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On Asymptotic Series Expansions of Solutions to the Riccati Equation

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Abstract—We consider scalar real Riccati equations with coefficients expanding in convergent power series in a neighborhood of infinity. Continued solutions of such equations are studied. Power geometry methods are used to obtain conditions for expanding these solutions in asymptotic series.

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INTRODUCTION

The following Riccati equation is studied:

$$y' + \sum_{i=0}^{2} f_i(x)y^i = 0, \qquad y = y(x), \quad x, y \in \mathbb{R}. \tag{1}$$

Riccati [1] considered an equation of the form (1)

$$y' + ay^2 + bx^p = 0, \qquad a, b, p = \text{const}\,. \tag{2}$$

After trying to integrate this equation by quadratures, J. Bernoulli discovered the possibility of expanding the solutions of equation (2) in a series (see [2]). Later it turned out that the solutions of equation (2) can be expanded in convergent series using Bessel functions (see, for example, [2]). In recent years, power geometry methods have been developed that allow one to obtain series expansions of solutions of differential equations [3]. These methods were effectively used in relation to equation (1) in [4], [5], where expansions of certain solutions of equation (1) in convergent series were given. In this paper, we investigate solutions of equation (1) defined in a certain neighborhood of the point $x = +\infty$ (such solutions will be called *extended*). We describe the conditions for the expansion of such solutions in asymptotic series (not always convergent) in a neighborhood of the point $x = +\infty$. In this case, we assume that the functions $f_i(x)$, $i \in \{0, 1, 2\}$, can be represented as the following real power series uniformly and absolutely converging in a neighborhood of the point $x = +\infty$:

$$\begin{gathered} f_i(x) = \sum_{j=1}^{\infty} c_{ij}x^{p_{ij}}, \qquad c_{ij}, p_{ij} = \text{const} \in \mathbb{R}, \quad p_{ij+1} < p_{ij}, \\ \lim_{j\to\infty} p_{ij} = -\infty, \qquad i \in \{0, 1, 2\}. \end{gathered} \tag{3}$$

Note that, in contrast to traditional power series, the exponents in the power series in (3) and other equations, just as in [4]–[6], are real numbers and not necessarily integers.

The results obtained can easily be transferred to the case of the endpoint x_0 by replacing $t = (x_0 - x)^{-1}$.

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If the solution is not an extended one, then it is defined in some left half-neighborhood of the point $a < +\infty$, not at the point of a itself. For an asymptotic analysis of such solutions, we can again, use the substitution $t = (a - x)^{-1}$.

Further, we will use terms employed in power geometry [3]. The Newton polygon N of equation (1) subject to (3) represents the closed convex hull of the points

$$Q = (-1, 1), \quad Q_{ij} = (p_{ij}, i), \qquad i \in \{0, 1, 2\}, \quad j \in \{1, 2, \dots\}.$$

In studying the behavior of solutions in a neighborhood of the point $x = +\infty$, an important role is played by the location of the point Q relative to the right border of the polygon N.

In the case where the point Q belongs to the right boundary of the polygon N, i.e., the condition

$$\max(p_{01} + p_{21}, 2p_{11}) \le -2 \tag{4}$$

holds, the extended solutions of equation (1) can be represented as function series uniformly and absolutely converging in a neighborhood of the point $x = +\infty$. In general, each term of such a series is the product of a power of x by a Laurent series in powers of $\ln x$ (in most cases, these Laurent series turn out to be polynomials and sometimes constants). The results of the study of equation (1) under condition (4) were given in [4], [5]. Figure 1 corresponds to the case of inequality (4).

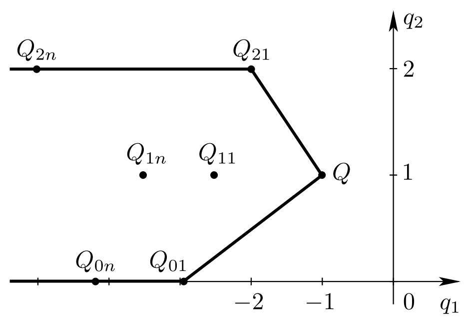

Fig. 1.

In this article, we will consider the case when the point Q does not belong to the right boundary of the polygon N, i.e., the following condition holds:

$$\max(p_{01} + p_{21}, 2p_{11}) > -2. \tag{5}$$

Figure 2 corresponds to the case of inequality (5).

