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Stabilization of a class of switched dynamic systems: the Riccati-equation-based Approach

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Our paper deals with the stabilization of a class of time-dependent linear autonomous complex systems with a switched structure. The initially given switched dynamic system is assumed to be controlled by a specific state feedback strategy associated with the linear quadratic regulator (LQR) type control. The proposed control design guarantees stabilization of the closed-loop system for all of the possible location transitions. In the solution procedure of the Algebraic Riccati Equation related to the LQR control strategy, only the knowledge of the algebraic structure related to the switched system are needed. We prove that the proposed optimal LQR type state feedback control design stabilizes the closed-loop switched system for every possible active location. The theoretical approach proposed in this paper is finally applied to a model of the *Single Wing Quadrotor Aircraft*, when changing from its *Quadrotor Flight Envelope* to its *Airplane Flight Envelope*.

Keywords: switched dynamic systems; implicit control systems; linear quadratic regulator (LQR); algebraic Riccati equation (ARE); Lyapunov stability.

1. Introduction

General switched dynamic systems (SDSs) belong to a wide family of the complex systems studied by many authors (see e.g. (Karcanias & Livada, 2020)). Because SDSs constitute a useful modelling approach for various engineering systems and processes, many useful control strategies for these dynamics have been developed. For example, in (Bonilla *et al.*, 2015a), it was shown that a wide class of time-dependent autonomous systems with a switched structure (as defined in (Liberzon, 2003)) can be adequately represented by the formal state space representation of the type

$$\frac{\mathrm{d}}{\mathrm{d}t} x = A_q x + Bu. \tag{1.1}$$

Here $B \in \mathbb{R}^{\bar{n}\times m}$ is an injective matrix and A_q has the common structure (see e.g. (Narendra & Balakrishnan, 1994)):

$$A_q = \overline{A}_0 + \overline{A}_1 \overline{D}(q), \tag{1.2}$$

where $\overline{A}_0 \in \mathbb{R}^{\bar{n}\times\bar{n}}$, $\overline{A}_1 \in \mathbb{R}^{\bar{n}\times\hat{n}}$ is an injective matrix and $\overline{D}(q) \in \mathbb{R}^{\hat{n}\times\bar{n}}$ are surjective matrices.[1] Let us denote $n = \bar{n} + \hat{n}$.

Because of the usual dynamics of a SDS, it remains in a specific location

$$q \in \mathscr{Q} \triangleq \left\{ q_1, \ldots, q_\eta \;\middle|\; q_i \in \mathbb{R}^\mu, \quad i \in \{1, \ldots, \eta\} \subset \mathbb{Z}^+ \right\}, \tag{1.3}$$

for all time instants $t \in \mathscr{I}_i$. Here $\mathscr{I}_i \triangleq [t_{i-1}, t_i)$, $t_i \in \mathbb{R}^+$, $t_0 = 0$, $t_{i-1} < t_i$, for all $i \in \mathbb{Z}^+$, $\lim_{i\to\infty} T_i = \infty$, and $\mathfrak{s} : \{\mathscr{I}_i \subset \mathbb{R}^+ \cup \{0\},\ i \in \mathbb{Z}^+\} \to \mathscr{Q}$, $\mathfrak{s}(\mathscr{I}_i) = q_j$, $j \in \{1, \ldots, \eta\}$.

We now assume that the locations set has a specific structure described by the following basic Hypothesis.

HYPOTHESIS 1.1. (Bonilla *et al.*, 2015b) Given $\bar{q}_0, \bar{q}_1, \ldots, \bar{q}_\ell \in \mathbb{R}^\mu$, the locations $q_j \in \mathscr{Q}$, $j \in \{1, \ldots, \eta\}$, belong to the set described as follows

$$q_j \in \overline{\mathscr{Q}}_{\bar{q}_0} \triangleq \left\{ q_j \in \mathscr{Q} \;\middle|\; q_j = \bar{q}_0 + \begin{bmatrix} \bar{q}_1 & \cdots & \bar{q}_\ell \end{bmatrix} \bar{\gamma}_j,\ \bar{\gamma}_j \in \mathbb{R}^\ell,\ j \in \{1, \ldots, \eta\} \right\}, \tag{1.4}$$

and additionally for each $\mathscr{I}_i \in \mathfrak{s}^{-1}(q_j)$, $\bar{\gamma}_j = \begin{bmatrix} \gamma_{j,1} & \cdots & \gamma_{j,\ell} \end{bmatrix}^T$ take constant values in $\mathbb{R}^\ell$; $i \in \mathbb{N}$, $j \in \{1, \ldots, \eta\}$.

---

[1] Recall that the celebrated rank-nullity theorem defines the concept of a surjective and an injective matrix. In the case of a $r_1 \times r_2$ matrix with a rank r_3, this fundamental theorem establishes that

$$\dim \ker \mathscr{A} = r_2 - r_3,$$

where $\mathscr{A}$ denotes the linear map associated with the given matrix. Injectivity of the matrix is defined as

$$\dim \ker \mathscr{A} = 0 \Rightarrow r_2 = r_3$$

and the surjectivity is equivalent to $r_3 = r_1$.

HYPOTHESIS 1.2. (Bonilla *et al.*, 2015b) There exist $\overline{\varDelta}_0, \overline{\varDelta}_{\overline{\gamma}_j} \in \mathbb{R}^{\hat{n}\times\overline{n}}$, $j \in \{1, \ldots, \eta\}$, such that

$$\overline{D}(q) = \overline{\varDelta}_0 - \sum_{i\in\mathbb{N}} \imath_{\mathscr{I}_i}(t)\, \overline{\varDelta}_{\overline{\mathfrak{s}}(\mathscr{I}_i)}, \tag{1.5}$$

where $\imath_{\mathscr{I}_i}(t)$ is the characteristic function of the time interval $\mathscr{I}_i$, and $\overline{\mathfrak{s}} : \{\mathscr{I}_i \subset \mathbb{R}^+ \cup \{0\},\ i \in \mathbb{Z}^+\} \to \{\overline{\gamma}_1, \ldots, \overline{\gamma}_\ell\}$, $\overline{\mathfrak{s}}(\mathscr{I}_i) = \overline{\gamma}_j$, $j \in \{1, \ldots, \eta\}$. $\mathfrak{s}(\mathscr{I}_i)$ and $\overline{\mathfrak{s}}(\mathscr{I}_i)$ follow the same index assignation rule, $i \mapsto j$.

We also assume

HYPOTHESIS 1.3. The pair $(\overline{A}_0 + \overline{A}_1\overline{\varDelta}_0, B)$ is controllable.

In (Bonilla *et al.*, 2015b), authors additionally propose a specific variable structure decoupling control strategy based on the ideal proportional and derivative (PD) feedback control strategy. As next a proper practical approximation of the above ideal PD feedback is developed. Such feedback control strategies reject the initially given 'variable structure' of the resulting system and make it possible to establish the required stability property for control strategies under consideration.

In this paper, we consider the stabilizing problem for a class of time-dependent switched dynamic systems equipped with a relative simple static state feedback. Our paper is organized as follows: in Section 2, we give a formal description of the *LQR-based stabilization of switched systems*. The initial system description is proposed in (Ortiz Castillo *et al.*, 2020). We next follow the more formal approach discussed in Bonilla *et al.* (2015a). In Section 3, we apply the celebrated Riccati stabilizing state feedback (see also Section 2) for a control design that stabilizes a *Single Wing Quadrotor Aircraft* in the case when changing from its *Quadrotor Flight Envelope* to its *Airplane Flight Envelope*. Section 4 summarizes our paper.

