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MATHEMATICAL PROBLEMS OF NONLINEARITY

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Omega-classification of Surface Diffeomorphisms Realizing Smale Diagrams

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The present paper gives a partial answer to Smale's question which diagrams can correspond to (A, B)-diffeomorphisms. Model diffeomorphisms of the two-dimensional torus derived by "Smale surgery" are considered, and necessary and sufficient conditions for their topological conjugacy are found. Also, a class G of (A, B)-diffeomorphisms on surfaces which are the connected sum of the model diffeomorphisms is introduced. Diffeomorphisms of the class G realize any connected Hasse diagrams (abstract Smale graph). Examples of diffeomorphisms from G with isomorphic labeled Smale diagrams which are not ambiently Ω-conjugated are constructed. Moreover, a subset $G_* \subset G$ of diffeomorphisms for which the isomorphism class of labeled Smale diagrams is a complete invariant of the ambient Ω-conjugacy is singled out.

Keywords: Smale diagram, (A,B)-diffeomorphism, Ω-conjugacy

1. Introduction and formulation of the results

Let f be a diffeomorphism of a connected closed n-manifold M^n. In 1967 S. Smale [1] introduced a concept of A-diffeomorphism, i.e., a diffeomorphism whose nonwandering set is hyperbolic and whose periodic points are dense in it. He proved that the nonwandering set NW_f of an A-diffeomorphism f is a union of pairwise disjoint subsets Λ_1, Λ_2, $\ldots$, Λ_r each of which is compact, invariant, topologically transitive and is called *basic set*. Moreover,

$$M^n = \bigcup_{\rho=1}^{r} W^s_{\Lambda_\rho} = \bigcup_{\rho=1}^{r} W^u_{\Lambda_\rho}.$$

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Basic sets Λ_i, Λ_j are said to be *in Smale relation* $\prec$ ($\Lambda_i \prec \Lambda_j$) if

$$W^s_{\Lambda_i} \cap W^u_{\Lambda_j} \neq \varnothing.$$

An A-diffeomorphism f *satisfies axiom B* (*is an* (A, B)*-diffeomorphism*) if from the condition $\Lambda_i \prec \Lambda_j$ it follows that there exist periodic points $p \in \Lambda_i$, $q \in \Lambda_j$ such that the manifolds W^s_p and W^u_q have a transverse intersection point. For (A, B)-diffeomorphisms the Smale relation $\prec$ is a partial order relation.

A sequence of pairwise distinct basic sets $\Lambda_i = \Lambda_{i_0}, \Lambda_{i_1}, \ldots, \Lambda_{i_m} = \Lambda_j$ ($m \geqslant 1$) such that $\Lambda_{i_0} \prec \Lambda_{i_1} \prec \ldots \prec \Lambda_{i_m}$ is called a *chain with the length* $m \in \mathbb{N}$ *connecting the basic sets* Λ_i *and* Λ_j. Such a chain is called *maximal* if no new basic set can be added to it.

The *Smale diagram* Δ_f of an (A, B)-diffeomorphism $f\colon M^n \to M^n$ is a graph whose vertices correspond to the basic sets and whose directed edges sequentially connect vertices of maximal chains. In fact, the Smale diagram is a special case of a Hasse diagram. Let us recall that a *Hasse diagram* of a partially ordered set $(X, \prec)$ is a graph whose vertices are elements of the set X, and a pair (x, y) forms an edge if $x \prec y$ and $\nexists z\colon x \prec z,\ z \prec y$.[1]

Lemma 1. *A Smale diagram of any* (A, B)*-diffeomorphism is a connected Hasse diagram.*

In [1] the following question is formulated as a problem (Problem 6.6a): which diagrams can correspond to (A, B)-diffeomorphisms?

This paper provides a partial answer to this question. Namely, let us define a model diffeomorphism $F_{C,\mathcal{O}^s,\mathcal{O}^u}\colon \mathbb{T}^2 \to \mathbb{T}^2$ derived from an algebraic Anosov diffeomorphism $\widehat{C}$ with a hyperbolic matrix C by "Smale surgery" along invariant manifolds of a finite (possibly empty) set of pairwise disjoint periodic orbits: along the stable manifolds of periodic orbits $\mathcal{O}^s = \{O^s_1, \ldots, O^s_k\}$ and along the unstable manifolds of periodic orbits $\mathcal{O}^u = \{O^u_1, \ldots, O^u_l\}$.

Theorem 1. *Model diffeomorphisms* $F_{C,\mathcal{O}^s,\mathcal{O}^u}$, $F_{C',\mathcal{O}'^s,\mathcal{O}'^u}\colon \mathbb{T}^2 \to \mathbb{T}^2$ *are topologically conjugate iff there exists a matrix* $H \in GL(2, \mathbb{Z})$ *such that* $HC = C'H$ *and* $\widehat{H}(\mathcal{O}^s) = \mathcal{O}'^s$, $\widehat{H}(\mathcal{O}^u) = \mathcal{O}'^u$, *where* $\widehat{H}$ *is the induced automorphism of the* 2*-torus.*

In Section 5, the class G of (A, B)-diffeomorphisms of closed orientable surfaces that are the connected sum of model diffeomorphisms is introduced. In Section 6 we prove the following result.

Theorem 2. *Any connected Hasse diagram can be realized by some diffeomorphism from the class* G.[2]

A *labeled Smale diagram* is the Smale diagram in which the topological conjugacy class of the restriction of the diffeomorphism to the corresponding basic set is additionally specified near each vertex. Two labeled diagrams are called *isomorphic* if there is an isomorphism of the corresponding graphs that preserves the incidence, the orientation of the edges, and the topological conjugacy classes of the vertices. Thus, the isomorphism of the labeled diagrams Δ_f, Δ_g of (A, B)-diffeomorphisms f, g is a criterion for an Ω*-conjugacy*, that is, for the existence of a homeomorphism $h\colon NW_f \to NW_g$ such that $hf|_{NW_f} = gh|_{NW_f}$. But in general this homeomorphism does not extend to the ambient manifold.

[1]For the first time this kind of visualization was systematically described by Birkhoff [2] in 1940; he named it in honor of Helmut Hasse, who used similar diagrams, however, such drawings were found in earlier works, for example, in the textbook of the French mathematician Henri Vogt [3], published in 1895.

[2]The idea of such a realization is developed in [13], but we present it here for completeness.

Recall that diffeomorphisms $f\colon M^n \to M^n$, $g\colon M'^n \to M'^n$ are called *ambiently Ω-conjugate* if there exists a homeomorphism $h\colon M^n \to M'^n$ such that $hf|_{NW_f} = gh|_{NW_f}$.

In Section 7 examples of diffeomorphisms from G with isomorphic labeled Smale diagrams which are not ambiently Ω-conjugated are constructed. Let us single out a subclass $G_* \subset G$ of diffeomorphisms in which any two model diffeomorphisms are connected by at most one orbit. For such diffeomorphisms, the isomorphic class of the labeled Smale diagram is a complete invariant of the ambient Ω-conjugacy.

Theorem 3. *Diffeomorphisms f, $f' \in G_*$ are ambiently Ω-conjugate iff their labeled diagrams are isomorphic.*

2. Every Smale diagram is a connected Hasse diagram

The connectivity of a Smale diagram will be proved in this section.

Proof of Lemma 1.

Let f be an (A, B)-diffeomorphism given on a connected closed manifold M^n and let NW_f be the nonwandering set of f. Let $\prec$ be a Smale partial order relation given on the set NW_f. Suppose the contrary: the Hasse diagram $(NW_f, \prec) = \Gamma$ is not connected, that is, there is a connected component Γ_i of Γ, different from Γ. Let $L_i = \{\Lambda_{i_1}, \ldots, \Lambda_{i_m}\}$ be the basic sets corresponding to the vertices of Γ_i. Since Γ_i is a connected component of the graph Γ, it follows that $\bigcup\limits_{j=1}^{m} W^s_{\Lambda_{i_j}} = \bigcup\limits_{j=1}^{m} W^u_{\Lambda_{i_j}}$. Let $M^n_i = \bigcup\limits_{j=1}^{m} W^s_{\Lambda_{i_j}} = \bigcup\limits_{j=1}^{m} W^u_{\Lambda_{i_j}}$. We show that M^n_i is an open subset of M^n.

To this end it is sufficient to show that every point $x \in M^n_i$ possesses an open neighborhood $U_x \subset M^n$ such that each point $y \in U_x$ belongs to an intersection of $W^s_{\Lambda_{i_{m_1}}}$ with $W^u_{\Lambda_{i_{m_2}}}$ for some $m_1(y)$, $m_2(y) \in \{1, \ldots, m\}$. Indeed, if we suppose the contrary, then there is a basic set Λ_* which does not belong to the set L_i and whose closure of the invariant manifolds contains the invariant manifold of some basic set from L_i. By virtue of the (A, B)-axiom we find that Λ_* is connected by the order relation $\prec$ with a basic set from L_i and, hence, belongs to L_i, which contradicts the assumption.

