



MATHEMATICAL PROBLEMS OF NONLINEARITY

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Cantor Type Basic Sets of Surface A -endomorphisms

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The paper is devoted to an investigation of the genus of an orientable closed surface M^2 which admits A -endomorphisms whose nonwandering set contains a one-dimensional strictly invariant contracting repeller Λ_r with a uniquely defined unstable bundle and with an admissible boundary of finite type. First, we prove that, if M^2 is a torus or a sphere, then M^2 admits such an endomorphism. We also show that, if Ω is a basic set with a uniquely defined unstable bundle of the endomorphism $f: M^2 \rightarrow M^2$ of a closed orientable surface M^2 and f is not a diffeomorphism, then Ω cannot be a Cantor type expanding attractor. At last, we prove that, if $f: M^2 \rightarrow M^2$ is an A -endomorphism whose nonwandering set consists of a finite number of isolated periodic sink orbits and a one-dimensional strictly invariant contracting repeller of Cantor type Ω_r with a uniquely defined unstable bundle and such that the lamination consisting of stable manifolds of Ω_r is regular, then M^2 is a two-dimensional torus \mathbb{T}^2 or a two-dimensional sphere \mathbb{S}^2 .

Keywords: A -endomorphism, regular lamination, attractor, repeller, strictly invariant set

1. Introduction and main results

By an *endomorphism* we mean a C^1 -smooth surjective map of a manifold to itself. If an endomorphism f is a one-to-one map and the inverse mapping f^{-1} is C^1 -smooth, then f becomes a *diffeomorphism*. We see that the notion of endomorphism generalizes the notion of diffeomorphism.

We denote by $\text{End}(M)$ the space of endomorphisms of a closed manifold M endowed with the standard C^1 -topology. Let $g \in \text{End}(M)$. The *orbit* or *g -orbit* of the point $x_0 \in M$ is a countable or finite set $\{x_i\}_{i=-\infty}^{\infty} = O(x_0)$ such that $g(x_i) = x_{i+1}$ for any $i \in \mathbb{Z}$. The set $\{x_i\}_{i=0}^{\infty} = O^+(x_0) \subset O(x_0)$ is called *the positive semi-orbit* of the point x_0 . A positive semi-orbit

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is uniquely defined by the original point x_0 , while the set of orbits passing through x_0 (which we denote $Q(x_0)$) is generally a set of cardinality continuum. To a fixed element $\alpha \in Q(x_0)$ there is uniquely appropriated the orbit $\{x_i^\alpha\}_{i=-\infty}^\infty = O^\alpha(x_0)$, where $g(x_i^\alpha) = x_{i+1}^\alpha$ for any $i \in \mathbb{Z}$ and $x_0^\alpha = x_0$. The set $\{x_i^\alpha\}_{i=-\infty}^0 = O^{\alpha-}(x_0)$ is called *the negative semi-orbit* of the orbit $O^\alpha(x_0)$. Below for simplicity we will omit index α when we consider an orbit of the point x_0 . A point $x \in M$ of the endomorphism $f: M \rightarrow M$ is called *nonwandering* if for any neighborhood U of a point x there is $i > 0$ such that $f^i(U) \cap U \neq \emptyset$. The set of nonwandering points forms the *nonwandering set* of the endomorphism f and is denoted by $NW(f)$. It is known that a nonwandering set is always forward-invariant, that is, $f(NW(f)) \subset NW(f)$.

An orbit $O(x_0)$ is called *hyperbolic* if there is a continuous splitting of the tangent space

$$\mathbb{T}_{O(x_0)}M = \bigcup_{i=-\infty}^{\infty} \mathbb{T}_{x_i}M = \mathbb{E}^s \oplus \mathbb{E}^u = \bigcup_{i=-\infty}^{\infty} \mathbb{E}_{x_i}^s \oplus \mathbb{E}_{x_i}^u$$

which is invariant with respect to Dg and such that

$$0 < \|Dg^m(v)\| \leq c\mu^m\|v\|, \quad \|Dg^m(w)\| \geq c^{-1}\mu^{-m}\|w\|$$

where $v \in \mathbb{E}^s \setminus \{0\}$, $w \in \mathbb{E}^u$, $\forall m \in \mathbb{N}$

for some constants $c > 0$, $0 < \mu < 1$ and a Riemannian metric on $\mathbb{T}M$.

Note that the unstable subbundle $\mathbb{E}^u(x_0)$ depends, in general, on the negative semi-orbit $\{x_i\}_{i=-\infty}^0$, so we will write $\mathbb{E}^u(x_0, \{x_i\}_{i=-\infty}^0)$. It may be $\mathbb{E}^u(x_0, \{x_i\}_{i=-\infty}^0) \neq \mathbb{E}^u(y_0, \{y_i\}_{i=-\infty}^0)$ for $x_0 = y_0$ with $\{x_i\}_{i=-\infty}^0 = O^-(x_0) \neq O^-(y_0) = \{y_i\}_{i=-\infty}^0$. Such an effect is impossible for a stable subbundle $\mathbb{E}^s(x_0)$ that depends only on the point x_0 .

The set $\Lambda \subset M$ is called *hyperbolic* if $f(\Lambda) = \Lambda$, any orbit lying in Λ is hyperbolic and the constants $c > 0$, $0 < \mu < 1$ in the above estimates do not depend on the choice of the orbit (therefore, one sometimes talks about *uniform hyperbolicity*). Note that the equality $f(\Lambda) = \Lambda$ guarantees that for any point $x_0 \in \Lambda$ there is at least one negative semi-orbit lying in Λ .

By analogy with a diffeomorphism satisfying the Smale axiom A , an endomorphism $f: M \rightarrow M$ is called an *A-endomorphism* if its nonwandering set $NW(f)$ is hyperbolic and periodic points are dense in $NW(f)$. Recall that a map $N \rightarrow N$ is called *transitive* if there exists a point $x \in N$ whose positive semi-orbit is dense in N .