2. Riccati Equation-Based Approach to the Stabilization Problem

Let us formulate the following problem.

PROBLEM 2.1 Find a constant state feedback that stabilizes system (1.1) and (1.2),

$$\frac{\mathrm{d}}{\mathrm{d}t}x = \left(\overline{A}_0 + \overline{A}_1\overline{D}(q)\right)x + Bu.$$

We next assume that the locations (1.3) are unknown

$$q \in \mathscr{Q} \triangleq \left\{q_1, \ldots, q_\eta \;\middle|\; q_i \in \mathbb{R}^{\mu}, \quad i \in \{1, \ldots, \eta\} \subset \mathbb{Z}^+\right\}.$$

We also assume the full knowledge of the essential parameters of constant system structure, (1.2) and (1.5), determined by the following triplet $(\overline{A}_0, \overline{A}_1, \overline{\varDelta}_0)$.

Solution of this problem involves the proposed state feedback with the linear quadratic regulator structure

$$F_{*0} = R^{-1}B^TP_0, \quad u = -F_{*0}\,x, \tag{2.1}$$

where R is a selected positive definite matrix and P_0 is a solution of the algebraic Riccati equation (ARE) (recall Assumption 1.3)

$$(\overline{A}_0 + \overline{A}_1\overline{\varDelta}_0)^T P_0 + P_0\,(\overline{A}_0 + \overline{A}_1\overline{\varDelta}_0) - P_0 B R^{-1} B^T P_0 + \overline{Q}_0 = 0, \tag{2.2}$$

and $\overline{Q}_0$ is a selected positive semidefinite matrix. The above ARE is determined by the essential parameters of system (1.1) and (1.2), namely $(\overline{A}_0, \overline{A}_1, \overline{\varDelta}_0)$ (recall that (1.2) and (1.5)).

Applying the state feedback (2.1) to the switched system representation (1.1), we next obtain the closed-loop state space form

$$\frac{\mathrm{d}}{\mathrm{d}t}x = (Aq_i - BR^{-1}B^T P_0)\,x \tag{2.3}$$

Taking into consideration the previously derived formulae (1.2) and (1.5) in (2.3), we also get

$$\frac{\mathrm{d}}{\mathrm{d}t}\,x = \left(\overline{A}_0 + \overline{A}_1\left(\overline{\varDelta}_0 - \sum_{i\in\mathbb{N}} \mathbb{1}_{\mathscr{I}_i}(t)\,\overline{\varDelta}_{\overline{\mathfrak{s}}(\mathscr{I}_i)}\right) - BR^{-1}B^T P_0\right) x, \tag{2.4}$$

In the same way as in Narendra & Balakrishnan (1994), let us define the following Lyapunov function

$$V(x) = x(t)^T P_0\, x(t). \tag{2.5}$$

The usual Lie derivative of (2.5) (the derivative along the trajectories of system (2.4)) can be calculated as follows

$$\begin{aligned}
\mathrm{d}V(t)/\mathrm{d}t = {}& \mathrm{d}x^T/\mathrm{d}t\,P_0\,x + x^T\,P_0\,\mathrm{d}x/\mathrm{d}t\\
= {}& x^T\left(\left(\overline{A}_0 + \overline{A}_1\overline{\varDelta}_0\right)^T P_0 - \sum_{i\in\mathbb{N}} \mathbb{1}_{\mathscr{I}_i}(t)\,\left(\overline{A}_1\overline{\varDelta}_{\overline{\mathfrak{s}}(\mathscr{I}_i)}\right)^T P_0 - \left(BR^{-1}B^T P_0\right)^T P_0\right) x\\
& + x^T\left(P_0\left(\overline{A}_0 + \overline{A}_1\overline{\varDelta}_0\right) - \sum_{i\in\mathbb{N}} \mathbb{1}_{\mathscr{I}_i}(t)\,P_0\left(\overline{A}_1\overline{\varDelta}_{\overline{\mathfrak{s}}(\mathscr{I}_i)}\right) - P_0\left(BR^{-1}B^T P_0\right)\right) x\\
\mathrm{d}V(t)/\mathrm{d}t = {}& x^T\Bigg(\left(\overline{A}_0 + \overline{A}_1\overline{\varDelta}_0\right)^T P_0 + P_0\left(\overline{A}_0 + \overline{A}_1\overline{\varDelta}_0\right) - P_0 BR^{-1}B^T P_0\\
& - \sum_{i\in\mathbb{N}} \mathbb{1}_{\mathscr{I}_i}(t)\left(\left(\overline{A}_1\overline{\varDelta}_{\overline{\mathfrak{s}}(\mathscr{I}_i)}\right)^T P_0 + P_0\left(\overline{A}_1\overline{\varDelta}_{\overline{\mathfrak{s}}(\mathscr{I}_i)}\right)\right) - P_0 BR^{-1}B^T P_0\Bigg) x.
\end{aligned}$$

From (2.2) and (2.1) we next deduce

$$
\begin{aligned}
\mathrm{d}V(t)/\mathrm{d}t &= -x^T\left(\overline{Q}_0 + \sum_{i\in\mathbb{N}} \mathbb{1}_{\mathscr{I}_i}(t)\left(\left(\overline{A}_1\overline{\Delta}_{\overline{\mathfrak{s}}(\mathscr{I}_i)}\right)^T P_0 + P_0\left(\overline{A}_1\overline{\Delta}_{\overline{\mathfrak{s}}(\mathscr{I}_i)}\right)\right) + F_{*0}^T R F_{*0}\right)x,\\
\mathrm{d}V(t)/\mathrm{d}t &= -x^T\left(Q_0 + \sum_{i\in\mathbb{N}} \mathbb{1}_{\mathscr{I}_i}(t)\left(\left(\overline{A}_1\overline{\Delta}_{\overline{\mathfrak{s}}(\mathscr{I}_i)}\right)^T P_0 + P_0\left(\overline{A}_1\overline{\Delta}_{\overline{\mathfrak{s}}(\mathscr{I}_i)}\right)\right)\right)x,\\
\mathrm{d}V(t)/\mathrm{d}t &= -x^T\left(Q_0 + \sum_{i\in\mathbb{N}} \mathbb{1}_{\mathscr{I}_i}(t)\,\overline{Q}_{\overline{\mathfrak{s}}(\mathscr{I}_i)}\right)x, \qquad (2.6)
\end{aligned}
$$

where

$$
\overline{Q}_{\overline{\mathfrak{s}}(\mathscr{I}_i)} = \left(\overline{A}_1\overline{\Delta}_{\overline{\mathfrak{s}}(\mathscr{I}_i)}\right)^T P_0 + P_0\left(\overline{A}_1\overline{\Delta}_{\overline{\mathfrak{s}}(\mathscr{I}_i)}\right), \quad i \in \{1, \ldots, \eta\}, \qquad (2.7)
$$

$$
Q_0 = \begin{bmatrix} \left(\sqrt{\overline{Q}_0}\right)^T & F_{*0}^T \end{bmatrix}\begin{bmatrix} \mathrm{I} & 0 \\ 0 & R \end{bmatrix}\begin{bmatrix} \sqrt{\overline{Q}_0} \\ F_{*0} \end{bmatrix}, \text{ and } \overline{Q}_0 = \left(\sqrt{\overline{Q}_0}\right)^T\left(\sqrt{\overline{Q}_0}\right).
$$

The analytic relations obtained above constitute in fact a formal proof of our main stability result

