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Calculation of libration point orbits in the circular restricted three-body problem

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Abstract. This study investigates possibilities for extension and improvement of algorithms for generation of libration point orbits in the framework of the circular restricted three body problem. Two algorithms for orbit generation based on bisection approach using different ways for evaluation of unstable component of motion are considered. The spacecraft's state vector is periodically adjusted in such a way that unstable component of motion is neutralized and the trajectory corresponding to the corrected state vector belongs to the central manifold associated with libration point. The first algorithm uses expression for unstable component derived from linearized equations of motion. The second one is based on the procedure of reduction to central manifold, utilizing canonical coordinate transformations to nullify high order monomials in the expansion of Hamiltonian of the system in terms of Legendre polynomials. This allows expressing unstable component as one of generalized coordinates of Hamiltonian system obtained as the result of aforementioned transformation. Evaluation of these techniques proved their applicability for orbit generation. However, the second approach allows generating orbits in greater vicinity of libration point.

1. Introduction

The circular restricted three-body problem (CRTBP) allows approximating the dynamics of the motion of bodies in the Solar System. This problem considers two massive bodies moving along circular paths around their center of mass. A third massless body moves in the gravitational field of two primaries. There are 5 Lagrange points (libration points) in the scope of CRTBP. Those are points in space where combined gravitational forces of primaries and centrifugal force felt by smaller body are in equilibrium. There are trajectories around libration points, which allow a spacecraft to remain on them for extended periods of time, spending limited amount of energy for corrections. This fact may motivate space missions to libration points. Sun-Earth libration point orbits (LPO) were successfully used in several space missions [1–4].

Full analytic solution of equations of motion in CRTBP is not known. To construct orbits that maintain the spacecraft in the vicinity of the libration point, methods for correcting the spacecraft's state vector are required. Numerous scientific works investigate methods for constructing LPO. Recent works utilize separation properties of stable and unstable manifolds associated with libration point. Study [4] introduced a bisection method, which allows separating trajectories belonging to positive and negative branches of unstable manifold, thus, localizing the state vector corresponding to the central manifold. This idea was further investigated in the work [5]. This work proposed using spheres as boundary surfaces for bisection method.



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The intersection of the trajectory with one of the spherical surfaces indicates belonging to the certain branch of unstable manifold. The halo orbit, vertical Lyapunov orbit and the quasi-periodic orbits close to them cannot be obtained by using this method. Study [6] introduced a generalized version of bisection method, which could handle the computations of various types of LPOs in a unified manner. This approach uses Lyapunov's horizontal orbit rotation body separated by zero velocity surfaces as boundary surfaces. Study [7] proposed using planar surfaces as boundaries for bisection. The main difficulty of approach is the choice of location for boundaries. All aforementioned techniques use geometrical criterion defined in 3-dimensional space to separate branches of unstable manifold. The goal of this study is to create simple alternative for this criterion defined in 6-dimensional phase space of CRTBP problem.

2. Equations of motion

Let the bodies P_1 , P_2 with masses M_1 , M_2 ($M_1 > M_2$) rotate along circular paths around the barycenter. Consider a rotating coordinate system defined by axes x, y, z centered at the barycenter of the system. The x -axis is directed from P_1 to P_2 . The z axis is directed along the angular momentum vector. The y axis compliments the coordinate system in such a way that it satisfies right-hand rule. The system rotates with a constant angular velocity $\vec{n} = (0, 0, n)^T$, where n is the value of the mean motion. Using this coordinate system, the equations of motion for a spacecraft may be expressed in the following form:

$$\begin{cases} \ddot{x} - 2\dot{y} = x + \frac{1-\mu}{r_1^3}(-\mu - x) + \frac{\mu}{r_2^3}(1 - \mu - x) \\ \ddot{y} + 2\dot{x} = y - \left(\frac{1-\mu}{r_1^3} + \frac{\mu}{r_2^3}\right)y \\ \ddot{z} = -\left(\frac{1-\mu}{r_1^3} + \frac{\mu}{r_2^3}\right)z \end{cases} \quad \begin{aligned} \mu &= \frac{M_2}{M_1 + M_2} \\ r_1 &= \sqrt{(x + \mu)^2 + y^2 + z^2} \\ r_2 &= \sqrt{(x - 1 + \mu)^2 + y^2 + z^2} \end{aligned} \quad (1)$$

