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ON A MIXED SINGULAR/SWITCHING CONTROL PROBLEM WITH MULTIPLE REGIMES

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Abstract

This paper studies a mixed singular/switching stochastic control problem for a multidimensional diffusion with multiple regimes on a bounded domain. Using probabilistic partial differential equation and penalization techniques, we show that the value function associated with this problem agrees with the solution to a Hamilton–Jacobi–Bellman equation. In this way, we see that the regularity of the value function is $\mathrm{C}^{0,1}\cap \mathrm{W}^{2,\infty}_{\mathrm{loc}}$.

Keywords: Singular/switching stochastic control problem; Hamilton–Jacobi–Bellman equations; nonlinear partial differential system

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Secondary 93E20; 93C30

1. Introduction

Singular and switching stochastic control problems have been of great research interest in control theory owing to their applicability to diverse problems of finance, economy, biology, and other fields; see, e.g., [12, 16] and the references therein. For that reason, new techniques and problems are continuously being developed. One of these problems, called a mixed singular/switching stochastic control problem, concerns the application of both singular and switching controls in an optimal way on some stochastic process that can change regime. Within a regime, a singular control is executed.

This paper is mainly concerned with determining the regularity of the value function in a mixed singular/switching stochastic control problem for a multidimensional diffusion with multiple regimes on a bounded domain. Our study is focused on the stochastically controlled process $\left(X^{\xi,\varsigma}, I^{\varsigma}\right)$ that evolves as

$$\begin{aligned} X^{\xi,\varsigma}_t &= X^{\xi,\varsigma}_{\tilde{\tau}_i} - \int_{\tilde{\tau}_i}^t b\Big(X^{\xi,\varsigma}_s, \ell_i\Big)\mathrm{d}s + \int_{\tilde{\tau}_i}^t \sigma\Big(X^{\xi,\varsigma}_s, \ell_i\Big)\mathrm{d}W_s - \int_{[\tilde{\tau}_i,t)} \mathrm{n}_s \mathrm{d}\zeta_s, \\ I^{\varsigma}_t &= \ell_i \quad \text{for } t\in\big[\tilde{\tau}_i, \tilde{\tau}_{i+1}\big) \text{ and } i\geq 0, \end{aligned} \tag{1.1}$$

where $X^{\xi,\varsigma}_0=\tilde{x}\in\overline{\mathcal{O}}\subset\mathbb{R}^d$, $I^{\varsigma}_{0-}=\tilde{\ell}\in\mathbb{I}:=\{1,2,\dots,m\}$, $\tilde{\tau}_i:=\tau_i\wedge\tau$, and τ represents the first exit time of the process $X^{\xi,\varsigma}$ from the set $\mathcal{O}$. Here $W=\{W_t:t\geq 0\}$ is a k-dimensional standard

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Brownian motion. The pair (ξ, ς) is a stochastic control strategy, defined later on (see (2.1), (2.2)), which consists of a singular control $\xi := (\mathbb{n}, \zeta) \in \mathbb{R}^d \times \mathbb{R}_+$ and a switching control $\varsigma := (\tau_i, \ell_i)_{i\geq 0}$ with τ_i a stopping time and $\ell_i \in \mathbb{I}$ for $i \geq 1$. The process (ξ, ς) is chosen in such a way that it will minimize the cost criterion

$$V_{\xi,\varsigma}\big(\tilde{x}, \tilde{\ell}\big) := \sum_{i\geq 0} \mathbb{E}_{\tilde{x},\tilde{\ell}}\big[\mathrm{e}^{-r(\tau_{i+1})}\vartheta_{\ell_i,\ell_{i+1}}\mathbb{1}_{\{\tau_{i+1}<\tau\}}\big] + \sum_{i\geq 0} \mathbb{E}_{\tilde{x},\tilde{\ell}}\bigg[\int_{\left[\tilde{\tau}_i,\tilde{\tau}_{i+1}\right)} \mathrm{e}^{-r(s)}\Big[h_{\ell_i}\Big(X_s^{\xi,\varsigma}\Big)\mathrm{d}s + g_{\ell_i}\Big(X_{s-}^{\xi,\varsigma}\Big)\circ \mathrm{d}\zeta_s\Big]\bigg]. \tag{1.2}$$

Under the assumption that there is no *loop of zero cost* (see (2.7)), one of the main goals of this paper is to verify that the value function

$$V_{\tilde{\ell}}(\tilde{x}) := \inf_{\xi,\varsigma} V_{\xi,\varsigma}\big(\tilde{x}, \tilde{\ell}\big), \text{ for } \big(\tilde{x}, \tilde{\ell}\big) \in \overline{\mathcal{O}} \times \mathbb{I}, \tag{1.3}$$

is in $\mathrm{C}^{0,1} \cap \mathrm{W}^{2,\infty}_{\mathrm{loc}}$; see Theorem 2.1.

Previous to this paper, the problem (1.3) has been studied extensively (in both bounded and unbounded set domains) for separate cases: when the process $X^{\xi,\varsigma}$ does not change its regime or when the singular control ξ is not executed on $X^{\xi,\varsigma}$; see, e.g., [6, 12, 13, 16, 19, 18] and the references therein. It is natural to study stochastic control problems where both controls are involved, and see how they can be applied.

For example, both (singular and switching) controls have been used in the study of interactions between dividend and investment policies. In that context, a firm that operates under an uncertain environment and risk constraints wants to determine an optimal control on its dividend and investment policies (see [2, 4, 3, 15]). The authors assumed that the cash reserve process of a firm switches between m-regimes governed by Brownian motions with the same volatility but different drifts [4, 15], a two-dimensional Brownian motion with the same stochastic volatility but different stochastic drifts [3], or different compound Poisson processes with drifts [2]. The costs and benefits of switching regimes are made automatically in the firm's cash reserve and are not considered in the expected returns.

The optimal dividend/investment policy strategy problem mentioned above was studied on the whole spaces $\mathbb{R}$ or $\mathbb{R}^2$; see [2, 4, 15] and [3], respectively. Using viscosity solution approaches, the authors obtained qualitative descriptions of the value function

$$\sup_{\zeta}\bigg\{\mathbb{E}_{\tilde{x},\tilde{\ell}}\bigg[\int_0^{\tau} \mathrm{e}^{-ct}\mathrm{d}\zeta_t\bigg]\bigg\}, \quad \text{with } c > 0 \text{ a fixed constant.} \tag{1.4}$$

It must be highlighted that the explicit or quasi-explicit solution of the optimal strategy for the first two cases mentioned in the previous paragraph (see [4, 3, 15]) has not yet been found and still remains an open problem. For the case when the cash reserve process switches between m-regimes governed by different Poisson processes with drifts, Azcue and Muler [2] proved that there exists an optimal dividend/switching strategy, which is stationary with a band structure.

In addition, Guo and Tomecek [11] studied a connection between singular controls of finite variation and optimal switching problems.

Notice that the mixed singular/switching stochastic control problem proposed in (1.1)–(1.3) is defined on an open bounded domain $\mathcal{O} \subset \mathbb{R}^d$, and the costs for switching regimes,

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represented by $\vartheta_{\ell,\kappa}$, are considered in the functional costs $V_{\xi,\varsigma}$. Additionally, it must be highlighted that our problem is given in a general way, meaning that for every regime ℓ, the drift and the volatility of the process $X^{\xi,\varsigma}$ are stochastic, and there are two types of costs: $g_\ell\big(X^{\xi,\varsigma}\big) \circ \mathrm{d}\zeta$ when the singular control ξ is exercised, and $h_\ell\big(X^{\xi,\varsigma}\big)$ if not. For more details about the cost $g_\ell\big(X^{\xi,\varsigma}\big) \circ \mathrm{d}\zeta$, see the next section.

The main contributions of this paper are the following:

1. We characterize the solution u to a suitable Hamilton–Jacobi–Bellman (HJB) equation, which is closely related to the value function V given in (1.3), as the limit of a sequence of solutions to another system of variational inequalities that is related to ε-penalized absolutely continuous/switching (ε-PACS) control problems; see Subsection 2.2.
2. We give an explicit representation of the optimal strategy of these ε-PACS control problems. Then, by probabilistic methods and construction of an approximating sequence of solutions as mentioned previously, we verify that the value function (1.3) and the solution u to the HJB equation (2.9) agree on $\overline{\mathcal{O}}$, showing that V_ℓ belongs to $\mathrm{C}^{0,1}\big(\overline{\mathcal{O}}\big) \cap \mathrm{W}^{2,\infty}_{\mathrm{loc}}(\mathcal{O})$; see Section 4.
3. We find explicitly the solution $\tilde{u}$ to a suitable HJB equation that is related to the value function V given in (1.3), when $\mathcal{O} = (0, l) \subset \mathbb{R}$, $l > 0$, and the parameters of the stochastic differential equation (SDE) (1.1) are given by $b(x, \ell) = b_\ell x$ and $\sigma(x, \ell) = \sigma_\ell x$ for $x \in (0, l)$, with $b_\ell, \sigma_\ell \in \mathbb{R}$ such that $\sigma_\ell > 0$. Additionally, c_ℓ and g_ℓ are positive constant functions and the running cost h_ℓ is taken as $h_\ell(x) := K_\ell x^{\gamma_\ell}$, where $\gamma_\ell \in (0, 1)$, $K_\ell > 0$ are fixed; see Section 5.

The rest of this document is organized as follows. In Section 2 we consistently formulate the stochastic control problem studied here (see (2.8)) and give the assumptions for obtaining the main results of this paper, Theorems 2.1 and 2.2. Also we introduce the ε-PACS control problem and its HJB equation (2.19). Then, in Section 3, we introduce a nonlinear partial differential system (NPDS) and give some a priori estimates. Afterwards, using Lemma 3.1, Proposition 3.1, the Arzelà–Ascoli compactness criterion, and the reflexivity of $\mathrm{L}^p_{\mathrm{loc}}(\mathcal{O})$ (see [8, Section C.8, p. 718] and [1, Theorem 2.46, p. 49], respectively), we prove the existence, regularity, and uniqueness of the solution u^ε to (2.19); see Subsection 3.1. In Subsection 3.2, we verify Theorem 2.1, which is proved by selecting a subsequence of $\{u^\varepsilon\}_{\varepsilon\in(0,1)}$. Then, in Section 4, we present a verification lemma for the ε-PACS control problem. This lemma is divided into two parts: Lemmas 4.1 and 4.2. Afterwards, we give the proof of Theorem 2.1. After that, in Section 5, numerical examples of Theorem 2.1 are given when $d = 1$. In Section 6, we draw our conclusions and discuss possible extensions of this paper. Finally, the proofs of Lemma 3.1 and Proposition 3.1 are given in the appendix. To finalize this section, we mention that the notation and the definitions of the function spaces that are used in this paper are standard; the reader can find them in [1, 5, 8, 9, 10].

2. Model formulation, assumptions, and main results

Let $W = \{W_t : t \geq 0\}$ be the k-dimensional standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ be the filtration generated by W. The process $\big(X^{\xi,\varsigma}, I^\varsigma\big)$ is governed by the SDE (1.1), where the parameters $b_\ell := b(\cdot, \ell) : \overline{\mathcal{O}} \longrightarrow \mathbb{R}^d$ and $\sigma_\ell := \sigma(\cdot, \ell) : \overline{\mathcal{O}} \longrightarrow \mathbb{R}^d \times \mathbb{R}^k$, with $\ell \in \mathbb{I}$ fixed, satisfy appropriate conditions to ensure that the SDE (1.1) is well-defined; see Subsection 2.1. The control process (ξ, ς) is in $\mathcal{U} \times \mathcal{S}$, where the singular control $\xi=(\mathrm{n},\zeta)$ belongs to the class $\mathcal{U}$ of admissible controls that satisfy

$$\begin{cases}(\mathrm{n}_t,\zeta_t)\in\mathbb{R}^d\times\mathbb{R}_+,\ t\geq 0,\ \text{such that } X^{\xi,\varsigma}_t\in\mathcal{O}\ \ t\in[0,\tau),\\ (\mathrm{n},\zeta)\text{ is adapted to the filtration }\mathbb{F},\\ \zeta_{0-}=0\text{ and }\zeta_t\text{ is nondecreasing and is right-continuous}\\ \text{with left-hand limits (càdlàg)}, t\geq 0,\ \text{and }\ |\mathrm{n}_t|=1\ \mathrm{d}\zeta_t\text{-almost surely (a.s.)},\ \ t\geq 0,\end{cases} \tag{2.1}$$

and the switching control process $\varsigma:=(\tau_i,\ell_i)_{i\geq 0}$ belongs to the class $\mathcal{S}$ of switching regime sequences that satisfy

$$\begin{cases}\varsigma\text{ is a sequence of }\mathbb{F}\text{-stopping times and regimes in }\mathbb{I},\text{ i.e,}\\ \varsigma=(\tau_i,\ell_i)_{i\geq 0}\text{ is such that }0=\tau_0\leq\tau_1<\tau_2<\cdots,\ \tau_i\uparrow\infty\text{ as }i\uparrow\infty\ \mathbb{P}\text{-a.s.,}\\ \text{and for each }i\geq 0,\ \ell_i\in\mathbb{I}=\{1,2,\ldots,m\}.\end{cases} \tag{2.2}$$

Notice that I^{ς} is a càdlàg process that starts in $\tilde{\ell}$ which has a possible jump at 0, i.e., if $\tau_1=0$, $I^{\varsigma}_{\tau_1}=\ell_1$. Without the influence of the singular control ξ in $X^{\xi,\varsigma}$, i.e. $\zeta\equiv 0$, the infinitesimal generator of $X^{\xi,\varsigma}$, within the regime $\ell\in\mathbb{I}$, is given by

$$\mathcal{L}_\ell u_\ell=\mathrm{tr}\big[a_\ell \mathrm{D}^2u_\ell\big]-\langle b_\ell,\mathrm{D}^1u_\ell\rangle, \tag{2.3}$$

where $a_\ell=\big(a_{\ell\, ij}\big)_{d\times d}$ is such that $a_{\ell\, ij}:=\frac{1}{2}\big[\sigma_\ell\sigma_\ell^\top\big]_{ij}$. Here $|\cdot|$, $\langle\cdot,\cdot\rangle$, and $\mathrm{tr}[\cdot]$ are the Euclidean norm, the inner product, and the matrix trace, respectively.

Remark 2.1. Taking $\Delta X^{\xi,\varsigma}_t:=X^{\xi,\varsigma}_t-X^{\xi,\varsigma}_{t-}$, with $(\xi,\varsigma)\in\mathcal{U}\times\mathcal{S}$, we observe $X^{\xi,\varsigma}_t=X^{\xi,\varsigma}_{t-}-\mathrm{n}_t\Delta\zeta_t\in\mathcal{O}$ for $t\in\big[\tilde{\tau}_i,\tilde{\tau}_{i+1}\big)$.

Given the initial state $\big(\tilde{x},\tilde{\ell}\big)\in\overline{\mathcal{O}}\times\mathbb{I}$ and the control $(\xi,\varsigma)\in\mathcal{U}\times\mathcal{S}$, the *functional cost* of the controlled process $\big(X^{\xi,\varsigma},I^{\varsigma}\big)$ is defined by (1.2) where $\mathbb{E}_{\tilde{x},\tilde{\ell}}$ is the expected value associated with $\mathbb{P}_{\tilde{x},\tilde{\ell}}$, the probability law of $\big(X^{\xi,\varsigma},I^{\varsigma}\big)$ when it starts at $\big(\tilde{x},\tilde{\ell}\big)$, and

$$r(t):=\int_0^t c\Big(X^{\xi,\varsigma}_s,I^{\varsigma}_s\Big)\mathrm{d}s=\sum_{i\geq 0}\int_{t\wedge\tilde{\tau}_i}^{t\wedge\tilde{\tau}_{i+1}}c_{\ell_i}\Big(X^{\xi,\varsigma}_s\Big)\mathrm{d}s, \tag{2.4}$$

$$\begin{aligned}\int_{[\tilde{\tau}_i,t\wedge\tilde{\tau}_{i+1})}\mathrm{e}^{-r(s)}g_{\ell_i}\Big(X^{\xi,\varsigma}_{s-}\Big)\circ\mathrm{d}\zeta_s:=&\int_{\tilde{\tau}_i}^{t\wedge\tilde{\tau}_{i+1}}\mathrm{e}^{-r(s)}g_{\ell_i}\big(X^{\xi,\varsigma}_s\big)\mathrm{d}\zeta^c_s\\ &+\sum_{\tilde{\tau}_i\leq s<t\wedge\tilde{\tau}_{i+1}}\mathrm{e}^{-r(s)}\Delta\zeta_s\int_0^1 g_{\ell_i}\Big(X^{\xi,\varsigma}_{s-}-\lambda\mathrm{n}_s\Delta\zeta_s\Big)\mathrm{d}\lambda,\end{aligned} \tag{2.5}$$

where ζ^c denotes the continuous part of ζ, $c_\ell:=c(\cdot,\ell)$ is a positive continuous function from $\overline{\mathcal{O}}$ to $\mathbb{R}$, and $h_\ell:=h(\cdot,\ell)$, $g_\ell:=g(\cdot,\ell)$ are nonnegative continuous functions from $\overline{\mathcal{O}}$ to $\mathbb{R}$. Notice that within the regime $\ell\in\mathbb{I}$ the singular control $\xi=(\mathrm{n},\zeta)$ generates two types of costs. One of them is when ξ continuously controls the process $X^{\xi,\varsigma}$ by ζ^c; the other is when ξ controls $X^{\xi,\varsigma}$ by jumps of ζ. While $X^{\xi,\varsigma}$ is in the regime ℓ, the term $\int_0^1 g_\ell\Big(X^{\xi,\varsigma}_{s-}-\lambda\mathrm{n}_s\Delta\zeta_s\Big)\mathrm{d}\lambda$ represents the cost for using the jump $\Delta\zeta_s\neq 0$ with direction $-\mathrm{n}_s$ on $X^{n,\zeta}_{s-}$ at time s.

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Remark 2.2. If $g_\ell \equiv a$, with a a positive constant, Equation (2.5) is reduced to the following form:

$$\int_{\left[\tilde{\tau}_i, t\wedge\tilde{\tau}_{i+1}\right)} \mathrm{e}^{-r(s)} g_{\ell_i}\left(X^{\xi,\varsigma}_{s-}\right) \circ \mathrm{d}\zeta_s = a \int_{\left[\tilde{\tau}_i, t\wedge\tilde{\tau}_{i+1}\right)} \mathrm{e}^{-r(s)} \mathrm{d}\zeta_s.$$

This means that the costs for controlling $X^{\xi,\varsigma}$ by ζ^{c} or $\Delta\zeta \neq 0$ are the same.

Remark 2.3. Notice that the cost for using the singular control ξ, defined in (2.5), was studied in the case that $X^{\xi,\varsigma}$ does not change regime; see, e.g., [6, 12, 19].

The cost for switching from the regime ℓ to κ is given by a constant $\vartheta_{\ell,\kappa} \geq 0$, and we assume that

$$\vartheta_{\ell_1,\ell_3} \leq \vartheta_{\ell_1,\ell_2} + \vartheta_{\ell_2,\ell_3}, \text{ for } \ell_3 \neq \ell_1, \ell_2, \tag{2.6}$$

which means that it is cheaper to switch directly from the regime ℓ_1 to ℓ_3 than by using the intermediate regime ℓ_2. Additionally, we assume that there is no *loop of zero cost*, i.e.,

$$\begin{aligned} &\text{no family of regimes } \{\ell_0, \ell_1, \ldots, \ell_n, \ell_0\} \\ &\text{such that } \vartheta_{\ell_0,\ell_1} = \vartheta_{\ell_1,\ell_2} = \cdots = \vartheta_{\ell_n,\ell_0} = 0. \end{aligned} \tag{2.7}$$

The *value function* is defined by

$$V_{\tilde{\ell}}(\tilde{x}) := \inf_{(\xi,\varsigma)\in\mathcal{U}\times\mathcal{S}} V_{\xi,\varsigma}\left(\tilde{x}, \tilde{\ell}\right), \text{ for } \left(\tilde{x}, \tilde{\ell}\right) \in \overline{\mathcal{O}} \times \mathbb{I}. \tag{2.8}$$

From (2.7) and by the dynamic programming principle, we identify heuristically that the value function V_ℓ is associated with the following HJB equation:

$$\max\left\{[c_\ell - \mathcal{L}_\ell]u_\ell - h_\ell, \left|\mathrm{D}^1 u_\ell\right| - g_\ell, u_\ell - \mathcal{M}_\ell u\right\} = 0 \text{ in } \mathcal{O}, \quad \text{s.t. } u_\ell = 0 \text{ on } \partial\mathcal{O}, \tag{2.9}$$

where $u = (u_1, \ldots, u_m) : \overline{\mathcal{O}} \longrightarrow \mathbb{R}^m$ and, for $\ell \in \mathbb{I}$, the operator $\mathcal{L}_\ell$ is as in (2.3) and $\mathcal{M}_\ell$ is defined by

$$\mathcal{M}_\ell u = \min_{\kappa\in\mathbb{I}\setminus\{\ell\}} \{u_\kappa + \vartheta_{\ell,\kappa}\}. \tag{2.10}$$

2.1. Assumptions and main results

First, let us give the necessary conditions to guarantee the existence and uniqueness of the solution u to the HJB equation (2.9) on the space $\mathrm{C}^{0,1}\left(\overline{\mathcal{O}}\right) \cap \mathrm{W}^{2,\infty}_{\mathrm{loc}}(\mathcal{O})$:

- *(H1) The switching costs sequence $\{\vartheta_{\ell,\kappa}\}_{\ell,\kappa\in\mathbb{I}}$ is such that $\vartheta_{\ell,\kappa} \geq 0$ and (2.6) and (2.7) hold.*
- *(H2) The domain set $\mathcal{O}$ is an open and bounded set such that its boundary $\partial\mathcal{O}$ is of class $\mathrm{C}^{4,\alpha'}$, with $\alpha' \in (0, 1)$ fixed.*

Let ℓ be in $\mathbb{I}$. Then the following hold:

- *(H3) The functions $h_\ell, g_\ell \in \mathrm{C}^{2,\alpha'}\left(\overline{\mathcal{O}}\right)$ are nonnegative, and $||h_\ell||_{\mathrm{C}^{2,\alpha'}\left(\overline{\mathcal{O}}\right)}$, $||g_\ell||_{\mathrm{C}^{2,\alpha'}\left(\overline{\mathcal{O}}\right)}$ are bounded by some finite positive constant Λ.*
- *(H4) Let $\mathcal{S}(d)$ be the set of $d \times d$ symmetric matrices. The coefficients of the differential part of $\mathcal{L}_\ell$, $a_\ell = (a_{\ell\, ij})_{d\times d} : \overline{\mathcal{O}} \longrightarrow \mathcal{S}(d)$, $b_\ell = (b_{\ell\, 1}, \ldots, b_{\ell\, d}) : \overline{\mathcal{O}} \longrightarrow \mathbb{R}^d$, and $c_\ell : \overline{\mathcal{O}} \longrightarrow \mathbb{R}$, are such that $a_{\ell\, ij}, b_{\ell\, i}, c_\ell \in \mathrm{C}^{2,\alpha'}\left(\overline{\mathcal{O}}\right)$, $c_\ell > 0$ on $\mathcal{O}$, and $||a_{\ell\, ij}||_{\mathrm{C}^{2,\alpha'}\left(\overline{\mathcal{O}}\right)}$,*

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$||b_{\ell\, i}||_{\mathrm{C}^{2,\alpha'}(\overline{\mathcal{O}})}$, $||c_\ell||_{\mathrm{C}^{2,\alpha'}(\overline{\mathcal{O}})}$ *are bounded by some finite positive constant* Λ. *We assume that there exists a real number* $\theta > 0$ *such that*

$$\langle a_\ell(x)\zeta, \zeta\rangle \geq \theta|\zeta|^2, \text{ for all } x \in \overline{\mathcal{O}},\ \zeta \in \mathbb{R}^d. \tag{2.11}$$

Under Assumptions (H1)–(H4), the first main goal obtained in this document is as follows.

Theorem 2.1. *The HJB equation* (2.9) *has a unique nonnegative strong solution (in the almost-everywhere (a.e.) sense)* $u = (u_1, \ldots, u_m)$ *where* $u_\ell \in \mathrm{C}^{0,1}(\overline{\mathcal{O}}) \cap \mathrm{W}^{2,\infty}_{\mathrm{loc}}(\mathcal{O})$ *for each* $\ell \in \mathbb{I}$.

Remark 2.4. Previous to this work, the HJB equation (2.9) was studied in 1983 by Lenhart and Belbas [13], in the absence of $|\mathrm{D}^1 u_\ell| - g_\ell$. They proved that the unique solution to their HJB equation belongs to $\mathrm{W}^{2,\infty}(\overline{\mathcal{O}})$. This HJB equation is related to optimal switching stochastic control problems. Afterwards, Yamada [18] analyzed the HJB equation for (2.8) when $\vartheta_{\ell,\kappa} = 0$ and $g_\ell = g$ for $\ell, \kappa \in \mathbb{I}$. This case is an example of a system with loops of zero cost whose HJB equation has the form

$$\max\left\{\max_{\ell\in\mathbb{I}}\{[c_\ell - \mathcal{L}_\ell]\tilde{u} - h_\ell\},\ |\mathrm{D}^1\tilde{u}| - g\right\} = 0 \text{ in } \mathcal{O}, \quad \text{s.t. } \tilde{u} = 0 \text{ on } \partial\mathcal{O}. \tag{2.12}$$

Yamada showed that there exists a solution $\tilde{u}$ to (2.12) in $\mathrm{C}^{0,1}(\overline{\mathcal{O}}) \cap \mathrm{W}^{2,\infty}_{\mathrm{loc}}(\mathcal{O})$ that does not depend on the elements of $\mathbb{I}$, and assuming $g > 0$, he guaranteed that $\tilde{u}$ belongs to $\mathrm{C}^1(\mathcal{O}) \cap \mathrm{C}(\overline{\mathcal{O}})$ and is the unique viscosity solution to (2.12). Comparing those papers with ours, we see that the results presented here are more general than those of [13], and under the assumption of the absence of loops of zero cost in the system, we guarantee the unique solution u to (2.9) in the a.e. sense.

In addition to the statement in (H2), we need to assume that the domain set is convex, which will permit us to verify that the value function V and the solution u to (2.9) agree on $\mathcal{O}$.

(H5) The domain set $\mathcal{O}$ *is an open, convex, and bounded set such that its boundary* $\partial\mathcal{O}$ *is of class* $\mathrm{C}^{4,\alpha'}$, *with* $\alpha' \in (0, 1)$ *fixed.*

Under Assumptions (H1) and (H3)–(H5), the second main goal obtained in this document is as follows.

Theorem 2.2. *Let V be the value function given by* (2.8). *Then* $V_{\tilde{\ell}}(\tilde{x}) = u_{\tilde{\ell}}(\tilde{x})$ *for* $(\tilde{x}, \tilde{\ell}) \in \overline{\mathcal{O}} \times \mathbb{I}$.

To finalize, let us comment on the assumptions mentioned above. Under (H1)–(H4) and using the Schaefer fixed point theorem (see, e.g., [8, Theorem 4, p. 539]), for each $\varepsilon, \delta \in (0, 1)$ fixed, we guarantee the existence and uniqueness of the classical nonnegative solution $u^{\varepsilon,\delta}$ to the NPDS (3.1); see Proposition 3.1 and the appendix for the proof. Also, those assumptions are required to show some a priori estimates of $u^{\varepsilon,\delta}$, which are independent of ε, δ; see Lemma 3.1. Since (H1) holds, $c_\ell > 0$ on $\overline{\mathcal{O}}$, and using Lemma 3.1, we verify that u^ε is the unique nonnegative solution to the HJB equation (2.20); see Subsection 3.1. Once again making use of (H1) and $c_\ell > 0$ on $\overline{\mathcal{O}}$, we prove that the solution u to the HJB equation (2.9) is unique; see Subsection 3.2. Finally, Assumption (H5) helps to check that the value function V, given in (2.18), and the solution u to the HJB equation (2.9) agree on $\overline{\mathcal{O}}$.

2.2. ε-PACS control problem

To prove the theorems above (Theorems 2.1 and 2.2), we first study an ε-PACS control problem that is closely related to the value function problem seen previously. At the same time, the solution to the HJB equation related to this stochastic control problem (see Proposition 2.1) helps to guarantee the existence and regularity of the solution to the system of variational inequalities (2.9). The verification lemma for this part, which is divided into two lemmas (Lemmas 4.1 and 4.2), will be presented below to provide the proofs of Theorem 2.1 and Proposition 2.1.