In [19], the Spectral Theorem for an A-endomorphisms is proved, which is a generalization of the Spectral Theorem by Smale [21] for A-diffeomorphisms. According to [19], the nonwandering set $NW(f)$ of the A-endomorphism $f: M \rightarrow M$ is represented in a unique way up to numbering as a union of closed and pairwise disjoint sets

$$NW(f) = \Omega_1 \cup \dots \cup \Omega_l, \quad \Omega_i \cap \Omega_j = \emptyset \quad \text{for } i \neq j,$$

such that $f(\Omega_j) = \Omega_j$ and $f|_{\Omega_j}: \Omega_j \rightarrow \Omega_j$ is transitive for all $1 \leq j \leq l$. The sets $\Omega_1, \dots, \Omega_l$ are called *basic sets*.

To date, there are a number of complete classification results for one-dimensional basic sets of A-diffeomorphisms on closed surfaces M^2 [2–5, 11] and manifolds of higher dimension [6, 7, 12, 13]. It is proved that one-dimensional basic sets on M^2 are Cantor laminations, that is, each basic set Ω is the union of pairwise disjoint curves locally arranged as a direct product of a segment and the Cantor set. Moreover, each leaf of such a lamination is homeomorphic in the intrinsic topology to the line \mathbb{R} , and the admissible boundary from the interior of any connected component of the set $M^2 \setminus \Omega$ consists of a finite number of periodic leaves.

For A-endomorphisms, which are not A-diffeomorphisms, there are few classes of systems for which it is possible to describe the structure of basic sets and obtain complete classification



results. Such classes include interval and circle endomorphisms, endomorphisms arising in complex dynamics on the Riemann sphere, and expanding endomorphisms of manifolds of higher dimension [20]. It should be noted that the structure of the basic sets of A -endomorphisms has not yet been studied so exhaustively even for endomorphisms of surfaces.

In this paper, we consider one-dimensional strictly hyperbolic basic sets of A -endomorphisms of closed surfaces. From the point of view of applications, an important role is played by basic sets which are attractors and repellers. We investigate the topology of orientable close surfaces which admits A -endomorphisms with such basic sets.

Let us start presenting the main results. A hyperbolic set Λ is called *uniquely hyperbolic*, or a set *with a uniquely defined unstable bundle*, if the unstable subbundle $\mathbb{E}^u(x_0, \{x_i\}_{i=-\infty}^0) = \mathbb{E}^u(x_0)$ does not depend on the negative semi-orbit $\{x_i\}_{i=-\infty}^0$ for any point $x_0 \in \Lambda$.

A basic set Ω_a is called an *attractor* if there exists a neighborhood U of the set Ω_a such that

$$f(\text{clos } U) \subset U, \quad \bigcap_{i \geq 0} f^i(U) = \Omega_a.$$

An attractor Ω_a is called *expanding* if its topological dimension is equal to the dimension of the unstable subbundle $\mathbb{E}^u(x_0, \{x_i\}_{i=-\infty}^0)$ for any point $x_0 \in \Omega_a$ and any negative semi-orbit $\{x_i\}_{i=-\infty}^0 \subset \Omega_a$.

A basic set Ω_r is called a *repeller* if there exists a neighborhood U of the set Ω_r such that

$$\text{clos } U \subset f(U), \quad \bigcap_{i \geq 0} f^{-i}(U) = \Omega_r.$$

A repeller Ω_r is called *contracting* if its topological dimension is equal to the dimension of the stable subbundle $\mathbb{E}^s(x_0)$ for any point $x_0 \in \Omega_r$.

A basic set Ω is called *strictly invariant* if $f^{-1}(\Omega) = \Omega = f(\Omega)$. It follows directly from the definition of hyperbolicity that, in a certain neighborhood of the basic set, an endomorphism is a local diffeomorphism. Below, saying that an endomorphism is not a diffeomorphism, we will assume for simplicity that **an endomorphism is a finitely multiple cover of a multiplicity at least two, with the exception of a finite number of branching points**.

Let K be an open set of a topological space N , and ∂K be the boundary of the set K . A subset of $\delta(K) \subset \partial K$ is called the admissible boundary from K (sometimes one says, admissible boundary from interior K) if for each point $z \in \partial K$ there exists an arc d with the end point z such that $d \setminus \{z\} \subset K$.

Let Ω be a strictly invariant one-dimensional basic set with a uniquely defined unstable bundle of an A -endomorphism $f: M^2 \rightarrow M^2$. It follows from the closeness of the set Ω and Proposition 2.7 [15] that, if Ω is an expanding attractor, then unstable manifolds of points from Ω form the lamination $\mathcal{L}(\Omega) = \bigcup_{x \in \Omega} W^u(x)$. Similarly, if Ω is a contracting repeller, then its stable manifolds also form a lamination, which we will denote by the same symbol \mathcal{L} .

We will say that $\mathcal{L}(\Omega)$ is a *Cantor lamination with an admissible boundary of finite type* if the following conditions hold:

- 1) all leaves of lamination $\mathcal{L}(\Omega)$ are the images of injective immersion of the line \mathbb{R} , and the union of leaves of $\mathcal{L}(\Omega)$ is locally homeomorphic to the product of a segment and Cantor set;
- 2) for any component K of the set $M^2 \setminus \mathcal{L}(\Omega)$ the boundary $\delta(K)$ admissible from K consists of a finite number of invariant manifolds (unstable manifolds if Ω is an expanding attractor, and stable manifolds if Ω is a contracting repeller).

We will say that Ω is a *basic set of Cantor type with an admissible boundary of finite type* if $\mathcal{L}(\Omega)$ is a Cantor lamination with an admissible boundary of finite type.

Denote by \mathbb{T}^2 the two-dimensional torus, and by \mathbb{S}^2 , the two-dimensional sphere.