The complete analytical solution of system (1) is unknown, the generation of trajectories in the scope of this problem requires the use of numerical integration. Upon linearization in the vicinity of the libration point, the equations of motion take the form:

$$\begin{cases} \ddot{x} - 2\dot{y} - (1 + 2c_2)x = 0 \\ \ddot{y} + 2\dot{x} - (1 - c_2)y = 0 \\ \ddot{z} + c_2z = 0 \end{cases} \quad (2)$$

Coefficient c_2 depends on the ratio of the masses of the Sun and the Earth and on the coordinates of the libration point used as the center of linearization. Solution of system (2) may be written as:

$$\begin{cases} x(t) = A_1 k_2 e^{\lambda t} + A_2 k_2 e^{-\lambda t} + A_3 k_1 \cos(\omega t) + A_4 k_1 \sin(\omega t) \\ y(t) = -A_1 k_4 e^{\lambda t} + A_2 k_4 e^{-\lambda t} + A_3 k_3 \sin(\omega t) - A_4 k_3 \cos(\omega t) \\ z(t) = A_5 k_5 \sin(\nu t) - A_6 k_5 \cos(\nu t) \end{cases} \quad (3)$$

$A_1 - A_6$ continuously depend on the initial conditions, while $k_1 - k_5, \lambda, \omega, \nu$ depend only on the constant c_2 . The solution (3) indicates the existence of center manifold ($A_1 = A_2 = 0$) containing periodical and quasi-periodical solutions as well as asymptotic stable ($A_1 = 0, A_2 \neq 0$) and unstable ($A_1 \neq 0, A_2 = 0$) manifolds. Nonlinear dynamics described by the equations (1) has a similar structure in the vicinity of libration point [8].

3. Modified bisection method

We propose using an estimate for unstable component of motion to create boundary surfaces for bisection method. We utilized the Hamiltonian normalization technique and canonical

transformations, previously used in [9] for semi-analytical construction of LPOs, to obtain an expression for the unstable component. Hamiltonian corresponding to nonlinear system (1) may be written as:

$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + yp_x - xp_y - \frac{1-\mu}{r_1} - \frac{\mu}{r_2} \quad (4)$$

$$p_x = \dot{x} - y; p_y = \dot{y} + x; p_z = \dot{z}$$

Hamiltonian (4) can be expressed as an expansion in terms of Legendre polynomials around a libration point:

$$H = H_2 - \sum_{n>2} c_n(\mu)\rho^n P_n\left(\frac{x}{\rho}\right) \quad (5)$$

$$H_2 = \frac{1}{2}(c_2(-2x^2 + y^2 + z^2) + 2yp_x - 2xp_y + p_x^2 + p_y^2 + p_z^2)$$

Where

$$c_n(\mu) = \frac{1}{\gamma^3} \left((\pm 1)^n \mu + (-1)^n \frac{(1-\mu)\gamma^{n+1}}{(1 \mp \gamma)^{n+1}} \right) \quad (\text{for } L_1, L_2) \quad (6)$$

P_n - degree n Legendre polynomial, γ - distance between libration point and closest massive body, $\rho^2 = x^2 + y^2 + z^2$. By applying transformation $(x, y, z, p_x, p_y, p_z)^T = C(q_1, q_2, q_3, p_1, p_2, p_3)^T$, where C - 6×6 symplectic matrix, Hamiltonian H_2 can be transformed to it's real normal form:

$$H_2 = \lambda_1 q_1 p_1 + \frac{\omega_1}{2}(q_2^2 + p_2^2) + \frac{\omega_2}{2}(q_3^2 + p_3^2) \quad (7)$$

$$\lambda_1 = \sqrt{\frac{c_2 - 2 + \sqrt{9c_2^2 - 8c_2}}{2}}; \omega_1 = \sqrt{-\frac{c_2 - 2 - \sqrt{9c_2^2 - 8c_2}}{2}}; \omega_2 = \sqrt{c_2}$$