Define the penalized controls set $\mathcal{U}^{\varepsilon}$ in the following way:

$$\mathcal{U}^{\varepsilon} := \left\{\xi = (\mathrm{n}, \zeta) \in \mathcal{U} : \zeta_t \text{ is absolutely continuous, } 0 \leq \dot{\zeta}_t \leq 2C/\varepsilon\right\}, \tag{2.13}$$

with $\varepsilon \in (0, 1)$ fixed, where C is some fixed positive constant independent of ε. For each $(\tilde{x}, \tilde{\ell}) \in \overline{\mathcal{O}} \times \mathbb{I}$ and $(\xi, \varsigma) \in \mathcal{U}^{\varepsilon} \times \mathcal{S}$, the process $X_t^{\xi,\varsigma} = \left\{X_t^{\xi,\varsigma} : t \geq 0\right\}$ evolves as

$$X_t^{\xi,\varsigma} = X_{\tilde{\tau}_i}^{\xi,\varsigma} - \int_{\tilde{\tau}_i}^{t} \left[b\left(X_s^{\xi,\varsigma}, \ell_i\right) + \mathrm{n}_s \dot{\zeta}_s\right] \mathrm{d}s + \int_{\tilde{\tau}_i}^{t} \sigma\left(X_s^{\xi,\varsigma}, \ell_i\right) \mathrm{d}W_s, \tag{2.14}$$

$$I_t = \ell_i \quad \text{for } t \in \left[\tilde{\tau}_i, \tilde{\tau}_{i+1}\right) \text{ and } i \geq 0, \tag{2.15}$$

where $\tilde{\tau}_i = \tau_i \wedge \tau$ and $\tau = \inf\left\{t > 0 : X_t^{\xi,\varsigma} \notin \mathcal{O}\right\}$. Before introducing the corresponding functional cost $\mathcal{V}_{\xi,\varsigma}$ of $(\xi, \varsigma) \in \mathcal{U}^{\varepsilon} \times \mathcal{S}$, let us define the penalization function ψ_{ε}. Consider φ as a function from $\mathbb{R}$ to itself that is in $\mathrm{C}^{\infty}(\mathbb{R})$ and satisfies

$$\begin{aligned} &\varphi(t) = 0,\ t \leq 0, \quad \varphi(t) > 0,\ t > 0, \\ &\varphi(t) = t - 1,\ t \geq 2, \quad \varphi'(t) \geq 0,\ \varphi''(t) \geq 0. \end{aligned} \tag{2.16}$$

Then, ψ_{ε} is taken as $\psi_{\varepsilon}(t) = \varphi(t/\varepsilon)$, for each $\varepsilon \in (0, 1)$. Also, for each $(x, \ell) \in \overline{\mathcal{O}} \times \mathbb{I}$ fixed, we define the Legendre transform of $H_{\ell}^{\varepsilon}(\gamma, x) := H^{\varepsilon}(\gamma, x, \ell) := \psi_{\varepsilon}\left(|\gamma|^2 - g_{\ell}(x)^2\right)$ by

$$l_{\ell}^{\varepsilon}(y, x) := l^{\varepsilon}(y, x, \ell) := \sup_{\gamma \in \mathbb{R}^d} \left\{\langle \gamma, y\rangle - H_{\ell}^{\varepsilon}(\gamma, x)\right\}, \quad \text{for } y \in \mathbb{R}^d.$$

Notice that, for each $(x, \ell) \in \overline{\mathcal{O}} \times \mathbb{I}$ fixed, $H_{\ell}^{\varepsilon}(\gamma, x)$ is a C^2 and convex function with respect to the variable $\gamma \in \mathbb{R}^d$, since $\psi_{\varepsilon} \in \mathrm{C}^{\infty}(\mathbb{R})$ is convex. The penalized cost of $(\xi, \varsigma) \in \mathcal{U}^{\varepsilon} \times \mathcal{S}$ is defined by

$$\begin{aligned} \mathcal{V}_{\xi,\varsigma}\left(\tilde{x}, \tilde{\ell}\right) = &\sum_{i \geq 0} \mathbb{E}_{\tilde{x},\tilde{\ell}}\left[\mathrm{e}^{-r(\tau_{i+1})} \vartheta_{\ell_i,\ell_{i+1}} \mathbb{1}_{\{\tau_{i+1} < \tau\}}\right] \\ &+ \sum_{i \geq 0} \mathbb{E}_{\tilde{x},\tilde{\ell}}\left[\int_{\tilde{\tau}_i}^{\tilde{\tau}_{i+1}} \mathrm{e}^{-r(s)}\left[h_{\ell_i}\left(X_s^{\xi,\varsigma}\right) + l_{\ell_i}^{\varepsilon}\left(\dot{\zeta}_s \mathrm{n}_s, X_s^{\xi,\varsigma}\right)\right] \mathrm{d}s\right]. \end{aligned} \tag{2.17}$$

The value function for this problem is given by

$$V_{\tilde{\ell}}^{\varepsilon}(\tilde{x}) := \inf_{(\xi,\varsigma) \in \mathcal{U}^{\varepsilon}} \mathcal{V}_{\xi,\varsigma}\left(\tilde{x}, \tilde{\ell}\right), \tag{2.18}$$

whose corresponding HJB equation is

$$\begin{aligned} \max\left\{[c_{\ell} - \mathcal{L}_{\ell}]u_{\ell}^{\varepsilon} + \sup_{y \in \mathbb{R}^d}\left\{\left\langle \mathrm{D}^1 u_{\ell}^{\varepsilon}, y\right\rangle - l_{\ell}^{\varepsilon}(y, \cdot)\right\} - h_{\ell},\ u_{\ell}^{\varepsilon} - \mathcal{M}_{\ell} u^{\varepsilon}\right\} = 0,\ \text{in } \mathcal{O}, \\ \text{s.t. } u_{\ell}^{\varepsilon} = 0,\ \text{on } \partial\mathcal{O}, \end{aligned} \tag{2.19}$$

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where $\mathcal{L}_\ell$, $\mathcal{M}_\ell$ are as in (2.3), (2.10), respectively. Observe that (2.19) can be rewritten as

$$\max\left\{[c_\ell-\mathcal{L}_\ell]u_\ell^\varepsilon+\psi_\varepsilon\left(\left|\mathrm{D}^1u_\ell^\varepsilon\right|^2-g_\ell^2\right)-h_\ell,\, u_\ell^\varepsilon-\mathcal{M}_\ell u^\varepsilon\right\}=0,\ \text{in } \mathcal{O}, \\ \text{s.t. } u_\ell^\varepsilon=0,\ \text{on } \partial\mathcal{O}, \tag{2.20}$$

because of $H_\ell^\varepsilon(\gamma,x)=\sup_{y\in\mathbb{R}^d}\left\{\langle\gamma,y\rangle-l_\ell^\varepsilon(y,x)\right\}$. Under Assumptions (H1)–(H4), the following result is obtained, whose proof is given in the next section; see Subsection 3.1.

Proposition 2.1. *For each $\varepsilon\in(0,1)$ fixed, there exists a unique nonnegative strong solution $u^\varepsilon=\left(u_1^\varepsilon,\dots,u_m^\varepsilon\right)$ to the HJB equation (2.20) where $u_\ell^\varepsilon\in\mathrm{C}^{0,1}\left(\overline{\mathcal{O}}\right)\cap\mathrm{W}^{2,\infty}_{\mathrm{loc}}(\mathcal{O})$ for each $\ell\in\mathbb{I}$.*

3. Existence and uniqueness of the solution to the HJB equations

This section is devoted to guarantee existence and uniqueness of the solution to the HJB equations (2.9) and (2.20). The solution u^ε to (2.20), with $\varepsilon\in(0,1)$ fixed, will be constructed as the limit of a sequence of functions $\left\{u^{\varepsilon,\delta}\right\}_{\delta\in(0,1)}$, when δ goes to zero, which are solutions to the following NPDS:

$$[c_\ell-\mathcal{L}_\ell]u_\ell^{\varepsilon,\delta}+\psi_\varepsilon\left(\left|\mathrm{D}^1u_\ell^{\varepsilon,\delta}\right|^2-g_\ell^2\right)+\sum_{\kappa\in\mathbb{I}\setminus\{\ell\}}\psi_\delta\left(u_\ell^{\varepsilon,\delta}-u_\kappa^{\varepsilon,\delta}-\vartheta_{\ell,\kappa}\right)=h_\ell,\ \text{in } \mathcal{O}, \\ \text{s.t. } u_\ell^{\varepsilon,\delta}=0,\ \text{on } \partial\mathcal{O}. \tag{3.1}$$

The Schaefer fixed point theorem is employed to guarantee the existence of the classic solution to $u^{\varepsilon,\delta}$ to (3.1). For that aim we need some a priori estimates of $u^{\varepsilon,\delta}$, which will also help to verify the theorems seen in the section above.

Remark 3.1. From now on, we consider cut-off functions $\omega\in\mathrm{C}_c^\infty(\mathcal{O})$ which satisfy $0\le\omega\le 1$, $\omega=1$ on the open ball $B_{\beta r}\subset B_{\beta' r}\subset\mathcal{O}$ and $\omega=0$ on $\mathcal{O}\setminus B_{\beta' r}$, with $r>0$, $\beta'=\frac{\beta+1}{2}$, and $\beta\in(0,1]$. It is also assumed that $||\omega||_{\mathrm{C}^2\left(\overline{B}_{\beta r}\right)}\le K_1$, where $K_1>0$ is a constant independent of ε and δ.

Under Assumptions (H1)–(H4), the following results are obtained. Their proofs can be found in the appendix.

Lemma 3.1. *Let $u^{\varepsilon,\delta}=\left(u_1^{\varepsilon,\delta},\dots,u_m^{\varepsilon,\delta}\right)$ be a vector solution to the NPDS (3.1), whose components are in $\mathrm{C}^4\left(\overline{\mathcal{O}}\right)$. Then, for each $\ell\in\mathbb{I}$, there exist positive constants C_1, C_2, C_3 independent of ε, δ such that if $x\in\overline{\mathcal{O}}$, then*

$$0\le u_\ell^{\varepsilon,\delta}(x)\le C_1, \tag{3.2}$$

$$\left|\mathrm{D}^1u_\ell^{\varepsilon,\delta}(x)\right|\le C_2, \tag{3.3}$$

$$\omega(x)\left|\mathrm{D}^2u_\ell^{\varepsilon,\delta}(x)\right|\le C_3. \tag{3.4}$$

Proposition 3.1. *Let $\varepsilon,\delta\in(0,1)$ be fixed. There exists a unique nonnegative solution $u^{\varepsilon,\delta}=\left(u_1^{\varepsilon,\delta},\dots,u_m^{\varepsilon,\delta}\right)$ to the NPDS (3.1) where $u_\ell^{\varepsilon,\delta}\in\mathrm{C}^{4,\alpha'}\left(\overline{\mathcal{O}}\right)$ for each $\ell\in\mathbb{I}$.*

Remark 3.2. The NPDS (3.1) has been studied in a number of similar problems. One of them is when the second term on the left-hand side of (3.1) does not appear. This equation was

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considered by Lenhart and Belbas [13] to study a HJB equation related to a switching stochastic control problem when there is no loop of zero cost. Later, Yamada [18] used an NPDS similar to (3.1) to study the HJB equation (2.12). Equations of this type also appear in some stochastic singular control problems; see [12] and the references therein.

3.1. Proof of Proposition 2.1

Let $\varepsilon \in (0,1)$ and $\ell \in \mathbb{I}$ be fixed. Since $u_\ell^{\varepsilon,\delta}$, $\partial_{ij} u_\ell^{\varepsilon,\delta}$ are bounded, uniformly in δ, on the spaces $\left(\mathrm{C}^1(\overline{\mathcal{O}}), ||\cdot||_{\mathrm{C}^1(\mathcal{O})}\right)$ and $\left(\mathrm{C}(\overline{B}_r), ||\cdot||_{\mathrm{C}(B_r)}\right)$, respectively, where $B_r \subset \mathcal{O}$ (see Lemma 3.1), and using the Arzelà–Ascoli compactness criterion (see [8, p. 718]) and the fact that for each $p \in (1,\infty)$, $\left(\mathrm{L}^p(B_{\beta r}), ||\cdot||_{\mathrm{L}^p(B_r)}\right)$, with $B_r \subset \mathcal{O}$, is a reflexive space (see [1, Theorem 2.46, p. 49]), it can be proven that there exist a subsequence $\left\{u_\ell^{\varepsilon,\delta_{\hat{n}}}\right\}_{\hat{n}\geq 1}$ of $\{u_\ell^{\varepsilon,\delta}\}_{\delta\in(0,1)}$ and a function u_ℓ^ε in $\mathrm{C}^{0,1}(\overline{\mathcal{O}}) \cap \mathrm{W}^{2,\infty}_{\mathrm{loc}}(\mathcal{O})$ such that

$$\begin{aligned} &u_\ell^{\varepsilon,\delta_{\hat{n}}} \underset{\delta_{\hat{n}}\to 0}{\longrightarrow} u_\ell^\varepsilon \text{ in } \mathrm{C}(\overline{\mathcal{O}}),\ \partial_i u_\ell^{\varepsilon,\delta_{\hat{n}}} \underset{\delta_{\hat{n}}\to 0}{\longrightarrow} \partial_i u_\ell^\varepsilon \text{ in } \mathrm{C}_{\mathrm{loc}}(\mathcal{O}),\\ &\partial_{ij} u_\ell^{\varepsilon,\delta_{\hat{n}}} \underset{\delta_{\hat{n}}\to 0}{\longrightarrow} \partial_{ij} u_\ell^\varepsilon, \text{ weakly } \mathrm{L}^p_{\mathrm{loc}}(\mathcal{O}), \text{ for each } p\in(1,\infty). \end{aligned} \tag{3.5}$$

We proceed to proving Proposition 2.1.

Proof of Proposition 2.1. *Existence.* Taking $\kappa \in \mathbb{I}\setminus\{\ell\}$, and using (2.3), (3.1), and Lemma 3.1, we have that $\psi_\delta\left(u_\ell^{\varepsilon,\delta} - u_\kappa^{\varepsilon,\delta} - \vartheta_{\ell,\kappa}\right)$ is locally bounded, uniformly in δ. From here and (3.5), we obtain that $u_\ell^\varepsilon - u_\kappa^\varepsilon - \vartheta_{\ell,\kappa} \leq 0$ in $\mathcal{O}$. Then,

$$u_\ell^\varepsilon - \mathcal{M}_\ell u^\varepsilon \leq 0, \quad \text{in } \mathcal{O}. \tag{3.6}$$

Note that the previous inequality is true on the boundary set $\partial\mathcal{O}$, since $u_\ell^{\varepsilon,\delta} = u_\kappa^{\varepsilon,\delta} = 0$ on $\partial\mathcal{O}$ and $\vartheta_{\ell,\kappa} \geq 0$. Recall that the operator $\mathcal{M}_\ell$ is defined in (2.10). On the other hand, since $u_\ell^{\varepsilon,\delta_{\hat{n}}}$ is the unique solution to (3.1), when $\delta = \delta_{\hat{n}}$, it follows that

$$\int_{B_r}\left\{[c_\ell - \mathcal{L}_\ell]u_\ell^{\varepsilon,\delta_{\hat{n}}} + \psi_\varepsilon\left(\left|\mathrm{D}^1 u_\ell^{\varepsilon,\delta_{\hat{n}}}\right|^2 - g_\ell^2\right)\right\}\varpi\,\mathrm{d}x \leq \int_{B_r} h_\ell \varpi\,\mathrm{d}x, \quad \text{for } \varpi \in \mathcal{B}(B_r), \tag{3.7}$$

where

$$\mathcal{B}(A) := \left\{\varpi \in \mathrm{C}^\infty_{\mathrm{c}}(A) : \varpi \geq 0 \text{ and } \mathrm{supp}[\varpi] \subset A \subset \mathcal{O}\right\}. \tag{3.8}$$

By (3.5) and letting $\delta_{\hat{n}} \to 0$ in (3.7), we obtain that

$$[c_\ell - \mathcal{L}_\ell]u_\ell^\varepsilon + \psi_\varepsilon\left(\left|\mathrm{D}^1 u_\ell^\varepsilon\right|^2 - g_\ell^2\right) \leq h_\ell \quad \text{a.e. in } \mathcal{O}. \tag{3.9}$$

From (3.6) and (3.9), $\max\left\{[c_\ell - \mathcal{L}_\ell]u_\ell^\varepsilon + \psi_\varepsilon\left(\left|\mathrm{D}^1 u_\ell^\varepsilon\right|^2 - g_\ell^2\right) - h_\ell,\ u_\ell^\varepsilon - \mathcal{M}_\ell u^\varepsilon\right\} \leq 0$ a.e. in $\mathcal{O}$. We shall prove that if

$$u_\ell^\varepsilon(x^*) - \mathcal{M}_\ell u^\varepsilon(x^*) < 0, \quad \text{for some } x^* \in \mathcal{O}, \tag{3.10}$$

then there exists a neighborhood $\mathcal{N}_{x^*} \subset \mathcal{O}$ of x^* such that

$$[c_\ell - \mathcal{L}_\ell]u_\ell^\varepsilon + \psi_\varepsilon\left(\left|\mathrm{D}^1 u_\ell^\varepsilon\right|^2 - g_\ell^2\right) = h_\ell, \quad \text{a.e. in } \mathcal{N}_{x^*}. \tag{3.11}$$

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Assume (3.10) holds. Then, taking $\kappa \in \mathbb{I} \setminus \{\ell\}$, we see that $u^{\varepsilon}_{\ell} - u^{\varepsilon}_{\kappa} - \vartheta_{\ell,\kappa} \leq u^{\varepsilon}_{\ell} - \mathcal{M}_{\ell} u^{\varepsilon} < 0$ at x^*. Since $u^{\varepsilon}_{\ell} - u^{\varepsilon}_{\kappa}$ is a continuous function, there exists a ball $B_{r_\kappa} \subset \mathcal{O}$ such that $x^* \in B_{r_\kappa}$ and $u^{\varepsilon}_{\ell} - u^{\varepsilon}_{\kappa} - \vartheta_{\ell,\kappa} < 0$ in B_{r_κ}. From here, and defining $\mathcal{N}_{x^*}$ as $\bigcap_{\kappa \in \mathbb{I}\setminus\{\ell\}} B_{r_\kappa}$, we have that $\mathcal{N}_{x^*} \subset \mathcal{O}$ is a neighborhood of x^* and

$$u^{\varepsilon}_{\ell} - u^{\varepsilon}_{\kappa} - \vartheta_{\ell,\kappa} < 0, \qquad \text{in } \mathcal{N}_{x^*}, \text{ for } \kappa \in \mathbb{I} \setminus \{\ell\}. \tag{3.12}$$

Meanwhile, observe that

$$\left|\left| u^{\varepsilon,\delta_{\hat{n}}}_{\ell} - u^{\varepsilon,\delta_{\hat{n}}}_{\kappa} - \left(u^{\varepsilon}_{\ell} - u^{\varepsilon}_{\kappa}\right) \right|\right|_{\mathrm{C}(\mathcal{O})} \underset{\delta_{\hat{n}} \to 0}{\longrightarrow} 0, \qquad \text{for } \kappa \in \mathbb{I} \setminus \{\ell\}, \tag{3.13}$$

since (3.5) holds. Then, by (3.12)–(3.13), we have that for each $\kappa \in \mathbb{I} \setminus \{\ell\}$, there exists a $\delta^{(\kappa)} \in (0, 1)$ such that if $\delta_{\hat{n}} \leq \delta^{(\kappa)}$, then $u^{\varepsilon,\delta_{\hat{n}}}_{\ell} - u^{\varepsilon,\delta_{\hat{n}}}_{\kappa} - \vartheta_{\ell,\kappa} < 0$ in $\mathcal{N}_{x^*}$. Taking $\delta' := \min_{\kappa \in \mathbb{I}\setminus\{\ell\}}\{\delta^{(\kappa)}\}$, it follows that $u^{\varepsilon,\delta_{\hat{n}}}_{\ell} - u^{\varepsilon,\delta_{\hat{n}}}_{\kappa} - \vartheta_{\ell,\kappa} < 0$ in $\mathcal{N}_{x^*}$, for all $\delta_{\hat{n}} \leq \delta'$ and $\kappa \in \mathbb{I} \setminus \{\ell\}$. From here and since for each $\delta_{\hat{n}} \leq \delta'$, $u^{\varepsilon,\delta_{\hat{n}}}_{\ell}$ is the unique solution to (3.1), when $\delta = \delta_{\hat{n}}$, this implies that

$$\int_{\mathcal{N}_{x^*}} \left\{ [c_\ell - \mathcal{L}_\ell] u^{\varepsilon,\delta_{\hat{n}}}_{\ell} + \psi_\varepsilon\left(\left| \mathrm{D}^1 u^{\varepsilon,\delta_{\hat{n}}}_{\ell} \right|^2 - g^2_\ell \right) \right\} \varpi \, \mathrm{d}x = \int_{\mathcal{N}_{x^*}} h_\ell \varpi \, \mathrm{d}x, \qquad \text{for } \varpi \in \mathcal{B}(\mathcal{N}_{x^*}).$$

Therefore, (3.11) holds. Hence, we get that for each $\varepsilon \in (0, 1)$, $u^{\varepsilon} = \left(u^{\varepsilon}_1, \ldots, u^{\varepsilon}_m\right)$ is a solution to the HJB equation (2.19). □

Proof of Proposition 2.1. *Uniqueness.* Let $\varepsilon \in (0, 1)$ be fixed. Suppose that $u^{\varepsilon} = \left(u^{\varepsilon}_1, \ldots, u^{\varepsilon}_m\right)$ and $v^{\varepsilon} = \left(v^{\varepsilon}_1, \ldots, v^{\varepsilon}_m\right)$ are two solutions to the HJB equation (2.20) whose components belong to $\mathrm{C}^{0,1}\left(\overline{\mathcal{O}}\right) \cap \mathrm{W}^{2,\infty}_{\mathrm{loc}}(\mathcal{O})$. Take $(x_0, \ell_0) \in \overline{\mathcal{O}} \times \mathbb{I}$ such that

$$u^{\varepsilon}_{\ell_0}(x_0) - v^{\varepsilon}_{\ell_0}(x_0) = \sup_{(x,\ell) \in \overline{\mathcal{O}} \times \mathbb{I}} \left\{ u^{\varepsilon}_{\ell}(x) - v^{\varepsilon}_{\ell}(x) \right\}. \tag{3.14}$$

Notice that by (3.14), we only need to verify that

$$u^{\varepsilon}_{\ell_0}(x_0) - v^{\varepsilon}_{\ell_0}(x_0) \leq 0, \tag{3.15}$$

which is trivially true if $x_0 \in \partial\mathcal{O}$, since $u^{\varepsilon}_{\ell_0} - v^{\varepsilon}_{\ell_0} = 0$ on $\partial\mathcal{O}$. Let us assume $x_0 \in \mathcal{O}$. We shall verify (3.15) by contradiction. Suppose that $u^{\varepsilon}_{\ell_0} - v^{\varepsilon}_{\ell_0} > 0$ at x_0. Then, by continuity of $u^{\varepsilon}_{\ell_0} - v^{\varepsilon}_{\ell_0}$, there exists a ball $B_{r_1}(x_0) \subset \mathcal{O}$ such that

$$c_{\ell_0}\left[u^{\varepsilon}_{\ell_0} - v^{\varepsilon}_{\ell_0}\right] \geq \min_{x \in B_{r_1}(x_0)} \left\{ c_{\ell_0}(x)\left[u^{\varepsilon}_{\ell_0}(x) - v^{\varepsilon}_{\ell_0}(x)\right]\right\} > 0, \quad \text{in } B_{r_1}(x_0). \tag{3.16}$$

The last inequality is true because of $c_{\ell_0} > 0$ in $\mathcal{O}$. Taking $\ell_1 \in \mathbb{I}$ such that

$$\mathcal{M}_{\ell_0} v^{\varepsilon}(x_0) = v^{\varepsilon}_{\ell_1}(x_0) + \vartheta_{\ell_0,\ell_1}, \tag{3.17}$$

by (2.20) and (3.14), we get that $v^{\varepsilon}_{\ell_0} - \left(v^{\varepsilon}_{\ell_1} + \vartheta_{\ell_0,\ell_1}\right) = v^{\varepsilon}_{\ell_0} - \mathcal{M}_{\ell_0} v^{\varepsilon} \leq u^{\varepsilon}_{\ell_0} - \mathcal{M}_{\ell_0} u^{\varepsilon} \leq 0$ at x_0. If $v^{\varepsilon}_{\ell_0}(x_0) - \mathcal{M}_{\ell_0} v^{\varepsilon}(x_0) < 0$, there exists a ball $B_{r_2}(x_0) \subset \mathcal{O}$ such that $v^{\varepsilon}_{\ell_0} - \mathcal{M}_{\ell_0} v^{\varepsilon} < 0$ in $B_{r_2}(x_0)$. Moreover, from (2.20),

$$\left.\begin{aligned} &[c_{\ell_0} - \mathcal{L}_{\ell_0}] v^{\varepsilon}_{\ell_0} + \psi_\varepsilon\left(\left|\mathrm{D}^1 v^{\varepsilon}_{\ell_0}\right|^2 - g^2_{\ell_0} \right) - h_{\ell_0} = 0, \\ &[c_{\ell_0} - \mathcal{L}_{\ell_0}] u^{\varepsilon}_{\ell_0} + \psi_\varepsilon\left(\left|\mathrm{D}^1 u^{\varepsilon}_{\ell_0}\right|^2 - g^2_{\ell_0} \right) - h_{\ell_0} \leq 0, \end{aligned}\right\} \quad \text{in } B_{r_2}(x_0). \tag{3.18}$$

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Notice that $\psi_\varepsilon\Big(\Big|\mathrm{D}^1u^\varepsilon_{\ell_0}\Big|^2-g^2_{\ell_0}\Big)-\psi_\varepsilon\Big(\Big|\mathrm{D}^1v^\varepsilon_{\ell_0}\Big|^2-g^2_{\ell_0}\Big)$ is a continuous function in $\mathcal{O}$, since $\partial_i u^\varepsilon_{\ell_0}, \partial_i v^\varepsilon_{\ell_0}\in \mathrm{C}^0(\mathcal{O})$, which satisfies $\psi_\varepsilon\Big(\Big|\mathrm{D}^1u^\varepsilon_{\ell_0}\Big|^2-g^2_{\ell_0}\Big)-\psi_\varepsilon\Big(\Big|\mathrm{D}^1v^\varepsilon_{\ell_0}\Big|^2-g^2_{\ell_0}\Big)=0$ at x_0, since x_0 is the point where $u^\varepsilon_{\ell_0}-v^\varepsilon_{\ell_0}$ attains its maximum. Meanwhile, by Bony's maximum principle (see [14]), it is known that for every $r\leq r_3$, with $r_3>0$ small enough,

$$\mathrm{tr}\big[a_{\ell_0}\mathrm{D}^2\big[u^\varepsilon_{\ell_0}-v^\varepsilon_{\ell_0}\big]\big]\leq 0,\quad \text{a.e. in } B_r(x_0). \tag{3.19}$$

So, from (3.16), (3.18), and (3.19), we have that for every $r\leq\hat{r}:=\min\{r_1,r_2,r_3\}$,

$$\begin{aligned}0&\geq \mathrm{tr}\big[a_{\ell_0}\mathrm{D}^2\big[u^\varepsilon_{\ell_0}-v^\varepsilon_{\ell_0}\big]\big]\\&\geq c_{\ell_0}\big[u^\varepsilon_{\ell_0}-v^\varepsilon_{\ell_0}\big]+\langle b_{\ell_0},\mathrm{D}^1\big[u^\varepsilon_{\ell_0}-v^\varepsilon_{\ell_0}\big]\rangle+\psi_\varepsilon\Big(\Big|\mathrm{D}^1u^\varepsilon_{\ell_0}\Big|^2-g^2_{\ell_0}\Big)-\psi_\varepsilon\Big(\Big|\mathrm{D}^1v^\varepsilon_{\ell_0}\Big|^2-g^2_{\ell_0}\Big)\\&\geq\min_{x\in B_{r_1}(x_0)}\Big\{c_{\ell_0}(x)\Big[u^\varepsilon_{\ell_0}(x)-v^\varepsilon_{\ell_0}(x)\Big]\Big\}+\langle b_{\ell_0},\mathrm{D}^1\big[u^\varepsilon_{\ell_0}-v^\varepsilon_{\ell_0}\big]\rangle\\&\quad+\psi_\varepsilon\Big(\Big|\mathrm{D}^1u^\varepsilon_{\ell_0}\Big|^2-g^2_{\ell_0}\Big)-\psi_\varepsilon\Big(\Big|\mathrm{D}^1v^\varepsilon_{\ell_0}\Big|^2-g^2_{\ell_0}\Big),\quad \text{a.e. in } B_r(x_0).\end{aligned}$$

Then,

$$\begin{aligned}&\lim_{r\to0}\Big\{\inf_{B_r(x_0)}\mathrm{ess}\Big[\psi_\varepsilon\Big(\Big|\mathrm{D}^1u^\varepsilon_{\ell_0}\Big|^2-g^2_{\ell_0}\Big)-\psi_\varepsilon\Big(\Big|\mathrm{D}^1v^\varepsilon_{\ell_0}\Big|^2-g^2_{\ell_0}\Big)\Big]\Big\}\\&\qquad\qquad<-\min_{x\in B_{r_1}(x_0)}\Big\{c_{\ell_0}(x)\Big[u^\varepsilon_{\ell_0}(x)-v^\varepsilon_{\ell_0}(x)\Big]\Big\}<0.\end{aligned} \tag{3.20}$$

This means $\psi_\varepsilon\Big(\Big|\mathrm{D}^1u^\varepsilon_{\ell_0}\Big|^2-g^2_{\ell_0}\Big)-\psi_\varepsilon\Big(\Big|\mathrm{D}^1v^\varepsilon_{\ell_0}\Big|^2-g^2_{\ell_0}\Big)$ is not continuous at x_0 which is a contradiction. Thus,

$$0=v^\varepsilon_{\ell_0}-\big(v^\varepsilon_{\ell_1}+\vartheta_{\ell_0,\ell_1}\big)=v^\varepsilon_{\ell_0}-\mathcal{M}_{\ell_0}v^\varepsilon\leq u^\varepsilon_{\ell_0}-\mathcal{M}_{\ell_0}u^\varepsilon\leq0\quad\text{at } x_0. \tag{3.21}$$

This implies that

$$\begin{aligned}&u^\varepsilon_{\ell_1}(x_0)-v^\varepsilon_{\ell_1}(x_0)\geq u^\varepsilon_{\ell_0}(x_0)-v^\varepsilon_{\ell_0}(x_0)>0,\\&v^\varepsilon_{\ell_0}(x_0)=v^\varepsilon_{\ell_1}(x_0)+\vartheta_{\ell_0,\ell_1}.\end{aligned} \tag{3.22}$$