Theorem 1. *There is an endomorphism $f: M^2 \rightarrow M^2$, where $M^2 = \mathbb{T}^2$ or $M^2 = \mathbb{S}^2$, such that the nonwandering set of f contains a one-dimensional strictly invariant contracting repeller Ω_r with uniquely defined unstable bundle and with an admissible boundary of finite type.*

It is known that on any closed surface M^2 there is an A-diffeomorphism $f: M^2 \rightarrow M^2$ with a one-dimensional expanding attractor and a one-dimensional contracting repeller of Cantor type (in both cases, the basic sets have an admissible boundary of finite type). The following statement contrasts with the situation where an endomorphism is not a diffeomorphism, even if the attractor of the endomorphism is strictly invariant and has a uniquely defined unstable bundle (for a diffeomorphism, this is done automatically).

Theorem 2. *Let Ω be a basic set with a uniquely defined unstable bundle of endomorphism $f: M^2 \rightarrow M^2$ of a closed orientable surface M^2 , and suppose f is not a diffeomorphism. Then Ω cannot be a Cantor expanding attractor with an admissible boundary of finite type.*

The A-endomorphisms satisfying Theorem 1 on the two-dimensional torus \mathbb{T}^2 and the two-dimensional sphere \mathbb{S}^2 have nonwandering sets consisting of a finite number of isolated sinks and a one-dimensional strictly invariant contracting repeller of Cantor type. In addition, the laminations corresponding to one-dimensional repellers continue on \mathbb{S}^2 to foliations with a finite number of singularities of a nonzero index, and on \mathbb{T}^2 to foliations without singularities. It is natural to consider the existence of A-endomorphisms with similar properties on other closed surfaces. Note that the index of a singularity of foliation in the component C of the set $M^2 \setminus \mathcal{L}(\Lambda_r)$ was related to the number of stable manifolds belonging to the admissible boundary of C as follows.

Let ν be the number of leaves (stable manifolds) in the boundary of the component C admissible from C . Then in C the foliation has the singularity of the index $1 - \frac{\nu}{2}$. On \mathbb{S}^2 , four components of the set $\mathbb{S}^2 \setminus \Omega$ have exactly one admissible leaf. Therefore, in each such component the singularity has the index $\frac{1}{2}$. On \mathbb{T}^2 , according to the construction, $\nu = 2$ for all components of the set $\mathbb{T}^2 \setminus \Omega$. Therefore, there are no singularities in each component.

We call the lamination \mathcal{L} *regular* if it continues to a foliation \mathcal{F} with the following property. Let C be a component of the set $M^2 \setminus \mathcal{L}$, and $\delta(C)$ be its boundary admissible from C , consisting of ν leaves. Then \mathcal{F} has exactly one singularity of the index $1 - \frac{\nu}{2}$ in C provided $\nu \neq 2$. If $\nu = 2$, then there are no singularities in C . The following statement holds.

Theorem 3. *Let $f: M^2 \rightarrow M^2$ be an A-endomorphism on a closed orientable surface M^2 , such that the nonwandering set $NW(f)$ consists of a finite number of isolated periodic sink orbits and a one-dimensional strictly invariant contracting repeller of Cantor type Λ_r with a uniquely defined unstable bundle. Suppose that the lamination $\mathcal{L}(\Lambda_r)$ is regular. Then M^2 is the two-dimensional torus \mathbb{T}^2 or the two-dimensional sphere \mathbb{S}^2 .*

Note that the analog of Theorem 2 does not hold, in general, for endomorphisms of three-dimensional manifolds. Indeed, consider a 3-dimensional torus \mathbb{T}^3 , which we represent as the product of $\mathbb{T}^2 \times \mathbb{S}^1$ of the two-dimensional torus \mathbb{T}^2 and the circle \mathbb{S}^1 . On the first multiplier \mathbb{T}^2 , we define a DA-diffeomorphism $f_0: \mathbb{T}^2 \rightarrow \mathbb{T}^2$ with an isolated source and a nontrivial one-dimensional expanding attractor Λ_1 , and on the second multiplier \mathbb{S}^1 , we define the expanding endomorphism $E_2: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ of the form $x \mapsto 2x \pmod{1}$. The endomorphism $(z, t) \rightarrow (f_0(z); E_2(t))$, $z \in \mathbb{T}^2$, $t \in \mathbb{S}^1$, has the expanding strictly invariant two-dimensional attractor $\Lambda_1 \times \mathbb{S}^1$ with a uniquely defined unstable Cantor type bundle, that is, $\Lambda_1 \times \mathbb{S}^1$ is locally homeomorphic to the

product of a Cantor set on a two-dimensional plane. At the same time, each connected component of the set $\mathbb{T}^3 \setminus (\Lambda_1 \times \mathbb{S}^1)$ has a boundary admissible from $\mathbb{T}^3 \setminus (\Lambda_1 \times \mathbb{S}^1)$ consisting of two lamination leaves $\mathcal{L}(\Lambda_1 \times \mathbb{S}^1)$, that is, an expanding attractor $\Lambda_1 \times \mathbb{S}^1$ has an admissible boundary of finite type.

Let us give some other examples of one-dimensional attractors and repellers of endomorphisms of two-dimensional manifolds. Let $h: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be a Morse–Smale diffeomorphism with one source s_r and one sink s_a . Then the nonwandering set of the endomorphism $(h; E_2): \mathbb{S}^1 \times \mathbb{S}^1 = \mathbb{T}^2 \rightarrow \mathbb{T}^2$ contains the strictly invariant one-dimensional expanding attractor $\{s_a\} \times \mathbb{S}^1$ with a uniquely defined unstable bundle and a strictly invariant one-dimensional repeller $\{s_r\} \times \mathbb{S}^1$ with a uniquely defined unstable bundle (both are homeomorphic to \mathbb{S}^1). Note that the repeller is not strictly contacting, since its stable bundle is zero-dimensional.

The nonwandering set of the endomorphism $SH: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ constructed in [20] consists of an isolated sink s_1 and a nontrivial (zero-dimensional) repeller Σ_r , homeomorphic to the Cantor set C . Then the nonwandering set of the endomorphism $(SH; E_2): \mathbb{S}^1 \times \mathbb{S}^1 = \mathbb{T}^2 \rightarrow \mathbb{T}^2$ consists of a one-dimensional expanding attractor $\{s_1\} \times \mathbb{S}^1$, which is not strictly invariant, and a one-dimensional repeller R which is noncontracting and is homeomorphic to $C \times \mathbb{S}^1$. Both basic sets have uniquely defined unstable bundles.

