by $\delta(N)$ the accessible from interior $\mathbb{T}^2 \setminus N$ boundary of the set $\mathbb{T}^2 \setminus N$, where N is a nontrivial minimal set of a flow f^t. By Proposition 6.2, $\delta(N)$ is an invariant set of f^t that consists of a finite or a countable family of trajectories from N. Therefore, $h(\delta(N))$ is a finite or a countable family of geodesics from $G(N)$. This family of geodesics is called a *distinguished family of the minimal set* N and is denoted by $R(N, h)$, $h(\delta(N)) = R(N, h)$. Of course, this family depends on the blowing-down operation h from Proposition 6.2. By Theorem 2.8, $R(N, h)$ is determined up to a *translation*, i. e., up to a homeomorphism $\mathbb{T}^2 \to \mathbb{T}^2$ whose covering is given by $x \mapsto x + x_0$, $y \mapsto y + y_0$, where x_0 and y_0 are some constants.

Two sets A, $B \subset \mathbb{T}^2$ are called *equivalent* if there is a translation that takes A to B. Thus, given any blowing-down operations h_1 and h_2, the sets $R(N, h_1)$, $R(N, h_2)$ are equivalent. The following theorem solves partially the problem of the topological classification under homotopic to the identity homeomorphisms of nontrivial minimal sets on $\mathbb{T}^2$.

Theorem 6.13 *Let N_1 and N_2 be nontrivial minimal sets of flows f_1^t and f_2^t respectively on $\mathbb{T}^2$. If N_1 and N_2 are topologically equivalent via a homeomorphism $\mathbb{T}^2 \to \mathbb{T}^2$ homotopic to the identity then N_1 and N_2 have the same geodesic frameworks $G(N_1) = G(N_2)$ and their distinguished families are equivalent. The geodesic framework $G(N)$ of any nontrivial minimal set $N \subset \mathbb{T}^2$ is a (linear) irrational geodesic lamination. Given any irrational geodesic lamination G and any finite or countable family $N_0 \subset G$ of geodesics, there exists a flow f^t with a nontrivial minimal set N such that $G(N) = G$ and $R(N, h) = N_0$ for some blowing-down operation h.*

Proof. Suppose that N_1 and N_2 are topologically equivalent via a homeomorphism $\varphi\colon \mathbb{T}^2 \to \mathbb{T}^2$ homotopic to the identity. By Proposition 6.1, $\operatorname{rot}(f_1^t) = \operatorname{rot}(f_2^t) = \mu$, since φ is homotopic to the identity. Then $G(N_1) = = G(N_2) = G$, and G is a (linear) irrational geodesic lamination because μ is irrational. Let h_1, h_2 be blowing-down operations defined by Proposition 6.2 for the minimal sets N_1, N_2 respectively. Obviously, the covering geodesic lamination $\overline{G}$ for G is invariant under any translation of $\mathbb{R}^2$. We have to prove that there is translation that takes $\overline{R}(N_1, h_1)$ onto $\overline{R}(N_2, h_2)$, where $\pi(\overline{R}(N_i, h_i)) = R(N_i, h_i)$ is the distinguished family of the minimal set N_i, $i = 1, 2$.

Take some component w of $\mathbb{T}^2 \setminus N_1$. Then $\varphi(w)$ is a component of $\mathbb{T}^2 \setminus N_2$. Let $\overline{w}$ be a lift of w. Obviously, $\overline{w}$ is a component of $\mathbb{R}^2 \setminus \overline{N}_1$. Then $\overline{\varphi}(\overline{w})$ is a component of $\mathbb{R}^2 \setminus \overline{N}_2$, and $\overline{h}_1(\overline{w})$, $\overline{h}_2(\overline{\varphi}(\overline{w}))$ are geodesics of $\overline{G}$. Since $\overline{h}_1(\overline{w})$ and $\overline{h}_2(\overline{\varphi}(\overline{w}))$ are parallel straight lines, there is a translation $\overline{F}\colon \mathbb{R}^2 \to \mathbb{R}^2$ that takes $\overline{h}_1(\overline{w})$ onto $\overline{h}_2(\overline{\varphi}(\overline{w}))$, $\overline{F}(\overline{h}_1(\overline{w})) = \overline{h}_2(\overline{\varphi}(\overline{w}))$. Clearly that $\overline{F}$ projects to the shift $F\colon \mathbb{T}^2 \to \mathbb{T}^2$ which takes G onto itself, $F(G) = G$.

Since h_1, and h_2, and φ are homotopic to the identity, one gets

$$\overline{F} \circ \overline{h}_1 \circ \gamma(\overline{w}) = \gamma \circ \overline{F} \circ \overline{h}_1(\overline{w}) = \gamma \circ \overline{h}_2 \circ \overline{\varphi}(\overline{w}) = \overline{h}_2 \circ \overline{\varphi} \circ \gamma(\overline{w})$$

for every integer translation $\gamma\colon \mathbb{Z}^2 \to \mathbb{Z}^2$. Since the integer translations form the covering group Γ_π, the components $\{\gamma(\overline{w})\colon \gamma \in \Gamma_\pi\}$ are dense in the family of all components of the set $\mathbb{R}^2 \setminus \overline{N}_1$. Hence, $\overline{F} \circ \overline{h}_1(\overline{v}) = \overline{h}_2 \circ \overline{\varphi}(\overline{v})$ for any component $\overline{v}$ of $\mathbb{R}^2 \setminus \overline{N}_1$. It follows that $\overline{F}(\overline{R}(N_1, h_1)) = \overline{R}(N_2, h_2)$, and $F(R(N_1, h_1)) = R(N_2, h_2)$.

Let G be an irrational geodesic lamination and $N_0 \subset G$ be a finite or countable family of geodesics. We can consider G as irrational linear flow f_0^t because of orientability of G. The circle $S_0^1 = \pi(x = 0)$ is a closed transversal of f_0^t. Since f_0^t is a linear irrational flow, the forward Poincaré mapping $S_0^1 \to S_0^1$ induced by f_0^t is the rotation R_α with irrational α. Without loss of generality, we can assume that f_0^t is a suspension $\mathrm{sus}^t(R_\alpha)$ over R_α. Clearly that the intersection $\chi = S_0^1 \cap N_0$ is an invariant set of R_α. By Theorem 2.9, there is a Denjoy homeomorphism f such that $\chi = \chi(f, h_0)$, where h_0 is a semi-conjugacy map between f and R_α. Denote by f^t the suspension over f. Then f^t is a Denjoy flow with a nontrivial minimal set denoted by N. The semi-conjugacy h_0 can be naturally extended to the blowing-down operation h that takes f^t to f_0^t. By construction, $h(N) = N_0 = R(N, h)$. □

Two sets $R_1, R_2 \subset \mathbb{T}^2$ are said to be *commensurable* if there is a homeomorphism $F\colon \mathbb{T}^2 \to \mathbb{T}^2$ such that $F(R_1) = R_2$, and F is of the form

$$x' = ax + by + \xi \bmod 1, \quad y' = cx + dy + \eta \bmod 1,$$

where the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is integer and unimodular, and $\xi, \eta \in \mathbb{R}$. The following theorem solves the problem of the topological classification of nontrivial minimal sets on $\mathbb{T}^2$.

Theorem 6.14 *Let N_1 and N_2 be nontrivial minimal sets of flows f_1^t and f_2^t respectively on $\mathbb{T}^2$. Then N_1 and N_2 are topologically equivalent if and only if their distinguished families $R(N_1, h_1)$, $R(N_2, h_2)$ are commensurable, where h_1 and h_2 are blowing-down operations.*

Proof. Suppose that N_1 and N_2 are topologically equivalent via a homeomorphism $\varphi\colon \mathbb{T}^2 \to \mathbb{T}^2$. Then the minimal sets N, $\varphi_*^{-1} \circ \varphi(N_2)$ are topologically equivalent via a homeomorphism $\varphi_*^{-1} \circ \varphi$ homotopic to the identity. It follows from Theorem 6.13 that N, $\varphi_*^{-1} \circ \varphi(N_2)$ are equivalent. Hence, N_1 and N_2 are commensurable.

Suppose that $R(N_1, h_1)$, $R(N_2, h_2)$ are commensurable. By definition, there is the composition $T \circ A\colon \mathbb{T}^2 \to \mathbb{T}^2$ of the translation T and linear mapping A such that $T \circ A(R(N_1, h_1)) = R(N_2, h_2)$. The flow $A(f_1^t)$ has the nontrivial minimal set $A(N_1)$. Note that the mapping $A \circ h_1 \circ A^{-1}$ is homotopic to the identity. Since h_1 is a blowing-down operation, $A \circ h_1 \circ A^{-1}$ is the blowing-down operation for the flow $A(f_1^t)$. Moreover, the distinguished families of the flows $A(f_1^t)$, f_2^t are equivalent. Clearly, $A(f_1^t)$ is topologically equivalent to f_1^t. We can continue the proof replacing $A(f_1^t)$ by f_1^t, and assuming that the distinguished families $R(N_1, h_1)$, $R(N_2, h_2)$ are equivalent.

Recall that the construction of blowing-down operation begins with the choosing a closed simple transversal. Let C_1, C_2 be closed simple transversals corresponding to the blowing-down operations h_1, h_2 respectively. Without loss of generality, we can assume that f_1^t and f_2^t are Denjoy flows which are the suspensions over Denjoy homeomorphisms $f_1\colon C_1 \to C_1$ and $f_2\colon C_2 \to C_2$ respectively. Since $R(N_1, h_1)$, $R(N_2, h_2)$ are equivalent, the characteristic sets of the Denjoy homeomorphisms f_1 and f_2 are also equivalent. By Theorem 2.8, f_1 and f_2 are conjugate via an orientation preserving homeomorphism $C_1 \to C_2$ which can be extended to a homeomorphism taking the suspension $\mathrm{sus}^t(f_1)$ to the suspension $\mathrm{sus}^t(f_2)$. As a consequence, the flows f_1^t, f_2^t are topologically equivalent. □

REMARK. The homeomorphism ψ that takes $\mathrm{sus}^t(f_1)$ to $\mathrm{sus}^t(f_2)$ is homotopic to the identity as a homeomorphism of ambient manifolds of suspensions. But the homeomorphism $\mathbb{T}^2 \to \mathbb{T}^2$ induced by ψ that takes f_1^t to f_2^t is not in general homotopic to the identity because the curves C_1, C_2 are not necessarily homotopic. Note that since C_1, C_2 are simple closed curves, there is a homeomorphism $\mathbb{T}^2 \to \mathbb{T}^2$ taking C_1 to C_2. This homeomorphism can be non-homotopic to the identity.

Clearly, a distinguished family as a part of geodesic framework of nontrivial minimal set determines this geodesic framework. Thus, after Theorem 6.14, we see that a distinguished family itself up to a commensurability is a complete topological invariant of nontrivial minimal set.

The cardinality of the set of distinguished geodesics $R(N, h)$ of nontrivial minimal N does not depend on a blowing-down operation h. The cardinality of $R(N, h)$ is called the *characteristic of minimal set* N. The following result shows that a characteristic itself is not a complete topological invariant.

Theorem 6.15 *There exists a continuum of pairwise topologically nonequivalent nontrivial minimal sets with an identical geodesic framework and the same* (*any prescribed*) *characteristic more than one.*

Proof immediately follows from Theorems 2.11, 6.13. □

Nontrivial minimal sets on a hyperbolic surface

Let N be a nontrivial minimal set of a flow f^t on an orientable closed hyperbolic surface M_p^2 of genus $p \geqslant 2$. First, using the universal covering space that is the hyperbolic plane Δ, we construct a closed set containing N which is said to be a canonical region of nontrivial minimal set.

Since N consists of nontrivially recurrent trajectories, there is a simple closed transversal C that intersects all trajectories from N, see Corollary 3.5. Take a lift $\overline{C} \in \pi^{-1}(C)$ and congruent points $a, b \in \overline{C} \cap \overline{N}$ such that there are no other pairs of congruent points on the arc $ab \subset \overline{C}$. Denote by $\overline{C}_1$, $\overline{C}_2$ the first curves from $\pi^{-1}(C)$ that are intersected by the positive semitrajectories $\overline{l}^+(a)$, $\overline{l}^+(b)$ after the points a and b respectively.

Proposition 6.3 $\overline{C}_1 \neq \overline{C}_2$.

Proof. Assume that $\overline{C}_1 = \overline{C}_2$. Then the points $a_1 = \overline{C}_1 \cap \overline{l}^+(a)$, $b_1 = \overline{C}_1 \cap \overline{l}^+(b)$ are congruent, and there are no other pairs of congruent points on the arc $a_1 b_1 \subset \overline{C}_1$. The arcs $aa_1 \subset \overline{l}^+(a)$, $bb_1 \subset \overline{l}^+(b)$ of the semitrajectories $\overline{l}^+(a)$, $\overline{l}^+(b)$ respectively are also congruent. Therefore, the curvilinear quadrangle bounded by the arcs ab, a_1b_1, aa_1, bb_1 projects into a torus that is a closed surface of genus 1. This contradicts the surface M_p^2 is of genus $p \geqslant 2$. ♢

Let us introduce the positive orientation on $\overline{C}$ from a to b. Note that $\overline{N} \cap \overline{C}$ is a Cantor set. Without loss of generality, one can assume that a is not the left point of an adjacent interval of $\overline{N} \cap \overline{C}$. This means that the arc $ab \subset \overline{C}$ contains points from $(\overline{N} \cap \overline{C}) \setminus \{a\}$ arbitrary close to a.

Given any point $x \in \overline{N} \cap ab$, denote by $\overline{C}(x)$ the first curve from $\pi^{-1}(C)$ intersected by the positive semitrajectory $\overline{l}^+(x)$ after x. In particular, $\overline{C}(a) = \overline{C}_1$ and $\overline{C}(b) = \overline{C}_2$. The continuous dependence of trajectories on the initial conditions implies that the set of points $y \in \overline{N} \cap ab$ with the same $\overline{C}_* = \overline{C}(y)$ is relatively open. Since $\overline{N} \cap ab$ is compact, there are only finitely many open intervals $U_1, \ldots, U_k$ covering $\overline{N} \cap ab$ such that for any $z_1, z_2 \in U_i \cap \overline{N}$ the curves $\overline{C}(z_1)$, $\overline{C}(z_2)$ coincide, while $\overline{C}(z_1) \neq \overline{C}(z_2)$ provided $z_1 \in U_i \cap \overline{N}$ and $z_2 \in U_j \cap \overline{N}$ with $i \neq j$. By Proposition 6.3, $k \geqslant 2$. It follows from the structure of the Cantor set that the intersection $U_i \cap \overline{N}$ is a closed interval $[a_i; b_i]$ for every $i = 1, \ldots, k$ where $a_i, b_i \in \overline{N}$. Thus, $\overline{N} \cap ab \subset \bigcup_{i+1}^{k} [a_i; b_i]$ and $\overline{C}(a_i) = \overline{C}(b_i)$ for any $i = 1, \ldots, k$.

Let $A_i = \overline{l}^+(a_i) \cap \overline{C}(a_i)$ and $B_i = \overline{l}^+(b_i) \cap \overline{C}(a_i)$. Denote by $\overline{d}_i$ the closed quadrangle bounded by the segments $[a_i; b_i] \subset \overline{C}$, $[A_i; B_i] \subset \overline{C}(a_i)$ and by the arcs $a_iA_i \subset \overline{l}^+(a_i)$, $b_iB_i \subset \overline{l}^+(b_i)$. Let $\overline{D}(\overline{C}) = \bigcup_{i=1}^{k} \overline{d}_i$, see Fig. 6.13.

Proposition 6.4 *Set* $\pi\left(\overline{D}(\overline{C})\right) = D(C)$. *Then*

- $N \subset D(C)$;

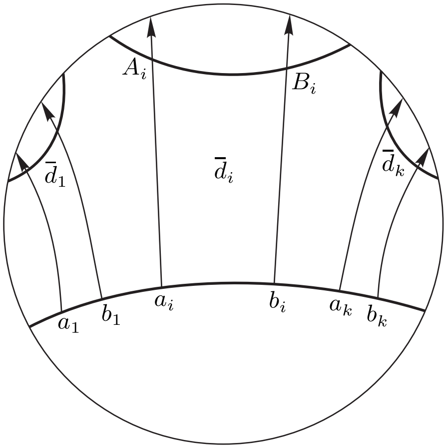


Figure 6.13. The set $\overline{D}(\overline{C}) = \bigcup_{i=1}^{k} \overline{d}_i$.

- *there are no pairs of congruent points in the interior* $\operatorname{int} \overline{D}(\overline{C})$;
- *each component of* $\operatorname{int} D(C)$ *is simply connected.*

Proof. Since $\pi(ab) = C$, every trajectory from N has a lift intersecting ab. This follows that $N \subset D(C)$. Suppose that there is a pair of congruent points inside $\operatorname{int} \overline{D}(\overline{C})$. Then there are congruent points $z_1, z_2 \in \overline{N} \cap \overline{D}(\overline{C})$ such that $z_1, z_2 \notin \overline{l}^+(a)$ since a is not a left point of adjacent interval. The trajectories $\overline{l}(z_1)$, $\overline{l}(z_2)$ are also congruent. This implies that the points $\overline{l}(z_1) \cap \overline{C}$, $\overline{l}(z_2) \cap \overline{C}$ are congruent as well. By construction of the set $\overline{D}(\overline{C})$, $\overline{l}(z_1) \cap \overline{C}$, $\overline{l}(z_2) \cap \overline{C} \subset ab$. This contradicts that there are no congruent points on ab except a and b. According to the construction of the set $\overline{D}(\overline{C})$, each component of $\operatorname{int} D(C)$ has a lift that is the interior of some $\overline{d}_i$ which is simply connected. It implies the last item. $\diamondsuit$

The set $D(C)$ is called a *canonical region* of N based on C. The set $\overline{D}(\overline{C})$ is called the *lift of canonical region* $D(C)$ *with the bottom* $ab \subset \overline{C}$.

A component of the set $M_p^2 \setminus N$ is called a *Denjoy cell* if it is simply connected and its boundary accessible from interior $M_p^2 \setminus N$ consists of exactly two trajectories of the minimal set N. These trajectories are said to be *special*. One can see that special trajectories of Denjoy cell have the same co-asymptotic geodesic called a *distinguished* geodesic. Similar to the case of the torus, we will call a family of distinguished geodesics a *distinguished family of the geodesic framework of the minimal set* N.

Since the generation and elimination of Denjoy cells do not change the geodesic framework of nontrivial minimal set, the presence of these cells can be considered, in a sense, artificial. Therefore, we first consider a classification of nontrivial minimal sets without Denjoy cells.

Theorem 6.16 *Let* N *be a nontrivial minimal set of a flow* f^t *on a closed orientable hyperbolic surface* M_p^2 *of genus* $p \geqslant 2$. *Suppose that* N *does not contain Denjoy cells. Then* N *is topologically equivalent via a homeomorphism* $M_p^2 \to M_p^2$ *homotopic to the identity to its own geodesic framework* $G(N)$ *that is an orientable weakly irrational geodesic lamination,* $G(N) \in \Lambda_{or}(M_p^2)$. *For any orientable weakly irrational geodesic lamination* $G_0 \in \Lambda_{or}(M_p^2)$, *there exists a nontrivial minimal set* N *without Denjoy cells of some flow* f^t *such that* $G(N) = G_0$. *Moreover, one can construct* f^t *so that* $N = G_0$.

Proof. Suppose that N is a nontrivial minimal set. By Corollary 3.5, there is a closed simple transversal C intersecting with all trajectories from N. Moreover, C is non-homotopic to zero. Therefore, there exists the closed simple geodesic g freely homotopic to C. Take a lift $\overline{C}$ of C. Due to Lemma 5.2, there is the co-asymptotic geodesic $\overline{g}(\overline{C}) = \overline{g}$ that is a lift of g. By definition, $\overline{C}$ and $\overline{g}$ have the same ideal endpoints $\sigma_1, \sigma_2 \in S_\infty$. We introduce the orientation on $\overline{C}$ and $\overline{g}$ with the positive direction from σ_1 to σ_2 that induces the orientation on C and g.

Let $\overline{m} \in \overline{C}$ be a point such that $\pi(\overline{m}) = m \in N$, and $l \in N$ be a trajectory through m. Take the lift $\overline{l}$ of l through $\overline{m}$. It follows from Corollary 5.4 that $\overline{l}$ has irrational asymptotic directions $\omega(\overline{l}), \alpha(\overline{l}) \in S_\infty$ with $\omega(\overline{l}) \neq \alpha(\overline{l})$. Clearly that the ideal endpoints $c_1, c_2 \in S_\infty$ of $\overline{C}$ are separated by the points $\omega(\overline{l}), \alpha(\overline{l})$, see Section 5.1. Therefore, the ideal endpoints of $\overline{g}$, which are c_1 and c_2, are also separated by $\omega(\overline{l}), \alpha(\overline{l})$. Hence, the co-asymptotic geodesic $\overline{g}(\overline{l})$ intersects $\overline{g}$ at a unique point denoted by $\overline{m}'$. Let $\overline{\theta}\colon \overline{N} \cap \overline{C} \to \overline{G}(\overline{N}) \cap \overline{g}$ be the mapping defined by $\overline{\theta}(\overline{m}) = \overline{m}'$ where $\overline{N} = \pi^{-1}(N)$ and $\overline{G}(\overline{N}) = \pi^{-1}(G(N))$, Fig. 6.14. Since $\overline{G}(\overline{N})$ is a geodesic lamination, the mapping $\overline{\theta}$ is a monotone homeomorphism.

Step 6.1 *Let* $\gamma \in \Gamma$ *be an axis of* $\overline{g}$, $\gamma(\overline{g}) = \overline{g}$. *Then given any* $\overline{m} \in \overline{C} \cap \overline{N}$, $\overline{\theta} \circ \gamma(\overline{m}) = \gamma \circ \overline{\theta}(\overline{m})$.

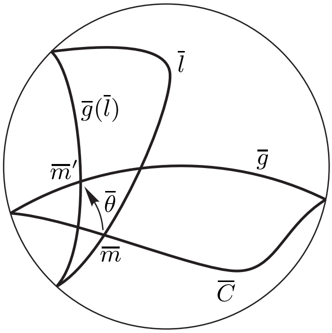


Figure 6.14. The mapping $\overline{\theta}$.

Proof of Step 6.1. Clearly, $\overline{N}$ and $\overline{G}(\overline{N})$ are invariant under γ. Since C is without self-intersections, $\gamma(\overline{C}) = \overline{C}$ and $\gamma(\overline{m}) \in \overline{C}$. This implies the result. $\Diamond$

Step 6.1 implies that $\overline{\theta}$ projects to the monotone mapping $\theta\colon N \cap C \to \to G(N) \cap g$,

$$\theta \circ \pi|_{\overline{N} \cap \overline{C}} = \pi \circ \overline{\theta}|_{\overline{N} \cap \overline{C}}.$$

Take any lift $\overline{C}^*$ of C. This means that there is an element $\gamma^* \in \Gamma$ such that $\overline{C}^* = \gamma^*(\overline{C})$. This implies that the co-asymptotic geodesic $\overline{g}(\overline{C}^*) = \overline{g}^*$ equals to $\gamma^*(\overline{g}) = \gamma^*\left(\overline{g}(\overline{C})\right)$. Similarly to the definition of $\overline{\theta}\colon \overline{N} \cap \overline{C} \to \to \overline{G}(\overline{N}) \cap \overline{g}$, one can define the mapping $\overline{\theta}^*\colon \overline{N} \cap \overline{C}^* \to \overline{G}(\overline{N}) \cap \overline{g}^*$.

Step 6.2 *One holds*

$$\gamma^* \circ \overline{\theta}|_{\overline{N} \cap \overline{C}} = \overline{\theta}^* \circ \gamma^*\Big|_{\overline{N} \cap \overline{C}}, \quad \textit{where} \quad \gamma^* \in \Gamma \textit{ with } \overline{C}^* = \gamma^*(\overline{C}).$$

Proof of Step 6.2 immediately follows from the invariantness of $\overline{N}$ and $\overline{G}(\overline{N})$ under γ^*. $\Diamond$

The intersections $N \cap C$, $G(N) \cap g$ are Cantor sets. The monotonicity of θ and Step 6.1 imply that θ can be continuously extended to a homeomorphism $C \to g$ denoted again by θ. So, $\overline{\theta}$ has the continuous extension to the homeomorphism $\overline{C} \to \overline{g}$ denoted again by $\overline{\theta}$ such that $\theta \circ \pi|_{\overline{C}} = = \pi \circ \overline{\theta}|_{\overline{C}}$. It follows from Step 6.2 that all $\overline{\theta}^*$ project to the same mapping $\theta\colon N \cap C \to G(N) \cap g$ induced by $\overline{\theta}|_{\overline{N} \cap \overline{C}}$. One can assume that $\overline{\theta}^*$ is continuously extended to the homeomorphism $\overline{C}^* \to \overline{g}^*$ denoted by $\overline{\theta}^*$ such that $\theta \circ \pi|_{\overline{C}^*} = \pi \circ \overline{\theta}^*|_{\overline{C}^*}$ for all $\overline{C}^* \in \pi^{-1}(C)$.

Due to Proposition 6.4, there is a canonical region $D(C)$ of N. Let $\overline{D}(\overline{C}) = \bigcup_{i=1}^{k} \overline{d}_i$ be the lift of $D(C)$ with a bottom $ab \subset \overline{C}$. By construction, each $\overline{d}_i$ is the closed quadrangle bounded by the segments $[a_i;\, b_i] \subset \overline{C}$, $[A_i;\, B_i] \subset \overline{C}(a_i)$ and by the arcs $a_iA_i \subset \overline{l}^+(a_i)$, $b_iB_i \subset \overline{l}^+(b_i)$, see Fig. 6.13. Recall that $\overline{l}^+(a_i)$, $\overline{l}^+(b_i)$, $i = 1, \ldots, k$, are lifts of nontrivially recurrent trajectories. According to Corollary 5.4, all these trajectories have co-asymptotic geodesics. Every lift $\overline{C}_i$ of C also has the co-asymptotic geodesic $\overline{g}(\overline{C}_i)$. Using the co-asymptotic geodesics $\overline{g}(\overline{l}^+(a_1))$, $\overline{g}(\overline{l}^+(b_1))$, $\ldots$, $\overline{g}(\overline{l}^+(a_k))$, $\overline{g}(\overline{l}^+(b_k))$, and the co-asymptotic geodesics $\overline{g}(\overline{C}_i)$, one can construct the set $\overline{Q}(\overline{g}) = \bigcup_{i=1}^{k} \overline{q}_i$ similarly to $\overline{D}(\overline{C}) = \bigcup_{i=1}^{k} \overline{d}_i$. Actually, $\pi\left(\overline{Q}(\overline{g})\right) = Q(g)$ is a canonical region of the geodesic framework $G(N)$. It follows from Proposition 6.4 that there exists a homeomorphism $\overline{\varphi}\colon \overline{D}(\overline{C}) \to \overline{Q}(\overline{g})$ that takes the arcs $a_iA_i \subset \overline{l}^+(a_i)$, $b_iB_i \subset \overline{l}^+(b_i)$ of the trajectories $\overline{l}^+(a_i)$, $\overline{l}^+(b_i)$ to arcs of the co-asymptotic geodesics $\overline{g}(\overline{l}^+(a_i))$, $\overline{g}(\overline{l}^+(b_i))$, and the segments $[a_i;\, b_i] \subset \overline{C}$, $[A_i;\, B_i] \subset$ $\subset \overline{C}(a_i)$ to arcs of the co-asymptotic geodesics $\overline{g}(\overline{C}_i)$, $i = 1, \ldots, k$. Moreover, $\overline{\varphi}$ projects to a homeomorphism $\varphi\colon D(C) \to Q(g)$ with the similar properties. This homeomorphism takes any simple closed curve S of the boundary $\partial D(C)$ to a simple closed curve $\mathscr{S}$ of the boundary $\partial Q(g)$. Given any component R of $M_p^2 \setminus D(C)$, there is a unique component $\mathscr{R}$ of $M_p^2 \setminus Q(g)$ such that φ realizes a one-to-one correspondence between the components of the boundaries ∂R and $\partial \mathscr{R}$. By construction, the curves $S \subset \partial R$ and $\mathscr{S} \subset \partial \mathscr{R}$ are homotopic. Therefore, φ can be extended to a homeomorphism $R \cup D(C) \to \mathscr{R} \cup Q(g)$. Going through this procedure for each component of $M_p^2 \setminus D(C)$, we get an orientation preserving homeomorphism $\varphi\colon M_p^2 \to M_p^2$ carrying the trajectories of N onto the geodesics of $G(N)$.

Let $\overline{\varphi}\colon \Delta \to \Delta$ be a lift of $\varphi\colon M_p^2 \to M_p^2$. It follows from Theorem 6.1 that $\overline{\varphi}$ extends to unique homeomorphism $\Delta \cup S_\infty \to \Delta \cup S_\infty$ denoted again by $\overline{\varphi}$. The set of points attained by the trajectories from $\overline{N}$ coincides with $\overline{G}_\infty(\overline{N}) \subset S_\infty$. By construction of $\overline{\varphi}$, $\overline{\varphi}|_{\overline{G}_\infty(\overline{N})} = id$. Since the set $\overline{G}_\infty(\overline{N})$ is dense on S_∞, $\overline{\varphi}|_{\overline{G}_\infty(\overline{N})} = id$. According to Lemma 6.7, φ is homotopic to the identity. This proves that N is topologically equivalent via a homeomorphism $M_p^2 \to M_p^2$ homotopic to the identity to its own geodesic framework $G(N)$.

Clearly that a nontrivial minimal set is a Maier quasiminimal set. Hence, by Theorem 5.32, the geodesic framework $G(N)$ is a weakly irrational geodesic lamination. Since N is a minimal set of flow, $G(N)$ is an orientable geodesic lamination, $G(N) \in \Lambda_{or}$.

Take an orientable weakly irrational geodesic lamination $G_0 \in \Lambda_{or}(M_p^2)$. Let g_0 be a simple closed geodesic transversally intersected by geodesics from G_0. Similarly to a nontrivial minimal set, one can construct a canonical region $D(g_0)$ for G_0. By Proposition 6.4, the interior of every component of $D(g_0)$ is simply connected. Moreover, every component of $D(g_0)$ looks like a neighborhood with the structure of linear local lamination (closed trivially foliated box). The orientability of G_0 implies that one can endow some neighborhood of $D(g_0)$ with the structure of flow f^t such that G_0 becomes a nontrivial minimal set of f^t. Outside of the neighborhood of $D(g_0)$, the flow f^t can be extended, for example, by fixed points. This completes the proof. □

Now, consider nontrivial minimal sets with Denjoy cells and describe the type of geodesics that form distinguished families of these minimal sets. Recall that a nontrivially recurrent geodesic may be either left or right improper; i. e., it may approach itself from either the left or the right side. If a nontrivially recurrent geodesic is improper only from one side, then it is called a *boundary* one. Otherwise (i. e., if a geodesic is improper from both sides), it is called *internal*.

A weakly irrational geodesic lamination on a closed hyperbolic surface has a finite nonzero number of boundary nontrivially recurrent geodesics and a continuum set of internal ones. The definition of a Denjoy cell and the density of each geodesic in a weakly irrational geodesic lamination imply that each distinguished geodesic is internal. The following two theorems solve the problem of the topological classification of nontrivial minimal sets of flows on a closed orientable hyperbolic surface. Proofs are slightly modification of the proof above of Theorem 6.16. We omit details to the Reader.

Theorem 6.17 *Let N be a nontrivial minimal set of a flow f^t on a closed orientable hyperbolic surface M_p^2 of genus $p \geqslant 2$. Then the geodesic framework $G(N)$ is an orientable weakly irrational geodesic lamination that contains at most a countable distinguished family that consists of internal geodesics. Conversely, let $G_0 \in \Lambda_{or}(M_p^2)$ be an orientable weakly irrational geodesic lamination and let $\mathcal{N}$ be at most a countable family of internal geodesics from G_0. Then, there exists a nontrivial minimal set N of a flow f^t such* *that $G(N) = G_0$ and the distinguished family of the geodesic framework $G(N)$ coincides with $\mathscr{N}$.*

Theorem 6.18 *Let N_1 and N_2 be nontrivial minimal sets of flows f_1^t and f_2^t, respectively, on a closed orientable hyperbolic surface M_p^2 of genus $p \geqslant 2$. Then N_1 and N_2 are orbitally topologically equivalent via a homeomorphism $M_p^2 \to M_p^2$ homotopic to the identity if and only if they have identical geodesic frameworks* (*with regard to the orientation of geodesics*) *and the same family of distinguished geodesics.*

6.6. Surface AP-homeomorphisms

The classification of discrete-time dynamical systems formed by iterations of maps reduces to the problem of (topological) conjugacy for maps itself that generate corresponding dynamical systems. Before solving this problem, it is natural to study the conjugacy for restrictions of maps under consideration to their invariant sets. There are two ways to do that. The first way is to ask, when the restrictions of two maps to their invariant sets are conjugate? The second way is to ask, when two maps are conjugate in neighborhoods of their invariant sets? Mainly, we will pay attention to the second way.

Definition 6.1 *Suppose that mappings f, $f'\colon M \to M$ have invariant sets N and N', respectively,*

$$f(N) = N \quad \textit{and} \quad f'(N') = N'.$$

Then f, f' are said to be conjugate on N and N' if there exists a homeomorphism $\phi\colon M \to M$ such that

$$\phi(N) = N' \quad \textit{and} \quad \phi \circ f|_N = f' \circ \phi|_N.$$

Sometimes, one says that N and N' are *topologically equivalent*. For $N = N' = M$, we get the definition of conjugacy.

One says that f and f' are *conjugate* via the homeomorphism ϕ. If ϕ is homotopic to the identity, we'll say that f, f' are *conjugate via a homotopy trivial homeomorphism*.

Let $f\colon M \to M$ be a homeomorphism of a surface M and $\mathscr{F}$ be a foliation on M that is invariant under f (i.e. $f(\mathrm{Sing}(\mathscr{F})) = \mathrm{Sing}(\mathscr{F})$ and f maps every leaf onto a leaf). $\mathscr{F}$ is said to be *contracting* if, given any points a and b that belong to the same leaf, the distance between $f^n(a)$ and $f^n(b)$ tends to zero as $n \to +\infty$ in the interior metric on the leaves. A foliation $\mathscr{F}$ is called *expanding* if it is contractive under f^{-1}.

A homeomorphism $f\colon M \to M$ is called *almost pseudo-Anosov* (in short, *AP-homeomorphism*) if it satisfies the conditions:

- f has invariant foliations $\mathscr{F}^s$, $\mathscr{F}^u$ that form a strongly irrational 2-web.
- $\mathscr{F}^s$ is contracting and $\mathscr{F}^u$ is expanding under f.

AP-homeomorphisms are in sense non-uniform pseudo-Anosov homeomorphisms. The class of AP-homeomorphisms includes pseudo-Anosov ones for which the contraction and expansion satisfy some uniform estimates.

Here, we demonstrate how the study of maps "at infinity" (induced by homeomorphisms of a surface) helps to solve the classification problem for AP-homeomorphisms. We'll see that AP-homeomorphisms are conjugate via a homotopy trivial homeomorphism if and only if they act alike on the circle at infinity S_∞. Note that the classification results are presented here in the spirit of the method developed in the present monograph. Not all the results were originally formulated as we formulate them here. For the interested Reader, we provide references to the original studies.

Homeomorphisms of 2-torus

Recall that on $\mathbb{T}^2$, a strongly irrational 2-web is actually a 2-web consisting of a pair of transversal irrational foliations without singularities. The following theorem says that an AP-homeomorphism $\mathbb{T}^2 \to \mathbb{T}^2$ is Anosov hyperbolic automorphism up to conjugacy.

Theorem 6.19 *Let $f\colon \mathbb{T}^2 \to \mathbb{T}^2$ be an AP-homeomorphism. Then f is conjugate to Anosov toral hyperbolic automorphism.*

Proof. Let $(\mathscr{F}_1, \mathscr{F}_2)$ be an invariant 2-web and $\overline{f}\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a lift of f. Then $\overline{f}$ has the invariant 2-web $(\overline{\mathscr{F}}_1, \overline{\mathscr{F}}_2)$ that is a lift of $(\mathscr{F}_1, \mathscr{F}_2)$. For definiteness, suppose that $\mathscr{F}_1$ is a contracting foliation and $\mathscr{F}_2$ is an expanding one. By Theorem 6.11, $(\mathscr{F}_1, \mathscr{F}_2)$ is topologically equivalent via a homeomorphism homotopic to the identity to the 2-web consisting of a pair of linear transversal irrational foliations. Therefore, any leaf of $\overline{\mathscr{F}}_1$ intersects every leaf of $\overline{\mathscr{F}}_2$.

Let $f_*\colon \mathbb{Z}^2 \to \mathbb{Z}^2$ be the automorphism of the integer 2-dimensional lattice $\mathbb{Z}^2$ induced by f. Since f is a homeomorphism, f_* can be given by an integer unimodular matrix $A_* \in SL(2, \mathbb{Z}^2)$ that defines the linear mapping L_A such that its lift $\overline{L}_A$ coincides with f_* on $\mathbb{Z}^2$. Let us show that A_* is hyperbolic, that is A_* has no eigenvalues of absolute value 1. Suppose the contrary.

Take a periodic point $p \in \mathrm{Per}(f)$. Passing to an iteration, if necessary, one can assume that p is a fixed point such that the origin $O \in \mathbb{Z}^2 \subset \mathbb{R}^2$ is the lift of p and a fixed point of $\overline{f}$. Since f_* is induced by f, $f_* = \overline{f}\big|_{\mathbb{Z}^2}$. Because of A_* is not hyperbolic, $\overline{L}_A$ has a periodic point, say O_1, different from O. Taking some iteration, if necessary, we can assume that O_1 is an integer fixed point. Therefore, O_1 is a fixed point of f_*, hence for $\overline{f}$. Let $W_1^s(O)$ be the contracting leaf of $\overline{\mathscr{F}}_1$ and $W_2^u(O_1)$ be the expanding leaf of $\overline{\mathscr{F}}_2$. The intersection $W_1^s(O) \cap W_2^u(O_1)$ consists of a unique point, say z. Since $\overline{f}(W_1^s(O)) = W_1^s(O)$ and $\overline{f}(W_2^u(O_1)) = W_2^u(O_1)$, z is a fixed point of $\overline{f}$. On the other side, z belongs to a contracting leaf (as well as, expanding one). Therefore, z can not be a fixed point. This contradiction implies that $\overline{L}_A|_{\mathbb{Z}^2}$ is hyperbolic. Hence, f_* is also hyperbolic. Hyperbolicity of L_A implies that L_A has the invariant 2-web (W^s, W^u) consisting of stable W^s and unstable W^u manifolds. Each of W^s, W^u is an irrational linear foliation. Let $(\overline{W}^s, \overline{W}^u)$ be a lift of (W^s, W^u). Geometrically, each of $\overline{W}^s$, $\overline{W}^u$ consists of straight lines with irrational slope.

By Proposition 2.1 [72], there is a continuous trivially homotopic mapping $h\colon M \to M$ such that $L_A \circ h = h \circ f$. Let us show that h is a homeomorphism. Denote by $\overline{h}\colon \mathbb{R}^2 \to \mathbb{R}^2$ a lift of h such that $\overline{L}_A \circ \overline{h} = \overline{h} \circ \overline{f}$. Hence,

$$\overline{L}_A^n \circ \overline{h} = \overline{h} \circ \overline{f}^n, \quad n \in \mathbb{Z}. \tag{6.5}$$

First, we prove that the restriction of $\overline{h}$ on a leaf of $\overline{\mathscr{F}}_1$ is a homeomorphism. Take any $x, y \in \overline{W}_1^s$. Suppose that $\overline{h}(x) = \overline{h}(y)$. Hence, $d(\overline{L}_A^n \circ \overline{h}(x), \overline{L}_A^n \circ \circ \overline{h}(y)) = 0$ for every $n \in \mathbb{Z}$. It follows from (6.5) that

$$d(\overline{L}_A^n \circ \overline{h}(x), \overline{L}_A^n \circ \overline{h}(y)) = d(\overline{h} \circ \overline{f}^n(x), \overline{h} \circ \overline{f}^n(y)) \to \infty$$

as $n \to +\infty$ because of the leaf $\overline{W}_1^s$ is contracting and $\overline{h}$ is a lift of homotopy trivial mapping. This contradiction shows that the restriction of $\overline{h}$ on $\overline{W}_1^s$ is a homeomorphism. Similarly, one can prove that the restriction of $\overline{h}$ on a leaf of $\overline{\mathscr{F}}_2$ is a homeomorphism. Since any leaf of $\overline{\mathscr{F}}_1$ intersects every leaf of $\overline{\mathscr{F}}_2$ at a unique point, it follows that $\overline{h}$ is a homeomorphism. As a consequence, h is a conjugacy map between f and L_A. $\square$

Homeomorphisms of hyperbolic surfaces

Let $\overline{f}\colon \Delta \to \Delta$ be a lift for $f\colon M \to M$, where M is a closed hyperbolic surface. Recall that $\overline{f}$ extends continuously to the homeomorphism $\Delta \cup S_\infty \to \to \Delta \cup S_\infty$ denoted again by $\overline{f}$.

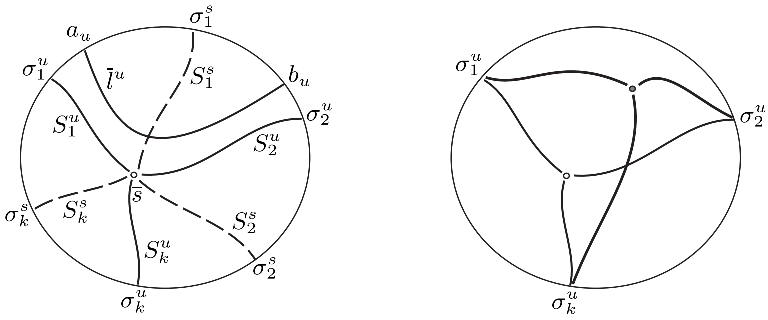
Lemma 6.15 *Let $f\colon M \to M$ be an AP-homeomorphism of closed orientable hyperbolic surface $M = \Delta/\Gamma$ with invariant contracting and expanding foliations $\mathscr{F}^s$, $\mathscr{F}^u$ respectively that form a strongly irrational 2-web. Suppose that a saddle s is a fixed point, and $\overline{f}$ is a lift of f such that a lift $\overline{s}$ of s is a fixed point of $\overline{f}$. Let $\overline{S}_1^{u(s)}, \ldots, \overline{S}_k^{u(s)}$ be separatrices of $\overline{s}$ that belong to the lift $\overline{\mathscr{F}}^{u(s)}$ of the expanding (contracting) foliation $\mathscr{F}^{u(s)}$. Then*

- *every $\overline{S}_j^{u(s)}$ reaches a point $\sigma_j^{u(s)}$ that is an isolated attractive (repelling) point of $\overline{f}\big|_{S_\infty}$;*
- *the points $\sigma_1^u, \sigma_1^s, \ldots, \sigma_k^u, \sigma_k^s$ alternatively appear on S_∞;*
- *$\overline{s}$ is a unique fixed point of $\overline{f}\big|_\Delta$, and $\sigma_1^u, \sigma_1^s, \ldots, \sigma_k^u, \sigma_k^s$ are unique fixed points of $\overline{f}\big|_{S_\infty}$.*

Proof. Since the both $\mathscr{F}^u$ and $\mathscr{F}^s$ are strongly irrational foliations, every separatrix $\overline{S}_j^{u(s)}$ reaches a point, say $\sigma_j^{u(s)}$ (see Theorem 5.22), at the circle at infinity S_∞. Because of $\overline{\mathscr{F}}^u$, $\overline{\mathscr{F}}^s$ are transversal and every singularity has negative index, $\overline{S}_i^u \cap \overline{S}_j^s = \varnothing$ for any i, j. Therefore, the points $\sigma_1^u, \sigma_1^s, \ldots, \sigma_k^u, \sigma_k^s$ alternatively appear on S_∞. Since $\overline{s}$ is a fixed point and the foliations $\overline{\mathscr{F}}^u$, $\overline{\mathscr{F}}^s$ are invariant, every point $\sigma_j^{u(s)}$ is fixed under $\overline{f}\big|_{S_\infty}$.