By (3.14) and (3.22), we have that $u^\varepsilon_{\ell_1}-v^\varepsilon_{\ell_1}$ attains its maximum at $x_0\in\mathcal{O}$, where its value agrees with $u^\varepsilon_{\ell_0}(x_0)-v^\varepsilon_{\ell_0}(x_0)$. Then, replacing $u^\varepsilon_{\ell_0}-v^\varepsilon_{\ell_0}$ by $u^\varepsilon_{\ell_1}-v^\varepsilon_{\ell_1}$ above and repeating the same arguments seen in (3.17)–(3.21), we get that there is a regime $\ell_2\in\mathbb{I}$ such that

$$\begin{aligned}&u^\varepsilon_{\ell_2}(x_0)-v^\varepsilon_{\ell_2}(x_0)=u^\varepsilon_{\ell_1}(x_0)-v^\varepsilon_{\ell_1}(x_0)=u^\varepsilon_{\ell_0}(x_0)-v^\varepsilon_{\ell_0}(x_0)>0,\\&v^\varepsilon_{\ell_1}(x_0)=v^\varepsilon_{\ell_2}(x_0)+\vartheta_{\ell_1,\ell_2}.\end{aligned}$$

Recursively, we obtain a sequence of regimes $\{\ell_i\}_{i\geq0}$ such that

$$\begin{aligned}&u^\varepsilon_{\ell_i}(x_0)-v^\varepsilon_{\ell_i}(x_0)=u^\varepsilon_{\ell_{i-1}}(x_0)-v^\varepsilon_{\ell_{i-1}}(x_0)=\cdots=u^\varepsilon_{\ell_0}(x_0)-v^\varepsilon_{\ell_0}(x_0)>0,\\&v^\varepsilon_{\ell_i}(x_0)=v^\varepsilon_{\ell_{i+1}}(x_0)+\vartheta_{\ell_i,\ell_{i+1}}.\end{aligned} \tag{3.23}$$

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Since $\mathbb{I}$ is finite, there is a regime ℓ' that will appear infinitely often in $\{\ell_i\}_{i\geq 0}$. Let $\ell_n = \ell'$, for some $n > 1$. After $\hat{n}$ steps, the regime ℓ' reappears, i.e. $\ell_{n+\hat{n}} = \ell'$. Then, by (3.23), we get

$$v^{\varepsilon}_{\ell'}(x_0) = v^{\varepsilon}_{\ell'}(x_0) + \vartheta_{\ell',\ell_{n+1}} + \vartheta_{\ell_{n+1},\ell_{n+2}} + \cdots + \vartheta_{\ell_{n+\hat{n}-1},\ell'}. \tag{3.24}$$

Notice that (3.24) contradicts the assumption that there is no loop of zero cost (see Equation (2.7)). From here we conclude that (3.15) must hold. Taking $v^{\varepsilon} - u^{\varepsilon}$ and proceeding in the same way as before, it follows that for each $\ell \in \mathbb{I}$, $v^{\varepsilon}_{\ell} - u^{\varepsilon}_{\ell} \leq 0$ in $\mathcal{O}$, and hence we conclude that the solution u^{ε} to the HJB equation (2.20) is unique. □

3.2. Proof of Theorem 2.1

In view of Lemma 3.1 and by (3.5), the following inequalities hold for each $\ell \in \mathbb{I}$:

$$0 \leq u^{\varepsilon}_{\ell} \leq C_1 \quad \text{and} \quad \left|\mathrm{D}^1 u^{\varepsilon}_{\ell}\right| \leq C_4, \quad \text{in } \overline{\mathcal{O}}, \tag{3.25}$$

for some positive constant $C_4 = C_4(d, \Lambda, \alpha')$. The constant C_1 is as in (3.2). Moreover, for each $B_{\beta r} \subset \mathcal{O}$, there exists a positive constant $C_5 = C_5(d, \Lambda, \alpha')$ such that

$$\left|\left|\mathrm{D}^2 u^{\varepsilon}_{\ell}\right|\right|_{L^p\left(B_{\beta r}\right)} \leq C_5, \quad \text{for each } p \in (1, \infty). \tag{3.26}$$

Then, from (3.25)–(3.26) and using again the Arzelà–Ascoli compactness criterion and the fact that $\left(\mathrm{L}^p\left(B_{\beta r}\right), ||\cdot||_{L^p\left(B_{\beta r}\right)}\right)$ is a reflexive space, we have that there exist a subsequence $\left\{u^{\varepsilon_n}_{\ell}\right\}_{n\geq 1}$ of $\left\{u^{\varepsilon}_{\ell}\right\}_{\varepsilon\in(0,1)}$ and a u_{ℓ} in $\mathrm{C}^{0,1}\left(\overline{\mathcal{O}}\right) \cap \mathrm{W}^{2,\infty}_{\mathrm{loc}}(\mathcal{O})$ such that

$$\begin{aligned} &u^{\varepsilon_n}_{\ell} \underset{\varepsilon_n\to 0}{\longrightarrow} u_{\ell} \text{ in } \mathrm{C}\left(\overline{\mathcal{O}}\right),\ \partial_i u^{\varepsilon_n}_{\ell} \underset{\varepsilon_n\to 0}{\longrightarrow} \partial_i u_{\ell} \text{ in } C_{\mathrm{loc}}(\mathcal{O}),\\ &\partial_{ij} u^{\varepsilon_n}_{\ell} \underset{\varepsilon_n\to 0}{\longrightarrow} \partial_{ij} u_{\ell}, \text{ weakly } \mathrm{L}^p_{\mathrm{loc}}(\mathcal{O}), \text{ for each } p \in (1, \infty). \end{aligned} \tag{3.27}$$

Remark 3.3. We use the notation $C = C(*, \ldots, *)$ and $K = K(*, \ldots, *)$ to represent positive constants that depend only on the quantities appearing in parentheses.

We proceed to proving Theorem 2.1.

Proof of Theorem 2.1. *Existence.* Now, let $\ell \in \mathbb{I}$ be fixed. Since $u^{\varepsilon_n}_{\ell}$ is the unique strong solution to the HJB equation (2.20) when $\varepsilon = \varepsilon_n$, which belongs to $\mathrm{C}^{0,1}\left(\overline{\mathcal{O}}\right)$, it follows that for each $\kappa \in \mathbb{I} \setminus \{\ell\}$, $u^{\varepsilon_n}_{\ell} - \left(u^{\varepsilon_n}_{\kappa} + \vartheta_{\ell,\kappa}\right) \leq u^{\varepsilon_n}_{\ell} - \mathcal{M}_{\ell} u^{\varepsilon_n} \leq 0$ in $\mathcal{O}$. From here and (3.27), we have that $u_{\ell} - u_{\kappa} - \vartheta_{\ell,\kappa} \leq 0$ in $\mathcal{O}$. Then, $u_{\ell} - \mathcal{M}_{\ell} u \leq 0$, in $\mathcal{O}$. Also, we know that $[c_{\ell} - \mathcal{L}_{\ell}]u^{\varepsilon_n}_{\ell} + \psi_{\varepsilon_n}\left(\left|\mathrm{D}^1 u^{\varepsilon_n}_{\ell}\right|^2 - g^2_{\ell}\right) \leq h_{\ell}$ a.e. in $\mathcal{O}$. Thus,

$$0 \leq \psi_{\varepsilon_n}\left(\left|\mathrm{D}^1 u^{\varepsilon_n}_{\ell}\right|^2 - g^2_{\ell}\right) \leq h_{\ell} - [c_{\ell} - \mathcal{L}_{\ell}]u^{\varepsilon_n}_{\ell}, \quad \text{a.e. in } \mathcal{O}. \tag{3.28}$$

Consequently, by (H3), (3.25), (3.26), and (3.28), there exists a positive constant $C_6 = C_6(d, \Lambda, \alpha')$ such that

$$0 \leq \int_{B_r} \psi_{\varepsilon_n}\left(\left|\mathrm{D}^1 u^{\varepsilon_n}_{\ell}\right|^2 - g^2_{\ell}\right)\varpi\, \mathrm{d}x \leq \int_{B_r} \left\{h_{\ell} - [c_{\ell} - \mathcal{L}_{\ell}]u^{\varepsilon_n}_{\ell}\right\}\varpi\, \mathrm{d}x \leq C_6$$

for each $\varpi \in \mathcal{B}(B_r)$, with $\mathcal{B}(\cdot)$ as in (3.8). Thus, using definition of ψ_{ε} (see (2.16)), and since $\left|\mathrm{D}^1 u^{\varepsilon_n}_{\ell}\right|^2 - g^2_{\ell}$ is continuous in $\mathcal{O}$, we have that for each $B_r \subset \mathcal{O}$, there exists $\varepsilon' \in (0, 1)$ small enough so that for all $\varepsilon_n \leq \varepsilon'$, $\left|\mathrm{D}^1 u_\ell^{\varepsilon_n}\right| - g_\ell \leq 0$ in B_r. Then, since (3.27) holds, it follows that $|\mathrm{D}^1 u_\ell| \leq g_\ell$ in $\mathcal{O}$. From (3.28), we get $\int_{B_r} \left\{[c_\ell - \mathcal{L}_\ell]u_\ell^{\varepsilon_n} - h_\ell\right\}\varpi \,\mathrm{d}x \leq 0$, for each $\varpi \in \mathcal{B}(B_r)$. From here and (3.27), we obtain that $[c_\ell - \mathcal{L}_\ell]u_\ell - h_\ell \leq 0$ a.e. in $\mathcal{O}$. Therefore, by what we saw previously,

$$\max\left\{[c_\ell - \mathcal{L}_\ell]u_\ell - h_\ell,\, \left|\mathrm{D}^1 u_\ell\right| - g_\ell,\, u_\ell - \mathcal{M}_\ell u\right\} \leq 0, \quad \text{a.e. in } \mathcal{O}. \tag{3.29}$$

Without loss of generality we assume that $u_\ell(x^*) - \mathcal{M}_\ell u(x^*) < 0$, for some $x^* \in \mathcal{O}$. Otherwise, equality holds in (3.29). Then, for each $\kappa \in \mathbb{I}$ such that $\kappa \neq \ell$, $u_\ell - (u_\kappa + \vartheta_{\ell,\kappa}) \leq u_\ell - \mathcal{M}_\ell u < 0$ at x^*. There exists a ball $B_{r_1}(x^*) \subset \mathcal{O}$ such that

$$u_\ell - (u_\kappa + \vartheta_{\ell,\kappa}) \leq u_\ell - \mathcal{M}_\ell u < 0, \quad \text{in } B_{r_1}(x^*), \tag{3.30}$$

by the continuity of $u_\ell - u_\kappa$ in $\overline{\mathcal{O}}$. Now, consider that $\left|\mathrm{D}^1 u_\ell\right| - g_\ell < 0$ for some $x_1^* \in B_{r_1}(x^*)$. Otherwise, equality again holds in (3.29). By continuity of $\left|\mathrm{D}^1 u_\ell\right| - g_\ell$, we have that for some $B_{r_2}(x_1^*) \subset \mathcal{O}$, $\left|\mathrm{D}^1 u_\ell\right| - g_\ell < 0$ in $B_{r_2}(x_1^*)$. From here, using (3.27), (3.30) and taking $\mathcal{N} := B_{r_1}(x^*) \cap B_{r_2}\left(x_1^*\right)$, it can be verified that there exists an $\varepsilon' \in (0, 1)$ small enough so that for each $\varepsilon_n \leq \varepsilon'$, $|\mathrm{D}^1 u_\ell^{\varepsilon_n}| - g_\ell < 0$ and $u_\ell^{\varepsilon_n} - \mathcal{M}_\ell u^{\varepsilon_n} < 0$ in $\mathcal{N}$. Thus, $[c_\ell - \mathcal{L}_\ell]u_\ell^{\varepsilon_n} = h_\ell$ a.e. in $\mathcal{N}$, since u^{ε_n} is the unique solution to the HJB equation (2.20), when $\varepsilon = \varepsilon_n$. Then, $\int_{\mathcal{N}} \left\{[c_\ell - \mathcal{L}_\ell]u_\ell^{\varepsilon_n} - h_\ell\right\}\varpi \,\mathrm{d}x = 0$, for each $\varpi \in \mathcal{B}(\mathcal{N})$. Hence, letting $\varepsilon_n \to 0$ and again using (3.27), we get that $u = (u_1, \ldots, u_m)$ is a solution to the HJB equation (2.9). □

Proof of Theorem 2.1. *Uniqueness.* Suppose that

$$u = (u_1, \ldots, u_m) \quad \text{and} \quad v = (v_1, \ldots, v_m)$$

are two solutions to the HJB equation (2.9) whose components belong to $\mathrm{C}^{0,1}\left(\overline{\mathcal{O}}\right) \cap \mathrm{W}^{2,\infty}_{\mathrm{loc}}(\mathcal{O})$. Take $(x_0, \ell_0) \in \overline{\mathcal{O}} \times \mathbb{I}$ such that

$$u_{\ell_0}(x_0) - v_{\ell_0}(x_0) = \sup_{(x,\ell) \in \overline{\mathcal{O}} \times \mathbb{I}} \{u_\ell(x) - v_\ell(x)\}. \tag{3.31}$$

As before (see Subsection 3.1), we only need to verify that

$$u_{\ell_0} - v_{\ell_0} \leq 0, \quad \text{at } x_0 \in \mathcal{O}. \tag{3.32}$$

Assume that $u_{\ell_0} - v_{\ell_0} > 0$ at x_0. Then there exists a ball $B_{r_1}(x_0) \subset \mathcal{O}$ such that

$$c_{\ell_0}[u_{\ell_0} - v_{\ell_0}] \geq \min_{x \in B_{r_1}(x_0)} \left\{c_{\ell_0}(x)\left[u_{\ell_0}(x) - v_{\ell_0}(x)\right]\right\} > 0$$

in $B_{r_1}(x_0)$, by the continuity of $u_{\ell_0} - v_{\ell_0}$ in $\overline{\mathcal{O}}$ and the fact that $c_{\ell_0} > 0$ in $\mathcal{O}$. Meanwhile, from (3.31), $v_{\ell_0} - \mathcal{M}_{\ell_0} v \leq u_{\ell_0} - \mathcal{M}_{\ell_0} u \leq 0$ at x_0. If $v_{\ell_0} - \mathcal{M}_{\ell_0} v < 0$ at x_0, there exists a ball $B_{r_2}(x_0) \subset \mathcal{O}$ such that $v_{\ell_0} - \mathcal{M}_{\ell_0} v < 0$ in $B_{r_2}(x_0)$. Now, consider the auxiliary function $f_\varrho := u_{\ell_0} - v_{\ell_0} - \varrho u_{\ell_0}$, with $\varrho \in (0, 1)$. Notice that $f_\varrho = 0$ on $\partial\mathcal{O}$, for $\varrho \in (0, 1)$, and

$$f_\varrho \uparrow u_{\ell_0} - v_{\ell_0} \text{ uniformly in } \mathcal{O}, \text{ when } \varrho \downarrow 0. \tag{3.33}$$

In addition, there is a $\varrho' \in (0, 1)$ small enough so that $\sup_{x \in B_{r_2}(x_0)}\{f_\varrho(x)\} > 0$ for all $\varrho \in (0, \varrho')$, since $u_{\ell_0} - v_{\ell_0} > 0$ at x_0. By (3.31) and (3.33), there exists $\hat{\varrho} \in (0, \varrho')$ small enough so that $f_{\hat{\varrho}}$ has a local maximum at $x_{\hat{\varrho}} \in B_{r_1}(x_0) \cap B_{r_2}(x_0)$. It follows that

$$\left|\mathrm{D}^1 v_{\ell_0}\left(x_{\hat{\varrho}}\right)\right| = [1 - \hat{\varrho}]\left|\mathrm{D}^1 u_{\ell_0}\left(x_{\hat{\varrho}}\right)\right| < \left|\mathrm{D}^1 u_{\ell_0}\left(x_{\hat{\varrho}}\right)\right| \leq g\left(x_{\hat{\varrho}}\right).$$

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Thus, there exists a ball $B_{r_3}(x_{\hat{\varrho}}) \subset B_{r_1}(x_0) \cap B_{r_2}(x_0)$ such that $[c_{\ell_0} - \mathcal{L}_{\ell_0}]v_{\ell_0} - h_{\ell_0} = 0$ and $[c_{\ell_0} - \mathcal{L}_{\ell_0}]u_{\ell_0} - h_{\ell_0} \leq 0$ in $B_{r_3}(x_{\hat{\varrho}})$. Then, by Bony's maximum principle, we have that

$$0 \geq \lim_{r \to 0} \left\{ \inf_{B_r(x_{\hat{\varrho}})} \operatorname{ess} \operatorname{tr}\left[a_{\ell_0} \mathrm{D}^2 f_{\hat{\varrho}}\right] \right\} \geq c_{\ell_0} f_{\hat{\varrho}} + \hat{\varrho} h_{\ell_0}$$

at $x_{\hat{\varrho}}$, which is a contradiction since $\hat{\varrho} h_\ell \geq 0$, $f_{\hat{\varrho}} > 0$, and $c_{\ell_0} > 0$ at $x_{\hat{\varrho}}$. We conclude that $0 = v_{\ell_0} - \mathcal{M}_{\ell_0} v \leq u_{\ell_0} - \mathcal{M}_{\ell_0} u \leq 0$ at x_0. Using the same arguments seen in the proof of uniqueness of the solution to the HJB equation (2.9) (see Subsection 3.1), it can be verified that there is a contradiction with the assumption that there is no loop of zero cost (see Equation (2.7)). From here we conclude that ((3.32)) must hold. Taking $v - u$ and proceeding in the same way as before, we see that u is the unique solution to the HJB equation (2.9). □

4. Verification lemma for ε-PACS control problem and proof of Theorem 2.2

In this section, under Assumptions (H1) and (H3)–(H5), we shall verify that the value function V given in (2.8) agrees with the solution u to the HJB equation (2.9) on $\overline{\mathcal{O}}$. To that end, let us start by showing that the value function V^ε given in (2.18), with $\varepsilon \in (0, 1)$ fixed, agrees with the solution u^ε to (2.20). The proof of this result is presented in two parts; see Lemmas 4.1 and 4.2.

Let us assume that $(X^{\xi,\varsigma}, I^\varsigma)$ evolves as (2.14)–(2.15), with $(\xi, \varsigma) \in \mathcal{U}^\varepsilon \times \mathcal{S}$ and initial state $(\tilde{x}, \tilde{\ell}) \in \mathcal{O} \times \mathbb{I}$. Recall that $\mathcal{U}^\varepsilon$ is defined in (2.13) and $\mathcal{S}$ is the set of elements $\varsigma = (\tau_i, \ell_i)_{i \geq 0}$ that satisfy (2.2). The functional cost $\mathcal{V}_{\xi,\varsigma}$ is as in (2.17).

Under Assumptions (H1)–(H4), Lemmas 4.1 and 4.3 will be proven.

Lemma 4.1. *(Verification lemma for ε-PACS control problem, first part) Let $\varepsilon \in (0, 1)$ be fixed. Then $u^\varepsilon_{\tilde{\ell}}(\tilde{x}) \leq \mathcal{V}_{\zeta,\varsigma}(\tilde{x}, \tilde{\ell})$ for each $(\tilde{x}, \tilde{\ell}) \in \overline{\mathcal{O}} \times \mathbb{I}$ and $(\xi, \varsigma) \in \mathcal{U}^\varepsilon \times \mathcal{S}$. From now on, for simplicity of notation, we replace $X^{\xi,\varsigma}$ by X in the proofs of the results.*

Proof of Lemma 4.1. Let $\{u^{\varepsilon,\delta_{\hat{n}}}\}_{\hat{n} \geq 1}$ be the sequence of unique solutions to the NPDS (3.1), when $\delta = \delta_{\hat{n}}$, which satisfy (3.5). By Proposition 3.1, it is known that $u^{\varepsilon,\delta_{\hat{n}}}_\ell \in \mathrm{C}^{4,\alpha'}(\overline{\mathcal{O}})$ for $\ell \in \mathbb{I}$. Then, using integration by parts and Itô's formula (see Corollary 2 (p. 68) and Theorem 33 (p. 81), respectively, in [17]) in $\mathrm{e}^{-r(t)} u^{\varepsilon,\delta_{\hat{n}}}_{\ell_i}(X_t)$ on $[\tilde{\tau}_i, \tilde{\tau}_{i+1})$, $i \geq 0$, where $\tilde{\tau}_i = \tau_i \wedge \tau$, $\tau = \inf\{t \geq 0 : X_t \in \mathcal{O}\}$ and $r(t)$ is as in (2.4), and taking the expected value, we have that

$$\begin{aligned}
\mathbb{E}_{\tilde{x},\tilde{\ell}}\left[\mathrm{e}^{-r(\tilde{\tau}_i)} u^{\varepsilon,\delta_{\hat{n}}}_{\ell_i}(X_{\tilde{\tau}_i})\right] = \mathbb{E}_{\tilde{x},\tilde{\ell}}\Bigg[&\left[u^{\varepsilon,\delta_{\hat{n}}}_{\ell_1}(X_{\tau_1}) + \vartheta_{\ell_0,\ell_1}\right] \mathbb{1}_{\{\tau_1 = 0, \tau_1 < \tau, i = 0\}} \\
&+ \left[u^{\varepsilon,\delta_{\hat{n}}}_{\ell_0}(X_{\tau_1}) - u^{\varepsilon,\delta_{\hat{n}}}_{\ell_1}(X_{\tau_1}) - \vartheta_{\ell_0,\ell_1}\right] \mathbb{1}_{\{\tau_1 = 0, \tau_1 < \tau, i = 0\}} \\
&+ \Bigg[\mathrm{e}^{-r(t \wedge \tilde{\tau}_1)} u^{\varepsilon,\delta_{\hat{n}}}_{\ell_0}(X_{t \wedge \tilde{\tau}_1}) - \widetilde{\mathcal{M}}_{\ell_0}[0, t \wedge \tilde{\tau}_1; X, u^{\varepsilon,\delta_{\hat{n}}}] \\
&+ \int_0^{t \wedge \tilde{\tau}_1} \mathrm{e}^{-r(s)} \Big[c_{\ell_0}(X_s) u^{\varepsilon,\delta_{\hat{n}}}_{\ell_0}(X_s) \\
&- \mathcal{L}_{\ell_0} u^{\varepsilon,\delta_{\hat{n}}}_{\ell_0}(X_s) + \left\langle \mathrm{D}^1 u^{\varepsilon,\delta_{\hat{n}}}_{\ell_0}(X_s), \dot{\zeta}_s \mathrm{n}_s \right\rangle\Big] \mathrm{d}s\Bigg] \mathbb{1}_{\{\tau_1 \neq 0, i = 0\}}
\end{aligned}$$

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$$+\Big[\mathrm{e}^{-r\left(t\wedge\tilde{\tau}_{i+1}\right)}u^{\varepsilon,\delta_{\hat{n}}}_{\ell_i}\big(X_{t\wedge\tilde{\tau}_{i+1}}\big)-\widetilde{\mathcal{M}}_{\ell_i}\big[\tilde{\tau}_i,\, t\wedge\tilde{\tau}_{i+1}; X,\, u^{\varepsilon,\delta_{\hat{n}}}\big]$$

$$+\int_{\tilde{\tau}_i}^{t\wedge\tilde{\tau}_{i+1}}\mathrm{e}^{-r(s)}\Big[c_{\ell_i}(X_s)u^{\varepsilon,\delta_{\hat{n}}}_{\ell_i}(X_s)$$

$$-\mathcal{L}_{\ell_i}u^{\varepsilon,\delta_{\hat{n}}}_{\ell_i}(X_s)+\Big\langle \mathrm{D}^1u^{\varepsilon,\delta_{\hat{n}}}_{\ell_i}(X_s),\,\dot{\zeta}_s\mathrm{n}_s\Big\rangle\Big]\mathrm{d}s\Big]\mathbb{1}_{\{i\neq 0\}}\Big],\tag{4.1}$$

where

$$\widetilde{\mathcal{M}}_{\ell_i}\big[\tilde{\tau}_i,\, t\wedge\tilde{\tau}_{i+1}; X,\, u^{\varepsilon,\delta_{\hat{n}}}\big]:=\int_{\tilde{\tau}_i}^{t\wedge\tilde{\tau}_{i+1}}\mathrm{e}^{-r(s)}\Big\langle \mathrm{D}^1u^{\varepsilon,\delta_{\hat{n}}}_{\ell_i}(X_s),\,\sigma_{\ell_i}(X_s)\Big\rangle\mathrm{d}W_s.\tag{4.2}$$

Since $\widetilde{\mathcal{M}}_{\ell_i}\big[\tilde{\tau}_i,\, t\wedge\tilde{\tau}_{i+1}; X,\, u^{\varepsilon,\delta_{\hat{n}}}\big]$ is a square-integrable martingale,

$$\langle\gamma,\, y\rangle\le\psi_\varepsilon\big(|\gamma|^2-g_\ell(x)^2\big)+l^\varepsilon_\ell(y,\, x),$$

and

$$[c_\ell-\mathcal{L}_\ell]u^{\varepsilon,\delta_{\hat{n}}}_\ell+\psi_\varepsilon\Big(\big|\mathrm{D}^1u^{\varepsilon,\delta_{\hat{n}}}_\ell\big|^2-g^2_\ell\Big)\le h_\ell,$$

it follows that

$$\mathbb{E}_{\tilde{x},\tilde{\ell}}\Big[\mathrm{e}^{-r\left(\tilde{\tau}_i\right)}u^{\varepsilon,\delta_{\hat{n}}}_{\ell_i}\big(X_{\tilde{\tau}_i}\big)\Big]\le\mathbb{E}_{\tilde{x},\tilde{\ell}}\Big[\Big[u^{\varepsilon,\delta_{\hat{n}}}_{\ell_1}\big(X_{\tau_1}\big)+\vartheta_{\ell_0,\ell_1}\Big]\mathbb{1}_{\{\tau_1=0,\tau_1<\tau,i=0\}}$$

$$+\mathcal{D}\Big[\tau_1,\,\ell_0,\,\ell_1; X,\, u^{\varepsilon,\delta_{\hat{n}}}\Big]\mathbb{1}_{\{\tau_1=0,\tau_1<\tau,i=0\}}$$

$$+\mathrm{e}^{-r(t\wedge\tilde{\tau}_1)}\Big[u^{\varepsilon,\delta_{\hat{n}}}_{\ell_1}\big(X_{t\wedge\tilde{\tau}_1}\big)+\vartheta_{\ell_0,\ell_1}\Big]\mathbb{1}_{\{\tau_1\neq 0,i=0\}}$$

$$+\mathrm{e}^{-r(t\wedge\tilde{\tau}_{i+1})}\Big[u^{\varepsilon,\delta_{\hat{n}}}_{\ell_{i+1}}\big(X_{t\wedge\tilde{\tau}_{i+1}}\big)+\vartheta_{\ell_i,\ell_{i+1}}\Big]\mathbb{1}_{\{i\neq 0\}}$$

$$+\mathcal{D}\Big[t\wedge\tilde{\tau}_1,\,\ell_0,\,\ell_1; X,\, u^{\varepsilon,\delta_{\hat{n}}}\Big]\mathbb{1}_{\{\tau_1\neq 0,i=0\}}$$

$$+\mathcal{D}\Big[t\wedge\tilde{\tau}_{i+1},\,\ell_i,\,\ell_{i+1}; X,\, u^{\varepsilon,\delta_{\hat{n}}}\Big]\mathbb{1}_{\{i\neq 0\}}$$

$$+\mathbb{1}_{\{\tau_1\neq 0,i=0\}}\int_0^{t\wedge\tilde{\tau}_1}\mathrm{e}^{-r(s)}\Big[h_{\ell_0}(X_s)+l^\varepsilon_{\ell_0}\big(\dot{\zeta}_s\mathrm{n}_s,\, X_s\big)\Big]\mathrm{d}s$$

$$+\mathbb{1}_{\{i\neq 0\}}\int_{\tilde{\tau}_i}^{t\wedge\tilde{\tau}_{i+1}}\mathrm{e}^{-r(s)}\Big[h_{\ell_i}\big(X_s\big)+l^\varepsilon_{\ell_i}\big(\dot{\zeta}_s\mathrm{n}_s,\, X_s\big)\Big]\mathrm{d}s\Big],\tag{4.3}$$

where

$$\mathcal{D}\big[t\wedge\tilde{\tau}_{i+1},\,\ell_i,\,\ell_{i+1}; X,\, u^{\varepsilon,\delta_{\hat{n}}}\big]:=\mathrm{e}^{-r\left(t\wedge\tilde{\tau}_{i+1}\right)}\Big[u^{\varepsilon,\delta_{\hat{n}}}_{\ell_i}\big(X_{t\wedge\tilde{\tau}_{i+1}}\big)-\Big[u^{\varepsilon,\delta_{\hat{n}}}_{\ell_{i+1}}\big(X_{t\wedge\tilde{\tau}_{i+1}}\big)+\vartheta_{\ell_i,\ell_{i+1}}\Big]\Big].\tag{4.4}$$

We have

$$\max\Big\{\mathrm{e}^{-r(t\wedge\tilde{\tau}_i)}u^{\varepsilon,\delta_{\hat{n}}}_{\ell_i}\big(X_{t\wedge\tilde{\tau}_i}\big),\,\big|\mathcal{D}\big[t\wedge\tilde{\tau}_{i+1},\,\ell_i,\,\ell_{i+1},\, X,\, u^{\varepsilon,\delta_{\hat{n}}}\big]\big|\Big\}\le 2C_1+\max_{(x,\ell)\in\mathcal{O}\times\mathbb{I}}c_\ell(x)$$

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by Lemma 3.1. Meanwhile, we know that

$$\max_{(x,\ell)\in\mathcal{O}\times\mathbb{I}} \left|u_\ell^{\varepsilon,\delta_{\hat{n}}}(x) - u_\ell^\varepsilon(x)\right| \underset{\delta_{\hat{n}}\to 0}{\longrightarrow} 0.$$

Then, letting first $\delta_{\hat{n}}\to 0$ and then $t\to\infty$ in (4.3), and by the dominated convergence theorem, we obtain that

$$\begin{aligned}
&\mathbb{E}_{\tilde{x},\tilde{\ell}}\Big[\mathrm{e}^{-r\left(\tilde{\tau}_i\right)} u_{\ell_i}^\varepsilon(X_{\tilde{\tau}_i})\Big] \\
&\leq \mathbb{E}_{\tilde{x},\tilde{\ell}}\Bigg[\Big\{\mathrm{e}^{-r(\tau_{i+1})} u_{\ell_{i+1}}^\varepsilon\big(X_{\tau_{i+1}}\big) + \mathcal{D}\big[\tau_{i+1},\,\ell_i,\,\ell_{i+1};X,\,u^\varepsilon\big]\Big\}\mathbb{1}_{\{\tau_{i+1}<\tau\}} \\
&\quad + \int_{\tilde{\tau}_i}^{\tilde{\tau}_{i+1}} \mathrm{e}^{-r(s)}\Big[h_{\ell_i}(X_s) + l_{\ell_i}^\varepsilon\big(\dot{\zeta}_s \mathrm{n}_s,\,X_s\big)\Big]\mathrm{d}s + \mathrm{e}^{-r(\tau_{i+1})}\vartheta_{\ell_i,\ell_{i+1}}\mathbb{1}_{\{\tau_{i+1}<\tau\}}\Bigg],
\end{aligned} \tag{4.5}$$

for $i\geq 0$. Observe that $\mathcal{D}\big[\tau_{i+1},\,\ell_i,\,\ell_{i+1},\,X,\,u^\varepsilon\big]\mathbb{1}_{\{\tau_{i+1}<\tau\}}\leq 0$ for $i\geq 0$ since $u_\ell^\varepsilon - \big(u_\kappa^\varepsilon + \vartheta_{\ell,\kappa}\big)\leq u_\ell^\varepsilon - \mathcal{M}_\ell u^\varepsilon \leq 0$. From this remark and (4.5), we deduce the statement of the lemma above. □

4.1. ε-PACS optimal control problem

Before presenting the second part of the verification lemma and proving it, we first construct the control $\big(\xi^{\varepsilon,*},\,\varsigma^{\varepsilon,*}\big)$ which turns out to be the optimal strategy for the ε-PACS control problem. To that end, let us introduce the switching regions.

