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REPRESENTATIONS OF FINITE-DIMENSIONAL QUOTIENT ALGEBRAS OF THE 3-STRING BRAID GROUP

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Abstract. We consider quotients of the group algebra of the 3-string braid group B_3 by p-th order generic polynomial relations on the elementary braids. If $p = 2, 3, 4, 5$, these quotient algebras are finite dimensional. We give semisimplicity criteria for these algebras and present explicit formulas for all their irreducible representations.

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Introduction

A classical theorem by H. S. M. Coxeter states that the quotient of the n-string braid group B_n by the p-th order relation $\sigma^p = 1$ on its elementary braid generator σ is finite if and only if

$$1/n + 1/p > 1/2. \tag{0.1}$$

In case of B_3 we obtain finite quotient groups of orders 6, 24, 96, and 600, for $p = 2$, 3, 4, and 5, respectively [9]. Generalizing this setting one can consider quotients of the group algebra $\mathbb{C}[B_n]$ obtained by imposing a p-th order monic polynomial relation on the elementary braids. Under condition (0.1) the resulting quotient algebras are finite dimensional and, by Tits deformation theorem (see [10, Section 68] or [13, Section 5]) in the generic situation these algebras are isomorphic to the group algebras of the corresponding Coxeter's quotient groupsso they are semisimple. As a next step it would be interesting to find semisimplicity conditions and to describe explicitly irreducible representations of these finite dimensional quotients.

A significant progress in this direction was made by I. Tuba and H. Wenzl. In the paper [23] they classified all the irreducible representations of B_3 in dimensions $d \leqslant 5$. Their classification scheme in dimensions $d \leqslant 4$ yields all the irreducible representations for the quotients in cases $n = 3$, $p = 2$, 3, 4, and describes their

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semisimplicity conditions. However, for $p = 5$ the above-mentioned quotients of $\mathbb{C}[B_3]$ admit irreducible representations of dimensions up to 6 and the classification in [23] does not cover them. In this note we construct all the irreducible representations of these algebras of dimension $d \leqslant 6$ and find criteria for their semisimplicity. In dimensions $d \leqslant 5$ we reproduce the classification of irreducible representations of B_3 from [23]. In dimension $d = 6$ our list gives all the irreducible representations of B_3 that factor through representations of the quotients of $\mathbb{C}[B_3]$ that we denote Q_X (their definition is given in the next section, see (1.7)). The latter factorization means that the spectrum of the elementary braid in these representations contains 5 different eigenvalues, one of them with multiplicity 2. We are working in the diagonal basis for the first elementary braid generator g_1, and we restrict our considerations to the case where all p roots of its minimal polynomial are distinct. For the sake of completeness we present formulas for representations from I. Tuba and H. Wenzl list in this basis too.

Our paper is organized as follows. In the next section we fix notation and derive preliminary results on possible values of the central element of B_3 in low dimensional irreducible representations ($d \leqslant 6$). Section 2 contains our main results: criteria of semisimplicity of the $p = 2, 3, 4, 5$ quotients of $\mathbb{C}[B_3]$(Theorem 4) and explicit formulae for all their irreducible representations(Proposition 2).

Let us describe briefly some related approaches and results. In [24] B. Westbury suggested an approach to representation theory of B_3 that uses representations of a particular quiver. It was subsequently used by L. Le Bruyn to construct Zariski dense rational parameterizations of the irreducible representations of B_3 of any dimension [15], [14]. This approach proved to be effective in treating a problem of braid reversion (see [15]). However it does not provide semisimplicity criteria for the representations constructed. A 5-dimensional variety of irreducible 6-dimensional representations of B_3 constructed below is contained in an 8-dimensional family of B_3-representations of type 6b (see Fig. 1 in [15]).

For the more general case of B_n, $n > 3$, series of irreducible representations related to Iwahori–Hecke algebras (the $p = 2$ case) and Birman–Murakami–Wenzl algebras (the $p = 3$ case, with additional restrictions) are well investigated (for a review, see [16]). Some other particular families of the B_n-representations were found in [12], [1].

In another line of research M. Broué, C. Malle and R. Rouquier [3], [4] generalized the notions of the braid group and of the Hecke algebra associated not only to Coxeter group, but to an arbitrary finite complex reflection group W. Their *generic Hecke algebra* is defined over certain polynomial ring $R = \mathbb{Z}[\{u_i\}]$. Broué, Malle and Rouquer conjectured that generic Hecke algebra is a free module of rank $|W|$ over its ring of definition. This conjecture is now proved (see [18], [20], [21], [11] and the list of references to Theorem 3.5 in [2]). The algebras Q_X (1.7) we are dealing with in this work are specializations of generic Hecke algebras of the groups $\mathfrak{S}_3$ and G_4, G_8, G_{16} (in Shephard and Todd's notations) under homomorphism $R \to \mathbb{C}$ that assigns certain complex values to the variables u_i. The freeness conjecture in these cases is proved in [17], [5], [6], so the dimensions of the algebras Q_X coincide with the cardinalities of their corresponding Coxeter groups.

1. Braid Group B_3 and its Quotients: Spectrum of Elementary Braids

The 3-string braid group B_3 is generated by a pair of *elementary braids* — g_1 and g_2 — satisfying the braid relation

$$g_1g_2g_1 = g_2g_1g_2. \tag{1.1}$$

Alternatively it can be given in terms of generators

$$a = g_1g_2, \quad b = g_1g_2g_1, \tag{1.2}$$

and relations

$$a^3 = b^2 = c, \tag{1.3}$$

where $c = (g_1g_2)^3 = (g_1g_2g_1)^2$ is a central element of B_3 which generates the center $\mathbb{Z}(B_3)$ [8]. Thus, the quotient group $B_3/\mathbb{Z}(B_3) = \langle a, b | a^3 = b^2 = 1\rangle$ is the free product $\mathbb{Z}_3 * \mathbb{Z}_2$ of two cyclic groups, which is known to be isomorphic to $\mathrm{PSL}(2, \mathbb{Z})$.

Let X be a finite set of pairwise different nonzero complex numbers:

$$X = \{x_1, x_2, \dots, x_n\}, \quad x_i \in \mathbb{C} \setminus \{0\}, \;\; x_i \neq x_j \;\forall i \neq j. \tag{1.4}$$

We set

$$P_X(g) := \prod_{i=1}^{n=|X|} (g - x_i 1), \quad \text{where } g \in \{g_1, g_2\}. \tag{1.5}$$

In this paper we consider finite dimensional quotient algebras of the group algebra $\mathbb{C}[B_3]$ obtained by imposing the following polynomial conditions on the elementary braids:[1]

$$P_X(g) = 0. \tag{1.6}$$

As was already mentioned in the introduction the quotient algebras

$$Q_X := \mathbb{C}[B_3]/\langle P_X(g)\rangle \tag{1.7}$$

are finite dimensional if and only if $|X| = n < 6$. With a particular choice of polynomials $P_X(g) = g^n - 1$ they are the group algebras of the quotient groups $B_3/\langle g^n\rangle$ and, by the Tits deformation argument, $Q_X \simeq \mathbb{C}[B_3/\langle g^n\rangle]$ for $n < 6$ and for generic choice of $x_i \in X$ and, therefore, in a generic situation Q_X is semisimple.

In the next section we will construct irreducible representations of these algebras. We use the Artin–Wedderburn theorem to prove that for quotient algebras Q_X we obtained the complete classification of the irreducible representations. It turns out that their dimensions do not exceed 6. In the rest of this section we will show that in these irreducible representations the spectra of the central element c (1.1) and of generators a and b (1.2) are, up to a discrete factor, defined by the eigenvalues x_i of the elementary braids.