There are repellers that are not submanifolds. It is known, for example, that endomorphisms of surfaces arising in complex dynamics can have one-dimensional repellers having a fractal structure.

The structure of the article is as follows. Section 2 provides preliminary information and proves auxiliary results. The main theorems are proved in Section 3.

2. Auxiliary results

Let Λ be the hyperbolic set of a C^k -endomorphism $f: M \rightarrow M$ ($k \geq 1$), and let $p \in \Lambda$. It is known [15, 18, 19] that for a sufficiently small $\delta > 0$ set

$$W_{\delta,loc}^s(p) = \{x \in M: \varrho(f^i(p); f^i(x)) < \delta \forall i \geq 0\}$$

is a C^k -embedded open disk $\mathbb{B}^{\dim \mathbb{E}^s(p)}$ with the tangent space $\mathbb{E}^s(p)$ at the point p , and for any point $x \in W_{\delta,loc}^s(p)$

$$\varrho(f^i(p); f^i(x)) \rightarrow 0 \text{ exponentially fast as } i \rightarrow \infty.$$

The set $W_{\delta,loc}^s(p)$ is called a *local stable manifold* at the point p . A *globally stable manifold* (or, simply, a *stable manifold*) at the point p is called a set

$$W^s(p) = \{x \in M: \varrho(f^i(p); f^i(x)) \rightarrow 0 \text{ as } i \rightarrow \infty\}.$$

If Λ is a compact hyperbolic set, then the number δ can be taken so that local stable manifolds $W_{\delta,loc}^s(z)$ are defined for all points $z \in \Lambda$. Then the global stable manifold $W^s(p)$ can be represented as follows:

$$W^s(p) = \bigcup_{i \geq 0} \text{Comp}_p f^{-i} [W_{\delta,loc}^s(p_i)], \text{ where } p_i = f^i(p),$$

where $\text{Comp}_z K$ denotes the connected component of the set K containing z .

Unlike stable bundles \mathbb{E}^s , unstable bundles depend not only on the point, but also on the negative half-orbit of the point. Let $O^-(p) \subset \Lambda$ be the negative semi-orbit of the point $p \in \Lambda$. According to [15, 18, 19], for a sufficiently small $\delta > 0$ set

$$W_{\delta,loc}^u(O^-(p)) = \{x \in M: \exists O^-(x), \varrho(p_{-i}; x_{-i}) < \delta \forall i \geq 0, \text{ where } p_{-i} \in O^-(p), x_{-i} \in O^-(x)\}$$

is a C^k -embedded open disk $\mathbb{B}^{\dim \mathbb{E}^u(O^-(p))}$ with a tangent space, which we will denote by $\mathbb{E}^u(O^-(p))$, at the point p . Moreover, for any point above $x \in W_{\delta,loc}^u(O^-(p))$

$$\varrho(p_{-i}; x_{-i}) \rightarrow 0 \text{ exponentially fast as } i \rightarrow \infty.$$

The set $W_{\delta,loc}^u(O^-(p))$ is called a *local unstable manifold* with respect to the negative semi-orbit $O^-(p)$ at the point p . A *globally unstable manifold* (or, simply, an unstable manifold) with respect to the negative half-orbit $O^-(p)$ at the point p is called a set

$$W^u(O^-(p)) = \{x \in M: \exists O^-(x), \varrho(p_{-i}; x_{-i}) \rightarrow 0 \text{ as } i \rightarrow \infty, \\ \text{where } p_{-i} \in O^-(p), x_{-i} \in O^-(x)\},$$

which can be represented in the form

$$W^u(O^-(p)) = \bigcup_{i \geq 0} f^i(W_{\delta,loc}^u(O^-(p_{-i}))), \text{ where } p_{-i} \in O^-(p).$$

Stable and unstable manifolds are called invariant manifolds.

We will need some auxiliary statements of a topological nature.

Lemma 1. *Let X be a topological space and \mathcal{B} its base of topology consisting of connected sets. Let $N \subset X$ be a closed subset. Then each connected component of the complement $X \setminus N$ is an open set.*

Proof. Consider an arbitrary point $x \in X \setminus N$. Since $X \setminus N$ is an open set, there is a connected open set $U \in \mathcal{B}$ such that $x \in U$. Let K be a connected component $X \setminus N$ containing the point x . Since K is the largest connected subset of $X \setminus N$ containing the point x , and U is connected, the inclusion of $U \subset K$ takes place. The proved statement now immediately follows from the arbitrariness of the choice of the point x . \square

Corollary 1. *Let M be a closed manifold, and $N \subset M$ be its closed subset. Then each connected component of $M \setminus N$ is an open set.*

Proof. Let us take a topology base \mathcal{B} for the manifold M as the union of open sets $\varphi(B_r(p))$, where $\varphi: \mathbb{R}^n \rightarrow M$, a coordinate map of atlas of manifolds M and $B_r(p)$, is the open ball of radius r . Now the required statement follows from Lemma 1. \square

Lemma 2. *Let M be a simply connected topological space, and let the continuous map $f: M \rightarrow M$ be a cover (in particular, a local homeomorphism). Then f is a homeomorphism.*

Proof.