For any $\ell\in\mathbb{I}$, let $\mathcal{S}_\ell^\varepsilon$ be the set defined by

$$\mathcal{S}_\ell^\varepsilon = \Big\{x\in\mathcal{O} : u_\ell^\varepsilon(x) - \mathcal{M}_\ell u^\varepsilon(x) = 0\Big\}.$$

Notice that $\mathcal{S}_\ell^\varepsilon$ is a closed subset of $\mathcal{O}$ and corresponds to the region where it is optimal to switch regimes. The complement $\mathcal{C}_\ell^\varepsilon$ of $\mathcal{S}_\ell^\varepsilon$ in $\mathcal{O}$, where it is optimal to stay in the regime ℓ, is the so-called continuation region

$$\mathcal{C}_\ell^\varepsilon = \Big\{x\in\mathcal{O} : u_\ell^\varepsilon(x) - \mathcal{M}_\ell u^\varepsilon(x) < 0\Big\}.$$

Remark 4.1. Observe that $u_\ell^\varepsilon \in W_{\mathrm{loc}}^{2,\infty}\big(\mathcal{C}_\ell^\varepsilon\big) = C_{\mathrm{loc}}^{1,1}\big(\mathcal{C}_\ell^\varepsilon\big)$. This implies that

$$h_\ell - \psi_\varepsilon\Big(\Big|\mathrm{D}^1 u_\ell^\varepsilon\Big|^2 - g_\ell^2\Big) \in \mathrm{C}_{\mathrm{loc}}^{0,\alpha'}\big(\mathcal{C}_\ell^\varepsilon\big),$$

since $h_\ell \in \mathrm{C}^{2,\alpha'}(\overline{\mathcal{O}}))$ and $\psi_\varepsilon\Big(\big|\mathrm{D}^1 u_\ell^\varepsilon\big|^2 - g_\ell^2\Big) \in \mathrm{C}_{\mathrm{loc}}^{0,\alpha'}\big(\mathcal{C}_\ell^\varepsilon\big)$. From here and using Theorem 9.19 of [10], we have that $u_\ell^\varepsilon \in \mathrm{C}_{\mathrm{loc}}^{2,\alpha'}\big(\mathcal{C}_\ell^\varepsilon\big)$.

Remark 4.2. Notice that there are no isolated points in a switching region $\mathcal{S}_\ell^\varepsilon$.

Also, the set $\mathcal{S}_\ell^\varepsilon$ satisfies the following property.

Lemma 4.2. *Let ℓ be in $\mathbb{I}$. Then*

$$\mathcal{S}_\ell^\varepsilon = \widetilde{\mathcal{S}}_\ell^\varepsilon := \bigcup_{\kappa\in\mathbb{I}\setminus\{\ell\}} \mathcal{S}_{\ell,\kappa}^\varepsilon,$$

where $\mathcal{S}_{\ell,\kappa}^\varepsilon := \big\{x\in\mathcal{C}_\kappa^\varepsilon : u_\ell^\varepsilon(x) = u_\kappa^\varepsilon(x) + \vartheta_{\ell,\kappa}\big\}$.

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Proof. We obtain trivially that $\widetilde{\mathcal{S}}_\ell^\varepsilon \subset \mathcal{S}_\ell^\varepsilon$, since $u_\ell^\varepsilon - u_\kappa^\varepsilon - \vartheta_{\ell,\kappa} \leq u_\ell^\varepsilon - \mathcal{M}_\ell u^\varepsilon \leq 0$ on $\mathcal{O}$ for $\kappa \in \mathbb{I} \setminus \{\ell\}$. If $x \in \mathcal{S}_\ell^\varepsilon$, there is an $\ell_1 \neq \ell$ where $u_\ell^\varepsilon(x) = u_{\ell_1}^\varepsilon(x) + \vartheta_{\ell,\ell_1}$. Notice that x must belong to either $\mathcal{C}_{\ell_1}^\varepsilon$ or $\mathcal{S}_{\ell_1}^\varepsilon$. If $x \in \mathcal{C}_{\ell_1}^\varepsilon$, then $x \in \mathcal{S}_{\ell,\ell_1}^\varepsilon \subset \widetilde{\mathcal{S}}_\ell^\varepsilon$. Otherwise, there is an $\ell_2 \neq \ell_1$ such that $u_{\ell_1}^\varepsilon(x) = u_{\ell_2}^\varepsilon(x) + \vartheta_{\ell_1,\ell_2}$. This implies

$$u_\ell^\varepsilon(x) = u_{\ell_2}^\varepsilon(x) + \vartheta_{\ell,\ell_1} + \vartheta_{\ell_1,\ell_2} \geq u_{\ell_2}^\varepsilon(x) + \vartheta_{\ell,\ell_2},$$

since (2.6) holds. Then, $u_\ell^\varepsilon(x) = u_{\ell_2}^\varepsilon(x) + \vartheta_{\ell,\ell_2}$. Again x must belong to either $\mathcal{C}_{\ell_2}^\varepsilon$ or $\mathcal{S}_{\ell_2}^\varepsilon$. If $x \in \mathcal{C}_{\ell_2}^\varepsilon$, we have $x \in \mathcal{S}_{\ell,\ell_2}^\varepsilon \subset \widetilde{\mathcal{S}}_\ell^\varepsilon$. Otherwise, by the same argument as before, and since the number of regimes is finite, it must be that there is some $\ell_i \neq \ell$ such that $x \in \mathcal{C}_{\ell_i}^\varepsilon$ and $u_\ell^\varepsilon(x) = u_{\ell_i}^\varepsilon(x) + \vartheta_{\ell,\ell_i}$. Therefore $x \in \mathcal{S}_{\ell,\ell_i}^\varepsilon \subset \widetilde{\mathcal{S}}_\ell^\varepsilon$. □

Now we construct the optimal control $\left(\xi^{\varepsilon,*}, \varsigma^{\varepsilon,*}\right)$ to the problem (2.18). Let $\left(\tilde{x}, \tilde{\ell}\right)$ be in $\overline{\mathcal{O}} \times \mathbb{I}$. The dynamics of the process $X_t^{\varepsilon,*} := \left\{X_t^{\varepsilon,*} : t \geq 0\right\}$ and $\left(\xi^{\varepsilon,*}, \varsigma^{\varepsilon,*}\right)$ is given recursively in the following way:

(i) Define $\tau_0^* = 0$ and $\ell_{0-}^* = \tilde{\ell}$. If $\tilde{x} \notin \mathcal{C}_{\tilde{\ell}}^\varepsilon$, take $\tau_1^* := 0$ and pass to Item (ii) because of Lemma 4.2. Otherwise, the process $X^{\varepsilon,*}$ evolves as

$$X_{t\wedge\tilde{\tau}_1^*}^{\varepsilon,*} = \tilde{x} - \int_0^{t\wedge\tilde{\tau}_1^*} \left[b\left(X_s^{\varepsilon,*}, \ell_0^*\right) + \mathrm{n}_s^{\varepsilon,*}\dot{\zeta}_s^{\varepsilon,*}\right]\mathrm{d}s + \int_0^{t\wedge\tilde{\tau}_1^*} \sigma\left(X_s^{\varepsilon,*}, \ell_0^*\right)\mathrm{d}W_s, \quad \text{for } t > 0, \tag{4.6}$$

with $X_0^{\varepsilon,*} = \tilde{x}$, $\tau^* := \inf\left\{t > 0 : X_t^{\varepsilon,*} \notin \mathcal{O}\right\}$,

$$\tilde{\tau}_1^* := \tau_1^* \wedge \tau^*, \quad \text{and} \quad \tau_1^* := \inf\left\{t \geq 0 : X_t^{\varepsilon,*} \in \mathcal{S}_{\ell_0^*}^\varepsilon\right\}. \tag{4.7}$$

The control process $\xi^{\varepsilon,*} = \left(\mathrm{n}^{\varepsilon,*}, \zeta^{\varepsilon,*}\right)$ is defined by

$$\mathrm{n}_t^{\varepsilon,*} = \begin{cases} \dfrac{\mathrm{D}^1 u_{\ell_0^*}^\varepsilon\left(X_t^{\varepsilon,*}\right)}{\left|\mathrm{D}^1 u_{\ell_0^*}^\varepsilon\left(X_t^{\varepsilon,*}\right)\right|}, & \text{if } \left|\mathrm{D}^1 u_{\ell_0^*}^\varepsilon\left(X_t^{\varepsilon,*}\right)\right| \neq 0 \text{ and } t \in \left[0, \tilde{\tau}_1^*\right), \\ \gamma_0, & \text{if } \left|\mathrm{D}^1 u_{\ell_0^*}^\varepsilon\left(X_t^{\varepsilon,*}\right)\right| = 0 \text{ and } t \in \left[0, \tilde{\tau}_1^*\right), \end{cases} \tag{4.8}$$

where $\gamma_0 \in \mathbb{R}^d$ is a fixed unit vector, and $\zeta_t^{\varepsilon,*} = \int_0^t \dot{\zeta}_s^{\varepsilon,*}\mathrm{d}s$, with $t \in \left[0, \tilde{\tau}_1^*\right)$ and

$$\dot{\zeta}_s^{\varepsilon,*} = 2\psi_\varepsilon'\left(\left|\mathrm{D}^1 u_{\ell_0^*}^\varepsilon\left(X_s^{\varepsilon,*}\right)\right|^2 - g_{\ell_0^*}\left(X_s^{\varepsilon,*}\right)^2\right)\left|\mathrm{D}^1 u_{\ell_0^*}^\varepsilon\left(X_s^{\varepsilon,*}\right)\right|. \tag{4.9}$$

(ii) Recursively, letting $i \geq 1$ and defining

$$\begin{aligned} &\ell_i^* \in \underset{\kappa\in\mathbb{I}\setminus\left\{\ell_{i-1}^*\right\}}{\arg\min} \left\{u_\kappa^\varepsilon\left(X_{\tau_i^*}^{\varepsilon,*}\right) + \vartheta_{\ell_{i-1}^*,\kappa}\right\}, \\ &\tilde{\tau}_{i+1}^* = \tau_{i+1}^* \wedge \tau^*, \quad \tau_{i+1}^* = \inf\left\{t > \tau_i^* : X_t^{\varepsilon,*} \in \mathcal{S}_{\ell_i^*}^\varepsilon\right\}, \end{aligned} \tag{4.10}$$

if $\tau_i^* < \tau^*$, the process $X^{\varepsilon,*}$ evolves as

$$\begin{aligned} X_{t\wedge\tilde{\tau}_{i+1}^*}^{\varepsilon,*} = X_{\tau_i^*}^{\varepsilon,*} &- \int_{\tau_i^*}^{t\wedge\tilde{\tau}_{i+1}^*} \left[b\left(X_s^{\varepsilon,*}, \ell_i^*\right) + \mathrm{n}_s^{\varepsilon,*}\dot{\zeta}_s^{\varepsilon,*}\right]\mathrm{d}s \\ &+ \int_{\tau_i^*}^{t\wedge\tilde{\tau}_{i+1}^*} \sigma\left(X_s^{\varepsilon,*}, \ell_i^*\right)\mathrm{d}W_s, \quad \text{for } t \geq \tau_i^*, \end{aligned} \tag{4.11}$$

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where

$$\mathfrak{n}_t^{\varepsilon,*} = \begin{cases} \dfrac{\mathrm{D}^1 u^{\varepsilon}_{\ell_i^*}\left(X_t^{\varepsilon,*}\right)}{\left|\mathrm{D}^1 u^{\varepsilon}_{\ell_i^*}\left(X_t^{\varepsilon,*}\right)\right|}, & \text{if } \left|\mathrm{D}^1 u^{\varepsilon}_{\ell_i^*}\left(X_t^{\varepsilon,*}\right)\right| \neq 0 \text{ and } t \in \left[\tau_i^*, \tilde{\tau}_{i+1}^*\right), \\ \gamma_0, & \text{if } \left|\mathrm{D}^1 u^{\varepsilon}_{\ell_i^*}\left(X_t^{\varepsilon,*}\right)\right| = 0 \text{ and } t \in \left[\tau_i^*, \tilde{\tau}_{i+1}^*\right), \end{cases} \tag{4.12}$$

with $\gamma_0 \in \mathbb{R}^d$ being a fixed unit vector, and $\zeta_t^{\varepsilon,*} = \int_{\tau_i^{\varepsilon,*}}^{t} \dot{\zeta}_s^{\varepsilon,*} \mathrm{d}s$, with $t \in \left[\tau_i^*, \tilde{\tau}_{i+1}^*\right)$ and

$$\dot{\zeta}_s^{\varepsilon,*} = 2\psi_\varepsilon'\left(\left|\mathrm{D}^1 u^{\varepsilon}_{\ell_i^*}\left(X_s^{\varepsilon,*}\right)\right|^2 - g_{\ell_i^*}\left(X_s^{\varepsilon,*}\right)^2\right)\left|\mathrm{D}^1 u^{\varepsilon}_{\ell_i^*}\left(X_s^{\varepsilon,*}\right)\right|. \tag{4.13}$$

Remark 4.3. Suppose that $\tau_i^* < \tau^*$ for some $i > 0$. We notice that for $t \in \left[\tau_i^*, \tau_{i+1}^*\right)$,

$$\mathfrak{n}_t^{\varepsilon,*}\dot{\zeta}_t^{\varepsilon,*} = 2\psi_\varepsilon'\left(\left|\mathrm{D}^1 u^{\varepsilon}_{\ell_i^*}(X_t^{\varepsilon,*})\right|^2 - g(X_t^{\varepsilon,*})^2\right)\mathrm{D}^1 u^{\varepsilon}_{\ell_i^*}(X_t^{\varepsilon,*}),$$

$\Delta\zeta_t^{\varepsilon,*} = 0$, $|\mathfrak{n}_t^{\varepsilon,*}| = 1$, and, by (2.16) and (3.25), this yields that $\dot{\zeta}_t^{\varepsilon,*} \leq \frac{2C_4}{\varepsilon}$. Also we see that $X_t \in \mathcal{C}^{\varepsilon}_{\ell_i^*}$ if $t \in \left[\tau_i^*, \tau_{i+1}^*\right)$, by Lemma 4.2.

Remark 4.4. On the event $\{\tau^* = \infty\}$, $\tilde{\tau}_i^* = \tau_i^*$ for $i \geq 0$. From here and by (4.8)–(4.9) and (4.12)–(4.13), we have that the control process $\left(\xi^{\varepsilon,*}, \varsigma^{\varepsilon,*}\right)$ belongs to $\mathcal{U}^{\varepsilon} \times \mathcal{S}$. On the event $\{\tau^* < \infty\}$, let $\hat{\imath}$ be defined as $\hat{\imath} = \max\left\{i \in \mathbb{N} : \tau_i^* \leq \tau^*\right\}$. Then, if we take $\tau_i^* := \tau^* + i$ and $\ell_i^* = \hat{\ell}$ for $i > \hat{\imath}$, where $\hat{\ell} \in \mathbb{I}$ is fixed, it follows that $\varsigma^{\varepsilon,*} = \left(\tau_i^*, \ell_i^*\right)_{i \geq 1} \in \mathcal{S}$. Meanwhile, since (3.25) holds on $\overline{\mathcal{O}}$ and $u_\ell^\varepsilon = 0$ on $\partial\mathcal{O}$, we take $\dot{\zeta}_t^{\varepsilon,*} \equiv 0$ and $\mathfrak{n}_t^{\varepsilon,*} := \gamma_0$, for $t > \tau^*$. In this way, we have that $\left(\mathfrak{n}^{\varepsilon,*}, \zeta^{\varepsilon,*}\right) \in \mathcal{U}^{\varepsilon}$.

Remark 4.5. Taking

$$I_t^{\varepsilon,*} = \tilde{\ell}\,\mathbb{1}_{\left[0,\tau_1^*\right)}(t) + \sum_{i \geq 1} \ell_i^*\,\mathbb{1}_{\left[\tau_i^*,\tau_{i+1}^*\right)}(t),$$

we see that it is a càdlàg process.

Lemma 4.3. (Verification lemma for ε-PACS control problem, second part) *Let $\varepsilon \in (0, 1)$ be fixed and let $(X^{\varepsilon,*}, I^{\varepsilon,*})$ be the process that is governed by (4.6)–(4.13). Then $u_\ell^\varepsilon(\tilde{x}) = \mathcal{V}_{\xi^{\varepsilon,*},\varsigma^{\varepsilon,*}}\left(\tilde{x}, \tilde{\ell}\right) = V_{\tilde{\ell}}^{\varepsilon}(\tilde{x})$ for each $\left(\tilde{x}, \tilde{\ell}\right) \in \overline{\mathcal{O}} \times \mathbb{I}$.*

Proof. Take $\hat{\tau}_i^{*,q} := \tau_i^* \wedge \inf\left\{t > \tau_{i-1}^* : X_t^{\varepsilon,*} \notin \mathcal{O}_q\right\}$, with $\mathcal{O}_q := \{x \in \mathcal{O} : \operatorname{dist}(x, \partial\mathcal{O}) > 1/q\}$ and q a positive integer large enough. By Remarks 4.1 and 4.3, it is known that u_ℓ^ε is a C^2 function on $\mathcal{C}_\ell^\varepsilon$ and that $X_t^{\varepsilon,*} \in \mathcal{C}^{\varepsilon}_{\ell_i^*}$ if $t \in \left[\hat{\tau}_i^{*,q}, \hat{\tau}_{i+1}^{*,q}\right)$. Then, using integration by parts and Itô's formula in $\mathrm{e}^{-r(t)} u^{\varepsilon}_{\ell_i^*}\left(X_t^{\varepsilon,*}\right)$ on the interval $\left[\hat{\tau}_i^{*,q}, \hat{\tau}_{i+1}^{*,q}\right)$, and taking expected value on it, we obtain a similar expression to that of (4.1). Now, considering that $\widetilde{\mathcal{M}}_{\ell_i^*}\left[\hat{\tau}_i^{*,q}, t \wedge \hat{\tau}_{i+1}^{*,q}; X^{\varepsilon,*}, u^{\varepsilon}\right]$, which was defined in (4.2), is a square-integrable martingale, and since

$$[c_{\ell_i^*} - \mathcal{L}_{\ell_i^*}]u^{\varepsilon}_{\ell_i^*} = h_{\ell_i^*} - \psi_\varepsilon\left(\left|\mathrm{D}^1 u^{\varepsilon}_{\ell_i^*}\right|^2 - g^2_{\ell_i^*}\right)$$

on $\mathcal{C}^{\varepsilon}_{\ell_i^*}$ and the supremum of $l_\ell^\varepsilon(\eta, x)$ is attained if γ is related to η by $\eta = 2\psi_\varepsilon'(|\gamma|^2 - g_\ell(x)^2)\gamma$, i.e.,

$$l_\ell^\varepsilon\left(2\psi_\varepsilon'\left(|\gamma|^2 - g_\ell(x)^2\right)\gamma, x\right) = 2\psi_\varepsilon'\left(|\gamma|^2 - g_\ell(x)^2\right)|\gamma|^2 - \psi_\varepsilon\left(|\gamma|^2 - g_\ell(x)^2\right),$$

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it can be checked that

$$
\begin{aligned}
&\mathbb{E}_{\tilde{x},\tilde{\ell}}\Big[\mathrm{e}^{-r(\hat{\tau}_i^{*,q})}u^{\varepsilon}_{\ell_i^*}\Big(X^{\varepsilon,*}_{\hat{\tau}_i^{*,q}}\Big)\Big]\\
&=\mathbb{E}_{\tilde{x},\tilde{\ell}}\Bigg[\big[u^{\varepsilon}_{\ell_1^*}(X^{\varepsilon,*}_{\tau_1^*})+\vartheta_{\ell_0^*,\ell_1^*}\big]\mathbb{1}_{\{\tau_1^*=0,\tau_1^*<\tau^*,i=0\}}\\
&\quad+\big[u^{\varepsilon}_{\ell_0^*}(X^{\varepsilon,*}_{\tau_1^*})-u^{\varepsilon}_{\ell_1^*}(X^{\varepsilon,*}_{\tau_1^*})-\vartheta_{\ell_0^*,\ell_1^*}\big]\mathbb{1}_{\{\tau_1^*=0,\tau_1^*<\tau^*,i=0\}}\\
&\quad+\mathrm{e}^{-r\left(t\wedge\hat{\tau}_1^{*,q}\right)}u^{\varepsilon}_{\ell_0^*}\Big(X^{\varepsilon,*}_{t\wedge\hat{\tau}_1^{*,q}}\Big)\mathbb{1}_{\{\tau_1^*\neq0,i=0\}}+\mathrm{e}^{-r\left(t\wedge\hat{\tau}_{i+1}^{*,q}\right)}u^{\varepsilon}_{\ell_i^*}\Big(X^{\varepsilon,*}_{t\wedge\hat{\tau}_{i+1}^{*,q}}\Big)\mathbb{1}_{\{i\neq0\}}\\
&\quad+\mathbb{1}_{\{\tau_1^*\neq0,i=0\}}\int_0^{t\wedge\hat{\tau}_1^{*,q}}\mathrm{e}^{-r(s)}[h_{\ell_0^*}(X^{\varepsilon,*}_s)+l^{\varepsilon}_{\ell_0^*}(\dot{\zeta}^{\varepsilon,*}_s\mathrm{n}^{\varepsilon,*}_s,X^{\varepsilon,*}_s)]\mathrm{d}s\\
&\quad+\mathbb{1}_{\{i\neq0\}}\int_{\hat{\tau}_i^{*,q}}^{t\wedge\hat{\tau}_{i+1}^{*,q}}\mathrm{e}^{-r(s)}[h_{\ell_i^*}(X^{\varepsilon,*}_s)+l^{\varepsilon}_{\ell_i^*}(\dot{\zeta}^{\varepsilon,*}_s\mathrm{n}^{\varepsilon,*}_s,X^{\varepsilon,*}_s)]\mathrm{d}s\Bigg].
\end{aligned}
\tag{4.14}
$$

Notice that $\hat{\tau}_i^{*,q}\uparrow\tilde{\tau}_i^*$ as $q\to\infty$, $\mathbb{P}_{\tilde{x}}$-a.s. Consequently, letting first $q\to\infty$ and then $t\to\infty$ in (4.14), we see that

$$
\begin{aligned}
&\mathbb{E}_{\tilde{x},\tilde{\ell}}\Big[\mathrm{e}^{-r(\tilde{\tau}_i^*)}u^{\varepsilon}_{\ell_i^*}\Big(X^{\varepsilon,*}_{\tilde{\tau}_i^*}\Big)\Big]\\
&=\mathbb{E}_{\tilde{x},\tilde{\ell}}\Bigg[\Big\{\mathrm{e}^{-r(\tau_{i+1}^*)}u^{\varepsilon}_{\ell_{i+1}^*}\Big(X^{\varepsilon,*}_{\tau_{i+1}^*}\Big)+\mathcal{D}[\tau_{i+1}^*,\ell_i^*,\ell_{i+1}^*,X^{\varepsilon,*},u^{\varepsilon}]\Big\}\mathbb{1}_{\{\tau_{i+1}^*<\tau^*\}}\\
&\quad+\int_{\tilde{\tau}_i^*}^{\tilde{\tau}_{i+1}^*}\mathrm{e}^{-r(s)}[h_{\ell_i^*}(X^{\varepsilon,*}_s)+l^{\varepsilon}_{\ell_i^*}(\dot{\zeta}^{\varepsilon,*}_s\mathrm{n}^{\varepsilon,*}_s,X^{\varepsilon,*}_s)]\mathrm{d}s+\mathrm{e}^{-r(\tau_{i+1}^*)}\vartheta_{\ell_i^*,\ell_{i+1}^*}\mathbb{1}_{\{\tau_{i+1}^*<\tau^*\}}\Bigg]
\end{aligned}
\tag{4.15}
$$

for $i\geq0$, with $\mathcal{D}\big[\tau_{i+1}^*,\ell_i^*,\ell_{i+1}^*,X^{\varepsilon,*},u^{\varepsilon}\big]$ as in (4.4). By (4.7) and (4.10),

$$
\mathcal{D}\big[\tau_{i+1}^*,\ell_i^*,\ell_{i+1}^*,X^{\varepsilon,*},u^{\varepsilon}\big]\mathbb{1}_{\left\{\tau_{i+1}^*<\tau^*\right\}}=0.
$$

Therefore, from here and (4.15), we obtain the desired result that was given in the lemma above. □

4.2. Proof of Theorem 2.2

To conclude this section, we present the proof of Theorem 2.2, which will be proven under Assumptions (H1) and (H3)–(H5). Recall that $\mathcal{U}$ and $\mathcal{S}$ are the families of controls $\xi=(\mathrm{n},\zeta)$ and $\varsigma=(\tau_i,\ell_i)_{i\geq0}$ that satisfy (2.1) and (2.2), respectively. The functional cost $V_{\xi,\varsigma}$ is given by (1.2) for $(\xi,\varsigma)\in\mathcal{U}\times\mathcal{S}$.