Let V be a finite dimensional linear space, with $\dim V = d$ and let $\rho_{X,V} : Q_X \to \mathrm{End}(V)$ be an irreducible representation of Q_X. We will assume that the character of $\rho_{X,V}$ is a continuous function of parameters $x_i \in X$.[2] We recall $|X| = n$.

[1] In the braid group elementary braids g_1 and g_2 are conjugate to each other and, hence, conditions on them are identical.

[2] All representations constructed in the next section satisfy the continuity condition.

Without loss of generality we can assume that the minimal polynomial of $\rho_{X,V}(g)$, where $g \in \{g_1, g_2\}$ is P_X. Indeed, from equation (1.6) we obtain $P_X(\rho_{X,V}(g)) = 0$. Therefore, the minimal polynomial of $\rho_{X,V}(g)$ divides P_X and, hence, the eigenvalues of $\rho_{X,V}(g)$ belong to X. Let $X' \subset X$ be the set of these eigenvalues. We can consider the irreducible representation of $Q_{X'}$, by removing from X all the elements which do not belong to X'.

By the assumption that the minimal polynomial of $\rho_{X,V}(g)$ is P_X, the definition of P_X and the fact that the elements of X are distinct we also have that $d \geqslant n$. The characteristic polynomial of elementary braids g_1, g_2 in representation $\rho_{X,V}(g)$ then has the form

$$\Pi_\rho(g) := \prod_{i=1}^{n=|X|} (g - x_i)^{m_i}, \quad \text{where } m_i \in \mathbb{N}^+ \text{ such that } \sum_{i=1}^{n} m_i = d. \tag{1.8}$$

In particular, $\det \rho_{X,V}(g) = \prod_{i=1}^n x_i^{m_i}$.

We recall that $a = g_1 g_2$, $b = g_1 g_2 g_1$, and $c = (g_1 g_2)^3$. Since c is central, we apply Schur's lemma and we have that c acts in the irreducible representation $\rho_{X,V}$ as a scalar operator. We denote

$$A := \rho_{X,V}(a), \quad B := \rho_{X,V}(b), \quad \rho_{X,V}(c) := C_\rho \mathrm{Id}_V. \tag{1.9}$$

By the definition of c we have $\det \rho_{X,V}(c) = (\det \rho_{X,V}(g_1))^3 (\det \rho_{X,V}(g_2))^3 = (\prod_{i=1}^n x_i^{m_i})^6$. Hence, we obtain the following relation:

$$\left(\prod_{i=1}^n x_i^{m_i}\right)^6 = (C_\rho)^d\,. \tag{1.10}$$

By (1.3) operators A and B satisfy equalities

$$A^3 = B^2 = C_\rho \mathrm{Id}_V. \tag{1.11}$$

Notice that A and B cannot be scalar, otherwise the matrices $\rho_{X,V}(g_1)$ and $\rho_{X,V}(g_2)$ would have common eigenvectors, meaning that the representation $\rho_{X,V}$ would be reducible. Thus, A and B should have at least two different eigenvalues, lying in the sets

$$\mathrm{Spec}\, A \subset C_\rho^{1/3} \cdot \{1, \nu, \nu^{-1}\}, \quad \nu := e^{2\pi\, \mathrm{i}/3}, \quad \mathrm{Spec}\, B \subset C_\rho^{1/2} \cdot \{1, -1\}. \tag{1.12}$$

The following proposition describes explicitly the spectrum of operators A and B in low dimensional representations, where $\lambda^{\#k}$ denotes the multiplicity k of the eigenvalue λ.

Proposition 1. *Let $\rho_{X,V} : Q_X \to \mathrm{End}(V)$ be a family of irreducible representations of algebras Q_X (1.7) such that*

a) *their characters are continuous functions of parameters $x_i \in X$;*
b) *the characteristic and minimal polynomials of the matrices $\rho_{X,V}(g_1)$ and $\rho_{X,V}(g_1)$ are given by Π_ρ (1.8) and P_X (1.5), respectively.*

Let A, B, C_ρ be as defined in (1.9). Denote $\nu := e^{2\pi\, \mathrm{i}/3}$, and introduce notation $e_k(X)$ for k-th elementary symmetric polynomial in the set of variables $X = \{x_i\}_{i=1,\dots,n}$.

Then for $n = |X| \leqslant 5$ and $d = \dim V \leqslant 6$ the coefficient C_ρ and eigenvalues of operators A and B can take the following values.

If $d = n = 2$, then $C_\rho = -e_2(X)^3$,

$$\operatorname{Spec} A = -e_2(X)\cdot\{\nu,\, \nu^{-1}\}, \quad \operatorname{Spec} B = \mathrm{i}\, e_2(X)^{\frac{3}{2}}\cdot\{1,\, -1\}; \tag{1.13}$$

If $d = n = 3$, then $C_\rho = e_3(X)^2$,

$$\operatorname{Spec} A = e_3(X)^{\frac{2}{3}}\cdot\{1,\, \nu,\, \nu^{-1}\}, \quad \operatorname{Spec} B = e_3(X)\cdot\{1,\, -1^{\#2}\}, \tag{1.14}$$

where the symbol $m^{\#n}$ means that element m appears n times in the set;

If $d = n = 4$, then for any root $h(X) := \sqrt[2]{e_4(X)}$, $C_\rho = h(X)^3$,

$$\operatorname{Spec} A = h(X)\cdot\{1^{\#2},\, \nu,\, \nu^{-1}\}, \quad \operatorname{Spec} B = h(X)^{\frac{3}{2}}\cdot\{1^{\#2},\, -1^{\#2}\}; \tag{1.15}$$

If $d = n = 5$, then for any root $f(X) := \sqrt[5]{e_5(X)}$, $C_\rho = f(X)^6$,

$$\operatorname{Spec} A = f(X)^2\cdot\{1,\, \nu^{\#2},\, (\nu^{-1})^{\#2}\}, \quad \operatorname{Spec} B = f(X)^3\cdot\{1^{\#3},\, -1^{\#2}\}; \tag{1.16}$$

If $d = 6$, $n = 5$, $m_i = 2$, $1 \leqslant i \leqslant 5$, then $C_\rho = -x_i e_5(X)$,

$$\begin{aligned} \operatorname{Spec} A &= -\sqrt[3]{x_i e_5(X)}\cdot\{1^{\#2},\, \nu^{\#2},\, (\nu^{-1})^{\#2}\},\\ \operatorname{Spec} B &= \mathrm{i}\,\sqrt[2]{x_i e_5(X)}\cdot\{1^{\#3},\, -1^{\#3}\}. \end{aligned} \tag{1.17}$$

Proof. Denote Tr_V (respectively, Tr) an operation of taking trace in representation $\rho_{X,V}$ (respectively, taking traces of matrices A, B and of their powers). To prove assertions of the proposition we calculate functions $\mathrm{Tr}_V(g_1^k g_2)$, for $k = 2, \ldots, 5$, in two different ways. The first way is to apply cyclic property of the trace, the braid relation and to take into account scalarity of the central element $\rho_{X,V}(c)$. The second way is to use minimal and characteristic polynomials of elementary braids g_1 and g_2. For illustration purposes the simplest case $d = n = 2$ is considered in detail.