THEOREM 2.2. Assume that all the technical assumptions of this section are fulfilled. Then the system (2.3) is stable in the sense of Lyapunov if one of the two following conditions is satisfied

$$
\lambda_{min}\left\{Q_0 + \left(\overline{A}_0 + \overline{A}_1\overline{\Delta}_0 - A_{q_j}\right)^T P_0 + P_0\left(\overline{A}_0 + \overline{A}_1\overline{\Delta}_0 - A_{q_j}\right)\right\} > 0, \quad \forall\, j \in \{1, \ldots, \eta\}, \qquad (2.8)
$$

or if the pair $\left(\sqrt{\overline{Q}_0},\, (\overline{A}_0 + \overline{A}_1\overline{\Delta}_0)\right)$ is observable and moreover,

$$
\lambda_{min}\left\{Q_0 + \left(\overline{A}_0 + \overline{A}_1\overline{\Delta}_0 - A_{q_j}\right)^T P_0 + P_0\left(\overline{A}_0 + \overline{A}_1\overline{\Delta}_0 - A_{q_j}\right)\right\} \geq 0, \quad \forall\, j \in \{1, \ldots, \eta\}, \qquad (2.9)
$$

Proof. Let us first note that (1.5), (1.2) and (2.7), imply:

$$
\overline{Q}_{\overline{\gamma}_j} = \left(\overline{A}_1\overline{\Delta}_{\overline{\gamma}_j}\right)^T P_0 + P_0\left(\overline{A}_1\overline{\Delta}_{\overline{\gamma}_j}\right) = \left(\overline{A}_0 + \overline{A}_1\overline{\Delta}_0 - A_{q_j}\right)^T P_0 + P_0\left(\overline{A}_0 + \overline{A}_1\overline{\Delta}_0 - A_{q_j}\right),\ j \in \{1, \ldots, \eta\}. \qquad (2.10)
$$

Application of Theorem 5.10 from Chapter 6, Section 5 of Stewart (1973), and Corollary 2.6-2 of Kailath (1980)) concludes the proof. □

REMARK 2.1. The obtained result provides a stability criterion for the switched systems in the absence of the exact (*a priori* given) information about a concrete switching mechanism.

Starting from a model given in the form (1.1), the feedback design procedure can next be summarized as follows.

1. Identify the essential parameters of the constant system structure, (1.2) and (1.5), $(\overline{A}_0, \overline{A}_1, \overline{\Delta}_0)$.
2. Choose matrices R and $\overline{Q}_0$ and solve the Riccati equation (2.2).

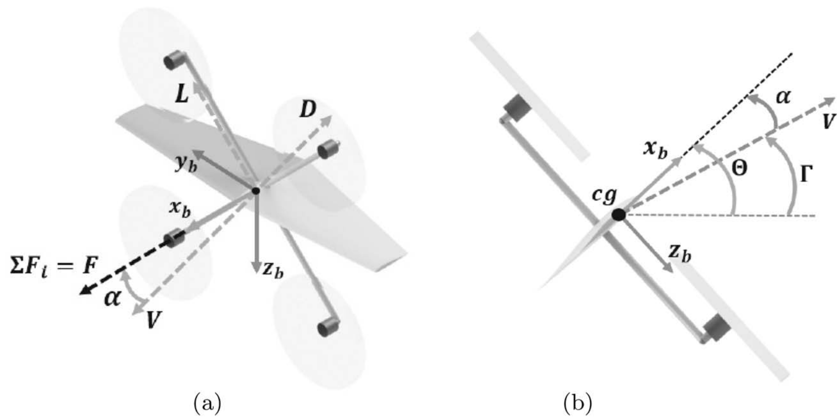

FIG. 1. Single wing quadrotor aircraft. (a) Perspective view. (b) Tangential variables, Γ: flight path angle, Θ: pitch angle, $\boldsymbol{\alpha}$: angle of attack, V: longitudinal flight speed.

3. If one of the two conditions of Theorem 2.2, (2.8) or (2.9), are satisfied go to next item, otherwise return to the previous item.
4. Define the specific feedback control law by (2.1).

3. A Practical Example

Let us consider the *Single Wing Quadrotor Aircraft* (*SWQA*) as shown in Fig. 1. The mechanical motion of the *SWQA* is studied in a fixed orthogonal axis set (earth axes) ($O\mathbf{XYZ}$), where $O\mathbf{Z}$ is a vertical axis, along the gravity vector $\begin{bmatrix} 0 & 0 & g \end{bmatrix}^T$.

Let Φ, Θ and Ψ be the conventional Euler angles, roll, pitch and yaw, measured with respect to the axis $O_B\mathbf{X}_B$, $O_B\mathbf{Y}_B$ and $O_B\mathbf{Z}_B$. Here ($O_B\mathbf{X}_B\mathbf{Y}_B\mathbf{Z}_B$) is the body axis system with its origin O_B fixed at the centre of gravity of the *SWQA* (Cook, 2013). The total mass of the quadrotor of the *SWQA* is equal to $m = 1.6$ [kg] and the moments of inertia with respect to the axis $O_B\mathbf{X}_B$, $O_B\mathbf{Y}_B$ and axis $O_B\mathbf{Z}_B$ are $\mathrm{I}_{xx} = 0.058$ [kg m^2], $\mathrm{I}_{yy} = 0.048$ [kg m^2] and $\mathrm{I}_{zz} = 0.052$ [kg m^2], respectively. We consider here some concrete values: for the gravity $g = 9.81$ [m s^{-2}] and the air density parameter $\rho = 1.2$ [kg/m^3].

Moreover, we assume that the single wing has a S5010 low-speed airfoil for flying wings (Selig *et al.*, 1996), with the aspect ratio: $AR = 6$, span $b = 1.35$ [m] and mean aerodynamic chord $\bar{\bar{c}} = 0.165$. Let [m] be a distance of the c.g. along $\bar{\bar{c}}$: $h = 0.1$ and location of the aerodynamic centre $h_0 \approx 0.25$.

3.1. *Description of the longitudinal directional behaviour*

We next assume that the lateral directional dynamics (**X–Y** plane) has been already asymptotically stabilized. Thus, we only need to consider the longitudinal directional movement (**X–Z** plane). The

SWQA is represented by the following differential equations:

$$
\begin{aligned}
\begin{bmatrix} \mathrm{d}^2X/\mathrm{d}t^2 \\ \mathrm{d}^2Z/\mathrm{d}t^2 \end{bmatrix} &= \begin{bmatrix} 0 \\ g \end{bmatrix} - (1/m)\begin{bmatrix} \cos\Gamma & \sin\Gamma \\ -\sin\Gamma & \cos\Gamma \end{bmatrix}\begin{bmatrix} D(V,\boldsymbol{\alpha}) \\ L(V,\boldsymbol{\alpha}) \end{bmatrix} + (F/m)\begin{bmatrix} \cos\Theta \\ -\sin\Theta \end{bmatrix}, \\
\mathrm{d}^2\Theta/\mathrm{d}t^2 &= (1/\mathrm{I}_{yy})(T_q + M(V,\boldsymbol{\alpha}) + \bar{\bar{c}}(h-h_0)L(V,\boldsymbol{\alpha})),
\end{aligned} \tag{3.1}
$$

where the lift, drag and pitching moment equations of the wing are given by the following relations (see (Cook, 2013) for details):

$$
L(V,\boldsymbol{\alpha}) = \frac{1}{2}\rho\frac{b^2}{AR}V^2C_L(\boldsymbol{\alpha}), \;\; D(V,\boldsymbol{\alpha}) = \frac{1}{2}\rho\frac{b^2}{AR}V^2C_D(\boldsymbol{\alpha}), \;\; M(V,\boldsymbol{\alpha}) = \bar{\bar{c}}\frac{1}{2}\rho\frac{b^2}{AR}V^2C_M(\boldsymbol{\alpha}). \tag{3.2}
$$