In the same way the complement of M^n_i in M^n is also open. Hence, the manifold M^n_i is open and closed simultaneously and, therefore, coincides with the ambient manifold M^n. Thus, the connected component Γ_i coincides with Γ, which contradicts the assumption. □

3. Model diffeomorphisms on the torus

In this section model diffeomorphisms on the two-dimensional torus whose connected sums realize an arbitrary Smale diagram will described.

3.1. Smale surgery

Let $C \in SL(2, \mathbb{Z})$ be a hyperbolic matrix with eigenvalues λ_1, λ_2 such that $\lambda = |\lambda_1| > 1$ and $|\lambda_2| = 1/\lambda$. Since the matrix C has a determinant equal to 1, it induces a hyperbolic automorphism $\widehat{C}\colon \mathbb{T}^2 \to \mathbb{T}^2$ with a fixed point O. According to [4], this diffeomorphism is an Anosov diffeomorphism and has two transverse invariant foliations (stable and unstable), each of which is dense on the torus. Moreover, the set of periodic points of the diffeomorphism $\widehat{C}$ is also dense on $\mathbb{T}^2$.

S. Smale [1] proposed a so-called "surgery" to obtain a diffeomorphism with a one-dimensional attractor and a fixed source point. Let us provide one of possible versions of such a surgery (for details, see [5]).

Let (x, y) be local coordinates in some neighborhood $U(O)$ of the point $O \in \mathbb{T}^2$ such that the diffeomorphism $\widehat{C}\colon \mathbb{T}^2 \to \mathbb{T}^2$ in this coordinates can be represented as $\widehat{C}(x, y) = (x/\lambda, \lambda y)$. Then $\{y = \mathrm{const}\}$ and $\{x = \mathrm{const}\}$ are stable and unstable foliations of $\widehat{C}$. Let $\mu\colon \mathbb{R} \to [0, 1]$ be a C^∞-smooth function defined by the formula

$$\mu(x) = \begin{cases} 0, & x \leqslant \lambda^{-3}, \\ \widetilde{\mu}(x), & \lambda^{-3} < x < 1, \\ 1, & x \geqslant 1, \end{cases}$$

where $\widetilde{\mu}(x)\colon (\lambda^{-3}, 1) \to (0, 1)$ strongly monotonically increases (see Fig. 1a).

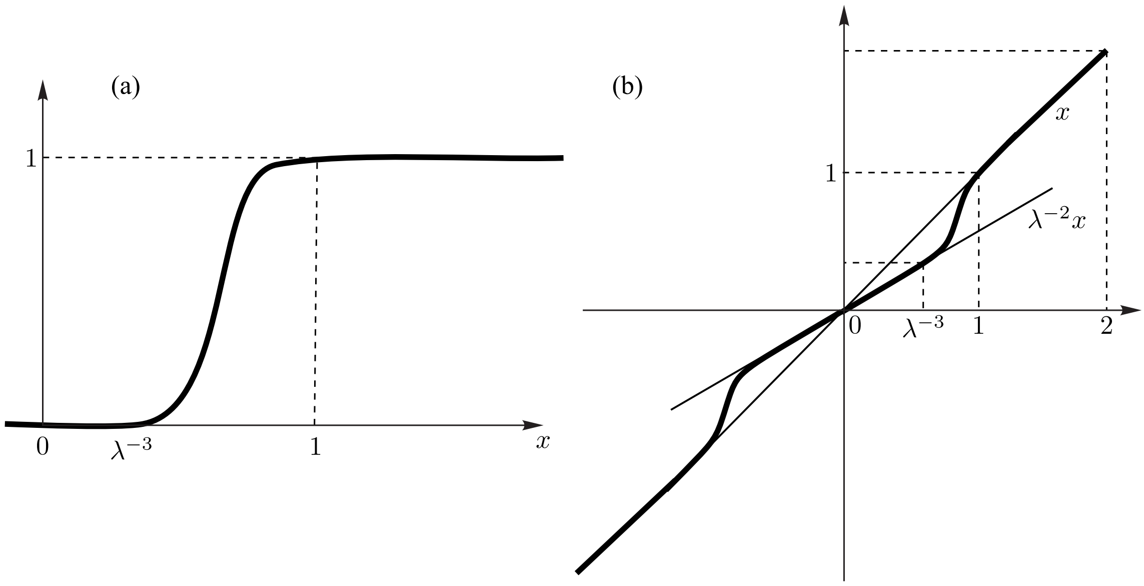

Fig. 1. (a) The graph of the function $\mu(x)$; (b) The graph of the function $\nu(x)$

Define the function $\nu\colon \mathbb{R} \to \mathbb{R}$ by the formula $\nu(x) = \lambda^{-2(1-\mu(|x|))}x$ (see Fig. 1b). Let $\mathbb{D}^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leqslant 4\}$. A diffeomorphism $B_{C,O^s}\colon \mathbb{D}^2 \to \mathbb{D}^2$ defined by the formula $B_{C,O^s}(x, y) = (\nu^{-1}(x), y)$ has the form $B_{C,O^s}(x, y) = (\lambda^2 x, y)$ if $x^2 + y^2 \leqslant \lambda^{-6}$ and is identical on $\partial\mathbb{D}^2$.

Let $\widehat{B}_{C,O^s}\colon \mathbb{T}^2 \to \mathbb{T}^2$ be a diffeomorphism which coincides with B_{C,O^s} in $U(0)$ and is identical outside it. Then, according to [6], the diffeomorphism $F_{C,O^s} = \widehat{B}_{C,O^s} \circ \widehat{C}$ is a DA-diffeomorphism whose nonwandering set consists of a one-dimensional attractor Λ_{C,O^s} with the so-called *bunch* of degree 2 formed by two different boundary fixed points p_{O^s} and q_{O^s}, and a fixed source α_{O^s} (see Fig. 2).

The construction described above is called the *Smale surgery* along the stable manifold of a fixed point. Since in the neighborhood of a source point in the local coordinates x, y the diffeomorphism F_{C,O^s} is a linear extension, it follows that in the basin $W^u_{\alpha_{O^s}}$ there exists a pair of transversal F_{C,O^s}-invariant foliations having the form $\{y = \mathrm{const}\}$ and $\{x = \mathrm{const}\}$ in the local coordinates x, y. Thus, the diffeomorphism F_{C,O^s} has a pair of global transversal invariant

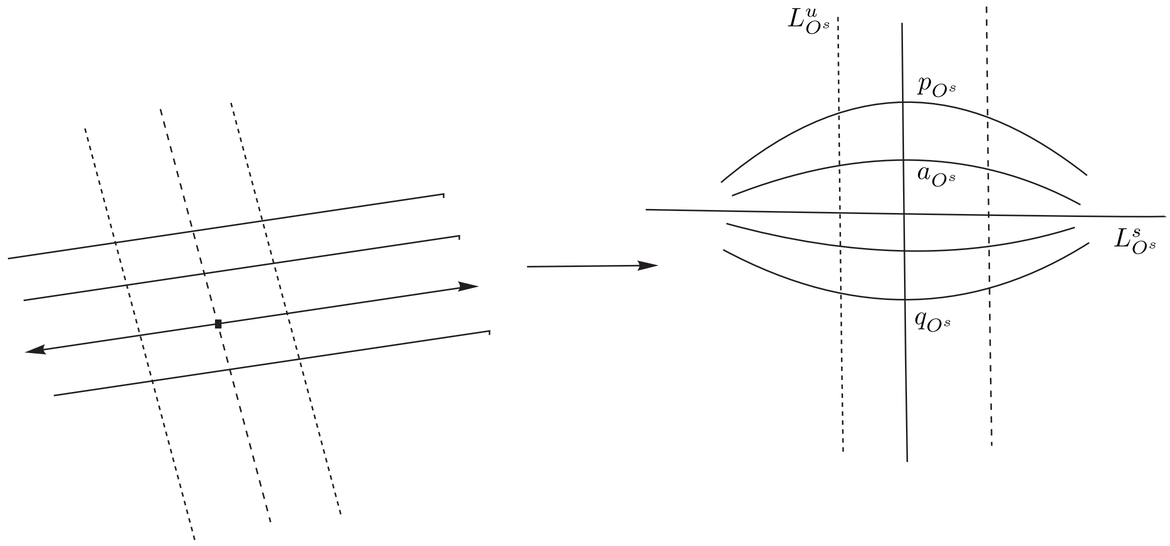

Fig. 2. DA-diffeomorphism F_{C,O^s}

foliations L^s_{C,O^s}, L^u_{C,O^s} containing the manifolds W^s_x, $x \in \Lambda_{C,O^s}$, W^u_x, $x \in \Lambda_{C,O^s}$, respectively, as leaves (see Fig. 2).