Case $d = n = 2$. First of all, we apply cyclic property of the trace to notice that $\mathrm{Tr}_V(g_1^2 g_2) = \mathrm{Tr}\, B$. From the other side we have $P_X(g_1) = 0$. Therefore, $g_1^2 = (x_1 + x_2)g_1 - x_1x_2 1$. Hence,

$$\begin{aligned} \mathrm{Tr}_V(g_1^2 g_2) &= \mathrm{Tr}_V\big((x_1 + x_2)g_1g_2 - x_1x_2g_2\big)\\ &= (x_1 + x_2)\,\mathrm{Tr}\, A - x_1x_2(x_1 + x_2) = e_1(X)\big(\mathrm{Tr}\, A - e_2(X)\big). \end{aligned}$$

Noticing that spectral condition (1.12) for the non-scalar 2×2 matrix B assumes $\mathrm{Tr}\, B = 0$ and taking into account $e_1(X) \neq 0$ and the continuity of $\mathrm{Tr}\, A$ as a function of $x_{1,2}$, we conclude that $\mathrm{Tr}\, A = e_2(X)$. From (1.10) we have $C_\rho = \pm e_2(X)^3$, which together with spectral condition on A (1.12) leaves us the only possibility to fulfill relations for the traces of A and B, namely the one presented in (1.13).

Case $d = n = 3$. Acting similarly, we shall evaluate $\mathrm{Tr}_V\big(g_1^3 g_2\big)$ in two different ways. First, we use cyclic property of the trace and the braid relation (1.1):

$$\mathrm{Tr}_V(g_1^3 g_2) = \mathrm{Tr}_V(g_1^2 g_2 g_1) = \mathrm{Tr}_V(g_1 g_2)^2 = \mathrm{Tr}\, A^2. \tag{1.18}$$

Second, we apply minimal polynomial for g_1 and characteristic polynomial for g_2:

$$\mathrm{Tr}_V(g_1^3 g_2) = e_1(X)\,\mathrm{Tr}\, B - e_2(X)\,\mathrm{Tr}\, A + e_3(X)e_1(X).$$

Comparing the results of these calculations and taking into account that, by (1.12) and (1.10), traces of powers of A and B can be expressed in terms of (roots of) $e_3(X)$ and, hence, are algebraically independent from $e_1(X)$ and $e_2(X)$ we find that $\mathrm{Tr}\, A = \mathrm{Tr}\, A^2 = 0$, $\mathrm{Tr}\, B = -e_3(X)$. On the other hand from (1.10) one finds $C_\rho = \sqrt[3]{1} e_3(X)^2$, which, together with the spectral conditions (1.12), gives (1.14) as the only possibility to satisfy the above relations for traces.

Case $d = n = 4$. Similarly to the case $d = n = 3$ we calculate $\mathrm{Tr}_V(g_1^4 g_2)$ in two ways:

$$\begin{aligned} \mathrm{Tr}_V(g_1^4 g_2) &= \mathrm{Tr}_V\big((g_1 g_2)^2 g_1\big) = C_\rho\, \mathrm{Tr}_V\big((g_1 g_2)^{-1} g_1\big) = C_\rho e_3(X)/e_4(X), \\ \mathrm{Tr}_V(g_1^4 g_2) &= e_1(X)\, \mathrm{Tr}\, A^2 - e_2(X)\, \mathrm{Tr}\, B + e_3(X)\, \mathrm{Tr}\, A - e_4(X) e_1(X), \end{aligned} \tag{1.19}$$

where in the last line we take additionally into account Equation (1.18). Hence, using an algebraic independence of C_ρ and thus of $\mathrm{Tr}\, A$, $\mathrm{Tr}\, A^2$ and $\mathrm{Tr}\, B$ from the elementary symmetric polynomials $e_i(X)$, $i = 1, 2, 3$, one concludes: $\mathrm{Tr}\, A = C_\rho/e_4(X)$, $\mathrm{Tr}\, A^2 = e_4(X)$, $\mathrm{Tr}\, B = 0$. The latter conditions are only compatible with equations (1.10) and (1.12) in two cases given in (1.15).

Case $d = n = 5$. Here we calculate $\mathrm{Tr}_V(g_1^5 g_2)$:

$$\begin{aligned} &\mathrm{Tr}_V(g_1^5 g_2) C_\rho\, \mathrm{Tr}_V\big((g_1 g_2)^{-1} g_1^2\big) = C_\rho\, \mathrm{Tr}_V(g_1^{-1} g_2) \\ &= \frac{C_\rho}{e_5(X)}\Big(C_\rho \frac{e_4(X)}{e_5(X)} - e_1(X)\, \mathrm{Tr}\, A^2 + e_2(X)\, \mathrm{Tr}\, B - e_3(X)\, \mathrm{Tr}\, A + e_4(X) e_1(X)\Big), \end{aligned}$$

where passing to the second line we expressed g_1^{-1} in terms of positive powers of g_1 using its minimal polynomial and then used $d = 5$ analogue of formula (1.19).

Calculating $\mathrm{Tr}_V(g_1^5 g_2)$ in another way we obtain

$$\begin{aligned} \mathrm{Tr}_V(g_1^5 g_2) = e_1(X)\left(C_\rho \frac{e_4(X)}{e_5(X)}\right) - e_2(X)\, \mathrm{Tr}\, A^2 + e_3(X)\, \mathrm{Tr}\, B - e_4(X)\, \mathrm{Tr}\, A \\ + e_5(X) e_1(X). \end{aligned}$$

Now collecting coefficients in the independent polynomials $e_i(X)$, $i = 1, 2, 3, 4$, and taking into account Equation (1.10) we find $C_\rho = e_5(X)^{6/5}$, $\mathrm{Tr}\, A = -e_5(X)^{2/5}$, $\mathrm{Tr}\, A^2 = -e_5(X)^{4/5}$, $\mathrm{Tr}\, B = e_5(X)^{3/5}$, which in combination with (1.12) finally leads to conditions (1.16).

Case $d = 6$, $n = 5$: We calculate $\mathrm{Tr}_V(g_1^5 g_2)$ in two ways similarly to the previous case, but using now different expressions $\mathrm{Tr}_V(g_1) = e_1(X) + x_i$, $\mathrm{Tr}_V(g_1^{-1}) = e_4(X)/e_5(X) + x_i^{-1}$, following from the characteristic polynomial (1.8). Collecting then coefficients in independent polynomials we derive $C_\rho = -x_i e_5(X)$, $\mathrm{Tr}\, A = \mathrm{Tr}\, A^2 = \mathrm{Tr}\, B = 0$, which in combination with (1.12) proves (1.17). □

2. Low Dimensional Representations of q_x and Semisimplicity

In this section we construct explicitly representations of algebras Q_X whose data coincide with those given in Proposition 1. Investigating reducibility conditions for these representations we obtain semisimplicity criteria for algebras Q_X and classify their irreducible representations. We derive formulas for the representations in the basis of eigenvectors of g_1.