Suppose $\overline{S}_1^u$, $\overline{S}_2^u$ are adjacent separatrices and $\overline{S}_1^s$ is between them as shown in Fig. 6.15 (a). Take a leaf $\overline{l}^u$ of $\overline{\mathscr{F}}^u$ intersecting $\overline{S}_1^s$ at a (unique) point c_u. We can assume also that $\overline{l}^u$ is not a separatrix. By Theorem 5.22, $\overline{l}^u$ reaches points $a_u, b_u \in S_\infty$ in the both directions. Clearly that if $c_u \to \overline{s}$ then $a_u \to \sigma_1^u$ and $b_u \to \sigma_2^u$. It follows that any σ_j^u is an isolated attractive fixed point. Similarly, any σ_j^s is an isolated repelling fixed point. Since a leaf can not approach a fixed arc of positive length of S_∞, $a_u \to \sigma_1^s$ and $b_u \to \sigma_2^s$ as $c_u \to \sigma_1^s$. This implies that the arc $(\sigma_1^u, \sigma_2^u) \subset S_\infty$ containing σ_1^s has a unique fixed point σ_1^s. It follows that $\overline{s}$ is a unique fixed point of the mapping $\overline{f}\big|_\Delta$, and $\sigma_1^u, \sigma_1^s, \ldots, \sigma_k^u, \sigma_k^s$ are unique fixed points of $\overline{f}\big|_{S_\infty}$. □

The crucial step to classify AP-homeomorphisms is the following theorem.



(a) (b)

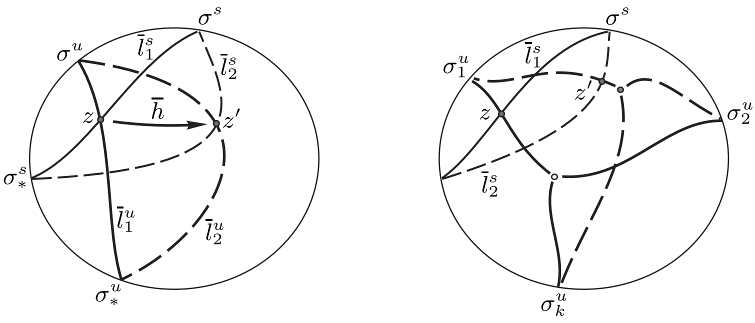
Figure 6.15

Theorem 6.20 *Let f_1, f_2: $M \to M$ be AP-homeomorphisms, where $M = \Delta/\Gamma$ is a closed orientable hyperbolic surface. Then f_1 and f_2 are conjugate via a homotopy trivial homeomorphism if and only if there exist the lifts $\overline{f}_1$, $\overline{f}_2$: $\Delta \to \Delta$ of f_1, f_2 respectively whose extensions on S_∞ coincide, $\overline{f}_1|_{S_\infty} = \overline{f}_2|_{S_\infty}$.*

Proof. Suppose that f_1 and f_2 are conjugate via a homotopy trivial homeomorphism $M \to M$. Then f_1 is homotopic to f_2. By Theorem 6.7, there are the lifts $\overline{f}_1$, $\overline{f}_2$ of f_1, f_2 respectively such that $\overline{f}_1|_{S_\infty} = \overline{f}_2|_{S_\infty}$.

Suppose now that f_1 and f_2 have the lifts $\overline{f}_1$, $\overline{f}_2$ respectively such that $\overline{f}_1|_{S_\infty} = \overline{f}_2|_{S_\infty}$. Let $\mathscr{F}_i^{s(u)}$ be a contracting (expanding) foliation of f_i ($i = 1, 2$) and $\overline{\mathscr{F}}_i^{s(u)}$ be a lift of $\mathscr{F}_i^{s(u)}$. Recall that the both $\mathscr{F}_i^s$ and $\mathscr{F}_i^u$ are strongly irrational foliations with common set of singularities consisting of saddles with negative indices. Take a saddle s_{i0}. Then s_{i0} is a periodic point of f_i because of finitely many singularities which are mapped onto itself. Taking an iteration, if necessary, one can assume that s_{i0} is a fixed point. Without loss of generality, we can suppose that there is a lift $\overline{s}_{i0}$ of s_{i0} such that $\overline{s}_{i0}$ is a fixed point of $\overline{f}_i$. For $\overline{\mathscr{F}}_i^u$, denote by $\overline{S}_{i1}^u, \ldots, \overline{S}_{ij(i)}^u$ separatrices of the saddle $\overline{s}_{i0}$. By Lemma 6.15, each $\overline{S}_{ik}^u$ reaches S_∞ at a point σ_{ik}^u. Moreover, the points $\sigma_{i1}^u, \ldots, \sigma_{ij(i)}^u$ are unique sink points of $\overline{f}_i|_{S_\infty}$. Since $\overline{f}_1|_{S_\infty} = \overline{f}_2|_{S_\infty}$, the homeomorphisms $\overline{f}_1|_{S_\infty}$ and $\overline{f}_2|_{S_\infty}$ have the same sets of sink and source points. In particular, $j(1) = j(2)$. As a consequence, the foliations $\overline{\mathscr{F}}^u_1$, $\overline{\mathscr{F}}^u_2$ have leaves with the same asymptotical direction, Fig. 6.15 (b). By Lemma 5.7 and Theorem 5.34, the geodesic frameworks $G(\mathscr{F}^u_1)$, $G(\mathscr{F}^u_2)$ of $\mathscr{F}^u_1$, $\mathscr{F}^u_2$ respectively are coincident. Similarly, $G(\mathscr{F}^s_1) = G(\mathscr{F}^s_2)$.

Now we are going to construct a homeomorphism $\overline{h} : \Delta \to \Delta$ that conjugates $\overline{f}_1$ with $\overline{f}_2$. Take a point $z \in \Delta$. There are two cases: 1) z is not a saddle of $\overline{\mathscr{F}}^u_1$ (hence, not a saddle of $\overline{\mathscr{F}}^s_1$); 2) z is a saddle of $\overline{\mathscr{F}}^u_1$ (hence, a saddle of $\overline{\mathscr{F}}^s_1$). In the case 1), there are two semileaves $\overline{l}^u_1$, $\overline{l}^s_1$ through $\overline{z}$ of $\overline{\mathscr{F}}^u_1$, $\overline{\mathscr{F}}^s_1$ respectively that reach points σ^u, $\sigma^s \in S_\infty$. Here, case 1) is divided into two subcases: 1a) the both $\overline{l}^u_1$ and $\overline{l}^s_1$ do not belong to a separatrix; 1b) at least one of $\overline{l}^u_1$, $\overline{l}^s_1$ belongs to a separatrix. In the subcase 1a), $\overline{l}^u_1$ and $\overline{l}^s_1$ are lifts of non-trivially leaves in positive and negative directions. Therefore, $\overline{l}^u_1$ and $\overline{l}^s_1$ reach points σ^u_*, $\sigma^s_* \in S_\infty$ different from σ^u, σ^s respectively, Fig. 6.16 (a). Because of transversality, the pair (σ^u, σ^u_*) separates the pair (σ^s, σ^s_*).



(a) (b)

Figure 6.16

It follows from Theorem 5.34 and the equalities $G(\mathscr{F}^u_1) = G(\mathscr{F}^u_2)$, $G(\mathscr{F}^s_1) = G(\mathscr{F}^s_2)$ that there are semileaves $\overline{l}^u_2$, $\overline{l}^s_2$ of $\overline{\mathscr{F}}^u_2$, $\overline{\mathscr{F}}^s_2$ respectively such that $\overline{l}^u_2$ reaches σ^u, σ^u_* and $\overline{l}^s_2$ reaches σ^s, σ^s_*. Since (σ^u, σ^u_*) separates (σ^s, σ^s_*), the intersection $\overline{l}^u_2 \cap \overline{l}^s_2$ consists of a unique point, say z'. Put by definition $\overline{h}(z) = z'$.

Note that the foliations $\overline{\mathscr{F}}^{s(u)}_i$ $(i = 1, 2)$ are invariant under action of the group Γ. Since $\overline{\mathscr{F}}^{s(u)}_i$ is invariant under $\overline{f}_i$ and $\overline{f}_1|_{S_\infty} = \overline{f}_2|_{S_\infty}$,

$$\overline{h} \circ \overline{f}_1(z) = \overline{f}_2 \circ \overline{h}(z), \quad \overline{h}(\gamma(z)) = \gamma(\overline{h}(z)), \quad \forall \gamma \in \Gamma. \tag{6.6}$$

In the subcase 1b), suppose for definiteness that $\overline{l}_1^u$ is a separatrix of a saddle $\overline{s}$. By Theorem 5.34, $\overline{l}_2^u$ is also a separatrix of some saddle, say $\overline{s}'$. Denote by $\overline{S}_2$, $\overline{S}_k$ the Bendixson extension of $\overline{l}_1^u$. Then $\overline{S}_2$ and $\overline{S}_k$ reach points σ_1^u, $\sigma_k^u \in S_\infty$ respectively. The equality $G(\mathscr{F}_1^u) = G(\mathscr{F}_2^u)$ and Theorem 5.34 imply that there are Bendixson extensions $\overline{S}_2'$ and $\overline{S}_k'$ of $\overline{l}_2^u$ that reach the same points σ_1^u, σ_k^u respectively. First, assume that $\overline{l}_1^s$ is not a separatrix. Then $\overline{l}_1^s$ reaches points σ^s, $\sigma_*^s \in S_\infty$. It follows from Theorem 5.34 and the equality $G(\mathscr{F}_1^s) = G(\mathscr{F}_2^s)$ that there is the leaf $\overline{l}_2^s$ of $\overline{\mathscr{F}}_2^s$ such that $\overline{l}_2^s$ reaches σ^s, σ_*^s. Because of $\overline{\mathscr{F}}_1^s$ and $\overline{\mathscr{F}}_1^u$ are transversal, the pair (σ^s, σ_*^s) separates (σ^u, σ_1^u) and (σ^u, σ_k^u). Therefore, $\overline{l}_2^s$ intersects $\overline{l}_2^u$ at some point, say z', Fig. 6.16 (b). Similarly, one can define z', if $\overline{l}_1^s$ is a separatrix. Again, put by definition $\overline{h}(z) = z'$.

In the case 2), when z is a saddle of $\overline{\mathscr{F}}_1^u$, $z = \overline{s}$, there exists a saddle $\overline{s}'$ of $\overline{\mathscr{F}}_2^u$ such that separatrix of $\overline{s}'$ reaches the same points with separatrix of $\overline{s}$, see Fig. 6.15 (b). It is easy to check that in both cases the relations (6.6) hold. These relations imply that $\overline{h}$ is a lift of a homeomorphism $h\colon M \to M$ that is a homotopy trivial conjugacy homeomorphism between f_1 and f_2. □

Let G be a group and ϕ_1, ϕ_2 be automorphisms of G. Recall that ϕ_1, ϕ_2 are conjugate if there is an automorphism $\xi\colon G \to G$ such that $\phi_2 \circ \xi = = \xi \circ \phi_1$. It is well-known that a homeomorphism $f\colon M \to M$ induces an automorphism $f_*\colon \pi_1(M) \to \pi_1(M)$ of the fundamental group $\pi_1(M)$. Two homeomorphisms $f_1, f_2\colon M \to M$ are called π_1-*conjugate* if f_{1*}, f_{2*} are conjugate automorphisms of the group $\pi_1(M)$.

If $h \circ f_1 = f_2 \circ h$ then $h_* \circ f_{1*} = f_{2*} \circ h_*$. Therefore, two conjugate homeomorphisms are necessarily π_1-conjugate. We are going to show that Theorem 6.20 and results of Nielsen [173] imply that a π_1-conjugacy is also a sufficient condition of conjugacy for AP-homeomorphisms.

Theorem 6.21 *Let $f_1, f_2\colon M \to M$ be AP-homeomorphisms of a closed orientable hyperbolic surface $M = \Delta/\Gamma$. Then f_1 and f_2 are conjugate if and only if they are π_1-conjugate.*

Proof. It is enough to prove that a π_1-conjugacy implies a conjugacy of f_1, f_2. Let $\xi\colon \pi_1(M) \to \pi_1(M)$ be an automorphism such that $f_{2*} \circ \xi = = \xi \circ f_{1*}$. Due to [173], there is a homeomorphism $\Psi\colon M \to M$ such that $\Psi_* = \xi$. The homeomorphism f_1 is conjugate to $\Psi^{-1} \circ f_1 \circ \Psi$ via the conjugacy map Ψ^{-1}. Therefore, $\Psi^{-1} \circ f_1 \circ \Psi$ is an AP-homeomorphism. Since $\left(\Psi^{-1} \circ f_1 \circ \Psi\right)_* = \xi^{-1} \circ f_{1*} \circ \xi = f_{2*}$, f_2 and $\Psi^{-1} \circ f_1 \circ \Psi$ induce the same automorphism of $\pi_1(M)$. It is sufficient to prove that $\Psi^{-1} \circ f_1 \circ \Psi$ conjugates to f_2. Therefore, we can consider f_1 such that $f_{1*} = f_{2*}$ and try to prove that f_1 and f_2 are conjugate.

Let $\overline{f}_1$ and $\overline{f}_2$ be lifts of f_{1*} and f_{2*} respectively. By Lemma 6.2, $\overline{f}_1$ and $\overline{f}_2$ induce automorphisms $\overline{f}_{1*}, \overline{f}_{2*} \colon \Gamma \to \Gamma$. Fixing a base point $z \in M$ and its lift $\overline{z} \in \Delta$, one establishes an isomorphism between Γ and $\pi_1(M)$. Hence, f_{1*} and f_{2*} induce the same automorphism of Γ. It follows that $\overline{f}_{1*}, \overline{f}_{2*}$ are conjugate via interior automorphism i. e., $\overline{f}_{2*} = \delta^{-1} \circ \overline{f}_{1*} \circ \delta$ for some $\delta \in \Gamma$. By Lemma 6.1, $\delta \circ \overline{f}_1$ is a lift of f_1 that acts identically with $\overline{f}_{2*}$. As a consequence, $\delta \circ \overline{f}_1\big|_{S_\infty} = \overline{f}_2\big|_{S_\infty}$. It follows from Theorem 6.20 that f_1 and f_2 are conjugate via a homotopy trivial homeomorphism. □

6.7. Nielsen Theory revisited

Here, we quote some results from brilliant series of articles by Jacob Nielsen [173]. Later on, f is an orientation preserving homeomorphism of closed orientable hyperbolic surface M^2.

Nielsen classes and numbers

Let $f \colon M^2 \to M^2$ be a homeomorphism of M^2, and $\mathrm{Fix}(f)$ be the set of fixed points of f. For simplicity, we'll assume that the set $\mathrm{Fix}(f)$ is finite (actually, in the frame of homotopy class of f, one can find a mapping with finite set of fixed points). It follows that any fixed point $x \in \mathrm{Fix}(f)$ has an index $\mathrm{ind}(x) \in \mathbb{Z}$ because of every fixed point is an isolated one, see Section 3.6.

Following Nielsen [174], Section 37, we decompose the set $\mathrm{Fix}(f)$ into classes as follows. We say that two fixed points are *homotopically related* if they can be joined by a path w such that w and $f(w)$ together constitute the path which is homotopic to zero on M^2. This relation splits $\mathrm{Fix}(f)$ into pairwise disjoint *classes* where fixed points belong to the same class iff they are homotopically related. The path w is called a *relating* path.

For simplicity we assume that a class contains finitely many fixed points. Let us define an *index of the class* being the sum of indices of its fixed points.

Definition 6.2 *A class with nonzero index is called a* ***Nielsen class****. The number $m = m(f)$ of Nielsen classes is called a* ***Nielsen number*** *of a homeomorphism $f \colon M^2 \to M^2$.*

Definition 6.3 *A homeomorphism $f\colon M^2 \to M^2$ is called unreducible if there is no non-homotopic to zero closed curve on M^2 that is mapped to freely homotopic curve. Otherwise, f is a reducible homeomorphism.*

There is a necessary condition of unreducibility through the action of a lift $\overline{f}$ on the group Γ of deck transformations. Recall that $\overline{f}$ induces the automorphism $\overline{f}_*\colon \Gamma \to \Gamma$, $\overline{f}_*\colon \gamma \to \overline{f} \circ \gamma \circ \overline{f}^{-1}$, see Lemma 6.2. Put by definition,

$$H(\overline{f}_*) = \{\gamma \in \Gamma \mid \overline{f}_*(\gamma) = \gamma \Leftrightarrow \overline{f} \circ \gamma = \gamma \circ \overline{f}\}.$$

Proposition 6.5 *If a homeomorphism $f\colon M^2 \to M^2$ is unreducible then $H(\overline{f}_*)$ consists of the identity mapping. If f is reducible then there is a lift $\overline{f}$ such that $H(\overline{f}_*) \neq \{id\}$.*

Proof. Suppose that there is a non-identity $\gamma_0 \in H(\overline{f}_*)$. We know that $\pi(\overline{A}_{\gamma_0}) = A_{\gamma_0}$ is the geodesic which is non-homotopic to zero where $\overline{A}_{\gamma_0}$ is the axis of γ_0. Since $\overline{f} \circ \gamma_0 = \gamma_0 \circ \overline{f}$, $\overline{f}(\overline{A}_{\gamma_0}) = \gamma_0 \circ \overline{f}(\overline{A}_{\gamma_0})$. Hence, $f(A_{\gamma_0})$ is freely homotopic to A_{γ_0}. This contradicts the unreducibility of f. Therefore, if $f\colon M^2 \to M^2$ is unreducible then $H(\overline{f}_*) = \{id\}$.

Suppose f is reducible. Then there is a closed geodesic $A_0 \subset M^2$ such that $f(A_0)$ is freely homotopic to A_0. Take a lift $\overline{A}_0$ of A_0 and a lift $\overline{f}_1$ of f. It follows that $\overline{A}_0$ and $\overline{f}_1(\overline{A}_0)$ are congruent. Hence, there exists $\gamma \in \Gamma$ such that $\gamma(\overline{f}_1(\overline{A}_0)) = \overline{A}_0$. The geodesic $\overline{A}_0$ is the axis of some hyperbolic translation, say $\gamma_0 \in \Gamma$. Hence, $\gamma_0(\gamma \circ \overline{f}_1(\overline{A}_0)) = \gamma \circ \overline{f}_1(\overline{A}_0)$. Obviously, $\gamma \circ$ $\circ\, \overline{f}_1 = \overline{f}$ is a lift of f. It follows from

$$\gamma_0 \circ \overline{f}(\overline{A}_0) = \overline{f}(\overline{A}_0) = \overline{f}(\gamma(\overline{A}_0)) = \overline{f} \circ \gamma(\overline{A}_0)$$

that the automorphism $\overline{f}_* = \overline{f} \circ \gamma \circ \overline{f}^{-1}\colon \Gamma \to \Gamma$ has $H(\overline{f}_*)$ containing γ_0. Hence, $H(\overline{f}_*) \neq \{id\}$. □

Proposition 6.6 *If a homeomorphism $f\colon M^2 \to M^2$ is unreducible and* $\mathrm{Fix}(f)$ *is finite then the set* $\mathrm{Fix}(\overline{f}) \in \Delta$ *of a lift $\overline{f}$ is also finite.*

Proof. Assume the contrary. Then there are points $\overline{x}$, $\gamma(\overline{x}) \in \mathrm{Fix}(\overline{f})$ for some non-identity $\gamma \in \Gamma$. Take a path $\overline{w}$ connecting $\overline{x}$ and $\gamma(\overline{x})$. Then $\overline{f}(\overline{w})$ is the path also connecting $\overline{x}$, $\gamma(\overline{x})$. It follows that $\pi(\overline{w})$ and $\pi\left(\overline{f}(\overline{w})\right)$ are non-homotopic to zero closed curves such that $f\left(\pi(\overline{w})\right) = \pi\left(\overline{f}(\overline{w})\right)$. This contradicts the unreducibility of f. □

Let $\Lambda_1, \ldots, \Lambda_m$ be the Nielsen classes of f. Denote by $i_s = \sum\limits_{x \in \Lambda_s} \operatorname{ind}(x)$ the index of Λ_s. By definition, $i_s \neq 0$, $s = 1, \ldots, m$.

Proposition 6.7 *Let Λ_s be a Nielsen class of homeomorphism $f \colon M^2 \to M^2$ and $\overline{f} \colon \Delta \to \Delta$ be a lift of f i.e., $f \circ \pi = \pi \circ \overline{f}$. Then the preimage $\pi^{-1}(\Lambda_s)$ contains the set $\overline{\Lambda}_s$ such that*

1) *the restriction $\pi|_{\overline{\Lambda}_s} : \overline{\Lambda}_s \to \Lambda_s$ is a one-to-one mapping*;
2) *$\overline{\Lambda}_s \subset \operatorname{Fix}(\gamma_s \circ \overline{f})$ for some $\gamma_s \in \Gamma$. In addition, $\operatorname{Fix}(\gamma_s \circ \overline{f}) = \pi^{-1}(\Lambda_s)$.*

Moreover, if f is unreducible then $\overline{\Lambda}_s = \operatorname{Fix}(\gamma_s \circ \overline{f})$.

Proof. Take a point $x_0 \in \operatorname{Fix}(f)$ and some $\overline{x}_0 \in \pi^{-1}(x_0)$. Since $f(x_0) = x_0$, there is $\gamma_s \in \Gamma$ such that $\gamma_s\left(\overline{f}(\overline{x}_0)\right) = \overline{x}_0$. Hence, $\overline{x}_0 \in \operatorname{Fix}(\gamma_s \circ \overline{f})$. Given any $x \in \Lambda_s$, there is a relating path w_x connecting x and x_0. Let $\overline{w}_x$ be the lift of w_x starting at $\overline{x}_0$. Then $\overline{w}_x$ connects $\overline{x}_0$ with the point $\overline{x} \in \pi^{-1}(x)$. By definition, w and $f(w)$ together constitute a homotopic to zero path on M^2. Since the lift of homotopic to zero closed curve is a countable family of closed curves in Δ, the path $\overline{f}(\overline{w}_x)$ connects $\overline{x}_0$ with $\overline{x}$. It follows that $\overline{x} \in \operatorname{Fix}(\gamma_s \circ \overline{f})$. Continuing this procedure with each point of Λ_s, we get the desired set $\overline{\Lambda}_s$.

Take any point $\overline{y}_0 \in \operatorname{Fix}(\gamma_s \circ \overline{f})$ and a path $\overline{w}_0$ connecting $\overline{x}_0$ with $\overline{y}_0$. Then $\overline{w}_0$ and $\overline{f}(\overline{w}_0)$ form the closed curve $\overline{C}$ in Δ. Hence, $\pi(\overline{C})$ is a homotopic to zero curve on M^2. It follows that $\pi(\overline{y}_0)$ belongs to Λ_s. Moreover, if f is unreducible then $\overline{y}_0 \in \overline{\Lambda}_s$. This completes the proof. □

Corollary 6.4 *Suppose Λ_s and $\overline{\Lambda}_s$ satisfy the condition of Proposition* 6.7*. Then Λ_s and $\overline{\Lambda}_s$ have the same index*:

$$\operatorname{ind}(\Lambda_s) = \operatorname{ind}(\overline{\Lambda}_s).$$

Lefschetz numbers and Nielsen numbers

Let $g \colon M \to M$ be a homeomorphism of a compact manifold M. It is well-known that given any $i \in \{0, 1, \ldots, \dim M\}$, g induces the linear map $g_{*i} \colon H_i(M, \mathbb{R}) \to H_i(M, \mathbb{R})$ of the i^{th} homology group $H_i(M, \mathbb{R})$ that can be considered as a finitely dimensional $\mathbb{R}$-space. For a fixed basis in $H_i(M, \mathbb{R})$, g_{*i} is defined by a matrix with the trace denoted by $\operatorname{Trace}(g_{*i})$. The *Lefschetz number* $L(g)$ is defined as follows:

$$L(g) = \sum_{i=0}^{\dim M} (-1)^i \operatorname{Trace}(g_{*i}). \tag{6.7}$$

Let $f\colon M^2 \to M^2$ be an orientation preserving homeomorphism of closed orientable hyperbolic surface M^2 of genus $p \geqslant 2$. Then f_{*1} is represented by an unimodular $(2p \times 2p)$ matrix denoted by A. Let $\lambda_1, \ldots, \lambda_{2p}$ be the eigenvalues of the matrix A (taking account a multiplicity). Since $\mathrm{Trace}(f_{*0}) = 1$ and $\mathrm{Trace}(f_{*2}) = 1$, the Lefschetz number $L(f)$ equals to

$$L(f) = 2 - \sum_{s=1}^{2p} \lambda_s.$$

Well-known Lefschetz Index Theorem says that the Lefschetz number equals to the sum of indices of fixed points. Thus, the Lefschetz formula (6.7) and Lefschetz Index Theorem imply the following Nielsen [175] version of the Lefschetz number for surface homeomorphisms.

$$L(f) = 2 - \sum_{s=1}^{2p} \lambda_s = \sum_{s=1}^{m} i_s \tag{6.8}$$

where m is the Nielsen number, and i_s is the index of Nielsen class Λ_s, $s = 1, \ldots, m$.

Indices of Nielsen classes and fixed points at circle at infinity

Recall that according to Lemma 6.2, a lift $\overline{f}\colon \Delta \to \Delta$ of f induces the automorphism $\overline{f}_*\colon \gamma \to \overline{f} \circ \gamma \circ \overline{f}^{-1}$ of the group Γ. Due to Theorem 6.1, $\overline{f}$ extends to unique homeomorphism $\Delta \cup S_\infty \to \Delta \cup S_\infty$ again denoted by $\overline{f}$. Thus, $\overline{f}\big|_{S_\infty} : S_\infty \to S_\infty$ is an orientation preserving circle homeomorphism. If the set $\mathrm{Fix}\left(\overline{f}\big|_{S_\infty}\right)$ is finite, we'll denote by μ the number of node fixed points (sinks and sources) of the homeomorphism $\overline{f}\big|_{S_\infty}$. The following statement is a core of the Nielsen Theory.

Theorem 6.22 *Let $f\colon M^2 \to M^2$ be an unreducible homeomorphism and $\overline{f}$ be a lift of f. Then*

- *the number of node fixed points μ of $\overline{f}\big|_{S_\infty}$ is even;*
- $\sum\limits_{\overline{x} \in \Delta \cap \mathrm{Fix}(\overline{f})} \mathrm{ind}(\overline{x}) = 1 - \frac{\mu}{2}.$

Moreover, all node fixed points of $\mathrm{Fix}\left(\overline{f}\big|_{S_\infty}\right)$ *are irrational.*

Scheme of proof. Let us glue two copies of $\Delta \cup S_\infty$ along the boundary S_∞ to get a 2-sphere S^2. Then the mapping $\overline{f}$ defines the homeomorphism $S^2 \to S^2$ denoted by $\overline{f}_0$. Clearly, μ is even because of sinks and sources are alternatively disposed on S_∞. One can prove that a node fixed point of the restriction $\overline{f}|_{S_\infty}$ becomes a node fixed point of $\overline{f}$. Note that the index of any node at S_∞ equals to 1.

According to Proposition 6.6, the set $\mathrm{Fix}(\overline{f})$ is finite. Thus, $\sum\limits_{\overline{x}\in\Delta\cap\mathrm{Fix}(\overline{f})} \mathrm{ind}(\overline{x})$ is also finite.

Due to Proposition 6.7, the set $\mathrm{Fix}(\overline{f}_0)$ is the union of the double set $\mathrm{Fix}(\overline{f})$ and $\mathrm{Fix}\left(\overline{f}|_{S_\infty}\right)$. Since the Euler characteristic of S^2 equals to 2, one gets

$$2 = 2 \cdot \left(\sum_{\overline{x}\in\Delta\cap\mathrm{Fix}(\overline{f})} \mathrm{ind}(\overline{x}) \right) + \mu.$$

Let $g_0 \in \mathrm{Fix}(\overline{f}) \cap S_\infty$ be a node fixed point. Assume that g_0 is a rational point. Then there is a non-identity $\gamma \in \Gamma$ such that g_0 is a fixed point of γ. By Lemma 6.6, $\overline{f} \circ \gamma = \gamma \circ \overline{f}$. On the other hand, due to Proposition 6.5, the unreducibility of f implies the triviality of the group $H(\overline{f})$. This contradiction shows that g_0 is an irrational point. □

As a consequence, one gets the following statement for a Nielsen class.

Proposition 6.8 *Let Λ be a Nielsen class of f and $\overline{\Lambda} \subset \pi^{-1}(\Lambda)$ satisfy the conditions of Proposition 6.7 including the unreducibility of f. Let μ be the number of node fixed points in* $\mathrm{Fix}\left(\overline{f}|_{S_\infty}\right)$. *Then*

- *μ is even;*
- $\mathrm{ind}(\overline{\Lambda}) = \mathrm{ind}(\Lambda) = 1 - \frac{\mu}{2}$.

Moreover, all node fixed points of $\mathrm{Fix}\left(\overline{f}|_{S_\infty}\right)$ *are irrational.*

Bibliographic Notes and Panoramas

Chapter 6. The main results of this chapter are based on the deep theories developed originally by Henri Poincaré, Jakob Nielsen and William Thurston.

(6.1). Theorem 6.1 was proved by Nielsen [173]. Our proof follows [108].

(6.2). Theorem 6.5 was proved by Poincaré [192] in terms of the rotation numbers (see [26, 66, 124, 168]).

Note that for a flow on $\mathbb{T}^2$ Poincaré rotation number does not generally give us a complete topological invariant (sometimes, it simply does not exist).

In the absence of nontrivially recurrent trajectories or nontrivial limit sets, a complete topological invariant is usually of combinatorial origin (a graph, a scheme, etc.). An invariant of this kind was obtained by Leontovich and Maier [136] for vector fields with a finite set of singular trajectories on the two-dimensional sphere S^2 and on a compact set of a plane i. e., on surfaces that do not admit nontrivially recurrent trajectories.

For the Morse – Smale flows, all recurrent trajectories are trivial by definition, and a complete topological invariant on compact surfaces is given by Peixoto graph [182]. This list of combinatorial invariants for dynamical systems and foliations without nontrivial recurrent sets may be continued (see, for example, Chapter 7 in [172], which is specially devoted to invariants). Invariants fall into three major classes: homology (or cohomology), homotopy, and combinatorial. Poincaré rotation number is at the same time homology and homotopy invariant. Homology and homotopy invariants (exm., fundamental class of Katok and homotopy rotation class of Aranson – Grines respectively) are convenient for description of flows with nontrivially recurrent trajectories. A homotopy invariant that is most related to the Riemannian structure of surface is a geodesic framework. In Section 6.3 using terms of the geodesic frameworks, we reformulate the Aranson – Grines's classification of irrational flows and nontrivial minimal sets of flows [27, 28].

(6.3). In 1973, Aranson and Grines [27] have got the classification of irrational flows on a closed orientable hyperbolic surface. V. M. Alekseev, in his review of [27], figuratively called the domain $\overline{\Phi I_0}, \overline{C})$ the "Aranson – Grines trousers".

Theorems 6.6–6.9 was proved by S. Aranson [24, 25]. Note that an irrational foliation may have saddle singularities of positive index (thorns). The classification of such foliations involves branched 2-sheeted covering projections (see [24, 25, 37] for details).

(6.4). Theorems 6.11 and 6.12 was proved in [33].

(6.5). Proposition 6.2 and Theorem 6.13 was proved in [36] (see also [226]). Theorems 6.16–6.17 was proved in [28].

(6.6). Theorem 6.19 was proved in [94]. Theorems 6.20, 6.21 was proved in [85].

(6.7). The results of this section was mainly obtained by Nielsen [173], and revisited in [28, 29, 80, 108, 162].

Let us recall some of them. Let $f: M \to M$ be an orientation preserving homeomorphism and T the set of iterations of f, $T = \{f^n : n \in \mathbb{Z}\}$. Denote by $L(T)$ the set of lifts of all elements of T. By Lemma 6.4, an element of $L(T)$ can be represented as $\gamma \circ \overline{f}^n : \Delta \to \Delta$ for some $\gamma \in \Gamma$, $n \in \mathbb{Z}$, and a lift $\overline{f}: \Delta \to \Delta$ of f. Due to Theorem 6.1, every element $\overline{g} \in L(T)$ is uniquely extended to a homeomorphism $\overline{g}: \Delta \cup S_\infty \to \Delta \cup S_\infty$. The restriction $\overline{g}|_{S_\infty} : S_\infty \to S_\infty$ will be denoted sometimes by $\overline{g}^*$.

Let $\overline{h}^* : S_\infty \to S_\infty$ be a homeomorphism and $s_0 \in S_\infty$ an isolated fixed point of $\overline{h}^*$. The point s_0 is called *neutral* if s_0 is neither a source nor sink. Bellow, given any $\gamma \in \Gamma$, one denotes by γ^+ and $\gamma^- \in S_\infty$ a sink and source respectively of γ.

Take a homeomorphism $\overline{h} \in L(T)$, $h \neq id$. Set $F_{\overline{h}} = \{\gamma \in \Gamma : \overline{h}\gamma\overline{h}^{-1} = \gamma\}$. Clearly, $F_{\overline{h}}$ is a subgroup of Γ. One can prove that $F_{\overline{h}}$ is a finitely generated subgroup.

The *Nielsen type* of $\overline{h}$ is a pair $(v_{\overline{h}}, u_{\overline{h}})$ of integers where $v_{\overline{h}}$ is the minimum number of generators of $F_{\overline{h}}$. If $F_{\overline{h}} = \{id\}$, we set $v_{\overline{h}} = 0$. Define $u_{\overline{h}}$ as follows:

1) if $v_{\overline{h}} = 0$, then $u_{\overline{h}}$ is the number of isolated sink fixed points of the homeomorphism $\overline{h}^* : S_\infty \to S_\infty$;

2) if $v_{\overline{h}} = 1$, i. e. $F_{\overline{h}}$ is a free cyclic group with a generator $\gamma_0 \in \Gamma$, then $u_{\overline{h}}$ is the number of F_h-orbits of the isolated sink fixed points of $\overline{h}^*$ distinct from γ_0^+, γ_0^-. The particular type $(1, 0)$ is divided into two subtypes:

$(1, 0)^+$ provided the points γ^+, γ^- are neutral fixed points of $\overline{h}^*$;

$(1, 0)^{++}$ provided one of the points γ^+, γ^- is a sink and other is a source of $\overline{h}^*$;

3) if $v_{\overline{h}} \neq 1$, then $u_{\overline{h}}$ is the number of $F_{\overline{h}}$-orbits of the isolated sink fixed points of the homeomorphism $\overline{h}^*$.

A homeomorphism $h: X \to X$ of topological space X is called *periodic* if there exists $m > 0$ such that $f^m = Id$. A homeomorphism $f: M \to M$ is called a *homeomorphism of algebraically finite type* if on M there exists a finite f-invariant family of disjoint closed cylinders $\sigma_1, \ldots, \sigma_k$ such that the restriction of f to the set $\widetilde{M} = M \setminus \bigcup_{i=1}^{i=k} \operatorname{int} \sigma_i$ is a periodic homeomorphism. A homeomorphism $f: M \to M$ is called *reducible* by means family of disjoint simple closed non homotopy to zero and non homotopy each to other curves $C_1, \ldots, C_k$ if the system is invariant under homeomorphism f.

The following Theorem 6.23 was proved by Nilesen [173].

Theorem 6.23 *The set of all homotopy classes* $\{f\}$ *is represented as a union of four disjoint classes* N_1, N_2, N_3, N_4, *distinguished by the following conditions*:

1) $\{f\} \in N_1$ *provided any* $\overline{h} \in L(T)$, $\overline{h} \neq id$ *has either the type* $(0, 0)$, *or the type*[4] $(1, 0)$;
2) $\{f\} \in N_2$ *provided any* $h \in L(T)$ *has either the type* $(0, 1)$ *or the type* $(v_h, 0)$. *In the latter case, there is* $h' \in L(T)$, $h' \neq id$ *such that either* a_1) $v_{h'} \geqslant 2$, *or* a_2) h' *has the type* $(1, 0)^+$ *with* $h' = h''^2$, *where* $h'' \in L(T)$ *has the type* $(0, 0)$;
3) $\{f\} \in N_3$ *provided there are* $h, h' \in L(T)$ $(h \notin \Gamma)$ *with respectively types* (v_h, u_h), $(v_{h'}, u_{h'})$ *such that* $v_h \neq 0$, $u_{h'} \neq 0$, *and if* $h \neq h'$, *then* $u_{h'} \geqslant 2$;
4) $\{f\} \in N_4$ *provided any* $h \in L(T)$ *(*$h \notin \Gamma$*) has the type* $(0, u_h)$ *and there exists* $h' \in L(T)$ *of the type* $(0, u_{h'})$, *where*[5] $u_{h'} \geqslant 2$.

The classes N_1, N_2 studied in detail by Nielsen [177, 179]. He showed that every homotopy class $\{f\}$ belongs to N_1 (N_2) if and only if $\{f\}$ contains a periodic homeomorphism (a non-periodic homeomorphism of algebraically finite type).

The modern representation of Nielsen's Theorem 6.23 was done by Thurston as follows.

Theorem 6.24 *The set of all homotopy classes* $\{f\}$ *is represented as a union of four disjoint classes* T_1, T_2, T_3, T_4, *distinguished by the following conditions*:

1) *if* $\{f\} \in T_1$, *then* $\{f\}$ *contains a periodic homeomorphism*;
2) *if* $\{f\} \in T_2$, *then* $\{f\}$ *contains a reducible nonperiodic homeomorphism of algebraically finite type*;
3) *if* $\{f\} \in T_3$, *then* $\{f\}$ *contains a reducible homeomorphism that is not a homeomorphism of algebraically finite type*;
4) *if* $\{f\} \in T_4$, *then* $\{f\}$ *contains a pseudo-Anosov homeomorphism.*

[4] If $\overline{h} \in L(T)$ has in this case the type $(1, 0)$, then this type is either $(1, 0)^{++}$ or $(1, 0)^+$, if h, respectively, preserves or changes the orientation of the universal covering Δ.

[5] In fact, to highlight the class N_4 the existence of an element $h' \in L(T)$ of type $(0, u_{h'})$, where $u_{h'} \geqslant 2$, is superfluous, since, by virtue of Lemma 7.1, it follows only from the condition that any $h \in L(t)$, $h \notin \Gamma$ has the type $(0, u_h)$.

J. Nielsen proved that, given any automorphism τ of the group Γ that is isomorphic to fundamental group $\pi_1(M^2)$ of closed orientable closed surface M^2, there is a homeomorphism $h\colon M^2 \to M^2$ such that $h_* = \tau$. Actually, J. Nielsen divided the set of automorphisms of Γ into four classes using inducing actions of automorphisms on the circle at infinity S_∞, see Theorem 6.23. He exhaustively studied the classes N_1, N_2 [177, 179], and he understood that the classes N_3, N_4 contains homeomorphisms with chaotic dynamics. Nevertheless, he had no representatives of such homeomorphisms.

The first examples that shed light on the complex dynamics of homeomorphisms whose homotopy classes belong to N_4 (and with a small modification of N_3) were constructed in 1970 by T. Brien and W. Reddy [55, 56] in connection with solving the problem of the existence of expansive homeomorphisms on two-dimensional manifolds. These homeomorphisms have positive topological entropy, a pair of contracting and expanding transitive foliations with singularities, and are semi-conjugate to Anosov diffeomorphisms of the torus T^2.

Later on, W. Thurston [71, 214] also has obtained classification of the set of all homotopy classes $\{f\}$ of homeomorphisms on a closed orientable two-dimensional manifold of genus $g \geqslant 2$, see Theorem 6.24. There is a remarkable exposition of Nielsen – Thurston theory in [80] where, in particular, was proved that $N_i = T_i$, $i = \{1, 2, 3, 4\}$.

The canonical representatives in each homotopic class from (1)–(4) of B, Theorem 6.24 are not structurally stable. At the same time, each homotopic class contains a structurally stable diffeomorphism with zero-dimensional basic sets (see [204] for reference). However, a priori this diffeomorphism can have a very complicated topological structure. It is therefore of interest to indicate the simplest structurally stable representatives in these classes. Namely, the simplest representative in homotopy class from N_1, can be taken to be a gradient-like diffeomorphism while the simplest representative in homotopy class $\{f\} \in N_2$, can be taken to be Morse – Smale diffeomorphism with oriented heteroclinic set. The ambient surface M^2 is representable as a union of two compact invariant sets. The first set is the union of $k \geqslant 2$ compact surfaces of genus greater than zero and does not contain heteroclinic orbits; the second set is a union of $m \geqslant 1$ subsets homeomorphic to a closed annulus whose interiors consist of wandering points and contain an orientable heteroclinic set. The concepts which we used here can be find in the survey [32].

In each homotopy class from N_3 it is natural to regard as the a simplest structurally stable diffeomorphism of the following type. The closed orientable surface M^2 of genus $g \geqslant 2$ is representable as a union of three compact sets A_1, A_2, A_3. The sets A_1 and A_2 are unions of compact surfaces of genus greater than zero, where A_1 contains finitely many periodic points and does not contain heteroclinic orbits, while A_2 contains finitely many widely disposed one-dimensional attractors and there are no other non-wandering points in the interior of this set. The set A_3 is a union of subsets homeomorphic to a closed annulus whose interiors contain an orientable heteroclinic set. Here A_1 can be empty, but A_2 and A_3 are not empty.

The simplest structurally stable diffeomorphism in each homotopy class from N_4 can be taken to be a diffeomorphism whose non-wandering set consists of finitely many isolated periodic points and a single one-dimensional widely disposed attractor Λ such that $M^2 \setminus \Lambda$ consists of the union of finitely many domains homeomorphic to a disc. Moreover, $M^2 \setminus \Lambda$ does not contain heteroclinic orbits.

Chapter 7

Chaotic Dynamical Systems with Minimal Entropy

In this chapter, we keep the notation of Section 5.1, see also Section 3.2. Later on, M^2 is a closed orientable hyperbolic surface. This means that M^2 topologically is a closed orientable surface of genus $p \geqslant 2$. Applying Δ-model of hyperbolic plane, one can represent M^2 as the quotient space Δ/Γ where Δ is the hyperbolic plane and Γ is a properly discontinuous subgroup of the covering group for the natural covering projection $\pi\colon \Delta \to \Delta/\Gamma = M^2$. In other words, Γ is a Fuchsian crystallographic group consisting of non-Euclidean translations of Δ. The covering map π induces the Riemannian metric d of constant negative curvature on M^2.

In Section 7.1, we introduce hyperbolic automorphisms acting on a group of deck transformations for a closed orientable hyperbolic surface, see exact definitions and properties. The action of hyperbolic automorphism is similar to action of Anosov diffeomorphisms in the fundamental group of a 2-torus. The notion of hyperbolic automorphism allows us to introduce Aranson – Grines hyperbolic homeomorphisms (in short AG-hyperbolic homeomorphism) on closed hyperbolic surfaces (see Definition 7.4). In Section 7.2, we use the properties of homeomorphism of the circle at infinity induced by a hyperbolic automorphism to construct a two-web of strongly irrational geodesic laminations. Properties of this web is studied in Section 7.3. Using these properties we prove that AG-hyperbolic homeomorphisms satisfy to Axiom A^* on the intersection of geodesic laminations generating two-webs (see Section7.4). At last, in Section 7.5 we give the formula for calculation of topological entropy of AG-homeomorphism. We prove that this entropy is minimal for homeomorphisms of homotopic class containing the AG-homeomorphism constructed. For simplicity bellow, we restricted ourself by orientation preserving homeomorphisms $M^2 \to M^2$.