3.2. Generalization of the surgery

The surgery described above is performed in a neighborhood of any periodic orbit of the Anosov diffeomorphism $\widehat{C}$. It can also be generalized to a surgery along the unstable manifold of a periodic orbit.

A model diffeomorphism of the torus is a diffeomorphism $F_{C,\mathcal{O}^s,\mathcal{O}^u}\colon \mathbb{T}^2 \to \mathbb{T}^2$ obtained from the algebraic Anosov diffeomorphism $\widehat{C}$ by Smale surgery along invariant manifolds of a finite (possibly empty) set of pairwise distinct periodic orbits: along the stable manifolds of orbits $\mathcal{O}^s = \{O^s_1, \ldots, O^s_k\}$ of periods $m_1, \ldots, m_k$, respectively, (*s-orbits*) and along the unstable manifolds of periodic orbits $\mathcal{O}^u = \{O^u_1, \ldots, O^u_l\}$ of periods $n_1, \ldots, n_l$, accordingly, (*u-orbits*).

By construction, the nonwandering set $NW_{F_{C,\mathcal{O}^s,\mathcal{O}^u}}$ contains a unique nontrivial basic set $\Lambda_{C,\mathcal{O}^s,\mathcal{O}^u}$ and the diffeomorphism $F_{C,\mathcal{O}^s,\mathcal{O}^u}$ has a pair of transversal invariant foliations $L^s_{C,\mathcal{O}^s,\mathcal{O}^u}$, $L^u_{C,\mathcal{O}^s,\mathcal{O}^u}$ containing, as leaves, stable and unstable manifolds, respectively, of points from $\Lambda_{C,\mathcal{O}^s,\mathcal{O}^u}$. The set $\alpha_{\mathcal{O}^s} = \{\alpha_{\mathcal{O}^s_1}, \ldots, \alpha_{\mathcal{O}^s_k}\}$ is a set of source periodic orbits and the set $\omega_{\mathcal{O}^u} = \{\omega_{\mathcal{O}^u_1}, \ldots, \omega_{\mathcal{O}^u_l}\}$ is a set of sink periodic orbits of the diffeomorphism $F_{C,\mathcal{O}^s,\mathcal{O}^u}$. Sets $p_{\mathcal{O}^s} = \{p_{\mathcal{O}^s_1}, \ldots, p_{\mathcal{O}^s_k}\}$, $q_{\mathcal{O}^s} = \{q_{\mathcal{O}^s_1}, \ldots, q_{\mathcal{O}^s_k}\}$, $p_{\mathcal{O}^u} = \{p_{\mathcal{O}^u_1}, \ldots, p_{\mathcal{O}^u_l}\}$, $q_{\mathcal{O}^u} = \{q_{\mathcal{O}^u_1}, \ldots, q_{\mathcal{O}^u_l}\}$ consist of boundary periodic points, such that for each $i \in \{1, \ldots, k\}$ there exist stable separatrices $l^{\varnothing}_{p_{\mathcal{O}^s_i}}$ and $l^{\varnothing}_{q_{\mathcal{O}^s_i}}$ of the orbits $p_{\mathcal{O}^s_j}$ and $q_{\mathcal{O}^s_j}$ which belong to the basin of the same source periodic orbit $\alpha_{\mathcal{O}^s_i}$ and for each $j \in \{1, \ldots, l\}$ there exist unstable separatrices $l^{\varnothing}_{p_{\mathcal{O}^u_j}}$ and $l^{\varnothing}_{q_{\mathcal{O}^u_j}}$ of the orbits $p_{\mathcal{O}^u_j}$ and $q_{\mathcal{O}^u_j}$ which belong to a basin of the same sink periodic orbit $\omega_{\mathcal{O}^u_j}$.

In this case, the basic set $\Lambda_{C,\mathcal{O}^s,\mathcal{O}^u}$ has one of the following possible structures:

- has the topological dimension 2 and is an attractor and a repeller simultaneously if $\mathcal{O}^s = \mathcal{O}^u = \varnothing$;

- has the topological dimension 1 (see Fig. 3) and is an attractor (repeller) if $\mathcal{O}^s \neq \varnothing$, $\mathcal{O}^u = \varnothing$ ($\mathcal{O}^s = \varnothing$, $\mathcal{O}^u \neq \varnothing$);
- has the topological dimension 0 (see Fig. 4) and is a saddle basic set if $\mathcal{O}^s \neq \varnothing$ and $\mathcal{O}^u \neq \varnothing$.

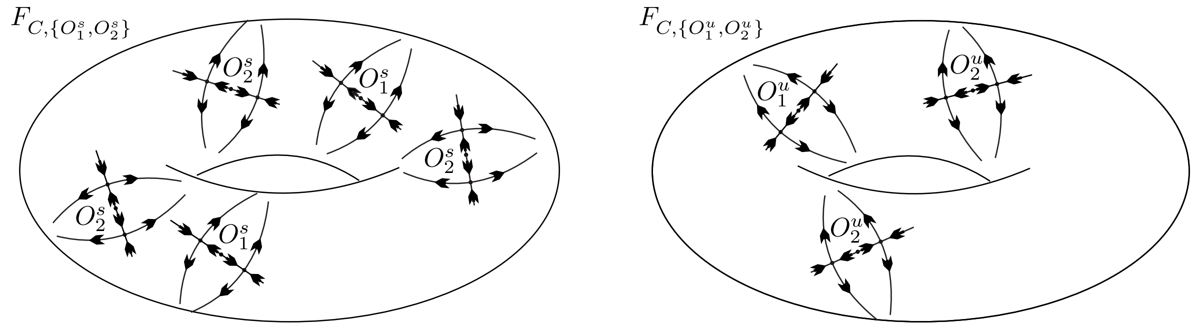

Fig. 3. The diffeomorphism $F_{C,\{O_1^s,O_2^s\}}$ with s-orbits with periods 2, 3 and the attractor $\Lambda_{C,\{O_1^s,O_2^s\}}$; the diffeomorphism $F_{C,\{O_1^u,O_2^u\}}$ with u-orbits with periods 1, 2 and the repeller $\Lambda_{C,\{O_1^u,O_2^u\}}$

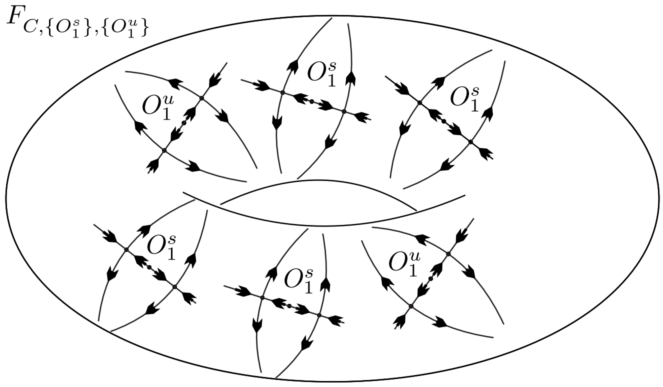

Fig. 4. Diffeomorphism $F_{C,\{O_1^s\},\{O_1^u\}}$ with s-orbit of period 4, u-orbit of period 2, and saddle basic set $\Lambda_{C,\{O_1^s\},\{O_1^u\}}$

4. Topological conjugacy of the model diffeomorphisms

In this section, we prove Theorem 1. Firstly, notice that necessary and sufficient conditions for model Anosov diffeomorphisms follow from [7–9]. For the case where exactly one of the sets $\mathcal{O}^s$, $\mathcal{O}^u$ is empty, Theorem 1 follows from [7].

Thus, everywhere below we suppose that both sets $\mathcal{O}^s$, $\mathcal{O}^u$ are not empty and, hence, $\Lambda_{C,\mathcal{O}^s,\mathcal{O}^u}$ is a zero-dimensional basic set. Let us provide some necessary information and prove auxiliary lemmas using ideas of the paper [10] and the book [11].

We say a point $x \in \Lambda_{C,\mathcal{O}^s,\mathcal{O}^u}$ is s-dense (u-dense) if both connected components of the set $W^s_x \setminus x$ ($W^u_x \setminus x$) contain sets which are dense in a periodic component of the $\Lambda_{C,\mathcal{O}^s,\mathcal{O}^u}$ containing the point x. A point $x \in \Lambda_{C,\mathcal{O}^s,\mathcal{O}^u}$ is said to be an s-boundary point (u-boundary point) if one of the connected components of the set $W^s_x \setminus x$ ($W^u_x \setminus x$) is disjoint from $\Lambda_{C,\mathcal{O}^s,\mathcal{O}^u}$.