Proposition 2. *The algebras Q_X in cases $|X| \leqslant 5$ have the following representations of dimensions $\dim V \leqslant 6$.*

If $|X| = \dim V = 1$, *there exists a unique representation,*

$$\rho_X^{(1)}(g_1) = \rho_X^{(1)}(g_2) = x_1. \tag{2.1}$$

If $|X| = \dim V = 2$, *there exists a unique representation,*

$$\rho_X^{(2)}(g_1) = \mathrm{diag}\{x_1,\, x_2\}, \quad \rho_X^{(2)}(g_2) = \frac{1}{x_1 - x_2}\begin{pmatrix} -x_2^2 & -x_1x_2 \\ x_1^2 - x_1x_2 + x_2^2 & x_1^2 \end{pmatrix}. \tag{2.2}$$

If $|X| = \dim V = 3$, *there exists a unique representation,*

$$\rho_X^{(3)}(g_1) = \mathrm{diag}\{x_1,\, x_2,\, x_3\}, \quad \rho_X^{(3)}(g_2) = \begin{pmatrix} \frac{x_2x_3(x_2+x_3)}{\Delta_1(X)} & \frac{x_3(x_1^2+x_2x_3)}{\Delta_1(X)} & \frac{x_2(x_1^2+x_2x_3)}{\Delta_1(X)} \\ \frac{x_3(x_2^2+x_1x_3)}{\Delta_2(X)} & \frac{x_1x_3(x_1+x_3)}{\Delta_2(X)} & \frac{x_1(x_2^2+x_1x_3)}{\Delta_2(X)} \\ \frac{x_2(x_3^2+x_1x_2)}{\Delta_3(X)} & \frac{x_1(x_3^2+x_1x_2)}{\Delta_3(X)} & \frac{x_1x_2(x_1+x_2)}{\Delta_3(X)} \end{pmatrix}, \tag{2.3}$$

where

$$\Delta_i(X) := \prod_{j=1,\, j\neq i}^{|X|} (x_j - x_i). \tag{2.4}$$

If $|X| = \dim V = 4$, *there exist two inequivalent representations depending on the choice of the square root* $h = \sqrt{e_4(X)}$:

$$\rho_{h,X}^{(4)}(g_1) = \mathrm{diag}\{x_1,\, x_2,\, x_3,\, x_4\},$$
$$\rho_{h,X}^{(4)}(g_2) = \begin{pmatrix} \frac{\alpha_1}{\Delta_1(X)} & \frac{\beta_1\gamma_3\gamma_4}{\Delta_1(X)} & \frac{\beta_1\gamma_2\gamma_4}{\Delta_1(X)} & \frac{\beta_1\gamma_2\gamma_3}{\Delta_1(X)} \\ \frac{\beta_2}{\Delta_2(X)} & \frac{\alpha_2}{\Delta_2(X)} & \frac{\beta_2\gamma_2}{\Delta_2(X)} & \frac{\beta_2\gamma_2}{\Delta_2(X)} \\ \frac{\beta_3}{\Delta_3(X)} & \frac{\beta_3\gamma_3}{\Delta_3(X)} & \frac{\alpha_3}{\Delta_3(X)} & \frac{\beta_3\gamma_3}{\Delta_3(X)} \\ \frac{\beta_4}{\Delta_4(X)} & \frac{\beta_4\gamma_4}{\Delta_4(X)} & \frac{\beta_4\gamma_4}{\Delta_4(X)} & \frac{\alpha_4}{\Delta_4(X)} \end{pmatrix}. \tag{2.5}$$

Here

$$\begin{aligned} &\alpha_i(h,\, X) := e_3(X^{\setminus i})e_1(X^{\setminus i}) - he_2(X^{\setminus i}), \quad X^{\setminus i} := X \setminus \{x_i\}, \\ &\beta_i(h,\, X) := e_4(X)/x_i^2 - h, \quad i = 1,\, 2,\, 3,\, 4, \\ &\gamma_a(h,\, X) := x_1x_a + x_bx_c - h, \quad a,\, b,\, c \in \{x_2,\, x_3,\, x_4\} \text{ are pairwise distinct.} \end{aligned} \tag{2.6}$$

If $|X| = \dim V = 5$, *there exist five inequivalent representations corresponding to different values of the root* $f(X) := \sqrt[5]{e_5(X)}$:

$$\rho_{f,X}^{(5)}(g_1) = \mathrm{diag}\{x_1,\, x_2,\, x_3,\, x_4,\, x_5\}, \quad \rho_{f,X}^{(5)}(g_2) = \|m_{ij}\|_{1\leqslant i,j\leqslant 5}, \tag{2.7}$$

$$m_{ii}(f,\, X) := \frac{e_4(X^{\setminus i})e_1(X^{\setminus i}) + fx_ie_3(X^{\setminus i}) + f\prod_{k=1,k\neq i}^{5}(f + x_k)}{\Delta_i(X)}, \tag{2.8}$$

$$m_{ij}(f,\, X) := \frac{(x_i^2 + fx_i + f^2)\prod_{k=1,k\neq i,j}^{5}(f^2 + x_ix_k)}{fx_ix_j\Delta_i(X)}, \quad \forall i \neq j. \tag{2.9}$$

If $|X| = 5$, $\dim V = 6$, there exist five inequivalent representations $\rho^{(6)}_{i,X}$, $i = 1, \ldots, 5$, corresponding to all admissible values $C_\rho = -x_i e_5(X)$ of the central element c. Formulas for $\rho^{(6)}_{5,X}$ are given in Table 1. Formulas for the other representations can be obtained by the transposition of the eigenvalues x_5 and x_i, that is, $\rho^{(6)}_{i,X} = \sigma_{i5} \circ \rho^{(6)}_{5,X}$, $i = 1, \ldots, 4$.

TABLE 1. 6-dimensional representation of Q_X, $|X| = 5$

$\rho^{(6)}_{5,X}(g_1) = \mathrm{diag}\{x_1, x_2, x_3, x_4, x_5, x_5\}$, $\quad \rho^{(6)}_{5,X}(g_2) = \lVert g_{ij} \rVert_{1 \leqslant i,j \leqslant 6}$,	
$G := \lVert g_{ij} \rVert_{1 \leqslant i,j \leqslant 4}$	$g_{ii} = \dfrac{e_4(X^{\setminus i}) e_1(X^{\setminus i}) - x_i x_5 e_3(X^{\setminus i})}{\Delta_i(X)}$, $X^{\setminus i} := X \setminus \{x_i\}$, $i = 1, \ldots, 4$; $\quad g_{1a} = \dfrac{p_a q_b q_c}{x_1^2 \Delta_a(X)}$, $g_{a1} = \dfrac{p_1}{x_a^2 \Delta_1(X)}$, $g_{ab} = \dfrac{q_a p_b}{x_a^2 \Delta_b(X)}$, where indices a, b, $c \in \{2, 3, 4\}$ are pairwise distinct, and $q_a(X) := x_1 x_a + x_b x_c$, $p_i(X) := e_5(X) - x_i^3 x_5^2$
$G_{31} := \begin{pmatrix} g_{51} & g_{52} \\ g_{61} & g_{62} \end{pmatrix}$	$\mathrm{diag}\left\{\dfrac{1}{\Delta_1(X)}, \dfrac{1}{\Delta_2(X)}\right\}$
$G_{32} := \begin{pmatrix} g_{53} & g_{54} \\ g_{63} & g_{64} \end{pmatrix}$	$\begin{pmatrix} q_4 r & q_3(\sigma_{34} \circ r) \\ (\sigma_{12} \circ r) & (\sigma_{12}\sigma_{34} \circ r) \end{pmatrix}$, where $r(X) := \dfrac{x_3}{x_1(x_2 - x_1)\Delta_3(X^{\setminus 2})}$, and $\sigma_{ij} \circ f(\ldots x_i \ldots x_j \ldots) := f(\ldots x_j \ldots x_i \ldots)$ for any $f(X)$
$G_{33} := \begin{pmatrix} g_{55} & g_{56} \\ g_{65} & g_{66} \end{pmatrix}$	$\begin{pmatrix} u & q_3 q_4 v \\ (\sigma_{12} \circ v) & (\sigma_{12} \circ u) \end{pmatrix}$, where $v(X) := \dfrac{p_2(X)}{x_1 x_5 (x_2 - x_1)\Delta_5(X^{\setminus 2})}$, and $u(X) := \dfrac{x_1 x_2 (x_3 + x_4)(x_3 x_4 - x_1 x_5) + x_3 x_4 (x_2 - x_1)(x_1^2 + x_2 x_5)}{(x_2 - x_1)\Delta_5(X^{\setminus 2})}$
$G_{23} := \begin{pmatrix} g_{35} & g_{36} \\ g_{45} & g_{46} \end{pmatrix}$	$\dfrac{1}{x_5 \Delta_5(X)} \begin{pmatrix} \frac{w}{x_3^2} & \frac{q_3(\sigma_{12} \circ w)}{x_3^2} \\ \frac{(\sigma_{34} \circ w)}{x_4^2} & \frac{q_4(\sigma_{12}\sigma_{34} \circ w)}{x_4^2} \end{pmatrix}$, $w(X) := p_1(X)\big(x_1 x_2 x_3 x_4 \{x_1 x_3 + x_5(x_2 + x_4)\} - x_5^3 \{x_1 x_3 (x_2 + x_4) + x_5 x_2 x_4\}\big)$
$G_{13} := \begin{pmatrix} g_{15} & g_{16} \\ g_{25} & g_{26} \end{pmatrix}$	$\dfrac{1}{\Delta_5(X)} \begin{pmatrix} \frac{z}{x_1} & \frac{q_3 q_4 (\sigma_{12}\sigma_{23} \circ w)}{x_1^2 x_5} \\ \frac{(\sigma_{23} \circ w)}{x_2^2 x_5} & \frac{(\sigma_{12} \circ z)}{x_2} \end{pmatrix}$, $z(X) := (e_1 e_3 - x_1^2 e_2)(x_1 e_1 e_3 - e_2 x_5^3) x_1 x_5 + e_3 (x_1 - x_5)\big(x_1^2 (e_1 - x_1)\{e_3(x_1 - x_5) - e_1 x_5^3\} + (x_1 e_2 - e_3)\{x_1 e_2 + (x_1 - x_5) x_5^2\} x_5\big)$, where e_i are elementary symmetric polynomials in variables x_2, x_3, x_4.