Here V is the flying speed calculated by

$$
V = \sqrt{(\mathrm{d}\mathbf{X}/\mathrm{d}t)^2 + (\mathrm{d}\mathbf{Z}/\mathrm{d}t)^2}. \tag{3.3}
$$

Assuming now the absence of wind and denote by $\boldsymbol{\alpha}$ the angle of attack, defined as the angle between the chord line of the airfoil and the velocity vector. Clearly, it is related to the pitch angle Θ

$$
\Theta = \boldsymbol{\alpha} + \Gamma, \quad \Gamma = \arctan\left(\frac{-\mathrm{d}\mathbf{Z}/\mathrm{d}t}{\mathrm{d}\mathbf{X}/\mathrm{d}t}\right), \tag{3.4}
$$

where Γ is called the flight path angle and defined as the angle between the velocity vector and the horizontal plane (Cook, 2013, p. 18) (*c.f* Fig. 1). The dimensionless aerodynamic coefficients C_L, C_D, C_M in (3.2) have the classic expressions:

$$
C_L(\boldsymbol{\alpha}) = C_{L_0} + C_{L_1}\boldsymbol{\alpha}, \;\; C_D(\boldsymbol{\alpha}) = C_{D_0} + C_{D_1}\boldsymbol{\alpha} + C_{D_2}\boldsymbol{\alpha}^2, \;\; C_M(\boldsymbol{\alpha}) = C_{M_0} + C_{M_1}\boldsymbol{\alpha}, \tag{3.5}
$$

where $(C_{L_0}, C_{L_1}) = (0.1875, 0660)$, $(C_{D_0}, C_{D_1}, C_{D_2}) = (0.0212, 0.0014, 0.0004)$ and $(C_{M_0}, C_{M_1},) = (-0.0134, 0.0092)$.

The input forces and moments in the body frame are obtained from the angular velocities of the existing four rotors: ω_i, $i = 1, \ldots 4$ according to

$$
\begin{bmatrix} F \\ T_p \\ T_q \\ T_r \end{bmatrix} = \begin{bmatrix} k_f & k_f & k_f & k_f \\ k_\tau & k_\tau & -k_\tau & -k_\tau \\ -k_q & k_q & k_q & -k_q \\ -k_r & k_r & -k_r & k_r \end{bmatrix}\begin{bmatrix} \omega_1^2 \\ \omega_2^2 \\ \omega_3^2 \\ \omega_4^2 \end{bmatrix} \tag{3.6}
$$

where F is the input thrust, T_p, T_q and T_r are the input torques of the rotors along the body axes and k_f, k_τ are the force and torque coefficients of the rotors, respectively. Additionally, k_q and k_r are torque coefficients derived from the rotor configuration shown in Fig. 1 (Powers *et al.*, 2014).

3.2. *Problem statement*

Consider the *SWQA* equipped with the four rotors vertically oriented in a hover flight $\varTheta = 90°$ (as shown in Fig. 1b). In fact, we are dealing with the *Quadrotor Flight Envelope* (*QFE*) and want to change the quadrotor mode into *Airplane Flight Envelope* (*AFE*). That means the *SWQA* has to fly as an airplane in a horizontal flying path $\varGamma = 0°$ with a positive angle of attack $\boldsymbol{\alpha}$. This angle needs to be not larger than $10°$ and the flying speed V is predefined to be not larger than 5 [m/s]. The required change between these two flight modes guarantees that the *SWQA* will continue the flight and not fall down.

To solve the problem described above, we have to track a take-off path in terms of its nominal tangential velocity. Given a desired flying speed, $\overline{V}$ and a required flight path angle $\overline{\varGamma}$, the *SWQA* has to track a prescribed trajectory ($\overline{\mathbf{X}}$, $\overline{\mathbf{Z}}$) such that

$$\mathrm{d}\overline{\mathbf{X}}/\mathrm{d}t = \overline{V}\cos(\overline{\varGamma}), \quad \mathrm{d}\overline{\mathbf{Z}}/\mathrm{d}t = -\overline{V}\sin(\overline{\varGamma}). \tag{3.7}$$

In the *QFE,* the thrust F of the rotors is the main lift force, and in the *AFE,* these are the wing aerodynamic forces. By L and D, we denote the main lift forces. During the transition phase between two flight envelopes mentioned above, the importance between two different types of forces changes gradually.

The nominal values of F and L are obtained from (3.1), (3.3), (3.4) and (3.7) as

$$\begin{aligned}\begin{bmatrix}\overline{F}\\ \overline{L}\end{bmatrix} &= \begin{bmatrix} D(\overline{V},\,\overline{\boldsymbol{\alpha}})/\cos(\overline{\boldsymbol{\alpha}})\\ -D(\overline{V},\,\overline{\boldsymbol{\alpha}})\tan(\overline{\boldsymbol{\alpha}})\end{bmatrix} + (m\,g)\begin{bmatrix}\sin(\overline{\varGamma})/\cos(\overline{\boldsymbol{\alpha}})\\ \cos(\overline{\varGamma}) - \sin(\overline{\varGamma})\tan(\overline{\boldsymbol{\alpha}})\end{bmatrix}\\ &\quad - m\begin{bmatrix} -(\mathrm{d}\overline{V}/\mathrm{d}t)\big/\cos(\overline{\boldsymbol{\alpha}})\\ (\mathrm{d}\overline{V}/\mathrm{d}t)\tan(\overline{\boldsymbol{\alpha}}) - \overline{V}\,(\mathrm{d}\overline{\varGamma}/\mathrm{d}t)\end{bmatrix},\end{aligned} \tag{3.8}$$

where: $\overline{V} = \sqrt{(\mathrm{d}\overline{\mathbf{X}}/\mathrm{d}t)^2 + (\mathrm{d}\overline{\mathbf{Z}}/\mathrm{d}t)^2}$ and ($\cos(\overline{\boldsymbol{\alpha}}) \approx 1$ and $\sin(\overline{\boldsymbol{\alpha}}) \approx \overline{\boldsymbol{\alpha}}$):

$$\overline{\boldsymbol{\alpha}} \approx \frac{(m\,g)\ \cos(\overline{\varGamma}) + m\,\overline{V}\,\mathrm{d}\overline{\varGamma}/\mathrm{d}t - (1/2)\rho(b^2/AR)\,\overline{V}^2\,C_{L_0}}{(1/2)\rho(b^2/AR)\,\overline{V}^2\left(C_{L_1} + C_{D_0}\right) + (m\,g)\ \sin(\overline{\varGamma}) + m\,\mathrm{d}\overline{V}/\mathrm{d}t}. \tag{3.9}$$

Note that the nominal value of the pitching moment T_q is given by

$$\overline{T}_q = \mathrm{I}_{yy}\,\mathrm{d}^2\overline{\varGamma}/\mathrm{d}t^2 - M(\overline{V},\,\overline{\boldsymbol{\alpha}}) - \bar{\bar{c}}\,(h - h_0)\,\overline{L}(\overline{V},\,\overline{\boldsymbol{\alpha}}). \tag{3.10}$$

Using these predetermined nominal values we are able to define the longitudinal incremental variables given as follows

$$\mathbf{x} = \overline{\mathbf{X}} - \mathbf{X},\ \ \mathbf{z} = \overline{\mathbf{Z}} - \mathbf{Z},\ \ \theta = \overline{\varTheta} - \varTheta,\ \ f = \overline{F} - F,\ \ \tau_q = \overline{T}_q - T_q. \tag{3.11}$$