7.1. Properties of hyperbolic automorphisms

Recall that the Δ-model of hyperbolic plane is geometrically the unit disk $|z| < 1$ restricted by the circle at infinity (absolute) $S_\infty\colon |z| = 1$. Every non-identity $\gamma \in \Gamma$ has two fixed points that belong to S_∞. Each fixed point of non-identity mapping of Γ is a rational point, see Section 5.1. The set of rational points $\mathscr{R}(\Gamma)$ is countable and everywhere dense in S_∞. The complement to the set $\mathscr{R}(\Gamma)$ on S_∞ forms the set of irrational points $\mathscr{I}(\Gamma)$.

Hyperbolic automorphisms of deck group

Let $f\colon M^2 \to M^2$ be a homeomorphism and $\overline{f}\colon \Delta \to \Delta$ a lift of f. By Theorem 6.1, $\overline{f}$ extends to a unique homeomorphism $\Delta \cup S_\infty \to \Delta \cup S_\infty$, so that the restriction $\overline{f}\big|_{S_\infty} = \overline{f}^{*}\colon S_\infty \to S_\infty$ is a homeomorphism. It follows from Lemma 6.3 that the sets $\mathscr{R}(\Gamma)$, $\mathscr{I}(\Gamma)$ are invariant under $\overline{f}^{*}$.

Due to Lemma 6.2, the mapping

$$\overline{f}_{*}\colon \zeta \longmapsto \zeta' = \overline{f} \circ \zeta \circ \overline{f}^{-1}, \quad \zeta, \zeta' \in \Gamma$$

is an automorphism of the group Γ. We'll say that $\overline{f}_{*}$ *corresponds* to f while f and $\overline{f}$ *generate* $\overline{f}_{*}$. According to Corollary 6.1, if $\overline{f}_1$ and $\overline{f}_2$ are lifts of f, then

$$\overline{f}_{2*}(\zeta) = \gamma \circ \overline{f}_{1*}(\zeta) \circ \gamma^{-1}, \quad \zeta \in \Gamma, \tag{7.1}$$

where $\gamma \in \Gamma$ is such that $\overline{f}_2 = \gamma \circ \overline{f}_1$ according to (6.4). Denote by $\mathrm{Auto}(\Gamma)$ the set of automorphisms of the group Γ. Clearly, $\mathrm{Auto}(\Gamma)$ is a group itself with the group operation being a composition.

Definition 7.1 *An automorphism $\tau \in \mathrm{Auto}(\Gamma)$ is called **hyperbolic** provided $\tau^n(\zeta) \neq \gamma \circ \zeta \circ \gamma^{-1}$ for every $\gamma \in \Gamma$, $\zeta \in \Gamma \setminus \{id\}$, and $n \in \mathbb{Z} \setminus \{0\}$.*

Since Γ is isomorphic to the fundamental group $\pi_1(M^2)$ of M^2, this definition can be reformulated applying $\pi_1(M^2)$. Note that τ induces uniquely the automorphism $\widetilde{\vartheta}\colon \pi_1(M^2) \to \pi_1(M^2)$. By Proposition 1.2, $\widetilde{\vartheta}$ is induced by some homeomorphism $\vartheta\colon M^2 \to M^2$ such that $\tau = \overline{\vartheta}_{*}$. Then τ is hyperbolic if and only if $\widetilde{\vartheta}$ has no periodic elements different from trivial element. This formulation is independent of the choice of base point on M^2.

Example 7.1 *Hyperbolic automorphisms.*

Let $\psi\colon \mathbb{T}^2 \to \mathbb{T}^2$ be the diffeomorphism induced by the linear mapping $\mathbb{R}^2 \to \mathbb{R}^2$ with matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^3 = \begin{pmatrix} 5 & 8 \\ 8 & 13 \end{pmatrix}.$$

One can check that ψ has at least two fixed points, say p_1 and p_2. Let M_2 be a closed orientable surface of genus 2 (pretzel). Recall that the genus of $\mathbb{T}^2$ equals 1. Due to Lemma 3.9, there is the 2-sheeted branched covering map $\pi_2\colon M_2 \to \mathbb{T}^2$ with the branched set $\{p_1, p_2\}$ each point of whose has the branched order 2. According to [55], there is the lift $f\colon M_2 \to M_2$ of ψ such that $\psi \circ \pi_2 = \pi_2 \circ f$. Since ψ is an Anosov diffeomorphism, $\overline{f}_*$ is a hyperbolic automorphism. Applying finitely many coverings, one can get examples of hyperbolic automorphism for any closed hyperbolic orientable surface. ♢

Denote by $\mathrm{Hyp}^{auto}(\Gamma)$ the set of hyperbolic automorphisms of the group Γ. Example 7.1 shows that $\mathrm{Hyp}^{auto}(\Gamma) \neq \varnothing$.

Given any $\gamma \in \Gamma$, the mapping $\zeta \mapsto \gamma \circ \zeta \circ \gamma^{-1}$ denoted by $A_\gamma \in \mathrm{Auto}(\Gamma)$ is an inner automorphism. It follows from definition

$$A_\gamma \circ \eta(\zeta) = \gamma \circ \eta(\zeta) \circ \gamma^{-1} \text{ where } \eta \in \mathrm{Auto}(\Gamma),\ \zeta \in \Gamma.$$

For $\tau \in \mathrm{Auto}(\Gamma)$, let us denote by $H(\tau) \subset \Gamma$ the set of elements $\gamma \in \Gamma$ such that $\tau(\gamma) = \gamma$. It is obvious that $H(\tau)$ is a subgroup of Γ. The subgroup $H(\tau)$ is *trivial* if $H(\tau)$ consists of the identity mapping.

Definition 7.1 can be reformulated now as follows. We can say that an automorphism τ is hyperbolic if $H(A_\gamma \circ \tau^n)$ is trivial for every $\gamma \in \Gamma$ and $n \in \mathbb{Z} \setminus \{0\}$.

As a consequence of Definition 7.1 and Proposition 6.5, one gets the following statement we are formulating for references (see Definition 6.3).

Proposition 7.1 *Let $\overline{f}$ be a lift of f. Suppose that $\tau \in \mathrm{Hyp}^{auto}(\Gamma)$ corresponds to $\overline{f}$. Then f^n is an unreducible homeomorphism for any $n \in \mathbb{Z} \setminus \{0\}$.*

Action of hyperbolic automorphism on the circle at infinity

Given any $\tau \in \mathrm{Hyp}^{auto}(\Gamma)$ and $n \in \mathbb{N}$, denote by $\tau_{n,\gamma}$ the automorphism $A_\gamma \circ \tau^n = \gamma \circ \tau^n \circ \gamma^{-1}$. By Proposition 1.2, one can assume that τ is generated by a lift $\overline{f}$ of a homeomorphism $f\colon M^2 \to M^2$ i. e., $\tau = \overline{f}_*$.

Therefore, $\tau_{n,\gamma}$ is generated by $\gamma \circ \overline{f}^n$: $\tau_{n,\gamma} = A_\gamma \circ \overline{f}^n_*$. Due to Theorem 6.1, $\gamma \circ \overline{f}^n$ extends to a unique homeomorphism $\Delta \cup S_\infty \to \Delta \cup S_\infty$. Put by definition,

$$\tau^*_{n,\gamma} = \gamma \circ \overline{f}^n \Big|_{S_\infty} : S_\infty \to S_\infty.$$

It follows from Proposition 7.1 that f^n is unreducible for any $n \in \mathbb{Z} \setminus \{0\}$. Recall that due to Theorem 6.22, the homeomorphism $\tau^*_{n,\gamma} = \gamma \circ \overline{f}^n \Big|_{S_\infty}$ has even number $\mu_{n,\gamma}$ of node fixed points.

Theorem 7.1 *Suppose* $\tau \in \mathrm{Hyp}^{auto}(\Gamma)$ *is generated by a lift* $\overline{f}$ *of homeomorphism* $f\colon M^2 \to M^2$ *i.e.,* $\tau = \overline{f}_*$. *Then there are* $\gamma \in \Gamma$ *and an even number* $n \in \mathbb{N}$ *such that the homeomorphism* $\tau^*_{n,\gamma}$ *has at least four node fixed points.*

Proof. Assume the contrary. Then for any even number $n \in \mathbb{N}$ and any $\gamma \in \Gamma$, the number $\mu_{n,\gamma}$ of node fixed points of $\tau^*_{n,\gamma}$ is less than four, $0 \leqslant \mu_{n,\gamma} < 4$. Since $\mu_{n,\gamma}$ is even, $\mu_{n,\gamma}$ is either 0 or 2. Due to (6.8) and Theorem 6.22, the following relation holds

$$L(f^n) = 2 - \sum_{s=1}^{2p} \lambda_s^n = 1 - \frac{\mu_{n,\gamma}}{2} \tag{7.2}$$

where $\lambda_1, \ldots, \lambda_{2p}$ are the eigenvalues (taking account a multiplicity) of the unimodular $(2p \times 2p)$ matrix A that represents the linear mapping $H_1(M^2, \mathbb{R}) \to H_1(M^2, \mathbb{R})$ of the first homology group $H_1(M^2, \mathbb{R})$.

First, suppose $\mu_{n,\gamma} = 2$. The relation (7.2) becomes

$$2 - \sum_{s=1}^{2p} \lambda_s^n = 0 \tag{7.3}$$

It follows

$$\sum_{s=1}^{2p} \frac{\lambda_s^n}{\lambda^n} - \frac{2}{\lambda^n} = 0 \tag{7.4}$$

Here λ is the largest of the absolute values of the eigenvalues $\lambda_1, \ldots, \lambda_{2p}$. Let us show that there is a subsequence $n_j \to +\infty$ $(j = 1, 2, \ldots)$ so that the limit on the left-hand side of (7.4) is positive. Indeed, the left-hand side of (7.4) can be written in the form

$$\sum_{s=1}^{2p} \frac{\lambda_s^n}{\lambda^n} - \frac{2}{\lambda^n} = 2\sum_{s=1}^{r/2} \cos n\varphi_s + \sum_{s=k+1}^{k} \nu_s^n + \sum_{s=k+1}^{2p} \mu_s^n - \frac{2}{\lambda^n} \tag{7.5}$$

where $\varphi_s \neq \pi m$ $(m \in Z)$, $\nu_s = \pm 1$, r $(0 \leqslant r \leqslant 2p)$ is even, and $|\mu_s| < 1$ if $\lambda \neq 1$ while $\mu_s = 0$ if $\lambda = 1$. If $r = 0$, then for any sequence of even numbers $n_j \to +\infty$ $(j = 1, 2, \ldots)$, the left-hand side of (7.4) equal to $k > 0$. If $r > 0$, then from [9] (Theorem 6) one can find an infinite set of even numbers n_j and integers $p_s^{(n_j)}$ such that for φ_s $(s = 1, 2, \ldots, r/2)$ we have

$$\left| n_j \frac{\varphi_s}{2\pi} - p_s^{(n_j)} \right| < \left(\frac{1}{2} n_j \right)^{-r/2} \tag{7.6}$$

Since $\cos n_j\varphi_s = 1$, for $n_j \to +\infty$ the limit of the expression $2\sum_{s=1}^{r/2} \cos n_j\varphi_s$ equals r. Since $\nu s = \pm 1$, $|\mu_s|$ is less than one or equal to zero, respectively, for $\lambda \neq 1$ or $\lambda = 1$. Since n_j is even, as $n_j \to +\infty$, the left-hand side of (7.4) tends to $r + (k - r) = k > 0$ if $\lambda \neq 1$, or to $r + (k - r) - 2 = k - 2$ if $\lambda = 1$. In the latter case, from $\lambda = 1$ and $\Pi_{s=1}^{2p} |\lambda_s| = 1$, we see that $k = 2p$, where $p \geqslant 2$, so that $k - 2 > 0$. This contradiction proves the result.

The case $\mu_{n,\gamma} = 0$ is considered similarly. This completes the proof. □

Below, we'll use the next statement proved by Nielsen [175] (Sections 3–6, 14–15), [176] (Sections 1–2). We formulate this statement for references although there are intersections with previous results.

Proposition 7.2 *Let τ be an automorphism of the group Γ inducing orientation preserving homeomorphism $\tau^* \colon S_\infty \to S_\infty$ such that the group $H(\tau)$ is trivial. Then the following statements hold*:

1) *τ^* has an even number $0 \leqslant \mu \leqslant 8p - 4$ of fixed points*; *furthermore, if $\mu \neq 0$, then all fixed points are irrational and half of them being sinks, while the other half are sources*;
2) *If $\ell \subset \Delta$ is the geodesic connecting two neighboring sink* (*source*) *points of τ^*, then $L = \pi(\ell) \subset M^2$ is an nonclosed non-self-intersecting curve*;
3) *Let Π^+ (Π^-) be the polygon such that each its side is geodesic connecting two neighboring sink* (*source*) *points of τ^*. Then the interior of Π^+ (Π^-) does not contain congruent points*;

4) *Any homeomorphism* $\overline{f}\colon \Delta \to \Delta$ *which induces* τ ($\overline{f}_* = \tau$) *has its fixed points in a compact part of* Δ *and there are no congruent points among them*;
5) *If* $\mu = 0$, *then there is an integer* $n > 0$ *and* $\gamma \in \Gamma$ *such that a homeomorphism* $\gamma \circ \tau^{*n}|_{S_\infty}$ *has at least four fixed points.*

7.2. Geodesic laminations and hyperbolic automorphisms

Let τ be a hyperbolic automorphism of the group Γ such that the homeomorphism $\tau^*\colon S_\infty \to S_\infty$ induced by τ preserves the orientation and has an even number $4 \leqslant \mu \leqslant 8p-4$ of fixed points. According Proposition 7.2, all the fixed points are irrational, half of them being sinks and the other half sources. By Theorem 7.1, one can always assume that this is the case, since we can always find some even n and some element $\gamma \in \Gamma$ so that the automorphism $A_\gamma \circ \tau^n$ has these properties.

Choose two sink fixed points u_1, u_2 and two source fixed points s_1, s_2 of τ^* such that:

1) for a positive circuit of the absolute (in the counterclockwise direction), they are encountered in the following order: s_1, u_1, s_2, u_2;
2) the arc s_1, s_2 which contains the point u_1 does not contain any other fixed point of τ^*.

Denote by g^u (g^s) the directed geodesic on Δ with endpoints u_1, u_2 (s_1, s_2). It follows from item (2) of Proposition 7.2 that the geodesic $G^u = \pi(g^u)$ ($G^s = \pi(g^s)$) is an unclosed non-self-intersecting curve on M^2.

Let W be the space of unit tangent vectors, that is, the set of points $w = (m, \xi)$, where $m \in M^2$, $\xi \in T_m M$, $||\xi|| = 1$, and q is the natural projection of W on M^2 ($q(w) = m$). We denote the geodesic flow on W by S^t, and the trajectory of this flow which passes through w by $S_w = \bigcup_{t=-\infty}^{t=+\infty} S^t(w)$. Choose any point $\overline{m}$ on g^u (g^s) and take the vector $\xi \in T_{\overline{m}}\Delta$, $||\xi|| = 1$ tangent to g^u (g^s) and with the direction corresponding to the positive direction of the trajectory S^u_w (S^s_w) of the flow S^t on W passing through the point $w = (m, \xi)$, where $m = \pi(\overline{m})$. Then $G^u = q(S^u_w)$ ($G^s = q(S^s_w)$). Let us denote the sets of all ω, α limit points of the trajectory S^u_w by $\omega(S^u_w)$, $\alpha(S^u_w)$ respectively, and put

$$\Omega^u = G^u \cup q(\omega(S^u_w)) \cup q(\alpha(S^u_w)), \quad \overline{\Omega}^u = \pi^{-1}(\Omega^u)$$

We define the sets Ω^s, $\overline{\Omega}^s$ similarly.

Definition 7.2 *A geodesic $L^u \subset \Omega^u$ ($L^s \subset \Omega^s$) is said to be a boundary geodesic if L^u (L^s) is a part of the boundary of $M^2 \setminus \Omega^u$ ($M^2 \setminus \Omega^s$) accessible from the interior of $M^2 \setminus \Omega^u$ ($M^2 \setminus \Omega^s$). A geodesic from Ω^u (Ω^s) which is not a boundary geodesic is called an interior geodesic.*

Theorem 7.2 *The set Ω^u (Ω^s) has the following properties*:

1) *Ω^u (Ω^s) is a minimal geodesic lamination consisting of nontrivially recurrent geodesics* (*that is weakly irrational geodesic lamination, see Definition* 5.4);
2) *every geodesic in $\overline{\Omega}^u$ ($\overline{\Omega}^s$) has two irrational boundary points on the absolute*;
3) *no more than two geodesics in $\overline{\Omega}^u$ ($\overline{\Omega}^s$) can share a boundary point on the absolute*;
4) *any two geodesics $l^u \subset \overline{\Omega}^u$, $l^s \subset \overline{\Omega}^s$ have no a common boundary point on the absolute*;
5) *There are finitely many boundary geodesics in Ω^u (Ω^s).*

Proof. By item (2) of Proposition 7.2, $G^u = \pi(g^u)$ is a nonclosed geodesic with no self-intersections. Due to the theorem which states that the flow S^t on W depends continuously on the initial conditions and the definition of the projection q, we immediately obtain that any geodesic on M^2 in $q(\omega(S^u_w))$ has no self-intersections and can not intersect G^u. Furthermore, no two geodesics of $q(\omega(S^u_w))$ can intersect.

Let us show that every geodesic $G'^u \subset q(\omega(S^u_w))$ is unclosed. Assume the contrary. The endpoints u'_1, u'_2 of any preimage $g'^u \subset \Delta$ of the geodesic G'^u correspond to the stable and unstable fixed points of some element $\beta \in \Gamma$. Choose $\varepsilon > 0$ so that the Euclidean ε-neighborhoods $\varepsilon(u'_1)$, $\varepsilon(u'_2)$ of the points u'_1, u'_2 in the absolute S_∞ do not intersect. Then there is a $\delta > 0$ such that for any point $x \in \delta(u'_1)$ ($x \in \delta(u'_2)$), the point $\beta(x)$ satisfies $\beta(x) \in \in \varepsilon(u'_1)$ ($\beta(x) \in \varepsilon(u'_2)$). Since $G'^u \subset q(\omega(S^u_w))$, invoking the theorem on the continuous dependence on initial conditions of the flow S^t, and using the properties of the covers $q\colon W \to M^2$ and $\pi\colon \Delta \to M^2$ it follows that there is a sequence of pre-images g^u_n ($n = 1, 2, \ldots$) of geodesics G^u in Δ so that g'^u is the topological limit of the geodesics g^u_n and such that the endpoints $u_1^{(n)}$, $u_2^{(n)}$ of the geodesics g^u_n tend to u'_1 and u'_2, respectively, on the absolute. Since the endpoints u_1, u_2 of the geodesic g^u are irrational and g^u_n is congruent to g^u, the points $u_1^{(n)}$, $u_2^{(n)}$ are irrational, and therefore are different from the points u'_1, u'_2 for all n. But then there will be an $n_0 > 0$ such that $u_1^{(n_0)} \in \in \delta(u'_1)$, $u_2^{(n_0)} \in \delta(u'_2)$. Since u'_1 is sink and u'_2 is source (both being fixed points of the element β), the points $u_1^{(n_0)}$, $u_2^{(n_0)}$ will separate $u_2^{n_0}$, $\beta(u_2^{n_0})$, $u_1^{n_0}$, $\beta(u_1^{n_0})$ the points $\beta(u_1^{(n_0)})$, $\beta(u_2^{(n_0)})$[1] therefore the congruent geodesics $g^u_{n_0}$ and $\beta(g^u_{n_0})$ intersect in Δ, which is impossible since their common image G^u does not have any self-intersections.

Let us now show that the endpoints u'_1, u'_2 of the geodesic g'^u are irrational. Assume the contrary. Let us suppose u'_2 is rational. Then there is an element $\beta \in \Gamma$ having u'_2 as a sink fixed point. Any other fixed point ν of β, since it must be source, cannot coincide with u'_1, since, as we showed above, the geodesic G'^u cannot be closed. Consider three congruent curves $\beta(g'^u)$, g'^u, $\beta^{-1}(g'^u)$ having u'_2 as common endpoint and the three distinct endvalues $\beta(u'_1)$, u'_1, $\beta^{-1}(u'_1)$ at their free ends. Because of the way we have chosen β, the point u'_1 will lie on the arc of the absolute bounded by the points $\beta(u'_1)$ and $\beta^{-1}(u'_1)$ and not containing u'_2. Since g'^u is the topological limit of the geodesics g^u_n ($\pi(g^u_n) = G^u$) with irrational endpoints $u_1^{(n)}$, $u_2^{(n)}$ there is an n_* such that the curve $g^u_{n_*}$ will intersect either $\beta(g'^u)$ or $\beta^{-1}(g'^u)$. Therefore $G'^u \cap G^u = \varnothing$ so that $G'^u = G^u$, which is impossible.

We now show that G^u is a recurrent geodesic. Again, let us assume the contrary. Since S^t is a geodesic flow on W, the set $\omega(S^u_w)$ does not contain any equilibrium states (there are no such states in geodesic flows). From what was proved above, it follows that there are no closed trajectories in $\omega(S^u_w)$. Since $\omega(S^u_w)$ is a closed invariant set, it must contain at least one minimal set ω'. Every trajectory of ω' will be recurrent and everywhere dense in ω' (see, for example, [168], pp. 402–404). Using the results of [175] (section 4, Lemma 4) we can find a simple closed geodesic C such that $C \cap \Omega' \neq \varnothing$, where $\Omega' = = q(\omega')$. Since G^u is a nonrecurrent geodesic, and ω' is a minimal set of the flow S^t, we see that $K = C \cap \Omega'$ is a Cantor set. As $t \to +\infty$ the geodesic G^u will intersect a countable set of geometrically distinct intervals (α_n, β_n), $n = = 1, 2, \ldots$ of the Cantor set K. Since C is a closed curve, the lengths of the intervals (α_n, β_n) will tend to zero as $n \to +\infty$.

Denote by G^u_{+n} the ray of the geodesic G^u which begins at the point where G^u intersects the interval (α_n, β_n). We can find $n_0 > 0$ so

[1] In other words, if we make a circuit of the absolute in the positive direction (counterclockwise), we will encounter these points either in the order $u_2^{(n_0)}$, $\beta(u_2^{(n_0)})$, $u_1^{(n_0)}$, $\beta(u_1^{(n_0)})$ or $u_2^{(n_0)}$, $\beta(u_1^{(n_0)})$, $u_1^{(n_0)}$, $\beta(u_2^{(n_0)})$.

that $G^u_{+n_0}$ will lie entirely within a region D which is homeomorphic to a disk, as $t \to +\infty$. It follows from construction that the boundary of D that is accessible from the interior[2] consists of the interval $[\alpha_{n_0}, \beta_{n_0}]$ and rays of the geodesics $G_1, G_2 \subset \Omega'$ passing through the points $\alpha_{n_0}, \beta_{n_0}$. Thus we can find pre-images g^u_1, g^u_2 of geodesics G_1, G_2 such that the stable fixed point u_2 of τ^* is a common endpoint of the geodesics g^u_1, g^u, g^u_2, so that g^u will lie in the region bounded by the geodesics g^u_1, g^u_2 and the arc $[\alpha, \beta]$ of the absolute not containing the point u_2, where α and β are endpoints of the geodesics g^u_1, g^u_2.

Since the endpoints u_1, u_2 of the geodesic g^u are adjacent stable fixed points of τ^*, the arc (u_2, u_1) containing the point α will contain only a single unstable fixed point s_1, of τ^*. There are now two cases: $s_1 = \alpha$ and $s_1 \neq \neq \alpha$. We will show that both these cases lead to a contradiction, so that G^u is a recurrent geodesic. Suppose $s_1 = \alpha$. Since G^u_1 is a recurrent geodesic, $G^u_1 \cap G^u_2 = \varnothing$ and G^u_1, G^u_2 do not have any self-intersections, we see that on Δ we can find pre-images $\widetilde{g}^u_1 \neq g^u_1$ ($\widetilde{g}^u_1 = \gamma(g^u_1)$), $\gamma \in \Gamma$) of G^u_1 endpoints $\widetilde{u}_1 = \gamma(u_1)$, $\widetilde{u}_2 = \gamma(u_2)$ which lie on the arc (u_2, s_1) not containing the point u_1, so that $\widetilde{u}_1, \widetilde{u}_2$ are different from s_1, u_2 yet belong to arbitrarily small neighborhoods of s_1 and u_2, so that $\widetilde{u}_1$ and $\widetilde{u}_2$ separate the points $\tau^*(\widetilde{u}_1)$ and $\tau^*(\widetilde{u}_2)$. Since τ is an automorphism of Γ, for $\gamma \in \Gamma$ we can find a unique element $\gamma' \in \Gamma$ such that

$$\tau^*(\widetilde{u}_1) = \tau^*(\gamma(s_1)) = \gamma'(\tau^*(s_1)) = \gamma'(s_1),$$
$$\tau^*(\widetilde{u}_2) = \tau^*(\gamma(u_2)) = \gamma'(\tau^*(u_2)).$$

Thus, the geodesic $\widetilde{\widetilde{g}}^u_1$ with boundary points $\tau^*(\widetilde{u}_1)$ and $\tau^*(\widetilde{u}_2)$ is congruent to $\widetilde{u}^u_1$, and yet intersects it, which is impossible.

Now suppose that $s_1 \neq \alpha$. For definiteness let us suppose that α belongs to the arc (s_1, u_2) not containing the point u_2 (if $\alpha \in (u_2, s_1)$ and we understand (u_2, s_1) to be the arc not containing the point u_1, the proof is the same). Since G^u has G^u_1 in its ω limit set, on Δ there will be a geodesic $\widetilde{g}^u$ congruent to g^u, such that its endpoints $\widetilde{u}_1$ and $\widetilde{u}_2$ do not coincide with α and u_2 and belong to arbitrarily small neighborhoods of α and u_2, respectively, in the arc (u_2, α) not containing the point u_1, and such that the point $\tau^*(\widetilde{u}_1)$ belongs to the arc (α, u_1) not containing u_2, and the point $\tau^*(\widetilde{u}_2)$ belongs

[2]The boundary is said to be accessible (from the interior) if any point x of the boundary can be connected to any point of D by a path entirely contained in $D \cup \{x\}$.

to the arc (u_2, s_1) not containing u_1. But then the geodesic $\widetilde{\widetilde{g}}_1$ with endpoints $\tau^*(\widetilde{u}_1)$, $\tau^*(\widetilde{u}_2)$ is congruent to g^u and intersects g_1^u, which is impossible. Let us now consider the set

$$\Omega^u = G^u \cup q(\omega(S_w^u)) \cup q(\alpha(S_w^u)).$$

Since G^u is a recurrent geodesic, $\Omega^u = q(w(S_w^u)) = q(\alpha(S_w^u))$ and Ω^u can be partitioned into a continuum of nonintersecting recurrent geodesics which are not closed and do not intersect themselves, and so that each is dense in Ω^u. We will show that there cannot be more than two geodesics with a common endpoint on the absolute in $\widetilde{\Omega}^u = \phi^{-1}(\Omega^u)$. Suppose we could find three geodesics g_1^u, g_2^u, $g_3^u \in \widetilde{\Omega}^u$ having a common endpoint u_0 on the absolute, with g_2^u belonging to the region in Δ bounded by the geodesics g_1^u and g_3^u and an arc on the absolute not containing the point u_0. Since $g_2^u \subset \widetilde{\Omega}^u$ and since it has irrational endpoints, we can find a geodesic $\widetilde{g}_2^u$ congruent to g_2^u, such that its endpoints are separated by either the endpoints of g_1^u or g_3^u on the absolute, which is impossible. We will next show that any geodesics $l^u \subset \overline{\Omega}^u$ and $l^u \subset \overline{\Omega}^s$ must have distinct boundary points on the absolute. Suppose this is not true. Let us denote the common endpoint of l^u and I^s by σ and denote the other endpoints of these geodesics by α and β. Consider the directed segments from α to σ and from β to σ on l^u and I^s, respectively. There are then two recurrent trajectories S^u and S^s in the geodesic flow $\{S^t\}$ on W such that $q(S^u) = L^u$ and $q(S^s) = L^s$, where $L^u = \pi(l^u) \subset \Omega^u$, $L^s = \phi(l^s) \subset \Omega^s$.

Since the closures $\overline{S}^s$, $\overline{S}^u$ of the trajectories S^s, S^u are distinct minimal sets (because $\Omega^s \neq \Omega^u$), no point $w^* \in \overline{S}^u$ can ever be a limit point of any trajectory of $\overline{S}^s$. On the other hand, since we have assumed that σ is a common endpoint of l^u and I^s, it follows from [81] (page 14) that for any sequence of points $w_n \in S^u$ such that $\lim_{n\to\infty} w_n' = w^*$, we can find a sequence $w_n' \in S^s$ such that $\lim_{n\to\infty} w_n' = w^*$, a contradiction. Furthermore, since Ω^u and Ω^s are projections of minimal sets of the flow $\{S^t\}$ to W, they must be connected and perfect.

We will now show that Ω^u is nowhere dense in M^2 (the proof for Ω^s is similar). If Ω^u did not have this property, because it can be partitioned into a continuum of disjoint geodesics without self-intersections, none of which are closed, and each of which is dense in Ω^u, then Ω^u would be open in M^2. Since Ω^u is also closed in M^2, this would give $\Omega^u = M^2$, which is not possible, since for M^2 of genus $p \geqslant 2$ there is no such partition without singularities [127], pp. 251–258.

Let us prove the last item, that is the set of boundary geodesics for Ω^u is finite (the proof is similar for Ω^s). According to [175] (Lemma 4 from section 4) we can find a simple closed geodesic C such that $C \cap \Omega^u \neq \varnothing$. From proved item 5 above the set $K = C \cap \Omega^u$ is a Cantor perfect set, and any boundary geodesic $L^u \subset \Omega^u$ will pass through the endpoints of the contiguous intervals of the set K.

Since M^2 is an oriented manifold, we can speak of the positive and negative sides of C. For any contiguous interval (α_n, β_n) of the set K, let us denote the geodesics of Ω^u passing through α_n and β_n by $L^u_{\alpha_n}$ and $L^u_{\beta_n}$.

Using the fact that the geodesics of Ω^u are recurrent (statement from item 1 above), of the theorem which states that trajectories of the geodesic flow $\{S^t\}$ on W depend continuously on the initial conditions, and the properties of the cover $q\colon W \to M^2$, we can see that there is only a finite set Σ of contiguous intervals of K such that for any pair of geodesics L^u_α and L^u_β which pass through the endpoints of the interval $(\alpha, \beta) \in \Sigma$ as we travel along L^u_α and L^u_β in the direction of motion from the negative side of C to the positive side, or the opposite direction, the next set of intersection points of L^u_α, L^u_β with C will either consists of the endpoints of other distinct contiguous intervals of K, or of the endpoints of a single interval α', β' of K; in this latter case the region $Q \subset M^2$ bounded by the arcs $(\alpha, \alpha') \in L^u_\alpha$ and $(\beta, \beta') \in L^u_\beta$ and the intervals (α, β) and $(\alpha', \beta') \subset C$ is not homeomorphic to a disk.

Consider any boundary geodesic $L^u \subset \Omega^u$. We will show that it passes through one endpoint of a contiguous interval of Σ. Assume the contrary. Choose a direction along L^u and denote the set of all points where L^u intersects C by $\{\xi_s\}$ ($s \in Z$); thus, for any $s \in Z$, the arc $(\xi_s, \xi_{s+1}) \subset L^u$ will not have any points in common with C. For any ξ_s, let us denote by η_s the point on C such that (ξ_s, η_s) is a contiguous interval of K, and let us denote the boundary geodesic of Ω^u which passes through η_s by $L^u_{\eta_s}$. The hypothesis asserts that for any $s_0 \in Z$, the geodesic $L^u_{\eta_s}$ passes through the points $\eta_{s_0-1}, \eta_{s_0+1}$ and that the regions $Q_{s_0+1}, Q_{s_0} \subset M^2$, which are bounded by the corresponding contiguous intervals (ξ_{s_0}, η_{s_0}), $(\xi_{s_0+1}, \eta_{s_0+1})$ and the arcs $(\xi_{s_0}, \xi_{s_0+1}) \subset L^u$, $(\eta_{s_0}, \eta_{s_0+1}) \subset L^u_{\eta_{s_0}}$, and also by the contiguous intervals (ξ_{s_0}, η_{s_0}), $(\xi_{s_0-1}, \eta_{s_0-1})$ and the arcs $(\xi_{s_0}, \eta_{s_0}) \subset L^u$, $(\xi_{s_0-1}, \eta_{s_0-1}) \subset$ $\subset L^u_{\eta_{s_0}}$, are homeomorphic to the disk. But then the geodesics L^u and $L^u_{\eta_{s_0}}$ bound the region $Q = \bigcup\limits_{s=-\infty}^{\infty} Q_s$ in M^2 which is homeomorphic to the disk, and this is impossible. This completes the proof. □

Lemma 7.1 *Suppose the homeomorphism τ^* of the absolute is induced by the automorphism τ of the group Γ. Let $\widetilde{u}_1$ and $\widetilde{u}_2$ ($\widetilde{s}_1$ and $\widetilde{s}_2$) be endpoints of the geodesic $l^u \subset \overline{\Omega}^u$ ($l^s \subset \overline{\Omega}^s$). Suppose that the geodesic l'^u (l'^s) has endpoints $\tau^*(\widetilde{u}_1)$, $\tau^*(\widetilde{u}_2)$ ($\tau^*(\widetilde{s}_1)$, $\tau^*(\widetilde{s}_2)$. Then*

1) $l'^u \subset \overline{\Omega}^u$ ($l'^s \subset \overline{\Omega}^s$);
2) *if l^u (l^s) is a preimage of a boundary geodesic of the set Ω^u (Ω^s), then l'^u (l'^s) is also the preimage of a boundary geodesic of the set Ω^u (Ω^s).*

Proof. Let us show that (1) holds for l'^u (the proof is similar for l'^s). Since the geodesic $G^u = \pi(g^u)$ is dense in Ω^u and the geodesic $L^u = \pi(l^u) \subset \Omega^u$, we see that l^u is the topological limit of the geodesics $\gamma_n(g^u)$ ($\gamma_n \in \Gamma$, $n = 1, 2, \ldots$), where the geodesic g^u has endpoints u_1 and u_2, these being stable fixed points of τ^*. Since τ^* is an automorphism of Γ and u_1, u_2 are fixed points of τ^*, we have $\tau^*(\gamma_n(u_i)) = \gamma'_n(\tau^*(u_i)) = \gamma'_n(u_i)$ where $\gamma'_n = \tau(\gamma_n) \in \Gamma$, $i = 1, 2$. But then the geodesic l'^u, which has $\tau^*(\widetilde{u}_1)$ and $\tau^*(\widetilde{u}_2)$ as endpoints is the topological limit of the geodesics $\gamma'_n(g^u)$. Thus, $L'^u = \pi(l'^u) \subset \Omega^u$. Whence $l'^u \subset \overline{\Omega}^u$.

Statement (2) is proved in the same way. One should notice that, since l^u is the preimage of a boundary geodesic L^u of the set Ω^u, the geodesics $\gamma_n(g^u)$ (which have topological limit l^u) lie entirely on one side of l^u. This completes the proof. □

7.3. Strongly irrational transversal geodesic lamination

The main goal of this section is to investigate interrelations between the geodesic laminations Ω^u, Ω^s.

Theorem 7.3 *The set $M^2 \setminus \Omega^u$ ($M^2 \setminus \Omega^s$) has the following properties:*

1) *the set $M^2\setminus\Omega^u$ ($M^2\setminus\Omega^s$) is the union of k^u (k^s) disjoint regions D_i^u (D_i^s), $i = 1, \ldots, k^u$ ($i = 1, \ldots, k^s$) each of which is homeomorphic to the disk, where $1 \leqslant k^{u(s)} \leqslant 4(p-1)$;*
2) *the boundary of D_i^u (D_i^s) which is accessible from the interior of D_i^u (D_i^s) consists of r_i^u (r_i^s) boundary geodesics belonging to Ω^u (Ω^s), where*

$$\sum_{i=1}^{i=k^u} r_i^u = 4(p-1) + 2k^u \quad \left(\sum_{i=1}^{i=k^s} r_i^s = 4(p-1) + 2k^s\right);$$

3) *let* $\overline{D}_i^u$ $(\overline{D}_i^s)$ *be a lift of* D_i^u (D_i^s) *then the closer of* $\overline{D}_i^u$ $(\overline{D}_i^s)$ *on* Δ *is a convex* $r_i^{u(s)}$*-polygon whose sides are boundary geodesics of* $\overline{\Omega}^u$ $(\overline{\Omega}^s)$*;*
4) *the set* Ω^u (Ω^s) *is a strongly irrational geodesic lamination*[3]*.*

Proof. We shall items of theorem only for Ω^u (for Ω^s cosideration is similar). Since the set Ω^u is closed, the set $M^2 \setminus \Omega^u$ is the union of disjoint open connected components. By Theorem 7.2, the accessible boundary from interior of the set $M^2 \setminus \Omega^u$ consists of finite number boundary geodesics (see Definition 7.2). So the set $M^2 \setminus \Omega^u$ is the union of finite number $k^u \geqslant 1$ disjoint open connected components:

$$M^2 \setminus \Omega^u = D_1^u \cup \ldots \cup D_{k^u}^u.$$

Denote by r_i the number of all boundary geodesics belonging to the accessible from interior boundary of the component D_i^u.

Let $L^u \subset \Omega^u$ be a boundary geodesic belonging to the accessible boundary from the interior of D_i^u ($i \in \{1, \ldots, k^u\}$). Since the set of all boundary geodesics of Ω^u is finite, from item (2) of Lemma 7.1 we see that for any lift $l^u \subset \Delta$ of L^u with endpoints $\widetilde{u}_1$ and $\widetilde{u}_2$ on the absolute, we can find an integer $m_i > 0$ and an element $\beta_i \in \Gamma$ such that the geodesic l'^u with endpoints $\tau^{*m_i}(\widetilde{u}_1)$ and $\tau^{*m_i}(\widetilde{u}_2)$ will be congruent to l^u using β_i. Consider the automorphism

$$A_{\gamma_i} \circ \tau^{n_i} = (A_{\beta_i}^{-1} \circ \tau^{m_i})^2 \text{ where } n_i = 2m_i,\ \gamma_i = \beta_i^{-1}.$$

Then the points $\widetilde{u}_1$ and $\widetilde{u}_2$ are fixed for the induced homeomorphism $\tau_i^* = \gamma_i \tau^{*n_i} = (\beta_i^{-1}\tau^{*m_i})^2$ and have the same type (either both are sinks or both are sources). If we suppose that types of fixed points $\widetilde{u}_1$ and $\widetilde{u}_2$ are different then using the fact that L^u is recurrent, we can find a preimage l'^u with endpoints $\widetilde{u}_1'$ and $\widetilde{u}_2'$ sufficiently close to $\widetilde{u}_1$ and $\widetilde{u}_2$ accordingly and such that the pair of points $\tau_i^*(\widetilde{u}_1')$, $\tau_i^*(\widetilde{u}_2')$ are separates the points $\widetilde{u}_1'$ and $\widetilde{u}_2'$ which is impossible, since Theorem 7.2 guarantees that there are no self-intersections in L^u. For the sake of definiteness let us suppose that $\widetilde{u}_1$ and $\widetilde{u}_2$ are sink fixed points of τ_i^*. It follows from the proof given above on the finite number of boundary geodesics, that for every boundary geodesic L^u there is exactly one boundary geodesic $L_1^u \neq L^u$ such that L_1^u is a part of the boundary of

[3] Let us recall that a strongly irrational geodesic lamination is an irreducible and weakly irrational geodesic lamination.

the region D_i^u that is accessible from the interior and for any lift l^u of L^u there is a lift l_1^u of the geodesic L_1^u having a common boundary point with l^u on the absolute S_∞. Let $\widetilde{u}_1$ be this common boundary point. Let us denote the endpoint of l_1^u different from $\widetilde{u}_1$ by $\widetilde{\sigma}_1$. By virtue of Lemma 7.1 and assertion (2) of Theorem 7.2, we see that $\widetilde{\sigma}_1$ is a fixed point of τ_i^*. Since $\widetilde{u}_1$ is a sink fixed point of τ_i^*, then $\widetilde{\sigma}_1$ is also a sink fixed point of τ_i^*.

Furthermore, there is a lift l_2^u of the boundary geodesic L_2^u belonging to the accessible (from the interior) boundary of D_i^u, such that $\widetilde{\sigma}_1$ is an endpoint of l_2^u. Since there are r_i^u boundary geodesics in the accessible boundary of D_i^u, if we continue this process for r_i^u steps, we will obtain a convex r_i^u-gon $\overline{\overline{D}}_i^u$ whose vertices are stable fixed points of τ_i^*.

Let us now show that any sink fixed point of τ_i^* is one of the r_i^u vertices of $\overline{\overline{D}}_i^u$. Assume the contrary. Then there is a sink fixed point $\widetilde{\sigma}$ of τ_i^* which is not one of these vertices and such that one of these vertices $\widetilde{\alpha}$ is the closest sink fixed point to $\widetilde{\sigma}$ on the absolute. Consider the geodesic $\widetilde{l}^u$ having $\widetilde{\alpha}$ and $\widetilde{\sigma}$ as its endpoints.

Let us denote the sides of the polygon $\overline{\overline{D}}_i^u$ that have the vertex $\widetilde{\alpha}$ in common by $\widetilde{l}'^u$ and $\widetilde{l}''^u$. Since $\widetilde{l}'^u$ and $\widetilde{l}''^u$ come asymptotically close to one another, and, by virtue of (1) of Theorem 7.2, the geodesic $\widetilde{L}^u = \pi(\widetilde{l}^u)$ is recurrent and does not intersect itself. Similar to proof of item (2) of Theorem 7.2, we get that $\widetilde{l}^u \subset \overline{\Omega}^u$ that is $\widetilde{L}^u \subset \Omega^u$. But then the point $\widetilde{\alpha}$ will be an endpoint for three distinct geodesics, namely $\widetilde{l}'^u$, $\widetilde{l}''^u$ and $\widetilde{l}^u$, which contradicts (2) of Theorem 7.2.

It follows from Theorem 7.2 (see also Proposition 7.2) that the r_i^u-gon $\overline{\overline{D}}_i^u$ does not contain a pair of congruent points, and thus the interior $\overline{D}_i^u$ of the r_i^u-gon $\overline{\overline{D}}_i^u$ is a one-fold cover of the region D_i^u; therefore D_i^u is homeomorphic to the disk and the set $\overline{\overline{D}}_i^u \cap \Delta$ is a one-fold cover of D_i^u, together with boundary geodesics accessible from the interior of D_i^u.

Let us prove $k^u \leqslant 4(p-1)$ and $\sum\limits_{i=1}^{i=k^u} r_i = 4(p-1) + 2k^u$, where $k^u \geqslant 1$ is the number of regions D_i^u in M^2. Since D_i^u is homeomorphic to the disk, and the accessible boundary of D_i^u is the union of the boundary geodesics of Ω^u, we have $r_i^u \geqslant 3$, whence $3k^u \leqslant \sum\limits_{i=1}^{i=k^u} r_i^u$.