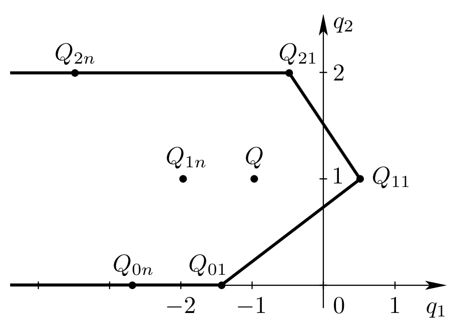

Fig. 2.

It will be shown that, under condition (5), equation (1) either has no extended solutions (these cases will be described in what follows) or this equation can be reduced by a power transformation to a form where condition (4) holds or two formal power series (formal solutions of equation (1)) that are the asymptotic representations (expansions) of all existing nontrivial extended solutions of this equation, can be calculated. These series, in general, diverge in a neighborhood of the point $x = +\infty$.

In what follows, we will assume that, in (3), $c_{21} \neq 0$ and also that if conditions (4) or (5) hold, then $c_{11} \neq 0$. The latter condition does not lead to a loss of generality. Indeed, if the function $f_1(x)$ is identically zero, then, in (4) and (5), we assume that $p_{11} = -\infty$ and, at the same time, $c_{11} = 1$.

If equation (1) is homogeneous (the function $f_0(x)$ is identically zero), then it can be integrated directly. However, for such an equation, we can use the same methods as in the general case. To do this, in the case of a homogeneous equation, we assume that

$$p_{01} = -p_{21} - 3, \qquad c_{0i} = 0, \quad i \in \{1, 2, \dots\}.$$

Then, in this situation, we have the inequality $p_{01} + p_{21} < -2$ and conditions (4) or (5) will hold, depending on the value of p_{11}.

Note that the the first approximations of all extended solutions of equation (1) were calculated in [6] for the case in which either condition (4) or condition (5) holds.

The paper [7] contains interesting results on the asymptotic analysis of solutions of the Riccati equation. The so-called stabilizing solutions of some types of the above equation were studied in [7] by other methods.

Geometric methods for the study of the Riccati equation are actively developed; in particular, they are used in the calculus of variations and optimal control theory [8].

1. FORMULATION OF THE MAIN RESULTS

Remark 1. To formulate the results, it is convenient to use another expression for the function $f_1(x)$, in which we will obtain rid of the condition $c_{11} \neq 0$.

If condition (5) holds, then, without loss of generality, we will assume that, in the series expansion of the function $f_1(x) = \sum_{j=1}^{\infty} c_{1j} x^{p_{1j}}$, the inequality $2p_{11} \geq p_{01} + p_{21}$ holds (in the case $2p_{11} = p_{01} + p_{21}$, the coefficient c_{11} may be equal to zero, but if $2p_{11} > p_{01} + p_{21}$, then $c_{11} \neq 0$).

Under conditions (3) and (5), the form of the set M of extended solutions of equation (1) depends on the relative position of the numbers $p_{01} + p_{21}$ and $2p_{11}$. To formulate the results, we recall the concept of asymptotic expansion (representation) of a function.

Definition. A power series $\sum_{i=1}^{\infty} c_i x^{\alpha_i}$, $\alpha_{i+1} < \alpha_i$, $\lim_{i\to\infty} \alpha_i = -\infty$, is called the *asymptotic expansion* (*representation*) of the function $f(x)$ as $x \to +\infty$, if, for all $m \geq 1$,

$$\lim_{x\to+\infty} x^{-\alpha_m}\Big(f(x) - \sum_{i=1}^{m} c_i x^{\alpha_i}\Big) = 0.$$

Theorem 1. *If conditions* (3), (5) *and one of the following two conditions:*

1) $p_{01} + p_{21} < 2p_{11}$,

2) $p_{01} + p_{21} = 2p_{11}$, $c_{11}^2 - 4c_{01}c_{21} > 0$

hold, then there exist two formal series

$$\begin{gathered} z_j(x) = \sum_{i=1}^{\infty} a_{ji} x^{s_{ji}}, \qquad j = 1, 2, \\ s_{11} = p_{01} - p_{11}, \quad s_{21} = p_{11} - p_{21}, \quad s_{ji+1} < s_{ji}, \\ \lim_{i\to\infty} s_{ji} = -\infty, \qquad a_{ji} = \text{const}, \end{gathered} \tag{6}$$