Where $\lambda_1, \omega_1, \omega_2$ - positive real numbers. Hamiltonian system corresponding to H_2 and it's solution may be expressed as:

$$\begin{cases} \dot{q}_1(t) = \lambda_1 q_1(t); q_1(0) = q_{10} \\ \dot{q}_2(t) = \omega_1 p_2(t); q_2(0) = q_{20} \\ \dot{q}_3(t) = \omega_2 p_3(t); q_3(0) = q_{30} \\ \dot{p}_1(t) = -\lambda_1 p_1(t); p_1(0) = p_{10} \\ \dot{p}_2(t) = -\omega_1 q_2(t); p_2(0) = p_{20} \\ \dot{p}_3(t) = -\omega_2 q_3(t); p_3(0) = p_{30} \end{cases} \quad \begin{cases} q_1(t) = q_{10} e^{\lambda_1 t} \\ q_2(t) = q_{20} \cos(\omega_1 t) + p_{20} \sin(\omega_1 t) \\ q_3(t) = p_{30} \sin(\omega_2 t) \\ p_1(t) = p_{10} e^{-\lambda_1 t} \\ p_2(t) = p_{20} \cos(\omega_1 t) - q_{10} \sin(\omega_1 t) \\ p_3(t) = p_{30} \cos(\omega_2 t) \end{cases} \quad (8)$$

The form of components of solution in (8) indicates that q_1 corresponds to hyperbolic motion. Component q_1 can be expressed through generalized coordinates of Hamiltonian (5) in the following way:

$$q_1(x, p_x, y, p_y) = \frac{u_1 \sqrt{2\lambda_1((4 + 3c_2)\lambda_1^2 + 4 + 5c_2 - 6c_2^2)}}{4\lambda_1(1 + 2c_2)(\lambda_1^2 + \omega_1^2)} \quad (9)$$

$$u_1 = (2(1 + 2c_2)p_y + (1 + 2c_2 + \omega_1^2)((1 + 2c_2)x + \lambda_1 p_x) + \lambda_1(1 + 2c_2 - \omega_1^2)y)$$

In course of this work we established possibility of construction LPO generation method using (9) as an expression for unstable component. However, this approach only allows generation orbits with small amplitudes (see section 4). Therefore, alternative expression for unstable component

is required. It is convenient to introduce the following complex change of coordinates to simplify further computations:

$$\begin{cases} q_1 \rightarrow q_1, & q_2 \rightarrow \frac{q_2+ip_2}{\sqrt{2}}, & q_3 \rightarrow \frac{q_3+ip_3}{\sqrt{2}} \\ p_1 \rightarrow p_1, & p_2 \rightarrow \frac{p_2+iq_2}{\sqrt{2}}, & p_3 \rightarrow \frac{p_3+iq_3}{\sqrt{2}} \end{cases} \quad (10)$$

After applying change (10) Hamiltonian takes form:

$$H = H_2(q, p) + \sum_{n \geq 3} H_n(q, p) \quad (11)$$

$$H_2(q, p) = \lambda_1 q_1 p_1 + i\omega_1 q_2 p_2 + i\omega_2 q_3 p_3; \quad H_n(q, p) = h_{i_1, j_1, i_2, j_2, i_3, j_3} q_1^{i_1} p_1^{j_1} q_2^{i_2} p_2^{j_2} q_3^{i_3} p_3^{j_3}$$

At this point we can utilize Birkhoff normalization method [10]. To exclude monomials of order 3 from expression for Hamiltonian (11) we apply the following canonical transformation:

$$H^3 = H + \{H, G_3\} + \frac{1}{2!} \{\{H, G_3\}, G_3\} + \frac{1}{3!} \{\{\{H, G_3\}, G_3\}, G_3\} + \dots \quad (12)$$

Where H^3 is transformed Hamiltonian, G_3 - generation function of transformation and $\{\cdot, \cdot\}$ denotes Poisson bracket. Given that P, Q are homogeneous polynomials of degrees r, s , respectively, $\{P, Q\}$ is a homogeneous polynomial of degree $r + s - 2$. Therefore the Hamiltonian H^3 satisfies the relations:

$$H_2^3 = H_2; \quad H_3^3 = H_3 + \{H_2, G_3\}; \quad H_4^3 = H_4 + \{H_3, G_3\} + \frac{1}{2!} \{\{H_2, G_3\}, G_3\}; \dots \quad (13)$$

Where H_i^3 is a homogeneous polynomial of degree i in the expansion of the Hamiltonian H^3 . G_3 must satisfy equation $H_3 + \{H_2, G_3\} = 0$ for H_3^3 to be equal zero after transformation. Therefore, G_3 takes form:

$$G_3(q, p) = \sum_{i_1+j_1+i_2+j_2+i_3+j_3=3} \frac{-h_{i_1, j_1, i_2, j_2, i_3, j_3}}{(j_1 - i_1)\lambda + (j_2 - i_2)\omega_1 + (j_3 - i_3)\omega_2} q_1^{i_1} p_1^{j_1} q_2^{i_2} p_2^{j_2} q_3^{i_3} p_3^{j_3} \quad (14)$$