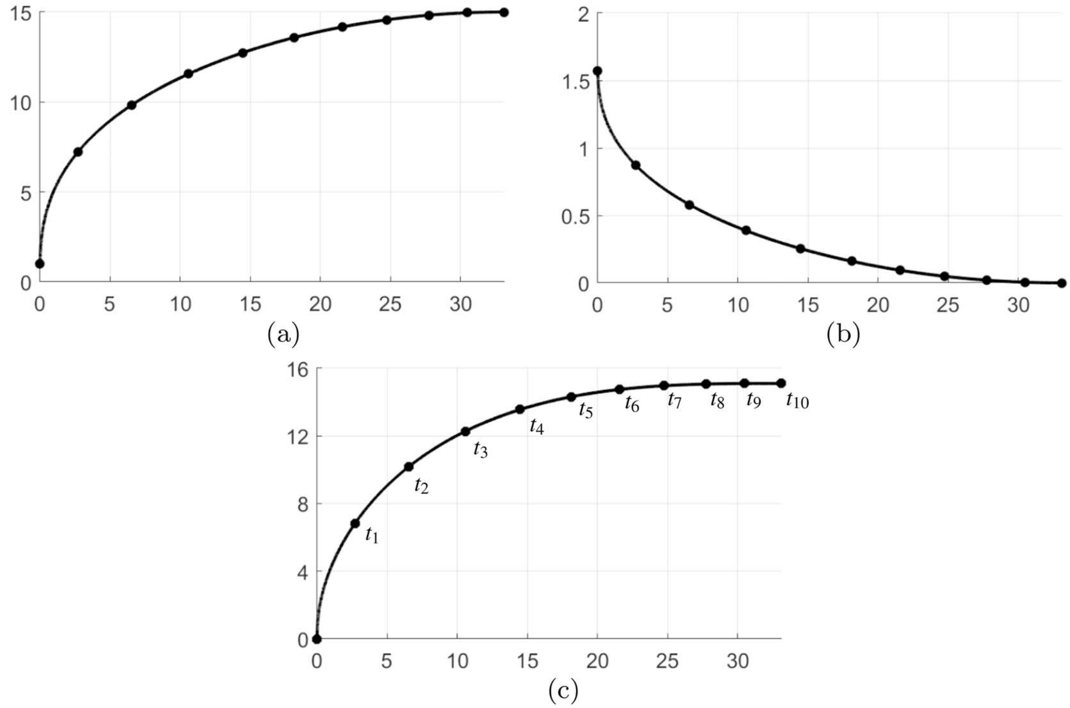

FIG. 2. Take-off path (3.12) and (3.7); $\overline{V}_0$=1 [m/s], $\overline{V}_N$=15 [m/s], t_N=5 [m/s]. (a) $\overline{V}$[m/s] *v.s.* $\overline{\mathbf{X}}$[m]. (b) $\overline{\varGamma}$[rad] *v.s.* $\overline{\mathbf{X}}$[m]. (c) $-\overline{\mathbf{Z}}$[m] *v.s.* $\overline{\mathbf{X}}$[m].

3.3. *Problem solution method*

In order to use the constructive treatment of the problem established in the previous section, let us first define the following smooth take-off path

$$\overline{V} = \overline{V}_0 + \frac{\overline{V}_N - \overline{V}_0}{2}\left(1 - \cos\left(\pi\frac{t}{t_N}\right)\right) \text{ [m/s]}, \quad \overline{\varGamma} = \frac{\pi}{2} - \frac{\pi}{4}\left(1 - \cos\left(\pi\frac{t}{t_N}\right)\right) \text{ [rad]}, \tag{3.12}$$

where t_N is the transition time, $\overline{V}_0$ is the initial speed and $\overline{V}_N$ denotes the desired cruise speed (see Fig. 2).

We next consider a necessary time partition: $\mathscr{P}_{10} = \{\mathscr{I}_1, \ldots, \mathscr{I}_N\}$, $\mathscr{I}_k = [t_{k-1}, t_k)$, $t_0 = 0$, $t_{k-1} < t_k$, $k \in \{1, \ldots, N\}$, and introduce the specific change of variable:

$$\begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{z}} \end{bmatrix} = P \begin{bmatrix} \mathbf{x} \\ \mathbf{z} \end{bmatrix}, \quad P = \begin{bmatrix} \cos\left(\widehat{\varTheta}(t)\right) & -\sin\left(\widehat{\varTheta}(t)\right) \\ \sin\left(\widehat{\varTheta}(t)\right) & \cos\left(\widehat{\varTheta}(t)\right) \end{bmatrix}; \quad \widehat{\varTheta}(t) = \sum_{k=1}^{N} \mathbb{1}_{\mathscr{I}_k}(t)\,\overline{\varTheta}(t_k). \tag{3.13}$$

From (3.1), (3.11) and (3.13), we can deduce the useful state space description for the input and state vectors

$$\frac{\mathrm{d}}{\mathrm{d}t} x = A_q\, x + B\, u + S\, \mathrm{q}(x), \tag{3.14}$$

$u = \left[\, f \quad \tau_q \,\right]^T$ $x = \left[\, x_{\mathbf{x}}^T \quad x_{\mathbf{z}}^T \quad x_{\theta}^T \,\right]^T$. Here: $x_{\mathbf{x}} = \left[\, \hat{\mathbf{x}} \quad \mathrm{d}\hat{\mathbf{x}}/\mathrm{d}t \,\right]^T$, $x_{\mathbf{z}} = \left[\, \hat{\mathbf{z}} \quad \mathrm{d}\hat{\mathbf{z}}/\mathrm{d}t \,\right]^T$ and $x_{\theta} = \left[\, \theta \quad \mathrm{d}\theta/\mathrm{d}t \,\right]^T$; $\mathrm{q}(x) = \left[\, \mathrm{q}_{\mathbf{x}k} \quad \mathrm{q}_{\mathbf{z}k} \quad \mathrm{q}_{\theta k} \,\right]^T$ is the uncertainty vector (nonlinear perturbation signal) (Bonilla *et al.*, 2020). This vector describes the terms that are neglected when a linearization around the equilibrium points $(x, u) = (0, 0)$ is implemented. Moreover, we need to introduce matrices A_q, B and S

$$A_q = \sum_{k=1}^{N} \mathbb{1}_{\mathscr{I}_k}(t) \left[\begin{array}{cc|cc|cc} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \mathring{a}_{\hat{\mathbf{x}}\hat{\mathbf{x}}k} & 0 & \mathring{a}_{\hat{\mathbf{x}}\hat{\mathbf{z}}k} & \mathring{a}_{\hat{\mathbf{x}}\theta k} & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \mathring{a}_{\hat{\mathbf{z}}\hat{\mathbf{x}}k} & 0 & \mathring{a}_{\hat{\mathbf{z}}\hat{\mathbf{z}}k} & \mathring{a}_{\hat{\mathbf{z}}\theta k} & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \mathring{a}_{\theta\hat{\mathbf{x}}k} & 0 & \mathring{a}_{\theta\hat{\mathbf{z}}k} & \mathring{a}_{\theta\theta k} & 0 \end{array}\right], \quad B = \left[\begin{array}{c|c} 0 & 0 \\ 1/m & 0 \\ \hline 0 & 0 \\ 0 & 0 \\ \hline 0 & 0 \\ 0 & 1/\mathrm{I}_{yy} \end{array}\right], \quad S = \left[\begin{array}{c|c|c} 0 & 0 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right]. \tag{3.15}$$

The $\mathring{a}_{ijk}$, $i, j \in \{\hat{\mathbf{x}}, \hat{\mathbf{z}}, \theta\}$ are constant coefficients determined over the intervals $\mathscr{I}_k$ for $k \in \{1, \ldots, N\}$. These coefficients are results of the linearizing procedure (3.1), (3.11) and (3.13) realized over the take-off path (3.12) and (3.7). This linearization is considered at the time instants $\{t_1, \ldots, t_N\}$ as shown in Fig. 2. Let us note that Appendix A contains the concrete necessary algebraic expressions we applied here.