We will now construct a continuous foliation (A) on the union of D_i^u with singular points such that the boundary geodesics of D_i^u are its leaves, and such that the foliation (A) has exactly k^u singularities: there is one singularity O_i in each region D_i^u $(i = 1, 2, \ldots, k^u)$. Since the region D_i^u is homeomorphic to the disk, and the closure $\overline{\overline{D}}_i^u$ of its lift $\overline{D}_i^u$ in Δ is an r_i^u-gon not containing any pair of congruent points, this reduces to the problem of constructing such a foliation in $\overline{\overline{D}}_i^u$ for each $(i = 1, 2, \ldots, k^u)$.

Fix an arbitrary point $\overline{O}_i^u$ in each $\overline{\overline{D}}_i^u$ and connect $\overline{O}_i^u$ to the vertices of $\overline{\overline{D}}_i^u$ by subsegments of geodesics. This will decompose $\overline{\overline{D}}_i^u$ into r_i^u geodesic triangles $\overline{\Delta}_j^{(i)}$ $(j = 1, \ldots, r_i^u)$ with common vertex $\overline{O}_i^u$. Furthermore, the interior of every triangle $\overline{\Delta}_j^{(i)}$ can be partitioned into disjoint smooth curves without self-intersections, in such a way that each set of endpoints of the geodesics, which will be two vertices of $\overline{\Delta}_j^{(i)}$, will belong to the absolute, and such that the curves we have constructed, when taken together with the boundaries of the triangles, will form a continuous foliation with singularities $\overline{O}_i^u$ in $\overline{\overline{D}}_i^u$. The projection of leaves of this foliation gives the foliation (A) on on the union of D_i^u with singular points $O_i^u = \pi(\overline{O}_i^u)$ $(i = 1, 2, \ldots, k^u)$. Let us denote the index of O_i^u by I_i^u. If we add to the foliation (A) all leaves of the lamination Ω^u we get the foliation on M^2 which we denote by $(\widetilde{A})$ possessing by k^u singularities O_i^u. From (3.11) we have $I_i^u = 1 - \frac{1}{2} r_i^u$. Theorems 3.7 implies $\sum_{i=1}^{i=k^u} I_i^u = 2 - 2p$. From this it follows that $\sum_{i=1}^{i=k^u} r_i^u = 4(p-1) + 2k^u$ and since $3k^u \leqslant \sum_{i=1}^{i=k^u} r_i^u$ we see that $k^u \leqslant 4(p-1)$.

According to Theorem 7.2 Ω^u is a weakly irrational geodesic lamination. Sense each component of $M^2 \setminus \Omega^u$ is homeomorphic to the disk, Ω^u is strongly irrational geodesic lamination. Theorem 7.3 is proved. □

Theorem 7.4 *Suppose the conditions of Theorem* 7.3 *hold. Then* $k^u = k^s = k$. *Moreover, there are integers* $n > 0$ *and the unique decomposition of the set of regions* $D_1^u, \ldots, D_k^u$; $D_1^s, \ldots, D_k^s$ *into pairs* (D_i^u, D_i^s), $i = 1, \ldots, k$, *such that*:

a) *given any pair* (D_i^u, D_i^s), $r_i^u = r_i^s = r_i$ *and there is an element* $\gamma_i \in \Gamma$ *and pair lifts* $(\overline{D}_i^u, \overline{D}_i^s)$ *such that the* $(\overline{\overline{D}}_i^u, \overline{\overline{D}}_i^s)$ *is pair of* r_i*-gons so that* $\overline{\overline{D}}_i^u$ $(\overline{\overline{D}}_i^s)$ *has vertices which are sink* (*source*) *fixed points of* $\widetilde{\tau}_i^*$ *induced by the automorphism* $\widetilde{\tau}_i = A_{\gamma_i} \circ \tau^n$;

b) *the vertices of* $\overline{\overline{D}}{}_i^u$ *and* $\overline{\overline{D}}{}_i^s$ *coincide with the set of all fixed points of* τ_i^*;

c) *the geodesic laminations* Ω^u, Ω^s *are transversal.*

Proof. Let us keep the notations of Theorem 7.3 and set $n = \prod_{i=1}^{i=k} n_i$.Take any D_i^u $i \in \{1, \ldots, k^u\}$ and consider a lift $\overline{D}_i^u$. The vertices of the r_i^u-gon $\overline{\overline{D}}{}_i^u$ are sink fixed points of the homeomorphism $\widetilde{\tau}_i^* = \gamma_i \tau^{*n}$ of the absolute which is induced by the automorphism $A_{\gamma_i} \circ \tau^n$; furthermore, the homeomorphism $\widetilde{\tau}_i^*$ has no any other sink fixed points. Let us denote by $\overline{\overline{D}}{}_i^s$ the closed r_i^s-gon whose sides are the closure of geodesics having as end points the adjacent source fixed points of $\widetilde{\tau}_i^*$. Since the sinks and sources fixed points of $\widetilde{\tau}_i^*$ appear alternatively on S_∞, $r_i^s = r_i^u = r_i$.

We will now show that the geodesics which form the sides of the r_i-gon $\overline{\overline{D}}{}_i^s$ belong to $\overline{\Omega}^s$. Consider any geodesic l^s which is a side of $\overline{\overline{D}}{}_i^s$ with endpoints $\widetilde{s}_1$ and $\widetilde{s}_2$. Since $\overline{\overline{D}}{}_i^u \cap \overline{\overline{D}}{}_i^s \neq \varnothing$, we can find a geodesic l^u which is one of the sides of $\overline{\overline{D}}{}_i^u$, such that $l^u \cap l^s \neq \varnothing$. We will denote the endpoints of l^u by $\widetilde{u}_1$ and $\widetilde{u}_2$, so that if we execute a positive circuit of the absolute (i. e., counterclockwise), we encounter the points in the following order: $\widetilde{u}_1$, $\widetilde{s}_2$, $\widetilde{u}_2$, $\widetilde{s}_1$.

Take any geodesics $g^u \subset \overline{\Omega}^u$ and $g^s \subset \overline{\Omega}^s$ which have a nonempty intersection. As they are lifts of nontrivially recurrent geodesics, we can find some $\widetilde{g}^s$ congruent to g^s such that $\widetilde{g}^s \cap l^u \neq \varnothing$ and the endpoints $\widetilde{\sigma_1}$, $\widetilde{\sigma_2}$ of the geodesic $\widetilde{g}^s$ belong to the arcs $(\widetilde{s}_1, \widetilde{u}_1)$ and $(\widetilde{u}_1, \widetilde{s}_2)$ respectively. Since $\widetilde{u}_1$, $\widetilde{u}_2$ and $\widetilde{s}_1$, $\widetilde{s}_2$ are respectively sink and source fixed point with respect to τ_i^*, the geodesic l^s is the topological limit of the congruent geodesics $\widetilde{g}_k^s$ whose endpoints are $\tau_i^{*-k}(\widetilde{\sigma}_1)$, $\tau_i^{*-k}(\widetilde{\sigma}_2)$, whence we see that $l^s \subset \overline{\Omega}^s$. Thus all sides of r_i-gon $\overline{\overline{D}}{}_i^s$ belong to $\overline{\Omega}^s$. Similraly $\overline{D}_i^u$ one can prove that the r_i-gon $\overline{D}_i^s$ does not contain of congruent points. Therefore the projection $\pi(\overline{D}_i^s)$ is homeomorphic to the disk, denoted by D_i^s. Clearly the correspondence between the domains D_i^u and D_i^s is one to one. Hence $k^u = k^s = k$. Thus we get the unique decomposition of connected components from $M^2 \setminus \Omega^u$ and $M^2 \setminus \Omega^s$ into pairs (D_i^u, D_i^s), $i = 1, \ldots, k$. As we already proved above the vertices of the r_i-gons $\overline{\overline{D}}{}_i^u$, $\overline{\overline{D}}{}_i^s$ are, in fact, all the sink and source fixed points of $\widetilde{\tau}_i^*$. There are geodesics $g^u \subset \Omega^u$, $g^s \subset \Omega^s$ with nonempty transversal intersection. Since Ω^u, Ω^s are minimal geodesic laminations, they are are transversal geodesic laminations. □

Corollary 7.1 *Let $\gamma \in \Gamma$ and $n \in \mathbb{Z}$ be such that the homeomorphism $\widetilde{\tau}^*$ induced by $A_\gamma \circ \tau^n$ has no less than four fixed points. Then the geodesic l^u (l^s) with endpoints which are two adjacent sink (source) fixed points of $\widetilde{\tau}^*$ belongs to $\overline{\Omega}^u$ ($\overline{\Omega}^s$).*

Proof. We will show that $l^u \subset \overline{\Omega}^u$ (the proof for l^s is analogous). Let us denote the endpoints of l^u by $\widetilde{u}_1$, $\widetilde{u}_2$, and the source fixed points of $\widetilde{\tau}^*$ by s_1, s_2, in such a way that we encounter these points in the order $\widetilde{s}_1$, $\widetilde{u}_1$, $\widetilde{s}_2$, $\widetilde{u}_2$ in a counterclockwise circuit of the absolute, and such that the arc $(\widetilde{s}_2, \widetilde{u}_1)$ containing $\widetilde{u}_2$, $\widetilde{s}_1$ dose not contain any other fixed points of $\widetilde{\tau}^*$. Let $\widetilde{l}^s$ be the geodesic having s_1 and s_2 as its endpoints and write $\overline{x}_0 = l^u \cap \widetilde{l}^s$. Let $\widetilde{l}^s_+$ be the ray of $\widetilde{l}^s$ with initial point $\overline{x}_0$ and endpoint $\widetilde{s}_2$.

Let us show that $\widetilde{l}^s_+ \cap \overline{\Omega}^u \neq \varnothing$ and $\widetilde{l}^s_+ \subsetneq \overline{\Omega}^u$. Suppose the contrary. Then either $\widetilde{l}^s_+ \cap \overline{\Omega}^u = \varnothing$ or $\widetilde{l}^s_+ \subset \overline{\Omega}^u$.

In the first case, $\widetilde{l}^s_+ \cap \overline{\Omega}^u = \varnothing$, the geodesic $\widetilde{l}^s$ belongs to some lift $\overline{D}^u_i$ of the D^u_i, and then the point $\widetilde{s}_2$ will be an endpoint for some geodesic $l^u_* \subset \overline{\Omega}^u$ belonging to the boundary of the region $\overline{D}^u_i$. It follows from item 1 of Theorem 7.2 that the image $\widetilde{L}^s = \pi(l^s)$ is a nontrivially recurrent geodesic. Since $\widetilde{s}_2$ is the common endpoint of the geodesics $\widetilde{l}^s$ and l^u_* then $\widetilde{L}^s \subset \Omega^u$, which cannot occur.

In the second case, when $\widetilde{l}^s_+ \subset \overline{\Omega}^u$, we get contradiction with fact that $\widetilde{s}_2$ is a source fixed point of $\widetilde{\tau}^*$.

Therefore we can find a geodesic $\widetilde{l}^u \subset \overline{\Omega}^u$ such that $\widetilde{l}^u \cap \widetilde{l}^s_+ \neq \varnothing$. Let us denote the endpoints of $\widetilde{l}^u$ by $\widetilde{\sigma}_1$ and $\widetilde{\sigma}_2$. Since the pair $\widetilde{\sigma}_1$, $\widetilde{\sigma}_2$ separates the pair $\widetilde{s}_1$, $\widetilde{s}_2$, and since the arc $(\widetilde{u}_1, \widetilde{u}_2)$ not containing $\widetilde{s}_1$ contains only a single unstable point of $\widetilde{\tau}^*$ (namely, $\widetilde{s}_2$), it follows that $\widetilde{l}^u$ is the topological limit as $k \to \infty$ of the geodesics $\widetilde{l}^u_k$ with endpoints $\widetilde{\tau}^{*k}(\widetilde{\sigma}_1)$, $\widetilde{\tau}^{*k}(\widetilde{\sigma}_2)$. According Lemma 7.1 $\widetilde{l}^u_k \subset \overline{\Omega}^u$ for every k. Since Ω^u is a closed set, we have $\widetilde{l}^u \subset \overline{\Omega}^u$. □

Lemma 7.2 *The set Ω^u (Ω^s) has Lebesgue measure zero in M^2.*

Proof. Denote by κ the constant negative curvature of M^2. It follows from the Gauss–Bonnet formula [4], that area S_i of D^u_i (D^s_i) equals to $S_i = \frac{\pi}{\kappa}(2 - - r_i)$, and the area S_{M^2} of the manifold M^2 is $\frac{4\pi}{\kappa}(1-p)$. Since $D^u_i \cap D^u_j = = \varnothing$ (and $D^s_i \cap D^s_j = \varnothing$ $(i \neq j)$, the area S of all the polygons D^u_i (D^s_i)

$(i = 1, \ldots, k)$ is equal to:

$$S = \sum_{i=1}^{i=k} S_i = -\frac{\pi}{\kappa}\left(\sum_{i=1}^{i=k} r_i - 2k\right)$$

It follows from Theorem 7.3, $\sum\limits_{i=1}^{i=k} r_i = 4(p-1)+2k$, whence $S = S_M$ and therefore the set Ω^u (Ω^s) has Lebesgue measure zero. The statement is proved. □

Corollary 7.2 *Let* $[\alpha, \beta]$ *be any closed arc which belongs to geodesic* $L^u \subset \subset \Omega^u$ $(L^s \subset \Omega^s)$*. Then the set* $[\alpha, \beta] \cap \Omega^s$ $([\alpha, \beta] \cap \Omega^u)$ *has Lebesgue measure zero in the inner metric on* $[\alpha, \beta]$.

The proof follows from Lemma 7.2 and Theorem 7.3, and the fact that Ω^u (Ω^s) has the local structure of a direct product of the Cantor set and an interval.

7.4. Hyperbolic homeomorphisms induced by hyperbolic automorphisms

We keep the notation of previous Sections 7.2, 7.3. Set $\Omega_0 = \Omega^u \cap \Omega^s$, $\overline{\Omega}_0 = \overline{\Omega}^u \cap \overline{\Omega}^s$. It follows from Theorem 7.3 that the set Ω_0 is non empty, perfect and nowhere dense in M^2. Moreover, given point $x \in \Omega_0$ there are exactly two geodesics $L_x^u \subset \Omega^u$, $L_x^s \subset \Omega^s$ passing through x such that:

$$\overline{L_x^u \cap L_x^s} = \Omega_0.$$

Take τ to be a hyperbolic automorphism the group Γ (see Definition 7.1). Let $\overline{x}$ be any point of $\overline{\Omega}_0$, and let $l_{\overline{x}}^u$ and $l_{\overline{x}}^s$ be the geodesics of $\overline{\Omega}^u$ and $\overline{\Omega}^s$ passing through $\overline{x}$. The homeomorphism τ^* induced by τ maps the endpoints $\widetilde{u}_1$, $\widetilde{u}_2$ and $\widetilde{s}_1$, $\widetilde{s}_2$ of $l_{\overline{x}}^u, l_{\overline{x}}^s$ to $\widetilde{u}'_1$, $\widetilde{u}'_2$, $\widetilde{s}'_1$, $\widetilde{s}'_2$. Since the $\widetilde{u}_1$, $\widetilde{u}_2$ separate the $\widetilde{s}_1$, $\widetilde{s}_2$, then $\widetilde{u}'_1$, $\widetilde{u}'_2$ will separate $\widetilde{s}'_1$, $\widetilde{s}'_2$. Therefore, the geodesics l'^u, l'^s with endpoints $\widetilde{u}'_1$, $\widetilde{u}'_2$ and $\widetilde{s}'_1$, $\widetilde{s}'_2$ will intersect at a point $\overline{y}$. Let us denote this correspondence between the initial point $\overline{x}$ and the point $\overline{y}$ by $\overline{f}_0$.

Lemma 7.3 *The map* $\overline{f}_0$ *is a homeomorphism* $\overline{\Omega}_0 \to \overline{\Omega}_0$ *satisfying the condition*

$$\overline{f}_0 \circ \gamma = \tau(\gamma) \circ \overline{f}_0, \quad \gamma \in \Gamma.$$

Proof follows from the definition of the set $\overline{\Omega}_0$ and Lemma 7.1. □

Corollary 7.3 *There is the homeomorphism the* $f_0\colon \Omega_0 \to \Omega_0$ *such that* $\pi \circ f_0 = \overline{f}_0 \circ \pi$.

The first aim of this section is to prove that homeomorphism $f_0\colon \Omega_0 \to \Omega_0$ satisfies to axiom A^* (whose definition we will recall below). The second one is to extend the homeomorphism f_0 to a homeomorphism $f\colon M^2 \to M^2$ such way that the restriction $f|_{M^2\setminus\Omega_0}$ has finite number of periodic points.

Let X be a compact metric space with a metric d, $f\colon X \to X$ a homeomorphism. For $x \in X$ and $\delta > 0$, let us write

$$W^s_\delta(x) = \{y \in X \mid d(f^n(x),\, f^n(y)) \leqslant \delta,\ n \geqslant 0\}, \tag{7.7}$$
$$W^u_\delta(x) = \{y \in X \mid d(f^{-n}(x),\, f^{-n}(y)) \leqslant \delta,\ n \geqslant 0\}.$$

Definition 7.3 *One says that a homeomorphism* $f\colon X \to X$ *satisfies to Axiom* A^* *if*:

1) *the set of periodic points of* f *is dense in the nonwandering set* $NW(f)$;
2) *there are* λ $(0 < \lambda < 1)$, $C > 0$, $\varepsilon > 0$, $\gamma > 0$, *and* $n_0 \geqslant 0$ *such that for any* $x \in NW(f)$ *the following conditions hold*:
   - *if* $y,\, z \in W^s_\gamma(x)$, *then* $d(f^n(y),\, f^n(z)) \leqslant C\lambda^n d(y,\, z)$, $n \geqslant n_0$;
   - *if* $y,\, z \in W^u_\gamma(x)$, *then* $d(f^{-n}(y),\, f^{-n}(z)) \leqslant C\lambda^n d(y,\, z)$, $n \geqslant n_0$;
   - *if* $d(x,\, y) \leqslant \varepsilon$, *then* $W^s_\gamma(x) \cap W^u_\gamma(y)$ *consists of a single point denoted by* $[x,\, y]$ *which belongs to* $NW(f)$;
   - *the map* $(x,\, y) \to [x,\, y]$ *is continuous where* $(x,\, y) \in NW(f) \times \times NW(f)$ *and* $d(x,\, y) \leqslant \varepsilon$.

A homeomorphism which satisfies Axiom A^* *is called an* A^*-homeomorphism.

Theorem 7.5 *The homeomorphism* $f_0\colon \Omega_0 \to \Omega_0$ *satisfies to Axiom* A^*.

The proof of this theorem will be follow from the series of statements below.

For $x \in \Omega_0$, let us denote L^u_x and L^s_x the geodesics from Ω^u and Ω^s respectively passing through the point x. For $\delta > 0$, set

$$L^s_{x,\delta} = \{y \in L^s_x \mid \rho(x,\, y) \leqslant \delta\},$$
$$L^u_{x,\delta} = \{y \in L^u_x \mid \rho(x, y) \leqslant \delta\}.$$

where $\rho(x,\, y)$ is the distance between x and y in the inner metric on L^s_x (L^u_x).

Definition 7.1 *A periodic point p of $f_0 \colon \Omega_0 \to \Omega_0$ is said to be a boundary periodic point of the set Ω_0 if there is a $\delta > 0$ such that exactly one of the connected components of each set from $L^u_{p,\delta} \setminus \{p\}$, $L^s_{p,\delta} \setminus \{p\}$ does not intersect Ω_0. A non boundary periodic point of the homeomorphism f_0 is called interior.*

Lemma 7.4 *Every boundary periodic point p of Ω_0 belongs to the intersection of the boundary geodesics $L^u_p \subset \Omega^u$, $L^s_p \subset \Omega^s$. Moreover on every a boundary geodesic of Ω^u (Ω^s) there are exactly two boundary periodic points of Ω_0 and no other periodic points of f_0.*

Proof follows from the construction of f_0 and Theorem 7.3. □

Remark 7.1 *Let $\overline{p}$ be the preimage of the interior periodic point p of period n of the set Ω_0, and let $\overline{f}_{0,n} \colon \overline{\Omega}_0 \to \overline{\Omega}_0$ be the homeomorphism which covers $f_0^n \colon \Omega_0 \to \Omega_0$. The definition of f_0 implies that the endpoints of the geodesics $l^u_{\overline{p}} \subset \overline{\Omega}^u$ and $l^s_{\overline{p}} \subset \overline{\Omega}^s$ which pass through $\overline{p}$ are fixed points of $\tau'^* = \gamma\tau^{*2n}$ for some $\gamma \in \Gamma$. It follows from Theorem 7.3, that the homeomorphism τ'^* has no other fixed points on the absolute. Therefore p is the only periodic point of f_0 on $L^u_p \subset \Omega^u$ $(L^s_p \subset \Omega^s)$. Thus, every interior periodic point p of Ω_0 lies in the intersection of the interior geodesics $L^u_p \subset \Omega^u$ and $L^s_p \subset \Omega^s$, and on every interior geodesic of Ω^u (Ω^s) there is at most one interior periodic point of Ω_0 and no other periodic points of f_0.*

Lemma 7.5 *There are real numbers K, $\widetilde{C} > 0$, and λ $(0 < \lambda < 1)$ such that for any arc $[a, b] \subset L^u \subset \Omega^u$, $a, b \in \Omega_0$, $\rho(a, b) > K$, we have*

$$\rho(f_0^n(a), f_0^n(b)) > \frac{\widetilde{C}}{\lambda^n}\rho(a, b) \quad \textit{for all } n \geqslant 0.$$

Proof. Let $p \in \Omega_0$ be a periodic point, let $r_0 \geqslant 1$ be its period ($f_0^{r_0}(p) = p$), and let L^u_p and L^s_p be geodesics of Ω^u and Ω^s passing through p. Consider any lift $\overline{p} \in \Delta$ of p, and any geodesics $l^u_{\overline{p}}$, $l^s_{\overline{p}}$ passing through $\overline{p}$. Then for $r = 2r_0$ there is a homeomorphism $\overline{f}_{0,r} \colon \overline{\Omega}_0 \to \overline{\Omega}_0$ covering $f_0^r \colon \Omega_0 \to \Omega_0$ and such that $\overline{f}_{0,r}(\overline{p}) = \overline{p}$ and the endpoints of $l^u_{\overline{p}}$, $l^s_{\overline{p}}$ are fixed points of the homeomorphism τ'^*, induced by $\overline{f}_{0,r}$.

Consider the endpoints σ^u, σ^s of the geodesics $l^u_{\overline{p}}$, $l^s_{\overline{p}}$ such that on one of the arcs (σ^u, σ^s) having endpoints of σ^u and σ^s there are no fixed points of τ'^{*}. Let $\overline{q} \in \overline{\Omega}_0$ be any point in $(\overline{p}, \sigma^s) \subset l^s_{\overline{p}}$. For any $\overline{x} \in [\overline{p}, \overline{q}] \cap \overline{\Omega}^u$ we will denote by $l^{u+}_{\overline{x}}$ the connected component on the set $l^u_{\overline{x}} \setminus \{\overline{x}\}$ ($l^u_{\overline{x}} \subset \overline{\Omega}^u$) such that the endpoint $\sigma^u_{\overline{x}}$ of this component belongs to the arc $[\sigma^u, \sigma^s)$ of the absolute.

For any such point $\overline{x}$, let us denote by $\widetilde{l}^s(\overline{x})$ the geodesic congruent to $l^s_{\overline{p}}$ such that $\widetilde{l}^s(\overline{x}) \cap l^{u+}_{\overline{x}} \neq \varnothing$, and such that the point $\overline{y}$ satisfies $\overline{y} = \widetilde{l}^s(\overline{x}) \cap \cap\, l^{u+}_{\overline{x}} \in [\gamma(\overline{p}), \gamma(\overline{q})] \subset \widetilde{l}^s(\overline{x})$ for some $\gamma \in \Gamma$ and the arc $[\overline{x}, \overline{y}] \subset l^{u+}_{\overline{x}}$ does not contain any points congruent to any point of $[p, q]$. Furthermore, let us write $S = \bigcup\limits_{\overline{x} \in [\overline{p}, \overline{q}]} \widetilde{l}^s(\overline{x})$.

For $\overline{x}_1 \neq \overline{x}_2$, $\overline{x}_1, \overline{x}_2 \in [\overline{p}, \overline{q}] \cap \overline{\Omega}^u$, the geodesics $\widetilde{l}^s(\overline{x}_1)$ and $\widetilde{l}^s(\overline{x}_2)$ will either coincide or will not intersect. Since the geodesics $l^u_{\overline{x}} \subset \Omega^u$ depend on $\overline{x} \in [\overline{p}, \overline{q}]$ in a continuous fashion, we see that the set S consists of a finite number $\widetilde{l}^s_1, \ldots, \widetilde{l}^s_k$ of disjoint geodesics, congruent to $l^s_{\overline{p}}$. Define

$$\Delta_i = \{\overline{x} \in [\overline{p}, \overline{q}] \cap \overline{\Omega}^u \mid l^{u+}_{\overline{x}} \cap \widetilde{l}^s_i \neq \varnothing\} \quad (i = 1, 2, \ldots, k).$$

Each set $\Delta_i = [\alpha_i, \beta_i] \cap \overline{\Omega}^u$ where $\alpha_1 = \overline{p}$, $\beta_k = \overline{q}$, $[\alpha_i, \beta_i] \subset l^s_{\overline{p}}$. Let us write $\widetilde{\alpha}_i = l^{u+}_{\alpha_i} \cap \widetilde{l}^s_i$, $\widetilde{\beta}_i = l^{u+}_{\beta_i} \cap \widetilde{l}^s_i$ and denote by P_i the quadrangle of geodesics with sides $[\alpha_i, \beta_i]$, $[\widetilde{\alpha}_i, \widetilde{\beta}_i]$, $[\alpha_i, \widetilde{\alpha}_i]$, $[\beta_i, \widetilde{\beta}_i]$. By construction, the set $\pi\left(\bigcup\limits_{i=1}^{k} P_i \cap \overline{\Omega}^u\right)$ coincides with Ω^u. On the Figure 7.1, it is shown the set $\bigcup\limits_{i=1}^{k} P_i \cap \overline{\Omega}^u \subset \Delta$.

Let us denote the maximal distance between $[\alpha_i, \beta_i]$ and $[\widetilde{\alpha}_i, \widetilde{\beta}_i]$ by d_i, and denote $\max\limits_{1 \leqslant i \leqslant k} d_i$ by d_*. Consider a neighborhood $U(\sigma^u)$ of σ^u, chosen to be small enough ("small" in the Euclidean sense) so that the non-Euclidean distance from any point $\overline{z} \in U(\sigma^u) \cap \Delta$ to any point $\overline{x} \in [\overline{p}, \overline{q}] \cap \overline{\Omega}^u$ will be larger than $2d_*$.

Now σ^u and σ^s are (respectively) sink and source fixed points of τ'^{*} and there are no fixed points of τ'^{*} in (σ^u, σ^s). Hence we see that because of the way we have constructed $\overline{f}_{0,r}$, for any point $\overline{w}$ in $\overline{\Omega}_0$ in the union of the region Q in Δ (which is bounded by the arcs $[\overline{p}, \sigma^u) \subset l^u_{\overline{p}}$, $[\overline{p}, \sigma^s) \subset l^s_{\overline{p}}$, and the arc $[\sigma^u, \sigma^s]$ on the absolute) with the arc $(\overline{p}, \sigma^u]$, we have $\overline{f}^{\,n}_{0,r}(\overline{w}) \to \sigma^u$ for $n \to +\infty$.

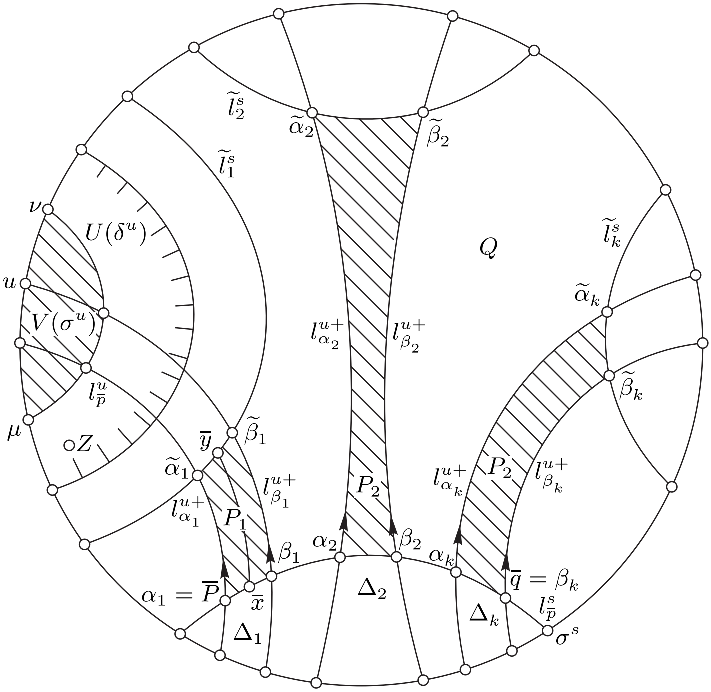


Figure 7.1. Fundamental set on $\overline{\Omega}^u$

Fix a geodesic $\widetilde{l}^s \subset \overline{\Omega}^s$ such that $\widetilde{l}^s \cap l^u_{\overline{p}} \neq \varnothing$ and $\widetilde{l}^s \subset U(\sigma^u)$; denote the endpoints of the geodesic $\widetilde{l}^s$ on the absolute by μ and ν, and denote the region in $U(\sigma^u)$ bounded by $\widetilde{l}^s$ and the arc $[\mu, \nu]$ containing σ^u by $V(\sigma^u)$, see Fig. 7.1.

Choose an integer $k_0 > 1$ such that $\overline{f}^{k_0}_{0,r}(\widetilde{\alpha}_i), \overline{f}^{k_0}_{0,r}(\widetilde{\beta}_i) \in V(\sigma^u)$ $(i = 1, 2, \dots, k)$. Then the definition of $V(\sigma^u)$ implies that $\rho(\overline{f}^{k_0}_{0,r}(\overline{x}), \overline{f}^{k_0}_{0,r}(\overline{y})) > 2d_*$, where $\overline{x} \in \Delta_i$ $(i = 1, 2, \dots, k)$, $\overline{y} = l^{u+}_{\overline{x}} \cap [\widetilde{\alpha}_i, \widetilde{\beta}_i]$. Set

$$\lambda_0(\overline{x}) = \frac{\rho(\overline{x}, \overline{y})}{\rho(\overline{f}^{k_0}_{0,r}(\overline{x}), \overline{f}^{k_0}_{0,r}(\overline{y}))}$$

Since $0 < \rho(\overline{x}, \overline{y}) \leqslant d_*$ and $\rho(\overline{f}^{k_0}_{0,r}(\overline{x}), \overline{f}^{k_0}_{0,r}(\overline{y})) > 2d_*$, we have $0 < \lambda_0(\overline{x}) \leqslant$

$\leqslant \frac{1}{2}$. Let $\lambda_0 = \max\limits_{\overline{x}\in \bigcup\limits_{i=1}^{i=k} \Delta_i} \lambda_0(\overline{x})$. Then $0 < \lambda_0 < 1$ and

$$\rho(\overline{f}_{0,r}^{k_0}(\overline{x}),\, \overline{f}_{0,r}^{k_0}(\overline{y})) = \frac{\rho(\overline{x},\, \overline{y})}{\lambda_0(\overline{x})} > \frac{\rho(\overline{x},\, \overline{y})}{\lambda_0}.$$

Put $m = k_0 r$. Since the homeomorphism $\overline{f}_{0,r}^{k_0}$ is a lift of f_0^m, the inequality above can be written for M^2 as follows

$$\rho(f_0^m(x),\, f_0^m(y)) > \frac{\rho(x,\, y)}{\lambda_0},$$

where $x,\, y \in [p,\, q] \cap \Omega^u$, $(x,\, y) \cap [p,\, q] = \varnothing$, $[p,\, q] = \pi([\overline{p},\, \overline{q}])$, $(x,\, y) \subset \subset L_x^u \subset \Omega^u$.

Let $l \geqslant 1$ be an arbitrary integer, and let $a_l = f_0^{(l-1)m}(x)$, $b_l = = f_0^{(l-1)m}(y)$. Because of the way we have constructed f_0^m, the arc $[a_l,\, b_l]$ can be written as a finite union of arcs $[a_l,\, b_l] = \bigcup\limits_{i=1}^{s_l} [x_i^{(l)},\, x_{i+1}^{(l)}]$ such that:

- $(x_i^{(l)},\, x_{i+1}^{(l)}) \in L_{a_l}^u$;
- $(x_i^{(l)},\, x_{i+1}^{(l)}) \cap (x_j^{(l)},\, x_{j+1}^{(l)}) = \varnothing$, if $i \neq j$;
- $x_1^{(l)} = a_l$, $x_{s_l+1} = b_l$;
- $x_i^{(l)} \in [p,\, q] \cap \Omega^u$ for any $i \in \{1,\, 2,\, \ldots,\, s_l + 1\}$;
- $(x_i^{(l)},\, x_{i+1}^{(l)}) \cap [p,\, q] = \varnothing$.

Therefore,

$$\rho(f_0^{lm}(x),\, f_0^{lm}(y)) = \rho(f_0^m(a_l),\, f_0^m(b_l)) = \sum_{i=1}^{s_l} \rho(f_0^m(x_i^{(l)}),\, f_0^m(x_{i+1}^{(l)})) >$$

$$> \frac{1}{\lambda_0} \sum_{i=1}^{s_l} \rho(x_i^{(l)} x_{i+1}^{(l)}) = \frac{1}{\lambda_0}\rho(a_l,\, b_l) = \frac{1}{\lambda_0}\rho(f_0^{(l-1)m}(x),\, f_0^{(l-1)m}(y))$$

By induction, one gets

$$\rho(f_0^{lm}(x),\, f_0^{lm}(y)) > \frac{1}{\lambda_0^l}\rho(x,\, y) = \frac{1}{\lambda^{lm}}\rho(x,\, y),$$

where $\lambda = (\lambda_0)^{1/m}$. Given any integer $n \geqslant 1$, there are integers $k_n \geqslant 0$ and s_n such that $n = k_n m + s_n$, $0 \leqslant s_n \leqslant m-1$. For all points $x', y' \in [p, q] \cap \Omega^u$, $(x', y') \cap [p, q] = \varnothing$, $(x', y') \subset L^u_{x'} \subset \Omega^u$, and for all $s = 0, 1, \dots, m-1$, set

$$C_0 = \min\left\{\frac{\rho(f_0^s(x'), f_0^s(y'))}{\rho(f_0(x'), f_0(y'))}\right\}, \quad C_1 = C_0\lambda^{m-1}.$$

Since $\lambda^{-k_n m} > \lambda^{m-n-1}$ and $\rho(f_0^{k_n m}(x), f_0^{k_n m}(y)) > \lambda^{-k_n m}\rho(x, y)$ for all pairs of points $x, y \in [p, q] \cap \Omega^u$, $(x, y) \subset L^u_x \subset \Omega^u$, $(x, y) \cap (p, q) = \varnothing$, we have the following estimates

$$\begin{aligned}\rho(f_0^n(x), f_0^n(y)) &= \rho(f_0^{s_n}(f_0^{k_n m}(x)), f_0^{s_n}(f_0^{k_n m}(y))) > \\ &> C_0\rho(f_0^{k_n m}(x), f_0^{k_n m}(y)) > C_0\lambda^{-k_n m}\rho(x, y) > \\ &> C_0\lambda^{m-n-1}\rho(x, y) = C_1\lambda^{-n}\rho(x, y).\end{aligned}$$

Let us now write $K = 4d_*$, $\widetilde{C} = \frac{1}{2}C_1$, and consider the arc

$$[a, b] \subset L^u \subset \Omega^u, \quad a, b \in \Omega_0;\ \rho(a, b) > K.$$

Since $[a, b] \subset \pi\left(\bigcup_{i=1}^{k}(P_i \cap \overline{\Omega}^u)\right) = \Omega^u$, there are points $x, y \in [a, b]$ such that the arc $[x, y] \subset [a, b]$ is the union of a finite number of arcs $[x_i, x_{i+1}]$ $(i = 1, 2, \dots, l)$ with properties:

- $(x_i, x_{i+1}) \subset L^u$;
- $(x_i, x_{i+1}) \cap (x_j, x_{j+1}) = \varnothing$ $(i \neq j)$;
- $x_i, x_{i+1} \in [p, q]$, $(x_i, x_{i+1}) \cap [p, q] = \varnothing$;
- $\rho(a, x_1) \leqslant d_*$, $\rho(x_{l+1}, b) \leqslant d_*$.

Therefore

$$\begin{aligned}\rho(f_0^n(a), f_0^n(b)) &= \rho(f_0^n(a), f_0^n(x_1)) + \rho(f_0^n(x_{l+1}), f_0^n(b)) + \\ &+ \sum_{i=1}^{l}\rho(f_0^n(x_i), f_0^n(x_{i+1})) > C_1\lambda^{-n}\sum_{i=1}^{l}\rho(x_i, x_{i+1}) = \\ &= C_1\lambda^{-n}\rho(x_1, x_{l+1}) = C_1\lambda^{-n}(\rho(a, b) - \rho(a, x_1) - \rho(x_{l+1}, b)) > \\ &> C_1\lambda^{-n}(\rho(a, b) - 2d_*) > \widetilde{C}\lambda^{-n}\rho(a, b).\end{aligned}$$

The lemma is proved. □

Let d the distance in a metric of constant negative curvature $\kappa = -1$ on M^2.

Lemma 7.6 *There are integer $n_0 \geqslant 1$ and real numbers $C > 0$, $\widetilde{\gamma} > 0$, $0 < \lambda < 1$ such that for any arc $[\alpha, \beta] \subset L^s \subset \Omega^s$ with $\alpha, \beta \in \Omega_0$, $d(\alpha, \beta) \leqslant \widetilde{\gamma}$, the following condition holds*:

$$d(f_0^n(\alpha), f_0^n(\beta)) \leqslant C\lambda^n d(\alpha, \beta), \quad n \geqslant n_0.$$

Proof. Since Ω^s is geodesic lamination, there is a $\gamma_0 > 0$ such that for any arc $[\alpha, \beta]$ of a geodesic of Ω^s with $d(\alpha, \beta) \leqslant \gamma_0$, we have $d(\alpha, \beta) = \rho(\alpha, \beta)$, where ρ is the interior metric of the given geodesic containing $[\alpha, \beta]$.

Let D^u be any connected component of the set $M^2 \setminus \Omega^u$. Choose an integer $m > 0$ such that all boundary periodic points of $f_0^m\colon \Omega_0 \to \Omega_0$ are fixed. According Theorem 7.4 there are a connected component D^s of $M^2 \setminus \Omega^s$ and an element $\beta \in \Gamma$ such that the closures $\overline{\overline{D}}^u$ and $\overline{\overline{D}}^s$ of the lifts $\overline{D}^u$ and $\overline{D}^s \subset \Delta$ of D^u and D^s are convex r-gons whose vertices are the sink and source fixed points of $\tau'^* = \beta\tau^{*m}$, induced by $A_\beta\tau^m$.

Let σ^u be any vertex of the r-gon $\overline{D}^u$, and let l_1^u and l_2^u be the sides of this polygon having σ^u as an endpoint. Let us denote by σ_1^s and σ_2^s the vertices of $\overline{\overline{D}}^s$ which have the property that the arc between them containing the point σ^u will not contain any other vertices of $\overline{\overline{D}}^s$.

Let l_*^s be the edge of $\overline{\overline{D}}^s$ containing σ_1^s and σ_2^s, and let $\overline{p}_1$, $\overline{p}_2$ be the intersection points of l_*^s with l_1^u and l_2^u, respectively.

Consider any geodesic $\widetilde{l}_*^s \subset \overline{\Omega}^s$ such that $\widetilde{l}_*^s$ intersects the arc $(\overline{p}_1, \sigma^u) \subset l_1^u$ at a point $\overline{a}$ such that the geodesic passing through $\overline{a}$ and orthogonal to l_1^u intersects l_2^u at $\overline{b}'$. Since $\widetilde{l}_*^s \cap l_*^s = \varnothing$, then $\widetilde{l}_*^s$ must intersect the arc $(\overline{p}_2, \sigma^u) \subset l_2^u$ at the point $\overline{b}$. Let Π_{σ^u} denote the geodesic quadrangle with vertices $\overline{p}_1$, $\overline{a}$, $\overline{b}$, and $\overline{p}_2$, and let D_{σ^u} be the distance between the arcs $[\overline{p}_1, \overline{a}]$ and $[\overline{p}_2, \overline{b}]$. If we now carry out this construction for all connected components $\{D^u\}$ and all vertices of $\{\overline{\overline{D}}^u\}$ we will obtain a finite collection of quadrangles $\{\Pi_{\sigma^u}\}$. Let us denote the maximum value of the D_{σ^u} by D, and write $\gamma_1 = \min\{\gamma_0, D\}$.

Consider any arc $[\alpha, \beta] \subset L^s \subset \Omega^s$ such that $\alpha, \beta \in \Omega_0$, $d(\alpha, \beta) \leqslant \gamma_1$. Two cases are possible:

1) $(\alpha, \beta) \cap \Omega_0 = \varnothing$;
2) $(\alpha, \beta) \cap \Omega_0 \neq \varnothing$.

In case 1, we see that (α, β) must belong to one of the connected components of the set $M^2 \setminus \Omega^u$. Without loss of generality we may suppose that $(\alpha, \beta) \subset D^u$ and that α, β lie on the boundary geodesics L^u, $L'^u \subset \Omega^u$ accessible from the interior of D^u, and with $L^u = \pi(l_1^u)$, $L'^u = \pi(l_2^u)$. Since $d(\alpha, \beta) \leqslant D$, we can find a lift $[\overline{\alpha}, \overline{\beta}]$ of the arc $[\alpha, \beta]$ such that $\overline{\alpha} \in [\overline{\alpha}, \sigma^u) \subset l_1^u$, $\overline{\beta} \in [\overline{\beta}, \sigma^u) \subset l_2^u$, and $[\overline{\alpha}, \overline{\beta}] \subset l^s_{\overline{\alpha}} \subset \Omega^s$, where $l^s_{\overline{\alpha}}$ is a lift of L^s passing through the point $\overline{\alpha}$.

Let us construct the perpendicular to l_1^u at $\overline{\alpha}$. Let $\overline{\beta}'$ be the point of intersection of this perpendicular with l_2^u. Let d_*, d, Δ_*, Δ be, respectively, the distances between the points $\overline{a}$, $\overline{b}'$; $\overline{\alpha}$, $\overline{\beta}'$; $\overline{a}$, $\overline{b}$; $\overline{\alpha}$, $\overline{\beta}$ and q be the distance between the points $\overline{a}$ and $\overline{\alpha}$. Consider quadrangle G with the vertexes $\overline{\alpha}$, $\overline{\beta}'$, $\overline{b}'$, $\overline{a}$, and denote by μ the angle at the vertex $\overline{\beta}'$ and ν the angle at the vertex $\overline{b}'$. Let us notice that $\mu = \pi - \Pi(d)$ and $\nu = \Pi(d^*)$ where $\Pi(d)$, $\Pi(d^*)$ are angles of parallelism at the points $\overline{\beta}'$, $\overline{b}'$ respectively.