Continuing the above process, we get transformed Hamiltonian $H^N = H_2(q^N, p^N) + R_N(q^N, p^N)$, where $R_N(q^N, p^N)$ - degree $N + 1$ polynomial, $q^N = (q_1^N, q_2^N, q_3^N)$. Note that N in q_i^N indicates the step of normalization process and does not have meaning of exponentiation. Transformation of form (12) yields recurrent formula which allows expressing generalized coordinates of transformed Hamiltonian through coordinates of Hamiltonian before transformation:

$$\begin{aligned} q_1^N &= q_1^{N-1} + \{q_1^{N-1}, G_N\} + \frac{1}{2!} \{\{q_1^{N-1}, G_N\}, G_N\} + \dots \\ &\vdots \end{aligned} \quad (15)$$

$$q_1^3 = q_1 + \{q_1, G_3\} + \frac{1}{2!} \{\{q_1, G_3\}, G_3\} + \frac{1}{3!} \{\{\{q_1, G_3\}, G_3\}, G_3\} + \dots \quad (16)$$

In the scope of this study we stopped Hamiltonian normalization process after nullifying monomials of order 3. By applying change inverse to (10) and using (9) we can express q_1^3 as a function of coordinates of initial Hamiltonian (5):

$$q_1^3 = q_1^3(x, y, z, p_x, p_y, p_z) \quad (17)$$

Given expressions for unstable component, we can formulate LPO generation technique. Consider spacecraft's initial state vector $\vec{s}_0 = (x_0, y_0, z_0, v_{x_0}, v_{y_0}, v_{z_0})^T$ and velocity correction

vector $\vec{v}_c = \Delta v(0, 0, 0, v_{c_x}, v_{c_y}, v_{c_z})$, $\|\vec{v}_c\| = \Delta v$. By the means of numerical integration of nonlinear system (1) with initial conditions:

$$(x(0), y(0), z(0), \dot{x}(0), \dot{y}(0), \dot{z}(0))^T = (x_0, y_0, z_0, v_{x_0} + \Delta v v_{c_x}, v_{y_0} + \Delta v v_{c_y}, v_{z_0} + \Delta v v_{c_z})^T \quad (18)$$

we obtain discrete trajectory $\vec{s}_i = (x_i, y_i, z_i, v_{x_i}, v_{y_i}, v_{z_i})^T, i \in [1, M]$. To correct initial state vector so that it corresponds to motion along LPO we need to find the value of Δv which acts as a point of discontinuity of the following function:

$$F(\Delta v) = \begin{cases} 1, & \exists i \in [1, M] : (\hat{q}(\vec{s}_i) - q_{max})(\hat{q}(\vec{s}_{i-1}) - q_{max}) < 0 \\ -1, & \exists i \in [1, M] : (\hat{q}(\vec{s}_i) - q_{min})(\hat{q}(\vec{s}_{i-1}) - q_{min}) < 0 \end{cases} \quad (19)$$

Where q_{min}, q_{max} - are chosen boundary values and \hat{q} - is an estimate for unstable component either defined by expression (9) or expressions (16)-(17). After the value of Δv is obtained, an orbit can be generated by numerical integration of nonlinear system. Since the orbits around the collinear libration points (L_1, L_2, L_3) are unstable, the above corrections must be performed periodically to calculate the orbits over a long time interval.

4. Results and discussion

In the course of this work, we tested technique described in section 3 for calculating orbits around the L_1 point of the Sun-Earth system. Examples of generated orbits are shown in Figures 1, 2. To calculate these orbits, the boundary values $q_{min} = -1, q_{max} = 1$ were chosen. Evaluation of this approach showed that both expressions for unstable component can be used to generate orbits, however, expression (16)-(17), obtained by applying Hamiltonian normalization technique, allows generating orbits in greater vicinity of libration point than expression (9). Bisection procedure did not converge with expression (9) when trying to calculate orbit shown in Figure 2. In this case, the impossibility of convergence is due to the fact that function (19) has multiple points of discontinuity. Precise determination of the area of applicability of this method is the goal of further research.