3.4. *Riccati stabilization*

In order to tackle the *Riccati Stabilization* of (3.14) and (3.15), we first define the locations set (*cf.* (1.3))

$$\mathscr{Q}_N \triangleq \left\{ q_1, \ldots, q_N \,\middle|\, q_k = \left[\, \mathring{a}_{\hat{\mathbf{x}}\hat{\mathbf{x}}k}\ \mathring{a}_{\hat{\mathbf{x}}\hat{\mathbf{z}}k}\ \mathring{a}_{\hat{\mathbf{x}}\theta k}\ \mathring{a}_{\hat{\mathbf{z}}\hat{\mathbf{x}}k}\ \mathring{a}_{\hat{\mathbf{z}}\hat{\mathbf{z}}k}\ \mathring{a}_{\hat{\mathbf{z}}\theta k}\ \mathring{a}_{\theta\hat{\mathbf{x}}k}\ \mathring{a}_{\theta\hat{\mathbf{z}}k}\ \mathring{a}_{\theta\theta k} \,\right]^T,\ k \in \{1, \ldots, N\} \right\}. \tag{3.16}$$

Thus, matrix A_q takes the following form (*cf.* (1.2a)):

$$A_q = \overline{A}_0 + \overline{A}_1\, \overline{D}(q), \quad \overline{D}(q) = \sum_{k=1}^{N} \mathbb{1}_{\mathscr{I}_k}(t)\, \overline{D}(q_k), \quad A_{q_k} = \overline{A}_0 + \overline{A}_1\, \overline{D}(q_k),$$

$$\overline{A}_0 = \left[\begin{array}{cc|cc|cc} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \quad \overline{A}_1 = \left[\begin{array}{ccc} 0 & 0 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right], \tag{3.17}$$

TABLE 1 *Numerical values of the components* $\mathring{a}_{ijk}$, $i,\ j \in \{\hat{\mathbf{x}},\ \hat{\mathbf{z}},\ \theta\}$, *of the locations* $q_k \in \mathscr{Q}_{10}$, $k \in \{1,\ \ldots,\ 10\}$ *(cf. (3.16); see also Appendix A). The column* $(\overline{\mathbf{X}},\ -\overline{\mathbf{Z}})$ *was rounded to hundredth and the columns of the* $\mathring{a}_{ijk}$ *were rounded to thousandth.*

$(\overline{\mathbf{X}},\ -\overline{\mathbf{Z}})$[m]	k	$-\mathring{a}_{\hat{\mathbf{x}}\hat{\mathbf{x}}k}$	$\mathring{a}_{\hat{\mathbf{x}}\hat{\mathbf{z}}k}$	$-\mathring{a}_{\hat{\mathbf{x}}\theta k}$	$-\mathring{a}_{\hat{\mathbf{z}}\hat{\mathbf{x}}k}$	$-\mathring{a}_{\hat{\mathbf{z}}\hat{\mathbf{z}}k}$	$-\mathring{a}_{\hat{\mathbf{z}}\theta k}$	$-\mathring{a}_{\theta\hat{\mathbf{x}}k}$	$-\mathring{a}_{\theta\hat{\mathbf{z}}k}$	$-\mathring{a}_{\theta\theta k}$
(5.79, 9.67)	1	0.041	0.279	0.435	0.525	4.049	48.149	0.429	2.221	2.207
(10.68, 12.32)	2	0.052	0.274	0.846	0.592	4.969	64.947	0.538	3.345	3.333
(14.80, 13.64)	3	0.058	0.292	1.071	0.647	5.499	76.010	0.592	4.097	4.085
(18.36, 14.35)	4	0.061	0.325	1.161	0.698	5.842	83.623	0.616	4.622	4.608
(21.48, 14.74)	5	0.063	0.367	1.157	0.748	6.075	88.890	0.621	4.994	4.977
(24.26, 14.94)	6	0.064	0.414	1.088	0.796	6.233	92.458	0.612	5.253	5.232
(26.77, 15.04)	7	0.064	0.463	0.974	0.842	6.340	94.756	0.594	5.429	5.405
(29.05, 15.09)	8	0.064	0.515	0.829	0.887	6.409	96.079	0.571	5.540	5.512
(31.14, 15.10)	9	0.063	0.567	0.664	0.929	6.449	96.644	0.544	5.600	5.569
(33.07, 15.10)	10	0.062	0.619	0.486	0.969	6.467	96.614	0.515	5.621	5.586

$$\overline{D}(q_k) = \begin{bmatrix} \mathring{a}_{\hat{\mathbf{x}}\hat{\mathbf{x}}k} & \mathring{a}_{\hat{\mathbf{x}}\hat{\mathbf{z}}k} & \mathring{a}_{\hat{\mathbf{x}}\theta k} \\ \mathring{a}_{\hat{\mathbf{z}}\hat{\mathbf{x}}k} & \mathring{a}_{\hat{\mathbf{z}}\hat{\mathbf{z}}k} & \mathring{a}_{\hat{\mathbf{z}}\theta k} \\ \mathring{a}_{\theta\hat{\mathbf{x}}k} & \mathring{a}_{\theta\hat{\mathbf{z}}k} & \mathring{a}_{\theta\theta k} \end{bmatrix} \left[\begin{array}{cc|cc|cc} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

We next define the time partition ($N = 10$ and $t_N = 5$ [s]) as

$$\mathscr{P}_{10} \triangleq \left\{ \mathscr{I}_k = [t_{k-1},\ t_k) \;\middle|\; t_0 = 0,\ t_k = 5\,(k/10)^{(475/1918)},\ k \in \{1,\ \ldots,\ 10\} \right\}. \tag{3.18}$$

Using the introduced time partition, we get the locations shown in Table 1 (*cf.* (3.16). We also refer to Appendix A for the necessary technical details.

From (3.17) and Table 1, we next obtain the spectra of the matrix A_{q_k}:

$$\sigma(A_{q_k}) = \begin{cases} q_1 : & \{0,\ 0,\ 3.4869,\ -0.099159,\ -3.7391 + 3.7709\,\imath,\ -3.7391 - 3.7709\,\imath\} \\ q_2 : & \{0,\ 0,\ 4.4614,\ -0.099919,\ -4.6913 + 4.8079\,\imath,\ -4.6913 - 4.8079\,\imath\} \\ q_3 : & \{0,\ 0,\ 5.0611,\ -0.10296,\ -5.2577 + 5.4413\,\imath,\ -5.2577 - 5.4413\,\imath\} \\ q_4 : & \{0,\ 0,\ 5.4595,\ -0.10605,\ -5.6286 + 5.8614\,\imath,\ -5.6286 - 5.8614\,\imath\} \\ q_5 : & \{0,\ 0,\ 5.7299,\ -0.10869,\ -5.8795 + 6.1473\,\imath,\ -5.8795 - 6.1473\,\imath\} \\ q_6 : & \{0,\ 0,\ 5.9117,\ -0.11077,\ -6.0491 + 6.3408\,\imath,\ -6.0491 - 6.3408\,\imath\} \\ q_7 : & \{0,\ 0,\ 6.0291,\ -0.11231,\ -6.1606 + 6.4674\,\imath,\ -6.1606 - 6.4674\,\imath\} \\ q_8 : & \{0,\ 0,\ 6.0979,\ -0.11332,\ -6.2287 + 6.544\,\imath,\ -6.2287 - 6.544\,\imath\} \\ q_9 : & \{0,\ 0,\ 6.1292,\ -0.11386,\ -6.2636 + 6.582\,\imath,\ -6.2636 - 6.582\,\imath\} \\ q_{10} : & \{0,\ 0,\ 6.1312,\ -0.11397,\ -6.2728 + 6.59\,\imath,\ -6.2728 - 6.59\,\imath\} \end{cases} \tag{3.19}$$