Any model diffeomorphism $F_{C,\mathcal{O}^s,\mathcal{O}^u}$ corresponds to a hyperbolic matrix C and by construction this diffeomorphism has the form $F_{C,\mathcal{O}^s,\mathcal{O}^u} = \widehat{B}_{C,\mathcal{O}^s,\mathcal{O}^u} \circ \widehat{C}$, where $\widehat{B}_{C,\mathcal{O}^s,\mathcal{O}^u}\colon \mathbb{T}^2 \to \mathbb{T}^2$ is isotopic to the identical diffeomorphism. Thus, the diffeomorphism $F_{C,\mathcal{O}^s,\mathcal{O}^u}$ is isotopic to the diffeomorphism $\widehat{C}$.

Proposition 1 ([11, Lemma 9.8]). *Let $f\colon \mathbb{T}^2 \to \mathbb{T}^2$ be a diffeomorphism such that the induced isomorphism $f_*\colon \pi_1(\mathbb{T}^2) \to \pi_1(\mathbb{T}^2)$ is hyperbolic and let C be the matrix that defines the isomorphism f_*. Then there is a unique continuous map h of the torus $\mathbb{T}^2$ which is homotopic to the identity and such that it semiconjugates the diffeomorphism f to the algebraic automorphism $\widehat{C}$.*

Let $\pi\colon \mathbb{R}^2 \to \mathbb{T}^2$ be a universal covering $\pi^{-1}(\Lambda_{C,\mathcal{O}^s,\mathcal{O}^u}) = \overline{\Lambda}_{C,\mathcal{O}^s,\mathcal{O}^u}$. If $x \in \mathbb{T}^2$, let $\overline{x} \in \mathbb{R}^2$ denote the point in the preimage $\pi^{-1}(x)$. Denote by $w^\delta_{\overline{x}}$, $\delta \in \{s, u\}$ the curve on $\mathbb{R}^2$ containing $\overline{x}$ such that $\pi(w^\delta_{\overline{x}}) = W^\delta_x$, $x \in \Lambda_{C,\mathcal{O}^s,\mathcal{O}^u}$. For points $\overline{y}, \overline{z} \in w^\delta_{\overline{x}}$ $(\overline{y} \neq \overline{z})$ let $[\overline{y}, \overline{z}]^\delta$, $[\overline{y}, \overline{z})^\delta$, $(\overline{y}, \overline{z}]^\delta$, $(\overline{y}, \overline{z})^\delta$ denote the connected arcs on the manifold $w^\delta_{\overline{x}}$ with the boundary points $\overline{y}$, $\overline{z}$.

By Proposition 1, there exists a unique continuous map $h\colon \mathbb{T}^2 \to \mathbb{T}^2$ which is homotopic to the identity and such that $hF_{C,\mathcal{O}^s,\mathcal{O}^u} = \widehat{C}h$. Let $\overline{h}\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a lift of the map h. For every $x \in \mathbb{T}^2$ let $\widehat{W}^s_x$ $(\widehat{W}^u_x)$ denote the stable (unstable) manifold of the point x with respect to the automorphism $\widehat{C}$ and let $\widehat{w}^\delta_{\overline{x}}$ denote the connected component of the preimage of the manifold $\widehat{W}^\delta_x$ of the automorphism $\widehat{C}$ which passes through the point $\overline{x}$.

Lemma 2. *Let $\overline{x}, \overline{y} \in \overline{\Lambda}_{C,\mathcal{O}^s,\mathcal{O}^u}$, $w^s_{\overline{x}} = w^s_{\overline{y}}$ $(w^u_{\overline{x}} = w^u_{\overline{y}})$ and $(\overline{x}, \overline{y})^s \cap \overline{\Lambda}_{C,\mathcal{O}^s,\mathcal{O}^u} \neq \varnothing$ $((\overline{x}, \overline{y})^u \cap \overline{\Lambda}_{C,\mathcal{O}^s,\mathcal{O}^u} \neq \varnothing)$. Then $\overline{h}(\overline{x}) \neq \overline{h}(\overline{y})$.*

Proof.

We consider the case $w^s_{\overline{x}} = w^s_{\overline{y}}$ (the case $w^u_{\overline{x}} = w^u_{\overline{y}}$ is similar). By [11, Lemma 9.10] there is $r > 0$ such that $d(\overline{x}_1, \overline{x}_2) < r$ for any two points $\overline{x}_1, \overline{x}_2 \in \overline{\Lambda}_{C,\mathcal{O}^s,\mathcal{O}^u}$ for which $\overline{h}(\overline{x}_1) = \overline{h}(\overline{x}_2)$, where d is the Euclidean metric on $\mathbb{R}^2$. Let $p \in \Lambda_{C,\mathcal{O}^s,\mathcal{O}^u}$ be an s-dense periodic point and let m be the natural number for which $F^m_{C,\mathcal{O}^s,\mathcal{O}^u}(p) = p$. Then there are points $x_*, y_* \in (x, y)^s$ such that $x_*, y_* \in W^{u+}_p$, where W^{u+}_p is one of the connected components of $W^u_p \setminus p$. Let $\overline{p} \in \overline{\Lambda}_{C,\mathcal{O}^s,\mathcal{O}^u}$ be a point in the set $\pi^{-1}(p)$ and let ϕ, $\phi_{\widehat{C}}$ be the lifts of the diffeomorphisms $F^m_{C,\mathcal{O}^s,\mathcal{O}^u}$, $\widehat{C}^m$ such that $\phi(\overline{p}) = \overline{p}$, $\overline{h}\phi = \phi_{\widehat{C}}\overline{h}$. Denote by $\overline{y}_*$, $\overline{x}_*$ the preimages of the points y_*, x_* belonging to the curves $w^{u+}_{\overline{p}}$, $w^s_{\overline{y}_*}$, respectively. On the curve $w^s_{\overline{y}_*}$ there are points $\overline{x}'$, $\overline{y}'$ congruent to the points $\overline{x}$, $\overline{y}$, respectively, with respect to integer plane shifts. Since the arc $(\overline{x}', \overline{y}')^s \subset w^s_{\overline{y}_*}$ contains the arc $(\overline{x}_*, \overline{y}_*)^s \subset w^s_{\overline{y}_*}$, the points $\overline{x}'$, $\overline{y}'$ are separated by the curve $w^u_{\overline{p}}$ on $\mathbb{R}^2$. By the λ-lemma (see, for example, [12, λ-lemma]), there is $N > 0$ such that $d(\phi^{-N}(x'), \phi^{-N}(\overline{y}')) > r$. Suppose that, contrary to the assumption, $\overline{h}(\overline{x}) = \overline{h}(\overline{y})$. Then $\overline{h}(\overline{x}') = \overline{h}(\overline{y}')$, $\overline{h}(\phi^{-N}(\overline{x}')) = \phi^{-N}_{\widehat{C}}(\overline{h}(\overline{x}')) = \phi^{-N}_{\widehat{C}}(\overline{h}(\overline{y}')) = \overline{h}(\phi^{-N}(\overline{y}'))$ and, therefore, $d(\phi^{-N}(\overline{x}'), \phi^{-N}(\overline{y}')) < r$, which is impossible. □

Lemma 3. *If for a point $x \in \Lambda_{C,\mathcal{O}^s,\mathcal{O}^u}$ its manifold $W^\delta_x \setminus x$ contains no δ-boundary points, then $\overline{h}(w^\delta_{\overline{x}}) = \widehat{w}^\delta_{\overline{h}(\overline{x})}$.*

Proof.

Let us provide the proof for the case $\delta = u$ (the proof for $\delta = s$ is similar). First we show that $\overline{h}(w^u_{\overline{x}}) \subset \widehat{w}^u_{\overline{h}(\overline{x})}$. Let $\overline{y}$ $(\overline{y} \neq \overline{x})$ be an arbitrary point on the curve $w^u_{\overline{x}}$. The definition of an unstable manifold implies $\lim\limits_{n\to-\infty}\rho(F^n_{C,\mathcal{O}^s,\mathcal{O}^u}(x),\,F^n_{C,\mathcal{O}^s,\mathcal{O}^u}(y))=0$ for $x=p(\overline{x})$, $y=p(\overline{y})$ where ρ is the metric on the torus. Continuity of h implies $\lim\limits_{n\to-\infty}\rho(h(F^n_{C,\mathcal{O}^s,\mathcal{O}^u}(x)),\,h(F^n_{C,\mathcal{O}^s,\mathcal{O}^u}(y)))=$ $=\lim\limits_{n\to-\infty}\rho(\widehat{C}^n(h(x)),\,\widehat{C}^n(h(y)))=0$, therefore $h(y)\in\widehat{W}^u_{h(x)}$, that is, $h(W^u_x)\subset\widehat{W}^u_{h(x)}$. Since $\overline{h}$ is a lift of h, we have $\overline{h}(w^u_{\overline{x}})\subset\widehat{w}^u_{\overline{h}(\overline{x})}$. The curve $\overline{h}(w^u_{\overline{x}})$ is a connected set on the curve $\widehat{w}^u_{\overline{h}(\overline{x})}$ and it contains the point $\overline{h}(\overline{x})$.