formally satisfying equation (1)*. Further, the set M of extended solutions of equation* (1) *is the union of two nonempty sets $M = M_1 \cup M_2$, where the elements M_j, $j = 1, 2$, of the set M are the extended solutions whose asymptotic expansion as $x \to +\infty$ is the series $z_j(x)$. In* (6)*, if condition* 1) *holds, then*

$$a_{11} = -c_{01}(c_{11})^{-1}, \qquad a_{21} = -c_{11}(c_{21})^{-1},$$

and if condition 2) *holds, then*

$$a_{11} = (2c_{21})^{-1}(-c_{11} + b) \quad a_{21} = (2c_{21})^{-1}(-c_{11} - b) \qquad b = \sqrt{c_{11}^2 - 4c_{01}c_{21}}\,.$$

The meaning of condition 2) in Theorem 1 is revealed in Theorems 2 and 3.

Theorem 2 [6]. *If conditions* (3) *and* (5) *hold and* $p_{01} + p_{21} = 2p_{11}$, *then, for* $c_{11}^2 - 4c_{01}c_{21} < 0$, *equation* (1) *has no extended solutions.*

Remark 2. Condition (3) in Theorem 2 is redundant. For the theorem to be valid, it is sufficient that the functions $f_i(x)$, $i \in \{0, 1, 2\}$, have as $x \to +\infty$ the power-law asymptotic

$$f_i(x) = c_{i1}x^{p_{i1}}(1 + o(x^{-\varepsilon})), \qquad \varepsilon = \mathrm{const} > 0, \quad i \in \{0, 1, 2\}.$$

Let us now consider the case where conditions (3) and (5) are satisfied and $p_{01} + p_{21} = 2p_{11}$, $c_{11}^2 - 4c_{01}c_{21} = 0$. In this case, equation (1) will be called *degenerate* and, in the opposite case, *nondegenerate*.

Theorem 3. *If equation* (1) *is degenerate, then there exists a transformation of the form*

$$\begin{gathered} y = z + h(x), \\ h(x) = \sum_{i=1}^{m} h_i x^{r_i}, \qquad h_i, r_i = \mathrm{const}, \\ r_{i+1} < r_i, \quad r_1 = p_{01} - p_{11}, \qquad h_1 = -(2c_{21})^{-1}c_{11}, \end{gathered} \tag{7}$$

a result of which equation (1) *becomes nondegenerate.*

This theorem allows us to establish the existence of extended solutions of the degenerate equation (1) and obtain their general form.

The formal series (6) are generally divergent. In Sec. 3 , we give an example of an equation for which one of the series (6) is divergent, while the other series is convergent.

2. PROOF OF THE STATEMENTS

In the proofs given below, by $U = \{x : x > R > 1\}$ we we denote a neighborhood of the point $x = +\infty$ in which the series (3) converge absolutely and uniformly. By the norm $\|W\|$ of an absolutely convergent series $W = \sum_{i=1}^{\infty} w_i$ we will mean the series $\sum_{i=1}^{\infty} |w_i|$. In a neighborhood of U, the following estimates exist:

$$\|f_i(x)\| \le Dx^{p_{i1}}, \qquad D = \mathrm{const} > 0, \quad i \in \{0, 1, 2\}. \tag{8}$$

The following well-known scheme forms the basis for justifying the statements about the general form of solutions to equation (1). If two solutions of the above equation are known, it is not difficult to obtain a general solution using a single quadrature. Indeed, let $y_1(x) \ne y_2(x)$ be the partial solutions of equation (1). Replacing $y = y_1(x) + w$, we obtain the homogeneous equation

$$w' + w^2 f_2(x) + w(2f_2(x)y_1(x) + f_1(x)) = 0. \tag{9}$$

To find the nontrivial solutions of this equation, we will make the change $w = z^{-1}$, after which equation (9) will take the form

$$z' - z(2f_2(x)y_1(x) + f_1(x)) - f_2(x) = 0. \tag{10}$$

The function $z_1(x) = (y_2(x) - y_1(x))^{-1}$ is the solution of equation (10). After the change $z = z_1(x) + v$, this equation takes the form

$$v' - v(2f_2(x)y_1(x) + f_1(x)) = 0. \tag{11}$$

The family of functions $v = Ce^{g(x)}$, where C is an arbitrary constant, and $g(x)$ is one of the primitive functions $2f_2(x)y_1(x) + f_1(x)$, is the general solution of equation (11). It follows that the general solution of equation (1) has the form: either

$$y = y_1(x)$$

or

$$y = y_1(x) + ((y_2(x) - y_1(x))^{-1} + Ce^{g(x)})^{-1}, \tag{12}$$

where C is an arbitrary constant. If three different solutions of equation (1) are known, then the general solution can be obtained without quadratures. Indeed, if, apart from $y_1(x)$, $y_2(x)$, the solution $y_3(x)$, $y_3(x) \neq y_1(x)$, $y_3(x) \neq y_2(x)$ is also known, then, after substituting it into (12), we obtain

$$C_1e^{g(x)} = (y_3(x) - y_1(x))^{-1} - (y_2(x) - y_1(x))^{-1}, \qquad C_1 = \mathrm{const}\,.$$