We now consider the concrete solution procedure for ARE (2.2). In order to solve this ARE, we put

$$q_0 = q_{10} \quad \Rightarrow \quad \overline{\varDelta}_0 = \overline{D}(q_{10}), \tag{3.20}$$

and choose

$$\overline{Q}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 1/20 & 0 \\ 0 & 1 \end{bmatrix}. \tag{3.21}$$

From (3.17), (3.20) and taking into consideration Table 1 and (3.21), we can deduce that the solution of (2.2) has the following explicit form:

$$P_0 = \begin{bmatrix} 1.3097 & 0.35454 & 0.0033206 & 0.019828 & -0.25039 & -0.0064344 \\ 0.35454 & 0.45562 & -0.047959 & 0.015024 & -0.12392 & -0.0035059 \\ 0.0033206 & -0.047959 & 1.2772 & 0.13327 & -1.778 & -0.047567 \\ 0.019828 & 0.015024 & 0.13327 & 0.092803 & -0.8661 & -0.031908 \\ -0.25039 & -0.12392 & -1.778 & -0.8661 & 20.29 & 0.61192 \\ -0.0064344 & -0.0035059 & -0.047567 & -0.031908 & 0.61192 & 0.086183 \end{bmatrix}. \tag{3.22}$$

The control feedback (2.1) can now be defined and has the following expression

$$\begin{aligned} F_{*0} &= R^{-1}B^T P_0 = \begin{bmatrix} 4.4318 & 5.6952 & -0.59949 & 0.1878 & -1.5491 & -0.043824 \\ -0.13405 & -0.07304 & -0.99097 & -0.66474 & 12.748 & 1.7955 \end{bmatrix}, \\ u &= -F_{*0}\, x. \end{aligned} \tag{3.23}$$

Moreover, we have

$$Q_0 = \overline{Q}_0 + F_{*0}^T R F_{*0} \quad \Rightarrow \quad \lambda_{min}\left(Q_0\right) = \lambda_{min}\left(\overline{Q}_0\right) = 1 \qquad \left(\lambda_{min}\left(F_{*0}^T R F_{*0}\right) = 0\right). \tag{3.24}$$

Note that Table 2 includes the computation of the spectra of matrices $X_k = \left(\overline{A}_0 + \overline{A}_1 \overline{\Delta}_0 - A_{q_k}\right)^T P_0 + P_0\left(\overline{A}_0 + \overline{A}_1 \overline{\Delta}_0 - A_{q_k}\right)$. Additionally, we test the condition (2.8) for the values of $k \in \{1, \ldots, 10\}$.

3.5. *Simulation results*

This section includes the simulation results involving the proposed stabilizing feedback control. In Fig. 3, we present some simulation results when applying the LQR state feedback (3.23) to the given *SWQA* system represented by (3.1) – (3.5). We have followed here the smooth take-off path (3.12) and (3.7) with $\overline{V}_0 = 1$ [m/s], $\overline{V}_N = 15$ [m/s] and $t_N = 5$ [m/s]. Moreover, the state components are given as follows: $x = \begin{bmatrix} x_{\mathbf{x}}^T & x_{\mathbf{z}}^T & x_\theta^T \end{bmatrix}^T$, $x_{\mathbf{x}} = \begin{bmatrix} \hat{\mathbf{x}} & \mathrm{d}\hat{\mathbf{x}}/\mathrm{d}t \end{bmatrix}^T$, $x_{\mathbf{z}} = \begin{bmatrix} \hat{\mathbf{z}} & \mathrm{d}\hat{\mathbf{z}}/\mathrm{d}t \end{bmatrix}^T$, $x_\theta = \begin{bmatrix} \theta & \mathrm{d}\theta/\mathrm{d}t \end{bmatrix}^T$, and the control actions, f and τ_q, are obtained from (3.11), (3.13) and (3.8) – (3.11).

From the information presented on Fig. 3 we can conclude that the *SWQA* tracks correctly the smooth take-off path (3.12) and (3.7) and has a stable behaviour.

TABLE 2 *Spectra $\sigma(X_k)$, $X_k = \left(\overline{A}_0 + \overline{A}_1\overline{\Delta}_0 - A_{q_k}\right)^T P_0 + P_0\left(\overline{A}_0 + \overline{A}_1\overline{\Delta}_0 - A_{q_k}\right)$, and condition (2.8) for $k\in\{1, \ldots, 10\}$; values rounded to ten thousandth*

k	$\sigma(X_k)$	$1+\lambda_{min}(X_k)$
1	{−0.9998, −0.2296, −0.0172, 0.0030, 0.2252, 80.593}	0.0002
2	{−0.6573, −0.1834, −0.0031, 0.0007, 0.1882, 52.519}	0.3427
3	{−0.4519, −0.1427, −0.0008, 0.0026, 0.16481, 34.05}	0.5481
4	{−0.3221, −0.1040, −0.0016, 0.0042, 0.1449, 21.364}	0.6779
5	{−0.2380, −0.0689, −0.0019, 0.0040, 0.12401, 12.617}	0.7620
6	{−0.1812, −0.0407, −0.0018, 0.0031, 0.1000, 6.7195}	0.8188
7	{−0.1425, −0.0215, −0.0012, 0.0018, 0.0713, 2.9591}	0.8575
8	{−0.1296, −0.0117, −0.0002, 0.0002, 0.0367, 0.852}	0.8704
9	{−0.2192, −0.01162, −0.0006, 0.0006, 0.0074, 0.1055}	0.7808
10	{0, 0, 0, 0, 0, 0}	1

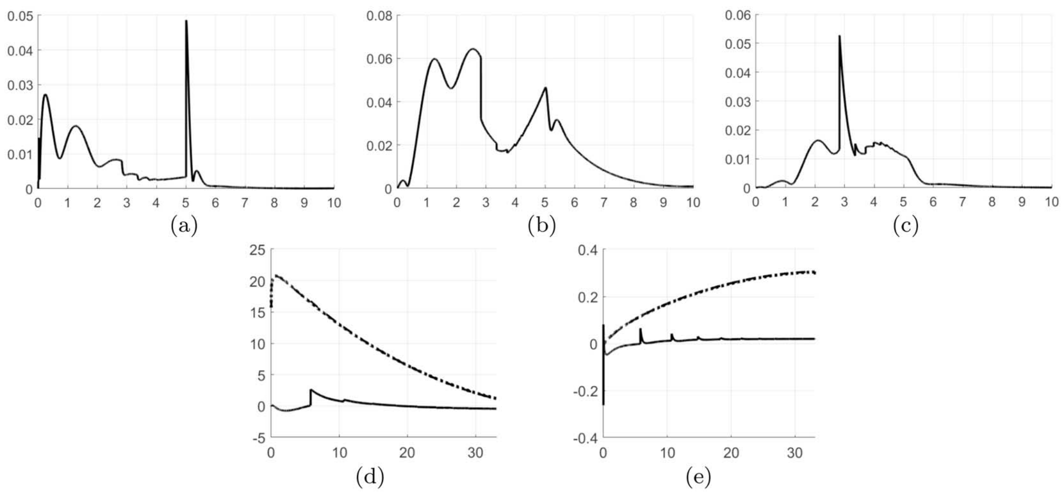

FIG. 3. Simulation results. (a) $\|x_\theta\|$. (b) $\|x_{\mathbf{z}}\|$. (c) $\|x_{\mathbf{x}}\|$. (d) Dotted line: $\overline{F}$, dashed line: $F = \overline{F} + f$, solid line: $10f$. (e) Dotted line: $\overline{T}_q$, dashed line: $T_q = \overline{T}_q + \tau_q$, solid line: $10\,\tau_q$.