We now show that $\overline{h}(w^u_{\overline{x}})=\widehat{w}^u_{\overline{h}(\overline{x})}$. Suppose the contrary, $\overline{h}(w^u_{\overline{x}})\neq\widehat{w}^u_{\overline{h}(\overline{x})}$. Then, by Lemma 2, the image of one of the connected components of the set $w^u_{\overline{x}}\setminus\overline{x}$ by the map $\overline{h}$ is a bounded set on the line $\widehat{w}^u_{\overline{h}(\overline{x})}$. But, by Lemma [11, Lemma 9.9], the map $\overline{h}$ is proper, i. e., the preimage of every compact set is compact, and this is a contradiction. $\square$

Lemma 4. *Let points $p,\,q\in\Lambda_{C,\mathcal{O}^s,\mathcal{O}^u}$ are δ-boundary periodic points from the same bunch. Then*

1) *the curves $w^{\overline{\delta}}_{\overline{p}}$, $w^{\overline{\delta}}_{\overline{q}}$, where $\overline{\delta}=s$ if $\delta=u$ and $\overline{\delta}=u$ if $\delta=s$, bound on $\mathbb{R}^2$ a domain $Q_{\overline{p},\overline{q}}$ disjoint from the set $\overline{\Lambda}_{C,\mathcal{O}^s,\mathcal{O}^u}$ and $\pi(Q_{\overline{p},\overline{q}})$ is an injective immersion of the open disk to $\mathbb{T}^2$;*

2) $\overline{h}(\overline{p})=\overline{h}(\overline{q})$;

3) $\overline{h}(Q_{\overline{p},\overline{q}})\subset\overline{h}(w^{\overline{\delta}}_{\overline{p}})$.

Proof.

Let us provide the proof for the case $\delta=s$. The other case is analogous. Notice that all the curves $w^u_{\overline{x}}$, where $\overline{x}\in\overline{\Lambda}_{C,\mathcal{O}^s,\mathcal{O}^u}$ and x is u-dense point, have the same asymptotic directions[3] [11, Corollary 9.5]. Since the domain $Q_{\overline{p},\overline{q}}$ is disjoint from the set $\overline{\Lambda}_{C,\mathcal{O}^s,\mathcal{O}^u}$ and since there are no congruent points on the curves $w^u_{\overline{p}}$, $w^u_{\overline{q}}$, the domain contains no congruent points either. Therefore, $\pi(Q_{\overline{p},\overline{q}})$ is an immersion of the open disk into $\mathbb{T}^2$.

We now prove items 2 and 3. Let m be the period of the points p. Since q and p are from the same bunch and $F_{C,\mathcal{O}^s,\mathcal{O}^u}$ is an orientation-preserving diffeomorphism, their periods are equal. Let $\phi\colon\mathbb{R}^2\to\mathbb{R}^2$ denote the lift of the diffeomorphism $F^m_{C,\mathcal{O}^s,\mathcal{O}^u}$ such that $\phi(\overline{p})=\overline{p}$ and $\phi(\overline{q})=$ $=\overline{q}$. Since $hF_{C,\mathcal{O}^s,\mathcal{O}^u}=\widehat{C}h$, we have $hF^m_{C,\mathcal{O}^s,\mathcal{O}^u}=\widehat{C}^mh$, therefore there is a lift $\phi_{\widehat{C}}$ of the diffeomorphism $\widehat{C}^m$ such that $\overline{h}\phi=\phi_{\widehat{C}}\overline{h}$. Suppose now $\overline{h}(\overline{p})\neq\overline{h}(\overline{q})$. Then the points $\overline{h}(\overline{p})$, $\overline{h}(\overline{q})$ are distinct fixed points of the diffeomorphism $\phi_{\widehat{C}}$, which is impossible.

We now show that $\overline{h}(Q_{\overline{p},\overline{q}})\subset\overline{h}(w^u_{\overline{p}})$. Suppose the contrary: there is a point $\overline{y}\in Q_{\overline{p},\overline{q}}$ such that $\overline{h}(\overline{y})\notin\overline{h}(w^u_{\overline{p}})$. Lemma 3 implies that $\overline{h}(\overline{y})\notin w^u_{\overline{h}(\overline{p})}$. Then $\lim\limits_{n\to+\infty}d(\phi^{-n}_{\widehat{C}}(\overline{h}(\overline{y})),\,\phi^{-n}_{\widehat{C}}(\overline{h}(\overline{p})))=$ $=\lim\limits_{n\to+\infty}d(\overline{h}(\phi^{-n}(\overline{y})),\,\overline{h}(\phi^{-n}(\overline{p})))=+\infty$ but $d(\phi^{-n}(\overline{y}),\,\phi^{-n}(\overline{p}))<r_0$ for all $n>0$, where r_0 is some positive number since y belongs to the basin of some source. This contradiction completes the proof of the lemma. $\square$

[3]Let $x\in\Lambda_{C,\mathcal{O}^s,\mathcal{O}^u}$, $t\in\mathbb{R}$ be a parameter on the curve $w^\delta_{\overline{x}}$ such that $w^\delta_{\overline{x}}(0)=\overline{x}$. A curve $w^{\delta+}_{\overline{x}}$ ($w^{\delta-}_{\overline{x}}$) has the asymptotic direction $\delta^+_{\overline{x}}$ ($\delta^-_{\overline{x}}$) $t\to+\infty$ ($t\to-\infty$) and there is a finite limit $\delta^+_{\overline{x}}=\lim\limits_{t\to+\infty}\frac{y^\delta(t)}{x^\delta(t)}$ $\left(\delta^-_{\overline{x}}=\lim\limits_{t\to-\infty}\frac{y^\delta(t)}{x^\delta(t)}\right)$ where $x^\delta(t)$, $y^\delta(t)$ are the Cartesian coordinates of the point $w^\delta_{\overline{x}}(t)$ on the plane $\mathbb{R}^2$.

Lemma 5. *The image of the set $\Lambda_{C,\mathcal{O}^s,\mathcal{O}^u}$ by h is the whole torus $\mathbb{T}^2$. The set $E_{F_{C,\mathcal{O}^s,\mathcal{O}^u}} = \{x \in \mathbb{T}^2 \colon h^{-1}(x)$ consists of more than one point$\}$ is the union of unstable manifolds of points from $\mathcal{O}^s$ and stable manifolds of points from $\mathcal{O}^u$ of the algebraic automorphism $\widehat{C}$. Moreover, $h^{-1}(W^u_{O^s_i}) = W^u_{\alpha_{O^s_i}} \cup W^u_{p_{O^s_i}} \cup W^u_{q_{O^s_i}}$ and $h^{-1}(W^s_{O^u_j}) = W^s_{\omega_{O^u_j}} \cup W^s_{p_{O^u_j}} \cup W^s_{q_{O^u_j}}$.*

Proof.

Let us prove that the image of the set $\Lambda_{C,\mathcal{O}^s,\mathcal{O}^u}$ by the map h is the whole torus $\mathbb{T}^2$. Let x be a point from $\Lambda_{C,\mathcal{O}^s,\mathcal{O}^u}$ both components of whose unstable manifold are dense in $\Lambda_{C,\mathcal{O}^s,\mathcal{O}^u}$. Lemma 3 implies $h(W^u_x) = \widehat{W}^u_{h(x)}$. Since the automorphism $\widehat{C}$ is hyperbolic, the manifold $\widehat{W}^u_{h(x)}$ is dense in $\mathbb{T}^2$. This and the continuity of h imply that $h(\Lambda_{C,\mathcal{O}^s,\mathcal{O}^u}) = \mathbb{T}^2$.

Let $\overline{x}$ be an arbitrary point from $\overline{\Lambda}_{C,\mathcal{O}^s,\mathcal{O}^u}$. Two cases are possible: 1) $w^u_{\overline{x}}$ is disjoint from the preimage of any s-boundary and $w^s_{\overline{x}}$ is disjoint from the preimage of any u-boundary periodic point of the set $\Lambda_{C,\mathcal{O}^s,\mathcal{O}^u}$, 2) $w^u_{\overline{x}}$ ($w^s_{\overline{x}}$) intersects the preimage of an s-boundary (u-boundary) periodic point p of the set $\Lambda_{C,\mathcal{O}^s,\mathcal{O}^u}$ at a point $\overline{p}$.