As the curvature κ of M^2 is equal to -1, then applying to quadrangle G the first formula XXV from [169] (page 33) we get:

$$\cos\mu = -\cosh q \cos\nu + \sinh q \sin\nu \sinh d^*.$$

It follows from the known formulas $\cos\Pi(\mu) = \tanh d$, $\sin\Pi(\nu) = \frac{1}{\cosh d}$ that

$$\begin{aligned} -\tanh d &= -\cosh q \tanh d^* + \sinh q \frac{1}{\cosh d^*} \sinh d^*, \\ -\tanh d &= -\cosh q \tanh d^* + \sinh q \tanh d^*, \\ -\tanh d &= \tanh d^* (-\cosh q + \sinh q), \\ \tanh d &= e^{-q} \tanh d_*. \end{aligned}$$

Since $0 \leqslant d \leqslant d_*$, there are constants C_0, C_1 such that

$$d = Qe^{-q} d_*,$$

where $C_0 \leqslant Q \leqslant C_1$ and C_0, C_1 depend on d_*. Let us show that there is a constant S such that

$$\Delta = Se^{-q}\Delta_*.$$

Applying the law of sines to the triangle $\overline{\alpha}\overline{\beta}\overline{\beta}'$, we get

$$\frac{\sinh\Delta}{\sin B'} = \frac{\sinh d}{\sin B}$$

where B, B' are the angles at the vertices $\overline{\beta}$ and $\overline{\beta}'$. Hence

$$\sinh\Delta \leqslant \frac{\sinh d}{\sin B}.$$

Since Ω_0 is compact there is a constant $R > 0$ such that $R\sin B \geqslant 1$ and therefore $\sinh\Delta \leqslant R\sinh d$. Since $0 \leqslant d \leqslant d_*$, there is a constant $R_1 > 0$ such that

$$\Delta \leqslant R_1 d.$$

Let us show that there is a constant R_0 such that

$$R_0 d \leqslant \Delta.$$

Applying once more the law of sines to the triangle $\overline{\alpha}\overline{\beta}\overline{\beta}'$, we get

$$d \leqslant \sinh d = \frac{\sinh\Delta\sin B}{\sin B'}, \quad \sinh\Delta \leqslant \frac{\sinh\Delta}{\sin B'}.$$

Since either B' or $\pi - B'$ is the angle of parallelism for the segment d,

$$\sin B' = \frac{1}{\cosh d}.$$

Hence

$$d \leqslant \cosh d \sinh\Delta.$$

Since $0 \leqslant d \leqslant d_*$, $0 \leqslant \Delta \leqslant R_1 d \leqslant R_1 d_*$, there is a constant $R_0 > 0$ such that $R_0 d \leqslant \Delta$. But then $\Delta = Pd$ where $R_0 \leqslant P \leqslant R_1$. Thus we have $\Delta = Se^{-q}\Delta_*$ where $T_0 \leqslant S \leqslant T_1$, $T_0 = R_0C_0d_*\Delta_*^{-1}$, $T_1 = R_1C_1d_*\Delta_*^{-1}$ (Figure 7.2).

Let $\overline{f}_{0,m}$ cover f_0^m and induce $\tau'^* = \beta\tau^{*m}$. For any $k \geqslant 0$ let us write Δ_{km} for the distance between the points $\overline{\alpha}_{k,m} = \overline{f}^k_{0,m}(\overline{\alpha})$, and $\overline{\beta}_{k,m} = \overline{f}^k_{0,m}(\overline{\beta})$, and write q_{km} for the distance between the points $\overline{a}$ and $\overline{\alpha}_{km}$. Then $\Delta_{km} = S_{km}e^{-q_{km}}\Delta_*$, where $T_0 \leqslant S_{km} \leqslant T_1$. Hence,

$$\Delta_{km} = S_{km}S^{-1}e^{-q\left(\frac{q_{km}}{q}-1\right)}\Delta = S'_{km}e^{-q\left(\frac{q_{km}}{q}-1\right)}\Delta$$

where $S'_{km} = S_{km}S^{-1} \in [T'_0, T'_1]$, $T'_0 = T_0T_1^{-1}$, $T'_1 = T'^{-1}_0$.

Let ω be the distance between the point $\overline{a}$ and $\overline{p}_1$. If we take $\omega > K$ where K is defined as in Lemma 7.5, then we find from Lemma 7.5 that there

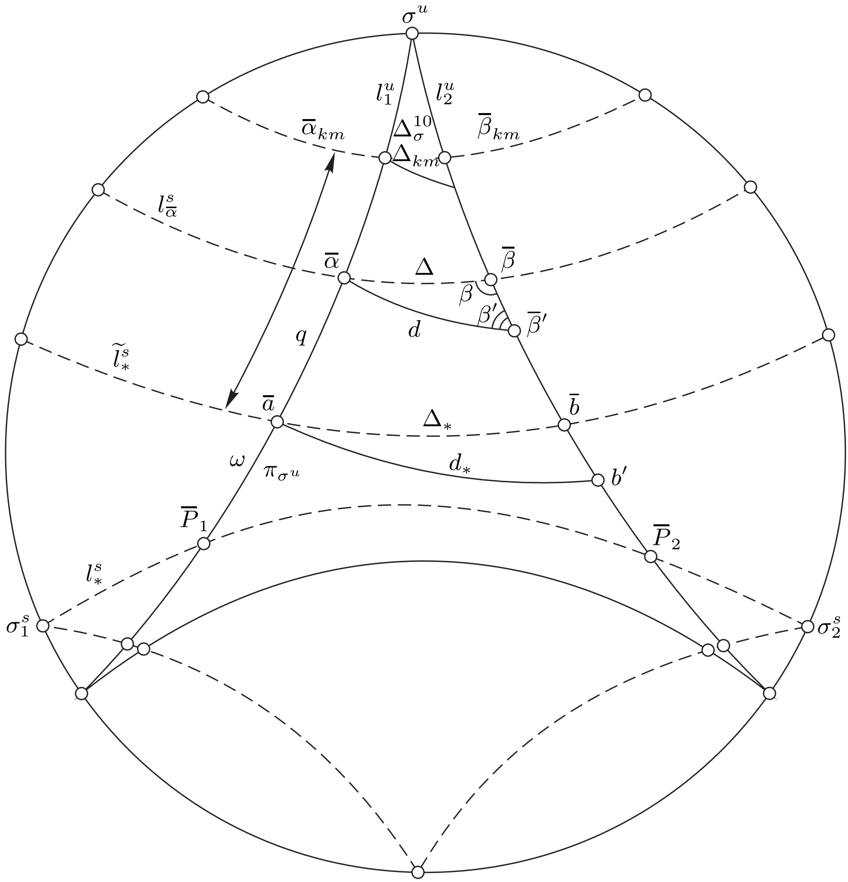


Figure 7.2. Exponential contraction

are a constant $\widetilde{C} > 0$, and some λ, $0 < \lambda < 1$, such that for any $k \geqslant 0$ we have the following estimate:

$$\omega + q_{km} \geqslant \widetilde{C}\lambda^{-km}(\omega + q).$$

But then there is a $k_0 \geqslant 0$ such that for all $k \geqslant k_0$ we have $\widetilde{C}\lambda^{-km} > 1$ and thus $q_{km} > \widetilde{C}\lambda^{-km} q$.

It follows that there is constant $G > 0$ such that for $k \geqslant k_0$

$$\Delta_{km} \leqslant G\lambda^{km}\Delta.$$

Indeed, $\Delta_{km} = S'_{km}e^{-q_{km}+q}\Delta = \widetilde{S}'_{km}e^{-q_{km}}\Delta$, where $\widetilde{S}'_{km} = e^q S'_{km}$. Without lost of generality we can suggest that for $k \geqslant k_0$ and some constant $A > 0$ we have from above

$$q_{km} > \widetilde{C}\lambda^{-km}q = \widetilde{C}'\frac{1}{\lambda} > \ln A + km\ln\frac{1}{\lambda},$$

where $\widetilde{C}' = qC'$.

Then

$$-q_{km} < -\ln A - km\ln\frac{1}{\lambda^{km}}, \quad \ln e^{-q_{mn}} < \ln\frac{\lambda^{km}}{A}, \quad e^{-q_{mn}} < G\lambda^{km},$$

where $G = \frac{1}{A}$.

Moreover, for $k \geqslant k_0$ we obtain

$$\Delta_{km} = S'_{km}e^{-q((q_{km}/q)-1)}\Delta < T'_1(e^{-q})^{\widetilde{C}\lambda^{-km}-1}\Delta < \\ < T_0^{-1}T_1^2\Delta_*(T_0^{-1}\Delta_*^{-1}\Delta)^{\widetilde{C}\lambda^{-km}}.$$

This estimate also shows that f_0 cannot be extended to a diffeomorphism on M^2.[4]

Let Δ_{σ^u} be the geodesic triangle with vertices $\overline{a}$, σ^u, $\overline{b}$ and denote by $\Delta^i_{\sigma^u}$ $(i = 0, 2, \ldots, m-1)$ the geodesic triangle with vertices $\overline{a}^{(i)}$, σ^u_i, $\overline{b}^{(i)}$, where $\overline{a}^{(i)} = \overline{f}^i_0(\overline{a})$, $\overline{b}^{(i)} = \overline{f}^i_0(\overline{b})$, and $\sigma^u_i = (\tau'^*)^i(\sigma^u)$. Let l^u_{1i} and l^u_{2i} be geodesics of $\overline{\Omega}^u$ passing through the points $\overline{a}^{(i)}$ and $\overline{b}^{(i)}$, respectively, and having σ^u_i as their common endpoint. Let $\overline{\alpha}^{(i)}$, $\overline{\beta}^{(i)}$, $\overline{\alpha}^{(i)}_{km}$, $\overline{\beta}^{(i)}_{km}$, $\overline{p}^{(i)}_1$, $\overline{p}^{(i)}_2$ be the images of the points $\overline{\alpha}$, $\overline{\beta}$, $\overline{\alpha}_{km}$, $\overline{\beta}_{km}$, $\overline{p}_1$, $\overline{p}_2$ under $\overline{f}^i_0$ and let $\omega^{(i)}$, $q^{(i)}$, $q^{(i)}_{km}$ be the distances between the following pairs of points: $\overline{p}^{(i)}_1$, $\overline{a}^{(i)}$; $\overline{a}^{(i)}$, $\overline{\alpha}^{(i)}$; $\overline{a}^{(i)}$, $\overline{\alpha}^{(i)}_{km}$. Denote by $\overline{b}'^{(i)}$, $\overline{\beta}'^{(i)}$ the point of intersections l^u_{2i} with perpendiculars to l^u_{1i} at the points $\overline{a}^{(i)}$, $\overline{\alpha}^{(i)}$ respectively and by $d^{(i)}_*$, $d^{(i)}$ the distances between pairs $\overline{a}^{(i)}$, $\overline{b}'^{(i)}$; $\overline{\alpha}^{(i)}$, $\overline{\beta}'^{(i)}$. Let us also denote by $\Delta^{(i)}_*$, $\Delta^{(i)}$, $\Delta^{(i)}_{km}$ the distances between the pairs $\overline{a}^{(i)}$, $\overline{b}^{(i)}$; $\overline{\alpha}^{(i)}$, $\overline{\beta}^{(i)}$; $\overline{\alpha}^{(i)}_{km}$, $\overline{\beta}^{(i)}_{km}$.

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[4]The authors thank V. S. Afraimovich for calling their attention to this fact.

From Lemma 7.5 we see that $q_{km}^{(i)} > \widetilde{C}\lambda^{-i}q_{km}$ for every $i \geqslant 0$ and $q_{km} \geqslant K$. As $e^{-q_{km}} = S_{km}^{-1}\Delta_k^{-1}\Delta_{km} \geqslant T_0^{-1}\Delta_*^{-1}\Delta_{km}$ for $k \geqslant 0$ and $\Delta_{km} < T_0^{-1}T_1^2\Delta_*(T_0^{-1}\Delta_*^{-1}\Delta)^{\widetilde{C}\lambda^{-km}}$ for $k \geqslant \overline{k}_0$, then for $k \geqslant \overline{k}_0$ we get

$$e^{-q_{km}} < T_0^{-1}T^2(T_0^{-1}\Delta_*^{-1}\Delta)^{\widetilde{C}\lambda^{-km}}.$$

Repeating this same set of inferences for $\Delta_{km}^{(i)}$, we can find constants $T_0^{(i)}$ and $T_1^{(i)}$ such that for all $k \geqslant 0$ we have $\Delta_{km}^{(i)} = S_{km}^{(i)}e^{-q_{km}^{(i)}}\Delta_*^{(i)}$, where $T_0^{(i)} < S_{km}^{(i)} \leqslant T_1^{(i)}$. Take Δ sufficiently small so that $T_0^{-1}\Delta_*^{-1}\Delta < e^{-1}$, and choose $k_0^* > k_0 > 0$ so that for all $k \geqslant k_0^*$, $0 \leqslant i \leqslant m-1$ we have

$$\widetilde{C}^2\lambda^{-(km+i)} > 1.$$

Then

$$(T_0^{-1}\Delta_*^{-1}\Delta)^{\widetilde{C}^2\lambda^{-(km+i)}} < \widetilde{C}^2T_0^{-1}\Delta_*^{-1}\lambda^{km+i}\Delta.$$

It follows

$$\begin{aligned}\Delta_{km}^{(i)} &< T_1^{(i)}\exp(-q_{km}^{(i)})\Delta_*^{(i)} < T_1^{(i)}\exp(-\widetilde{C}\lambda^{-1}q_{km})\Delta_*^{(i)} < \\ &< T_1^{(i)}\exp(-\lambda^{-1}\widetilde{C}^2\lambda^{-km}q)\Delta_*^{(i)} = T_1^{(i)}(e^{-q})^{\lambda^{-i}\widetilde{C}^2\lambda^{-km}}\Delta_*^{(i)} = \\ &= T_1^{(i)}(\Delta S^{-1}\Delta_*^{-1})^{\widetilde{C}^2\lambda^{-(km+i)}}\Delta_*^i < T_1^{(i)}(\Delta T_0^{-1}\Delta_*^{-1})^{\widetilde{C}^2\lambda^{-(km+i)}}\Delta_*^i < \\ &< T_1^{(i)}T_0^{-1}\Delta_*^{-1}\widetilde{C}^{-2}\Delta_*^{(i)}\lambda^{km+i}\Delta < C\lambda^{km+i}\Delta\end{aligned}$$

where $C = \max\limits_{0\leqslant i\leqslant m-1}\{\widetilde{C}^{-2}T_0^{-1}T_1^{(i)}\Delta_*^{(i)}\Delta_*^{-1}\}$.

Let us write $\widetilde{\gamma} = \min\{\gamma_1, T_0\Delta_*e^{-1}\}$ and $n_0 = (k_0+1)m-1$. Since we can write any integer $n > 0$ in the form $n = km+i$, $k \geqslant 0$, $0 \leqslant i \leqslant m-1$ and since the homeomorphism $\overline{f}_0^{(i)}\overline{f}_{0,m}^k\colon \overline{\Omega}_0 \to \overline{\Omega}_0$ covers $f_0^{km+i} = f_0^n\colon \Omega_0 \to \Omega_0$, then from the preceding discussion we see that for any arc $[\alpha, \beta] \subset L^s \subset \Omega^s$ such that $d(\alpha, \beta) < \widetilde{\gamma}$, $\alpha, \beta \in \Omega_0$ and $(\alpha, \beta) \cap \Omega_0 = \varnothing$ for all $n \geqslant n_0$, we have

$$d(f_0^n(\alpha), f_0^n(\beta)) \leqslant C\lambda^n d(\alpha, \beta).$$

In case (2), when $(\alpha, \beta) \cap \Omega_0 \neq \varnothing$, it follows from Lemma 7.2 and Corollary 7.2 that the set $K_0 = [\alpha, \beta] \cap \Omega^n$ is a Cantor set of Lebesgue measure zero, so that

$$K_0 = [\alpha, \beta] - \bigcup_{i=1}^{\infty}(\alpha_i, \beta_i)$$

where (α_i, β_i) $(i = 1, 2, \ldots)$ are contiguous intervals in K_0 and

$$d(\alpha, \beta) = \sum_{i=1}^{\infty} d(\alpha_i, \beta_i), \quad d(f_0^n(\alpha), f_0^n(\beta)) = \sum_{i=1}^{\infty} d(f_0^n(\alpha_i), f_0^n(\beta_i)).$$

From case 1, we obtain

$$d(f_0^n(\alpha), f_0^n(\beta)) \leqslant \sum_{i=1}^{\infty} d(f_0^n(\alpha_i), f_0^n(\beta_i)) \leqslant$$

$$\leqslant C\lambda^n \sum_{i=1}^{\infty} d(\alpha_i, \beta_i) = C\lambda^n d(\alpha, \beta)$$

for $n \geqslant n_0$. The Lemma is proved. □

Remark 7.2 *If we replace f_0 by f_0^{-1} and follow the steps in the proof of Lemma 7.6, we obtain the following inequality for any arc $[\alpha, \beta] \subset L^u \subset \Omega^u$ with $\alpha, \beta \in \Omega_0$, $d(\alpha, \beta) \leqslant \widetilde{\gamma}$*

$$d(f_0^{-n}(\alpha), f_0^{-n}(\beta)) \leqslant C\lambda^n d(\alpha, \beta), \quad n \geqslant n_0.$$

Lemma 7.7 *There are constants $\gamma, \gamma' > 0$ $(\gamma' < \gamma)$ such that for any $x \in \Omega_0$ one holds*

$$L^s_{x,\gamma'} \cap \Omega_0 \subset W^s_\gamma(x) \subset L^s_{x,\gamma} \cap \Omega_0.$$

Proof. First we prove the inclusion

$$W^s_\gamma(x) \subset L^s_{x,\gamma} \cap \Omega_0.$$

Let $\widetilde{\gamma}$, C, λ, n_0 be as in Lemma 7.6. It follows from Remark 7.2 that for any points $x, y \in \Omega_0$ with $y \in L^u_{x,\widetilde{\gamma}}$ we have

$$\rho(f_0^{-n}(x), f_0^{-n}(y)) \leqslant C\lambda^n \rho(x, y), \quad n \geqslant n_0.$$

Since $0 < \lambda < 1$, there is an fixed integer $m = m_0 > n_0$ such that $C\lambda^m < \frac{1}{4}$. Then

$$\rho(f_0^{-m}(x), f_0^{-m}(y)) < \frac{1}{4}\rho(x, y), \quad x, y \in \Omega_0, \quad y \in L^u_{x,\widetilde{\gamma}}.$$

We see that if $x', y' \in \Omega_0$, $y' \in L^u_{x',\widetilde{\gamma}}$ and $\rho(f_0^m(x'), f_0^m(y')) \leqslant \widetilde{\gamma}$, then $\rho(f_0^m(x'), f_0^m(y')) > 4\rho(x', y')$. Since Ω_0 is compact there is a $\delta > 0$ ($\delta < \widetilde{\gamma}$) such that for any two points $x, y \in \Omega_0$ and $y \in L^u_{x,\delta}$ we have

$$\rho(f_0^m(x), f_0^m(y)) < \widetilde{\gamma}.$$

Let us now show that if $y \in L^u_{x,\delta} \cap \Omega_0$ ($x \in \Omega_0$), then we can find a number $k \geqslant 1$ such that

$$\delta \leqslant \rho(f_0^{km}(x), f_0^{km}(y)) < \widetilde{\gamma}.$$

In fact, because of the way we have chosen m, we have $\widetilde{\gamma} > \rho(f_0^m(x), f_0^m(y)) > > 4\rho(x, y)$. If $\rho(f_0^m(x), f_0^m(y)) > \delta$, then $k = 1$. If $\rho(f_0^m(x), f_0^m(y)) \leqslant \delta$, then again we have

$$\widetilde{\gamma} > \rho(f_0^{2m}(x), f_0^{2m}(y)) > 4\rho(f_0^m(x), f_0^m(y)) > 16\rho(x, y).$$

Since $\rho(x, y) > 0$, we can find $k \geqslant 1$ so that

$$\widetilde{\gamma} > \rho(f_0^{km}(x), f_0^{km}(y)) > \delta.$$

Because of the properties of $\widetilde{\gamma}$ we have

$$\rho(f_0^{km}(x), f_0^{km}(y)) = d(f_0^{km}(x), f_0^{km}(y)).$$

Since Ω_0 is compact, we can infer from its construction that given $\delta > 0$ we can find $\gamma > 0$ ($\gamma < \delta/4$) such that the intersection $L^s_{x,\widetilde{\gamma}} \cap L^u_{y,\widetilde{\gamma}}$ consists of a single point which we will denote by $[x, y]$; this point belongs to a δ-neighborhood $\delta(x)$ of x, and the map

$$[\cdot, \cdot]\colon \{(x, y) \in \Omega_0 \times \Omega_0,\ d(x, y) \leqslant \gamma\} \to \Omega_0$$

is continuous.

Let y be any point in the γ-neighborhood $\gamma(x)$ of x such that $y \overline{\in} L^s_{x,\gamma} \cap \Omega_0$. We have to show that $y \overline{\in} W^s_\gamma(x)$. Indeed, because of our choice of γ there is a unique point

$$z = [y, x] = L^s_{y,\widetilde{\gamma}} \cap L^u_{x,\widetilde{\gamma}} \in L^u_{x,\gamma} \cap \Omega_0, \quad \rho(x, y) \leqslant \rho(x, z) + \rho(z, y) \leqslant 2\delta \leqslant \widetilde{\gamma}.$$

On the other hand, $\rho(x, y) \geqslant \rho(x, z) - \rho(z, y)$. Consequently there is a number k ($k \geqslant 1$) such that

$$\delta \leqslant \rho(f_0^{km}(x), f_0^{km}(z)) < \widetilde{\gamma}$$

Since $C\lambda^m < \frac{1}{2}$, we have $\rho(f_0^{km}(z),\, f_0^{km}(y)) < \frac{1}{2}\rho(z,\, y)$, whence

$$\rho(f_0^{km}(x),\, f_0^{km}(y)) \geqslant \rho(f_0^{km}(x),\, f_0^{km}(z)) - \rho(f_0^{km}(z),\, f_0^{km}(y)) > \\ > \delta - \frac{1}{4}\rho(z,\, y) > \delta - \frac{\delta}{2}.$$

By our choice of $\widetilde{\gamma}$, we have

$$\rho(f_0^{km}(x),\, f_0^{km}(y)) = d(f_0^{km}(x),\, f_0^{km}(y)),$$

and thus $y \overline{\in} W_\gamma^s(x)$. Hence,

$$W_\gamma^s(x) \subset L_{x,\gamma}^s \cap \Omega_0.$$

We will now show that there is a $\gamma' < \gamma$ such that

$$L_{x,\gamma'}^s \cap \Omega_0 \subset W_\gamma^s(x).$$

Let $\varepsilon > 0$ ($\varepsilon < \gamma$) be a real number such that if $d(x,\, y) < \varepsilon$ ($x,\, y \in \Omega_0$) then for all $n = 0,\, \ldots,\, n_0 - 1$ we have $d(f_0^n(x),\, f_0^n(y)) \leqslant \gamma$. Put $\gamma' = \min(\varepsilon,\, C^{-1}\gamma,\, \gamma)$. Let $y \in L_{x,\gamma'}^s \cap \Omega_0$. Then $\rho(x,\, y) \leqslant \gamma'$. Since $\gamma' < \gamma < \widetilde{\gamma}$, we have $\rho(x,\, y) = d(x,\, y) \leqslant \gamma'$. But then from our choice of γ' we see that $d(f_0^n(x),\, f_0^n(y)) < \gamma'$, $n = 0,\, 1,\, \ldots,\, n_0 - 1$. From Lemma 7.6, for $n \geqslant n_0$ we have

$$d(f_0^n(x),\, f_0^n(y)) \leqslant C\lambda^n d(x,\, y) < C\gamma' < \gamma.$$

Therefore we have $d(f_0^n(x),\, f_0^n(y)) \leqslant \gamma'$ for all $n \geqslant 0$, that is, $y \in W_\gamma^s(x)$, and thus $L_{x,\gamma'}^s \cap \Omega_0 \subset W_\gamma^s(x) \subset L_{x,\gamma}^s \cap \Omega_0$. Lemma is proved. □

Corollary 7.4 *The homeomorphism $f_0\colon \Omega_0 \to \Omega_0$ is expansive.*

Proof. Let us write $\delta' = \min\{\gamma,\, \delta/2\}$, where γ, δ are constants from Lemma 7.7. Then for any two points x, y ($x \neq y$) of Ω_0 we can find n such that $d(f^n(x),\, f^n(y)) > \delta'$. □

Corollary 7.5 *Homeomorphism is transitive on Ω_0 and the set of periodic points of f_0 is dense in Ω_0.*

Proof. Let x be a periodic point of f_0 and let L_x^u, L_x^s be geodesics of Ω^u, Ω^s passing through x. Since L_x^u and L_x^s are dense in Ω^u, Ω^s, the set of homoclinic points is dense in Ω_0. By a simple modification of the proof given in [202], we see that in a sufficiently small neighborhood of homoclinic points y there is some periodic point y_0.

Because the periodic points are dense in Ω_0 and for every point $x \in \Omega_0$ the intersection $L_x^u \cap \Omega_0$ is also dense in Ω_0, we see that for every neighborhood $V(x)$ in Ω_0 (in the induced topology) the set $\bigcup_{k \geqslant 0} f^k(V(x))$ is dense in Ω_0. But then from Birkhoff's theorem (see for example [208]) we see that we can find $y \in \Omega_0$ such that the half-trajectory $\{f^n(y)\}$ $(n \geqslant 0)$ is dense in Ω_0. Corollary is proved. □

Proof of Theorem 7.5

Let us recall that according to Definition 7.3, f_0 will be satisfy to axiom A^* if:

1) the set of periodic points of f_0 is dense in the nonwandering set $NW(f_0)$;
2) there are λ $(0 < \lambda < 1)$, $C > 0$, $\varepsilon > 0$, $\gamma > 0$, and $n_0 \geqslant 0$ such that for any $x \in NW(f_0)$ the following conditions hold:
   - if $y, z \in W_\gamma^s(x)$, then $d(f_0^n(y), f_0^n(z)) \leqslant C\lambda^n d(y, z)$, $n \geqslant n_0$;
   - if $y, z \in W_\gamma^u(x)$, then $d(f^{-n}(y), f^{-n}(z)) \leqslant C\lambda^n d(y, z)$, $n \geqslant n_0$;
   - if $d(x, y) \leqslant \varepsilon$, then $W_\gamma^s(x) \cap W_\gamma^u(y)$ consists of a single point denoted by $[x, y]$ which belongs to $NW(f_0)$;
   - the map $(x, y) \to [x, y]$ is continuous where $(x, y) \in NW(f_0) \times \times NW(f_0)$ and $d(x, y) \leqslant \varepsilon$.

Since due Corollary 7.5, the set of periodic points of f_0 is dense in Ω_0, the nonwandering set $NW(\Omega_0)$ coincides with Ω_0. It means that the first item of axiom A^* holds.

Let us prove the second item of axiom A^*.

The existence of $0 < \lambda < 1$, $C > 0$ and $n_0 \geqslant 1$ follows from Lemmas 7.5 and 7.6. The existence of γ follows from Lemma 7.7. The existence of $\varepsilon > 0$ such that the map $(x, y) \to [x, y]$ is continuous where $(x, y) \in NW(f_0) \times \times NW(f_0)$ and $d(x, y) \leqslant \varepsilon$ follows from the proof of Lemma 7.7. Thus Theorem is completely proved. □

Extension of homeomorphism $f_0 \colon \Omega_0 \to \Omega_0$ to AG-hyperbolic homeomorphism $f^{ag} \colon M^2 \to M^2$

Definition 7.4 *A homeomorphism $f^{ag} \colon M^2 \to M^2$ of a closed hyperbolic orientable surface M^2 is called Aranson–Grines hyperbolic homeomorphism* (*in short AG-hyperbolic homeomorphism*) *if the following conditions hold*:

1) *there are two invariant with respect to* f^{ag} *transversal geodesic laminations* Ω^u *and* Ω^s *such that* $\Omega^u \cap \Omega^s = \Omega_0$ *is not empty*;
2) Ω^u, Ω^s *are strongly irrational geodesic laminations*;
3) *the restriction* $f^{ag}|_{\Omega_0} : \Omega_0 \to \Omega_0$ *satisfies to the axiom* A^*;
4) *the unstable and stable manifolds of points from* Ω_0 *belongs to the geodesics of the laminations* Ω^u *and* Ω^s *respectively*;
5) *the non-wandering set* $NW(f^{ag})$ *consists of* Ω_0 *and finitely many isolated periodic points.*

Theorem 7.6 *Let* $M^2 = \Delta/\Gamma$ *be a closed orientable hyperbolic surfaces and* $\tau\colon \Gamma \to \Gamma$ *be a hyperbolic automorphism. Then there is an AG-hyperbolic homeomorphism* $f^{ag}\colon M^2 \to M^2$ *such that* $\overline{f}^{ag}_* = \tau$.

Proof. According to Theorems 7.2 and 7.5 for hyperbolic automorphism τ there is unique A^*-homeomorphism $f_0\colon \Omega_0 \to \Omega_0$ such that $\overline{f}^*_0 = \tau$. To prove the Theorem we have to extend $f_0\colon \Omega_0 \to \Omega_0$ to M^2 by a homeomorphism f^{ag} that has a finite number of periodic points on $M^2 \setminus \Omega_0$, such that $\overline{f}^{ag}_* = \tau$.

Let $m > 0$ be an integer such that all the boundary periodic points of $f_0^m\colon \Omega_0 \to \Omega_0$ are fixed. According to Theorem 7.4, given any connected component D^u of $M^2 \setminus \Omega^u$, there are exactly one connected component D^s of $M^2 \setminus \Omega^s$ and an element $\beta \in \Gamma$ such that the closures $\overline{\overline{D}}^u$, $\overline{\overline{D}}^s$ of the lifts $\overline{D}^u$, $\overline{D}^s$ are convex r-gons whose vertices are the respective sink and source fixed points of $\tau'^* = \beta\tau^{*m}$ which is induced by $A_\beta \tau^m$.

Let denote the convex $2r$-gon $\overline{D}^u \cap \overline{D}^s \subset \Delta$ by $\overline{\xi}$ and set $\xi = \pi(\overline{\xi}) \subset \subset D^u \cap D^s$. The vertices of ξ are fixed points of f_0^m. The sides of convex $2r$-gon ξ belonging to the geodesics of Ω^u and Ω^s will be denoted by a^u and a^s respectively. The lifts of the arcs a^u, $a^s \subset \xi$ which belong to $\overline{\xi}$ will be denoted by $\overline{a}^u$, $\overline{a}^s$.

Every point $\overline{x} \in \overline{a}^s$ $(\overline{a}^u)$ can be connected by a geodesic with a point σ^u (σ^s) of the absolute, which is the vertex of a geodesic triangle $\overline{\Delta}^u$ $(\overline{\Delta}^s)$ whose sides are arcs of the geodesics which bound the r-gons $\overline{D}^u$, $\overline{D}^s$. We shall assume that sides of triangle $\overline{\Delta}^u$ $(\overline{\Delta}^s)$ belonges to it. Fix any point $\overline{z}_0$ inside $\overline{\xi}$. Take some points $\overline{\alpha}^u$, $\overline{\alpha}^s$ inside the sides $\overline{a}^u$, $\overline{a}^s$. Connect the point $\overline{z}_0$ with $\overline{\alpha}^u$, $\overline{\alpha}^s$ by segments $\overline{h}^u$, $\overline{h}^s$ of geodesic lines. The convex $2r$-gon $\overline{\xi}$ is the union of pentagons $\overline{\xi}^s_1$, $\overline{\xi}^s_2$, ..., $\overline{\xi}^s_r$ which intersect along the sides $\{\overline{h}^s\}$ (similarly we could use pentagons $\overline{\xi}^u_1$, ..., $\overline{\xi}^u_r$ which intersect in $\{\overline{h}^u\}$). Foliate each pentagon $\overline{\xi}^s_i$ $(\overline{\xi}^u_i)$ by disjoint arcs whose boundary points belong to the sides $\{\overline{a}^s\}$ $(\{\overline{a}^u\})$ such that each arc of $\overline{\xi}^s_i$ intersects each arc of $\overline{\xi}_i^u$ transversely (Fig. 7.3). After projection by means of π these arcs and sets $\{\overline{h}^s\}$, $\{\overline{h}^s\}$ we get foliation on interior of polygon ξ with one singularity z_0 (all leaves of this foliation are transfersal to the sides of ξ).

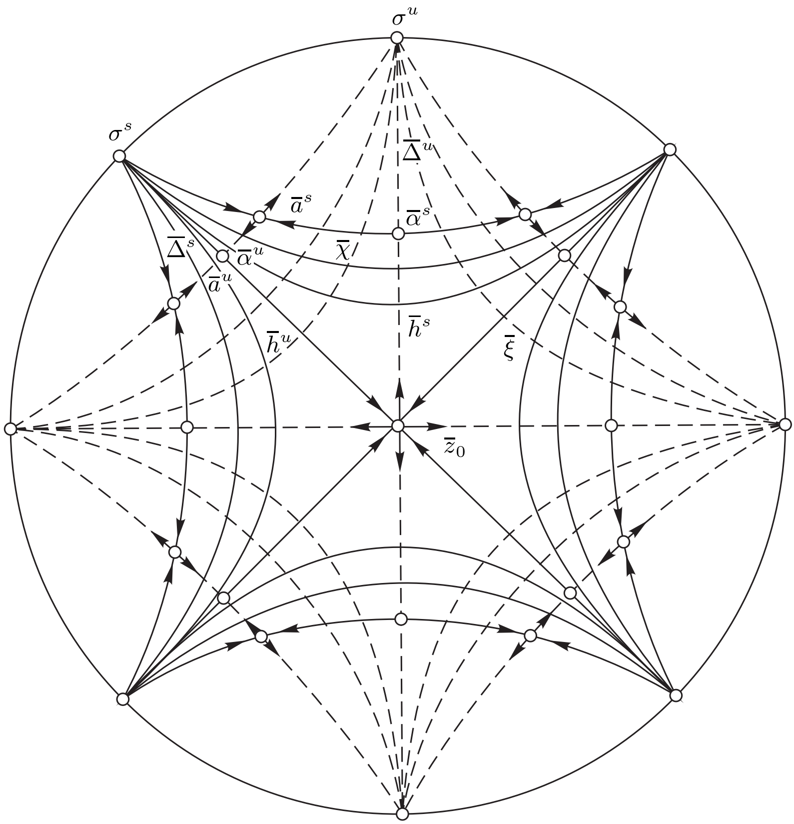

Figure 7.3. Construction of extension

Since the sets $\overline{D}^u, \overline{D}^s$ do not contain congruent points, the sets D^u, D^s are foliated by arcs with analogous properties.

Set $\Delta^u = \pi(\overline{\Delta}^u)$, $\Delta^s = \pi(\overline{\Delta}^s)$ and denote images of the points $\overline{\alpha}^s \in \overline{a}^s$, $\overline{\alpha}^u \in \overline{a}^u$, $\overline{z}_0 \in \overline{\xi}$ under π by $\alpha^s \in a^s$, $\alpha^u \in a^u$, $z_0 \in \xi$. The endpoints of each segment a^s, a^u will be denoted by ξ^s_-, ξ^s_+ and ξ^u_-, ξ^u_+ respectively. We will henceforth use the index i to denote different sets and points. Because of the way we have chosen m and the fact that ξ^s_-, ξ^s_+, ξ^u_-, $\xi^u_+ \in \Omega_0$ are boundary periodic points of $f_0 \colon \Omega_0 \to \Omega_0$, we see that these points are mapped by f_0^i to the points $(\xi^s_-)_i$, $(\xi^s_+)_i$, $(\xi^u_-)_i$, $(\xi^u_+)_i$, which are the endpoints of the intervals a^s_i and a^u_i of the $2r$-gon ξ_i; furthermore, these points are fixed under f_0^m. For any $a^s \subset \xi (a^u \subset \xi)$ let us write φ_s $(\varphi_u))$ for the homeomorphism with the following properties

1) $\varphi^i_s(\varphi^i_u)$ maps a^s (a^u) to a^s_i (a^u_i), φ^i_s (φ^i_u) maps the points ξ^s_-, ξ^s_+, α^s_i $(\xi^u_-$, ξ^u_+, $\alpha^u)$ to the points $(\xi^s_-)_i$, $(\xi^s_+)_i$, α^s_i $((\xi^u_-)_i$, $(\xi^u_+)_i$, $\alpha^u_i)$. In addition, the eqaiulity and $\varphi^m_s(a^s) = a^s$ $(\varphi^m_u(a^u) = a^u)$ φ^i_s (φ^i_u);
2) the homeomorphism φ^m_s (φ^m_u) has three fixed points in a^s (a^u): the sink fixed points ξ^s_-, ξ^s_+, and the source fixed point α^s (the source fixed points ξ^u_-, ξ^u_+ and the sink fixed point α^u).

Since the number of polygons ξ is finite, there are only a finite number of such homeomorphisms φ_s (φ_u). If $x \in M \setminus \Omega_0$, then we have two cases:

(1) either x belongs to one of the sets Δ^u (Δ^s);

(2) or x lies in the interior of one of the polygons ξ.

In case (1), let us suppose for definiteness that $x \in \Delta^u$ (if $x \in \Delta^s$, the construction is similar). Two subcases can now occur:

(a1) $x \in a^s$;

(a2) $x \notin a^s$.

In subcase (a1), set $y = \widetilde{f}(x) = \varphi_s(x)$.

Consider subcase (a2) $(x \notin a^s,\ x \in \Delta^u)$. We see from the construction given above that x belongs to the intersection of the two geodesic rays H^+, H^- whose initial points x^+, x^- belong to the intervals a^s, a'^u of ξ and ξ'.

The homeomorphisms φ_s, φ'_u send the points x^+, x^- to the points $x^+_* = \varphi_s(x^+)$ and $x^-_* = \varphi_u(x^-)$ which belong to the sides a^s_*, a'^u_* of the polygons ξ_*, ξ^-_*. The points x^+_*, x^-_* are the initial points of the geodesic rays H^+_*, H^-_* such that H^+_*, H^-_* belong to the sets $\Delta^u_* \in \{\Delta^u\}$, $\Delta^s_* \in \{\Delta^s\}$, respectively, and such that H^+_*, H^-_* intersect in a countable number of points. Since $x \in H^+ \cap H^-$, $x \notin \Omega_0$, there is a contiguous interval (α, β) outside the Cantor set $\Omega^s \cap L^u_{\xi^s_+}$ such that H^- intersects (α, β) in a single point z, so that the arc (z, x) has the property that $(z, x) \cap \Omega_0 = \varnothing$.

Since the points α, β satisfy $\alpha, \beta \in \Omega_0$, they are mapped by f_0 to $\alpha_*, \beta_* \in L^u_{(\xi^s_+)_*}$, where $(\xi^s_+)_* = f_0(\xi^s_+)$; the arc (α_*, β_*) is now a contiguous interval outside the Cantor set $\Omega^s \cap L^u_{(\xi^s_+)_*}$. The ray H^-_* intersects the arc (α_*, β_*)

in the single point z_*. Set $y = \widetilde{f}(x)$ as that point of $H_*^+ \cap H_*^-$ such that the arc $(z_*, y) \subset H_*^-$ has the property that $(z_*, y) \cap \Omega_0 = \varnothing$ (this point is unique).

In case (2), the point x belong to the interior of one of the polygons ξ, so that there are again two subcases: (b1) $x = z_0$; (b2) $x \neq z_0$.

In subcase (b1), we set $y = \widetilde{f}(x) = z_*$, where $z_* \in \{z_0\}$ and z_* belongs to the polygon $\xi_* \in \{\xi\}$ whose vertices are the images of the vertices of the polygon ξ under f_0 ($\{z_0\}$ means the set of all points chosen above in each polygon ξ).

In subcase (b2), x we have two subcases:

(b21) x is the intersection point of two arcs l_1, l_2 such that endpoints of l_1, belong to sides of ξ, and one of endpoints of l_2 belongs to some side of ξ and the other endpoint coincide with z_0.

(b22) x is the intersection point of two arcs l_1, l_2 such that endpoints of both arcs l_1 and l_2, belong to some sides of ξ.

In the subcase (b21) the homeomorphisms φ_s, φ_u act on the endpoints of l_1, l_2 and associate with the point z_0 the point $z_0^* \in \{z_0\}$ which belongs to the polygon $\xi_* \in \{\xi\}$ whose vertices are the images of the vertices of ξ under f_0; we thus obtain new arcs l_{1*}, l_{2*} passing through the images of the endpoints of l_1, l_2. The arcs l_{1*}, l_{2*} intersect in a single point y, by construction. Let us set $y = \widetilde{f}(x)$.

In the subcase (b22) the point $y = \widetilde{f}(x)$ is defined similarly.

Let $f^{ag} \colon M^2 \to M^2$ denote the following mapping

$$f^{ag} = \begin{cases} f_0,\ x \in \Omega_0, \\ \widetilde{f},\ x \in M^2 \setminus \Omega_0. \end{cases}$$

We can immediately verify that $f^{ag} \colon M^2 \to M^2$ is a homeomorphism, and intersection of nonwandering set of f^{ag} with the set $M^2 \setminus \Omega_0$ consists of a finite number of periodic points. All periodic points of f^{ag} belonging to Ω_0 are saddle points in the following sense. Let $p \in \Omega_0$ is periodic point of period m and $L_p^s \subset \Omega^s$, $L_p^u \subset \Omega^u$ geodesics passing through the point p. Then there is a $\delta > 0$ such that if $z \in L_{p,\delta}^s \setminus \{p\}$, then $f^{nm}(z) \to p$ as $n \to \infty$; if $z \in L_{p,\delta}^u \setminus \{p\}$, then $f^{-nm}(z) \to p$ as $n \to \infty$; if z belongs to the δ-neighborhood of p and does not lie on the arcs $L_{p,\delta}^s$, $L_{p,\delta}^u$, then the point $f^{mn}(z)$ leaves the δ-neighborhood of p for some n. $\square$

7.5. Topological entropy of hyperbolic homeomorphisms

Let $f\colon M^2 \to M^2$ be homeomorphism of a closed orientable surface M^2 of genus $g \geqslant 2$. Following J. Nielsen (see Section 6.7) the set $\mathrm{Fix}(f)$ of fixed points of f is decomposed into classes of fixed points. Two fixed points $x, y \in \mathrm{Fix}(f)$ belong to the same equivalence class if they can be joined by a path w such that w and $f(w)$ together constitute a null-homotopic path in M^2. Following to Definition 6.2, to each such class can be assigned a number known as the index of the class. If the number of fixed points in the class is finite, then the index of the class is the sum of the usual indices of these points. Moreover, the number of classes of fixed points with nonzero index is finite. Following to Definition 6.2 Nielsen number $\widetilde{N}(f)$ of homeomorphism $f\colon M^2 \to M^2$ is called the number of equivalence classes of fixed points from $\mathrm{Fix}(f)$ whose index is not equal to zero.

Let f^{ag} be AG-hyperbolic homeomorphism constructed in Section 7.4.

Lemma 7.8 *The topological entropy $h(f)$ of f^{ag} is calculated by the formula*

$$h(f) = \varlimsup_{n\to+\infty} \frac{\ln N_n}{n},$$

where N_n is the number of fixed points of $(f^{ag})^n$.

Proof. It follows from Theorem 7.5 that the restriction $f^{ag}|_{\Omega_0}$ satisfies to Axiom A^* for $n \geqslant n_0$. Then the homeomorphism $g|_{\Omega_0} = (f^{ag})^{n_0}|_{\Omega_0}$ satisfies to Axiom A^* for $n \geqslant 0$. Then by [52] and the fact that the number of periodic points of f in the set $M \setminus \Omega_0$ is finite, we find that the topological entropy $h(g)$ of g satisfies

$$h(g) = \varlimsup_{n\to+\infty} \frac{\ln \overline{N}_n}{n},$$

where $\overline{N}_n$ is the number of fixed points of g^n. Since $g = f^{n_0}$, we have $\overline{N}_n = N_{nn_0}$. From [1] it now follows that $h(f^{n_0}) = n_0 h(f)$, so that

$$h(f) = \frac{1}{n_0} h(f^{n_0}) = \varlimsup_{n\to+\infty} \frac{\ln N_{nn_0}}{nn_0} \leqslant \varlimsup_{n\to+\infty} \frac{\ln N_n}{n}.$$

According to Corollary 7.4, the restriction $f|_{\Omega_0}$ is an expansive homeomorphism. As f has a finite number of periodic points outside Ω_0 then by [62], the topological entropy $h(f)$ of f satisfies

$$\overline{\lim_{n\to+\infty}} \frac{\ln N_n}{n} \leqslant h(f) < +\infty$$

whence $h(f) = \overline{\lim\limits_{n\to+\infty}} \frac{\ln N_n}{n}$. This completes the proof. □

For natural number $n \in \mathbb{N}$ denote by $\widetilde{N}_n$ the Nielsen number for homeomorphism f^n.