Combining this with (12), we easily see that the general solution of equation (1) has the form: either

$$y = y_1(x)$$

or

$$y = y_1(x) + \frac{(y_2(x) - y_1(x))(y_3(x) - y_1(x))}{y_3(x) - y_1(x) + C(y_2(x) - y_3(x))}\,, \tag{13}$$

where C is an arbitrary constant.

To prove Theorem 1, we will need the following lemma (see [6, Lemma 2]).

Consider the following equation of the form (1):

$$z' + \sum_{i=0}^{2} G_i(x)z^i = 0, \qquad G_i(x) \in C^0, \quad x \in U, \tag{14}$$

where, for $i = 0, 2$ $|G_i(x)| \leq Ax^{\beta_i}$, $A, \beta_i = \mathrm{const}$, $A > 0$ and, as $x \to +\infty$, the function $G_1(x)$ satisfies the condition

$$G_1(x) = x^{\beta_1}(A_1 + o(x^{-\nu})), \qquad A_1, \beta_1, \nu = \mathrm{const}, \quad \beta_1 > -1, \quad A_1 \neq 0, \quad \nu > 0. \tag{15}$$

Lemma. *If $\beta_0 + \beta_2 < -2$, then, for any $\delta > 0$, there exists a number $S_1 = S_1(\delta) \geq S$ and a solution $z(x) = z_\delta(x)$ of equation* (14)*, defined for $x \geq S_1$, such that the following estimate holds*:

$$|z(x)| \leq x^{\beta_0 + 1 + \delta}, \qquad x \geq S_1. \tag{16}$$

Proof of Theorem 1. First, let condition 1 hold. Note that, in this case, by Remark 1, we will have $c_{11} \neq 0$. Consider the sequence of formal series ($k = 1, 2, \dots$):

$$\begin{gathered} u_1 = u_1(x) = -(c_{11}x^{p_{11}})^{-1}f_0(x), \\ u_{k+1} = u_{k+1}(x) = -(c_{11}x^{p_{11}})^{-1}f_{0k}(x), \\ f_{0k}(x) = (h_k^2 - h_{k-1}^2)f_2(x) + u_kg_1(x) + u_k', \qquad g_1(x) = f_1(x) - c_{11}x^{p_{11}}, \\ h_0 = 0, \qquad h_k = \sum_{i=1}^{k} u_i. \end{gathered} \tag{17}$$

Let us now put $z_1(x) = \sum_{i=1}^{\infty} u_i(x)$. It follows from (17) that the function $U_k(x) = \sum_{i=1}^{k} u_i(x)$ satisfies the equation

$$U_k'(x) + U_k^2(x)f_2(x) + U_k(x)f_1(x) + f_0(x) + c_{11}x^{p_{11}}u_{k+1}(x) = 0.$$

Below we will show that the greatest power of the terms making up the series $u_k(x)$ tends to $-\infty$ as $k \to +\infty$. Combining this with the previous equality, we see that the series $z_1(x) = \sum_{i=1}^{\infty} u_i(x)$ is the formal solution of equation (1).

Let us construct the series $z_2(x)$. To do this, we will first make the change

$$y = w + v_1, \qquad in (1) v_1 = v_1(x) = -c_{11}(c_{21})^{-1}x^{p_{11}-p_{21}},$$

after which the equation takes the form

$$w' + f_2(x)w^2 + \widetilde{f}_1(x)w + \widetilde{f}_0(x) = 0,$$
$$\widetilde{f}_1(x) = f_1(x) + 2v_1(x)f_2(x) = -c_{11}x^{p_{11}} + \dots,$$
$$\widetilde{f}_0(x) = f_0(x) + v_1^2(x)(f_2(x) - c_{21}x^{p_{21}}) + v_1(x)(f_1(x) - c_{11}x^{p_{11}}) + v_1'(x).$$