Remark 1. As it is noticed in Section 1 a representation of Q_X is also a representation of $Q_{X'}$ if $X \subset X'$.

Remark 2. Irreducible representations of B_3 of dimensions $d \leqslant 5$ were classified by Imre Tuba and Hans Wenzl in [23]. We reproduce their table of representations in the basis where g_1 takes a diagonal form. In their approach I. Tuba and H. Wenzl used a different basis, in which matrices of the braids g_1 and g_2 assume a special 'ordered' triangular from. This allows them analyzing also algebras whose minimal polynomials P_X have multiple roots and, hence, matrices of the braids $g_{1,2}$ are not diagonalizable. These cases are missed in our approach. Instead, our method is suitable for construction of the 6-dimensional representations for algebras Q_X, $|X| = 5$ and, thus, allows us classifying irreducible representations for these algebras and studying their semisimplicity.

Note also that formulas for representations of dimensions $d \leqslant 5$ have been reconstructed in [5] using different methods with the help of the CHEVIE package of GAP3 (see [18], [22]).

Proof. By our initial assumptions matrices of braids $g_{1,2}$ in any representation are diagonalizable. We choose a basis where $\rho_{X,V}(g_1) := D_g$ is diagonal. By (1.8) the diagonal components of D_g are x_i taken with multiplicities m_i.

Keeping in mind that in an irreducible representation matrices A and B of braids a and b are also diagonalizable (see Equation (1.11)) we use for them parameterization

$$A = U^{-1} D_a U, \quad B = V D_b V^{-1}. \tag{2.10}$$

Here D_a and D_b are diagonal matrices whose diagonal components are elements of $\operatorname{Spec} A$ and $\operatorname{Spec} B$. For irreducible representations of dimensions $\leqslant 6$ they were defined in Proposition 1. Due to relation $g_1 = a^{-1}b$ matrices U and V have to satisfy condition

$$U D_g V = D_a^{-1} U V D_b. \tag{2.11}$$

We solve this matrix equality for U and V in cases where diagonal matrices D_g, D_a and D_b are as described in Proposition 1. Formulae for representations given in Proposition 2 follow then, e.g., from relation $g_2 = g_1^{-1}a$: $\rho_{X,V}(g_2) = D_g^{-1}A$.

Solving (2.11) is straightforward but rather tedious computation. For an interested reader we give few details of it in cases $d = 2, 3, 4$.

Case $d = 2$. We choose

$$D_g = \operatorname{diag}\{x_1, x_2\}, \quad D_a = -e_2(X) \operatorname{diag}\{\nu, \nu^{-1}\}, \quad D_b = \mathrm{i} e_2(X)^{\frac{3}{2}} \operatorname{diag}\{1, -1\}.$$

Noticing that matrices U/V are defined up to left/right multiplication by a diagonal matrix we use for them the following ansatzes

$$U = \begin{pmatrix} 1 & * \\ * & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & * \\ * & 1 \end{pmatrix},$$

where stars stand for unknown components. With this settings Equation (2.11) defines U and V up to conjugation by a diagonal matrix. We choose a solution which gives a nice expression (2.2) for $\rho_X^{(2)}(g_2)$,

$$U = \begin{pmatrix} 1 & -\frac{x_1}{\nu^{-1}x_1 + \nu x_2} \\ -\frac{\nu x_1 + \nu^{-1} x_2}{x_1} & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & -\frac{\mathrm{i}\sqrt{e_2}}{x_1 - x_2 + \mathrm{i}\sqrt{e_2}} \\ \frac{x_1 - x_2 - \mathrm{i}\sqrt{e_2}}{\mathrm{i}\sqrt{e_2}} & 1 \end{pmatrix}.$$

Note that, unlike U and V, resulting expression for $\rho_X^{(2)}(g_2)$ is defined with the only restriction $x_1 \neq x_2$ and does not depend on a choice of root $\sqrt{e_2}$.

Case $d = 3$. We choose

$$D_g = \mathrm{diag}\{x_1, x_2, x_3\}, \quad D_a = e_3(X)^{\frac{2}{3}} \, \mathrm{diag}\{1, \nu^{-1}, \nu\},$$
$$D_b = e_3(X) \, \mathrm{diag}\{1, -1, -1\},$$

and use ansatzes

$$U = \begin{pmatrix} 1 & * & * \\ * & 1 & * \\ * & * & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & * & * \\ * & 1 & 0 \\ * & 0 & 1 \end{pmatrix}.$$

The solution of Equation (2.11) which gives formula (2.3) for $\rho_X^{(3)}(g_2)$ reads

$$U = \begin{pmatrix} 1 & \frac{x_1+h}{x_2+h} & \frac{x_1+h}{x_3+h} \\ \frac{x_2+\nu h}{x_1+\nu h} & 1 & \frac{x_2+\nu h}{x_3+\nu h} \\ \frac{x_3+\nu^{-1}h}{x_1+\nu^{-1}h} & \frac{x_3+\nu^{-1}h}{x_2+\nu^{-1}h} & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & -1 & -1 \\ -\frac{(x_1-x_3)(x_2^2+x_1x_3)}{(x_2-x_3)(x_1^2+x_2x_3)} & 1 & 0 \\ -\frac{(x_1-x_2)(x_3^2+x_1x_2)}{(x_3-x_2)(x_1^2+x_2x_3)} & 0 & 1 \end{pmatrix}.$$