Lemma 7.9 *There is some $k_0 > 0$ such that for all $n \in \mathbb{N}$ we have $0 \leqslant N_n - \widetilde{N}_n \leqslant k_0$.*

Proof. Let us write $m_n = N_n - \widetilde{N}_n$. By definition of the Nielsen number $\widetilde{N}_n$ we have $m_n \geqslant 0$. By the construction f^{ag}, there is only a finite number (call it k_1) of boundary periodic points in Ω_0 (see Remark 7.4 to Definition 7.1), while the set of interior periodic points is countable and everywhere dense in Ω_0 (see Theorem 7.5 and Lemma 7.7). The homeomorphism f^{ag} has a finite number of periodic points (denote this number k_2) outside Ω_0. Set $k_0 = k_1 + k_2$; then the number $N_n^{(1)}$ of boundary fixed points of $(f^{ag})^n$ in Ω_0 plus the number of fixed points in $M^2 \setminus \Omega_0$ satisfy $N_n^{(1)} \leqslant k_0$.

Let $N_n^{(2)}$ be the number of interior fixed points of $(f^{ag})^n$ in Ω_0. It follows from Remark 7.1 that each class of fixed points containing interior periodic points consists of only one point. Then $\widetilde{N}_n^{(2)} = N_n^{(2)}$ where $\widetilde{N}_n^{(2)}$ is the number of classes corresponding to the interior fixed points of $(f^{ag})^n$. Write $\widetilde{N}_n$ in the form $\widetilde{N}_n = \widetilde{N}_n^{(1)} + \widetilde{N}_n^{(2)}$, where $\widetilde{N}_n^{(1)}$ is the number classes of points which are boundary fixed points or are fixed points in $M^2 \setminus \Omega_0$. Since $\widetilde{N}_n^{(1)} \leqslant N_n^{(1)}$, $\widetilde{N}_n^{(2)} \leqslant N_n^{(2)}$, hence

$$m_n = N_n - \widetilde{N}_n = N_n^{(1)} + N_n^{(2)} - \widetilde{N}_n^{(1)} - \widetilde{N}_n^{(2)} = N_n^{(1)} - \widetilde{N}_n^{(1)} \leqslant k_0.$$

This completes the proof. □

Theorem 7.7 *Let $M^2 = \Delta/\Gamma$ be a closed orientable hyperbolic surfaces, $\tau\colon \Gamma \to \Gamma$ a hyperbolic automorphism and $f^{ag}\colon M^2 \to M^2$ an AG-hyperbolic homeomorphism such that $\overline{f}_* = \tau$. Then topological entropy $h(f^{ag})$ of f^{ag} equals to*

$$h(f^{ag}) = \overline{\lim_{n\to+\infty}} \frac{\ln \widetilde{N}_n}{n}.$$

Moreover, $h(f^{ag})$ is the minimal entropy for homeomorphisms of M^2 which are homotopic to f^{ag}.

Proof. From Lemmas 7.8 and 7.9 we have

$$h(f^{ag}) = \varlimsup_{n\to+\infty} \frac{\ln N_n}{n}, \quad 0 \leqslant N_n - \widetilde{N}_n \leqslant k_0.$$

Then $h(f^{ag}) = \varlimsup\limits_{n\to+\infty} ((\ln \widetilde{N}_n)/n)$.

Let $f'\colon M^2 \to M^2$ any homeomorphism which is homotopic to f^{ag}.

According to Theorem 2.7 from [49] topological entropy $h(f')$ satisfies to the inequality

$$h(f') \geqslant \varlimsup_{n\to+\infty} \frac{\ln \widetilde{N}'_n}{n},$$

where $\widetilde{N}'_n$ is the number of Nielsen classes for f'. As f' is homotopic to f^{ag}, then $\widetilde{N}'_n = \widetilde{N}_n$. Thus

$$h(f') \geqslant h(f^{ag})$$

and Theorem is completely proved. □

Bibliographic Notes and Panoramas

Chapter 7. One of important aspect of Modern Theory of Dynamical Systems is a construction of dynamical systems having properties intimately close to the topology of ambient manifolds. M. Shub [203] first posed this problem for diffeomorphisms satisfying Axiom A. In the article [204], it was proved that every homotopy class of homeomorphisms of closed manifold M^n ($n \geqslant 1$) contains an A-diffeomorphism. However, the method of construction of such diffeomorphisms does not allow to extract a diffeomorphism with minimal entropy. The reason is that Shub – Sullivan's method admits an existence of arbitrary number of nontrivial basic sets in a disk.

Construction of AG-hyperbolic homeomorphism is solution of the classical problem posed by J. Nilsen. Actually, Nielsen [173] understood complex dynamics of any homeomorphisms inducing hyperbolic action in fundamental group of surface, but he did not know methods of investigation such homeomorphisms. It is well known that such classes of homeomorphisms contain also pseudo-Anosov homeomorphisms introduced and constructed by Thurston [214] (see section 6.1).

Chapter 7 is devoted to solving the following extremal problem: for every homotopy class of homeomorphisms defined on a closed connected oriented two-dimensional manifold M^2 of genus $p \geqslant 2$, and inducing a hyperbolic automorphism of the fundamental group, construct a representative, so called AG-hyperbolic homeomorphism with minimal topological entropy. Such representative contains exactly one nontrivial (different from periodic orbit) locally maximal invariant zero-dimensional set Ω_0 whose stable and unstable manifolds of points lie on geodesics with respect to a metric of constant negative curvature. This construction was done firstly by S. Aranson and V. Grines [29] in 1980. As was noticed by R. Miller [162] in 1982, after some natural factorization it is possible to construct homeomorphism on given surface which is homotopic to pseudo-Anosov homeomorphism. Thus, these works was a bridge between Nilsen and Thurston theories.

(7.1). Example 7.1 was constructed in [55].

Proposition 6.8 was proved by Nielsen [173].

(7.2). Theorem 7.2 was proved in [29, 30]. The result of this theorem is a crucial observation on interrelation between of hyperbolic actions in a fundamental group and the existing of perfect geodesic laminations. Similar result was independently opened by Miller [162]. See also survey [32] and book [17].

(7.3). Theorem 7.3 was proved in [29, 30].

(7.4). Axiom A^* introduced by R. Bowen in [52] which is generalization of Axiom A by S. Smale. Bowen has noted that many of the important properties of A-diffeomorphisms can be proved by methods of topological dynamics without on an assumption of smoothness. These properties include for example the existence of Markov partitions and formulas relating the topological entropy with the growth of the periodic points [50–52].

Theorem 7.6 was proved in [29, 30].

(7.5). On the 2-sphere (surface of genus $p = 0$), the Morse – Smale diffeomorphisms are representatives of diffeomorphisms satisfying Axiom A with minimal topological entropy (this being zero) in each of the homotopy classes (there are only two such classes). Let us note that there are diffeomorphisms in these classes on the sphere satisfying Axiom A but having nonzero entropy [5, 208]. On the 2-torus (surface of genus $p = 1$) there is a countable set of homotopy classes of diffeomorphisms, each of which contains representative satisfying Axiom A and having minimal nonzero entropy. These representatives are in fact the algebraic Anosov automorphisms [5] for each of them nonwandering set is the whole torus. Any such homotopic class have else two representatives with minimal entropy such that nonwandering set of any diffeomorphism have either zero-dimensional or one-dimensional nontrivial basic sets. Recall that A-diffeomorphism of the torus constructed in [208, 221] has one nontrivial one-dimensional basic set. That its entropy is minimal follows from the results of the papers [72] and [84]. The method used to construct this diffeomorphism is obviously applicable to the construction of diffeomorphisms of the torus with nontrivial zero-dimensional basic sets. On a surface of genus $p \geqslant 2$, there are A-diffeomorphisms with zero- or one-dimensional nontrivial basic sets and minimal entropy. The topology of the one-dimensional nontrivial basic sets on M of genus $p \geqslant 0$ has been studied in [185–189] and [82–84].

CHAPTER 8

One-Dimensional Attractors and Aranson–Grines homeomorphisms

In this chapter, we consider an orientation preserving A-diffeomorphism f on a closed orientable hyperbolic surface M^2 such that non-wandering set $NW(f)$ contains an one-dimensional widely disposed attractor Λ. In Section 8.1, we study asymptotic behavior of lifts of stable and unstable manifolds of points from Λ on universal covering Δ of M^2 where Δ is hyperbolic plane. We suppose that diffeomorphism f induces hyperbolic automorphism of fundamental group $\pi_1(M^2)$. This allows us to construct in Section 8.2 the geodesic framework for the widely disposed attractor Λ. Actually, this geodesic framework coincides with geodesic lamination Ω^u constructed in chapter 7.

In Section 8.3 we prove that the restriction of f to Λ semi-conjugated with the restriction of AG-hyperbolic homeomorphism (see Defininition 7.4) to the set $\Omega_0 = \Omega^u \cap \Omega^s$, where Ω^u and Ω^s are transversal laminations constructed in Chapter 7. This semi-conjugacy can be extended to homeomorphism $M^2 \to$ $\to M^2$ which is homotopic to the identity map.

8.1. Asymptotic behaviors of stable and unstable manifolds

Let $\pi\colon \Delta \to M^2$ be a universal covering and let Γ be the group of its covering transformations, where Δ is Hyperbolic plain in Poincaré model ($\Delta =$ $= \{z \in \mathbb{C}\colon |z| < 1\}$). Let us recall that $S_\infty = \{z \in \mathbb{C}\colon |z| = 1\}$ is called the circle at infinity (see Section 5.1).

Let $f\colon M^2 \to M^2$ be an orientation preserving A-diffeomorphism of a closed orientable hyperbolic surface M^2. Suppose that the non-wandering set $NW(f)$ contains a one-dimensional attractor Λ. For the basic set Λ let $\pi^{-1}(\Lambda) = \overline{\Lambda}$. If $x \in \Lambda$ then let $\overline{x} \in \overline{\Lambda}$ denote the point in the preimage $\pi^{-1}(x)$. Let $\delta \in \{u, s\}$ and $\nu \in \{+, -\}$. Denote by $w^\delta_{\overline{x}}$ the curve on Δ such that $\pi(w^\delta_{\overline{x}}) = W^\delta_x$. For points $\overline{y}, \overline{z} \in w^\delta_{\overline{x}}$, ($\overline{y} \neq \overline{z}$) let $[\overline{y}, \overline{z}]^\delta$,

$[\overline{y}, \overline{z})^\delta$, $(\overline{y}, \overline{z}]^\delta$, $(\overline{y}, \overline{z})^\delta$ denote the connected arcs on the manifold $w^\delta_{\overline{x}}$ with the boundary points $\overline{y}$, $\overline{z}$.

Definition 8.1 *A nontrivial basic set* Λ *of an A-diffeomorphism* $f\colon M^2 \to M^2$ *is said to be* widely disposed on the manifold M^2 *if for every point* $x \in \Lambda$ *every simple closed curve formed by the arcs* $[x, y]^u$, $[x, y]^s$ $(y \in W^u_x \cap W^s_x$, $y \neq x)$ *is not contractible on* M^2.

Let Λ be a widely disposed basic set of an A-diffeomorphism $f\colon M^2 \to$ $\to M^2$ and $x \in \Lambda$. If $t \in \mathbb{R}$ is a parameter on the curve W^δ_x such that $W^\delta_x(0) = x$ then $w^\delta_{\overline{x}}(t)$ is the point on $w^\delta_{\overline{x}}$ such that $\pi(w^\delta_{\overline{x}}(t)) = W^\delta_x(t)$ and $w^{\delta+}_{\overline{x}}$, $w^{\delta-}_{\overline{x}}$ are the connected components of the curve $w^\delta_{\overline{x}} \setminus \overline{x}$ for $t > 0$, $t < 0$ respectively.

Recall that we say that a curve $w^{\delta\nu}_{\overline{x}}$ $(\nu \in \{+, -\})$ has the asymptotic direction $\delta^\nu_{\overline{x}}$ as $t \to \nu\infty$, if the set $\operatorname{clos}(w^{\delta\nu}_{\overline{x}}) \setminus w^{\delta\nu}_{\overline{x}}$ consists of the point $\overline{x}$ and the point $\delta^\nu_{\overline{x}}$ which belongs to S_∞.

Let Λ_* be a periodic component of Λ.

Definition 8.2 *A simple closed curve* $C_{\Lambda_*} \subset \operatorname{int} M^2$ *is called a* quasi-transversal of the periodic component Λ_*, *if*:

1) C_{Λ_*} *is the union of arcs* $C^u = [z, y]^u$ *and* $C^s = [y, z]^s$ *for some points* $z, y \in \Lambda_*$;
2) $(z, y)^u \cap \Lambda \neq \varnothing$ *and* $(y, z)^s \cap \Lambda_* \neq \varnothing$;
3) *the index of the intersection* $W^u_z \cap W^s_z$ *is the same at the points* z, y, *see Fig.* 8.1.

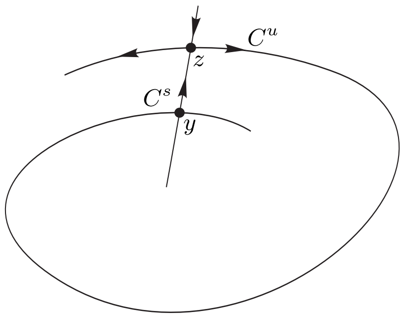

Figure 8.1. A quasi-transversal

Lemma 8.1 *Every periodic component Λ_* of widely disposed basic set Λ of an A-diffeomorphism $f\colon M^2 \to M^2$ has a quasi-transversal.*

Proof. Let $x \in \Lambda_*$. By Lemma 1.24, at least one of the connected components of each set $W_x^s \setminus x$ and $W_x^u \setminus x$ contains a set which is dense in Λ_*. To be definite let W_x^{s+} and W_x^{u+} be such components. Then W_x^{u+} intersects W_x^{s+} at countably many points $x_1 = W_x^{u+}(t_1), \ldots, x_m = W_x^{u+}(t_m), \ldots$ (we enumerate them as the parameter t increases on W_x^{u+}). Consider two cases: 1) indexes of intersection of W_x^u and W_x^s at the points x and x_1 are the same; 2) indexes of intersection of W_x^u and W_x^s at the points x and x_1 are different.

In the first case let the quasi-transversal be the curve composed of the arcs $[x, x_1]^u$, $[x, x_1]^s$. In the second case there is $N > 1$ such that $x_N \in \in (x, x_1)^s$ and for every $k < N$ the point x_k is not on the arc $(x, x_1)^s$. Then if the indexes of intersection of the points x and x_N are the same (are distinct) then the curve composed of the arcs $[x, x_N]^u$ and $[x, x_N]^s$ ($[x_2, x_N]^u$ and $[x_2, x_N]^s$) is the desired quasi-transversal. □

If a set Λ is widely disposed then a quasi-transversal C_{Λ_*} is not contractible on $\operatorname{int} M^2$. From the properties of a cover and from the properties of the group Γ it follows that the set $\overline{C}_{\Lambda_*} = \pi^{-1}(C_{\Lambda_*})$ consists of countably many curves such that:

1) every curve $c \in \overline{C}_{\Lambda_*}$ is not compact, it has no self-intersections and distinct curves $c, c' \in \overline{C}_{\Lambda_*}$ are disjoint;
2) for every curve $c \in \overline{C}_{\Lambda_*}$ there is an element $\gamma_c \in \Gamma$ (distinct from the identity) for which c is invariant and for every point $\overline{x} \in c$ the arc $(\overline{x}, \gamma_c(\overline{x})) \subset c$ contains no *congruent* points (i.e. belonging to the same Γ-orbit);
3) if M^2 is a surface of negative Euler characteristic then each curve $c \in \overline{C}_{\Lambda_*}$ has two distinct boundary points c^+, c^- which are the fixed points for the elements γ_c. The boundary points on the absolute for two distinct curves of the set $\overline{C}_{\Lambda_*}$ are distinct.

Lemma 8.2 *Let $W_x^{\delta\nu}$ be densely disposed in a periodic component Λ_* of a widely disposed basic set Λ, let C_{Λ_*} be a quasi-transversal for Λ_*, $c \in \overline{C}_{\Lambda_*}$, $c \cap w_{\overline{x}}^{\delta\nu} \neq \varnothing$ and let $\pi(c \cap w_{\overline{x}}^{\delta\nu}) \in (\operatorname{int} C^u \cup \operatorname{int} C^s)$. Then the intersection $c \cap w_{\overline{x}}^{\delta\nu}$ is the unique point.*

Proof. To be definite let $\delta = u$ and $\nu = +$. We now show that the intersection $c \cap w_{\overline{x}}^{u+}$ is a unique point $\overline{x}_c$.

Suppose the contrary, then there are points $\overline{x}_c$, $\overline{x}_* \in (w^{u+}_{\overline{x}} \cap c)$ such that $(\overline{x}_c, \overline{x}_*)^u \cap c = \varnothing$. Let $\overline{C}^s_{\overline{x}_c}$ be a lift of the arc C^s which contains the point $\overline{x}_c$. Denote by $\overline{z}$, $\overline{y}$ the boundary points of the arc $\overline{C}^s_{\overline{x}_c}$ which belong to the pre-images of the points z, y respectively. Denote by $\overline{C}^u_{\overline{x}_c}$ the lift of the arc C^u such that $\overline{z}$ is its boundary point. Let $\overline{y}'$ be the other boundary point of the arc $\overline{C}^u_{\overline{x}_c}$. Then the point $\overline{y}'$ is congruent to the point $\overline{y}$ by means of some element γ of the group Γ ($\overline{y}' = \gamma(\overline{y})$), on the arc $\overline{C}^s_{\overline{x}_c} \cup \overline{C}^u_{\overline{x}_c}$ there are no congruent points except $\overline{y}$, $\overline{y}'$ and $c = \bigcup\limits_{k\in\mathbb{Z}} \gamma^k(\overline{C}^s_{\overline{x}_c} \cup \overline{C}^u_{\overline{x}_c})$.

Denote by $\overline{C}^s_{\overline{x}_*}$ the lift of the arc C^s which contains the point $\overline{x}_*$. Then there is an integer $n \in \mathbb{Z}$ such that $\overline{C}^s_{\overline{x}_*} = \gamma^n(\overline{C}^s_{\overline{x}_c})$. Consider four possible cases: 1) $n = 0$; 2) $n = +1$; 3) $n = -1$; 4) $|n| > 1$. We now show that in each of these cases we come to a contradiction.

In the case 1) ($n = 0$) we get that the union $[\overline{x}_c, \overline{x}_*]^u \cup [\overline{x}_*, \overline{x}_c]^s$ is a simple closed curve (see Figure 8.2) and therefore the image of this curve on M^2 by the projection π is a simple closed contractible curve and this contradicts the fact that the set Λ_* is widely disposed.

In the case 2) ($n = 1$) the union $[\overline{x}_c, \overline{x}_*]^u \cup [\overline{x}_*, \overline{y}']^s \cup [\overline{y}', \overline{z}]^u \cup [\overline{z}, \overline{x}_c]^s$ is a simple closed curve which bounds an open disk D on Δ. Consider two subcases (see Figure 8.2):

2a) $w^u_{\overline{y}',\eta} \cap D \neq \varnothing$ for every $\eta > 0$;

2b) $w^u_{\overline{z},\eta} \cap D \neq \varnothing$ for every $\eta > 0$.

In the case 2a) we now show that the domain D contains no points congruent to the point $\overline{z}$. Assume the contrary, then there is an element $\beta \in \Gamma$ distinct from the identity and such that the arc $\beta(\overline{z}, \overline{y})^s$ intersects the boundary of D. But this is impossible because on M^2 the image of the curves $(\overline{x}_c, \overline{x}_*)^u$, $(\overline{y}', \overline{z})^u$ by the projection π contains no points of the arc C^s.

By the assumption, the connected component l^u of the set $w^u_{\overline{y}'} \setminus \overline{y}'$ disjoint from the point $\overline{z}$ intersects the disk D. Since the image of the curve l^u by the projection π contains a sequence of points converging to the point z the curve l^u intersects either the arc $(\overline{z}, \overline{y})^s$ or the arc $\gamma((\overline{z}, \overline{y})^s)$ and we come to contradiction to the fact that the set Λ_* is widely disposed in the same way as in case 1).

The case 2b) is similar to the case 2a). In the case 3) ($n = -1$) the contradiction arises in the same way as in the case 2).

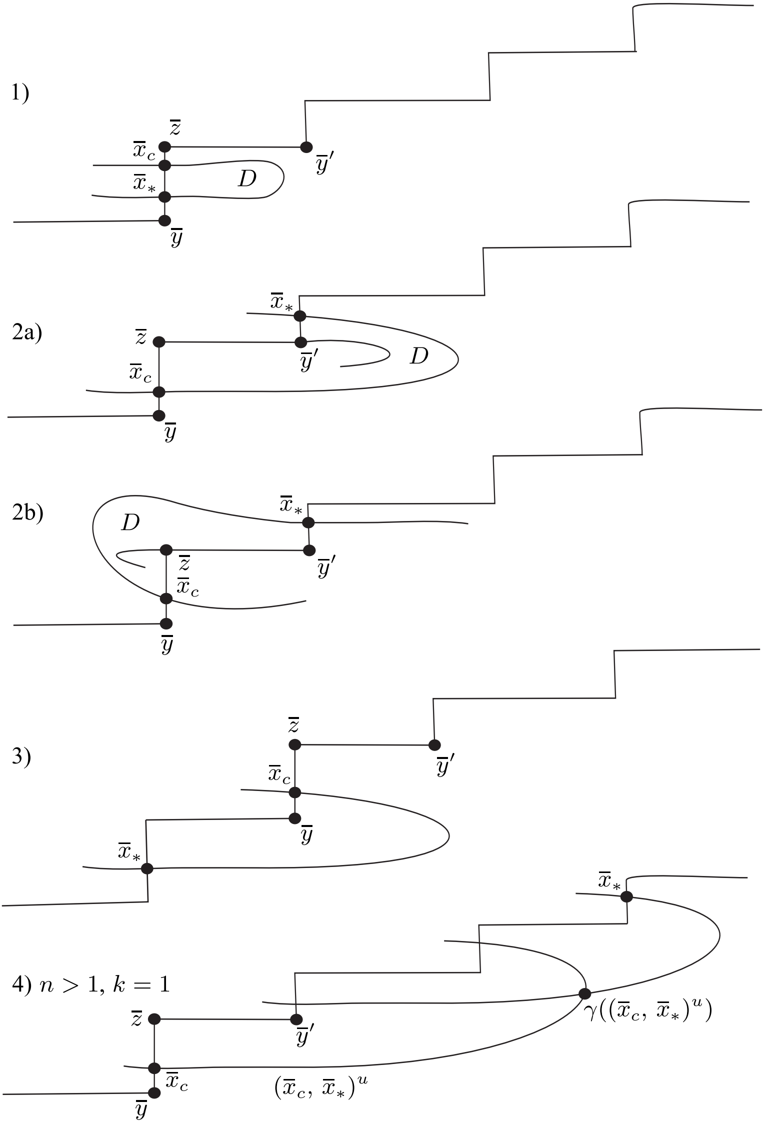

Figure 8.2. An illustration to the proof of Lemma 8.2

In the case 4) ($|n| > 1$) the intersection $\gamma^k((\overline{x}_m, \overline{x}_*)^u) \cap (\overline{x}_m, \overline{x}_*)^u$, where $k = 1$ if $n > 1$, and $k = -1$ if $n < -1$, contains at least one point (see Figure 8.2) and this is impossible. □

Corollary 8.1 *If a component $W_x^{\delta\nu}$, $x \in \Lambda_*$ is densely situated in Λ_* then $w_{\overline{x}}^{\delta\nu}$ intersects countably many curves of $\overline{C}_{\Lambda_*}$.*

Definition 8.3 *Let $x \in \Lambda_*$ and a component $W_x^{\delta\nu}$ be densely situated in the periodic component Λ_*. We say a sequence of curves $\mathscr{C}_{\overline{x}}^{\delta\nu} = \{c_1, \ldots, c_m, \ldots\} \subset \subset \overline{C}_{\Lambda_*}$ to be* directive for the curve *$w_{\overline{x}}^{\delta\nu}$ if there is a subsequence of parameters $\{t_1, \ldots, t_m, \ldots\}$ such that $|t_1| < \ldots, |t_m|, \ldots$ and $w_{\overline{x}}^{\delta\nu}(t_m) \in c_m$.*

Lemma 8.3 *Let for $x \in \Lambda_*$ the component $W_x^{\delta\nu}$ be densely situated in Λ_* and $\mathscr{C}_{\overline{x}}^{\delta\nu} = \{c_1, \ldots, c_m, \ldots\}$ be a directive sequence for $w_x^{\delta\nu}$. Let A_m be the closure of the part of the set $\mathrm{cl}(\Delta) \setminus \mathrm{cl}(c_m)$ which contains $\mathrm{cl}(c_k)$ for every $k > m$. Then the topological limits $\mathrm{Lim}\,\mathscr{C}_{\overline{x}}^{\delta\nu}$ and $\mathrm{Lim}\,A_m$ coincide and they consist of a unique point belonging to S_∞.*

Proof. Since $A_1 \supset A_2 \supset \ldots$ it follows that the set $A = \bigcap_{m=1}^{\infty} A_m$ is a nonempty connected compact subset of $\mathrm{cl}(\Delta)$. Hence, A is the topological limit of the sets $A_1, A_2, \ldots$.

We now show that $A \subset S_\infty$. Assume the contrary: there is a point $\overline{a} \in \in A \cap \Delta$. Then for $\varepsilon > 0$ arbitrary small the ε-neighborhood of the point $\overline{a}$ intersects countably many mutually distinct congruent curves c_k and therefore the quasitransversal C_{Λ_*} intersects arbitrary small neighborhood of the point $a = \pi(\overline{a})$ by countably many distinct arcs, which is impossible.

We now show that the set A consists of a unique point belonging to S_∞. Assume the contrary, then the set A is an arc belonging to S_∞. The set of the rational points is dense on the absolute S_∞ and therefore there is an element $\gamma \in \Gamma$ such that $\gamma(c_1^+)$ is an interior point of the arc A. But then there is $N > 1$ such that the curve c_N intersects (but not coincides with) the curve $\gamma(c_1)$ and that is impossible.

So the set A is the unique point which is the limit on S_∞ of both sequences c_m^+ and c_m^-. The definition of the topological limit implies that this point belongs to $\mathrm{Lim}\,c_m$. On the other hand since $c_m \subset A_m$ we have $\mathrm{Lim}\,c_m \subset \subset \mathrm{Lim}\,A_m$ and therefore $\mathrm{Lim}\,c_m$ consists of a unique point. □

Theorem 8.1 *Let for $x \in \Lambda_*$ the component $W_x^{\delta\nu}$ be densely situated in Λ_*. Then $w_{\overline{x}}^{\delta\nu}$ has an irrational asymptotic direction $\delta_{\overline{x}}^{\nu}$ for $t \to \nu\infty$.*

Proof. To be definite we consider the case $\delta = s$, $\nu = +$. Let $\mathscr{C}^{s+}_{\overline{x}} = \{c_1, \ldots, c_m, \ldots\}$ be the directive sequence for the curve $w^{s+}_{\overline{x}}$. Let $w_m = \mathrm{cl}(w^{s+}_{\overline{x}}) \cap A_m$, where A_m is the closure of the part of the set $\mathrm{cl}(\Delta) \setminus \mathrm{cl}(c_m)$ which contains $\mathrm{cl}(c_k)$ for every $k > m$. We see that the set $W = \bigcap_{m=0}^{\infty} w_m$ is a nonempty connected compact subset of $\mathrm{cl}(\Delta)$. By construction $\mathrm{cl}(w^{s+}_{\overline{x}}) \setminus w^{s+}_{\overline{x}} = W \cup \overline{x}$. Hence, W is the topological limit of the sets $w_1, w_2, \ldots$. On the other hand $\mathrm{Lim}\, w_m \subset \mathrm{Lim}\, A_m$ and since $\mathrm{Lim}\, A_m = \mathrm{Lim}\, c_m$ consists of a unique point (see Lemma 8.3) we have $W = \mathrm{Lim}\, c_m$. Therefore, the curve $w^{s+}_{\overline{x}}$ has an asymptotic direction which we denote by $s^{+}_{\overline{x}}$.

We now show that $s^{+}_{\overline{x}}$ is an irrational point on the absolute. Suppose the contrary. The definition of a rational point implies that there is an element $\gamma \in \Gamma$ such that the point $s^{+}_{\overline{x}}$ is its sink fixed point and any element of the group Γ, distinct from the identity and for which the point $s^{+}_{\overline{x}}$ is fixed, is a degree of the element γ. Let $w_1 = \gamma(w^{s+}_{\overline{x}})$, $w_2 = \gamma^2(w^{s+}_{\overline{x}})$. Since the curves $w^{s+}_{\overline{x}}$, w_1, w_2 have the common boundary point $s^{+}_{\overline{x}}$ on the absolute there is $\widetilde{n} > 1$ such that the curve $c_{\widetilde{n}} \in \mathscr{C}^{s+}_{\overline{x}}$ intersects the curves $w^{s+}_{\overline{x}}$, w_1, w_2. Denote the intersection points by $\overline{b}_+$, $\overline{b}_1$, $\overline{b}_2$ respectively (see Figure 8.3). The properties of the element γ imply that the domain D_1 on Δ bounded by the curves $w^{s+}_{\overline{x}}$, w_1 and the arc $(\overline{b}_+, \overline{b}_1) \subset c_{\widetilde{n}}$, as well as the domain D_2 bounded by the curves w_1, w_2 and the arc $(\overline{b}_1, \overline{b}_2) \subset c_{\widetilde{n}}$ (we mean D_1, D_2 to be the regions that do not contain the curve c_1) both contain no curves congruent to the curve $w^{s+}_{\overline{x}}$ and such that the point $s^{+}_{\overline{x}}$ is the boundary point for them.

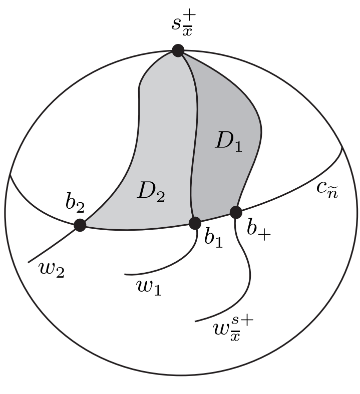

Figure 8.3. An illustration to Theorem 8.1

Since the image of the curve $w_{\overline{x}}^{s+}$ on M^2 by the projection π contains a dense in Λ set we have that at least one of the arcs $(\overline{b}_+, \overline{b}_1) \subset c_{\widetilde{n}}$, $(\overline{b}_1, \overline{b}_2) \subset$ $\subset c_{\widetilde{n}}$ has a point $\overline{b}_*$ such that $\pi(\overline{b}_*) \in W_x^{s+}$. But then there is a curve w_* congruent to $w_{\overline{x}}^{s+}$ which passes through the point $\overline{b}_*$. Since the curve w_* intersects the curve $c_{\widetilde{n}}$ at one point at most, the boundary point of the curve w_* which belongs to S_∞ coincides with the point $s_{\overline{x}}^+$ and this is impossible. $\square$

The technique of the proof of Theorem 8.1 implies the following propositions.

Corollary 8.2 *Let $x \in \Lambda_*$, $W_x^{\delta\nu}$ is densely situated in Λ_* and $\mathscr{C}_{\overline{x}}^{\delta\nu}$ is the directive sequence. Then $\delta_{\overline{x}}^{\nu} = \operatorname{Lim} \mathscr{C}_{\overline{x}}^{\delta\nu}$.*

Corollary 8.3 *Let $x \in \Lambda_*$ is a δ-dense point in Λ_* (see Definition 1.3). Then $\delta_{\overline{x}}^+ \neq \delta_{\overline{x}}^-$.*

As Λ is attractor then each point $x \in \Lambda_*$ is u-dense and the curve $w_{\overline{x}}^u$ $(\pi(\overline{x}) = x)$ has two distinct boundary points (asymptotic directions) $u_{\overline{x}}^+$, $u_{\overline{x}}^-$ which are irrational points of the absolute S_∞. Denote by P_{Λ_*} the set of all s-boundary periodic points from Λ_*. Let $\overline{P}_{\Lambda_*} = \pi^{-1}(P_{\Lambda_*})$. If $p \in P_{\Lambda_*}$ then let $W_p^{s\varnothing}$ $(w_{\overline{p}}^{s\varnothing})$ denote the connected component of the set $W_p^s \setminus p$ disjoint from Λ_* (the connected component of the set $w_{\overline{p}}^s \setminus \overline{p}$ such that $\pi(w_{\overline{p}}^{s\varnothing}) = W_p^{s\varnothing}$) and let $W_p^{s\infty}$ $(w_{\overline{p}}^{s\infty})$ denote the connected component of the set $W_p^s \setminus p$ which is dense in Λ_* (the connected component of the set $w_{\overline{x}}^s \setminus \overline{x}$ such that $\pi(w_{\overline{p}}^{s\infty}) =$ $= W_p^{s\infty}$). Let $W_{P_{\Lambda_*}}^{s\infty} = \bigcup\limits_{p \in P_{\Lambda_*}} W_p^{s\infty}$ $(w_{\overline{P}_{\Lambda_*}}^{s\infty} = \bigcup\limits_{\overline{p} \in \overline{P}_{\Lambda_*}} w_{\overline{p}}^{s\infty})$ and $W_{P_{\Lambda_*}}^u =$ $= \bigcup\limits_{p \in P_{\Lambda_*}} W_p^u$ $(w_{\overline{P}_{\Lambda_*}}^u = \bigcup\limits_{\overline{p} \in \overline{P}_{\Lambda_*}} w_{\overline{p}}^u)$. It follows from our construction that every point $x \in (\Lambda_* \setminus W_{P_{\Lambda_*}}^{s\infty})$ is s-dense and the curve $w_{\overline{x}}^s$ $(\pi(\overline{x}) = x)$ has two distinct boundary points (asymptotic directions) $s_{\overline{x}}^+$, $s_{\overline{x}}^-$ which are irrational points on the absolute. Finally, for every point $p \in P_{\Lambda_*}$ the curve $w_{\overline{p}}^{s\infty}$ has an asymptotic direction $s_{\overline{p}}^\infty$ which is an irrational point on the absolute.

It follows from Theorems 1.19 and 1.20 that for the 1-dimensional attractor Λ the accessible from inside boundary of the set $M^2 \setminus \Lambda$ consists of the unstable manifolds of all s-boundary points (and there are finitely many of them).

Definition 8.4 *A* bunch *b of the attractor Λ is the union of the maximal number r_b of the unstable manifolds $W_{p_1}^u, \ldots, W_{p_{r_b}}^u$ of the s-boundary points $p_1, \ldots, p_{r_b}$ of the set Λ accessible from some (the same for all) point $x \in (M^2 \setminus \Lambda)$. The number r_b is said to be the degree of the bunch.*

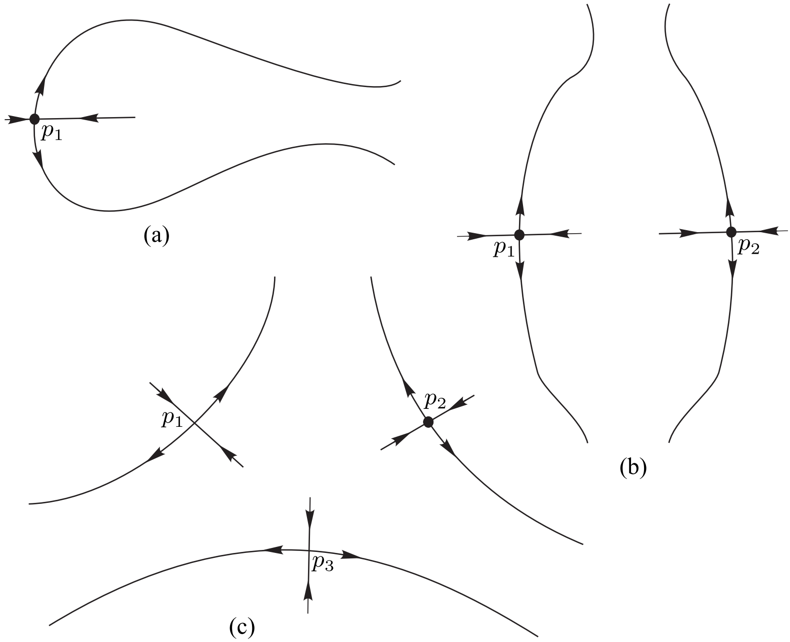

Figure 8.4. Bunches of 1-dimensional attractors

Fig. 8.4 (a), (b), (c) shows bunches of degrees 1, 2, 3 respectively.

Lemma 8.4 *Let Λ_* be a periodic component of one-dimensional widely disposed attractor of a diffeomorphism $f\colon M^2 \to M^2$. Then*

1) *for every point $\overline{x} \in \overline{\Lambda}_*$ the intersection $\mathrm{cl}(w^u_{\overline{x}}) \cap \mathrm{cl}(w^s_{\overline{x}})$ consists of a unique point $\overline{x}$;*
2) *if $w^s_{\overline{x}} \cap w^s_{\overline{y}} = \varnothing$ for some points $\overline{x}, \overline{y} \in \Lambda_*$, then $\mathrm{cl}(w^s_{\overline{x}}) \cap \mathrm{cl}(w^s_{\overline{y}}) = \varnothing$;*
3) *if $w^u_{\overline{x}} \cap w^u_{\overline{y}} = \varnothing$ for some points $\overline{x} \in \overline{\Lambda}_*$, $\overline{y} \in \left(\overline{\Lambda}_* \setminus w^u_{\overline{P}_{\Lambda_*}}\right)$, then $\mathrm{cl}(w^u_{\overline{x}}) \cap$ $\cap\, \mathrm{cl}(w^u_{\overline{y}}) = \varnothing$;*
4) *for every point $\overline{p} \in \overline{P}_{\Lambda_*}$ there are two points $\overline{p}_+, \overline{p}_- \in \overline{P}_{\Lambda_*}$ such that $w^u_{\overline{p}_+} \cap w^u_{\overline{p}} = \varnothing$, $w^u_{\overline{p}_-} \cap w^u_{\overline{p}} = \varnothing$, $\mathrm{cl}(w^u_{\overline{p}_+}) \cap \mathrm{cl}(w^u_{\overline{p}}) = u^+_{\overline{p}}$, $\mathrm{cl}(w^u_{\overline{p}_-}) \cap \mathrm{cl}(w^u_{\overline{p}}) = u^-_{\overline{p}}$ and the points $p = \pi(\overline{p})$, $p_- = \pi(\overline{p}_-)$, $p_+ =$ $= \pi(\overline{p}_+)$ are the s-boundary points of the same bunch of the attractor Λ_**

(*see Figure* 8.5). *Moreover* $\overline{p}_- = \overline{p}_+$ *iff* p_- *and* p_+ *belongs to a bunch of degree* 2;

5) *if* $w^u_{\overline{x}} \cap w^s_{\overline{y}} = \varnothing$ *for some points* $\overline{x}, \overline{y} \in \overline{\Lambda}_*$ *then* $\mathrm{cl}(w^u_{\overline{x}}) \cap \mathrm{cl}(w^s_{\overline{y}}) = \varnothing$.

Proof. (1) Let $\overline{x} \in \overline{\Lambda}_*$. Let us prove firstly that $w^u_{\overline{x}} \cap w^s_{\overline{x}} = \overline{x}$. Show that the conclusion holds if $x = \pi(\overline{x})$ is a periodic point. Without loss of generality we assume that x is a fixed point of f (otherwise the same reasoning applies for the diffeomorphism $f^{per(x)}$). Let $\overline{f}\colon \Delta \to \Delta$ be a covering map of the diffeomorphism f for which the point $\overline{x}$ is fixed. Suppose the contrary: there is a point $\overline{y} \in (w^s_{\overline{x}} \cap w^u_{\overline{x}})$ distinct from the point $\overline{x}$. Then the point $\overline{y}$ is a homoclinic point of the diffeomorphism $\overline{f}$ and therefore for every neighborhood $U_{\overline{x}}$ of the point $\overline{x}$ there is a sequence $t_1, t_2, \dots, t_m, \dots$ of the values of parameter t such that $\lim\limits_{m\to\infty} t_m = +\infty$ and the points $w^u_{\overline{x}}(t_m)$, $m \in \mathbb{N}$ are in the neighborhood $U_{\overline{x}}$. This contradicts the fact that each connected component of the curve $w^u_{\overline{x}} \setminus \overline{x}$ has an asymptotic direction. Let now $\overline{x}$ be an arbitrary point from Λ_* there is a point $\overline{y} \in (w^s_{\overline{x}} \cap w^u_{\overline{x}})$. Then there is a point $\overline{p}$ on the preimage of the periodic u-dense point p of the diffeomorphism f such that $w^s_{\overline{p}} \cap w^u_{\overline{p}}$ consist of more than one point (it follows from the facts that the periodic points are dense in the non-wandering set, the stable manifolds are C^1-close on compact sets and from the properties of the cover). And this contradicts the foregoing.

Suppose now the intersection $\mathrm{cl}(w^u_{\overline{x}}) \cap \mathrm{cl}(w^s_{\overline{x}})$ to contain points distinct from $\overline{x}$. To be definite let $u^+_{\overline{x}} = s^+_{\overline{x}} = \mu$. Denote by D the domain bounded by the curves $\mathrm{cl}(w^{u+}_{\overline{x}})$, $\mathrm{cl}(w^{s+}_{\overline{x}})$ and such that D is disjoint from the point $u^-_{\overline{x}}$. Since x is an u-dense point there is a point $\overline{z} \in w^{s+}_{\overline{x}}$ such that the curve $w^u_{\overline{z}}$ is congruent to the curve $w^u_{\overline{x}}$. Then $\mu \in \mathrm{cl}(w^u_{\overline{z}})$. Thus the point μ is a fixed point of some element of the group Γ and this is in contradiction with its irrationality.

The proofs of the propositions (2) and (3) are similar to one another, so we prove only (3). Suppose the contrary: $w^u_{\overline{x}} \cap w^u_{\overline{y}} = \varnothing$ for some points $\overline{x} \in \overline{\Lambda}_*$, $\overline{y} \in \left(\overline{\Lambda}_* \setminus w^u_{\overline{P}_{\Lambda_*}}\right)$ and $\mathrm{cl}(w^u_{\overline{x}}) \cap \mathrm{cl}(w^u_{\overline{y}}) \neq \varnothing$. To be definite let $u^+_{\overline{x}} = u^+_{\overline{y}}$. Then there is a curve $c \in \overline{C}_{\Lambda_*}$ such that $w^u_{\overline{x}} \cap c = \overline{a}$ and $w^u_{\overline{y}} \cap c = \overline{b}$. Then by item (3) of Theorem 1.19 on the arc $(\overline{a}, \overline{b}) \subset c$ there is a point $\overline{z} \in \overline{\Lambda}_*$ such that the curve $w^u_{\overline{z}}$ is congruent to the curve $w^u_{\overline{x}}$. By construction $u^+_{\overline{x}} \in \mathrm{cl}(w^u_{\overline{z}})$ and this is in contradiction with the irrationality of $u^+_{\overline{x}}$.