Proof of Theorem 2.2. Let $\{u^{\varepsilon_n}\}_{n\geq1}$ be the sequence of unique strong solutions to the HJB equation (2.20), when $\varepsilon=\varepsilon_n$, which satisfy (3.27). From Lemma 4.3, we know that

$$
u^{\varepsilon_n}_{\tilde{\ell}}(\tilde{x})=\mathcal{V}_{\xi^{\varepsilon_n,*},\varsigma^{\varepsilon_n,*}}\big(\tilde{x},\tilde{\ell}\big)=V^{\varepsilon_n}\big(\tilde{x},\tilde{\ell}\big)\quad\text{for }\big(\tilde{x},\tilde{\ell}\big)\in\overline{\mathcal{O}}\times\mathbb{I},
$$

with $(\xi^{\varepsilon_n,*},\varsigma^{\varepsilon_n,*})$ as in (4.7)–(4.10) and (4.12)–(4.13), when $\varepsilon=\varepsilon_n$. Notice that

$$
l_{\varepsilon_n}(x,\beta\gamma)\geq\langle\beta\gamma,g_{\ell_i^*}(x)\gamma\rangle-\psi_{\varepsilon_n}\big(|g_{\ell_i^*}(x)\gamma|^2-g_{\ell_i^*}(x)^2\big)=\beta g_{\ell_i^*}(x),
$$

with $\beta \in \mathbb{R}$ and $\gamma \in \mathbb{R}^d$ a unit vector. Then, from here and considering $(X^{\varepsilon_n,*}, I^{\varepsilon_n,*})$ governed by (4.6) and (4.11), it follows that

$$\begin{aligned} V_{\tilde{\ell}}(\tilde{x}) &\leq V_{\xi^{\varepsilon_n,*},\varsigma^{\varepsilon_n,*}}(\tilde{x}, \tilde{\ell}) \\ &= \sum_{i\geq 0} \mathbb{E}_{\tilde{x},\tilde{\ell}}\bigg[\int_{\tilde{\tau}_i^*}^{\tilde{\tau}_{i+1}^*} \mathrm{e}^{-r(s)}[h_{\ell_i^*}(X_s^{\varepsilon_n,*})\mathrm{d}s + \dot{\zeta}_s^{\varepsilon_n,*} g_{\ell_i^*}(X_{s-}^{\varepsilon_n,*})]\mathrm{d}s \\ &\quad + \mathrm{e}^{-r(\tau_{i+1}^*)} \vartheta_{\ell_i^*,\ell_{i+1}^*} \mathbb{1}_{\{\tau_{i+1}^*<\tau^*\}}\bigg] \\ &\leq \sum_{i\geq 0} \mathbb{E}_{\tilde{x},\tilde{\ell}}\bigg[\int_{\tilde{\tau}_i^*}^{\tilde{\tau}_{i+1}^*} \mathrm{e}^{-r(s)}[h_{\ell_i^*}(X_s^{\varepsilon_n,*}) + l_{\ell_i^*}^{\varepsilon_n}(\dot{\zeta}_s^{\varepsilon_n,*} \mathrm{n}_s^{\varepsilon_n,*}, X_s^{\varepsilon_n,*})]\mathrm{d}s \\ &\quad + \mathrm{e}^{-r(\tau_{i+1}^*)} \vartheta_{\ell_i^*,\ell_{i+1}^*} \mathbb{1}_{\{\tau_{i+1}^*<\tau^*\}}\bigg] = u_{\tilde{\ell}}^{\varepsilon_n}(\tilde{x}). \end{aligned} \tag{4.16}$$

Notice that $V_{\xi^{\varepsilon_n,*},\varsigma^{\varepsilon_n,*}}$ is the cost function given in (1.2) corresponding to the control $(\xi^{\varepsilon_n,*}, \varsigma^{\varepsilon_n,*})$ where the second term in the right-hand side of (2.5) is zero, since $\zeta^{\varepsilon_n,*}$ has the continuous part only. Letting $\varepsilon_n \to 0$ in (4.16), we have $u_{\tilde{\ell}}(\tilde{x}) \geq V_{\tilde{\ell}}(\tilde{x})$ for each $(\tilde{x}, \tilde{\ell}) \in \overline{\mathcal{O}} \times \mathbb{I}$.

Let $\{u^{\varepsilon_n,\delta_{\hat{n}}}\}_{n,\hat{n}\geq 1}$ be the sequence of unique solutions to the NPDS (3.1), when $\varepsilon = \varepsilon_n$ and $\delta = \delta_{\hat{n}}$, which satisfy (3.5) and (3.27). Let us assume (X, I) evolves as in (1.1) with initial state $(\tilde{x}, \tilde{\ell}) \in \overline{\mathcal{O}} \times \mathbb{I}$ and the control process (ξ, ς) belongs to $\mathcal{U} \times \mathcal{S}$. Take $\hat{\tau}_i^q := \tau_i \wedge \inf\{t > \tau_{i-1} : X_t \notin \mathcal{O}_q\}$, with $i \geq 1$, $\mathcal{O}_q := \{x \in \mathcal{O} : dist(x, \partial\mathcal{O}) > 1/q\}$, and q a positive integer large enough. Using integration by parts and Itô's formula in $\mathrm{e}^{-r(t)} u_{\ell_i}^{\varepsilon_n,\delta_{\hat{n}}}(X_t)$ on $[\hat{\tau}_i^q, \hat{\tau}_{i+1}^q)$, $i \geq 0$, we get that

$$\begin{aligned} &\mathrm{e}^{-r(\hat{\tau}_i^q)} u_{\ell_i}^{\varepsilon_n,\delta_{\hat{n}}}(X_{\hat{\tau}_i^q}) \\ &= \Big[u_{\ell_1}^{\varepsilon_n,\delta_{\hat{n}}}(X_{\tau_1}) + \vartheta_{\ell_0,\ell_1}\Big] \mathbb{1}_{\{\tau_1=0,\tau_1<\tau,i=0\}} \\ &\quad + \Big[u_{\ell_0}^{\varepsilon_n,\delta_{\hat{n}}}(X_{\tau_1}) - u_{\ell_1}^{\varepsilon_n,\delta_{\hat{n}}}(X_{\tau_1}) - \vartheta_{\ell_0,\ell_1}\Big] \mathbb{1}_{\{\tau_1=0,\tau_1<\tau,i=0\}} \\ &\quad + \mathrm{e}^{-r(t\wedge\hat{\tau}_1^q)} u_{\ell_0}^{\varepsilon_n,\delta_{\hat{n}}}\Big(X_{t\wedge\hat{\tau}_1^q}\Big) \mathbb{1}_{\{\tau_1\neq 0,i=0\}} + \mathrm{e}^{-r(t\wedge\hat{\tau}_{i+1}^q)} u_{\ell_i}^{\varepsilon_n,\delta_{\hat{n}}}\Big(X_{t\wedge\hat{\tau}_{i+1}^q}\Big) \mathbb{1}_{\{i\neq 0\}} \\ &\quad + \mathbb{1}_{\{\tau_1\neq 0,i=0\}} \int_0^{t\wedge\hat{\tau}_1^q} \mathrm{e}^{-r(s)}\Big[\Big[c_{\ell_0}(X_s) u_{\ell_0}^{\varepsilon_n,\delta_{\hat{n}}}(X_s) \\ &\quad - \mathcal{L}_{\ell_0} u_{\ell_0}^{\varepsilon_n,\delta_{\hat{n}}}(X_s)\Big]\mathrm{d}s + \Big\langle \mathrm{D}^1 u_{\ell_0}^{\varepsilon_n,\delta_{\hat{n}}}(X_s), \mathrm{n}_s \Big\rangle \mathrm{d}\zeta_s^c\Big] \\ &\quad + \mathbb{1}_{\{i\neq 0\}} \int_{\hat{\tau}_i^q}^{t\wedge\hat{\tau}_{i+1}^q} \mathrm{e}^{-r(s)}\Big[[c_{\ell_i}(X_s) u_{\ell_i}^{\varepsilon_n,\delta_{\hat{n}}}(X_s) \\ &\quad - \mathcal{L}_{\ell_i} u_{\ell_i}^{\varepsilon_n,\delta_{\hat{n}}}(X_s)]\mathrm{d}s + \Big\langle \mathrm{D}^1 u_{\ell_i}^{\varepsilon_n,\delta_{\hat{n}}}(X_s), \mathrm{n}_s \Big\rangle \mathrm{d}\zeta_s^c\Big] \\ &\quad - \sum_{0\leq s<t\wedge\hat{\tau}_1^q} \mathrm{e}^{-r(s)} \mathcal{J}\big[s; \ell_0, X, u^{\varepsilon_n,\delta_{\hat{n}}}\big] \mathbb{1}_{\{\tau_1\neq 0,i=0\}} \\ &\quad - \sum_{\hat{\tau}_i^q\leq s<t\wedge\hat{\tau}_{i+1}^q} \mathrm{e}^{-r(s)} \mathcal{J}\big[s; \ell_i, X, u^{\varepsilon_n,\delta_{\hat{n}}}\big] \mathbb{1}_{\{i\neq 0\}} \\ &\quad - \widetilde{\mathcal{M}}_{\ell_0}\Big[0, t\wedge\hat{\tau}_1^q; X, u^{\varepsilon_n,\delta_{\hat{n}}}\Big] \mathbb{1}_{\{\tau_1\neq 0,i=0\}} - \widetilde{\mathcal{M}}_{\ell_i}\big[\hat{\tau}_i^q, t\wedge\hat{\tau}_{i+1}^q; X, u^{\varepsilon_n,\delta_{\hat{n}}}\big] \mathbb{1}_{\{i\neq 0\}}, \end{aligned} \tag{4.17}$$

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where $\widetilde{\mathcal{M}}_{\ell_i}\big[\hat{\tau}_i^q,\, t\wedge\hat{\tau}_{i+1}^q; X,\, u^{\varepsilon_n,\delta_{\hat{n}}}\big]$ is as in (4.2) and

$$\mathcal{J}\big[s;\ell_i, X, u^{\varepsilon_n,\delta_{\hat{n}}}\big] := u_{\ell_i}^{\varepsilon_n,\delta_{\hat{n}}}\big(X_{s-}-\mathrm{n}_s\Delta\zeta_s\big) - u_{\ell_i}^{\varepsilon_n,\delta_{\hat{n}}}(X_{s-}), \qquad \text{for } i\geq 0.$$

Since $X_{s-}-\mathrm{n}_s\Delta\zeta_s\in\mathcal{O}$ for $s\in\Big[\hat{\tau}_i^q,\, t\wedge\hat{\tau}_{i+1}^q\Big)$, $i\geq 0$, and $\mathcal{O}$ is a convex set, by the mean value theorem, we have

$$\begin{aligned}-\mathcal{J}\big[s;\ell_i, X, u^{\varepsilon_n,\delta_{\hat{n}}}\big] &\leq \Big|u_{\ell_i}^{\varepsilon_n,\delta_{\hat{n}}}\big(X_{s-}-\mathrm{n}_s\Delta\zeta_s\big) - u_{\ell_i}^{\varepsilon_n,\delta_{\hat{n}}}(X_{s-})\Big| \\ &\leq \Delta\zeta_s\int_0^1\Big|\mathrm{D}^1u_{\ell_i}^{\varepsilon_n,\delta_{\hat{n}}}(X_{s-}-\lambda\mathrm{n}_s\Delta\zeta_s)\Big|\mathrm{d}\lambda.\end{aligned}$$

Taking the expected value in (4.17), and since $\widetilde{\mathcal{M}}_{\ell_i}\big[\hat{\tau}_i^q,\, t\wedge\hat{\tau}_{i+1}^q; X,\, u^{\varepsilon_n,\delta_{\hat{n}}}\big]$ is a square-integrable martingale and $\big[c_{\ell_i}-\mathcal{L}_{\ell_i}\big]u_{\ell_i}^{\varepsilon_n,\delta_{\hat{n}}}\leq h_{\ell_i}$, we obtain

$$\begin{aligned}&\mathbb{E}_{\tilde{x},\tilde{\ell}}\Big[\mathrm{e}^{-r\left(\hat{\tau}_i^q\right)}u_{\ell_i}^{\varepsilon_n,\delta_{\hat{n}}}\Big(X_{\hat{\tau}_i^q}\Big)\Big] \\ &\leq \mathbb{E}_{\tilde{x},\tilde{\ell}}\Big[\Big[u_{\ell_1}^{\varepsilon_n,\delta_{\hat{n}}}(X_{\tau_1})+\vartheta_{\ell_0,\ell_1}\Big]\mathbb{1}_{\{\tau_1=0,\tau_1<\tau,i=0\}} \\ &\quad+\Big[u_{\ell_0}^{\varepsilon_n,\delta_{\hat{n}}}(X_{\tau_1})-u_{\ell_1}^{\varepsilon_n,\delta_{\hat{n}}}(X_{\tau_1})-\vartheta_{\ell_0,\ell_1}\Big]\mathbb{1}_{\{\tau_1=0,\tau_1<\tau,i=0\}} \\ &\quad+\mathrm{e}^{-r\left(t\wedge\hat{\tau}_1^q\right)}u_{\ell_0}^{\varepsilon_n,\delta_{\hat{n}}}\Big(X_{t\wedge\hat{\tau}_1^q}\Big)\mathbb{1}_{\{\tau_1\neq0,i=0\}}+\mathrm{e}^{-r\left(t\wedge\hat{\tau}_{i+1}^q\right)}u_{\ell_i}^{\varepsilon_n,\delta_{\hat{n}}}\Big(X_{t\wedge\hat{\tau}_{i+1}^q}\Big)\mathbb{1}_{\{i\neq0\}} \\ &\quad+\mathbb{1}_{\{\tau_1\neq0,i=0\}}\int_0^{t\wedge\hat{\tau}_1^q}\mathrm{e}^{-r(s)}\Big[h_{\ell_0}(X_s)\mathrm{d}s+\Big|\mathrm{D}^1u_{\ell_0}^{\varepsilon_n,\delta_{\hat{n}}}(X_s)\Big|\mathrm{d}\zeta_s^c\Big] \\ &\quad+\mathbb{1}_{\{i\neq0\}}\int_{\hat{\tau}_i^q}^{t\wedge\hat{\tau}_{i+1}^q}\mathrm{e}^{-r(s)}\Big[h_{\ell_i}(X_s)\mathrm{d}s+\Big|\mathrm{D}^1u_{\ell_i}^{\varepsilon_n,\delta_{\hat{n}}}(X_s)\Big|\mathrm{d}\zeta_s^c\Big] \\ &\quad+\sum_{0\leq s<t\wedge\hat{\tau}^q}\mathrm{e}^{-r(s)}\Delta\zeta_s\int_0^1\Big|\mathrm{D}^1u_{\ell_0}^{\varepsilon_n,\delta_{\hat{n}}}(X_{s-}-\lambda\mathrm{n}_s\Delta\zeta_s)\Big|\mathrm{d}\lambda\,\mathbb{1}_{\{\tau_1\neq0,i=0\}} \\ &\quad+\sum_{\hat{\tau}_i^q\leq s<t\wedge\hat{\tau}_{i+1}^q}\mathrm{e}^{-r(s)}\Delta\zeta_s\int_0^1\Big|\mathrm{D}^1u_{\ell_i}^{\varepsilon_n,\delta_{\hat{n}}}(X_{s-}-\lambda\mathrm{n}_s\Delta\zeta_s)\Big|\mathrm{d}\lambda\,\mathbb{1}_{\{i\neq0\}}\Big].\end{aligned} \tag{4.18}$$

Notice that for $s\in\big[\hat{\tau}_i^q,\, t\wedge\hat{\tau}_{i+1}^q\big)$,

$$\begin{aligned}&\Big|u_{\ell_i}^{\varepsilon_n,\delta_{\hat{n}}}(X_s)-u_{\ell_i}(X_s)\Big|\leq\max_{(x,\ell)\in\mathcal{O}_q\times\mathbb{I}}\Big|u_{\ell}^{\varepsilon_n,\delta_{\hat{n}}}(x)-u_\ell(x)\Big|\underset{\varepsilon_n,\delta_{\hat{n}}\to0}{\longrightarrow}0, \\ &\Big|\partial_ju_{\ell_i}^{\varepsilon_n,\delta_{\hat{n}}}(X_s)-\partial_ju_{\ell_i}(X_s)\Big|\leq\max_{(x,\ell)\in\mathcal{O}_q\times\mathbb{I}}\Big|\partial_ju_{\ell}^{\varepsilon_n,\delta_{\hat{n}}}(x)-\partial_ju_\ell(x)\Big|\underset{\varepsilon_n,\delta_{\hat{n}}\to0}{\longrightarrow}0, \\ &\Big|\partial_ju_{\ell_i}^{\varepsilon_n,\delta_{\hat{n}}}\big(X_s-\lambda\mathrm{n}_s\Delta\zeta_s\big)-\partial_ju_{\ell_i}\big(X_s-\lambda\mathrm{n}_s\Delta\zeta_s\big)\Big| \\ &\qquad\qquad\leq\max_{(x,\ell)\in\mathcal{O}_q\times\mathbb{I}}\Big|\partial_ju_{\ell}^{\varepsilon_n,\delta_{\hat{n}}}(x)-\partial_ju_\ell(x)\Big|\underset{\varepsilon_n,\delta_{\hat{n}}\to0}{\longrightarrow}0,\end{aligned}$$

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with $\lambda \in [0, 1]$. Then, letting $\varepsilon_n, \delta_{\hat{n}} \to 0$ in (4.18), by the dominated convergence theorem and using $|D^1 u_{\ell_i}| \le g_{\ell_i}$, we have

$$\begin{aligned}
\mathbb{E}_{\tilde{x},\tilde{\ell}}\Big[\mathrm{e}^{-r\left(\hat{\tau}_i^q\right)} u_{\ell_i}\big(X_{\hat{\tau}_i^q}\big)\Big] \le \mathbb{E}_{\tilde{x},\tilde{\ell}}\Big[&[u_{\ell_1}(X_{\tau_1}) + \vartheta_{\ell_0,\ell_1}]\mathbb{1}_{\{\tau_1=0,\tau_1<\tau,i=0\}} \\
&+ \Big[u_{\ell_0}(X_{\tau_1}) - u_{\ell_1}(X_{\tau_1}) - \vartheta_{\ell_0,\ell_1}\Big]\mathbb{1}_{\{\tau_1=0,\tau_1<\tau,i=0\}} \\
&+ \mathrm{e}^{-r\left(t\wedge\hat{\tau}_1^q\right)} u_{\ell_0}\Big(X_{t\wedge\hat{\tau}_1^q}\Big)\mathbb{1}_{\{\tau_1\neq 0,i=0\}} + \mathrm{e}^{-r\left(t\wedge\hat{\tau}_{i+1}^q\right)} u_{\ell_i}\Big(X_{t\wedge\hat{\tau}_{i+1}^q}\Big)\mathbb{1}_{\{i\neq 0\}} \\
&+ \mathbb{1}_{\{\tau_1\neq 0,i=0\}}\int_0^{t\wedge\hat{\tau}_1^q} \mathrm{e}^{-r(s)}\big[h_{\ell_0}(X_s)\mathrm{d}s + g_{\ell_0}(X_{s-})\circ \mathrm{d}\zeta_s^c\big] \\
&+ \mathbb{1}_{\{i\neq 0\}}\int_{\hat{\tau}_i^q}^{t\wedge\hat{\tau}_{i+1}^q} \mathrm{e}^{-r(s)}\big[h_{\ell_i}(X_s)\mathrm{d}s + g_{\ell_i}(X_{s-})\circ \mathrm{d}\zeta_s\big]\Big],
\end{aligned} \tag{4.19}$$

with $g_{\ell_i}(X_{s-})\circ \mathrm{d}\zeta_{s-}$ as in (2.5). Again, letting $q\to\infty$ and $t\to\infty$ in (4.19), we have

$$\begin{aligned}
&\mathbb{E}_{\tilde{x},\tilde{\ell}}\Big[\mathrm{e}^{-r\left(\tilde{\tau}_i\right)} u_{\ell_i}\big(X_{\tilde{\tau}_i}\big)\Big] \\
&\le \mathbb{E}_{\tilde{x},\tilde{\ell}}\Big[\Big\{\mathrm{e}^{-r(\tau_{i+1})} u_{\ell_{i+1}}\big(X_{\tau_{i+1}}\big) + \mathcal{D}\big[\tau_{i+1}, \ell_i, \ell_{i+1}, X, u\big]\Big\}\mathbb{1}_{\{\tau_{i+1}<\tau\}} \\
&\quad + \int_{\tilde{\tau}_i}^{\tilde{\tau}_{i+1}} \mathrm{e}^{-r(s)}\Big[h_{\ell_i}(X_s)\mathrm{d}s + g_{\ell_i}(X_{s-})\circ \mathrm{d}\zeta_s\Big] + \mathrm{e}^{-r(\tau_{i+1})}\vartheta_{\ell_i,\ell_{i+1}}\mathbb{1}_{\{\tau_{i+1}<\tau\}}\Big]
\end{aligned} \tag{4.20}$$

for $i \ge 0$, with $\tilde{\tau}_i = \tau_i \wedge \tau$ and $\mathcal{D}[\tau_{i+1}, \ell_i, \ell_{i+1}, X, u]$ as in (4.4). Noticing that

$$\mathcal{D}[\tau_{i+1}, \ell_i, \ell_{i+1}, X, u]\mathbb{1}_{\{\tau_{i+1}<\tau\}} \le 0$$

and using (1.2), (4.20), we obtain $u_{\tilde{\ell}}(\tilde{x}) \le V_{\xi,\varsigma}\big(\tilde{x}, \tilde{\ell}\big)$ for each $\big(\tilde{x}, \tilde{\ell}\big) \in \overline{\mathcal{O}} \times \mathbb{I}$. Since the previous property is true for each control $(\xi, \varsigma) \in \mathcal{U} \times \mathcal{S}$, we conclude that $u_{\tilde{\ell}}(\tilde{x}) \le V_{\tilde{\ell}}(\tilde{x})$ for each $\big(\tilde{x}, \tilde{\ell}\big) \in \overline{\mathcal{O}} \times \mathbb{I}$. □

4.3. On ε-PACS optimal controls

Previously, we checked that the value function V_ℓ, defined in (2.8), satisfies the HJB equation (2.9). This means that for each $\ell \in \mathbb{I}$ fixed, the domain set $\mathcal{O}$ is divided into three parts. Consider $\mathcal{C}_\ell := \mathcal{O} \setminus (\mathcal{N}_\ell \cup \mathcal{S}_\ell)$ where

$$\mathcal{N}_\ell := \big\{x \in \mathcal{O} : |D^1 V_\ell| = g_\ell\big\} \quad \text{and} \quad \mathcal{S}_\ell = \{x \in \mathcal{O} : V_\ell = \mathcal{M}_\ell V\}.$$

The open set $\mathcal{C}_\ell$ is where V_ℓ satisfies the elliptic partial differential equation $[c_\ell - \mathcal{L}_\ell]V_\ell = h_\ell$, which suggests that the 'optimal control' (ξ^*, ς^*) corresponding to this problem will not be exercised when the process X^{ξ^*,ς^*} is in $\mathcal{C}_\ell$. Otherwise, either ξ^* or ς^* will be exercised on X^{ξ^*,ς^*} in the following way:

(i) if $X^{\xi^*,\varsigma^*} \in \mathcal{N}_\ell \setminus \mathcal{S}_\ell$, the singular control ξ^* will act on X^{ξ^*,ς^*} in such a way that X^{ξ^*,ς^*} will be pushed back to some point $y \in \partial\mathcal{C}_\ell$;

(ii) if $X^{\xi^*,\varsigma^*} \in \mathcal{S}_\ell$, the switching control ς^* will be executed in such a way that the process X^{ξ^*,ς^*} will switch to some regime $\kappa \neq \ell$ at time $\tau_\kappa \le \tau$.

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The construction of an optimal strategy for the problem (2.8) still remains an open problem of interest. A way to carry out this construction is to verify first that $\partial\mathcal{C}_\ell$ is at least of class C^1, which is not easy to do.

In the literature, we can see that the existence of an optimal dividend/switching strategy, which is an example of a mixed singular/switching control problem, has been solved when the payoff expected value is given by (1.4) and the cash reserve process switches between m-regimes governed by different Poisson processes with drifts; see [2], in which the authors proved that their solution is stationary with a band structure.

Another way to address the problem (2.8) is by means of ε-PACS optimal controls, which have been constructed in (4.6)–(4.13).

By Lemma 4.3 and the proof of Theorem 2.2, it is known that for each $\ell\in\mathbb{I}$, $V^{\varepsilon_n}_\ell\to V_\ell$ as $\varepsilon_n\downarrow 0$, and $V_\ell\leq V_{\xi^{\varepsilon_n,*},\varsigma^{\varepsilon_n,*}}(\cdot,\ell)\leq V^{\varepsilon_n}_\ell$ on $\overline{\mathcal{O}}$, with $\left(\xi^{\varepsilon_n,*},\varsigma^{\varepsilon_n,*}\right)$ as in (4.6)–(4.13), and $V_{\xi^{\varepsilon_n,*},\varsigma^{\varepsilon_n,*}}(\cdot,\ell)$, V_ℓ, $V^{\varepsilon_n}_\ell$ given by (1.2), (2.8), and (2.18), respectively.

Taking ε_n small enough and assuming that the process $X^{\varepsilon_n,*}_t$ is on the regime ℓ^*_i at time $t\in\left[\tau^*_i,\tilde{\tau}^*_{i+1}\right)$, we have that $X^{\varepsilon_n,*}_t\in\mathcal{C}^{\varepsilon_n}_{\ell^*_i}$ and the control $\left(\xi^{\varepsilon_n,*},\varsigma^{\varepsilon_n,*}\right)$ will be exercised as follows:

(i) if the controlled process $X^{\varepsilon_n,*}$ satisfies

$$\left|\mathrm{D}^1 V^{\varepsilon_n}_{\ell^*_i}\left(X^{\varepsilon_n,*}_t\right)\right|\leq g_{\ell^*_i}\left(X^{\varepsilon_n,*}_t\right),$$

then $\zeta^{\varepsilon_n,*}_t\equiv 0$ and $X^{\varepsilon_n,*}_t$ will stay in $\mathcal{C}_{\ell^*_i}$;

(ii) if $0<\left|\mathrm{D}^1 V^{\varepsilon_n}_{\ell^*_i}\left(X^{\varepsilon_n,*}_t\right)\right|^2-g_{\ell^*_i}\left(X^{\varepsilon_n,*}_t\right)^2<2\varepsilon_n$, the process $X^{\varepsilon_n,*}_t$ will be crossing $\partial\mathcal{C}_{\ell^*_i}$ persistently;

(iii) otherwise, $\xi^{\varepsilon_n,*}_t=\left(\mathrm{n}^{\varepsilon_n,*}_t,\zeta^{\varepsilon_n,*}_t\right)$ will exercise a force

$$\frac{2}{\varepsilon_n}\left|\mathrm{D}^1 V^{\varepsilon_n}_{\ell^*_i}\left(X^{\varepsilon_n,*}_t\right)\right|$$

and in the direction

$$-\frac{\mathrm{D}^1 V^{\varepsilon_n}_{\ell^*_i}\left(X^{\varepsilon_n,*}_t\right)}{\left|\mathrm{D}^1 V^{\varepsilon_n}_{\ell^*_i}\left(X^{\varepsilon_n,*}_t\right)\right|}$$

at $X^{\varepsilon_n,*}_t$ in such a way that it will be pushed back to $\partial\mathcal{C}_{\ell^*_i}$.

(iv) At time $t=\tau^*_{i+1}<\tau^*$, $X^{\varepsilon_n,*}_{\tau^*_{i+1}}$ will be in $\mathcal{S}^{\varepsilon_n}_{\ell^*_i}$ for the first time and will switch to the regime ℓ^*_{i+1}.

This procedure will be repeated until the time τ^*, which represents the first exit time of the process $X^{\varepsilon_n,*}$ from the set $\mathcal{O}$.

5. Explicit solution to the HJB equation (2.9) on open intervals of $\mathbb{R}$

In this section, we consider the controlled process $\left(X^{\xi,\varsigma},I^{\varsigma}\right)$ that evolves as (1.1) where $X^{\xi,\varsigma}_0=\tilde{x}\in(0,l)\subset\mathbb{R}$, for some $l>0$ fixed, and $I^{\varsigma}_{0-}=\tilde{\ell}\in\mathbb{I}:=\{1,\dots,m\}$. Here $W=\{W_t:t\geq 0\}$ is a one-dimensional standard Brownian motion. The parameters of the SDE (1.1) are given by $b(x,\ell)=b_\ell x$ and $\sigma(x,\ell)=\sigma_\ell x$ for $x\in(0,l)$, with $b_\ell,\sigma_\ell\in\mathbb{R}$ such that $\sigma_\ell>0$.

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Let us take the value function $V_{\tilde{\ell}}(\tilde{x})$ as in (1.3) where $c_\ell(x)$ and $g_\ell(x)$ are positive constant functions that are denoted by c_ℓ and g_ℓ, respectively, and the running cost h_ℓ is taken as $h_\ell(x) := K_\ell x^{\gamma_\ell}$ where $\gamma_\ell \in (0, 1)$, $K_\ell > 0$ are fixed. For simplicity, we assume that $\gamma_l \neq r_{2,l}$; cf. (5.3). By Theorem 2.2, it is known that $V_{\tilde{\ell}}(\tilde{x})$ satisfies (in the a.e. sense) the following HJB equation:

$$\max\left\{\left[c_{\tilde{\ell}} - \mathcal{L}_{\tilde{\ell}}\right]\tilde{u}_{\tilde{\ell}} - h_{\tilde{\ell}},\, \left|\tilde{u}'_{\tilde{\ell}}\right| - g_{\tilde{\ell}},\, \tilde{u}_{\tilde{\ell}} - \mathcal{M}_{\tilde{\ell}}\tilde{u}\right\} = 0 \text{ in } (0, l), \quad \text{s.t. } \tilde{u}_{\tilde{\ell}}(0) = \tilde{u}_{\tilde{\ell}}(l) = 0 \tag{5.1}$$

where $\mathcal{M}_\ell \tilde{u} = \min_{\kappa\in\mathbb{I}\setminus\{\ell\}}\{\tilde{u}_\kappa + \vartheta_{\ell,\kappa}\}$, and the parameters $\vartheta_{\ell,\kappa} \in \mathbb{R}$ are nonnegative and satisfy (2.6)–(2.7). Here

$$\mathcal{L}_\ell \tilde{u}_\ell(x) = \frac{1}{2}\sigma_\ell^2 x^2 \tilde{u}''_\ell(x) - b_\ell x \tilde{u}'_\ell,$$

and $\tilde{u}'_\ell$, $\tilde{u}''_\ell$ represent the first and second derivatives of $\tilde{u}_\ell$.

In the subsection below, we shall give an explicit solution to the HJB equation (5.1) with some additional restrictions on the constant parameters.