Case $d = 4$. We choose $D_g = \mathrm{diag}\{x_1, x_2, x_3, x_4\}$,

$$D_a = h(X) \, \mathrm{diag}\{1, 1, \nu, \nu^{-1}\}, \quad D_b = h(X)^{\frac{3}{2}} \, \mathrm{diag}\{1, 1, -1, -1\},$$

and ansatzes for U, V:

$$U = \begin{pmatrix} I & \Psi^+ \\ \Psi^- & \Phi \end{pmatrix}, \quad V = \begin{pmatrix} I & \Lambda^+ \\ \Lambda^- & I \end{pmatrix},$$

where I is 2×2 unit matrix, $\Phi^\pm$ and $\Lambda^\pm$ are arbitrary 2×2 matrices, and 2×2 matrix Φ has unit diagonal components. A particular solution of Equation (2.11) which gives expression (2.5) for $\rho_{h,X}^{(4)}(g_2)$ reads

$$\Psi^+ = \begin{pmatrix} \frac{x_1(x_3-x_2)\beta_1\gamma_4}{x_3(x_1-x_2)\beta_3} & \frac{x_1(x_4-x_2)\beta_1\gamma_3}{x_4(x_1-x_2)\beta_4} \\ \frac{x_2(x_3-x_1)\beta_2}{x_3(x_2-x_1)\beta_3} & \frac{x_2(x_4-x_1)\beta_2}{x_4(x_2-x_1)\beta_4} \end{pmatrix},$$

$$\Psi^- = \begin{pmatrix} \frac{x_1x_2}{(x_1x_2+\nu^{-1}h)(x_2x_3+\nu h)} & \frac{x_2x_4+\nu h}{x_3x_4+\nu h} \\ \frac{x_1x_2}{(x_1x_2+\nu h)(x_2x_4+\nu^{-1}h)} & \frac{x_2x_3+\nu^{-1}h}{x_3x_4+\nu^{-1}h} \end{pmatrix},$$

$$\Phi = \begin{pmatrix} 1 & \frac{x_2x_4+\nu h}{x_2x_3+\nu h} \\ \frac{x_2x_3+\nu^{-1}h}{x_2x_4+\nu^{-1}h} & 1 \end{pmatrix},$$

$$\Lambda^+ = -\begin{pmatrix} \frac{x_3(x_3-x_2)(x_1-\sqrt{h})\gamma_4}{x_1(x_1-x_2)(x_3-\sqrt{h})} & \frac{x_4(x_4-x_2)(x_1-\sqrt{h})\gamma_3}{x_1(x_1-x_2)(x_4-\sqrt{h})} \\ \frac{x_3(x_3-x_1)(x_2-\sqrt{h})}{x_2(x_2-x_1)(x_3-\sqrt{h})} & \frac{x_4(x_4-x_1)(x_2-\sqrt{h})}{x_2(x_2-x_1)(x_4-\sqrt{h})} \end{pmatrix},$$

$$\Lambda^- = -\frac{1}{\gamma_2}\begin{pmatrix} \frac{x_1(x_4-x_1)(x_3+\sqrt{h})}{x_3(x_4-x_3)(x_1+\sqrt{h})} & \frac{x_2(x_4-x_2)(x_3+\sqrt{h})\gamma_3}{x_3(x_4-x_3)(x_2+\sqrt{h})} \\ \frac{x_1(x_3-x_1)(x_4+\sqrt{h})}{x_4(x_3-x_4)(x_1+\sqrt{h})} & \frac{x_2(x_3-x_2)(x_4+\sqrt{h})\gamma_4}{x_4(x_3-x_4)(x_2+\sqrt{h})} \end{pmatrix}.$$

To get it we exclude consecutively matrices $\Lambda^{\pm}$, Ψ^{-}, Φ from Equations (2.11) expressing them finally in terms of Ψ^{+}. The only condition imposed by Equation (2.11) on the components of Ψ^{+} is

$$\frac{(\Psi^{+})_{11}(\Psi^{+})_{22}}{(\Psi^{+})_{12}(\Psi^{+})_{21}} = \frac{(x_3-x_2)(x_4-x_1)\gamma_4}{(x_4-x_2)(x_3-x_1)\gamma_3}.$$

The three remaining degrees of freedom are due to arbitrariness in conjugation of U and V by a diagonal matrix. We fix it to get the expression for $\rho^{(4)}_X(g_2)$ in the most suitable form.

Solving Equation (2.11) in cases $d = 5$, $\dim V = 5$, 6, is more lengthy. We skip it presenting final results of the calculations in equations (2.7)–(2.9) and in Table 1. For them the braid relation (1.1) can be checked directly. □

Proposition 3. *For the algebras Q_X (1.7) defined by the set of data X (1.4), the representations $\rho^{(d)}_{\dots}$, $d \leqslant 5$, described in Proposition 2 are irreducible if and only if the following conditions on their parameters are satisfied.*

If $|X| = 2$, $\rho^{(2)}_X$ is irreducible if

$$I^{(2)}_{ij} := x_i^2 - x_i x_j + x_j^2 \neq 0, \tag{2.12}$$

where indices i, $j \in \{1, 2\}$ are distinct.

If $|X| = 3$, $\rho^{(3)}_X$ is irreducible if

$$I^{(3)}_{ijk} := x_i^2 + x_j x_k \neq 0, \tag{2.13}$$

where i, j, $k \in \{1, 2, 3\}$ are pairwise distinct.

If $|X| = 4$, $\rho^{(4)}_{h,X}$ is irreducible if

$$I^{(4)}_{h,i} := x_i^2 - h \neq 0, \qquad J^{(4)}_{h,ijkl} := x_i x_j + x_k x_l - h \neq 0, \tag{2.14}$$

where i, j, k, $l \in \{1, 2, 3, 4\}$ are pairwise distinct.

If $|X| = 5$, $\rho^{(5)}_{f,X}$ is irreducible if

$$I^{(5)}_{f,i} := x_i^2 + x_i f + f^2 \neq 0, \qquad J^{(5)}_{f,ij} := x_i x_j + f^2 \neq 0, \tag{2.15}$$

where i, $j \in \{1, 2, 3, 4, 5\}$ are pairwise distinct.

Otherwise, they are reducible but indecomposable.

For representations $\rho^{(6)}_{s,X}$, $s = 1, \dots, 5$, also given in Proposition 2, we present a less detailed statement, which describes conditions under which all of them are irreducible.

If $|X| = 5$, $\rho^{(6)}_{s,X}$, $1 \leqslant s \leqslant 5$, are irreducible if

$$I^{(6)}_i := e_5(X) + x_i^5 \neq 0, \quad J^{(6)}_{ij} := e_5(X) - x_i^3 x_j^2 \neq 0, \quad K^{(6)}_{i,jklm} := x_j x_k + x_l x_m \neq 0, \tag{2.16}$$

where i, j, k, l, $m \in \{1, 2, 3, 4, 5\}$ are pairwise distinct.

Otherwise, among them there are reducible but indecomposable representations.

Proof. We will search for invariant subspaces in representations $\rho^{(d)}_{\dots}$ of Proposition 2. Note that for any $y \in Q_X$ such that $\operatorname{Spec} \rho_{X,V}(y)$ is multiplicity free an invariant subspace in V should be a linear span of some subset of a basis of eigenvectors of $\rho_{X,V}(y)$.