Then, just as above, we will consider the following sequence of formal series ($k = 2, 3, \dots$):

$$v_2(x) = (c_{11}x^{p_{11}})^{-1}\widetilde{f}_0(x),$$
$$v_{k+1} = v_{k+1}(x) = (c_{11}x^{p_{11}})^{-1}\widetilde{f}_{0k}(x),$$
$$\widetilde{f}_{0k}(x) = (\widetilde{h}_k^2 - \widetilde{h}_{k-1}^2)f_2(x) + v_k\widetilde{g}_1(x) + v_k', \qquad \widetilde{g}_1(x) = \widetilde{f}_1(x) + c_{11}x^{p_{11}}, \tag{18}$$
$$\widetilde{h}_1 = 0, \qquad \widetilde{h}_k = \sum_{i=2}^{k} v_i.$$

Let us put $z_2(x) = \sum_{i=1}^{\infty} v_i(x)$. Arguing as above, we readily see that the series $z_2(x)$ is a formal solution of equation (1).

For any formal series

$$w = \sum_{i=1}^{\infty} a_i x^{r_i}, \qquad a_1 \neq 0, \quad r_{i+1} < r_i, \quad r_i \to -\infty,$$

we introduce the number $P(w) = r_1$, which we will be called the *order* of the series w. We assume that the order $P(w)$ of the zero series

$$w = \sum_{i=1}^{\infty} a_i x^{r_i}, \qquad a_i = 0, \quad i \geq 1,$$

is $-\infty$. It follows from (17) that

$$P(u_j) \leq s_1 - (j-1)\delta, \qquad j \geq 1,$$

where $s_1 = p_{01} - p_{11}$, $\delta = \min(2p_{11} - p_{01} - p_{21}, p_{11} + 1, p_{11} - p_{12}) > 0$.

Let us now express $z_1(x)$ as the power series

$$z_1(x) = \sum_{i=1}^{\infty} a_{1i}x^{s_{1i}}, \qquad s_{1i+1} < s_{1i}, \quad s_{1i} \to -\infty.$$

This series is a formal solution of equation (1) and satisfies conditions (6). Let us fix an arbitrary finite sum $\sum_{i=1}^{m} a_{1i}x^{s_{1i}}$, $m \geq 1$. Let us now define the number $k = \min\{j : s_{1j} < s_{1m} - 2\}$ and consider the sum $\sum_{i=1}^{k} a_{1i}x^{s_{1i}}$. Define the integer $L > 1$ as follows:

$$L = \min\{l : (l-1)\delta > |p_{01}| + |p_{11}| + |p_{21}| + 2 + |s_{1k}|\}. \tag{19}$$

Hence we have

$$\sum_{i=1}^{\infty} u_i = \sum_{i=1}^{L} u_i + w_1 = \sum_{i=1}^{k} a_{1i}x^{s_{1i}} + w_2, \tag{20}$$

where w_1, w_2 are formal series with $P(w_j) < s_{1k}$, $j = 1, 2$. Let us now choose a number N such that the following inequalities hold:

$$N > 2|s_1| - s_{1k} + 1, \qquad N > 2|s_1| + p_{21} + 2. \tag{21}$$

In what follows, to show that a function is bounded by a constant, we will use a so-called "universal" constant $D > 0$ such that $D + D = D$, $D \cdot D = D$.

The functions $f_i(x)$, $i \in \{0, 1, 2\}$, can be expressed as

$$f_i(x) = f_i^*(x) + F_i(x),$$
$$f_i^*(x) = \sum_{p_{ij} > -N} c_{ij} x^{p_{ij}}, \qquad \|F_i(x)\| \le D x^{-N}. \tag{22}$$

Consider the sequence of functions ($j = 1, 2, \dots, L$):

$$q_1 = q_1(x) = -(c_{11} x^{p_{11}})^{-1} f_0^*(x),$$
$$q_{j+1} = q_{j+1}(x) = -(c_{11} x^{p_{11}})^{-1} f_{0j}^*(x),$$
$$f_{0j}^*(x) = (\varphi_j^2 - \varphi_{j-1}^2) f_2^*(x) + q_j g_1^*(x) + q_j', \qquad g_1^*(x) = f_1^*(x) - c_{11} x^{p_{11}}, \tag{23}$$
$$\varphi_0 = 0, \quad \varphi_j = \sum_{i=1}^{j} q_i, \quad F_{0j}(x) = \varphi_j^2 F_2(x) + \varphi_j F_1(x) + F_0(x).$$