5.1. Construction of the solution to the HJB equation (5.1)

Considering $\ell \in \mathbb{I}$ fixed, let us first construct the solution to the following system of variational inequalities:

$$\max\left\{\left[c_\ell - \mathcal{L}_\ell\right]\tilde{u}_\ell - h_\ell,\, \left|\tilde{u}'_\ell\right| - g_\ell\right\} = 0 \text{ in } (0, l), \qquad \text{s.t. } \tilde{u}_\ell(0) = \tilde{u}_\ell(l) = 0. \tag{5.2}$$

It is well known that the general solution to $[c_\ell - \mathcal{L}_\ell]\tilde{u}_\ell = h_\ell$ on $\left[x^*_\ell, l\right]$, with $x^*_\ell > 0$, has the form

$$\tilde{u}_\ell(x) = A_\ell x^{r_{1,\ell}} + B_\ell x^{r_{2,\ell}} + \overline{K}_\ell x^{\gamma_\ell},$$

where

$$\overline{K}_\ell = \frac{2K_\ell}{\sigma^2(r_{1,\ell} - \gamma_\ell)(\gamma_\ell - r_{2,\ell})}$$

and $r_{1,\ell}$, $r_{2,\ell}$ denote the negative and positive solutions to

$$r^2 - \left(\frac{2b_\ell}{\sigma_\ell^2} + 1\right)r - \frac{2c_\ell}{\sigma_\ell^2} = 0. \tag{5.3}$$

Taking

$$\tilde{u}_\ell(x) := \begin{cases} g_\ell x & \text{if } 0 \le x < x^*_\ell, \\ A_\ell x^{r_{1,\ell}} + B_\ell x^{r_{2,\ell}} + \overline{K}_\ell x^{\gamma_\ell} & \text{if } x^*_\ell \le x \le l, \end{cases} \tag{5.4}$$

we see easily that

$$\left|\tilde{u}'_\ell\right| - g_\ell = 0 \text{ on } \left[0, x^*_\ell\right) \quad \text{and} \quad [c_\ell - \mathcal{L}_\ell]\tilde{u}_\ell - h_\ell = 0 \text{ on } \left[x^*_\ell, l\right],$$

and if $c_\ell + b_\ell \le 0$, then

$$[c_\ell - \mathcal{L}_\ell]\tilde{u}_\ell(x) - h_\ell(x) = g_\ell(c_\ell + b_\ell)x - K_\ell x^{\gamma_\ell} < 0 \quad \text{for } x \in \left(0, x^*_\ell\right).$$

In order for $\tilde{u}$ to satisfy (5.2), we shall consider parameters b_ℓ, σ_ℓ, c_ℓ, and g_ℓ such that

$$\left|u'_\ell\right| - g_\ell \le 0 \quad \text{on } \left(x^*_\ell, l\right). \tag{5.5}$$

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TABLE 1. Parameter values for the equations (5.3) and (5.8).

ℓ	b_ℓ	σ_ℓ	c_ℓ	g_ℓ	K_ℓ	γ_ℓ
1	–1	0.9	1	1	1	0.5
2	0.5	1	1.2	1	1	0.5
3	–0.5	1.3	1	1	1	0.5

Additionally, they must satisfy

$$x_\ell^* < \left(\frac{K_\ell}{g_\ell(c_\ell + b_\ell)}\right)^{\frac{1}{1-\gamma_\ell}}, \quad \text{if } c_\ell + b_\ell > 0, \tag{5.6}$$

since if (5.6) holds and $x \in (0, x_\ell^*)$, then

$$\begin{aligned}[c_\ell - \mathcal{L}_\ell]\tilde{u}_\ell(x) - h_\ell(x) &= x^{\gamma_\ell}\left(g_\ell(c_\ell + b_\ell)x^{1-\gamma_\ell} - K_\ell\right) \\ &< x^{\gamma_\ell}\left(g_\ell(c_\ell + b_\ell)\left(x_\ell^*\right)^{1-\gamma_\ell} - K_\ell\right) < 0.\end{aligned} \tag{5.7}$$

In order for $\tilde{u}_\ell$ to belong to $\mathrm{C}^1([0, l])$, by smooth fit, the parameters A_ℓ, B_ℓ, and x_ℓ^* must satisfy the following system of equations:

$$\begin{aligned} &A_\ell l^{r_{1,\ell}} + B_\ell l^{r_{2,\ell}} + \overline{K}_\ell l^{\gamma_\ell} = 0, \\ &A_\ell + B_\ell\left(x_\ell^*\right)^{r_{2,\ell}-r_{1,\ell}} + \overline{K}\left(x_\ell^*\right)^{\gamma_\ell - r_{1,\ell}} - g_\ell\left(x_\ell^*\right)^{1-r_{1,\ell}} = 0, \qquad \text{s.t. (5.6).} \\ &r_{1,\ell}A_\ell + r_{2,\ell}B_\ell\left(x_\ell^*\right)^{r_{2,\ell}-r_{1,\ell}} + \gamma_\ell\overline{K}_\ell\left(x_\ell^*\right)^{\gamma_\ell - r_{1,\ell}} - g_\ell\left(x_\ell^*\right)^{1-r_{1,\ell}} = 0, \end{aligned} \tag{5.8}$$

By what we have seen previously, we have the next result.

Proposition 5.1. *Let b_ℓ, σ_ℓ, c_ℓ, and g_ℓ be parameters satisfying* (5.5) *and* (5.6). *Then the function $\tilde{u}_\ell$ given as in* (5.4) *is a solution to the variational inequalities* (5.2). *Moreover, if A_ℓ, B_ℓ, and x_ℓ^* are a solution to* (5.8), *then $\tilde{u}_\ell$ belongs to $\mathrm{C}^1([0, l]) \cap \mathrm{C}^2\left([0, l] \setminus \left\{x_\ell^*\right\}\right)$ and is unique in this space.*

The uniqueness of $\tilde{u}_\ell$ is guaranteed by [7, Theorem 1.1].

Numerical examples. Table 1 contains the values that are considered to find numerically the solution $\tilde{u}_\ell$ to the variational inequalities (5.2), when $\ell = 1, 2, 3$ and $l = 3$; see the left panel of Figure 1.

Table 2 presents the numerical solutions to the system of equations (5.3) and (5.8) for the parameters given in Table 1. Observe that when $\ell = 2, 3$, the condition (5.6) is satisfied. The left panel of Figure 3 shows that (5.5) and (5.7) are satisfied for the parameters seen in Table 1. From here observe that $\tilde{u}$ is the numerical solution to the variational inequality (5.2) for the parameters given in Table 1.

Let us now give two examples where the conditions (5.5) and (5.6) are not satisfied. Consider the following parameters:

Table 4 presents the numerical solutions to the system of equations (5.3) and (5.8). The left panel of Figure 2 shows that $\tilde{u}_1$ does not satisfy the condition (5.5). Notice that $x_2^* \approx 0.4411$ does not satisfy the condition (5.6), leaving a discontinuity of $\tilde{u}_2$ at x_2^*; see the right panel of Figure 1. Additionally, in the right panel of Figure 2, we see that $\tilde{u}_2$ violates (5.7).

TABLE 2. Approximate solution to the system of equations (5.3) and (5.8) with the parameters given in Table 1.

ℓ	$r_{1,\ell}$	$r_{2,\ell}$	A_ℓ	B_ℓ	x_ℓ^*
1	–3.0259	0.8160	–0.0221	–1.5657	0.5231
2	–0.8439	2.8439	–0.0215	–0.0480	0.2118
3	–0.9027	1.3110	–0.0622	–0.4213	0.3249

TABLE 3. Parameter values for the equations (5.3) and (5.8).

ℓ	b_ℓ	σ_ℓ	c_ℓ	g_ℓ	K_ℓ	γ_ℓ
1	0.1	0.5	0.5	1.5	1	0.5
2	–0.5	0.3	5	0.7	1	0.1

TABLE 4. Approximate solution to the system of equations (5.3) and (5.8) for the parameters given in Table 3.

ℓ	$r_{1,\ell}$	$r_{2,\ell}$	A_ℓ	B_ℓ	x_ℓ^*
1	–1.2932	3.0932	–0.2253	–0.0978	0.7138
2	–16.7461	6.6350	0	–0.0002	0.4411

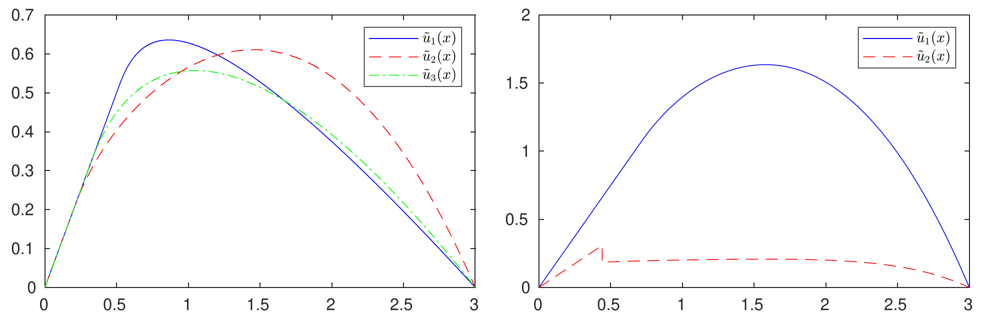

FIGURE 1. The left and right panels show plots of $\tilde{u}_\ell$ for the parameters given in Tables 2 and 4, respectively.

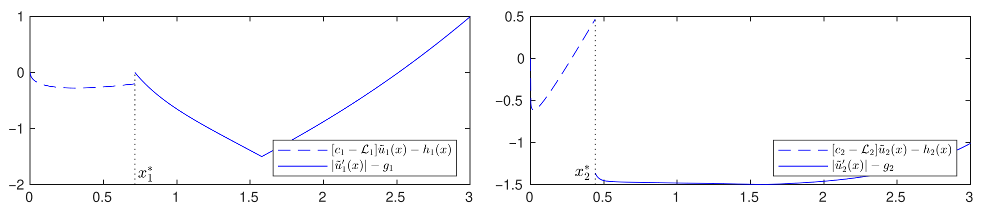

FIGURE 2. The left and right panels show plots of $[c_\ell - \mathcal{L}_\ell]\tilde{u}_\ell - h_\ell$ and $\left|\tilde{u}_\ell'\right| - g_\ell$ on $\left(0, x_\ell^*\right)$ and $\left(x_\ell^*, l\right)$ for the parameters given in Table 4.

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5.1.1. *Construction of the solution to* (5.1). In this part, we consider the vector function $\tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_m)$ such that its elements are given as (5.4) and satisfy the hypothesis of Proposition 5.1 in such a way that $\tilde{u}_\ell$ belongs to $\mathrm{C}^1([0, l]) \cap \mathrm{C}^2([0, l] \setminus \{x^*_\ell\})$ and is the unique solution of (5.2) for each $\ell \in \mathbb{I}$. Assume that the entrances of $\tilde{u}$ are different.

We will study the parameters $\vartheta_{\ell,\kappa} \geq 0$ where $\tilde{u}$ satisfies the HJB equation (5.1). For that, define first the sets $\mathcal{D}_{\ell,\kappa}$ and $\mathcal{U}_{\ell,\kappa}$, with $\ell \neq \kappa$, by

$$\mathcal{D}_{\ell,\kappa} = \{x \in (0, l) : u'_\ell(x) = u'_\kappa(x)\},$$
$$\mathcal{U}_{\ell,\kappa} = \{x \in \mathcal{D}_{\ell,\kappa} : 0 \leq [\tilde{u}_\ell - \tilde{u}_\kappa](y) \leq [\tilde{u}_\ell - \tilde{u}_\kappa](x) \text{ for any } y \in \mathcal{D}_{\ell,\kappa}\}.$$

Now let us take $\hat{\vartheta}_{\ell,\kappa}$, with $\ell, \kappa \in \mathbb{I}$ and $\ell \neq \kappa$, in the following way:

$$\hat{\vartheta}_{\ell,\kappa} := [\tilde{u}_\ell - \tilde{u}_\kappa](\bar{x}_{\ell,\kappa}), \quad \text{for some } \bar{x}_{\ell,\kappa} \in \mathcal{U}_{\ell,\kappa} \text{ fixed.} \tag{5.9}$$

If $\mathcal{U}_{\ell,\kappa} = \emptyset$, we set $\hat{\vartheta}_{\ell,\kappa} = 0$. We have the following result.

Lemma 5.1. *If $\tilde{u}_\ell < \tilde{u}_\kappa + \hat{\vartheta}_{\ell,\kappa}$ on $(0, l) \setminus \mathcal{U}_{\ell,\kappa}$, then $\tilde{u}_\ell$ is a solution to the following system of variational inequalities:*

$$\max\{[c_\ell - \mathcal{L}_\ell]\tilde{u}_\ell - h_\ell, |\tilde{u}'_\ell| - g_\ell, \tilde{u}_\ell - (\tilde{u}_\kappa + \hat{\vartheta}_{\ell,\kappa})\} = 0 \text{ in } (0, l), \quad \text{s.t. } \tilde{u}_\ell(0) = \tilde{u}_\ell(l) = 0. \tag{5.10}$$

Remark 5.1. Notice that if $\vartheta_{\ell,\kappa} > \hat{\vartheta}_{\ell,\kappa}$, $\tilde{u}_\ell$ satisfies (5.10) with $\hat{\vartheta}_{\ell,\kappa}$ replaced by $\vartheta_{\ell,\kappa}$, but there would be no opportunity to switch from the state ℓ to κ since $\tilde{u}_\ell < \tilde{u}_\kappa + \vartheta_{\ell,\kappa}$ on $(0,l)$.

To conclude this part, let us suppose that for each $\ell \in \mathbb{I}$ fixed, $\tilde{u}_\ell$ satisfies the following condition:

$$\tilde{u}_\ell < \tilde{u}_\kappa + \hat{\vartheta}_{\ell,\kappa} \quad \text{on } (0, l) \setminus \mathcal{U}_{\ell,\kappa} \text{ for each } \kappa \neq \ell. \tag{5.11}$$

From here, it is easy to see that

$$\tilde{u}_\ell - \mathcal{M}_\ell \tilde{u} = \tilde{u}_\ell - \min_{\kappa \neq \ell} \{\tilde{u}_\kappa + \hat{\vartheta}_{\ell,\kappa}\} < 0 \quad \text{on } (0, l) \setminus \bigcup_{\kappa \neq \ell} \mathcal{U}_{\ell,\kappa}.$$

Remark 5.2. Defining $\widetilde{\mathcal{S}}_\ell = \{x \in (0, l) : \tilde{u}_\ell = \mathcal{M}_\ell \tilde{u}\}$, we have that

$$\widetilde{\mathcal{S}}_\ell = \bigcup_{\kappa \neq \ell} \mathcal{U}_{\ell,\kappa} \quad \text{for } \ell \in \mathbb{I},$$

which is the zone where it is optimal to switch from the state ℓ to some other state κ.

Proposition 5.2. *Let $\tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_m)$ be the vector function whose entrances, given by* (5.4), *are different and satisfy the hypotheses of Proposition* 5.1 *and the condition* (5.11). *If $\hat{\vartheta}_{\ell,\kappa}$, defined in* (5.9), *satisfy the conditions* (2.6)–(2.7), *then $\tilde{u}$ is the unique solution to* (5.1).

Remark 5.3. Taking $\widetilde{\mathcal{N}}_\ell := \{x \in (0, l) : |\tilde{u}'_\ell| - g_\ell = 0\}$, we have that $\widetilde{\mathcal{C}}_\ell := (0, l) \setminus (\widetilde{\mathcal{N}}_\ell \cup \widetilde{\mathcal{S}}_\ell)$ is the zone where it is not optimal to switch to any other state.

Numerical example. Consider the parameters given in Table 1. The left panel of Table 5 presents the approximate values of $\hat{\vartheta}_{\ell,\kappa}$ which are found using (5.9).

The right panel of Table 5 presents the approximate values of $\hat{\vartheta}_{\ell,\bar{\kappa}} + \hat{\vartheta}_{\bar{\kappa},\kappa}$. Note that the inequalities (2.6)–(2.7) are always satisfied, say $\hat{\vartheta}_{1,2} \approx 0.1368 < \hat{\vartheta}_{1,3} + \hat{\vartheta}_{3,2} \approx 0.1444$. Table 6

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TABLE 5. The table on the left presents the approximate values of $\hat{\vartheta}_{\ell,\kappa}$. The table on the right presents the approximate values of $\hat{\vartheta}_{\ell,\bar{\kappa}} + \hat{\vartheta}_{\bar{\kappa},\kappa}$, with $\ell \neq \kappa \neq \bar{\kappa}$.

ℓ \ κ	1	2	3
1	–	0.1368	0.0981
2	0.1736	–	0.1518
3	0.0233	0.0462	–

ℓ \ κ	1	2	3
1	–	0.1444	0.2886
2	0.1750	–	0.2718
3	0.2199	0.1601	–

TABLE 6. Approximate values of $\bar{x}_{\ell,\kappa}$ where switching from the state ℓ to the state κ is recommended.

ℓ \ κ	1	2	3
1	–	0.6490	0.7025
2	2.1774	–	2.1330
3	0.5261	2.3187	–

presents the approximate values of $\bar{x}_{\ell,\kappa}$ where switching from the state ℓ to the state κ is recommended.

Finally, Figure 3 shows that the function $\tilde{u}_\ell$ indeed satisfies the HJB equation (5.1). The figures in the left column show the plots of $[c_\ell - \mathcal{L}_\ell]\tilde{u}_\ell - h_\ell$ and $|\tilde{u}'_\ell| - g_\ell$ on $(0, x^*_\ell)$ and (x^*_ℓ, l), respectively. Note that $[c_\ell - \mathcal{L}_\ell]\tilde{u}_\ell - h_\ell < 0$ on $(0, x^*_\ell)$, and $|\tilde{u}'_\ell| - g_\ell < 0$ on (x^*_ℓ, l). The figures in the right column show the plots of $\tilde{u}_\ell - \mathcal{M}_\ell\tilde{u}_\ell$ on $(0,l)$. Also, we observe that

$$\widetilde{\mathcal{N}}_\ell = (0, x^*_\ell) \quad \text{and} \quad \widetilde{\mathcal{S}}_\ell = \{\bar{x}_{\ell,\kappa}, \bar{x}_{\ell,\tilde{\kappa}}\},$$

for $\ell, \kappa, \tilde{\kappa} \in \mathbb{I} = \{1, 2, 3\}$ with $\ell \neq \kappa \neq \tilde{\kappa}$. Suppose, for illustration, that the process has started from the point $\tilde{x} = 2$ at the regime 1. Then the optimal strategy requests to switch to the regime 3 at the point $\bar{x}_{1,3}$. Then, following the figure at the bottom right corner, the optimal strategy requests either to switch to the regime 2 at the point $\bar{x}_{3,2}$ or to the regime 1 at the point $\bar{x}_{3,1}$, depending on which point is reached first, and so on.

6. Conclusions and some further work

The main contribution of our paper consists in combining singular and switching controls for the general diffusion model on a bounded domain. Under Assumptions (H1)–(H4), we showed that the existence and uniqueness of the strong solutions to the HJB equations (2.9) and (2.20) are guaranteed on the space $C^{0,1}(\overline{\mathcal{O}}) \cap W^{2,\infty}_{\text{loc}}(\mathcal{O})$. After that, we proved that the value function V^ε for the ε-PACS control problem seen in Subsection 2.2 satisfies (2.20), showing that for $\ell \in \mathbb{I}$, $V^\varepsilon_\ell \in C^{0,1}(\overline{\mathcal{O}}) \cap W^{2,\infty}_{\text{loc}}(\mathcal{O})$. Finally, assuming also that the domain set $\mathcal{O}$ is an open convex set and by probabilistic arguments, we verified that the value function V, given in (2.8), is characterized as a limit of V^ε as $\varepsilon \downarrow 0$, which permitted us to conclude that V satisfies (2.9) and, from here, to see that for $\ell \in \mathbb{I}$, $V_\ell \in C^{0,1}(\overline{\mathcal{O}}) \cap W^{2,\infty}_{\text{loc}}(\mathcal{O})$.

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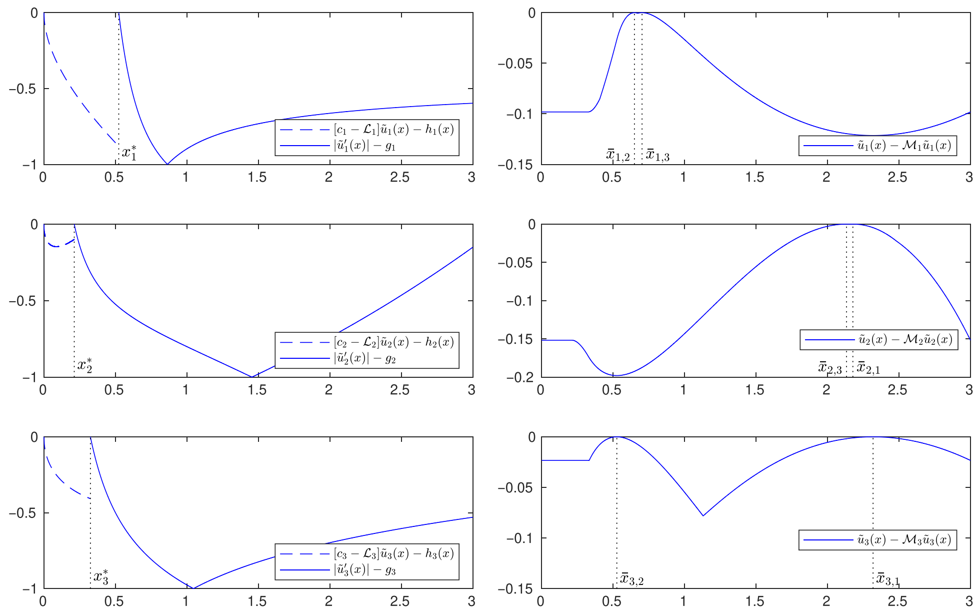

FIGURE 3. The figures in the left column show the plots of $[c_\ell - \mathcal{L}_\ell]\tilde{u}_\ell - h_\ell$ and $\left|\tilde{u}_\ell'\right| - g_\ell$ on $\left(0, x_\ell^*\right)$ and $\left(x_\ell^*, l\right)$, respectively. The figures in the right column show the plots of $\tilde{u}_\ell - \mathcal{M}_\ell \tilde{u}_\ell$ on $(0,l)$.

Although the optimal control process for the mixed singular/switching stochastic control problem (2.8) was not given explicitly, and this is still an open problem, we constructed a family of ε-PACS optimal control processes $\left\{\left(\xi^{\varepsilon_n,*}, \varsigma^{\varepsilon_n,*}\right)\right\}_{n\geq 1}$ (see (4.6)–(4.13)) such that the limit of their value functions V^{ε_n} (as $\varepsilon_n \to 0$) agrees with the value function V.

There are many extensions to be considered and directions for future research. Some of them could be as follows:

1. To study the value function V given in (2.8) when the infinitesimal generator of the process $X^{\xi,\varsigma}$, without the influence of the singular control ξ in $X^{\xi,\varsigma}$, within the regime ℓ, is given by

$$\mathcal{L}_\ell u_\ell = \operatorname{tr}\left[a_\ell \mathrm{D}^2 u_\ell\right] - \left\langle b_\ell, \mathrm{D}^1 u_\ell \right\rangle + \int_{\mathbb{R}^d_*} \left[u_\ell(x+z) - u_\ell(x) - \left\langle \mathrm{D}^1 u_\ell, z\right\rangle \mathbb{1}_{\{|z|\in(0,1)\}}\right] s_\ell(x, z)\nu(\mathrm{d}z),$$

where ν is a Radon measure on $\mathbb{R}^d_* := \mathbb{R}^d \setminus \{0\}$ satisfying $\int_{\mathbb{R}^d_*}\left[|z|^2 \wedge 1\right]\nu(\mathrm{d}z) < \infty$, and $s_\ell : \overline{\mathcal{O}} \times \mathbb{R}^d \longrightarrow [0, 1]$ is such that

$$\int_{\mathbb{R}^d_*} s_\ell(x, z)\mathbb{1}_{\{x+z\notin\mathcal{O}\}}\nu(\mathrm{d}z) < \infty$$

for $x \in \mathcal{O}$. In this case, the main difficulty lies in obtaining some a priori estimates of $\int_{\{|z|\in(0,1)\}}\left[\int_0^1 \left|\mathrm{D}^2 u^\varepsilon(\,\cdot + tz)\right|\mathrm{d}t\right]|z|^2 s_\ell(\cdot, z)\nu(\mathrm{d}z)$ that are independent of ε.

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2. To analyze the problem given in (1.1)–(1.3) on the whole space $\mathbb{R}^d$. In this direction, a technically involved problem could be studied when the controlled process can switch between m-regimes which are governed by Brownian motions with different drifts.
3. Additionally, this problem could be extended to verify that the value function V given in (2.8) is a viscosity solution to the HJB equation (2.9) on unbounded sets when the ellipticity condition (2.11) is omitted. Recall that the viscosity solutions for problems of this type were studied in [2, 4, 15] and [3] for the cases of $\mathbb{R}$ and $\mathbb{R}^2$, respectively.

Appendix A. Proofs of Lemma 3.1 and Proposition 3.1

Recall that Lemma 3.1 and Proposition 3.1 are under Assumptions (H1)–(H4). Let us first show (3.2) and (3.3) of Lemma 3.1, which helps to verify the existence of the classic solution $u^{\varepsilon,\delta}$ to the NPDS (3.1). Afterwards, to complete the proof of Proposition 3.1, we shall prove (3.4) of Lemma 3.1. From now on, for simplicity of notation, we replace $u^{\varepsilon,\delta}$ by u in the proofs.

A.1. Verification of Equations (3.2) and (3.3)

Proof of Lemma 3.1. *Equation* (3.2). Let $(x_0, \ell_0) \in \overline{\mathcal{O}} \times \mathbb{I}$ be such that

$$u_{\ell_0}(x_0) = \min_{x\in\overline{\mathcal{O}},\ell\in\mathbb{I}} u_\ell(x).$$

If $x_0 \in \partial\mathcal{O}$, it follows easily that $u_\ell(x) \geq u_{\ell_0}(x_0) = 0$ for all $(x, \ell) \in \overline{\mathcal{O}} \times \mathbb{I}$. Suppose that $x_0 \in \mathcal{O}$. Since $u_{\ell_0} = 0$ in $\partial\mathcal{O}$, we have that $u_{\ell_0}(x_0) \leq 0$. On the other hand, we know that

$$\begin{aligned} &\mathrm{D}^1 u_{\ell_0}(x_0) = 0, \quad \mathrm{tr}\big[a_{\ell_0}(x_0)\mathrm{D}^2 u_{\ell_0}(x_0)\big] \geq 0, \\ &u_{\ell_0}(x_0) - u_\kappa(x_0) \leq 0, \ \text{ for } \kappa \in \mathbb{I}\setminus\{\ell_0\}. \end{aligned} \tag{A.1}$$

Then, using (H3), (3.1), and (A.1),

$$0 \leq \mathrm{tr}\big[a_{\ell_0}\mathrm{D}^2 u_{\ell_0}\big] = c_{\ell_0}u_{\ell_0} - h_{\ell_0} \leq c_{\ell_0}u_{\ell_0}, \quad \text{at } x_0.$$

Since $c_{\ell_0} > 0$ on $\overline{\mathcal{O}}$, it follows that $u_{\ell_0}(x_0) \geq 0$. Therefore $u_\ell(x) \geq u_{\ell_0}(x_0) = 0$, for all $x \in \overline{\mathcal{O}}$ and $\ell \in \mathbb{I}$.