(4) Let $\check{C}^s = C^s \cap \Lambda_*$. By Lemma 1.23 the set $\check{C}^s$ is a perfect nowhere dense on C^s set. From Lemma 8.1 it follows that there are curves $c_{\pm 1}, c_{\pm 2}, \ldots \in \overline{C}_{\Lambda_*}$, each of which c_m intersects $w^u_{\overline{p}}$ at an unique

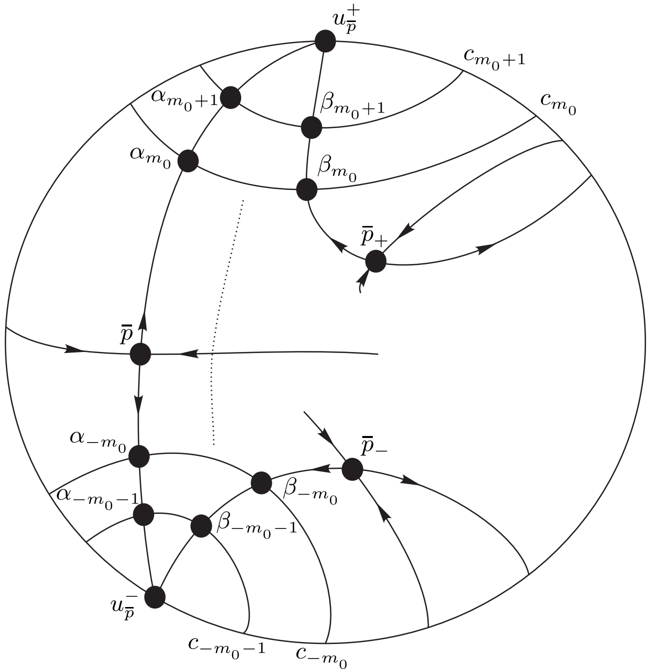

Figure 8.5. An illustration to the proof of Lemma 8.4 (4)

point $\alpha_m = w^u_{\overline{x}}(t_m)$, and the point $u^+_{\overline{p}}$ $(u^-_{\overline{p}})$ is the limit point of the sequence $\{\alpha_m\}$ for $m \to +\infty$ $(m \to -\infty)$. Let $\beta_m \in \overline{\Lambda}_*$ be the point on c_m such that $(\alpha_m, \beta_m) \cap \overline{\Lambda}_* = \varnothing$. Then (a_m, b_m), where $a_m = \pi(\alpha_m)$, $b_m = \pi(\beta_m)$ is an adjacent interval of the set $\check{C}^s$. Since $w^u_{\overline{p}}$ contains no congruent points $[a_m, b_m] \cap [a_k, b_k] = \varnothing$ for $m \neq k$ and $\lim\limits_{m\to\pm\infty} \operatorname{diam}[a_m, b_m] = 0$. From C^1-closeness of the unstable manifolds and from the properties of a covering it follows that there is $\eta > 0$ such that if $\operatorname{diam}[a_m, b_m] < \eta$ then the curve $w^u_{\beta_m}$ passing through the point β_m intersects the curve c_{m+1} for $m \geqslant 0$ and it intersects the curve c_{m-1} for $m < 0$.

Pick $m_0 > 0$ such that $\operatorname{diam}[a_m, b_m]$ is less then η for all $m > m_0$ $(m < -m_0)$. Then the curve $w^u_{\beta_{m_0}}$ $\left(w^u_{\beta_{-m_0}}\right)$ intersects all the curves c_m for $m > m_0$ $(m < -m_0)$ and therefore the point $u^+_{\overline{x}}$ $(u^-_{\overline{x}})$ is its boundary point.

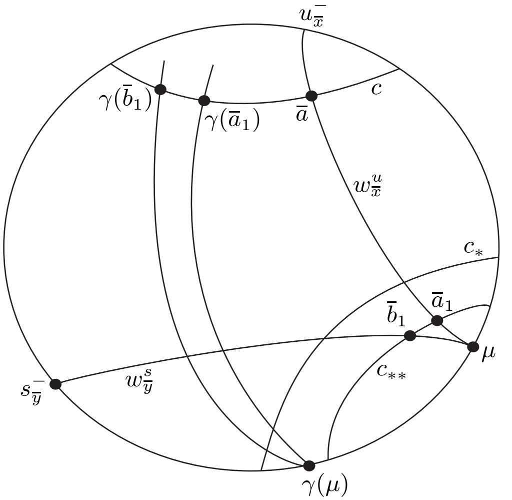

Figure 8.6. An illustration to the proof of Lemma 8.4 (5)

From Theorem 1.19 (2) it follows that the curve $w^u_{\beta_{m_0}}$ $\left(w^u_{\beta_{-m_0}}\right)$ contains the preimage of the s-boundary point p_+ (p_-) (see Figure 8.5). By construction the points p_+ (p_-) belong to the same bunch as the point p. Moreover it is easy to check that $\overline{p}_- = \overline{p}_+$ iff p_- and p_+ belongs to a bunch of degree 2.

(5) Let $w^u_{\overline{x}} \cap w^s_{\overline{y}} = \varnothing$ for some points $\overline{x}, \overline{y} \in \overline{\Lambda}_*$. Suppose $\mathrm{cl}(w^u_{\overline{x}}) \cap$ $\cap\, \mathrm{cl}(w^s_{\overline{y}}) \neq \varnothing$. Two subcases are possible: (5a) $\mathrm{cl}(w^u_{\overline{x}}) \setminus w^u_{\overline{x}} = \mathrm{cl}(w^s_{\overline{y}}) \setminus w^s_{\overline{y}}$ and (5b) the curves $w^u_{\overline{x}}$, $w^s_{\overline{y}}$ have a unique common boundary point μ.

In the case (5a) denote by Q the domain bounded by the curves $\mathrm{cl}(w^u_{\overline{x}})$, $\mathrm{cl}(w^s_{\overline{y}})$. Since x is an u-dense point there is a point $\overline{z} \in w^s_{\overline{y}}$ such that the curve $w^u_{\overline{z}}$ is congruent to the curve $w^u_{\overline{x}}$. Then one of the connected components of $w^u_{\overline{z}} \setminus \overline{z}$ belongs to the domain Q and therefore its boundary point on the absolute belongs to $\mathrm{cl}(w^s_{\overline{z}})$ which is in contradiction with item (1) of this lemma.

Consider case (5b). To be definite let $u^+_{\overline{x}} = s^+_{\overline{y}} = \mu$. Pick a curve $c \in \overline{C}_{\Lambda_*}$ such that its boundary points subtend an arc on S_∞ containing the point $u^-_{\overline{x}}$ and containing no other boundary points of the curves $w^u_{\overline{x}}$ and $w^s_{\overline{y}}$. Let $\overline{a}$ be the point of intersection of c and $w^u_{\overline{x}}$. Pick a curve $c_* \in \overline{C}_{\Lambda_*}$ such that $c_* \cap w^u_{\overline{x}} \neq$ $\neq \varnothing$, $c_* \cap w^s_{\overline{y}} \neq \varnothing$ and the boundary points of the curve c_* subtend the arc λ_*

on S_∞ containing the point μ and containing no other boundary points of the curves $w^u_{\overline{x}}$ and $w^s_{\overline{y}}$. Since the image of the curve $w^u_{\overline{x}}$ by the map π is dense in Λ_* there is a curve $c_{**} \in \overline{C}_{\Lambda_*}$ such that $c_{**} \cap w^u_{\overline{x}} = \overline{a}_1$, $c_{**} \cap w^s_{\overline{y}} = \overline{b}_1$ and for some element $\gamma \in \Gamma$, $\gamma(\overline{a}_1) \in c$, $\gamma(\mu)$ belongs to the curve λ_* and $\gamma(w^u_{\overline{x}})$ intersects[1] $w^s_{\overline{y}}$ (see Figure 8.6). Then the curve $\gamma(w^s_{\overline{y}})$ passes through the point $\gamma(\overline{b}_1)$ belonging to the curve c. By construction the curve $\gamma(w^s_{\overline{y}})$ either intersects $w^s_{\overline{y}}$ if y is an s-dense point or it intersects $w^u_{\overline{p}}$ at more than one point if $\overline{y} \in w^s_{\overline{p}}$, $\overline{p} \in \overline{P}_{\Lambda_*}$. Both cases are impossible. □

Let a diffeomorphism $\overline{f} : \Delta \to \Delta$ be a lift of the diffeomorphism $f : M^2 \to \to M^2$, T be a cyclic group generated by f (the elements of the group T are the transformations f^n, $n \in \mathbb{Z}$) and $L(T)$ be the group of liftings of elements from T to Δ (the elements of the group $L(T)$ are homeomorphisms on Δ of the form $\overline{h} = \gamma\overline{f}^n$, where $\gamma \in \Gamma$, $n \in \mathbb{Z}$, $\overline{f}$ is a homeomorphism, covering f, i. e. $f\pi = \pi\overline{f}$. For each $\overline{h} \in L(T)$ we denote by $\overline{h}^*$ the homeomorphism of the absolute S_∞ induced by $\overline{h}$ ($\overline{h}^*$ is an extension of $\overline{h}$ to S_∞, see section 6.1).

Lemma 8.5 *Let Λ_* be a periodic component of one-dimensional widely disposed attractor of a diffeomorphism $f : M^2 \to M^2$. Let $\overline{p} \in \overline{\Lambda}_*$ such that the point $p = \pi(\overline{p})$ is interior periodic point of f and $\overline{h} \in L(T)$ be a diffeomorphism for which the point $\overline{p}$ is fixed. Then the set* $\mathrm{Per}(\overline{h})$ *consists of exactly five points*: $\overline{p}$, $u^+_{\overline{p}}$, $u^-_{\overline{p}}$, $s^+_{\overline{p}}$, $s^-_{\overline{p}}$; *the points $u^+_{\overline{p}}$, $u^-_{\overline{p}}$ are sink and the points $s^+_{\overline{p}}$, $s^-_{\overline{p}}$ are source periodic points*[2].

Proof. Since $\overline{p}$ is a fixed point of the homeomorphism $\overline{h}$ we have $\overline{h}(w^\delta_{\overline{p}}) = = w^\delta_{\overline{p}}$ and consequently the points $s^+_{\overline{p}}$, $s^-_{\overline{p}}$, $u^+_{\overline{p}}$, $u^-_{\overline{p}}$ are periodic points of $\overline{h}$. Since $\mathrm{Per}_{\overline{h}} = \mathrm{Per}_{\overline{h}^2}$ without loss of generality we assume $s^+_{\overline{p}}$, $s^-_{\overline{p}}$, $u^+_{\overline{p}}$, $u^-_{\overline{p}}$ to be fixed points of the homeormophism $\overline{h}$.

Denote by $A_{s^+_{\overline{p}}}$ the subset of $\mathrm{cl}\,\Delta$ bounded by the curve $\mathrm{cl}(w^u_{\overline{p}})$ and the arc $(u^+_{\overline{p}}, u^-_{\overline{p}}) \subset S_\infty$ disjoint from the point $s^-_{\overline{p}}$. We now show that $\alpha(z) = s^+_{\overline{p}}$ for every point $z \in \left(A_{s^+_{\overline{p}}} \setminus \mathrm{cl}\, w^u_{\overline{p}}\right)$.

[1]If $w^u_{\overline{x}}$ contains the point $\overline{p} \in \overline{P}_{\Lambda_*}$ it is possible that there is no such element γ. Then instead of the curve $w^u_{\overline{x}}$ the same reasoning applies to the curve $w^u_{\overline{q}}$, $\overline{q} \in \overline{P}_{\Lambda_*}$, for which μ is one of its boundary points according to item (4).

[2]We say a periodic point $x \in X$ of a homeomorphism $\varphi : X \to X$ to be *sink (source)*, if there is a neighborhood $U(x) \subset X$ of x such that for any point $y \in U(x)$ the sequence $\varphi^{m\cdot\mathrm{per}(x)}(y)$ ($\varphi^{-m\cdot\mathrm{per}(x)}(y)$) converges to x as $m \to +\infty$.

Since W^u_p is dense in Λ_* and since the unstable manifolds are C^1-close on compact sets the properties of a covering give us that there is a point $\overline{z}_- \in w^{s+}_{\overline{p}}$ such that $z \in A_-$ where A_- is the subset of $\mathrm{cl}\,\Delta$ bounded by the curve $\mathrm{cl}(w^u_{\overline{z}_-})$ and by the arc $(u^+_{\overline{z}_-}, u^-_{\overline{z}_-}) \subset S_\infty$ such that the point $s^-_{\overline{p}}$ does not belong to $(u^+_{\overline{z}_-}, u^-_{\overline{z}_-})$. Let $\overline{z}_{-m} = \overline{h}^{-m}(\overline{z}_-)$, $m \in \mathbb{N}$. Denote by A_{-m} the subset $\mathrm{cl}\,\Delta$ bounded by the curve $\mathrm{cl}(w^u_{\overline{z}_{-m}})$ and the arc $(u^+_{\overline{z}_{-m}}, u^-_{\overline{z}_{-m}}) \subset S_\infty$ such that the point $s^-_{\overline{p}}$ does not belong to $(u^+_{\overline{z}_{-m}}, u^-_{\overline{z}_{-m}})$. Then to prove the lemma it suffices to show that the set $\mathscr{A}_- = \bigcap\limits_{m=1}^{\infty} A_{-m}$ consists of the unique point $s^+_{\overline{p}}$.

By construction $A_{-m} = \overline{h}^{-m}(A_-)$ and $A_{-1} \supset A_{-2} \supset \ldots \supset A_{-m} \supset \ldots$. Clearly, the set $\mathscr{A}_-$ is connected. By construction, it contains the point $s^+_{\overline{p}}$. We now show that $\mathscr{A}_- \subset S_\infty$. Suppose the contrary: there is a point $\overline{y} \in \mathscr{A}_-$ which belongs to Δ. Since Λ_* is a closed set the point $\pi(\overline{y})$ belongs to Λ_*. Then the curve $w^u_{\overline{y}}$ belongs to $\mathscr{A}_-$, it is disjoint from $w^s_{\overline{p}}$ and one of boundary points of $w^u_{\overline{y}}$ coincides with the point $s^+_{\overline{p}}$. But this contradicts to the item (5) of Lemma 8.4. We now show that $\mathscr{A}_-$ is not an arc and consequently it consists of a unique point $s^+_{\overline{p}}$. Suppose the contrary: $\mathscr{A}_-$ is an arc on S_∞. Since the Γ-orbit of any point from S_∞ is dense in S_∞ there is an element $\gamma \in \Gamma$ such that $\gamma(u^+_{\overline{z}_0}) \in \mathrm{int}\,\mathscr{A}_-$. Then the curve $\gamma(w^u_{\overline{z}_0})$ intersects the curves $w^u_{\overline{z}_m}$ while being distinct from them. But the curves $w^u_{\overline{z}_m}$ accumulate to the arc $\mathscr{A}_-$, which contradicts the fact that the unstable manifolds of the points of a basic set either coincide or they are disjoint.

We now show that $\omega(z) \subset \mathrm{cl}\, w^u_{\overline{p}}$ for every point $z \in \left(A_{s^+_{\overline{p}}} \setminus \mathrm{cl}\, w^u_{\overline{p}}\right)$. From the foregoing the topological limit of the curves $\mathrm{cl}(w^u_{\overline{z}_m})$ is the point $s^+_{\overline{p}}$. Then there is a point $\overline{z}_+ \in w^{s+}_{\overline{p}}$ such that $z \in A_+$ where A_+ is the subset of $\mathrm{cl}\,\Delta$ bounded by the curves $\mathrm{cl}(w^u_{\overline{z}_+})$, $\mathrm{cl}(w^u_{\overline{p}})$ and the arcs $\left(u^+_{\overline{z}_+}, u^+_{\overline{p}}\right)$, $\left(u^-_{\overline{z}_+, u^-_{\overline{p}}}\right) \subset$ $\subset S_\infty$ which do not contain the point $s^-_{\overline{p}}$. Let $\overline{z}_m = \overline{h}^m(\overline{z}_+)$, $m \in \mathbb{N}$. Denote by A_m the subset of $\mathrm{cl}\,\Delta$ bounded by the curves $\mathrm{cl}(w^u_{\overline{z}_m})$, $\mathrm{cl}(w^u_{\overline{p}})$ and the arcs $\left(u^+_{\overline{z}_m}, u^+_{\overline{p}}\right)$, $\left(u^-_{\overline{z}_m, u^-_{\overline{p}}}\right) \subset S_\infty$ which do not contain the point $s^-_{\overline{p}}$. By construction $A_m = \overline{h}^m(A_+)$. Since $\overline{p} = \overline{h}^m(\overline{z}_m)$ we have $\mathscr{A}_- = w^u_{\overline{p}}$ and therefore $\omega(z) \subset \mathrm{cl}\, w^u_{\overline{p}}$ for every point $z \in \left(A_{s^+_{\overline{p}}} \setminus \mathrm{cl}\, w^u_{\overline{p}}\right)$.

Analogously one proves that $\alpha(z) = s^{-}_{\overline{p}}$ and $\omega(z) \subset \operatorname{cl} w^{u}_{\overline{p}}$ for every point $z \in \left(A_{s^{-}_{\overline{p}}} \setminus \operatorname{cl} w^{u}_{\overline{p}}\right)$ where $A_{s^{-}_{\overline{p}}}$ is the subset of $\operatorname{cl}\Delta$ bounded by the curve $\operatorname{cl}(w^{u}_{\overline{p}})$ and the arc $(u^{+}_{\overline{p}}, u^{-}_{\overline{p}}) \subset S_{\infty}$ which do not contain the point $s^{+}_{\overline{p}}$. In the same way one shows that $\omega(z) \cap w^{u}_{\overline{p}} = \varnothing$ for every point $z \in (\operatorname{cl}\Delta \setminus \operatorname{cl} w^{s}_{\overline{p}})$ and from that the conclusion of the lemma follows. □

8.2. Geodesic frameworks of widely disposed attractors

We now assume that the automorphism $\overline{f}_{*}$ induced by the covering diffeomorphism $\overline{f}$ is hyperbolic.

By l^{u} (respectively, l^{s}) we denote the geodesic on Δ for which the points $u^{+}_{\overline{p}}$ and $u^{-}_{\overline{p}}$ (respectively, $s^{+}_{\overline{p}}$ and $s^{-}_{\overline{p}}$) introduced in conclusion of Lemma 8.5 are boundary points on S_{∞}. We set $L^{u} = \pi(l^{u})$ and $L^{s} = \pi(l^{s})$ and denote by Ω^{u} (Ω^{s}) the geodesic lamination that is the closure of the geodesic L^{u} (L^{s}) on M^{2}. We set $\Omega_{0} = \Omega^{u} \cap \Omega^{s}$ and denote by f^{ag} the AG-hyperbolic homeomorphism constructed in Chapter 7. By $\overline{\Omega}^{u}$, $\overline{\Omega}^{s}$, and $\overline{\Omega}_{0}$ we denote the preimages of the sets Ω^{u}, Ω^{s} and Ω_{0}, respectively, in Δ.

Theorem 8.2 *Let Λ be an widely disposed attractor that does not contain bunches of degree 2 of an A-diffeomorphism f for which the automorphism $\overline{f}_{*}$ is hyperbolic. Then*:

1) *for each curve $w^{u}(\overline{x}) \subset \overline{\Lambda}$ there exists a unique geodesic $\widetilde{l}^{u} \subset \overline{\Omega}^{u}$ having common boundary points with $w^{u}(\overline{x})$ at the absolute*;
2) *for each geodesic $\widetilde{l}^{u} \subset \overline{\Omega}^{u}$ there is a unique curve $w^{u}(\overline{x}) \subset \overline{\Lambda}$ having common boundary points with $\widetilde{l}^{u}$ at the absolute*;
3) *if a curve $w^{u}(\overline{x})$ contains the preimage of a boundary periodic point of the set Λ, then the geodesic $\widetilde{l}^{u}$ is the preimage of a boundary geodesic of Ω^{u}*;
4) *if the geodesic $\widetilde{l}^{u}$ is the preimage of a boundary geodesic of Ω^{u}, then the curve $w^{u}(\overline{x}) \subset \overline{\Lambda}$ contains the preimage of a boundary periodic point of the set Λ*;
5) *for each curve $w^{s}(\overline{x})$ belonging to the curve $w^{s\infty}(\overline{q})$, where $\overline{q}$ is the preimage of a boundary periodic point q of the attractor Λ, there exist exactly two boundary geodesics $\widetilde{l}^{s}_{1}$ and $\widetilde{l}^{s}_{2} \in \overline{\Lambda}^{s}$ for which the boundary point $s_{\infty}(\overline{q})$ of the curve $w^{s}(\overline{x})$ is their common boundary point at the absolute*;

6) *for each pair of boundary geodesics* $\widetilde{l}_1^s$ *and* $\widetilde{l}_2^s$ *having a common boundary point* $\widetilde{s}$ *on the absolute, there exists a point* $\overline{q} \in \overline{\Lambda}$ *such that the curve* $w_\infty^s(\overline{q})$ *has the point* $\widetilde{s}$ *as a boundary point lying on the absolute*;
7) *for each curve* $w^s(\overline{x}) \subset \overline{\Lambda}$ *that does not contain the preimage of a boundary periodic point of the attractor* Λ*, there exists a unique interior geodesic* $\widetilde{l}^s \subset \overline{\Lambda}^s$ *that has common boundary points with* $w^s(\overline{x})$ *at the absolute*;
8) *for each interior geodesic* $\widetilde{l}^s \subset \overline{\Omega}^s$*, there exists a unique curve* $w^s(\overline{x})$*,* $\overline{x} \subset \overline{\Lambda}$ *that does not contain the preimage of a boundary periodic point and that has common boundary points with* $\widetilde{l}^s$ *at the absolute.*

Proof. Let Λ_* be a periodic component of Λ and $\overline{\Lambda}_*$ be the preimage of Λ_* on the covering Δ. For $x \in \overline{\Lambda}_*$ let us introduce a parameter t, $-\infty < t < \infty$, on the curve $w^u(\overline{x}) \subset \overline{\Lambda}_*$ and denote by $w(t)$ the point on $w^u(\overline{x})$ corresponding to t ($w(0) = \overline{x}$).

Let us prove item (1). Since both connected components of the set $W^u(x) \setminus x$ ($x = \pi(\overline{x})$, $W^u(x) = \pi(w^u(\overline{x}))$) are dense in Λ_*, it follows that for $t \to +\infty$ ($t \to -\infty$) each of them intersects at countably many points with the quasi-transversal C_{Λ_*}. Then the curve $w^u(\overline{x})$ intersects countably many curves $\{c_n\}$, $n \in Z^+$ (respectively, $n \in Z^-$), $\pi(c_n) = C_{\Lambda_*}$, as $t \to +\infty$ (respectively, $t \to -\infty$). Let c_n^+ and c_n^- be the boundary points of the curve c_n. We have

$$\lim_{n\to+\infty} c_n^+ = \lim_{n\to+\infty} c_n^- = \sigma^{u+}, \quad \lim_{n\to-\infty} c_n^+ = \lim_{n\to+\infty} c_n^- = \sigma^{u-},$$

where σ^{u+} and σ^{u-} are the boundary points of the curve $w^u(\overline{x})$.

Set $\overline{y} = w^u(\overline{x}) \cap c_1$. Since the closure of the manifold $W^u(p) = \pi(w^u(\overline{p}))$ coincides with Λ_* and the curve C_{Λ_*} is a quasi-transversal for the unstable manifolds of the points from Λ_*, it follows that there exists a sequence of points $\overline{y}_k \in c_1$, $k \in Z^+$, converging to $\overline{y}$ such that $\pi(\overline{y}_k) \in W^u(p)$. But then each curve $w^u(\overline{y}_k)$ is congruent to the curve $w^u(\overline{p})$ by means of some element $\gamma_k \in G$. For $n > 0$ (respectively, $n < 0$) denote by λ_n^+ (respectively, λ_n^-) the arc of the absolute with boundary points c_n^+ and c_n^- containing the point σ^{u+} (respectively, σ^{u-}). It follows from the C^1-closeness on compact sets of the unstable manifolds of points from Λ_* and from the properties of the covering that for any $N > 0$ there exists an N_* such that for all n satisfying the inequality $|n| > N_*$, the boundary points of the curve $w^u(\overline{y}_n)$ belong to the union of the arcs $\lambda_N \cup \lambda_{-N}$. It follows that the topological limit of the curves $w^u(\overline{y}_k)$

as $k \to \infty$ is the curve $w^u(\overline{x})$. By construction, the geodesic $l_n^u = \gamma_n(l^u)$ has common boundary points with the curve $w^u(\overline{y}_n)$. But then the topological limit of the sequence l_n^u at $n \to \infty$ is the geodesic $\widetilde{l}^u \subset \overline{\Omega}^u$ with boundary points σ^{u+} and σ^{u-}.

Let us prove item (2). Let $\widetilde{l}^u$ be an arbitrary geodesic from $\overline{\Omega}^u$, and let $\overline{z}$ be any point belonging to $\widetilde{l}^u$. Since the closure of the geodesic $L^u = = \pi(l^u)$ coincides with the lamination Ω^u, it follows that there exists a sequence of points $\{\overline{z}_n\}$, $n \in Z^+$, converging to point $\overline{z}$ such that $\pi(\overline{z}_n) \in L^u$. Then each geodesic $l_n^u \in \overline{\Omega}^u$ passing through $\overline{z}_n$ is congruent to the geodesic l^u by means of some element $\widetilde{\gamma}_n \in G$. Since the geodesics from the sequence $\{l_n^u\}$ are pairwise disjoint, it follows that the geodesic $\widetilde{l}^u$ is the topological limit of the sequence $\{l_n^u\}$. Let $\widetilde{u}_1$ and $\widetilde{u}_2$ be the boundary points of the geodesic $\widetilde{l}^u$. We set $w_n^u = \overline{\gamma}_n(w^u(\overline{p}))$ and denote by w^u the topological limit of the sequence of curves $\{w_n^u\}$. By construction, the points $\widetilde{u}_1$ and $\widetilde{u}_2$ belong to the topological limit w^u. Let us show that the set w^u contains at least one point that does not belong to the absolute. Assume the contrary. Then the set w^u contains the open arc λ of the absolute with boundary points $\widetilde{u}_1$ and $\widetilde{u}_2$. Let $\widetilde{w}^u$ the curve congruent to $w^u(\overline{p})$ with boundary point belonging to λ. Then there exists a number n_0 such that the curve $w_{n_0}^u$ has a nonempty intersection with the curve $\widetilde{w}^u$, which is impossible.

Let $\overline{x}$ be any point from w^u not belonging to the absolute. Then $\overline{x}$ belongs to $\overline{\Lambda}_*$ and the curve $w^u(\overline{x})$ belongs to w^u. Let us show that the set of boundary points of the curve $w^u(\overline{x})$ coincides with the union of the points $\widetilde{u}_1$ and $\widetilde{u}_2$. Assume the converse; then at least one boundary point σ^u of the curve $w^u(\overline{x})$ does not coincide with one of the points $\widetilde{u}_1$ and $\widetilde{u}_2$. We denote by $w_{\sigma^u}^u(\overline{x})$ the connected component of the set $w^u(\overline{x}) \setminus \overline{x}$ for which the point σ^u is a boundary point of the absolute.

Since the closure of the set $\pi(w_{\sigma^u}^u(\overline{x}))$ coincides with Λ_*, it follows that there exists a sequence of curves $\{\widetilde{c}_n\}$, $n \in Z^+$, $\widetilde{c}_n \cap w_{\sigma^u}^u(\overline{x}) \neq \varnothing$, $\pi(\widetilde{c}_n) = C$, with boundary points $\widetilde{c}_n^+$ and $\widetilde{c}_n^-$ such that

$$\lim_{n\to+\infty} \widetilde{c}_n^+ = \lim_{n\to+\infty} \widetilde{c}_n^- = \sigma^u.$$

Let us choose a number $\widetilde{N}$ so that the arc $\widetilde{\lambda} = (\widetilde{c}_n^+, \widetilde{c}_n^-)$ of the absolute containing the point σ^u does not contain the points $\widetilde{u}_1$ and $\widetilde{u}_2$. Since the point $\overline{x}$ belongs to the topological limit of the curves $\{w_n^u\}$, it follows from the C^1-closeness of the unstable manifolds of the points from Λ on compact sets and the properties of the covering that there exists a subsequence $\{w_{n_k}^u\}$,

$k \in Z^+$, of distinct curves from the set $\{w_n^u\}$ such that one of the boundary points of each curve $\{w_{n_k}^u\}$ belongs to the arc $\widetilde{\lambda}$, which it is impossible, since the limit set of boundary points of all curves from $\{w_n^u\}$ is the union of the points $\widetilde{u}_1$ and $\widetilde{u}_2$.

It follows from the above proof that if the curve $w^u(\overline{x})$ contains the preimage of a boundary periodic point of the set Λ_*, then the geodesic $\widetilde{l}^u$ having common boundary points with $w^u(\overline{x})$ on the absolute is the preimage of a boundary geodesic of Ω^u. Conversely, if the geodesic $\widetilde{l}^u$ is the preimage of a boundary geodesic of Ω^u, then the corresponding curve $w^u(\overline{x}) \subset \overline{\Lambda}_*$ contains the preimage of a boundary periodic point of Λ_*. Thus item (3) and item (4) hold.

Let us prove item (5). Let $\overline{p}$ be the preimage of a boundary periodic point p of the attractor Λ with some period k. We denote by σ^{u-} and σ^{u+} the boundary points of the curve $w^u(\overline{p})$ and by $s_\infty(\overline{p})$ the boundary point of the curve $w^{s\infty}(\overline{p})$. By virtue of item (4) of Lemma 8.4, there exist points $\overline{q}^+$ and $\overline{q}^-$ that are the pre-images of boundary periodic points from Λ such that $w^u(\overline{q}^+)$ (respectively, $w^u(\overline{q}^-)$) has σ^{u+} (respectively, σ^{u-}) as one of the boundary points (since the set Λ does not contain special pairs of boundary periodic points, it follows that $\overline{q}^+ \neq \overline{q}^-$). We denote by $s_\infty(\overline{q}^+)$ and $s_\infty(\overline{q}^-)$ the boundary points of the curves $w^s\infty(\overline{q}^+)$ and $w^{s\infty}(\overline{q}^-)$ on the absolute. Consider the diffeomorphism $\overline{f}_{2k}$ covering f^{2k} for which the point $\overline{p}$ is fixed and both connected components of the set $w^u(\overline{p}) \setminus \overline{p}$ are invariant. Then the points σ^{u+} and σ^{u-} are fixed points of the homeomorphism $\overline{f}^*_{2k}$, and hence the points $\overline{q}^+$, $\overline{q}^-$ are fixed points of the diffeomorphism $\overline{f}_{2k}$. But then the points $s_\infty(\overline{p})$, $s_\infty(\overline{q}^+)$, and $s_\infty(\overline{q}^-)$ are also fixed points of the homeomorphism $\overline{f}^*_{2k}$. Consider the homeomorphism $\overline{f}_{02k}$ covering the homeomorphism f_0^{2k} such that $\overline{f}^*_{02k}$ coincides with the homeomorphism $\overline{f}^*_{2k}$. By Lemma 8.2, there exist boundary geodesics $\widetilde{l}^u$, $\widetilde{l}^u_1$, and $\widetilde{l}^u_2$ from the set $\overline{\Omega}^u$ having common boundary points with the curves $w^u(\overline{p})$, $w^u(\overline{q}^+)$, and $w^u(\overline{q}^-)$, respectively. By the construction of the hyperbolic homeomorphism f_0, there exist boundary geodesics $\widetilde{l}^s_1$ and $\widetilde{l}^s_2$ from the set $\overline{\Omega}^s$ for which, respectively, $s_\infty(\overline{q}^+)$, $s_\infty(\overline{p})$ and $s_\infty(\overline{p})$, $s_\infty(\overline{q}^-)$ are the boundary points on the absolute, see Fig. 8.7.

Let us prove item (6). Let $\widetilde{l}^s_1$ and $\widetilde{l}^s_2$ be boundary geodesics from $\overline{\Omega}^s$ for which the point $\widetilde{s}$ is a common boundary point on the absolute. By the construction of the hyperbolic homeomorphism f_0, there exist points u_1 and u_2 on the absolute, a number $r \in Z^+$, and homeomorphism $\overline{f}_{r0}$ covering f_0^r such

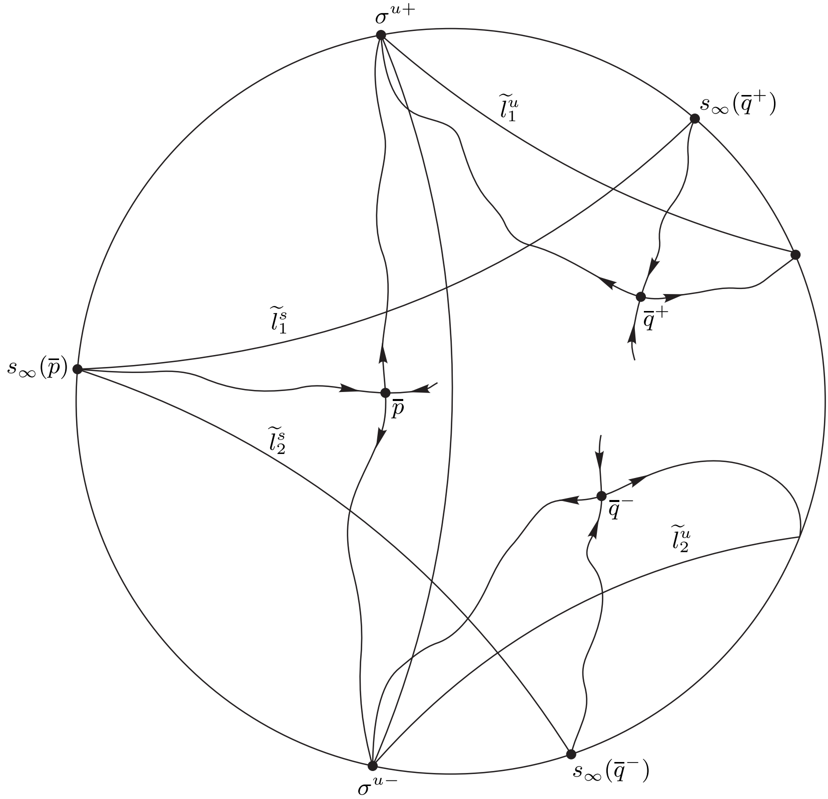

Figure 8.7

that the point $\widetilde{s}$ is repellent and the points u_1 and u_2 are attractive points of the homeomorphism $\overline{f}_{r0}^*$ and that the arc $[u_1, u_2]$ of the absolute containing the point $\widetilde{s}$ does not contain other fixed points of the homeomorphism $\overline{f}_{r0}^*$. But then the geodesic $\widetilde{l}^u$ with boundary points u_1 and u_2 belongs to $\overline{\Omega}^u$, and by Lemma 8.2, there exists a curve $w^u(\overline{x}) \subset \overline{\Lambda}$ that contains the preimage $\overline{q}$ of a boimdaxy periodic point of the set Λ and has common boundary points with $\widetilde{l}^u$ on the absolute. By construction, the point $\widetilde{s}$ is a boundary point of the curve $w^{s\infty}(\overline{q})$.

Items (7) and (8) can be proved by an arguments similar to the proof of items (1) and (2). The proof of Theorem 8.2 is complete. □

Definition 8.5 *An attractor Λ of A-diffeomorphism $f\colon M^2 \to M^2$ of genus more than zero is called perfect if $M^2 \setminus \Lambda$ consists of finite number domains each of which is homeomorphic to disk.*

Next Theorem is direct corollary of theorems 8.2 and Lemma 8.4.

Theorem 8.3 *Let Λ be an widely disposed attractor of A-diffeomorphism $f\colon M^2 \to M^2$, where M^2 closed orientable surface of genus more than 1, and $\overline{f}_*\colon \Gamma \to \Gamma$ be hyperbolic automorphism. Then the attractor Λ is connected and perfect.*

Proof. Suppose firstly that Λ does not contain bunches of degree less that 3. Since the set $M^2 \setminus \Omega^u$ consists of finitely many domains homeomorphic to disks, it follows from the proof above that the set $M^2 \setminus \Lambda_*$ has the same property. That is the set Λ consists of one periodic component Λ_* and is perfect attractor. It follows from hyperbolicity of $\overline{f}_*$, Theorem 8.1 and Lemma 8.4 that the attractor Λ does not contain a bunch of degree 1 and any bunch of degree 2 is accessible boundary from interior of some disk. □

8.3. Classification Theorem

Theorem 8.4 *Let f be an A-diffeomorphism for which the automorphism $\overline{f}_*$ is hyperbolic, and let $\Lambda \in NW(f)$ be an widely disposed attractor that does not contain bunches of degree 2. Then there exists a continuous mapping $h\colon M^2 \to M^2$ such that h is homotopic to the identity mapping and the following conditions are satisfied*:

1) $h(\Omega_0) = \Lambda$, $fh|_{\Omega_0} = hf^{ag}|_{\Omega_0}$;
2) *the set $B \subset \Lambda$ of points b whose preimage $h^{-1}(b)$ contains more than one point consists precisely of the points that belong to the union of stable manifolds of all boundary periodic points*; *moreover, $h^{-1}(b) \cap \Omega^s$ consists of exactly two points belonging to distinct boundary geodesics from Ω^s.*

Proof. Let $\overline{x}_0$ be any point belonging to the set $\overline{\Omega}_0$ and lying on the intersection of the geodesics $l^s(\overline{x}_0) \subset \overline{\Omega}^s$ and $l^u(\overline{x}_0) \subset \overline{\Omega}^u$. The following cases are possible:

1) $l^s(\overline{x}_0)$ is the preimage of an interior geodesic of the set Ω^s;
2) $l^s(\overline{x}_0)$ is the preimage of a boundary geodesic of the set Ω^s.

In the first case, by Theorem 8.2, there exist curves $w^s(\overline{x})$ and $w^u(\overline{y})$ possessing common boundary points with the geodesics $l^s(\overline{x}_0)$ and $l^u(\overline{x}_0)$, respectively. Since the pairs of boundary points of the geodesics $l^s(\overline{x}_0)$ and $l^u(\overline{x}_0)$ are separated on the absolute, it follows that the curves $w^s(\overline{x})$ and $w^u(\overline{y})$ intersect on Δ; moreover, by Lemma 8.4 (item 1.), the intersection contains a single point, which we denote by $\overline{x}'$.

In the second case we denote by s_1 and s_2 the boundary points of the geodesic $l^s(\overline{x}_0)$. By the construction of the lamination Ω^s, there exists a geodesic l_1^s (respectively, l_2^s) other than $l^s(\overline{x}_0)$ for which the point s_1 (respectively, s_2) is a boundary point on the absolute. By the construction of the hyperbolic homeomorphism f^{ag} (in chapter 7), there exists a number $k \in Z^+$ and a homeomorphism $\overline{f}_k^{ag}$ covering the homeomorphism $(f^{ag})^k$ such that the points s_1 and s_2 are source fixed points of the homeomorphism $(\overline{f}_k^{ag})^*$ of the absolute and the arc $\lambda = (s_1, s_2)$ of the absolute, containing no boundary points of the geodesics l_1^s and l_2^s other than s_1 and s_2 contains exactly one (sink) fixed point of the homeomorphism $(\overline{f}_k^{ag})^*$; we denote this point by u. Then, by the construction of the geodesic lamination Ω^u, there exist pre-images l_1^u and l_2^u of distinct boundary geodesics from Ω^u such that u is a common boundary point of these pre-images on the absolute. Moreover l_1^u, and l_2^u has a nonempty intersection with the geodesics $l^s(\overline{x}_0)$. Let $\overline{p}_0^1 = l_1^u \cap l^s(\overline{x}_0)$ and $\overline{p}_0^2 = l_2^u \cap l^s(\overline{x}_0)$. By Theorem 8.2, there exists a point $\overline{p}_1 \in \overline{\Lambda}$ (respectively, $\overline{p}_2 \in \overline{\Lambda}$) that is the preimage of the boundary periodic point p_1 (respectively, p_2) and such that the curve $w_\infty^s(\overline{p}_1)$ (respectively, $w_\infty^s(\overline{p}_2)$) has the point s_1 (respectively, s_2) as a boundary point on the absolute. It follows from the proof carried out for Theroem 8.2 that the curve $w^u(\overline{p}_1)$ (respectively, $w^u(\overline{p}_2)$) has common boundary points with the geodesic l_1^u (respectively, l_2^u) on the absolute. By Theorem 8.2, there exists a curve $w^u(\overline{x})$, $\overline{x} \in \overline{\Lambda}$ possessing common boundary points with the geodesic $l^u(x_0)$ on the absolute. Next, there are the following two possibilities:

2a) the point $\overline{x}_0$ belongs to the arc $[\overline{p}_0^1, s_1)^s \subset l^s$;

2b) the point $\overline{x}_0$ belongs to the arc $[\overline{p}_0^2, s_2)^s \subset l^s$.

Then in case 2a) we set $\overline{x}' = w^u(\overline{x}) \cap w^s(\overline{p}_1)$, and in case 2b) we set $\overline{x}' = = w^u(\overline{x}) \cap w^s(\overline{p}_2)$ (see Figure 8.8).

Let $\overline{\varphi}$ be the map which associates to each point $\overline{x}_0 \in \Omega_0$ the point $\overline{x}' \in \overline{\Lambda}$ which was defined found in the cases 1), 2a) and 2b).

Note that in case 2a) for the point $\overline{x}_0$ there exists a unique point $\overline{y}_0^1 = = l_1^s \cap l^u(\overline{x}_0)$ such that $\overline{\varphi}(\overline{y}_0^1) = \overline{\varphi}(\overline{x}_0) = \overline{x}'$, and in case 2b) for each point $\overline{x}_0$ there exists a unique point $\overline{y}_0^2 = l_2^s \cap l^u(\overline{x}_0)$ such that $\overline{\varphi}(\overline{y}_0^2) = \overline{\varphi}(\overline{x}_0) = \overline{x}'$.

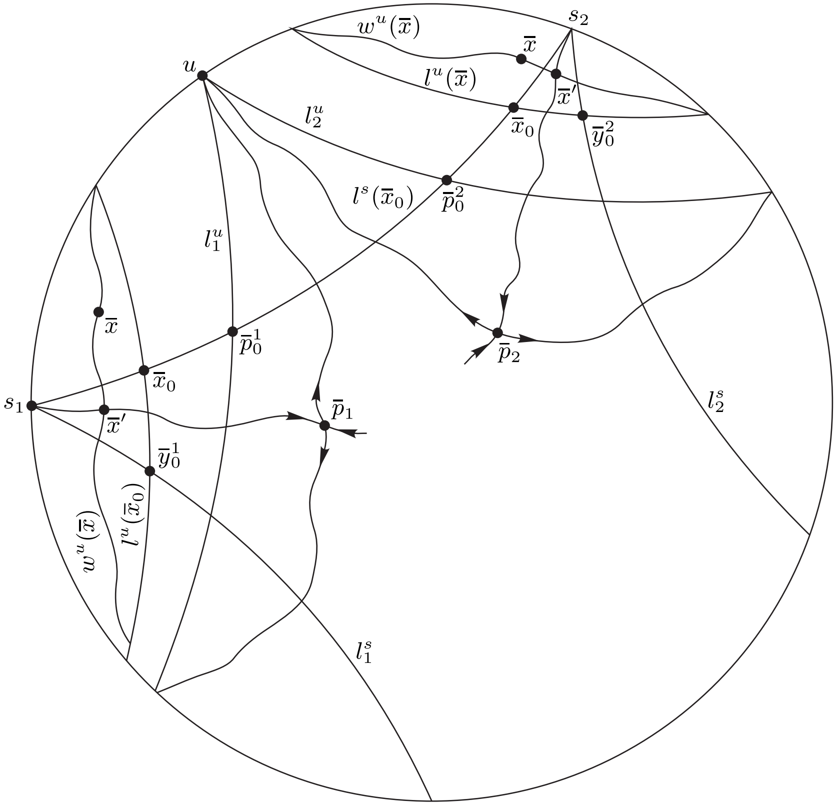

Figure 8.8

By Theorem 8.2 the mapping $\overline{\varphi}$ takes $\overline{\Omega}_0$ onto $\overline{\Lambda}$. By the construction of the homeomorphism $\overline{f}^{ag}$, the mapping $\overline{\varphi}$ satisfies the relations

$$\overline{\varphi}\overline{f}^{ag}\Big|_{\overline{\Omega}_0} = \overline{f}\overline{\varphi}|_{\overline{\Omega}_0}, \quad \overline{\varphi}\gamma|_{\overline{\Omega}_0} = \gamma\overline{\varphi}|_{\overline{\Omega}_0}, \quad \gamma \in \Gamma,$$

where the covering mappings $\overline{f}^{ag}$ and $\overline{f}$ satisfy the condition $(\overline{f}^{ag})^* = \overline{f}^*$. By construction, it readily follows that the mapping $\overline{\varphi}$ is continuous on the set $\overline{\Omega}_0$.