From (19)–(23), we obtain the estimates

$$\|q_j(x)\| \le D x^{s_1 - (j-1)\delta}, \qquad P(u_j - q_j) \le -N + 2|s_1| - p_{11},$$
$$\|F_{0j}(x)\| \le D x^{-N + 2|s_1|}, \qquad 1 \le j \le L. \tag{24}$$

After the change $y = z + \varphi_L(x)$, equation (1) takes the form

$$z' + z^2 f_2(x) + z(f_1(x) + 2\varphi_L(x) f_2(x)) + (\varphi_L^2(x) - \varphi_{L-1}^2(x)) f_2(x)$$
$$+ q_L(x) g_1(x) + q_L'(x) + F_{0L-1}(x) = 0. \tag{25}$$

It follows from (19), (21), (23), (24) that the resulting equation satisfies the assumptions of the lemma, where

$$\beta_0 < \max(p_{01} - L\delta, -N + 2|s_1|) < s_{1k},$$
$$\beta_0 + \beta_2 < \max(p_{01} - L\delta, -N + 2|s_1|) + p_{21} < -2.$$

Hence, equation (25) has the solution $z = \widetilde{z}(x)$, which, in a neighborhood $U_1 \subset U$ of the point $x = +\infty$ satisfies the estimate $|\widetilde{z}(x)| \le D x^{s_{1k}+1}$. Then equation (1) has the solution

$$y_1 = y_1(x) = \widetilde{z}(x) + \varphi_L(x).$$

The following inequality follows from (21) and (24):

$$P\biggl(\sum_{i=1}^{L} u_i - \varphi_L(x)\biggr) \le -N + 2|s_1| - p_{11} < s_{1k}.$$

But then, from (20), we obtain

$$\varphi_L(x) = \sum_{i=1}^{k} a_{1i} x^{s_{1i}} + o(x^{s_{1k}}) \qquad \text{at} \quad x \to +\infty.$$

It follows that the solution $y_1(x)$ has the form

$$y_1(x) = \sum_{i=1}^{m} a_{1i} x^{s_{1i}} + o(x^{s_{1m} - \mu_1}), \qquad \mu_1 = \text{const} > 0. \tag{26}$$

Let us now represent $z_2(x)$ as the power series $z_2(x) = \sum_{i=1}^{\infty} a_{2i} x^{s_{2i}}$ satisfying conditions (6) and fix an arbitrary finite sum $\sum_{i=1}^{m} a_{2i} x^{s_{2i}}$, $m \ge 1$. From the previous arguments, it follows that equation (1) has a solution $y_2 = y_2(x)$ of the form

$$y_2(x) = \sum_{i=1}^{m} a_{2i} x^{s_{2i}} + o(x^{s_{2m} - \mu_2}), \qquad \mu_2 = \text{const} > 0. \tag{27}$$

Let us now show that any extended solution of $y(x)$ of equation (1) satisfies one of the following conditions as $x \to +\infty$ to:

$$x^{-s_{1m}}\left(y(x) - \sum_{i=1}^{m} a_{1i}x^{s_{1i}}\right) = o(x^{-\mu_1}), \tag{28}$$

$$x^{-s_{2m}}\left(y(x) - \sum_{i=1}^{m} a_{2i}x^{s_{2i}}\right) = o(x^{-\mu_2}). \tag{29}$$

Obviously, the solution $y_1(x)$ satisfies condition (28) and the solution $y_2(x)$ satisfies condition (29). It follows from (12) that, for any other solution $y(x)$ (given that $C \neq 0$), either condition (28) (if $c_{11} > 0$) or condition (29) (if $c_{11} < 0$) will hold. Thus, it is proved that, under condition 1, for any extended solution of $y(x)$ of equation (1), one of the series $z_j(x)$, $j = 1, 2$, is its asymptotic expansion.