In the former case there is no point $\overline{y}$ ($\overline{y} \neq \overline{x}$) from $\overline{\Lambda}_{C,\mathcal{O}^s,\mathcal{O}^u}$ such that $\overline{h}(\overline{x}) = \overline{h}(\overline{y})$. Indeed, suppose the contrary. First, let $\overline{y} \in w^u_{\overline{x}}$. But for all $y \in W^u_x$, $(x, y)^u \cap \Lambda_{C,\mathcal{O}^s,\mathcal{O}^u} \neq \varnothing$ so $\overline{y} \neq \overline{x}$ by Lemma 2. So $\overline{y} \notin w^u_{\overline{x}}$. Consider the domain $Q_{\overline{x},\overline{y}}$ on $\mathbb{R}^2$ bounded by the curves $w^u_{\overline{x}}$, $w^u_{\overline{y}}$. Pick a point $\overline{z}$ on the curve $w^u_{\overline{x}}$ such that the curve $w^s_{\overline{z}}$ tends to infinity on $\mathbb{R}^2$ in both possible directions, then [11, Corollary 9.5] implies $w^s_{\overline{z}} \cap w^u_{\overline{y}} \neq \varnothing$. Let $z' = w^s_{\overline{z}} \cap w^u_{\overline{y}}$. By [11, Theorem 8.5], $\overline{\Lambda}_{C,\mathcal{O}^s,\mathcal{O}^u} \cap Q_{\overline{x},\overline{y}} \neq \varnothing$, therefore the open arc $(\overline{z}, \overline{z}')^u$ intersects $\overline{\Lambda}_{C,\mathcal{O}^s,\mathcal{O}^u}$ and, by Lemma 2, $\overline{h}(\overline{z}) \neq \overline{h}(\overline{z}')$. On the other hand, $\overline{z} \in w^u_{\overline{x}}$, $\overline{z}' \in w^u_{\overline{y}}$, by [11, Corollary 9.5], $\overline{h}(w^u_{\overline{x}}) = \overline{h}(w^u_{\overline{y}})$ and we get $\overline{h}(\overline{z}) = \overline{h}(\overline{z}')$, which is a contradiction.

Consider another case. Suppose that $w^u_{\overline{x}}$ intersects the preimage of an s-boundary periodic point p of the set $\Lambda_{C,\mathcal{O}^s,\mathcal{O}^u}$ at a point $\overline{p}$ (another case is proved in a similar way). Let $Q_{\overline{p},\overline{q}}$ be the domain on $\mathbb{R}^2$ satisfying item 1) of Lemma 4. If $\overline{x} = \overline{p}$, then by item 2) of Lemma 4 $\overline{h}(\overline{x}) = \overline{h}(\overline{y})$, where $\overline{y} = \overline{q}$. For all $y \in W^u_p$ $(x, y)^u \cap \Lambda_{C,\mathcal{O}^s,\mathcal{O}^u} \neq \varnothing$, so by Lemma 2 $\overline{y}$ is the unique point on $\mathbb{R}^2$ for which $\overline{x} = \overline{y}$. If $\overline{x} \neq \overline{p}$, then by [11, Corollary 9.5] $w^s_{\overline{x}} \cap w^u_{\overline{q}} \neq \varnothing$. Let $\overline{y} = w^s_{\overline{x}} \cap w^u_{\overline{q}}$, then, since $\overline{h}(w^u_{\overline{x}}) = \overline{h}(w^u_{\overline{y}})$, we have $\overline{h}(\overline{x}) = \overline{h}(\overline{y})$. If simultaneously $w^s_{\overline{x}}$ intersects the preimage of an u-boundary periodic point p' of the set $\Lambda_{C,\mathcal{O}^s,\mathcal{O}^u}$ at a point $\overline{p}'$, then there is another point $\overline{y}' = w^u_{\overline{x}} \cap w^s_{\overline{q}'}$, where q' is from the same bunch as p', such that $\overline{y}' = \overline{x}'$. In this case there are 4 points on $\mathbb{R}^2$ with equal values of $\overline{h}$. If $w^s_{\overline{x}}$ is disjoint from the preimage of any u-boundary periodic point, then $\overline{y}$ is the unique point such that $\overline{h}(\overline{y}) = \overline{h}(\overline{x})$ by Lemma 2.

Since $\overline{h}$ is a lift of h, it follows that $E_{F_{C,\mathcal{O}^s,\mathcal{O}^u}}$ is the union of unstable manifolds of points from $\mathcal{O}^s$ and stable manifolds of points from $\mathcal{O}^u$ of the algebraic automorphism $\widehat{C}$ and $h^{-1}(W^u_{O^s_i}) = W^u_{\alpha_{O^s_i}} \cup W^u_{p_{O^s_i}} \cup W^u_{q_{O^s_i}}$ and $h^{-1}(W^s_{O^u_j}) = W^s_{\omega_{O^u_j}} \cup W^s_{p_{O^u_j}} \cup W^s_{q_{O^u_j}}$. □

Now let us prove Theorem 1: model diffeomorphisms $F_{C,\mathcal{O}^s,\mathcal{O}^u}, F_{C',\mathcal{O}'^s,\mathcal{O}'^u} \colon \mathbb{T}^2 \to \mathbb{T}^2$ are topologically conjugate if and only if there exists a matrix $H \in GL(2, \mathbb{Z})$ such that $HC = C'H$ and $\widehat{H}(\mathcal{O}^s) = \mathcal{O}'^s$, $\widehat{H}(\mathcal{O}^u) = \mathcal{O}'^u$, where $\widehat{H}$ is the induced automorphism of a 2-torus.

Proof of Theorem 1.

For simplicity of notation, let $F = F_{C,\mathcal{O}^s,\mathcal{O}^u}$, $F' = F_{C',\mathcal{O}'^s,\mathcal{O}'^u}$, $\Lambda = \Lambda_{C,\mathcal{O}^s,\mathcal{O}^u}$, $\Lambda' = \Lambda_{C',\mathcal{O}'^s,\mathcal{O}'^u}$, $L^u = L^u_{C,\mathcal{O}^s,\mathcal{O}^u}$, $L'^u = L^u_{C',\mathcal{O}'^s,\mathcal{O}'^u}$, $L^s = L^s_{C,\mathcal{O}^s,\mathcal{O}^u}$, $L'^s = L^s_{C',\mathcal{O}'^s,\mathcal{O}'^u}$.

Necessity. Let diffeomorphisms $F, F' \colon \mathbb{T}^2 \to \mathbb{T}^2$ be topologically conjugate by a homeomorphism $h \colon \mathbb{T}^2 \to \mathbb{T}^2$. The induced isomorphism $h_* \colon \pi_1(\mathbb{T}^2) \to \pi_1(\mathbb{T}^2)$ is uniquely determined by a matrix $H \in GL(2, \mathbb{Z})$ which, by virtue of conjugacy, satisfies the condition $HC = C'H$ and, consequently, the condition $\widehat{H}\widehat{C} = \widehat{C}'\widehat{H}$ holds.

Since h is a conjugating homeomorphism, it follows that $h(W^u_{\alpha_{\mathcal{O}^s_i}}) = W^u_{\alpha_{\mathcal{O}'^s_i}}$, $i \in \{1, \ldots, k\}$ and $h(W^s_{\omega_{\mathcal{O}^u_j}}) = W^s_{\omega_{\mathcal{O}'^u_j}}$, $j \in \{1, \ldots, l\}$. By Proposition 1 and Lemma 5, there exist maps h_F and $h_{F'}$ such that $h_F F = \widehat{C} h_F$, $h_{F'}F' = \widehat{C}' h_{F'}$ and $h_F(W^u_{\alpha_{\mathcal{O}^s_i}}) = \mathcal{O}^u_i$, $h_F(W^s_{\omega_{\mathcal{O}^u_j}}) = \mathcal{O}^s_j$, $h_{F'}(W^u_{\alpha_{\mathcal{O}'^s_i}}) = \mathcal{O}'^u_i$, $h_{F'}(W^s_{\omega_{\mathcal{O}'^u_j}}) = \mathcal{O}'^u_j$. Due to the uniqueness of $h_{F'}$ and the equalities $hF = F'h$, $\widehat{H}\widehat{C} = \widehat{C}'\widehat{H}$ we get $h_{F'} = \widehat{H} h_F h^{-1}$. Then $h_{F'}h(W^u_{\alpha_{\mathcal{O}^s_i}}) = \widehat{H} h_F(W^u_{\alpha_{\mathcal{O}^s_i}})$, hence $h_{F'}(W^u_{\alpha_{\mathcal{O}'^s_i}}) = \widehat{H}(\mathcal{O}^u_i)$. Thus, $\mathcal{O}'^u_i = \widehat{H}(\mathcal{O}^u_i)$. In a similar way it is possible to show that $\mathcal{O}'^s_j = \widehat{H}(\mathcal{O}^s_j)$.