Let $\overline{B} \subset \overline{\Lambda}$ be the subset of points $\overline{b}$ whose preimage $\overline{\varphi}^{-1}(\overline{b})$ consists of more than one point. It follows from the construction of the mapping $\overline{\varphi}$ that the set $\overline{B}$ has the form

$$\bigcup_{\overline{p}\in\overline{G}} w^s(\overline{p}) \cap \overline{\Lambda},$$

where $\overline{G} = \pi^{-1}(G)$ is the preimage of the union G of all boundary points of the attractor Λ. If $\overline{b} \in w^s(\overline{p}) \cap \overline{\Lambda}$, then $\overline{\varphi}^{-1}(\overline{b})$ consists of exactly two points $\overline{b}_0^1$ and $\overline{b}_0^2$ that belong to the preimages of distinct boundary geodesics from Ω^s having a common boundary point on the absolute.

Now let us construct a continuous mapping $\overline{h}\colon \Delta \to \Delta$ such that $\overline{h}\gamma = \gamma\overline{h}$ and $\overline{h}|_{\overline{\Omega}_0} = \overline{\varphi}|_{\overline{\Omega}_0}$.

First, we construct a continuous mapping $\overline{h}_0$ of $\overline{\Omega}^u$ onto $\overline{\Lambda}$ such that $\overline{h}|_{\overline{\Omega}_0} = \overline{\varphi}|_{\overline{\Omega}_0}$. Let $\overline{z}$ be an arbitrary point belonging to the geodesic $l^u(\overline{z}) \subset \subset \overline{\Omega}^u$. If $\overline{z} \in \overline{\Omega}_0$, then we set $\overline{h}_0(\overline{z}) = \overline{\varphi}(\overline{z})$; if $z \neq \overline{\Omega}_0$, then by the construction of $\overline{\varphi}$ there exist points $\overline{z}_1, \overline{z}_2 \in l^u \cap \overline{\Omega}_0$ such that the arc $(\overline{z}_1, \overline{z}_2)^u \subset l^u$ does not contain points from $\overline{\Omega}_0$ and $\overline{\varphi}(\overline{z}_1) = \overline{\varphi}(\overline{z}_2)$. In this case, we set $\overline{h}_0(\overline{z}) = \overline{\varphi}(\overline{z}_1) = \overline{\varphi}(\overline{z}_2)$.

For each domain $D_i \subset M^2 \setminus \Omega^u$, $i = 1, \ldots, k$, we choose exactly one preimage $\overline{D}_i^u$ on the covering $\overline{M}^2$ and denote by l_i^{uj}, $j = 1, \ldots, r_i$, the geodesic from $\overline{\Omega}^u$ entering the boundary of $\overline{D}_i^u$. By Theorem 8.2, for each geodesic l_i^{uj} there exists a unique point $\overline{p}_i^j$ that is the preimage of a boundary periodic point of the set Λ such that the curve $w^u(\overline{p}_i^j)$ has common boundary points with the geodesic l_i^{uj} on the absolute. We denote by $\overline{D}_i'^u$ the domain on $\overline{M}^2$ bounded by the union of the curves $\bigcup_{j=1}^{j=r_i} w^u(\overline{p}_i^j)$ together with their boundary points lying on the absolute.

For each i, we can readily construct a continuous mapping $\overline{h}_i$ of the closure of the domain $\overline{D}_i^u$ on the closure of the domain $\overline{D}_i'^u$ such that $\overline{h}_i$ coincides with $\overline{h}_0$ on each geodesic l_i^{uj} belonging to the boundary of the domain $\overline{D}_i^u$. Now let $\overline{z}$ be any point on $\Delta \setminus \overline{\Omega}^u$. Then there exists an element $\gamma \in G$ such that the point $\gamma(\overline{z})$ belongs to the closure of one of the domains $\overline{D}_i^u$. Set $\overline{h}(\overline{z}) = = \gamma^{-1}(\overline{h}_i(\gamma(\overline{z})))$. By construction, $\overline{h}$ is a continuous mapping satisfying the conditions $\overline{h}\gamma = \gamma\overline{h}$ and $\overline{h}|_{\overline{\Omega}_0} = \overline{\varphi}|_{\overline{\Omega}_0}$.

Set $B = \pi(\overline{B})$. Then the mapping $h\colon M^2 \to M^2$ for which the mapping $\overline{h}$ is the covering is the desired one. The proof of Theorem is complete. □

Bibliographic Notes and Panoramas

Chapter 8. Asymptotic behavior of lifts of stable and unstable manifolds of points from one-dimensional attractor on universal covering of ambient surface was firstly investigated for orientable basic sets by V. Grines in 1974–1977 [82–84] and then was generalized by R. Plykin, V. Grines, X. Kalay for widely disposed basic sets [89, 188].

The representations of widely disposed attractors by means of geodesic laminations was done by V. Grines in [88] and where was also shown that dynamics in widely disposed attractors are factors of dynamics of restrictions AG-homeomorphisms constructed in Chapter 7.

(8.1). Theorem 8.1 was proved in [88].

(8.2). Theorem 8.2 was proved in [88].

(8.3). Theorem 8.4 was proved in [88].

CHAPTER 9

Structural Stability and Anosov–Weil Theory

In this chapter we mainly consider structurally stable diffeomorphism f on closed orientable surface M^2 of genus $g \geqslant 0$ whose nonwandering set contains at least one one-dimensional attractor (if nonwandering set contains one dimensional repeller, one can consider diffeomorphism f^{-1}).

In Section 9.1, we prove that if $f\colon M^2 \to M^2$ is structurally stable then the existence of one dimensional attractor implies the existence at least one periodic point which is a source.

In Section 9.2, we study asymptotic behaviors of stable and unstable manifolds of an one-dimensional widely disposed attractor Λ of A-diffeomorphism (not necessary structurally stable) from point of view of Anosov – Weil problem. We consider separately cases M^2 to be the torus and M^2 to be a hyperbolic surface. When M^2 is the torus we prove a bound deviation of stable and unstable invariant manifolds of Λ from co-asymptotic geodesics. When M^2 is hyperbolic surface we also prove a bound deviation of unstable invariant manifolds of Λ from co-asymptotic geodesics. If A-diffeomorphism is structurally stable we prove that stable invariant manifolds of Λ have property of bound deviation from co-asymptotic geodesics as well.

9.1. Asymptotic properties of invariant manifolds

Here, we keep the notation of Section 1.8. Let f be an A-diffeomorphism of a closed orientable surface M^2 such that the non-wandering set $NW(f)$ contains a one-dimensional attractor Λ. By Corollary 1.3, f has at least one s-boundary point belonging to Λ. Moreover, all s-boundary (in short, boundary) points are periodic.

Definition 9.1 *We say a* 1*-dimensional attractor of an A-diffeomorphism* f *to be* separable *if*

1) *There is finite set* R_Λ *of the saddle and source points from the trivial basic sets of the diffeomorphism* f *such that* $\mathrm{cl}(W^s_\Lambda) \setminus W^s_\Lambda$ *is the union of the stable manifolds of the points from* R_Λ;
2) *for every s-boundary point* $p \in \Lambda$ *holds* $\mathrm{cl}(W^{s\varnothing}_p) \setminus W^{s\varnothing}_p = p \cup \alpha$ *where* $\alpha \in R_\Lambda$ *is a source point*;
3) *for every saddle point* $\sigma \in R_\Lambda$ *the manifold* W^s_σ *contains no heteroclinic points and the unstable separatrix* ℓ^u_σ *is either disjoint from* W^s_Λ *or it is a subset of* W^s_Λ (*see Figure* 9.1).

The definition of a separable repeller is analogous.

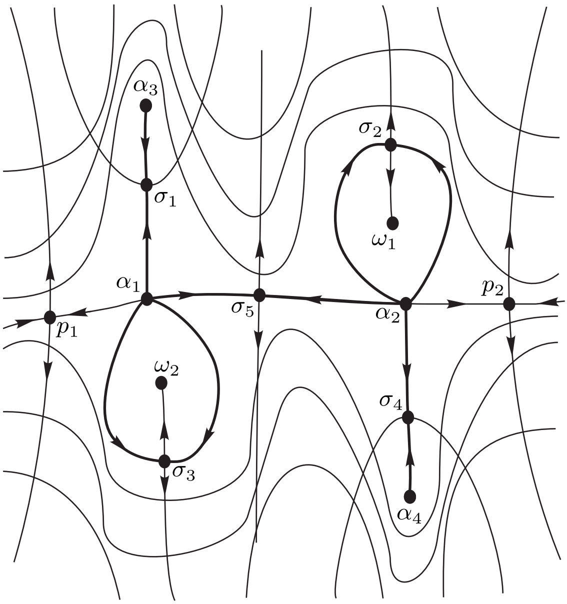

Figure 9.1. An illustration to Definition 9.1

The following theorem states the important property of 1-dimensional basic sets of structurally stable diffeomorphisms from which immediately follows Theorem 9.2 below.

Theorem 9.1 *Every* 1*-dimensional attractor* (*repeller*) *of a structurally stable diffeomorphism* $f\colon M^2 \to M^2$ *is separable.*

Proof. Let Λ be an attractor of the structurally stable diffeomorphism $f\colon M^2 \to M^2$ (for a repeller the proof is similar). We now prove that the three conditions of Definition 9.1 hold.

1) To prove item 1) of Definition 9.1 it suffices to prove that $W^u_{\Lambda'} \cap W^s_{\Lambda} = \varnothing$ holds for every nontrivial basic set Λ' distinct from Λ. Suppose the contrary: there are points $x \in \Lambda$, $x' \in \Lambda'$ such that $W^s_x \cap W^u_{x'} \neq \varnothing$. Since stable manifolds of the points of Λ (unstable manifolds of the points of Λ') are C^1-close on compact sets, without loss of generality one assumes that the manifold W^s_x contains no s-boundary periodic points of the basic set Λ and that the manifold $W^u_{x'}$ contains no u-boundary periodic points of the basic set Λ'.

Let $y \in (W^s_x \cap W^u_{x'})$. By Lemma 1.23 the point y belongs to an adjacent interval $(a,\, b)^s \subset W^s_x$ which consists of the wandering points of the diffeomorphism f and such that $a, b \in \Lambda$ and each of W^u_a, W^u_b contains exactly one s-boundary point of p_a, p_b, see Figure 9.2 left ($p_a = p_b$ if $W^u_a = W^u_b$ see Figure 9.2 right). Denote by L^u_a (L^u_b) the connected component of the set $W^u_a \setminus a$ ($W^u_b \setminus b$) disjoint from the point p_a (p_b). Then the curve $l_{ab} = L^u_a \cup L^u_b \cup [a,\, b]^s$ bounds a domain D_{ab}. This domain is a continuous immersion of the open disk into the manifold M^2, all its points are the wandering points of the diffeomorphism f and the curve l_{ab} is the boundary of D_{ab} which is accessible from inside.

Denote by W^{u*}_y the connected component of the set $W^u_y \setminus y$ disjoint from the point x'. The strong transversality condition implies $W^{u*}_y \cap D_{ab} \neq \varnothing$. By Theorem 1.19 the component W^{u*}_y contains a set which is dense in the periodic component of the set Λ'. Therefore, there are points in W^{u*}_y disjoint from the domain D_{ab}. Then there is a point $y' \in (a,\, b)^s$ distinct from the point y and such that the arc $(y,\, y')^u \subset W^u_{x'}$ belongs to the domain D_{ab}. Since for any point $\widetilde{a} \in L^u_a$ there is a unique point $\widetilde{b} \in L^u_b$ such that $\widetilde{a}, \widetilde{b} \in W^s_{\widetilde{x}}$, $\widetilde{x} \in \Lambda$ and $(\widetilde{a},\, \widetilde{b})^s \subset D_{ab}$ it follows that the arc $(\widetilde{a},\, \widetilde{b})^s$ is tangent to the arc $(y,\, y')^u$ and this contradicts the strong transversality condition, see Figure 9.2.

2) To prove the item 2) of Definition 9.1 it suffices to show that for every s-boundary point p of the basic set Λ there is no saddle point σ from the trivial basic set of the diffeomorphism f such that $W^u_\sigma \cap W^{s\varnothing}_p \neq \varnothing$. If we assume the contrary then similarly to the proof of the item 1) we come to contradiction to the strong transversality condition.

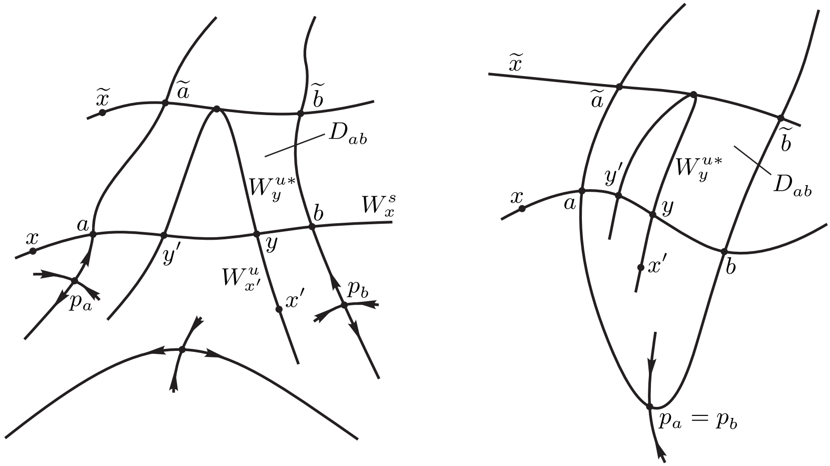

Figure 9.2

3) Assuming the contrary in this case we come to contradiction to the strong transversality condition as well. □

Let p be a boundary periodic point of Λ and let $a \neq p$ be any point of $W^u(p)$. We denote by $W^u_a(p)$ the connected component of the set $W^u(p) \setminus p$ which contains a. It follows from Theorems 1.19 and 1.20, that there is a unique boundary periodic point q (q may coincide with p) such that there is a unique point $b \neq a$ such that $b \in W^s(a) \cap W^u(q)$ and $(a, b)^s \cap \Lambda = \varnothing$. Note that if q coincides with p, then b belongs to the connected component of the set $W^u(p) \setminus p$ which does not contain a.

Denote by $W^u_b(q)$ the connected component of the set $W^u(q) \setminus q$ which contains b and call the connected components $W^u_a(p)$, $W^u_b(q)$ *s-connected*, see Fig. 9.3.

Corollary 9.1 *Let f be a structurally stable diffeomorphism of a closed orientable surface M^2 of genus $p \geqslant 0$ such that the non-wandering set $NW(f)$ contains a one-dimensional attractor Λ. Suppose that the components $W^u_a(p)$, $W^u_b(q)$ are s-connected, where p, q are the boundary periodic points of Λ and $a \in W^u(p)$, $b \in W^u(q)$ (if p coincides with q, then a and b belong to distinct connected components of the set $W^u(p) \setminus p$). Then there exist the source periodic points $\alpha_1, \ldots, \alpha_l$ and the saddle periodic points $p_1, \ldots, p_n$ of the trivial basic sets, where $l \geqslant 1$, $n \geqslant 0$, and if $n = 0$, then $l = 1$, so that*

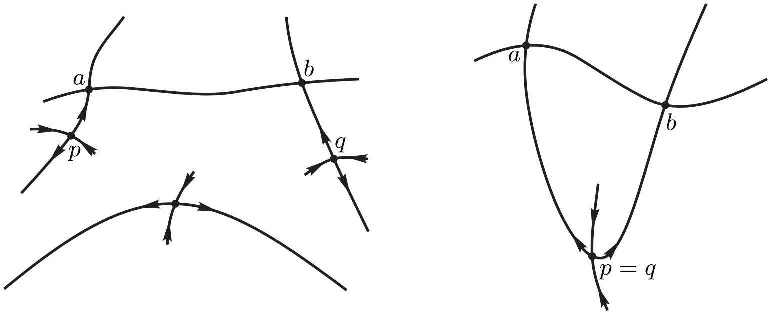

Figure 9.3

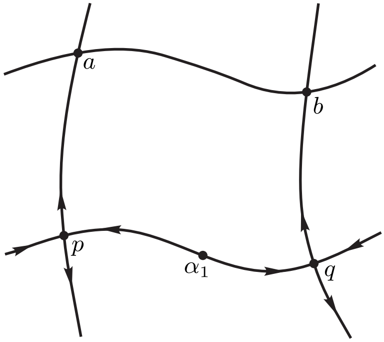

Figure 9.4. $n = 0$

the following conditions hold:

$$L_{pq} = \begin{cases} W^{s\varnothing}(p) \cup W^{s\varnothing}(q) \cup \alpha_1 \cup p \cup q, & n = 0 \\ & \text{(see Fig. 9.4)}, \\ W^{s\varnothing}(p) \cup W^{s\varnothing}(q) \cup \bigcup\limits_{i=1}^{i=n} W^s(p_i) \cup \bigcup\limits_{i=1}^{i=n} \alpha_i \cup p \cup q, & n \geqslant 1 \\ & \text{(see Fig. 9.5)}. \end{cases}$$

1) *is a connected one-dimensional complex*;
2) *each set* $\partial W^{s\varnothing}(p)$, $\partial W^{s\varnothing}(q)$ *consists of exactly one source periodic point of* L_{pq};

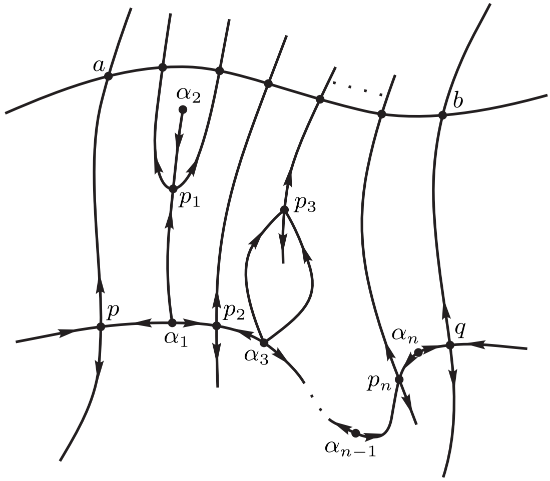

Figure 9.5. $n > 0$

3) *each set* $\partial W^s(p_i)$, $i \in \{l, n\}$, *consists of one or two periodic points of* L_{pq}.

As a direct consequence of the corollary 9.1, we obtain

Theorem 9.2 *If the non-wandering set of a structurally stable diffeomorphism* $f\colon M^2 \to M^2$ *contains a one-dimensional attractor, then it contains a source periodic point.*

9.2. Conditions of bound deviation of invariant manifolds from co-asymptotic geodesics

Here, we keep the notations of the previous section. Let Λ_* be a C-dense component of the attractor Λ and $L^\sigma(t) \subset W^\sigma(x)$, $x \in \Lambda$, $t \in [0, +\infty)$ $\sigma \in \{s, u\}$ be a ray on M^2 which contains a set dense in Λ_*. Let $l^\sigma(t)$ be one of pre-images of $L^\sigma(t)$ on the universal covering $\overline{M}^2$, $\pi(l^\sigma(t)) = L^\sigma(t)$.

Denote by g^σ any geodesic on $\overline{M}^2$ which have the same asymptotic direction as the ray $l^\sigma(t)$ in one of the directions (if $g = 1$, then we consider a metric of zero curvature, and if $g > 1$, we consider a metric of constant negative curvature). Let us drop a perpendicular from $l^\sigma(t)$ on g^σ and denote by $g^\sigma_t \in g^\sigma$

the foot of this perpendicular. Denote by d the distance in appropriate metric on $\overline{M}^2$.

We say that the preimage $l^\sigma(t)$ of the ray $L^\sigma(t)$ boundedly deviates from the geodesic g^σ if the distance $d(l^\sigma(t),\, g^\sigma_t)$ is bounded by a constant which does not depend on $t \in [0,\, +\infty)$.

Theorem 9.3 *Let Λ be an widely disposed one-dimensional attractor of an A-diffeomorphism f of an orientable closed two-dimensional manifold M^2 of genus $p \geqslant 1$ and let $L^\sigma(t) \subset W^\sigma(x)$, $x \in \Lambda$, be a ray which contains a set dense in some C-dense component of Λ. Then a preimage $l^\sigma(t)$ of the ray $L^\sigma(t)$ on the universal covering $\overline{M}^2$ boundedly deviates from any geodesic g^σ which has the same asymptotic direction as $l^\sigma(t)$ in one of two directions, where σ may take the following values*:

if $g = 1$, then σ equals s and u;

if $g > 1$, then $\sigma = u$;

if $g > 1$ and, in addition, f is structurally stable, then σ equals s and u.

Proof. First, we consider the case where the surface M^2 is the torus. If the non-wandering set of the A-diffeomorphism f contains a widely disposed attractor then f induces a hyperbolic automorphism f_* of the group Γ [84]. Let f_0 be an algebraic automorphism of the torus such that $f_{0*} = f_*$. Then according to [72] there is a continuous map $h\colon M^2 \to M^2$ such that $hf = f_0 h$ and h is homotopic to the identity mapping.

Let $\overline{h}\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the covering map of h ($\pi\overline{h} = h\pi$). For the curve $l^\sigma(t)$ there is a straight line g^σ on $\mathbb{R}^2$ such that some ray g^σ_1 of this straight line has the same asymptotic direction as $l^\sigma(t)$ and $g^\sigma_1 = \overline{h}(l^\sigma(t))$.

Suppose that the statement of the theorem does not hold. Then there exist a sequence $t_k \to +\infty$ of values of the parameter t and a sequence γ_k of elements of the group Γ such that for any $\sigma \in \{s,\, u\}$ we have $d(l^\sigma(t_k),\, g^\sigma_{t_k}) \to \to +\infty$ and the sequence $m_k = \gamma_k(l^\sigma(t_k))$ converges to some point $m_* \in \mathbb{R}$ as $t_k \to +\infty$.

Put $\widetilde{g}^\sigma_k = \overline{h}(l^\sigma(t_k))$, $n_k = \gamma_k(\widetilde{g}^\sigma_k)$. By construction $d(\widetilde{g}^\sigma_k,\, l^\sigma(t_k)) \to \infty$, therefore $d(m_k,\, n_k) \to \infty$. As $\overline{h}(m_k) = \overline{h}(\gamma_k(l^\sigma(t_k))) = \gamma_k(\overline{h}(l^\sigma(t_k))) = = \gamma_k(\widetilde{g}^\sigma_k) = n_k$, then $d(m_k,\, \overline{h}(m_k)) \to \infty$. We get a contradiction with continuity of the map $\overline{h}$.

Let now the genus of the surface M^2 be greater than 1 and let $L^\sigma(t) \subset \subset W^\sigma(x)$, $x \in \Lambda$, be a ray which contains a set dense in some C-dense component Λ_* of the set Λ.

Consider any point $y \in \Lambda_*$ for which the manifold $W^s(y)$ does not contain a boundary periodic point. The curve $w^\sigma(\overline{y})$ with $\pi(\overline{y}) = y$ has exactly two boundary points $\mu^{\sigma+}$, $\mu^{\sigma-}$ which are irrational points of the circle at infinity S_∞ (where each of the points μ^{s+}, μ^{s-} does not coincide with any of the points μ^{u+}, μ^{u-}).

Denote by g^σ the geodesic on Δ which has $\mu^{\sigma+}$, $\mu^{\sigma-}$ as its boundary points. Then $\pi(g^\sigma)$ is a nonclosed geodesic without self-intersections on M^2.

Introduce a parameter $t \in (-\infty, +\infty)$ on the curve $w^\sigma(\overline{y})$ and denote by $w^\sigma(t)$ the point on $w^\sigma(\overline{y})$ such that $w^\sigma(0) = \overline{y}$ and $w^\sigma(t)$ tends to the point $\mu^{\sigma+}$ as $t \to +\infty$. We show that the ray $w^{\sigma+}(t) \subset w^\sigma(t)$, $t \in [0, +\infty)$, boundedly deviates from the geodesic g^σ, where $\sigma = u$, if f is an A-diffeomorphism, and $\sigma = u$ and $\sigma = s$ if, in addition, f is structurally stable.

Suppose the contrary. Then there is a sequence $t_k \to +\infty$ of values of the parameter t and a sequence γ_k of elements of the group Γ such that $d(w^\sigma(t_k), g^\sigma_{t_k}) \to \infty$ and the sequence $n_k = \gamma_k(g^\sigma_{t_k})$ converges to some point $n_* \in \Delta$ (where $g^\sigma_{t_k}$ is the foot of the perpendicular dropped from the point $w^\sigma(t_k)$ on the geodesic g^σ).

Put $m_k = \gamma_k(w^\sigma(t_k))$, $g^\sigma_k = \gamma_k(g^\sigma)$, $w^\sigma_k = \gamma_k(w^\sigma(\overline{y}))$. Denote by g^σ_*, l^σ_* the topological limits of the sequences g^σ_k, w^σ_k respectively. As the boundary points of the geodesic g^σ are irrational and the geodesic $\pi(g^\sigma)$ has no self-intersections on M^2, all geodesics of the sequence g^σ_k are mutually disjoint and have no common boundary points on S_∞. Therefore the set g^σ_* is the geodesic passing through the point n_* and denote by $\mu^{\sigma+}_*$, $\mu^{\sigma-}_*$, the boundary points of g^σ_*. By construction the points $\mu^{\sigma+}_*$, $\mu^{\sigma-}_*$ are different from the boundary points of the geodesics g^σ_k for all k.

The topological limit l^σ_* is a connected set which contains the points $\mu^{\sigma+}_*$, $\mu^{\sigma-}_*$. Since elements of the group Γ are distance-preserving, the distance between the points n_k, m_k infinitely increases as $k \to \infty$. Then the sequence of the points m_k has at most one limit point which belongs to the circle at infinity S_∞ and is different from the points $\mu^{\sigma+}_*$, $\mu^{\sigma-}_*$.

We show that l^σ_* cannot contain any open arc λ of S_∞. Suppose the contrary. As the orbit of action of the group Γ on any point of S_∞ is dense in S_∞, there is an element $\gamma \in \Gamma$ such that the point $\gamma(\mu^{\sigma+})$ belongs to λ. But then there should be a number k_0 such that the curve $w^\sigma_{k_0}$ intersects the curve $\gamma(w^\sigma(\overline{y}))$. This is impossible as the curve $\pi(w^\sigma(\overline{y}))$ has no self-intersections.

Thus the set l_*^σ contains a connected subset $\widetilde{l}_*^\sigma$ which belongs to Δ and has the following properties:

(1) $\widetilde{l}_*^\sigma$ has exactly two boundary points $\nu_*^{\sigma+}$, $\nu_*^{\sigma-}$ which belong to S_∞;

(2) at most one of the points $\nu_*^{\sigma+}$, $\nu_*^{\sigma-}$ (for example, $\nu_*^{\sigma+}$) does not coincide with any point of $\mu_*^{\sigma+}$, $\mu_*^{\sigma-}$. Put $\widetilde{L}_*^\sigma = \pi(\widetilde{l}_*^\sigma)$ and consider separately the case $\sigma = u$ and the case $\sigma = s$.

In the case $\sigma = u$ the set $\widetilde{L}_*^u$ belongs entirely to the set Λ_* and for any point $z_* \in \widetilde{L}_*^u$ the unstable manifold $W^u(z_*)$ of the point z_* belongs to $\widetilde{L}_*^u$. Denote by $\overline{z}_*$ the preimage of the point z_* which belongs to $\widetilde{l}_*^u$. Let $w^u(\overline{z}_*)$ be the preimage of the curve $W^u(z_*)$ passing through the point $\overline{z}_*$. By construction $w^u(\overline{z}_*)$ belongs to $\widetilde{l}_*^u$ and has exactly two boundary points, one of which is the point ν_*^{u+}.

Denote by $\widetilde{l}_*^{u+}$ the connected component of the set $w^u(\overline{z}_*) \setminus z_*$ for which the point ν_*^{u+} is a boundary point. Introduce a parameter $t \in (0, +\infty)$ on the curve $\widetilde{l}_*^{u+}$ such that $\widetilde{l}_*^{u+}(t)$ tends to the point ν_*^{u+} as $t \to +\infty$.

The curve $\widetilde{L}_*^{u+}(t) = \pi(\widetilde{l}_*^{u+}(t))$ intersects the arc int λ^s of a quasi-transversal C in a countable set of points as $t \to +\infty$. Then there is a countable set of curves c_n, $c_n \in \{c\}$, $\pi(c_n) = C$, $n \in \mathbb{Z}^+$, such that the intersection $\widetilde{l}_*^{u+}(t) \cap c_n$ consists of exactly one point. Denote by c_n^+, $c_n^- \in S_\infty$ the boundary points of c_n.

The point ν_*^{u+} is the topological limit of the sequence $\{c_n\}$. Then there is a number N such that the arc (c_N^+, c_N^-) of S_∞ which contains the point ν_*^{u+} does not contain the points μ_*^{u+}, μ_*^{u-}. As the curve $\widetilde{l}_*^{u+}(t)$ belongs to the topological limit of the sequence w_k^u, there is a number N_* such that the curve $w_{N_*}^u$ intersects the curve c_N at most in two points, that is impossible, see Fig. 9.6.

Consider the case $\sigma = s$. In this case we use the statement of Theorem 9.1, which is true under the assumption that the diffeomorphism f is structurally stable. Let us show that there is a point $\overline{z}_* \in \widetilde{L}_*^s$ which has the following properties:

(a) $\overline{z}_* \in \Lambda_*$;

(b) the preimage $\widetilde{l}_*^{s+}$ of the connected component $\widetilde{L}_*^{s+}$ of the set $W^s(z_*) \setminus z_*$ belongs to $\widetilde{l}_*^s$ and has the point ν_*^{s+} as its boundary point.

Let x_* be any point of $\widetilde{L}_*^s$. There are two cases:

(1) x_* belongs to the stable manifold of a point of Λ_*;

(2) x_* does not belong to the stable manifold of any point of Λ_*.

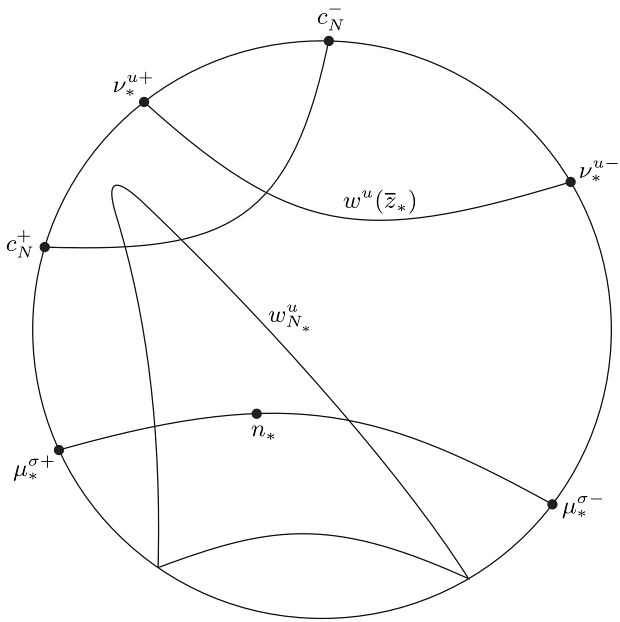

Figure 9.6

In the first case there are two subcases:

(a_1) $W^s(x_*)$ does not contain a boundary periodic point;

(a_2) $W^s(x_*)$ contains a boundary periodic point p of Λ_*.

In subcase (a_1) we show that the point $z_* = x_*$ is the required one.

Let $\overline{z}_*$ be the preimage of the point z_* belonging to the set $\widetilde{l}_*^s$. Then the curve $w^s(\overline{z}_*)$ $(\pi(w^s(\overline{z}_*)) = W^s(z_*))$ belongs to the set l_*^s.

As $w^s(\overline{z}_*)$ has distinct boundary points on S_∞, then one of the connected components of the set $w^s(\overline{z}_*) \setminus \overline{z}_*$ (denote it by $\widetilde{l}_*^{s+}$) has the point ν_*^{s+} as its boundary point.

In subcase (a_2) consider the complex L_{pq} which was constructed in Theorem 9.1, where q is a boundary periodic point of the set Λ (p may coincide with q).

By construction $L_{pq} \subset \widetilde{L}_*^s$. Then by properties of the universal covering there is a connected complex $l_{\overline{pq}}$ belonging to $\widetilde{l}_*^s$ such that $\pi(l_{\overline{pq}}) = L_{pq}$, $\pi(\overline{p}) = p$, $\pi(\overline{q}) = q$ (if $p = q$, then the points $\overline{p}$, $\overline{q}$ are congruent).

The curves $w^s(\overline{p})$, $w^s(\overline{q})$ belong to $\widetilde{l}^s_*$ have distinct boundary points on S_∞. Then one of these curves has the point ν^{s+}_* as its boundary point on S_∞ and hence either $\overline{p}$ or $\overline{q}$ is the required point $\overline{z}_*$.

In case (2) by Corollary 9.1 the point x_*, belongs to the complex L_{pq} for some boundary periodic points p, q, which belong to $\widetilde{L}^s_*$ by construction. Then we get subcase (a_2) of case (1).

Thus there is a point $z_* \in \widetilde{L}^s_*$ such that $z_* \in \Lambda_*$, and the preimage $\widetilde{l}^{s+}_*$ of one of the connected components $\widetilde{L}^{s+}_*$ of the set $W^s(z_*) \setminus z_*$ has the point ν^{s+}_* as its boundary point. The component $\widetilde{L}^{s+}_*$ contains a set dense in Λ_* and consequently intersects the arc int λ^u of a quasi-transversal C in a countable set of points. So we come to a contradiction as in the case $\sigma = u$.

Thus there is a constant $K^\sigma(\overline{y})$ such that $d(w^\sigma(t), g^\sigma_t) < K^\sigma(\overline{y})$ for all $t \in [0, +\infty)$, $\sigma \in \{s, u\}$.

Let $\overline{x}$ be an arbitrary preimage of the point x on Δ, $w^u(\overline{x})$ a preimage of the manifold $W^u(x)$, and $l^\sigma(t) \subset w^\sigma(\overline{x})$ a preimage of the ray $L^\sigma(t)$.

Denote by ξ^{u+}, ξ^{u-}, the boundary points of the curve $w^u(\overline{x})$. For the curve $w^s(\overline{x})$ there are exactly two possibilities:

(a) the manifold $W^s(x)$ does not contain a boundary periodic point; then the curve $w^s(\overline{x})$ has exactly two boundary points ξ^{s+}, ξ^{s-} which lie on the circle at infinity;

(b) the manifold $W^s(x)$ contains a boundary periodic point q; then the curve $w^s(\overline{x})$ has exactly two boundary points $\mu^s_\infty \in S_\infty$ and $\overline{q} \in \Delta$, where $\overline{q}$ is a preimage of the point q.

We show that the conclusion of the theorem is true for the ray $l^s(t)$ in case (b). The proof for the ray $l^u(t)$ and for the ray $l^s(t)$ in case (a) is analogous and easier.

Denote by $\widehat{\mu}^{u+}$, $\widehat{\mu}^{u-}$ the boundary points of the curve $w^u(\overline{q})$ such that during the clockwise movement along the circle at infinity starting from the point μ^s_∞, we obtain the following order of points: μ^s_∞, ξ^{u-}, $\widehat{\mu}^{u-}$, $\widehat{\mu}^{u+}$, ξ^{u+}.

As the projection of the ray $w^{s+}(t) \subset w^s(\overline{y})$, $t \in [0, +\infty)$, contains the set dense in Λ_*, there is a sequence of points $\{\overline{y}_n\}$ on the arc $(\overline{x}, \mu^{u+})^u \subset w^u(\overline{x})$ converging to the point $\overline{x}$ and such that $\pi(\overline{y}_n) \in \pi(w^{s+}(t))$ for each $n \in \mathbb{Z}^+$. There is a point $\overline{p}$ which is a preimage of a periodic point p of the set Λ_* such that $\widehat{\mu}^{u+}$ is the boundary point of the curve $w^u(\overline{p})$. Denote by $\widehat{\mu}^s_\infty$ the boundary point of the curve $w^s(\overline{p})$ belonging to the circle at infinity and denote by μ^s_n, $\widehat{\mu}^s_n$ the boundary points of the curve $w^s(\overline{y}_n)$. We assume that the notations were chosen so that μ^s_n belongs to an arc (ξ^{u+}, ξ^{u-}) of the circle at infinity containing the point μ^s_∞, and $\widehat{\mu}^s_n$ belongs to an arc $(\widehat{\mu}^{u-}, \widehat{\mu}^{u+})$ not containing the point μ^s_∞.

Since the sequence $\overline{y}_n$ converges to the point $\overline{x}$, then the sequence μ^s_n converges to the point μ^s_∞, the sequence $\widehat{\mu}^s_n$ converges to the point $\widehat{\mu}^s_\infty$, and the curve $w^s(\overline{x})$ belongs to the topological limit of the sequence of the curves $w^s(y_n)$.

Denote by g^s_n the geodesics with the boundary points μ^s_n, $\widehat{\mu}^s_n$ and denote by g^s_* the geodesic with the boundary points μ^s_∞, $\widehat{\mu}^s_\infty$. By construction the geodesic g^s_* is the topological limit of the geodesics g^s_n.

We show that the distance from any point on the curve $l^s(t)$ to the geodesic g^s_* is bounded above by a constant $K(\overline{y})$. Suppose the contrary.

Then there is a point $\overline{z} \in l^s(t)$ such that $d(\overline{z}, \overline{z}_*) > K(\overline{y})$, where $\overline{z}_*$ is the foot of the perpendicular dropped from the point $\overline{z}$ on the geodesic g^s_*. As the point $\overline{z}$ belongs to the topological limit of the sequence w^s_n, there is a sequence of points $\overline{z}_n \in w^s_n$ converging to the point $\overline{z}$.

Denote by $\overline{v}_n$ the foot of the perpendicular dropped from the point $\overline{z}_n$ on the geodesic g^s_n. Then, by construction, the sequence $\{\overline{v}_n\}$ converges to the point $\overline{z}_*$, and the sequence of the distances $d(\overline{z}_n, \overline{v}_n)$ converges to the distance $d(\overline{z}, \overline{v}_*)$.

Hence there is a number n_0 such that $d(\overline{z}_{n_0}, \overline{v}_{n_0}) > K(\overline{y})$. Since all curves w^s_n are congruent to the curve $w^s(\overline{y})$, there is an element γ_{n_0} of the group G such that the point $\gamma_{n_0}(\overline{z}_n)$ belongs to $w^{s+}(t)$ and $\gamma_{n_0}(\overline{v}_n)$ belongs to g^s. As the element γ_{n_0} preserves distance, then $d(\gamma_{n_0}(\overline{z}_n), \gamma_{n_0}(\overline{v}_n)) >$ $> K(\overline{y})$, which is impossible.

Since the geodesics having a common boundary point at the circle at infinity exponentially approach, the statement of the theorem is true for any geodesic for which the point μ^s_∞ is a boundary point. Theorem 9.3 is completely proved. □

Bibliographic Notes and Panoramas

Chapter 9. The problem considered in this Chapter arised from Anosov's question posed at the beginning of the 1970s: whether there exist lifts on a universal covering (hyperbolic plane or Euclidian plane) for curves without self-intersections on a surface of nonpositive Euler characteristic such that they have asymptotic directions but unboundedly deviates from the geodesic rays with the same asymptotic direction. Negative answer for Anosov's question was claimed by V. Pupko in 1967 [195]. However V. Grines noticed that from the example of R. Robinson and R. Williams [199] follows the existence of a curve with unbounded deviation from co-asymptotic geodesic. Such curve is a stable manifold a point belonging to one dimensional expanding attractor of A-diffeomorphism of pretzel (surface of genus 2).

Notice that the A-diffeomorphism of the example by R. Robinson and R. Williams is structurally unstable. So, it is natural to investigate the problem of bounded deviation for structurally stable surface diffeomorphisms.

As noticed V. Grines, the problem on unbounded deviation intimately close with an interrelation between structural stability and asymptotic behavior of stable and unstable manifolds of points of one-dimensional basic sets. In 1997 [86], he proved next statement: if p is a boundary periodic point of a one-dimensional attractor Ω and $W^{s\varnothing}(p)$ is the connected component of the set $W^s(p) \setminus p$ which does not intersect Ω, then the set $\partial W^{s\varnothing}(p))$ consists of exactly one source periodic point, where $\partial W^{s\varnothing}(p) = \overline{W}^{s\varnothing} \setminus (W^{s\varnothing}(p) \cup p)$. Thus, if the nonwandering set of a structurally stable diffeomorphism f contains a one-dimensional attractor then it must also contain a zero-dimensional source. Notice that in 1974 R. Plykin [186] proved that if nonwandering set of A-diffeomorphism f of sphere or torus, not necessary structural stable, contains a one-dimensional attractor then it also contains a source periodic point.

(9.1). Theorem 9.1 was proved in [86].

(9.2). Theorem 9.3 was proved in [86].

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List of standard notations

Here is the list of standard notations that are used (sometimes without commentary):

$\mathbb{R}$ (resp. $\mathbb{C}$) is the set of real (complex) numbers; $\mathbb{R}_+$ is the set of nonnegative real numbers;

$\mathbb{Z}_+ = \mathbb{Z} \cap \mathbb{R}_+$ is the set of non-negative integers.

$\mathbb{R}^n$ is the n-dimensional ($n \geqslant 2$) Euclidean space; $\mathbb{R}^n_+ = \mathbb{R}^+ \times \mathbb{R}^{n-1}$ is a half-space;

$\mathbb{Z}^n$ is the integer n-dimensional ($n \geqslant 1$) lattice (vectors of $\mathbb{R}^n$ with integer coordinates);

S^1 is a circle;

$f \circ g$ is the composition of functions f and g defined by the rule $f \circ g(x) = f(g(x))$.

If $f\colon N \to N$ is one-to-one, f^n means n-th iteration, $f^n = \underbrace{f \circ f \circ \ldots \circ f}_{n}$, $n \in \mathbb{Z}$.

If $f\colon N \to N$ is not invertable, $f^{-1}(D)$ means the full preimage of $D \subset N$.

The end of a proof of statements (theorems, lemmas, so on) is marked by $\square$.

The end of examples is sometimes marked by $\diamondsuit$.

$\operatorname{clos} N = \operatorname{clos}(N)$ is a topological closure of a set N; $\operatorname{int} N = \operatorname{int}(N)$ is an interior of N.

$U_\varepsilon(N)$ is an ε-neighborhood of subset N.

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