Let us now consider case 2, in which $p_{01} + p_{21} = 2p_{11}$, $c_{11}^2 - 4c_{01}c_{21} > 0$, and make the following substitution in equation (1):

$$\begin{gathered} y = w + u(x), \qquad u(x) = a_{11}x^{s_{11}}, \qquad s_{11} = p_{01} - p_{11} = p_{11} - p_{21}, \\ a_{11} = (2c_{21})^{-1}(-c_{11} + b), \quad b = \sqrt{c_{11}^2 - 4c_{01}c_{21}}\,. \end{gathered} \tag{30}$$

As a result, equation (1) will take the following form:

$$\begin{gathered} w' + w^2 f_2(x) + w\widetilde{g}_1(x) + \widetilde{g}_0(x) = 0, \qquad \widetilde{g}_i(x) = \sum_{j=1}^{\infty} \widetilde{c}_{ij}x^{\widetilde{p}_{ij}}, \\ \widetilde{p}_{11} = p_{11}, \quad \widetilde{c}_{11} = b, \quad \widetilde{p}_{01} < p_{01}, \qquad \widetilde{p}_{ij+1} < \widetilde{p}_{ij}, \quad \lim_{j\to\infty} \widetilde{p}_{ij} = -\infty, \quad i \in \{0, 1\}. \end{gathered} \tag{31}$$

It is obvious that condition 1 of our theorem holds for the above equation. Hence, there exist two formal series

$$\begin{gathered} w_j(x) = \sum_{i=1}^{\infty} \widetilde{a}_{ji}x^{\widetilde{s}_{ji}}, \qquad j = 1, 2, \\ \widetilde{s}_{11} = \widetilde{p}_{01} - p_{11}, \qquad \widetilde{s}_{21} = p_{11} - p_{21}, \qquad \widetilde{s}_{ji+1} < \widetilde{s}_{ji}, \\ \lim_{i\to\infty} \widetilde{s}_{ji} = -\infty, \qquad \widetilde{a}_{ji} = \text{const}, \end{gathered}$$

such that any solution of equation (31) has an asymptotic expansion in the form of one of these series. But then the following two series:

$$z_j(x) = u(x) + \sum_{i=1}^{\infty} \widetilde{a}_{ji}x^{\widetilde{s}_{ji}}, \qquad j = 1, 2,$$

have the corresponding properties with respect to equation (1). Note that $\widetilde{a}_{21} = -b(c_{21})^{-1}$. It follows that

$$u(x) + \widetilde{a}_{21}x^{\widetilde{s}_{21}} = (2c_{21})^{-1}(-c_{11} - b)x^{s_{21}}.$$

Thus, in case 2, condition (6) holds. Theorem 1 is proved. □

Proof of Theorem 2. In (1), let us make the change of variables

$$y = \frac{2w - c_{11}x^{p_{11}+1}}{2c_{21}x^{p_{21}+1}},$$

then, subject to the assumptions of the theorem, in a neighborhood of points $x = +\infty$, equation (1) will take the form

$$\begin{gathered} xw' + w^2(1 + o(x^{-\alpha})) + wo(x^{p_{11}+1-\alpha}) + A_1x^{2(p_{11}+1)}(1 + o(x^{-\alpha})) \\ + A_2x^{p_{11}+1} = 0, \qquad A_1, A_2, \alpha = \text{const}, \quad A_1, \alpha > 0. \end{gathered}$$

But then, in a neighborhood of the point $x=+\infty$, the following inequality holds:

$$xw' + A(w^2 + x^{2(p_{11}+1)}) < 0, \qquad A = \text{const} > 0.$$

This yields

$$xw' + Ax^{2(p_{11}+1)} < 0, \qquad xw' + Aw^2 < 0.$$

It follows from the first inequality that if the function $w(x)$ is defined in a neighborhood of the point $x=+\infty$, then $w(x)\to-\infty$ as $x\to+\infty$. But integrating the second inequality, we see that this is impossible. The resulting contradiction shows that all the solutions of equation (1) are nonextendable. Theorem 2 is proved. □

Proof of Theorem 3. First, let the series

$$f_2(x) = \sum_{i=1}^{\infty} c_{2i} x^{p_{2i}}$$

contain more than one nonzero term, i.e., $c_{22}\neq 0$. Let us fix the number $\alpha = p_{21} - p_{22} > 0$. Transformations of the form (30) will be called *standard*. Consider the sequence of standard transformations that leave equation (1) degenerate. Note that the number of such transformations is finite. Indeed, if the original equation is degenerate, then we fix a number $k\geq 1$ such that $p_{1k+1}\leq p_{11}-\alpha$, $p_{1k} > p_{11}-\alpha$. Obviously, by doing no more than k standard transformations, we will arrive at the equation of the form (1)

$$\widetilde{y}\,' + \sum_{i=0}^{2} \widetilde{f}_i(x)\widetilde{y}^{\,i} = 0, \tag{32}$$

in which $\widetilde{f}_i(x) = \widetilde{c}_{i1}x^{\widetilde{p}_{i1}} + \dots$, $i=0,1$, $\widetilde{f}_2(x) = f_2(x)$ and $\widetilde{p}_{01}\leq p_{01}-\alpha$, $\widetilde{p}_{11}\leq p_{11}-\alpha$. Hence it is clear that as a result of finitely many standard transformations, the condition $\widetilde{p}_{01}+p_{21} > -2$ will be violated and equation (32) will be nondegenerate.