Suppose that there are two different points $x_1, x_2 \in M$, $x_1 \neq x_2$, such that $f(x_1) = f(x_2) = x_0$. Let $G_0: [0; 1] \rightarrow M$ be a mapping of the segment $[0; 1]$ such that $G_0(0) = x_1$, $G_0(1) = x_2$. Then the mapping $F_0 = f \circ G_0$ has the property $F_0(0) = F_0(1)$. As M is simply connected, there is a homotopy $F_t: [0; 1] \rightarrow M$, $0 \leq t \leq 1$, $F_1([0; 1]) = x_0$. Since f is a covering, by virtue of the covering homotopy theorem, there exists a homotopy $G_t: [0; 1] \rightarrow M$, $0 \leq t \leq 1$, such that $F_t = f \circ G_t$ for all $0 \leq t \leq 1$. Therefore, $F_1([0; 1]) = f \circ G_1([0; 1]) = f(x_1) = f(x_2)$. Hence, the path $G_1([0; 1])$ connects in $f^{-1}(x_0)$ various points $G_0(0) = x_1$, $G_0(1) = x_2$, which contradicts the fact that the set $f^{-1}(x_0)$ consists of isolated points. The resulting contradiction proves the required statement, since f is a local homeomorphism. \square

Let us recall the definition of lamination. A detailed explanation of the theory of laminations can be found in [1]. $C^{r,l}$, a smooth d -dimensional lamination on an n -dimensional manifold M^n ($1 \leq d \leq n-1$, $0 \leq l \leq r \leq \infty$), is a closed set $\mathcal{L} \subset M^n$, which is the union of $\bigcup_{\alpha} L_{\alpha}$ pairwise disjoint images of L_{α} of some d -dimensional manifolds with respect to some C^r -smooth injective immersion, and for any point $x \in \mathcal{L}$ there is its neighborhood $U(x)$ and C^l -diffeomorphism $\psi: U(x) \rightarrow \mathbb{R}^n$ such that any connected component of the intersection $U(x) \cap L_{\alpha}$ is mapped by a diffeomorphism ψ to an open subset of a d -dimensional subspace $\{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_{d+1} = c_{d+1}, \dots, x_n = c_n\}$, while the restriction of ψ on any connected component of the intersection of $U(x) \cap L_{\alpha}$ is a C^r -diffeomorphism on the image.

3. Proof of the main results

Proof of Theorem 1. The existence of an A-endomorphism $f: \mathbb{T}^2 \rightarrow \mathbb{T}^2$ of torus \mathbb{T}^2 with a Cantor type one-dimensional contracting repeller satisfying Theorem 1 was proved in the work [8] of V. Z. Grines, E. V. Zhuzhoma and E. D. Kurenkov, see also [10]. The proof consists in constructing the so-called Smale surgery operation [21] for the algebraic Anosov endomorphism $\mathbb{T}^2 \rightarrow \mathbb{T}^2$.

For the sphere \mathbb{S}^2 , the proof is based on the method of constructing the Plykin attractor on \mathbb{S}^2 using a two-leaf branched cover $\mathbb{T}^2 \rightarrow \mathbb{S}^2$, see [16]. Consider the transformation group Γ of the plane \mathbb{R}^2 generated by the group \mathbb{Z}^2 of integer shifts $x \mapsto x+m$, $y \mapsto y+k$, $m, k \in \mathbb{Z}$, and the symmetry $(x; y) \mapsto (-x; -y)$. Then the quotient space \mathbb{R}^2/Γ is the two-dimensional sphere \mathbb{S}^2 , and the natural projection $p: \mathbb{R}^2 \rightarrow \mathbb{R}^2/\Gamma = \mathbb{S}^2$ is a branched cover with four branching points $\sigma_1, \sigma_2, \sigma_3, \sigma_4 \in \mathbb{S}^2$ of index two each. The set

$$\Sigma = \bigcup_{(m; k) \in \mathbb{Z}^2} \{(m; k)\} \cup \left\{ \left(m + \frac{1}{2}; k + \frac{1}{2} \right) \right\} \subset \mathbb{R}^2$$

is the preimage of the set $\Sigma_0 = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$, that is, $p(\Sigma) = \Sigma_0 = \bigcup_{i=1}^4 \sigma_i$.

It is known that, if the transformation $\tilde{f}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ satisfies the relations

$$\tilde{f}(\Sigma) = \Sigma, \quad \tilde{f} \circ \Gamma = \Gamma \circ \tilde{f}, \quad (3.1)$$

then \tilde{f} induces a transformation of $f: \mathbb{S}^2 \rightarrow \mathbb{S}^2$ such that $f \circ p = p \circ \tilde{f}$.

Consider the linear transformation $A: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, defined by the matrix $A = \begin{pmatrix} 5 & 6 \\ 2 & 3 \end{pmatrix}$. We denote the matrix and the transformation with one letter, which should not lead to confusion. It is not difficult to check that A satisfies the relations (3.1), and therefore A induces the transformation $\mathbb{S}^2 \rightarrow \mathbb{S}^2$, which we denote by f_0 and which has four fixed points $\sigma_1, \sigma_2, \sigma_3, \sigma_4 \in \mathbb{S}^2$. The following construction applies to each of these points.

For simplicity, we will consider the point $\sigma_1 = \sigma = p(0, 0)$ (and omit the index). The point σ has a single preimage $\tilde{\sigma}(0, 0) \in \mathbb{R}^2$, $p(\tilde{\sigma}) = \sigma$, such that $A(\tilde{\sigma}) = \tilde{\sigma}$. Since σ is a branching point of index two, there are neighborhoods \tilde{U}, U of points $\tilde{\sigma}$ and σ , respectively, such that $p(\tilde{U}) = U$ and the restriction $p|_{\tilde{U}}: \tilde{U} \rightarrow U$ is a natural projection of factor space $\tilde{U}/s_0 \rightarrow U$, where s_0 is the symmetry $(x; y) \mapsto (-x; -y)$. It is not difficult to verify that $\tilde{\sigma}$ is a hyperbolic saddle point of the linear transformation A with the eigenvectors \vec{e}_u, \vec{e}_s corresponding to the eigenvalues $\lambda_u > 1$ and $0 < \lambda_s < 1$, respectively. We will consider the pair \vec{e}_u, \vec{e}_s as the basis (unit) vectors of

the coordinate system $(v_1; v_2)$ so that $(1; 0)$ and $(0; 1)$ are the coordinates \vec{e}_u, \vec{e}_s , respectively, and $\tilde{\sigma}(0; 0) = O$.

For the set $N \subset \mathbb{R}^2$, we denote by $\mathbb{Z}^2(N)$ the family of sets obtained by the action of the group \mathbb{Z}^2 on N (sets from $\mathbb{Z}^2(N)$ are called congruent to the set N).