Consider representations $\rho^{(d)}_{\dots}$ of dimension $d = \dim V \leqslant 5$. Here the spectrum of $\rho^{(d)}_{\dots}(g_1)$ is simple. Choose a basis of eigenvectors of $\rho^{(d)}_{\dots}(g_1)$: $\{v_k := \delta_{ki},\ 1 \leqslant i \leqslant d\}_{k=1,\dots d}$. Denote

$$V_Y := \mathrm{Span}\{v_k \colon k \in Y\}, \quad \text{where } Y \subset \{1, 2, 3, 4, 5\}. \tag{2.17}$$

Obviously, any invariant subspace in the representation space V, if exists, should be of the form V_Y. Furthermore, if the representation is decomposable then the decomposition is

$$V = V_Y \oplus V_{\bar{Y}}, \quad \text{where } \bar{Y} := \{1, 2, 3, 4, 5\} \setminus Y. \tag{2.18}$$

Correspondingly, the matrix $\rho^{(d)}_{\dots}(g_2)$ is block-triangular (respectively, block-diagonal) with blocks labelled by indices from subsets Y and $\bar{Y}$ if and only if the representation is reducible (respectively, decomposable). Let us analyze the block structure of $\rho^{(d)}_{\dots}(g_2)$ in cases $d = 3, 4, 5$ (case $d = 2$ is trivial).

Case $d = 3$. Representation $\rho^{(3)}_X$ (2.3) has 2-dimensional invariant subspace $V_{\{1,2\}}$ if and only if $I^{(3)}_{312} = 0$. Its complementary 1-dimensional subspace $V_{\{3\}}$ exists under conditions $I^{(3)}_{123} = I^{(3)}_{231} = 0$. Altogether conditions $I^{(3)}_{312} = I^{(3)}_{123} = I^{(3)}_{231} = 0$ lead to $x_1 = x_2 = x_3 = 0$ and, hence, they are incompatible. Invariance conditions in two other cases — $V_{\{2,3\}}$, $V_{\{1\}}$, and $V_{\{1,3\}}$, $V_{\{2\}}$ — differ from the above by a cyclic permutation of the subscript indices. It follows that $\rho^{(3)}_X$ is irreducible if and only if inequalities (2.13) are fulfilled, and otherwise it is indecomposable.

Case $d = 4$. Conditions for existence of invariant subspaces in $\rho^{(4)}_{h,X}$ are

$$V_{\{1,2,3\}} \colon I^{(4)}_{h,4} = 0; \quad V_{\{4\}} \colon I^{(4)}_{h,3} = J^{(4)}_{h,1234} = 0 \text{ or } I^{(4)}_{h,2} = J^{(4)}_{h,1324} = 0; \tag{2.19}$$

$$V_{\{1,2\}} \colon I^{(4)}_{h,3} = I^{(4)}_{h,4} = 0; \quad V_{\{3,4\}} \colon J^{(4)}_{h,1234} = 0 \text{ or } I^{(4)}_{h,1} = I^{(4)}_{h,2} = 0. \tag{2.20}$$

For the rest of invariant subspaces their existence conditions can be obtained by a cyclic permutations of subscripts 1, 2, 3, 4 in (2.19) [3], or of subscripts 2, 3, 4 in (2.20). Altogether these conditions justify irreducibility criterium (2.14). Decomposability, e.g., $V = V_{\{1,2,3\}} \oplus V_{\{4\}}$ or $V = V_{\{1,2\}} \oplus V_{\{3,4\}}$, demands

$$I^{(4)}_{h,1} = I^{(4)}_{h,2} = I^{(4)}_{h,3} = I^{(4)}_{h,4} = 0 \quad \text{or } I^{(4)}_{h,3} = I^{(4)}_{h,4} = J^{(4)}_{h,1234} = 0$$

or similar sets of relations with permuted subscripts 2, 3, 4. One can check that these conditions are incompatible with initial settings for X (1.4).

[3]The only exception is subspace $V_{\{1\}}$, which cannot be invariant in this representation.

Case $d = 5$. Invariant subspaces in $\rho^{(5)}_{f,X}$ exist under conditions:

$$\begin{gathered} V_{\{1,2,3,4\}}\colon I^{(5)}_{f,5} = 0; \\ V_{\{5\}}\colon J^{(5)}_{f,12} = J^{(5)}_{f,34} = 0 \text{ (or any permutation of subscripts 2, 3, 4)} \\ \text{or } J^{(5)}_{f,12} = I^{(5)}_{f,3} = I^{(5)}_{f,4} = 0 \text{ (or any permutation of subscripts 1, 2, 3, 4);} \end{gathered} \tag{2.21}$$

$$\begin{gathered} V_{\{1,2,3\}}\colon J^{(5)}_{f,45} = 0 \text{ or } I^{(5)}_{f,4} = I^{(5)}_{f,5} = 0; \\ V_{\{4,5\}}\colon I^{(5)}_{f,3} = J^{(5)}_{f,12} = 0 \text{ (or any permutation of subscripts 1, 2, 3).} \end{gathered} \tag{2.22}$$

For the rest of invariant subspaces the existence conditions can be obtained by permutation of indices in formulas above. Taken together these conditions prove irreducibility criterium (2.15). On the other hand, an attempt to find decomposition into invariant subspaces, like $V = V_{\{1,2,3,4\}} \oplus V_{\{5\}}$ or $V = V_{\{1,2,3\}} \oplus V_{\{4,5\}}$, results in a set of conditions

$$\begin{gathered} I^{(5)}_{f,1} = J^{(5)}_{f,23} = J^{(5)}_{f,45} = 0 \text{ or } I^{(5)}_{f,1} = I^{(5)}_{f,2} = I^{(5)}_{f,3} = J^{(5)}_{f,45} = 0 \\ \text{(or any permutation of subscripts 1, 2, 3, 4, 5),} \end{gathered}$$

which are incompatible with (1.4). Thus, representations $\rho^{(5)}_{f,X}$ are always indecomposable.

Case $d = 6$ is more sophisticated. We carry out considerations for representation $\rho^{(6)}_{5,X}$ (see Table 1). For the other 6-dimensional representations results follow then by transpositions of arguments x_i.

Take a basis of eigenvectors of $\rho^{(6)}_{5,X}(g_1)$: $\{v_k := \delta_{ki},\ 1 \leqslant i \leqslant 6\}_{k=1,\dots 6}$. Assume there exists an invariant subspace $V_{\text{inv}} \subsetneq V$ and consider its subspace

$$W := V_{\text{inv}} \cup V_{\{1,2,3,4\}}.$$

Spectrum of $\rho^{(6)}_{5,X}(g_1)$ in this subspace is simple, so W has a form $W = V_Y$ (2.17) for some subset $Y \subset \{1, 2, 3, 4\}$. We consider separately cases with different Y.

Case $W = V_{\{1,2,3,4\}}$. Consider action of matrix $\rho^{(6)}_{5,X}(g_2)$ on W. Since components g_{51} and g_{62} of this matrix are always nonzero we conclude that vectors v_5 and v_6 belong to V_{inv} and hence, $V_{\text{inv}} = V$, which is a contradiction.

Case $W = V_{\{1\}}$. Considering action of $\rho^{(6)}_{5,X}(g_2)$ on $v_1 \in W \subset V_{\text{inv}}$ we obtain $v_5 \in V_{\text{inv}}$. Now let us assume that $V_{\text{inv}} = V_{\{1,5\}}$. Then the matrix $\rho^{(6)}_{5,X}(g_2)$ should take block-diagonal form with vanishing components $g_{21} = g_{31} = g_{41} = g_{61} = g_{25} = g_{35} = g_{45} = g_{65} = 0$. This happens if and ony if $p_1(X) \equiv J^{(6)}_{15} = 0$. Thus, we conclude that representation $\rho^{(6)}_{5,X}$ under condition $J^{(6)}_{15} = 0$ has the invariant subspace $V_{\{1,5\}}$. This subspace is not further reducible.