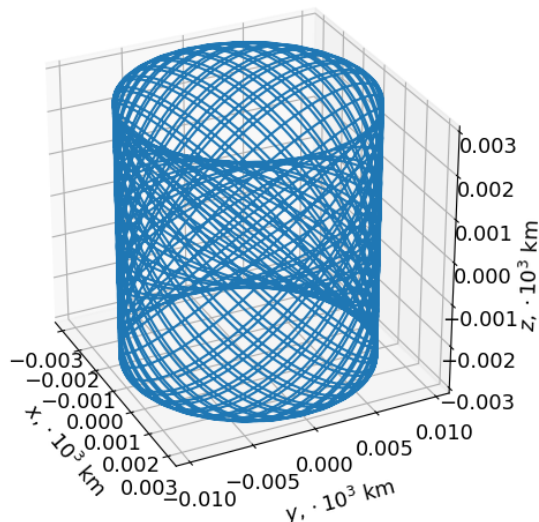


Figure 1. An example of a generated orbit belonging to the Lissajous family

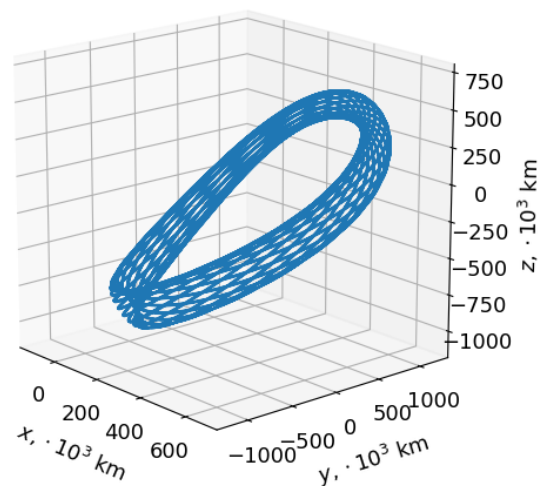


Figure 2. An example of a generated orbit belonging to the quasi-halo family

Figure 3 illustrates a comparison of the graphs of the unstable motion component corresponding to a small amplitude Lissajous orbit (Figure 1). Unstable component q_1^3 grows exponentially

between corrections and drops to zero at the moments of correction while q_1 has significant fluctuations. This comparison indicates that expression (17) keeps properties of unstable component in greater vicinity of libration point.

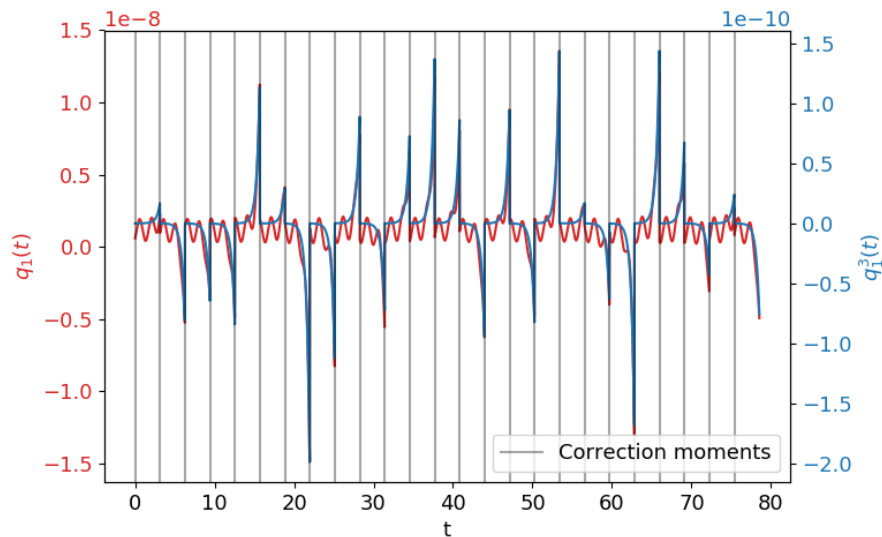


Figure 3. Comparison of plots of the unstable component q_1 (expression (17)) and q_1^3 (expressions (16)-(17))

5. Summary

This work proposes a method for LPO generation based on bisection using approximation for unstable component of motion. Two estimates of the unstable component of motion were obtained. The first is expressed from the solution of the linearized system, and the second is obtained by using Hamiltonian normalization procedure. Both estimates allow generating orbits, but the second is applicable in greater vicinity of libration point.

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