For each $\ell \in \mathbb{I}$, consider $\tilde{v}_\ell$ as the unique solution to the Dirichlet problem $[c_\ell - \mathcal{L}_\ell]\tilde{v}_\ell = h_\ell$ in $\mathcal{O}$, such that $\tilde{v}_\ell = 0$ on $\partial\mathcal{O}$. By Theorem 1.2.10 of [9], it is well known that $\tilde{v}_\ell \in \mathrm{C}^{2,\alpha'}\big(\overline{\mathcal{O}}\big)$ and

$$||\tilde{v}_\ell||_{\mathrm{C}^{2,\alpha'}\left(\overline{\mathcal{O}}\right)} \leq K_2||h_\ell||_{\mathrm{C}^{0,\alpha'}\left(\overline{\mathcal{O}}\right)} \leq K_2\Lambda \ =: C_1,$$

where $K_2 = K_2(d, \Lambda, \alpha')$, since (H2)–(H4) hold. Meanwhile, from Equation (3.1), it can be seen that for each $\ell \in \mathbb{I}$, $[c_\ell - \mathcal{L}_\ell]u_\ell \leq h_\ell$ in $\mathcal{O}$. Then, taking $\eta_\ell := u_\ell - \tilde{v}_\ell$, we get, for each $\ell \in \mathbb{I}$,

$$[c_\ell - \mathcal{L}_\ell]\eta_\ell \leq 0, \ \text{ in } \mathcal{O}, \quad \text{ s.t. } \eta_\ell = 0, \ \text{ on } \partial\mathcal{O}. \tag{A.2}$$

Let $x_\ell^* \in \overline{\mathcal{O}}$ be a maximum point of η_ℓ. If $x_\ell^* \in \partial\mathcal{O}$, trivially we have $u_\ell - \tilde{v}_\ell \leq 0$ in $\overline{\mathcal{O}}$. Suppose that $x_\ell^* \in \mathcal{O}$. Note that $\eta_\ell\big(x_\ell^*\big) \geq 0$ and

$$\mathrm{D}^1\eta_\ell\big(x_\ell^*\big) = 0, \quad \mathrm{tr}\big[a_\ell\big(x_\ell^*\big)\mathrm{D}^2\eta_\ell\big(x_\ell^*\big)\big] \leq 0. \tag{A.3}$$

Then, using (A.2)–(A.3), we have $0 \geq \mathrm{tr}\big[a_\ell \mathrm{D}^2\eta_\ell\big] \geq c_\ell\eta_\ell$ at x_ℓ^*. Hence, $\eta_\ell \leq 0$ at x_ℓ^*, since $c_\ell > 0$ on $\overline{\mathcal{O}}$. We conclude that $[u_\ell - \tilde{v}_\ell](x) \leq [u_\ell - \tilde{v}_\ell]\big(x_\ell^*\big) = 0$ for $x \in \overline{\mathcal{O}}$. Therefore, for each $\ell \in \mathbb{I}$, $0 \leq u_\ell \leq C_1$ on $\overline{\mathcal{O}}$. □

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Proof of Lemma 3.1. Equation (3.3). For each $\ell \in \mathbb{I}$, consider the auxiliary function $w_\ell := |\mathrm{D}^1 u_\ell|^2 - \lambda A_{\varepsilon,\delta} u_\ell$ on $\overline{\mathcal{O}}$, where

$$A_{\varepsilon,\delta} := \max_{(x,\ell)\in\overline{\mathcal{O}}\times\mathbb{I}} |\mathrm{D}^1 u_\ell(x)|$$

and $\lambda \geq 1$ is a constant that will be selected later on. Observe that if $A_{\varepsilon,\delta} \leq 1$, we obtain a bound for $A_{\varepsilon,\delta}$ that is independent of ε, δ. Hence, we obtain the statement given in the lemma above. We assume henceforth that $A_{\varepsilon,\delta} > 1$. Taking first and second derivatives to w_ℓ, it can be checked that

$$\begin{aligned} -\mathrm{tr}\big[a_\ell \mathrm{D}^2 w_\ell\big] = &-2\sum_i \big\langle a_\ell \mathrm{D}^1 \partial_i u_\ell, \mathrm{D}^1 \partial_i u_\ell \big\rangle \\ &- 2\sum_i \mathrm{tr}\big[a_\ell \mathrm{D}^2 \partial_i u_\ell\big]\partial_i u_\ell + \lambda A_{\varepsilon,\delta} \mathrm{tr}\big[a_\ell \mathrm{D}^2 u_\ell\big]. \end{aligned} \tag{A.4}$$

Meanwhile, from (2.3) and (3.1),

$$\lambda A_{\varepsilon,\delta}\mathrm{tr}\big[a_\ell \mathrm{D}^2 u_\ell\big] = \lambda A_{\varepsilon,\delta}\Big[\widetilde{D}_1 u_\ell + \psi_{\varepsilon,\ell}(\cdot) + \sum_{\kappa\in\mathbb{I}\setminus\{\ell\}} \psi_{\delta,\ell,\kappa}(\cdot)\Big], \tag{A.5}$$

where $\psi_{\varepsilon,\ell}(\cdot)$, $\psi_{\delta,\ell,\kappa}(\cdot)$ denote $\psi_\varepsilon\Big(|\mathrm{D}^1 u_\ell|^2 - g_\ell^2\Big)$, $\psi_\delta\big(u_\ell - u_\kappa - \vartheta_{\ell,\kappa}\big)$, respectively, and $\widetilde{D}_1 u_\ell := \langle b_\ell, \mathrm{D}^1 u_\ell\rangle + c_\ell u_\ell - h_\ell$. Now, differentiating (3.1), multiplying by $2\partial_i u_\ell$, and taking the summation over all i, we see that

$$\begin{aligned} -2\sum_i \mathrm{tr}\big[a_\ell \mathrm{D}^2 \partial_i u_\ell\big]\partial_i u_\ell = &\widetilde{D}_2 u_\ell - 2\psi'_{\varepsilon,\ell}(\cdot)\Big\langle \mathrm{D}^1 u_\ell, \mathrm{D}^1\Big[|\mathrm{D}^1 u_\ell|^2 - g_\ell^2\Big]\Big\rangle \\ &- 2\sum_{\kappa\in\mathbb{I}\setminus\{\ell\}} \psi'_{\delta,\ell,\kappa}(\cdot)\Big[|\mathrm{D}^1 u_\ell|^2 - \big\langle \mathrm{D}^1 u_\ell, \mathrm{D}^1 u_\kappa\big\rangle\Big], \end{aligned} \tag{A.6}$$

where

$$\widetilde{D}_2 u_\ell := 2\sum_k \partial_k u_\ell \mathrm{tr}\big[[\partial_k a_\ell]\mathrm{D}^2 u_\ell\big] - 2\big\langle \mathrm{D}^1 u_\ell, \mathrm{D}^1\big[\langle b_\ell, \mathrm{D}^1 u_\ell\rangle + c_\ell u_\ell - h_\ell\big]\big\rangle. \tag{A.7}$$

Then, from (A.4)–(A.6), it can be shown that

$$\begin{aligned} -\mathrm{tr}\big[a_\ell \mathrm{D}^2 w_\ell\big] = &-2\sum_i \langle a_\ell \mathrm{D}^1 \partial_i u_\ell, \mathrm{D}^1 \partial_i u_\ell\rangle + \widetilde{D}_2 u_\ell + \lambda A_{\varepsilon,\delta}\widetilde{D}_1 u_\ell \\ &- 2\psi'_{\varepsilon,\ell}(\cdot)\Big\langle \mathrm{D}^1 u_\ell, \mathrm{D}^1\Big[|\mathrm{D}^1 u_\ell|^2 - g_\ell^2\Big]\Big\rangle + \lambda A_{\varepsilon,\delta}\psi_{\varepsilon,\ell}(\cdot) \\ &- \sum_{\kappa\in\mathbb{I}\setminus\{\ell\}} \Big\{2\psi'_{\delta,\ell,\kappa}(\cdot)\Big[|\mathrm{D}^1 u_\ell|^2 - \big\langle \mathrm{D}^1 u_\ell, \mathrm{D}^1 u_\kappa\big\rangle\Big] - \lambda A_{\varepsilon,\delta}\psi_{\delta,\ell,\kappa}(\cdot)\Big\}, \end{aligned}$$

on $\mathcal{O}$. Notice that by (H3), (H4), and (3.2),

$$\begin{aligned} -2\sum_i &\big\langle a_\ell \mathrm{D}^1 \partial_i u_\ell, \mathrm{D}^1 \partial_i u_\ell\big\rangle + \widetilde{D}_2 u_\ell + \lambda A_{\varepsilon,\delta}\widetilde{D}_1 u_\ell \\ &\leq 2\Lambda d|\mathrm{D}^1 u_\ell|^2 - \theta|\mathrm{D}^2 u_\ell|^2 + 2\Lambda\big[1+d^3\big]|\mathrm{D}^1 u_\ell|\,|\mathrm{D}^2 u_\ell| \\ &\quad + 2\Lambda[1+C_1]|\mathrm{D}^1 u_\ell| + \lambda A_{\varepsilon,\delta}\Lambda\big[|\mathrm{D}^1 u_\ell| + C_1\big] \\ &\leq \Big[2\Lambda d + \frac{\Lambda^2[1+d^3]^2}{\theta}\Big]|\mathrm{D}^1 u_\ell|^2 \\ &\quad + 2\Lambda[1+C_1]|\mathrm{D}^1 u_\ell| + \lambda A_{\varepsilon,\delta}\Lambda\big[|\mathrm{D}^1 u_\ell| + C_1\big]. \end{aligned} \tag{A.8}$$

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Then, by (A.8) and using the convexity property of $\psi_{\cdot}$, i.e. $\psi_{\cdot}(r) \leq \psi'_{\cdot}(r)r$ for all $r \in \mathbb{R}$, we see that

$$\begin{aligned} -\operatorname{tr}\Big[a_\ell \mathrm{D}^2 w_\ell\Big] &\leq K_3 \big|\mathrm{D}^1 u_\ell\big|^2 + K_3[1+\lambda A_{\varepsilon,\delta}]\big|\mathrm{D}^1 u_\ell\big| + \lambda K_3 A_{\varepsilon,\delta} \\ &- \psi'_{\varepsilon,\ell}(\cdot)\Big\{2\Big\langle \mathrm{D}^1 u_\ell, \mathrm{D}^1\big|\mathrm{D}^1 u_\ell\big|^2\Big\rangle - K_3\big|\mathrm{D}^1 u_\ell\big| - \lambda A_{\varepsilon,\delta}\Big[\big|\mathrm{D}^1 u_\ell\big|^2 - g_\ell^2\Big]\Big\} \\ &- \sum_{\kappa\in\mathbb{I}\setminus\{\ell\}} \psi'_{\delta,\ell,\kappa}(\cdot)\Big\{2\Big[\big|\mathrm{D}^1 u_\ell\big|^2 - \big\langle \mathrm{D}^1 u_\ell, \mathrm{D}^1 u_\kappa\big\rangle\Big] - \lambda A_{\varepsilon,\delta}\big[u_\ell - u_\kappa - \vartheta_{\ell,\kappa}\big]\Big\} \end{aligned} \tag{A.9}$$

on $\mathcal{O}$, for some $K_3 = K_3(d, \Lambda, \alpha')$. Let $(x_\lambda, \ell_\lambda) \in \overline{\mathcal{O}} \times \mathbb{I}$ (depending on λ) be such that $w_{\ell_\lambda}(x_\lambda) = \max_{(x,\ell)\in\overline{\mathcal{O}}\times\mathbb{I}} w_\ell(x)$. From here, (3.2), and by definition of w_ℓ, we see that

$$\big|\mathrm{D}^1 u_\ell(x)\big|^2 \leq \big|\mathrm{D}^1 u_{\ell_\lambda}(x_\lambda)\big|^2 + \lambda A_{\varepsilon,\delta} C_1, \quad \text{for } (x, \ell) \in \overline{\mathcal{O}} \times \mathbb{I}. \tag{A.10}$$

So it suffices to bound $\big|\mathrm{D}^1 u_{\ell_\lambda}(x_\lambda)\big|^2$ by a positive constant $C = C(d, \Lambda, \alpha')$. If $x_\lambda \in \partial\mathcal{O}$, by (3.2), it can be verified that $\big|\mathrm{D}^1 u_{\ell_\lambda}\big| \leq d^{\frac{1}{2}} C_1$ on $\partial\mathcal{O}$. Then, from (A.10),

$$\big|\mathrm{D}^1 u_\ell(x)\big|^2 \leq C_1\big[dC_1 + \lambda A_{\varepsilon,\delta}\big], \quad \text{for } (x, \ell) \in \overline{\mathcal{O}} \times \mathbb{I}. \tag{A.11}$$

On the other hand, observe that for each $\varrho > 0$, there exists $x_\varrho \in \overline{\mathcal{O}}$ such that

$$\big[A_{\varepsilon,\delta} - \varrho\big]^2 \leq \big|\mathrm{D}^1 u_\ell\big(x_\varrho\big)\big|^2. \tag{A.12}$$

Using (A.11) in (A.12) and letting $\varrho \to 0$, we have

$$A_{\varepsilon,\delta}^2 \leq C_1\big[dC_1 + \lambda A_{\varepsilon,\delta}\big]. \tag{A.13}$$

Multiplying by $1/A_{\varepsilon,\delta}$ in (A.13), and since $A_{\varepsilon,\delta} > 1$, we get

$$\big|\mathrm{D}^1 u_\ell(x)\big| \leq A_{\varepsilon,\delta} \leq dC_1^2 + \lambda C_1, \quad \text{for } (x, \ell) \in \overline{\mathcal{O}} \times \mathbb{I}. \tag{A.14}$$

In this case, considering $\lambda > 1$ fixed and $C_2 := C_1[dC_1 + \lambda]$, we obtain the result that is proposed in the lemma. Let x_λ be in $\mathcal{O}$. It is known that $\partial_i\big|\mathrm{D}^1 u_{\ell_\lambda}\big|^2 - \lambda A_{\varepsilon,\delta}\partial_i u_{\ell_\lambda} = 0$ at x_λ. Then,

$$2\Big\langle \mathrm{D}^1 u_{\ell_\lambda}, \mathrm{D}^1\big|\mathrm{D}^1 u_{\ell_\lambda}\big|^2\Big\rangle = 2\lambda A_{\varepsilon,\delta}\big|\mathrm{D}^1 u_{\ell_\lambda}\big|^2, \ \text{ at } x_\lambda. \tag{A.15}$$

Also, since $\vartheta_{\ell_\lambda,\kappa} \geq 0$ and $w_{\ell_\lambda}(x_\lambda) - w_\kappa(x_\lambda) \geq 0$ for each $\kappa \in \mathbb{I}$, and

$$|y_1|^2 - |y_2|^2 = 2\big[|y_1|^2 - \langle y_1, y_2\rangle\big] - |y_1 - y_2|^2 \leq 2\big[|y_1|^2 - \langle y_1, y_2\rangle\big]$$

for $y_1, y_2 \in \mathbb{R}^d$, we have

$$2\Big[\big|\mathrm{D}^1 u_{\ell_\lambda}\big|^2 - \big\langle \mathrm{D}^1 u_{\ell_\lambda}, \mathrm{D}^1 u_\kappa\big\rangle\Big] - \lambda A_{\varepsilon,\delta}\big[u_{\ell_\lambda} - u_\kappa - \vartheta_{\ell_\lambda,\kappa}\big] \geq 0, \ \text{ at } x_\lambda. \tag{A.16}$$

By (A.9), (A.15), and (A.16), and since $\operatorname{tr}\big[a_{\ell_\lambda}(x_\lambda)\mathrm{D}^2 w_{\ell_\lambda}(x_\lambda)\big] \leq 0$, we have

$$\begin{aligned} 0 \leq -\operatorname{tr}\big[a_{\ell_\lambda}\mathrm{D}^2 w_{\ell_\lambda}\big] &\leq K_3\big|\mathrm{D}^1 u_{\ell_\lambda}\big|^2 + K_3\big[1 + \lambda A_{\varepsilon,\delta}\big]\big|\mathrm{D}^1 u_{\ell_\lambda}\big| + \lambda K_3 A_{\varepsilon,\delta} \\ &- \psi'_{\varepsilon,\ell_\lambda}(\cdot)\Big[\lambda A_{\varepsilon,\delta}\big|\mathrm{D}^1 u_{\ell_\lambda}\big|^2 - K_3\big|\mathrm{D}^1 u_{\ell_\lambda}\big| + \lambda A_{\varepsilon,\delta} g_{\ell_\lambda}^2\Big] \end{aligned} \tag{A.17}$$

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at x_λ. On the other hand, notice that either $\psi'_{\varepsilon,\ell_\lambda}(\cdot) < \frac{1}{\varepsilon}$ or $\psi'_{\varepsilon,\ell_\lambda}(\cdot) = \frac{1}{\varepsilon}$ at x_λ. If $\psi'_{\varepsilon,\ell_\lambda}(\cdot) < \frac{1}{\varepsilon}$ at x_λ, then by the definition of ψ_ε, given in (2.16), it follows that $\left|\mathrm{D}^1 u_{\ell_\lambda}(x_\lambda)\right|^2 - g_{\ell_\lambda}(x_\lambda)^2 \leq 2\varepsilon$. This implies that $\left|\mathrm{D}^1 u_{\ell_\lambda}(x_\lambda)\right|^2 \leq \Lambda^2 + 2$. Then, by (A.10) and arguing as in (A.14), we obtain $\left|\mathrm{D}^1 u_\ell(x)\right| \leq A_{\varepsilon,\delta} \leq 2 + \Lambda^2 + \lambda C_1$ for each $(x, \ell) \in \overline{\mathcal{O}} \times \mathbb{I}$. For $\lambda > 1$ fixed, this yields the result given in the lemma. Now, assume that $\psi'_{\varepsilon,\ell_\lambda}(\cdot) = \frac{1}{\varepsilon}$. Then, taking $\lambda > \max\{1, K_3\}$ fixed, using (A.17), and proceeding similarly as for (A.14), we get $0 \leq [K_3 - \lambda]\left|\mathrm{D}^1 u_{\ell_\lambda}\right|^2 + K_3[2+\lambda]\left|\mathrm{D}^1 u_{\ell_\lambda}\right| + \lambda K_3$ at x_λ. From here, we obtain that $\left|\mathrm{D}^1 u_{\ell_\lambda}(x_\lambda)\right| < K_4$ for some $K_4 = K_4(d, \Lambda, \alpha')$. Using (A.10) and by arguments similar to (A.13), we conclude that there exists $C_2 = C_2(d, \Lambda, \alpha')$ such that $\left|\mathrm{D}^1 u_\ell\right| \leq A_{\varepsilon,\delta} \leq C_2$ on $\overline{\mathcal{O}}$. □

A.2. Existence and uniqueness of the solution to the NPDS

Let $\mathcal{C}^k$, $\mathcal{C}^{k,\alpha'}$ be the sets given by $(\mathrm{C}^k(\overline{\mathcal{O}}))^m$, $(\mathrm{C}^{k,\alpha'}(\overline{\mathcal{O}}))^m$, respectively, with $k \in \mathbb{N}$ and $\alpha' \in (0, 1)$. Defining

$$||w||_{\mathcal{C}^k} = \max_{i\in\mathbb{I}}\{||w_i||_{\mathrm{C}^k(\overline{\mathcal{O}})}\}$$

for each $w = (w_1, \ldots, w_m) \in \mathcal{C}^k$, the reader can verify that $||\cdot||_{\mathcal{C}^k}$ is a norm on $\mathcal{C}^k$ and $(\mathcal{C}^k, ||\cdot||_{\mathcal{C}^k})$ is a Banach space.

Notice that for each $w \in \mathcal{C}^1$ fixed, there exists a unique solution $u \in \mathcal{C}^{2,\alpha'}$ to the NPDS

$$\begin{aligned}[c_\ell - \mathcal{L}_\ell]u_\ell + \psi_\varepsilon(|\mathrm{D}^1 u_\ell|^2 - g_\ell^2) = h_\ell - \sum_{\kappa\in\mathbb{I}\setminus\{\ell\}} \psi_\delta(w_\ell - w_\kappa - \vartheta_{\ell,\kappa}) \text{ in } \mathcal{O},\\ \text{s.t. } u_\ell = 0 \text{ on } \partial\mathcal{O},\end{aligned} \tag{A.18}$$

since (H2)–(H4) hold and

$$\sum_{\kappa\in\mathbb{I}\setminus\{\ell\}} \psi_\delta(w_\ell - w_\kappa - \vartheta_{\ell,\kappa}) \in \mathrm{C}^1(\overline{\mathcal{O}})$$

(see [10, Theorem 15.10, p. 380]). Additionally, from [1, Theorem 4.12, p. 85] and [9, Theorem 1.2.19], it follows that

$$\begin{aligned}||u_\ell||_{\mathrm{C}^{1,\alpha'}(\overline{\mathcal{O}})} \leq K_5\bigg[&||h_\ell||_{\mathrm{L}^{p'}(\mathcal{O})} + ||\psi_\varepsilon(|\mathrm{D}^1 u_\ell|^2 - g_\ell^2)||_{\mathrm{L}^{p'}(\mathcal{O})}\\ &+ \sum_{\kappa\in\mathbb{I}\setminus\{\ell\}} ||\psi_\delta(w_\ell - w_\kappa - \vartheta_{\ell,\kappa})||_{\mathrm{L}^{p'}(\mathcal{O})}\bigg], \quad \text{for } \ell \in \mathbb{I},\end{aligned} \tag{A.19}$$

for some $K_5 = K_5(d, \Lambda, \alpha')$, where $p' \in (d, \infty)$ is such that $\alpha' = 1 - d/p'$. Observe that for each $w \in \mathcal{C}^1$, we get $[c_\ell - \mathcal{L}_\ell]u_\ell \leq h_\ell$, with $\ell \in \mathbb{I}$. Then, arguing in the same way as in the proof of (3.2), it follows that for each $w \in \mathcal{C}^1$,

$$-\frac{2m}{\delta}\frac{\max_{\kappa\in\mathbb{I}}\{||w_\kappa||_{\mathrm{C}^1(\overline{\mathcal{O}})}\}}{\min_{(x,\kappa)\in\overline{\mathcal{O}}\times\mathbb{I}}\{c_\kappa(x)\}} \leq u_\ell \leq C_1 \quad \text{on } \overline{\mathcal{O}}, \text{ for } \ell \in \mathbb{I}, \tag{A.20}$$

with C_1 as in (3.2). Meanwhile, taking $\tilde{\eta}_\ell = |\mathrm{D}^1 u_\ell|^2 - \lambda\tilde{A}u_\ell$, with

$$\tilde{A} := \max_{(x,\ell)\in\overline{\mathcal{O}}\times\mathbb{I}} |\mathrm{D}^1 u_\ell(x)|,$$

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and following steps similar to those seen in the proof of (3.3), it can be verified that for each $w \in \mathcal{C}^1$,

$$|\mathrm{D}^1 u_\ell| \leq K_6 \left\{ 1 + \max_{\kappa \in \mathbb{I}} \{ ||w_\kappa||_{\mathrm{C}^1(\overline{\mathcal{O}})} \} \right\} \quad \text{on } \overline{\mathcal{O}}, \text{ for } \ell \in \mathbb{I}, \tag{A.21}$$

for some constant $K_6 = K_6(\Lambda, d, \theta, m, \lambda, 1/\delta)$. We proceed to give the proof of Proposition 3.1.

Proof of Proposition 3.1. *Existence.* Define the mapping

$$T:(\mathcal{C}^1, ||\cdot||_{\mathcal{C}^1}) \longrightarrow (\mathcal{C}^1, ||\cdot||_{\mathcal{C}^1})$$

by $T[w] = u$ for each $w \in \mathcal{C}^1$, where $u \in \mathcal{C}^{2,\alpha'} \subset \mathcal{C}^1$ is the unique solution to the NDPS (A.18). To use Schaefer's fixed point theorem (see e.g. [8, Theorem 4, p. 539]), we only need to verify the following: (i) the mapping T is continuous and compact; (ii) the set $\mathcal{A} := \{w \in \mathcal{C}^1 : w = \varrho T[w],\ \text{for some } \varrho \in [0, 1]\}$ is bounded uniformly, i.e. $||w||_{\mathcal{C}^1} \leq C$, for each $w \in \mathcal{A}$, where C is some positive constant that is independent of w and ϱ.

To verify the first item above, notice that, by (A.19)–(A.21) and by the Arzelà–Ascoli compactness criterion (see [8, Section C.8, p. 718]), T maps bounded sets in $\mathcal{C}^1$ into bounded sets in itself that are precompact in $\mathcal{C}^1$. With this remark and by the uniqueness of the solution to the NPDS (A.18), the reader can verify that T is a continuous and compact mapping from $\mathcal{C}^1$ into itself. Let us prove the second item above. Consider now $w \in \mathcal{A}$. Observe that if $\varrho = 0$, it follows immediately that $w \equiv \overline{0} \in \mathcal{C}^1$, where $\overline{0}$ is the null function. Assume that $w \in \mathcal{C}^1$ is such that $T[w] = \frac{1}{\varrho} w = (\frac{1}{\varrho} w_1, \ldots, \frac{1}{\varrho} w_m)$ for some $\varrho \in (0, 1]$, or, in other words, $w \in \mathcal{C}^{2,\alpha'}$ and

$$[c_\ell - \mathcal{L}_\ell] w_\ell = f_\ell, \text{ in } \mathcal{O}, \quad \text{s.t. } w_\ell = 0, \text{ on } \partial\mathcal{O}, \quad \text{for } \ell \in \mathbb{I}, \tag{A.22}$$

where

$$f_\ell := \varrho \left[h_\ell - \psi_\varepsilon((|\mathrm{D}^1 w_\ell|/\varrho)^2 - g_\ell^2) - \sum_{\kappa \in \mathbb{I} \setminus \{\ell\}} \psi_\delta(w_\ell - w_\kappa - \vartheta_{\ell,\kappa}) \right].$$

By Theorems 6.14 and 9.19 of [10], we have that $w \in \mathcal{C}^{3,\alpha'}$, since (H2)–(H4) hold and $f_\ell \in \mathrm{C}^{1,\alpha'}(\overline{\mathcal{O}})$. Then, $\mathcal{A} \subset \mathcal{C}^{3,\alpha'}$. Observe that $0 \leq w_\ell \leq C_1$ on $\overline{\mathcal{O}}$, because of $[c_\ell - \mathcal{L}_\ell] w_\ell \leq h_\ell$. Taking $\tilde{\eta}_\ell = |\mathrm{D}^1 w_\ell|^2 - \lambda \tilde{A} w_\ell$, with $\tilde{A} := \max_{(x,\ell) \in \overline{\mathcal{O}} \times \mathbb{I}} |\mathrm{D}^1 w_\ell(x)|$, using (A.22) and applying the same arguments seen in the proof of (3.3), one can check that for each $w = (w_1, \ldots, w_m) \in \mathcal{A}$, we have

$$|\mathrm{D}^1 w_\ell| \leq C_2, \quad \text{on } \overline{\mathcal{O}}, \text{ for } \ell \in \mathbb{I}, \tag{A.23}$$

where C_2 is a positive constant as in Lemma 3.1. Notice that C_1 and C_2 are independent of w and ϱ. It follows that $\mathcal{A}$ is bounded uniformly. From that, we see that the items above, (i) and (ii), are true, and by Schaefer's fixed point theorem, there exists a fixed point $u = (u_1, \ldots, u_m) \in \mathcal{C}^1$ to the problem $T[u] = u$ which satisfies the NPDS (3.1). In addition, we have $u = T[u] \in \mathcal{C}^{2,\alpha'}$, and by arguments similar to those seen previously, it can be shown that u is nonnegative and belongs to $\mathcal{C}^{3,\alpha'}(\overline{\mathcal{O}})$. Again, repeating the same arguments seen above, we can conclude that $u \in \mathcal{C}^{4,\alpha'}(\overline{\mathcal{O}})$. □

Proof of Proposition 3.1. *Uniqueness.* The uniqueness of the solution u to the NPDS (3.1) is obtained by contradiction. Assume that there are two solutions $u, v \in \mathcal{C}^{4,\alpha'}$ to the NPDS (3.1). Let $\nu = (\nu_1, \ldots, \nu_m) \in \mathcal{C}^{4,\alpha'}$ be such that $\nu_\ell := u_\ell - v_\ell$ for $\ell \in \mathbb{I}$. Let $(x_\circ, \ell_\circ) \in \overline{\mathcal{O}} \times \mathbb{I}$ be such that $\nu_{\ell_\circ}(x_\circ) = \max_{(x,\ell) \in \overline{\mathcal{O}} \times \mathbb{I}} \nu_\ell(x)$. If $x_\circ \in \partial\mathcal{O}$, trivially we have $u_\ell - v_\ell \leq 0$ in $\overline{\mathcal{O}}$, for $\ell \in \mathbb{I}$. Suppose that $x_\circ \in \mathcal{O}$. Then

$$\begin{aligned} &\mathrm{D}^1 \nu_{\ell_\circ}(x_\circ) = 0, \quad \operatorname{tr}[a_{\ell_\circ}(x_\circ) \mathrm{D}^2 \nu_{\ell_\circ}(x_\circ)] \leq 0, \\ &u_{\ell_\circ}(x_\circ) - u_\kappa(x_\circ) \geq v_{\ell_\circ}(x_\circ) - v_\kappa(x_\circ) \quad \text{for } \kappa \neq \ell_\circ. \end{aligned} \tag{A.24}$$

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Then, from (3.1) and (A.24),

$$\begin{aligned} 0 &\geq \mathrm{tr}[a_{\ell_\circ} \mathrm{D}^2 v_{\ell_\circ}] \\ &= c_{\ell_\circ} v_{\ell_\circ} + \sum_{\kappa \in \mathbb{I} \setminus \{\ell_\circ\}} \{\psi_\delta(u_{\ell_\circ} - u_\kappa - \vartheta_{\ell_\circ,\kappa}) - \psi_\delta(v_{\ell_\circ,\kappa} - v_\kappa - \vartheta_{\ell_\circ,\kappa})\} \geq c_{\ell_\circ} v_{\ell_\circ} \quad \text{at } x_\circ \end{aligned} \tag{A.25}$$

because of

$$0 \leq \psi_\delta(u_{\ell_\circ} - u_\kappa - \vartheta_{\ell_\circ,\kappa}) - \psi_\delta(v_{\ell_\circ} - v_\kappa - \vartheta_{\ell_\circ,\kappa})$$

at $x_\circ$, for $\kappa \in \mathbb{I}$. From (A.25) and since $c_{\ell_\circ} > 0$, we have that $u_\ell(x) - v_\ell(x) \leq u_{\ell_\circ}(x_\circ) - v_{\ell_\circ}(x_\circ) \leq 0$ for $(x, \ell) \in \overline{\mathcal{O}} \times \mathbb{I}$. Taking now $\nu := v - u$ and proceeding in the same way than before, we obtain immediately that $v_\ell - u_\ell \leq 0$ on $\overline{\mathcal{O}}$ for $\ell \in \mathbb{I}$. Therefore $u = v$, and from here we conclude that the NPDS (3.1) has a unique solution u, whose components belong to $\mathrm{C}^{4,\alpha'}(\overline{\mathcal{O}})$. □

A.3. Verification of Equation (3.4)

Let us define the auxiliary function ϕ_ℓ by

$$\phi_\ell := \omega^2 |\mathrm{D}^2 u_\ell|^2 + \lambda A^1_{\varepsilon,\delta} \omega \mathrm{tr}[\alpha_{\ell_0} \mathrm{D}^2 u_\ell] + \mu |\mathrm{D}^1 u_\ell|^2 \quad \text{on } \mathcal{O}, \tag{A.26}$$

with

$$A^1_{\varepsilon,\delta} := \max_{(x,\ell) \in \overline{\mathcal{O}} \times \mathbb{I}} \omega(x) |\mathrm{D}^2 u_\ell(x)|,$$

$\lambda \geq \max\{1, 2/\theta\}$, $\mu \geq 1$ fixed, and $\alpha_{\ell_0} = (\alpha_{\ell_0\, ij})_{d \times d}$ such that $\alpha_{\ell_0\, ij} := a_{\ell_0\, ij}(x_0)$, where $(x_0, \ell_0) \in \overline{\mathcal{O}} \times \mathbb{I}$ is fixed. Recall that ω is a cut-off function as in Remark 3.1. We shall show that ϕ_ℓ satisfies (A.25). In particular, (A.25) holds when ϕ_ℓ is evaluated at its maximum $x_\mu \in \mathcal{O}$, which helps to show that (3.4) is true.