Let us take $r_0 > 0$ so small that the $\lambda_u r_0$ -neighborhood \tilde{U}_0 of the point O belongs to \tilde{U} and $\lambda_u r_0$ -neighborhoods of all points from the set $p^{-1}(\sigma_1 \cup \sigma_2 \cup \sigma_3 \cup \sigma_4)$ have no intersection. Let $\delta: [0; \infty) \rightarrow [0; 1]$ be a C^∞ -function such that $\delta(r) \equiv 1$ when $0 \leq r \leq \frac{r_0}{2}$, and $\delta(r) \equiv 0$ for $r \geq r_0$. Consider a system of differential equations in the neighborhood \tilde{U}_0 and in all congruent to \tilde{U}_0 neighborhoods, assuming that the coordinate system in \tilde{U}_0 is transferred to the congruent neighborhood by the corresponding integer shift.

$$\begin{cases} \dot{v}_1 = -v_1 \cdot \ln \lambda_u \cdot \delta \left(\sqrt{v_1^2 + v_2^2} \right) \\ \dot{v}_2 = 0. \end{cases} \quad (3.2)$$

Denote by φ_r^t the t -time shift along the trajectories of the system (3.2). Let us fix some number $t_0 > 1$, and put

$$\tilde{f}_r = A \circ \varphi_r^{t_0}. \quad (3.3)$$

Since the system (3.2) is invariant under the symmetry of $(v_1; v_2) \mapsto (-v_1; -v_2)$, the diffeomorphism \tilde{f}_r of the form (3.2) satisfies equation (3.1), and hence induces an endomorphism of $f_r: \mathbb{S}^2 \rightarrow \mathbb{S}^2$. Using a method completely analogous that of [8], one can show that f_r is an A -endomorphism and the nonwandering set of f^r has a strictly invariant one-dimensional and uniquely hyperbolic contracting repeller $\Lambda_r = \Omega$, locally homeomorphic to the product of the segment and the Cantor set. In this case, any connected component of the set $\mathbb{S}^2 \setminus \Lambda_r$ has a boundary admissible from $\mathbb{S}^2 \setminus \Lambda_r$, which consists of exactly one stable manifolds of periodic points from Λ_r . \square

Proof of Theorem 2. Let us show that the basic set Λ cannot be an expanding attractor. Let us assume the contrary. Then there is an A -endomorphism $f: M^2 \rightarrow M^2$ with a Cantor type expanding attractor Λ such that any connected component of the set $M^2 \setminus \Lambda_a$ has a boundary admissible from $M^2 \setminus \Lambda$ consisting of a finite number of unstable manifolds of points from Λ_a . Consider an arbitrary point $x \in \Lambda$. Let us show that the restriction $f|_{W^u(x)}: W^u(x) \rightarrow f(W^u(x)) = W^u(f(x))$ of an endomorphism f on any unstable manifold $W^u(x)$, $x \in \Lambda$, is a homeomorphism in the intrinsic topology of the leaf $W^u(x)$.

First, we note that due to the hyperbolicity of Λ the restriction f on the set Λ is a local diffeomorphism. Since Λ is an attractor, it follows by virtue of Theorem 1 from [9] that $W^u(x)$ belongs to Λ . Therefore, $W^u(x)$ does not contain branch points of the endomorphism f and, therefore, the restriction $f|_{W^u(x)}$ is a cover without branch points. Now the required statement follows from the fact that $W^u(x)$ is simply connected and from Lemma 2.

We now show that in the complement $M^2 \setminus \Lambda$ there is some periodic point O of the endomorphism f . Since Λ is an attractor, there is a closed neighborhood $U \supset \Lambda$ such that $f(U) \subset \text{int}(U)$, and $\bigcap_{l=0}^{\infty} f^l(U) = \Lambda$. Consider an arbitrary point $y_0 \in M^2 \setminus U$ and its arbitrary negative semi-orbit $O^-(y_0)$. It follows directly from the inclusion of $f(U) \subset U$ that the intersection $O^-(y_0) \cap U$ is empty. Since the α -limit set of any point $x \in M^2$ is contained in the nonwandering set $NW(f)$, the closure $\text{clos}(O^-(y_0))$ contains a point $w \in NW(f) \setminus \Lambda$. Then, by virtue of the spectral decomposition theorem, there is a basic set $\Lambda_1 \subset M^2 \setminus U$ different from Λ containing the point w .

In the basic set Λ_1 , we choose an arbitrary periodic point O of some smallest period k . If $k > 1$, then further arguments can be carried out to map f^k . Thus, below it is possible to assume that point O is a fixed point of the f (without loss of generality).

Let K be the connected component of the complement $M^2 \setminus \Lambda$ containing the fixed point O . Denote by $\delta(K)$ the admissible boundary of the K from interior K and take a leaf $l \subset \Lambda$ of the lamination Λ such that l belongs to $\delta(K)$. By the condition, l is an unstable manifold of some point from Λ . We show that $f^r(l) = l$ for some $r \in \mathbb{N}$.

Indeed, recall that, as l belongs the admissible boundary from interior K , for any point $z \in l$ there exists an arc d with endpoints O and z such that $d \setminus \{z\} \subset K$. Since Λ is strictly invariant with respect to f , the point $f(z)$ belongs to Λ , and $f(d) \setminus \{f(z)\}$ belongs to $M^2 \setminus \Lambda$. Since $f(O) = O$, it follows that $f(d) \setminus \{f(z)\}$ belongs to K . Therefore, $f(z)$ belongs to the leaf $f(l)$ of lamination Λ such that $l \subset \delta(K)$. This means that the set $\delta(K)$ is invariant. Since $\delta(K)$ consists of a finite number of leaves, $f^r(l) = l$ for some $r \in \mathbb{N}$. Taking the iterations f^{nr} , if necessary, we will assume below that all leaves of the lamination Λ belonging to $\delta(K)$ are invariant with respect to f .