Sufficiency. Suppose there is a matrix $H \in GL(2, \mathbb{Z})$ such that $HC = C'H$ and $\widehat{H}(\mathcal{O}^s) = \mathcal{O}'^s$, $\widehat{H}(\mathcal{O}^u) = \mathcal{O}'^u$. Let us construct a homeomorphism $h\colon \mathbb{T}^2 \to \mathbb{T}^2$ which conjugates the diffeomorphisms F and F'.

Let $\widetilde{W} = W^u_{p_{\mathcal{O}^s}} \cup W^u_{q_{\mathcal{O}^s}} \cup W^s_{p_{\mathcal{O}^u}} \cup W^s_{q_{\mathcal{O}^u}}$, $\widetilde{W}' = W^u_{p_{\mathcal{O}'^s}} \cup W^u_{q_{\mathcal{O}'^s}} \cup W^s_{p_{\mathcal{O}'^u}} \cup W^s_{q_{\mathcal{O}'^u}}$ and $\widetilde{\Lambda} = \Lambda \setminus \widetilde{W}$, $\widetilde{\Lambda}' = \Lambda' \setminus \widetilde{W}'$. By Lemma 5, the maps $\widetilde{h}_F = h_F|_{\widetilde{\Lambda}} : \widetilde{\Lambda} \to \mathbb{T}^2 \setminus (W^u_{\mathcal{O}^s} \cup W^s_{\mathcal{O}^u})$, $\widetilde{h}_{F'} = h_{F'}|_{\widetilde{\Lambda}'} : \widetilde{\Lambda}' \to \mathbb{T}^2 \setminus \big(W^u_{\mathcal{O}'^s} \cup W^s_{\mathcal{O}'^u}\big)$ are bijections, moreover, since $\widehat{H}$ is the algebraic automorphism of a torus satisfying the conditions of the theorem, $\widehat{H}(W^u_{\mathcal{O}^s} \cup W^s_{\mathcal{O}^u}) = W^u_{\mathcal{O}'^s} \cup W^s_{\mathcal{O}'^u}$. This means that the map

$$\widetilde{h} = \widetilde{h}^{-1}_{F'} \widehat{H} \widetilde{h}_F \colon \widetilde{\Lambda} \to \widetilde{\Lambda}'$$

is a homeomorphism conjugating $F|_{\widetilde{\Lambda}}$ and $F'|_{\widetilde{\Lambda}'}$. Furthermore, by the invariance of the stable and unstable foliations, the homeomorphism $\widetilde{h}$ can be extended to the whole Λ such that $\widetilde{h}(p_{O^s_i}) = p_{O'^s_i}$ and $\widetilde{h}(p_{O^u_j}) = p_{O'^u_j}$ for all $i \in \{1, \ldots, k\}$, $j \in \{1, \ldots, l\}$. Let us show how to extend $\widetilde{h}$ to a homeomorphism $h\colon \mathbb{T}^2 \to \mathbb{T}^2$, conjugating F and F'.

To do this, let us denote by W^s_i the union of the closures of stable separatrices $l^{\varnothing}_{p_{\mathcal{O}^s_i}}$ and $l^{\varnothing}_{q_{\mathcal{O}^s_i}}$ lying in $W^u_{\alpha_{O^s_i}}$ and by W^u_j the union of the closures of unstable separatrices $l^{\varnothing}_{p_{\mathcal{O}^u_i}}$ and $l^{\varnothing}_{q_{\mathcal{O}^u_i}}$ lying in $W^s_{\omega_{O^u_j}}$. W'^s_i, W'^u_j are similar notations for the diffeomorphism F'. Let us continue the homeomorphism $\widetilde{h}\Big|_{p_{O^s_i}}$ to the homeomorphism $h^s_i\colon W^s_i \to W'^s_i$, conjugating $F|_{W^s_i}$ with $F'|_{W'^s_i}$ and the homeomorphism $\widetilde{h}\Big|_{p_{O^u_j}}$ to the homeomorphism $h^u_j\colon W^u_j \to W'^u_j$ conjugating $F|_{W^u_j}$ with $F'|_{W'^u_j}$ in an arbitrary way. Let $W^s = \bigcup\limits_{i=1}^{k} W^s_i$, $W'^s = \bigcup\limits_{i=1}^{k} W'^s_i$, $W^u = \bigcup\limits_{j=1}^{l} W^u_j$, $W'^u = \bigcup\limits_{j=1}^{l} W'^u_j$. Denote by $h^s\colon W^s \to W'^s$ a homeomorphism composed of h^s_i and by $h^u\colon W^u \to W'^u$ a homeomorphism composed of h^u_j.

Let $r_{i,j}$ be an arbitrary point belonging to the set $W^u_{p_{O^s_i}} \cap W^s_{p_{O^u_j}}$. Then (see Fig. 5) there is a unique path connected component of the set $W^u_{\alpha_{O^s_i}} \cap W^s_{\omega_{O^u_j}}$, containing $r_{i,j}$ in its closure, which we denote by $Q_{r_{i,j}}$. The closure of a similar component for the point $r'_{i,j} = \widetilde{h}(r_{i,j})$ is denoted by $Q_{r'_{i,j}}$. Let us show how to extend the homeomorphism $\widetilde{h}$ to the set $Q_{r_{i,j}}$, which completes the proof of the theorem.

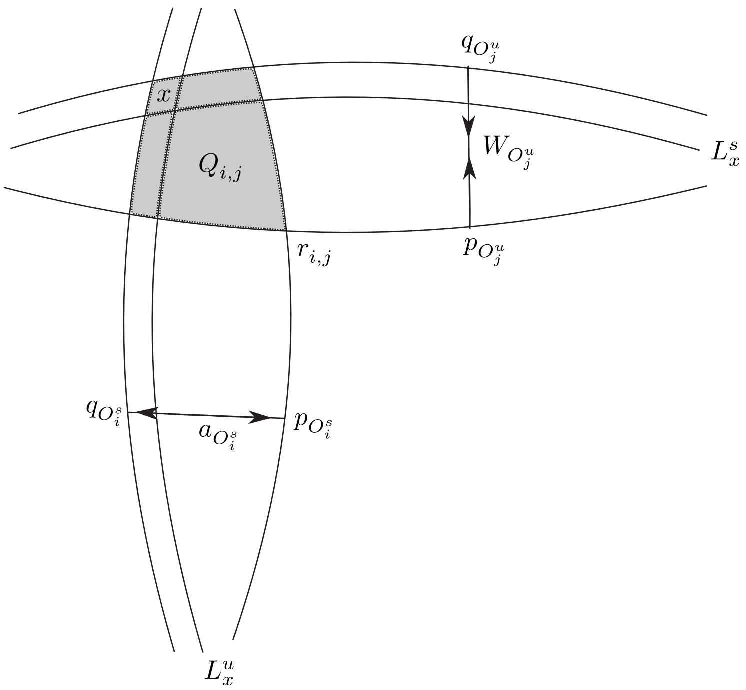

Fig. 5. Construction of a conjugating homeomorphism

Any point $x \in Q_{r_{i,j}}$ is the intersection point of the leaves $L^u_x \in L^u$ and $L^s_x \in L^s$. Define the homeomorphism $h\colon Q_{r_{i,j}} \to Q_{r'_{i,j}}$ by the formula $h(x) = \widehat{h}^s(L^u_x) \cap \widehat{h}^u(L^s_x) \cap Q_{r'_{i,j}}$, where $\widehat{h}^s\colon L^u \to L'^u$, $\widehat{h}^u\colon L^s \to L'^s$ is the map of leaves induced by the map h^s, h^u, respectively. $\square$

5. Connected sum of the model diffeomorphisms

Consider model diffeomorphisms $F_{C,\mathcal{O}^s,\mathcal{O}^u}$, $F'_{C',\mathcal{O}'^s,\mathcal{O}'^u}\colon \mathbb{T}^2 \to \mathbb{T}^2$ such that $\mathcal{O}^u \neq \varnothing$, $\mathcal{O}'^s \neq \varnothing$ and periodic orbits $O^u \in \mathcal{O}^u$, $O'^s \in \mathcal{O}'^s$ have the same period. Since O^u is a sink orbit and O'^s is a source orbit, there exists a diffeomorphism $\varphi\colon W^s_{O^u} \setminus O^u \to W^u_{O'^s} \setminus O'^s$ such that

$$\varphi \circ F^{-1}_{C,\mathcal{O}^s,\mathcal{O}^u}\Big|_{W^s_{O^u}\setminus O^u} = F'_{C',\mathcal{O}'^s,\mathcal{O}'^u} \circ \varphi\big|_{W^s_{O^u}\setminus O^u}.$$

Let us say

$$Q = (\mathbb{T}^2 \setminus O^u) \sqcup (\mathbb{T}^2 \setminus O'^s), \quad \widehat{Q} = (\mathbb{T}^2 \setminus O^u) \cup_\varphi (\mathbb{T}^2 \setminus O'^s)$$

and denote by $p\colon Q \to \widehat{Q}$ the natural projection.