Lemma A.1. *Let ϕ_ℓ be the auxiliary function given by (A.26). Then there exists a positive constant $C_7 = C_7(d, \Lambda, \alpha', K_1)$ such that on $(x, \ell) \in \overline{\mathcal{O}} \times \mathbb{I}$,*

$$\begin{aligned} \omega^2 \mathrm{tr}\big[a_\ell \mathrm{D}^2 \phi_\ell\big] &\geq 2\theta \Big[\omega^4 |\mathrm{D}^3 u_\ell|^2 + \mu \omega^2 |\mathrm{D}^2 u_\ell|^2\Big] - 2\lambda C_7 A^1_{\varepsilon,\delta} \omega^2 |\mathrm{D}^3 u_\ell| \\ &- \lambda C_7 [A^1_{\varepsilon,\delta}]^2 - C_7(\lambda + \mu) A^1_{\varepsilon,\delta} - C_7 \mu + \omega^2 \sum_{\kappa \in \mathbb{I} \setminus \{\ell\}} \psi'_{\delta,\ell,\kappa}(\cdot)[\phi_\ell - \phi_\kappa] \\ &+ A^1_{\varepsilon,\delta} \omega^2 \psi'_{\varepsilon,\ell}(\cdot) \Big\{2\omega[\lambda\theta - 2] |\mathrm{D}^2 u_\ell|^2 - 2\lambda C_7 |\mathrm{D}^2 u_\ell| - (\lambda + \mu) C_7 + \frac{2}{A^1_{\varepsilon,\delta}} \langle \mathrm{D}^1 u_\ell, \mathrm{D}^1 \phi_\ell \rangle \Big\}. \end{aligned} \tag{A.27}$$

Before providing the verification of the lemma above, let us first prove (3.4).

Proof of Lemma 3.1. *Equation* (3.4). Let ϕ_ℓ be as in (A.26), where $\lambda \geq \max\{1, 2/\theta\}$ is fixed and $\mu \geq 1$ will be determined later on, and $(x_0, \ell_0) \in \overline{\mathcal{O}} \times \mathbb{I}$ satisfies

$$\omega(x_0) |\mathrm{D}^2 u_{\ell_0}(x_0)| = A^1_{\varepsilon,\delta} = \max_{(x,\ell) \in \overline{\mathcal{O}} \times \mathbb{I}} \omega(x) |\mathrm{D}^2 u_\ell(x)|. \tag{A.28}$$

Notice that if $x_0 \in \overline{\mathcal{O}} \setminus B_{\beta' r}$, by Remark 3.1 and (A.26), we obtain $\partial_{ij} u_\ell(x) \equiv 0$, for each $(x, \ell) \times \overline{\mathcal{O}} \times \mathbb{I}$. From here, (3.4) is trivially true. So assume that x_0 is in $B_{\beta' r}$. Without loss of generality

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we also assume that $A^1_{\varepsilon,\delta} > 1$, since if $A^1_{\varepsilon,\delta} \leq 1$, we get that $\omega(x)\left|D^2 u_\ell(x)\right| \leq A^1_{\varepsilon,\delta} \leq 1$ for $(x,\ell) \in \overline{\mathcal{O}} \times \mathbb{I}$. Taking $C_3 = 1$, we obtain the result in (3.4). Let $(x_\mu, \ell_\mu) \in \overline{\mathcal{O}} \times \mathbb{I}$ be such that $\phi_{\ell_\mu}(x_\mu) = \max_{(x,\ell)\in\overline{\mathcal{O}}\times\mathbb{I}} \phi_\ell(x)$. If $x_\mu \in \overline{\mathcal{O}} \setminus B_{\beta' r}$, from (3.3) and (A.26) it follows that

$$\omega^2\left|D^2 u_\ell\right|^2 \leq -\lambda A^1_{\varepsilon,\delta}\omega \mathrm{tr}\left[\alpha_{\ell_0} D^2 u_\ell\right] + \mu C_2^2, \text{ for } (x,\ell) \in \overline{\mathcal{O}} \times \mathbb{I}. \tag{A.29}$$

Evaluating (x_0, l_0) in (A.29) and by (2.3), (H3), (2.16), (3.1), and (3.3), it can be verified that $\left[A^1_{\varepsilon,\delta}\right]^2 \leq \lambda\Lambda[1 + C_2]A^1_{\varepsilon,\delta} + \mu C_2^2$. From here and because $A^1_{\varepsilon,\delta} > 1$, we conclude that

$$\omega(x)\left|D^2 u_\ell(x)\right| \leq A^1_{\varepsilon,\delta} \leq \lambda\Lambda[1 + C_2] + \mu C_2^2 =: C_3, \quad \text{for } (x,\ell) \in \overline{\mathcal{O}} \times \mathbb{I}.$$

From now on, assume that $x_\mu \in B_{\beta' r}$. Then

$$D^1\phi_{\ell_\mu}(x_\mu) = 0, \quad \mathrm{tr}\left[a_{\ell_\mu}(x_\mu) D^2\phi_{\ell_\mu}(x_\mu)\right] \leq 0. \tag{A.30}$$

Noting that

$$2\theta\omega^4\left|D^3 u_\ell\right|^2 - 2\lambda C_7 A^1_{\varepsilon,\delta}\omega^2\left|D^3 u_\ell\right| \geq -\frac{\lambda^2 C_7^2}{\theta}\left[A^1_{\varepsilon,\delta}\right]^2,$$

with $C_7 > 0$ as in Lemma A.1, and using (A.25) and (A.30), we have that

$$\begin{aligned} 0 \geq\, & 2\theta\mu\omega^2\left|D^2 u_{\ell_\mu}\right|^2 - \lambda^2 C_7\left[1 + \frac{C_7}{\theta}\right]\left[A^1_{\varepsilon,\delta}\right]^2 - C_7(\lambda+\mu)A^1_{\varepsilon,\delta} - C_7\mu \\ & + A^1_{\varepsilon,\delta}\omega^2\psi'_{\varepsilon,\ell}(\cdot)\left\{2\omega[\lambda\theta - 2]\left|D^2 u_{\ell_\mu}\right|^2 - 2\lambda C_7\left|D^2 u_{\ell_\mu}\right| - (\lambda+\mu)C_7\right\}, \quad \text{at } x_\mu. \end{aligned}$$

From here, we have that at least one of the following two inequalities is true:

$$2\theta\mu\omega^2\left|D^2 u_{\ell_\mu}\right|^2 - \lambda^2 C_7\left[1 + \frac{C_7}{\theta}\right]\left[A^1_{\varepsilon,\delta}\right]^2 - C_7(\lambda+\mu)A^1_{\varepsilon,\delta} - C_7\mu \leq 0, \quad \text{at } x_\mu, \tag{A.31}$$

$$A^1_{\varepsilon,\delta}\omega^2\psi'_{\varepsilon,\ell}(\cdot)\left\{2\omega[\lambda\theta - 2]\left|D^2 u_{\ell_\mu}\right|^2 - 2\lambda C_7\left|D^2 u_{\ell_\mu}\right| - (\lambda+\mu)C_7\right\} \leq 0, \quad \text{at } x_\mu. \tag{A.32}$$

Suppose that (A.31) holds. Then, evaluating (x_μ, ℓ_μ) in (A.26), we get

$$\begin{aligned} \phi_{\ell_\mu} \leq\, & \frac{\lambda^2 C_7}{2\theta\mu}\left[1 + \frac{C_7}{\theta}\right]\left[A^1_{\varepsilon,\delta}\right]^2 + \frac{C_7(\lambda+\mu)}{2\theta\mu}A^1_{\varepsilon,\delta} + \frac{C_7}{2\theta} + \mu C_2^2 \\ & + \lambda\Lambda A^1_{\varepsilon,\delta}\left\{\frac{\lambda^2 C_7}{2\theta\mu}\left[1 + \frac{C_7}{\theta}\right]\left[A^1_{\varepsilon,\delta}\right]^2 + \frac{C_7(\lambda+\mu)}{2\theta\mu}A^1_{\varepsilon,\delta} + \frac{C_7}{2\theta}\right\}^{1/2}, \quad \text{at } x_\mu. \end{aligned} \tag{A.33}$$

Meanwhile, evaluating (x_0, ℓ_0) in (A.26) and using (2.3) and (3.1), we get

$$\phi_{\ell_0} \geq \left[A^1_{\varepsilon,\delta}\right]^2 - \lambda\Lambda A^1_{\varepsilon,\delta}[C_2 + 1], \quad \text{at } x_0. \tag{A.34}$$

Then, taking μ large enough so that

$$\frac{K_7^{(\lambda)}}{\mu} \leq \left[\frac{1}{\lambda\Lambda}\left[1 - \frac{K_7^{(\lambda)}}{\mu}\right]\right]^2,$$

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with

$$K_7^{(\lambda)} := \frac{\lambda^2 C_7}{2\theta}\left[1 + \frac{C_7}{\theta}\right],$$

using (A.33)–(A.34), and since $\phi_{\ell_0}(x_0) \leq \phi_{\ell_\mu}(x_\mu)$ and λ, $A^1_{\varepsilon,\delta} > 1$, we have that

$$\frac{1}{\lambda\Lambda}\left[1 - \frac{K_7^{(\lambda)}}{\mu}\right]A^1_{\varepsilon,\delta} - K_8^{(\mu)} \leq \left\{\frac{K_7^{(\lambda)}}{\mu}[A^1_{\varepsilon,\delta}]^2 + \frac{C_7(\lambda+\mu)}{2\theta\mu}A^1_{\varepsilon,\delta} + \frac{C_7}{2\theta}\right\}^{1/2},$$

with

$$K_8^{(\mu)} := \frac{C_7}{2\theta\Lambda}\left[\frac{1}{\mu} + 1\right] + \frac{C_7}{2\theta} + \mu C_2.$$

Then,

$$\left\{\frac{1}{\lambda^2\Lambda^2}\left[1 - \frac{K_7^{(\lambda)}}{\mu}\right]^2 - \frac{K_7^{(\lambda)}}{\mu}\right\}[A^1_{\varepsilon,\delta}]^2 \leq \left\{\frac{2K_8^{(\mu)}}{\lambda\Lambda}\left[1 - \frac{K_7^{(\lambda)}}{\mu}\right] + \frac{C_7(\lambda+\mu)}{2\theta\mu}\right\}A^1_{\varepsilon,\delta} + \frac{C_7}{2\theta}.$$

From here, we conclude there exists a constant $C_3 = C_3(d, \Lambda, \alpha', K_3)$ such that

$$\omega(x)\left|D^2 u_\ell(x)\right| \leq A^1_{\varepsilon,\delta} \leq C_3 \quad \text{for } (x, \ell) \in \overline{\mathcal{O}} \times \mathbb{I}.$$

Now, assume that (A.32) holds. Then

$$2\omega^2[\lambda\theta - 2]\left|D^2 u_{\ell_\mu}\right|^2 \leq 2\lambda C_7\omega\left|D^2 u_{\ell_\mu}\right| + (\lambda+\mu)C_7$$

at x_μ, since $\psi'_\varepsilon \geq 0$ and $\omega \leq 1$. From here, we have that $\omega\left|D^2 u_{\ell_\mu}\right| \leq K_9^{(\lambda,\mu)}$ at x_μ, where $K_9^{(\lambda,\mu)}$ is a positive constant independent of $A^1_{\varepsilon,\delta}$. Therefore,

$$[A^1_{\varepsilon,\delta}]^2 - \lambda\Lambda A^1_{\varepsilon,\delta}[C_2+1] \leq \phi_{\ell_0}(x_0) \leq \phi_{\ell_\mu}(x_\mu) \leq \left[K_9^{(\lambda,\mu)}\right]^2 + \lambda\Lambda A^1_{\varepsilon,\delta}K_9^{(\lambda,\mu)} + \mu C_2^2.$$

From here, we conclude that there exists a constant $C_3 = C_3(d, \Lambda, \alpha', K_1)$ such that $\omega\left|D^2 u_\ell\right| \leq A^1_{\varepsilon,\delta} \leq C_3$ for all $(x, \ell) \in \overline{\mathcal{O}} \times \mathbb{I}$. □

Proof of Lemma A.1. Taking first and second derivatives of ϕ_ℓ on $\overline{B}_{\beta' r}$, it can be verified that

$$\begin{aligned}
&\operatorname{tr}\left[a_\ell D^2\phi_\ell\right]\\
&= \left|D^2 u_\ell\right|^2 \operatorname{tr}\left[a_\ell D^2\omega^2\right] + 2\left\langle a_\ell D^1\omega^2, D^1\left|D^2 u_\ell\right|^2\right\rangle + \omega^2 \operatorname{tr}\left[a_\ell D^2\left|D^2 u_\ell\right|^2\right]\\
&\quad + \lambda A^1_{\varepsilon,\delta}\operatorname{tr}\left[\alpha_{\ell_0} D^2 u_\ell\right]\operatorname{tr}\left[a_\ell D^2\omega\right] + 2\lambda A^1_{\varepsilon,\delta}\left\langle a_\ell D^1\omega, D^1\operatorname{tr}\left[\alpha_{\ell_0} D^2 u_\ell\right]\right\rangle\\
&\quad + \lambda A^1_{\varepsilon,\delta}\omega\sum_{ji}\alpha_{\ell_0 ji}\operatorname{tr}\left[a_\ell D^2\partial_{ji}u_\ell\right] + \mu\operatorname{tr}\left[a_\ell D^2\left|D^1 u_\ell\right|^2\right].
\end{aligned}$$

From here and noticing that from (2.11),

$$\operatorname{tr}\left[a_\ell D^2\left|D^1 u_\ell\right|^2\right] \geq 2\theta\left|D^2 u_\ell\right|^2 + 2\sum_i \partial_i u_\ell \operatorname{tr}\left[a_\ell D^2\partial_i u_\ell\right],$$

$$\operatorname{tr}\left[a_\ell D^2\left|D^2 u_\ell\right|^2\right] \geq 2\theta|D^3 u_\ell|^2 + 2\sum_{ji}\partial_{ji}u_\ell \operatorname{tr}\left[a_\ell D^2\partial_{ji}u_\ell\right],$$

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we find that

$$\begin{aligned}\operatorname{tr}\big[a_\ell \mathrm{D}^2\phi_\ell\big] \geq 2\theta\Big[\omega^2\big|\mathrm{D}^3u_\ell\big|^2+\mu\big|\mathrm{D}^2u_\ell\big|^2\Big]+\big|\mathrm{D}^2u_\ell\big|^2\operatorname{tr}\big[a_\ell\mathrm{D}^2\omega^2\big]\\ +2\big\langle a_\ell\mathrm{D}^1\omega^2,\mathrm{D}^1\big|\mathrm{D}^2u_\ell\big|^2\big\rangle+\lambda A^1_{\varepsilon,\delta}\operatorname{tr}\big[\alpha_{\ell_0}\mathrm{D}^2u_\ell\big]\operatorname{tr}\big[a_\ell\mathrm{D}^2\omega\big]\\ +2\lambda A^1_{\varepsilon,\delta}\big\langle a_\ell\mathrm{D}^1\omega,\mathrm{D}^1\operatorname{tr}\big[\alpha_{\ell_0}\mathrm{D}^2u_\ell\big]\big\rangle+2\mu\sum_i\operatorname{tr}\big[a_\ell\mathrm{D}^2\partial_iu_\ell\big]\partial_iu_\ell\\ +\sum_{ji}\big[2\omega^2\partial_{ji}u_\ell+\lambda A^1_{\varepsilon,\delta}\omega\alpha_{\ell_0 ji}\big]\operatorname{tr}\big[a_\ell\mathrm{D}^2\partial_{ji}u_\ell\big].\end{aligned}\tag{A.35}$$

Meanwhile, differentiating twice in (3.1), we see that

$$\begin{aligned}\operatorname{tr}\big[a_\ell\mathrm{D}^2\partial_{ji}u_\ell\big]=\psi''_{\varepsilon,\ell}(\cdot)\bar\eta^{(i)}_\ell\bar\eta^{(j)}_\ell+\psi'_{\varepsilon,\ell}(\cdot)\partial_{ji}\big[\big|\mathrm{D}^1u_\ell\big|^2-g^2_\ell\big]+\sum_{\kappa\in\mathbb{I}\setminus\{\ell\}}\psi''_{\delta,\ell,\kappa}(\cdot)\bar\eta^{(i)}_{\ell,\kappa}\bar\eta^{(j)}_{\ell,\kappa}\\ +\sum_{\kappa\in\mathbb{I}\setminus\{\ell\}}\psi'_{\delta,\ell,\kappa}(\cdot)\partial_{ji}\big[u_\ell-u_\kappa\big]-\operatorname{tr}\big[\partial_j[a_\ell]\mathrm{D}^2\partial_iu_\ell\big]-\operatorname{tr}\big[[\partial_{ji}a_\ell]\mathrm{D}^2u_\ell\big]\\ -\operatorname{tr}\big[[\partial_ia_\ell]\mathrm{D}^2\partial_ju_\ell\big]-\partial_{ji}\big[h_\ell-\langle b_\ell,\mathrm{D}^1u_\ell\rangle-c_\ell u_\ell\big],\end{aligned}\tag{A.36}$$

where $\bar\eta_\ell=\Big(\bar\eta^{(1)}_\ell,\ldots,\bar\eta^{(d)}_\ell\Big)$ and $\bar\eta_{\ell,\kappa}=\Big(\bar\eta^{(1)}_{\ell,\kappa},\ldots,\bar\eta^{(d)}_{\ell,\kappa}\Big)$ with $\bar\eta^{(i)}_\ell:=\partial_i\Big[\big|\mathrm{D}^1u_\ell\big|^2-g^2_\ell\Big]$ and $\bar\eta^{(i)}_{\ell,\kappa}:=\partial_i[u_\ell-u_\kappa]$. From (A.6) and (2.35)–(2.36), it follows that

$$\begin{aligned}\omega^2\operatorname{tr}\big[a_\ell\mathrm{D}^2\phi_\ell\big]\geq 2\theta\Big[\omega^4\big|\mathrm{D}^3u_\ell\big|^2+\mu\omega^2\big|\mathrm{D}^2u_\ell\big|^2\Big]+\widetilde D_3+\widetilde D_4\\ +\omega^2\Big\{\psi''_{\varepsilon,\ell}(\cdot)\big\langle\big[2\omega^2\mathrm{D}^2u_\ell+\lambda A^1_{\varepsilon,\delta}\omega\alpha_{\ell_0}\big]\bar\eta_\ell,\bar\eta_\ell\big\rangle\\ +\sum_{\kappa\in\mathbb{I}\setminus\{\ell\}}\psi''_{\delta,\ell,\kappa}(\cdot)\big\langle\big[2\omega^2\mathrm{D}^2u_\ell+\lambda A^1_{\varepsilon,\delta}\omega\alpha_{\ell_0}\big]\bar\eta_{\ell,\kappa},\bar\eta_{\ell,\kappa}\big\rangle\Big\}\\ +\omega^2\psi'_{\varepsilon,\ell}(\cdot)\widetilde D_5+\omega^2\sum_{\kappa\in\mathbb{I}\setminus\{\ell\}}\psi'_{\delta,\ell,\kappa}(\cdot)\widetilde D_{6\,\kappa},\end{aligned}\tag{A.37}$$

where

$$\begin{aligned}\widetilde D_3&:=2\omega^2\big\langle a_\ell\mathrm{D}^1\omega^2,\mathrm{D}^1\big|\mathrm{D}^2u_\ell\big|^2\big\rangle+2\lambda A^1_{\varepsilon,\delta}\omega^2\big\langle a_\ell\mathrm{D}^1\omega,\mathrm{D}^1\operatorname{tr}\big[\alpha_{\ell_0}\mathrm{D}^2u_\ell\big]\big\rangle\\ &\quad-\sum_{ij}\big[2\omega^4\partial_{ij}u_\ell+\lambda A^1_{\varepsilon,\delta}\omega^3\alpha_{\ell_0\,ij}\big]\big[2\operatorname{tr}\big[\partial_ja_\ell\mathrm{D}^2\partial_iu_\ell\big]-\partial_{ij}\langle b_\ell,\mathrm{D}^1u_\ell\rangle\big],\\ \widetilde D_4&:=\omega^2\big|\mathrm{D}^2u_\ell\big|^2\operatorname{tr}\big[a_\ell\mathrm{D}^2\omega^2\big]-\mu\omega^2\widetilde D_2u_\ell+\lambda A^1_{\varepsilon,\delta}\omega^2\operatorname{tr}\big[\alpha_{\ell_0}\mathrm{D}^2u_\ell\big]\operatorname{tr}\big[a_\ell\mathrm{D}^2\omega\big]\\ &\quad-\sum_{ji}\big[2\omega^4\partial_{ji}u_\ell+\lambda A^1_{\varepsilon,\delta}\omega^3\alpha_{\ell_0\,ji}\big]\Big\{\operatorname{tr}\big[[\partial_{ji}a_\ell]\mathrm{D}^2u_\ell\big]+\partial_{ji}\big[h_\ell-c_\ell u_\ell\big]\Big\},\\ \widetilde D_5&:=2\mu\big\langle\mathrm{D}^1u_\ell,\bar\eta_\ell\big\rangle+\operatorname{tr}\big[\big[2\omega^2\mathrm{D}^2u_\ell+\lambda A^1_{\varepsilon,\delta}\omega\alpha_{\ell_0}\big]\mathrm{D}^2\big[\big|\mathrm{D}^1u_\ell\big|^2-g^2_\ell\big]\big],\\ \widetilde D_{6\,\kappa}&:=2\mu\big\langle\mathrm{D}^1u_\ell,\bar\eta_{\ell,\kappa}\big\rangle+\operatorname{tr}\big[\big[2\omega^2\mathrm{D}^2u_\ell+\lambda A^1_{\varepsilon,\delta}\omega\alpha_{\ell_0}\big]\mathrm{D}^2\big[u_\ell-u_\kappa\big]\big].\end{aligned}$$

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Recall that $\widetilde{D}_2 u_\ell$ is given in (A.7). To obtain the next inequalities, we shall recurrently use (H3), (H4), Remark 3.1, (3.2), (3.3), and $\lambda,\ \mu \geq 1$. Then,

$$\begin{aligned}\widetilde{D}_3 \geq &-2\left\{4\Lambda K_1 d^4 + \Lambda^2 d^4 + d^3[2+\Lambda][d+\Lambda]\right\}\lambda A^1_{\varepsilon,\delta}\omega^2\left|\mathrm{D}^3 u_\ell\right| \\ &- 2d^2\lambda A^1_{\varepsilon,\delta}[2+\Lambda]dC_2\Lambda - 4d^3\Lambda\lambda\left[A^1_{\varepsilon,\delta}\right]^2[2+\Lambda],\end{aligned} \tag{A.38}$$

and by (2.11),

$$\begin{aligned}\widetilde{D}_4 \geq &-\left\{2\Lambda d^2 K_1 + d^4\Lambda^2 K_1 + d^2\Lambda[2+\Lambda]\left[d^2+1\right]\right\}\lambda\left[A^1_{\varepsilon,\delta}\right]^2 \\ &- \left\{2\mu\left[2C_2\Lambda d^3 + 2d^{1/2}\Lambda C_2\right] + d^2\lambda\Lambda[2+\Lambda]\left[2C_2 + C_1\right]\right\}A^1_{\varepsilon,\delta} \\ &- 2\mu\left\{2C_2\Lambda d^2 + 2C_1C_2 d^{1/2}\Lambda - 2C_2\Lambda d^{1/2}\right\}.\end{aligned} \tag{A.39}$$

On the other hand, since $\lambda \geq \frac{2}{\theta}$ and using (2.11), we have that

$$\begin{aligned}\left\langle\left[2\omega^2\mathrm{D}^2 u_\ell + \lambda A^1_{\varepsilon,\delta}\omega\alpha_{\ell_0}\right]\gamma,\ \gamma\right\rangle &\geq \omega\left[\lambda A^1_{\varepsilon,\delta}\theta - 2\omega\left|\mathrm{D}^2 u_\ell\right|\right]|\gamma|^2 \\ &\geq \omega A^1_{\varepsilon,\delta}\left[\lambda\theta - 2\right]|\gamma|^2 \geq 0,\end{aligned} \tag{A.40}$$

for $\gamma \in \mathbb{R}^d$. From here and since $\psi''_{\varepsilon,\ell}(\cdot) \geq 0$ and $\psi''_{\delta,\ell,\kappa}(\cdot) \geq 0$, it follows that

$$\begin{aligned}&\psi''_{\varepsilon,\ell}(\cdot)\left\langle\left[2\omega^2\mathrm{D}^2 u_\ell + \lambda A^1_{\varepsilon,\delta}\omega\alpha_{\ell_0}\right]\bar{\eta}_\ell,\ \bar{\eta}_\ell\right\rangle \\ &\qquad + \sum_{\kappa\in\mathbb{I}\setminus\{\ell\}}\psi''_{\delta,\ell,\kappa}(\cdot)\left\langle\left[2\omega^2\mathrm{D}^2 u_\ell + \lambda A^1_{\varepsilon,\delta}\omega\alpha_{\ell_0}\right]\bar{\eta}_{\ell,\kappa},\ \bar{\eta}_{\ell,\kappa}\right\rangle \geq 0.\end{aligned} \tag{A.41}$$

It is easy to verify that

$$\begin{aligned}&\omega^2\left\langle\mathrm{D}^1 u_\ell,\ \mathrm{D}^1\left|\mathrm{D}^2 u_\ell\right|^2\right\rangle + \lambda A^1_{\varepsilon,\delta}\omega\left\langle\mathrm{D}^1 u_\ell,\ \mathrm{D}^1\mathrm{tr}\left[\alpha_{\ell_0}\mathrm{D}^2 u_\ell\right]\right\rangle + \mu\left\langle\mathrm{D}^1 u_\ell,\ \mathrm{D}^1\left|\mathrm{D}^1 u_\ell\right|^2\right\rangle \\ &\quad = \left\langle\mathrm{D}^1 u_\ell,\ \mathrm{D}^1\phi_\ell\right\rangle - \left\langle\mathrm{D}^1 u_\ell,\ \mathrm{D}^1\omega^2\right\rangle\left|\mathrm{D}^2 u_\ell\right|^2 - \lambda A^1_{\varepsilon,\delta}\mathrm{tr}[\alpha_{\ell_0}\mathrm{D}^2 u_\ell]\left\langle\mathrm{D}^1 u_\ell,\ \mathrm{D}^1\omega\right\rangle\end{aligned} \tag{A.42}$$

since

$$\begin{aligned}\partial_i\phi_\ell = &\left|\mathrm{D}^2 u_\ell\right|^2\partial_i\omega^2 + \omega^2\partial_i\left|\mathrm{D}^2 u_\ell\right|^2 \\ &+ \lambda A^1_{\varepsilon,\delta}\mathrm{tr}\left[\alpha_{\ell_0}\mathrm{D}^2 u_\ell\right]\partial_i\omega + \lambda A^1_{\varepsilon,\delta}\omega\mathrm{tr}\left[\alpha_{\ell_0}\mathrm{D}^2\partial_i u_\ell\right] + \mu\partial_i\left|\mathrm{D}^1 u_\ell\right|^2 \quad \text{on } B_{\beta r}.\end{aligned}$$

Then, by (A.40) and (A.42),

$$\begin{aligned}\widetilde{D}_5 \geq &\ 2\omega A^1_{\varepsilon,\delta}[\lambda\theta - 2]\left|\mathrm{D}^2 u_\ell\right|^2 + 2\left\langle\mathrm{D}^1 u_\ell,\ \mathrm{D}^1\phi_\ell\right\rangle \\ &- 4\lambda d^{1/2}C_2K_1A^1_{\varepsilon,\delta}\left|\mathrm{D}^2 u_\ell\right| - 2\lambda\Lambda d^{5/2}C_2K_1A^1_{\varepsilon,\delta}\left|\mathrm{D}^2 u_\ell\right| \\ &- 4\mu\Lambda^2 d^{1/2}A^1_{\varepsilon,\delta}C_2 - 2\lambda d\Lambda^2 A^1_{\varepsilon,\delta} - \lambda\Lambda^3 d^2 A^1_{\varepsilon,\delta} \\ &- 4d^2\lambda\Lambda^2 A^1_{\varepsilon,\delta} - 2\Lambda^3 d^2\lambda A^1_{\varepsilon,\delta}.\end{aligned} \tag{A.43}$$

Using the properties $|A|^2 - 2\mathrm{tr}[AB] + |B|^2 = \sum_{ij}\left(A_{ij} - B_{ij}\right)^2 \geq 0$ and $|y_1|^2 - 2\langle y_1, y_2\rangle + |y|^2 = \sum_i (y_{1,i} - y_{2,i})^2 \geq 0$, where $A = (A_{ij})_{d\times d}$, $B = (B_{ij})_{d\times d}$, and $y_1 = (y_{1,2}, \ldots, y_{1,d})$, $y_2 = (y_{2,1}, \ldots, y_{2,d})$ belong $\mathcal{S}(d)$ and $\mathbb{R}^d$, respectively, and by definition of ϕ_ℓ, it is easy to corroborate the following identity:

$$\widetilde{D}_{6\kappa} \geq \phi_\ell - \phi_\kappa,\ \text{ for } \kappa \neq \ell. \tag{A.44}$$

Applying (A.41)–(A.44) in (A.37) and considering that all constants that appear in those inequalities (i.e. (A.41)–(A.44)) are bounded by a universal constant $C_7 = C_7(d,\ \Lambda,\ \alpha',\ K_1)$, we obtain the desired result in the lemma above. With this remark, the proof is concluded. □

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