Consider an arbitrary point $p \in l$, where $l \in \delta(K)$. Since the restriction of f to Λ is a k -cover, $k \geq 2$, the complete preimage of $f^{-1}(p)$ contains some point $p_1 \neq p$. Since $f(\Lambda) = \Lambda = f^{-1}(\Lambda)$, $p_1 \in \Lambda$. Due to the injectivity of the restriction $f|_{W^u(p)}$, we have that $p_1 \notin l$. It is clear that p_1 does not belong to the boundary $\delta(K)$ admissible from K . Indeed, suppose $p_1 \in l_*$ where l_* is the leaf of the lamination Λ admissible from K . The leaf l_* is invariant respect to f , different from l and $f(l^*) = l$, which is impossible.

Consider the path γ_p connecting the points p and O such that $\text{int } \gamma_p \in K$. Since $p_1 \neq p$, there is a path γ'_p connecting the points p_1 and O_1 , such that $\gamma_p = f(\gamma'_p)$, $f(O_1) = O$. Thus, p_1 belongs to the admissible boundary of some connected component K_1 of the set $M^2 \setminus \Lambda$. Moreover, p_1 belongs to a leaf, say l_1 , of the lamination Λ such that $f(l_1) = l$. Since $l_1 \notin \delta(K)$, $K \neq K_1$. Similarly, it is shown that the complete preimage of $f^{-1}(p_1)$ contains a point $p_2 \notin \{p, p_1\}$ belonging to the component K_2 of the set $M^2 \setminus \Lambda$ such that $K_2 \notin \{K, K_1\}$, and there is a point $O_2 \in K_2$ such that $f(O_2) = O_1$. Continuing this process, we get a negative semi-orbit $O^-(p) = \{p = p_0, p_1, p_2, \dots\}$, $f(p_{j+1}) = p_j$, and the sequence $\{K = K_0, K_1, K_2, \dots\}$, $f(K_{j+1}) = K_j$, with the following properties:

- 1) the point p_i belongs to the boundary of the domain K_i admissible from K_i for all $i \in \mathbb{N} \cup \{0\}$;
- 2) K_i are pairwise distinct connected components of the set $M^2 \setminus \Lambda$;
- 3) there is a negative semi-orbit $O^-(O) = \{O, O_1, O_2, \dots\}$, $f(O_{j+1}) = O_j$ such that $O_i \in K_i$ for all $i \in \mathbb{N} \cup \{0\}$.

Since $O \in M^2 \setminus U$ and $f(U) \subset U$, there is an inclusion of $O^-(O) \subset M^2 \setminus \text{int}(U)$. Hence, taking into account the compactness of the set $M^2 \setminus \text{int}(U)$, it follows that the sequence $O^-(O)$ has a limit point $O^* \in M^2 \setminus \text{int}(U)$. Let K^* be the connected component of the set $M^2 \setminus \Lambda$ containing the point O^* . According to Corollary 1, the connected component K^* is an open set. Then there are two different numbers $n_1, n_2 \in \mathbb{N}$ such that $O_{n_1} \in K^*$ and $O_{n_2} \in K^*$, which contradicts the fact that K_{n_1} and K_{n_2} are different connected components of the set $M^2 \setminus \Lambda$. \square

Proof of Theorem 3. Due to the conditions, the stable manifolds of the repeller Λ_r form the lamination [15], which we denote by $\mathcal{L}(\Lambda_r)$. The set $M^2 \setminus \mathcal{L}(\Lambda_r)$ is the union of a countable number of pairwise disjoint open disks that are basins of isolated sink periodic orbits. In addition, the lamination $\mathcal{L}(\Lambda_r)$ can be extended to some foliation $F(\Lambda_r)$ on the surface of M^2 .

It is known that the sum of the indices of singularities of a foliation is equal to the Euler characteristic of the ambient surface (see, for example, [1, 14]). Suppose that $M^2 \neq \mathbb{T}^2, \mathbb{S}^2$. Since in this case the Euler characteristic of M^2 is negative, there is a component C of the set $M^2 \setminus \mathcal{L}(\Lambda_r)$ containing the singularity of index $\leq -\frac{1}{2}$. Denote by ν the number of leaves from $\delta(C)$. Since the index of the singularity from C satisfies the inequality $1 - \frac{\nu}{2} \leq -\frac{1}{2}$, we see that $\nu \geq 3$. It follows from the strict invariance of Λ_r that $f^{-1}(\delta(C)) \subset \Lambda_r$. Moreover, since the restriction $f|_{\Lambda_r}$ is a local diffeomorphism, the preimage $f^{-1}(\delta(C))$ belongs to the admissible boundary of some component of $M^2 \setminus \mathcal{L}(\Lambda_r)$. Since each component of the set $M^2 \setminus \mathcal{L}(\Lambda_r)$ is homeomorphic to an open disk, it is simply connected. Thus, $f^{-1}(\delta(C))$ is the union of the admissible boundaries of either one or several connected components of the set $M^2 \setminus \mathcal{L}(\Lambda_r)$. In any case, since the lamination $\mathcal{L}(\Lambda_r)$ is regular, the preimage of $f^{-1}(C)$ contains a component of the set $M^2 \setminus \mathcal{L}(\Lambda_r)$, different from C , with singularity of a negative index. Since f is not a diffeomorphism, the number of singularities of the negative index must be infinitely large. Since the surface M^2 is compact, the foliation $F(\Lambda_r)$ must have only a finite number of singularities of the nonzero index, by virtue of the Euler–Poincaré formula (see, e. g., [1, 14]). The contradiction completes the proof.