Case $W = V_{\{2,3\}}$. From the action of $\rho^{(6)}_{5,X}(g_2)$ on $v_2 \in V_{\text{inv}}$ we get $v_6 \in V_{\text{inv}}$, as $g_{26} \neq 0$. Assuming then $V_{\text{inv}} = V_{\{2,3,6\}}$ and checking block-triangularity of $\rho^{(6)}_{5,X}(g_2)$: $g_{12} = g_{13} = g_{16} = g_{42} = g_{43} = g_{46} = g_{52} = g_{53} = g_{56} = 0$, we find that this case is realized under condition $q_4(X) \equiv K^{(6)}_{5,1423} = 0$. Thus, $V_{\{2,3,6\}}$ is a minimal invariant subspace containing $W = V_{\{2,3\}}$.

Two cases considered above illustrate appearance of conditions like $J^{(6)}_{\dots} \neq 0$ and $K^{(6)}_{\dots} \neq 0$ in formulation of the proposition. Permuting arguments x_i, that is, considering all representations $\rho^{(6)}_{\dots}$ one can obtain all polynomials $J^{(6)}_{\dots}$, $K^{(6)}_{\dots}$ in the conditions of their reducibility. Consideration of the other cases with $W \neq \varnothing$ is similar. It does not result in any other independent reducibility conditions. In particular, for representation $\rho^{(6)}_{5,X}$ one obtains:

- in case $W = V_{\{2,3,4\}}$ minimal possible invariant subspace $V_{\text{inv}} = V_{\{2,3,4,5,6\}}$;
- in case $W = V_{\{1,4\}}$ minimal possible invariant subspace $V_{\text{inv}} = V_{\{1,4,5,6\}}$.

In searching for a decomposition of $\rho^{(6)}_{5,X}$ into a direct sum these invariant subspaces could be complements, respectively, for the subspaces $V_{\text{inv}} = V_{\{1,2\}}$ (case $W = V_{\{1\}}$) and $V_{\text{inv}} = V_{\{2,3,6\}}$ (case $W = V_{\{2,3\}}$). As we see, this does not happen. In all other reducible regimes with $W \neq \varnothing$ representations $\rho^{(6)}_{\dots}$ turn to be indecomposable.

It remains to consider the case $W = \varnothing$. Assuming that V_{inv} is 2-dimensional, i.e., $V_{\text{inv}} = V_{\{5,6\}}$, we get a contradiction since block-triangularity conditions for $\rho^{(6)}_{5,X}$: $G_{13} = G_{23} = 0$ do not have any solution.

Still, there is a possibility to find 1-dimensional space V_{inv}. This happens if 2×2 matrices G_{13}, G_{23} and G_{33} for certain values of parameters x_i have common eigenspace V_{inv}, which is a null space for G_{13} and G_{23}. Calculating determinants of G_{13} and G_{23}:

$$\det G_{13} \sim K^{(6)}_{5,1234} J^{(6)}_{35} J^{(6)}_{45} (e_5(X) + x_5^5), \quad \det G_{23} \sim J^{(6)}_{15} J^{(6)}_{25} (e_5(X) + x_5^5),$$

we see that the only new possible regime where one observes nontrivial invariant subspace is given by condition $I^{(6)}_5 = 0$. Indeed, in this case one finds common eigenvector

$$\{(x_5^2 + x_2x_3)(x_5^2 - x_1x_3)(x_2^2 - x_2x_5 + x_5^2),\ x_1x_3(x_1^2 - x_1x_5 + x_5^2)\},$$

with eigenvalues 0, 0 and x_5, respectively, for G_{13}, G_{23} and G_{33}. The invariant subspace generated by this vector does not have an invariant direct summand, as there is no invariant subspace containing $V_{\{1,2,3,4\}}$. □

Our main result follows as a direct consequence of Propositions 2 and 3:

Theorem 4. *For $|X| \leqslant 5$ the algebra Q_X (1.7) defined by a set of data X (1.4) is semisimple if and only if one of the following conditions hold.*

$$|X| = 2,\ I^{(2)}_{12} \neq 0; \tag{2.23}$$

$$|X| = 3,\ \{I^{(2)}_{ij},\ I^{(3)}_{ijk}\} \cap \{0\} = \varnothing \tag{2.24}$$

for all pairwise distinct indices $i, j, k \in \{1, 2, 3\}$;

$$|X| = 4,\ \{I^{(2)}_{ij},\ I^{(3)}_{ijk},\ I^{(4)}_{h,i},\ J^{(4)}_{h,ijkl}\} \cap \{0\} = \varnothing \tag{2.25}$$

for any h such that $h^2 = e_4(X)$ and for all pairwise distinct indices $i, j, k, l \in \{1, 2, 3, 4\}$;

$$|X| = 5, \ \{I_{ij}^{(2)}, I_{ijk}^{(3)}, I_{h,i}^{(4)}, J_{h,ijkl}^{(4)}, I_{f,i}^{(5)}, J_{f,ij}^{(5)}, I_{i}^{(6)}, J_{ij}^{(6)}, K_{i,jklm}^{(6)}\} \cap \{0\} = \varnothing \quad (2.26)$$

for any f such that $f^5 = e_5(X)$, for any h such that $h^2 = e_4(X^{\setminus i})$, and for all pairwise distinct indices i, j, k, l, $m \in \{1, 2, 3, 4, 5\}$.

In the semisimple case all irreducible representations of these algebras are described in Proposition 2.

Remark 3. For the algebras Q_X, $|X| = 2, 3, 4$, the statement of theorem was first proved in [23] (see Theorem 2.9 there). For the algebras Q_X, $|X| = 5$, polynomial conditions of the form $I_i^{(6)} = 0$, $J_{ij}^{(6)} = 0$, $K_{i,jklm}^{(6)} = 0$ have appeared recently in the investigations of the algebra decomposition matrices (see [7, Section 3.15]).

Proof. The existence of reducible but indecomposable representations serves as a criterion of nonsemisimplicity of an algebra. Proposition 3 provides such representations for all algebras Q_X which the theorem above declares to be nonsemisimple.

On the other hand, by the Artin–Wedderburn theorem an algebra over an algebraically closed field is semisimple if and only if sum of squares of dimensions of its inequivalent irreducible representations equals dimension of the algebra. Dimensions of the algebras Q_X for $|X| = 2$, 3, 4, and 5 are, respectively, 6, 24, 96, and 600 (see [19, Theorem 3.2(3)] and [5, Corollaries 3.4 and 4.11]). Then, Propositions 2 and 3 provide enough irreducible representations for algebras Q_X to guarantee their semisimplicity under conditions (2.23)–(2.26). For instance in case $|X| = 5$ the algebra Q_X under conditions (2.26) has the following inequivalent irreducible representations (see Proposition 2 and Remark 1): $\binom{5}{1} = 5$ times 1-dimensional, $\binom{5}{2} = 10$ times 2-dimensional, $\binom{5}{3} = 10$ times 3-dimensional, $2 \times \binom{5}{4} = 10$ times 4-dimensional, 5 times 5-dimensional, and 5 times 6-dimensional. Altogether: $5 * 1^2 + 10 * 2^2 + 10 * 3^2 + 10 * 4^2 + 5 * 5^2 + 5 * 6^2 = 600$, which fits the dimension of the algebra and proves its semisimplicity. $\square$

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