If all the $c_{2i}=0$, $i\in\{2,3,\dots\}$, then for $k\geq 1$ we take a number k such that $p_{1k}\leq -1$, $p_{1k-1} > -1$. Obviously, by doing no more than k standard transformations, we will arrive at at a nondegenerate equation of the form (32). Theorem 3 is proved. □

3. EXAMPLES

Example 1. Consider the equation

$$y' + y^2 - xy + 1 = 0. \tag{33}$$

Here one of the series (6) is convergent, while the second series is divergent. To calculate these series, we will consider shortened equations and corresponding transformations of the equation under consideration (see [3]), for which the assumptions of Theorem 1 hold.

The function $z_1(x) = u_1(x) = x^{-1}$ from (17) is the solution of the shortened equation corresponding to the lower right edge of the Newton polygon of the original equation, as well as is the solution of equation (33). Calculating the functions $v_k(x)$ sequentially using the scheme described in the proof of Theorem 1, we obtain

$$v_1(x) = x, \qquad v_k(x) = a_k x^{-2k+3}, \quad a_k \leq -k!, \quad k\geq 2,$$

which implies that the series $z_2(x) = \sum_{k=1}^{\infty} v_k(x)$ diverges in any neighborhood of the point $x=+\infty$.

Let us now consider an example illustrating Theorem 3.

Example 2. Consider

$$y' + xy^2 + (2x-2)y + x - 2 + x^{-1} + x^{-2} + x^{-4} = 0. \tag{34}$$

This equation is degenerate, because conditions (3), (5) and the following equalities hold:

$$p_{01} + p_{21} = 2p_{11}, \qquad c_{11}^2 - 4c_{01}c_{21} = 0.$$

The function $u_1(x) = -1$ is the solution of the shortened equation $xu_1^2 + 2u_1x + x = 0$, corresponding to the right vertical edge of the Newton polygon of equation (34). After the change $y = y_1 - 1$, equation (34) takes the form

$$y_1' + xy_1^2 - 2y_1 + x^{-1} + x^{-2} + x^{-4} = 0. \tag{35}$$

Here again, conditions (3), (5) and equalities $p_{01} + p_{21} = 2p_{11}$, $c_{11}^2 - 4c_{01}c_{21} = 0$ hold, i.e., equation (35) is degenerate. The function $u_2(x) = x^{-1}$ is the solution of the shortened equation $xu_2^2 - 2u_2 + x^{-1} = 0$ corresponding to the right edge of the Newton polygon of equation (35). After the change $y_1 = y_2 + x^{-1}$ equation (35) takes the form

$$y_2' + xy_2^2 + x^{-4} = 0. \tag{36}$$

This equation satisfies conditions (3) and (4) and is, therefore, nondegenerate. In view of Theorem 1 and Remark 2 from [5], we see that equation (36) has a solution of the form of the power series $y_2 = (1/3)x^{-3} + \cdots$, which is absolutely and uniformly convergent in a neighborhood of the point $x = +\infty$. To equation (36) also corresponds the shortened equation $v' + xv^2 = 0$, whose solution is the function $v(x) = 2x^{-2}$. Continuing the calculations using methods from [5], we obtain the solution of equation (36) that has the form of the series

$$y_2 = 2x^{-2} - x^{-3} - x^{-4}(\ln x + a) + \cdots,$$

which is absolutely and uniformly convergent in a neighborhood of the point $x = +\infty$ and whose summands are products of decreasing powers of the variable x by polynomials in $\ln x$. As a result, we find that the family of extended solutions of equation (34) includes the power series $y = -1 + x^{-1} + (1/3)x^{-3} + \cdots$, as well as the family of series

$$y = -1 + x^{-1} + 2x^{-2} - x^{-3} - x^{-4}(\ln x + a) + \cdots,$$

where a is an arbitrary constant. All series absolutely and uniformly converge in a neighborhood of the point $x = +\infty$. It follows from (13) that the equation under consideration has no other continuous solutions.

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