Let us illustrate our arguments above for the simplest saddle-type singularities, when the set of singularities is invariant with respect to f . Let us assume for simplicity that the singularity $s_0 \in C$ of the negative index has ν separatrix. Then its index $\text{ind } s_0$, according to [14], formula (3.11), equals $1 - \frac{\nu}{2}$, where $\nu \geq 3$. If f has no branch points in C , then the foliation $F(\Lambda_r)$ has in each component of $f^{-1}(C)$ a singularity of the index $1 - \frac{\nu}{2} < 0$. If s_0 is a branch point of degree $k \geq 2$, then each singularity of $f^{-1}(s_0)$ has $k \cdot \nu$ separatrix and therefore its index equals $1 - \frac{k \cdot \nu}{2} < 0$. In both cases, the index of a singularity from the preimage $f^{-1}(s_0)$ is negative. \square

REMARK. The Euler characteristic of the surfaces \mathbb{T}^2 and \mathbb{S}^2 equals zero and two, respectively. Therefore, except for singularities of zero index, there are possible singularities of the index $\frac{1}{2}$ for $\nu = 1$. However, their preimages for $k = 2$ have zero index. This shows that, for the surfaces \mathbb{T}^2 and \mathbb{S}^2 , the proof of Theorem 3 does not hold.

References

- [1] Anosov, D. V. and Zhuzhoma, E. V., Nonlocal Asymptotic Behavior of Curves and Leaves of Laminations on Universal Coverings, *Proc. Steklov Inst. Math.*, 2005, no. 2(249), pp. 1–219; see also: *Tr. Mat. Inst. Steklova*, 2005, vol. 249, pp. 3–239.
- [2] Grines, V. Z., The Topological Equivalence of One-Dimensional Basic Sets of Diffeomorphisms on Two-Dimensional Manifolds, *Uspekhi Mat. Nauk*, 1974, vol. 29, no. 6(180), pp. 163–164 (Russian).
- [3] Grines, V. Z., The Topological Conjugacy of Diffeomorphisms of a Two-Dimensional Manifold on One-Dimensional Orientable Basic Sets: 1, *Tr. Mosk. Mat. Obs.*, 1975, vol. 32, pp. 35–60 (Russian).
- [4] Grines, V. Z., The Topological Conjugacy of Diffeomorphisms of a Two-Dimensional Manifold on One-Dimensional Orientable Basic Sets: 2, *Tr. Mosk. Mat. Obs.*, 1977, vol. 34, pp. 243–252 (Russian).
- [5] Grines, V. Z., On the Topological Classification of Structurally Stable Diffeomorphisms of Surfaces with One-Dimensional Attractors and Repellers, *Sb. Math.*, 1997, vol. 188, no. 4, pp. 537–569; see also: *Mat. Sb.*, 1997, vol. 188, no. 4, pp. 57–94.
- [6] Grines, V. Z. and Zhuzhoma, E. V., The Topological Classification of Orientable Attractors on an n -Dimensional Torus, *Russian Math. Surveys*, 1979, vol. 34, no. 4, pp. 163–164; see also: *Uspekhi Mat. Nauk*, 1979, vol. 34, no. 4, pp. 185–186.



- [7] Grines, V. Z. and Zhuzhoma, E. V., On Rough Diffeomorphisms with Expanding Attractors or Contracting Repellers of Codimension One, *Dokl. Math.*, 2000, vol. 62, no. 2, pp. 274–276; see also: *Dokl. Ross. Akad. Nauk*, 2000, vol. 374, no. 6, pp. 735–737.
- [8] Grines, V. Z., Zhuzhoma, E. V., and Kurenkov, E. D., On DA -Endomorphisms of the Two-Dimensional Torus, *Sb. Math.*, 2021, vol. 212, no. 5, pp. 698–725; see also: *Mat. Sb.*, 2021, vol. 212, no. 5, pp. 102–132.
- [9] Grines, V. Z. and Kurenkov, E. D., On Hyperbolic Attractors and Repellers of Endomorphisms, *Nelin. Dinam.*, 2017, vol. 13, no. 4, pp. 557–571 (Russian).
- [10] Kurenkov, E. D., On Existence of an Endomorphism of 2-Torus with Strictly Invariant Repeller, *Zh. SVMO*, 2017, vol. 19, no. 1, pp. 60–66 (Russian).
- [11] Plykin, R. V., Sources and Sinks of A -Diffeomorphisms of Surfaces, *Math. USSR-Sb.*, 1974, vol. 23, no. 2, pp. 233–253; see also: *Mat. Sb. (N.S.)*, 1974, vol. 94(136), no. 2(6), pp. 243–264.
- [12] Plykin, R. V., On the Geometry of Hyperbolic Attractors of Smooth Cascades, *Russian Math. Surveys*, 1984, vol. 39, no. 6, pp. 85–131; see also: *Uspekhi Mat. Nauk*, 1984, vol. 39, no. 6(240), pp. 75–113.
- [13] Grines, V. and Zhuzhoma, E., On Structurally Stable Diffeomorphisms with Codimension One Expanding Attractors, *Trans. Amer. Math. Soc.*, 2005, vol. 357, no. 2, pp. 617–667.
- [14] Grines, V. Z. and Zhuzhoma, E. V., *Surface Laminations and Chaotic Dynamical Systems*, Izhevsk: R&C Dynamics, Institute of Computer Science, 2021.
- [15] Ikegami, G., Hyperbolic Sets and Axiom A for Endomorphisms, *Proc. of Inst. of Natural Sciences, Nihon Univ.*, 1991, vol. 26, pp. 69–86.
- [16] Katok, A., Bernoulli Diffeomorphisms on Surfaces, *Ann. of Math. (2)*, 1979, vol. 110, no. 3, pp. 529–547.
- [17] Newhouse, S., On Codimension One Anosov Diffeomorphisms, *Amer. J. Math.*, 1970, vol. 92, no. 3, pp. 761–770.
- [18] Przytycki, F., Anosov Endomorphisms, *Studia Math.*, 1976, vol. 58, no. 3, pp. 249–285.
- [19] Przytycki, F., On Ω -Stability and Structural Stability of Endomorphisms Satisfying Axiom A, *Studia Math.*, 1977, vol. 60, no. 1, pp. 61–77.
- [20] Shub, M., Endomorphisms of Compact Differentiable Manifolds, *Amer. J. Math.*, 1969, vol. 91, no. 1, pp. 175–199.
- [21] Smale, S., Differentiable Dynamical Systems, *Bull. Amer. Math. Soc.*, 1967, vol. 73, no. 6, pp. 747–817.