4. Concluding Remarks

In this paper, we have proposed a Riccati-equation-based stabilizing state feedback for a wide class of switched dynamic models of the flying objects type. The obtained matrices of the main state space representation (1.1) associated with the class of switched system possess a useful structure similar to (Narendra & Balakrishnan, 1994). Moreover, the admissible switching mechanisms have a generic hybrid nature studied in Azhmyakov (2019). These switched dynamic models make it possible to consider the useful formal state space representation (1.1)–(1.2), where the essential parameters of constant system structure is given by the simple model (1.2). Moreover, (1.5) is determined by the triplet $(\overline{A}_0, \overline{A}_1, \overline{\Delta}_0)$. Note that the combinatorial structure of the locations set in switched systems we examined is in fact represented by the matrix $\overline{D}(q_i)$ (*cf.* (1.5)).

In our contribution, we have developed a kind of 'robustness' result. This robustness is understood with respect to a possible (admissible) switching mechanism such that the Riccati-equation-based stabilization feedback (2.1) and (2.2) stabilize the initially given dynamic system (1.1)–(1.3). This stabilization is implemented under the assumption of an unknown dynamic location $q \in \mathscr{Q}$. We only assumed the knowledge of the essential parameters associated with the constant system structure (1.2) and (1.5), which was determined by the triple $(\overline{A}_0, \overline{A}_1, \overline{\Delta}_0)$.

The formal proof of Theorem 2.2 we presented involves some recent results from (Bonilla *et al.*, 2015b). Finally note that the proposed Riccati-equation-based stabilization feedback we developed not only guarantees the stability property of the *Single Wing Quadrotor Aircraft* in a concrete flying mode but also implies the stability during the mode change, namely, during the change from the *QFE* mode to the *AFE*.

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A. Coefficients of (3.15)

$$
\begin{aligned}
\mathring{a}_{\hat{\mathbf{x}}\hat{\mathbf{x}}k} &\approx \left(m\,\overline{V}(t_k)\right)^{-1}\Big(-\mathring{D}_{V,k}\,\overline{V}(t_k)+\big(\mathring{D}_{\alpha,k}-L_k+\mathring{L}_{V,k}\,\overline{V}(t_k)\big)\overline{\boldsymbol{\alpha}}(t_k)\Big),\\
\mathring{a}_{\hat{\mathbf{x}}\hat{\mathbf{z}}k} &\approx \left(m\,\overline{V}(t_k)\right)^{-1}\Big(L_k-\mathring{D}_{\alpha,k}+\Big(D_k+\mathring{L}_{\alpha,k}-\mathring{D}_{V,k}\,\overline{V}(t_k)\Big)\,\overline{\boldsymbol{\alpha}}(t_k)\Big),\\
\mathring{a}_{\hat{\mathbf{x}}\theta k} &\approx m^{-1}\left(-\mathring{D}_{\alpha,k}+\mathring{L}_{\alpha,k}\,\overline{\boldsymbol{\alpha}}(t_k)\right),\\
\mathring{a}_{\hat{\mathbf{z}}\hat{\mathbf{x}}k} &\approx \left(m\,\overline{V}(t_k)\right)^{-1}\Big(\mathring{D}_{\alpha,k}-\mathring{L}_{V,k}\,\overline{V}(t_k)+\Big(D_k+\mathring{L}_{\alpha,k}-\mathring{D}_{V,k}\,\overline{V}(t_k)\Big)\,\overline{\boldsymbol{\alpha}}(t_k)\Big),\\
\mathring{a}_{\hat{\mathbf{z}}\hat{\mathbf{z}}k} &\approx \left(2\,m\,\overline{V}(t_k)\right)^{-1}\Big(-D_k-\mathring{L}_{\alpha,k}+2\left(\mathring{D}_{\alpha,k}-L_k+\mathring{L}_{V,k}\,\overline{V}(t_k)\right)\overline{\boldsymbol{\alpha}}(t_k)\Big),\\
\mathring{a}_{\hat{\mathbf{z}}\theta k} &\approx -m^{-1}\left(\overline{F}(t_k)+\mathring{L}_{\alpha,k}+\mathring{D}_{\alpha,k}\,\overline{\boldsymbol{\alpha}}(t_k)\right),\\
\mathring{a}_{\theta\hat{\mathbf{x}}k} &\approx \mathrm{I}_{yy}^{-1}\left(\mathring{M}_{V,k}+\overline{\overline{c}}(h-h_0)\,\mathring{L}_{V,k}\right)-\mathring{a}_{\theta\theta k}\,\overline{\boldsymbol{\alpha}}(t_k),\\
\mathring{a}_{\theta\hat{\mathbf{z}}k} &\approx \mathring{a}_{\theta\theta k}+\mathrm{I}_{yy}^{-1}\left(\mathring{M}_{V,k}+\overline{\overline{c}}(h-h_0)\,\mathring{L}_{V,k}\right)\overline{\boldsymbol{\alpha}}(t_k),\\
\mathring{a}_{\theta\theta k} &= \mathrm{I}_{yy}^{-1}\left(\mathring{M}_{\alpha,k}+\overline{\overline{c}}(h-h_0)\,\mathring{L}_{\alpha,k}\right),
\end{aligned}
$$

where:

$$
\begin{aligned}
&L_k=L(\overline{V}(t_k),\,\overline{\boldsymbol{\alpha}}(t_k)), && \mathring{L}_{\alpha,k}=\partial L(\overline{V}(t_k),\,\overline{\boldsymbol{\alpha}}(t_k))/\partial\boldsymbol{\alpha}, && \mathring{L}_{V,k}=\partial L(\overline{V}(t_k),\,\overline{\boldsymbol{\alpha}}(t_k))/\partial V,\\
&D_k=D(\overline{V}(t_k),\,\overline{\boldsymbol{\alpha}}(t_k)), && \mathring{D}_{\alpha,k}=\partial D(\overline{V}(t_k),\,\overline{\boldsymbol{\alpha}}(t_k))/\partial\boldsymbol{\alpha}, && \mathring{D}_{V,k}=\partial D(\overline{V}(t_k),\,\overline{\boldsymbol{\alpha}}(t_k))/\partial V,\\
&M_k=M(\overline{V}(t_k),\,\overline{\boldsymbol{\alpha}}(t_k)), && \mathring{M}_{\alpha,k}=\partial M(\overline{V}(t_k),\,\overline{\boldsymbol{\alpha}}(t_k))/\partial\boldsymbol{\alpha}, && \mathring{M}_{V,k}=\partial M(\overline{V}(t_k),\,\overline{\boldsymbol{\alpha}}(t_k))/\partial V.
\end{aligned}
$$