A connected sum of model diffeomorphisms $F_{C,\mathcal{O}^s,\mathcal{O}^u}$, $F'_{C',\mathcal{O}'^s,\mathcal{O}'^u}$ along the orbits O^u, O'^s is a diffeomorphism $\phi\colon \widehat{Q} \to \widehat{Q}$ which coincides with the diffeomorphism $pF_{C,\mathcal{O}^s,\mathcal{O}^u}p^{-1}\Big|_{p(\mathbb{T}^2\setminus O^u)}$ on $p(\mathbb{T}^2 \setminus O^u)$ and with the diffeomorphism $pF'_{C',\mathcal{O}'^s,\mathcal{O}'^u}p^{-1}\Big|_{p(\mathbb{T}^2\setminus O'^s)}$ on $p(\mathbb{T}^2 \setminus O'^s)$.

An example of a connected sum of model diffeomorphisms is shown in Fig. 6. Here $\widehat{Q}$ is an orientable surface of genus two (pretzel). The connected sum operation is naturally generalized to the case of several model diffeomorphisms and several periodic orbits.

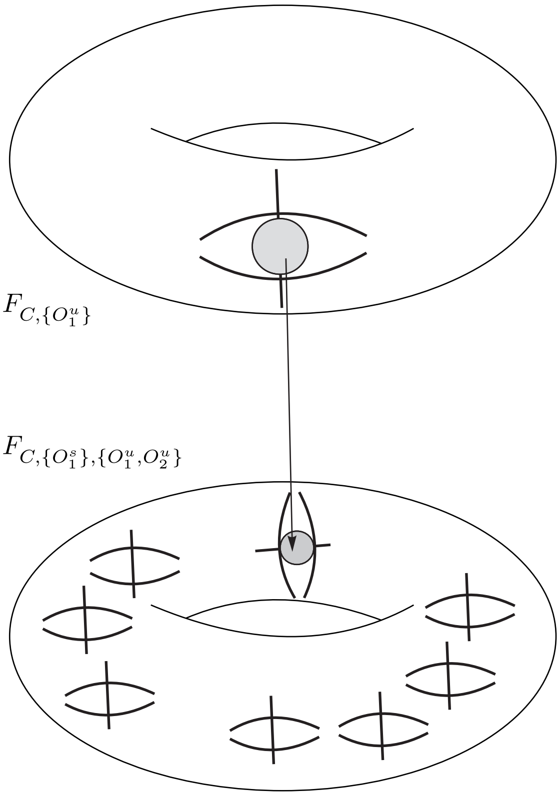

Fig. 6. Connected sum of model diffeomorphisms

Denote by G a class of $(A,\,B)$-diffeomorphisms which are a connected sum of model diffeomorphisms on a torus.

6. Realization of a connected Hasse diagram by a connected sum of the model diffeomorphisms

In this section, following [13], we realize any connected Hasse diagram by some diffeomorphism from class G. A scheme of the realization is provided below.

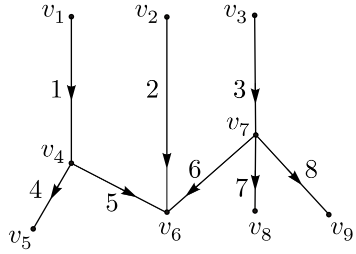

Fig. 7. A Hasse diagram with numbered edges

Proof of Theorem 2.

Let Γ be a connected Hasse diagram with vertices $v_1,\,v_2,\,\ldots,\,v_r$ and a set of oriented edges by the form $(v_i,\,v_j)$, $i\neq j$. Let us number all edges in an arbitrary order, that is, for each edge (v_i, v_j) let $k_{i,j}$ be a corresponding number, so that each edge is defined by triple $e_{i,j} = (v_i, v_j, k_{i,j})$ (see Fig. 7).

Let v_i be a vertex with l_i incoming edges $e^{in}_{i,j} = (v_i, v^{in}_j, k^{in}_{i,j})$, $j \in \{1, \ldots, l_i\}$ and m_i outgoing edges $e^{out}_{i,j} = (v_i, v^{out}_j, k^{out}_{i,j})$, $j \in \{1, \ldots, m_i\}$ (l_i and m_i can be zeros). Then consider a model diffeomorphism $F_{v_i} = F_{C,\mathcal{O}^s,\mathcal{O}^u}$, where the sets $\mathcal{O}^s$, $\mathcal{O}^u$ consist of l_i periodic orbits of the periods $k^{in}_{i,1}, \ldots, k^{in}_{i,l_i}$ and m_i periodic orbits of the periods $k^{out}_{i,1}, \ldots, k^{out}_{i,m_i}$, respectively.

The resulting diffeomorphim $f \in g$ realizing the graph Γ is the connected sum of F_{v_i}. Namely, if there is an edge $e^{in}_{i,j} = (v_i, v^{in}_j, k^{in}_{i,j})$ which coincides with an edge $e^{out}_{j,i} = (v_j, v^{out}_i, k^{out}_{j,i})$, then we connect F_{v_i} with F_{v_j} along the s- and u-orbits of the same period $k^{in}_{i,j} = k^{out}_{j,i}$. □

7. The labeled diagram is a complete invariant of the ambient Ω-conjugacy

In this section we prove Theorem 3.

By virtue of Theorem 1, the labeled Smale diagram for a diffeomorphism $f \in G$ is a Hasse diagram corresponding to the order of the basic sets of the diffeomorphism f, where each vertex is equipped with a triple C, $\mathcal{O}^s$, $\mathcal{O}^u$ corresponding to the basic set $\Lambda_{C,\mathcal{O}^s,\mathcal{O}^u}$ topological conjugacy class.

For diffeomorphisms f, $f' \in G$ the isomorphism of their Hasse diagrams and the fulfilment of the conditions of Theorem 1 on isomorphic labeled vertices is a necessary and sufficient condition for their Ω-conjugacy. However, in the general case the conjugating homeomorphism does not extend from the basic sets to the ambient surface. For example, Fig. 8 shows two diffeomorphisms obtained from the same set of model diffeomorphisms by different types of connected sums such that the diffeomorphisms are not ambiently Ω-conjugate (see Fig. 8).

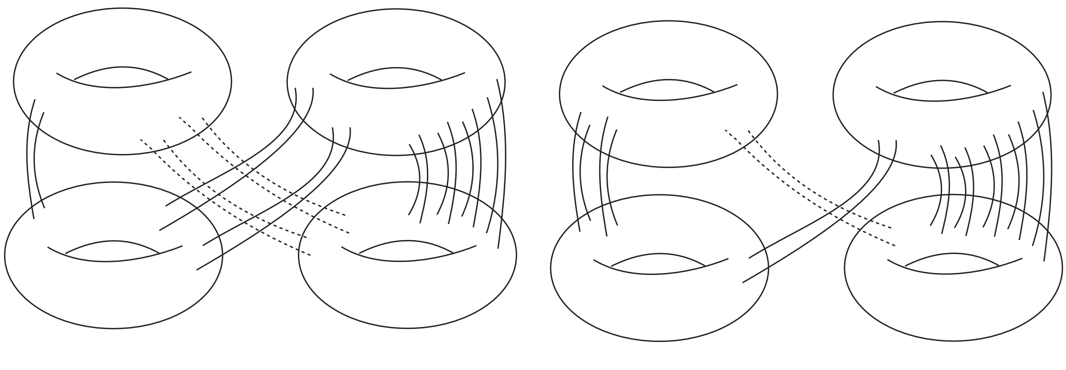

Fig. 8. Diffeomorphisms f and f' that are not ambiently Ω-conjugate

Let us single out a subclass $G_* \subset G$ of diffeomorphisms in which any two model diffeomorphisms are connected along at most one orbit.

Let us prove that the diffeomorphisms f, $f' \in G_*$ are ambiently Ω-conjugate if and only if their labeled diagrams are isomorphic.

Proof of Theorem 3.

Necessity. Since any two model diffeomorphisms from class G_* are a connected sums where any two model diffeomorphisms are connected along at most one orbit, it follows that the ambient Ω-conjugacy of the diffeomorphisms f, $f' \in G_*$ implies the isomorphism of their labeled Smale diagrams.

Sufficiency. The isomorphism of the labeled diagrams of the diffeomorphisms f, $f' \in G_*$ and Theorem 1 imply the existence of homeomorphisms conjugating model diffeomorphisms corresponding to isomorphic vertices. The map composed of these homeomorphisms is defined everywhere on the ambient surface except on the annuli connecting the two models. Continuing the map to the annuli in an arbitrary way, we get the desired homeomorphism which executes the ambient Ω-conjugacy of the diffeomorphisms f, f'. □

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