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V. A. ROKHLIN AND D. A. GUDKOV AGAINST THE BACKGROUND OF HILBERT'S 16TH PROBLEM (ACCORDING TO THEIR CORRESPONDENCE IN 1971–1982)

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The story of the friendship and collaboration between Vladimir Abramovich Rokhlin and the Nizhny Novgorod mathematician Dmitry Andreevich Gudkov during the last period of Rokhlin's mathematical biography, when he worked in the topology of real algebraic varieties, in which he obtained remarkable results. The paper is based on the correspondence of 1971–1982 *preserved in Gudkov's archive containing* 15 *letters by V. A. Rokhlin and* 8 *letters by D. A. Gudkov. Bibliography:* 19 *titles.*

Let me remind you that in the first part of his 16th problem (only this part is discussed in this note), D. Hilbert posed the problem of topological classification of projective real algebraic varieties, highlighting, in particular, the question about plane curves of degree 6: "The maximum number of closed and separate branches which a plane algebraic curve of the nth order can have has been determined by Harnack. There arises the further question as to the relative position of the branches in the plane. As to curves of the 6th order, I have satisfied myself – by a complicated process, it is true – that of the eleven branches which they can have according to Harnack, by no means all can lie external to one another, but that one branch must exist in whose interior one branch and in whose exterior nine branches lie, or inversely" [1].

In 1969, D. A. Gudkov [2] found an answer to this Hilbert's question: curves of degree 6 generate only schemes of oval arrangements that in the table in Fig. 1 lie below the broken line, where the notation $\frac{\alpha}{1}\beta$ encodes $\beta+1$ ovals lying outside each other one of which contains inside α ovals lying outside each other; $\frac{1}{\frac{1}{1}}$ denotes three "concentric" ovals, and $\langle\alpha\rangle$ denotes α ovals outside each other.

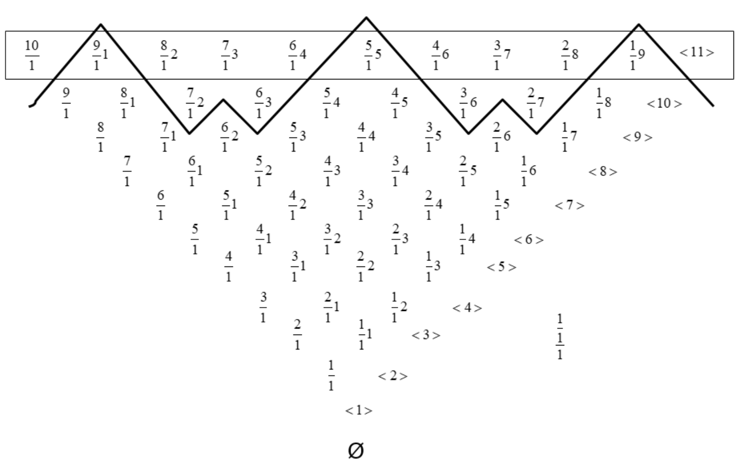

Fig. 1. Gudkov's classification of curves of degree 6.

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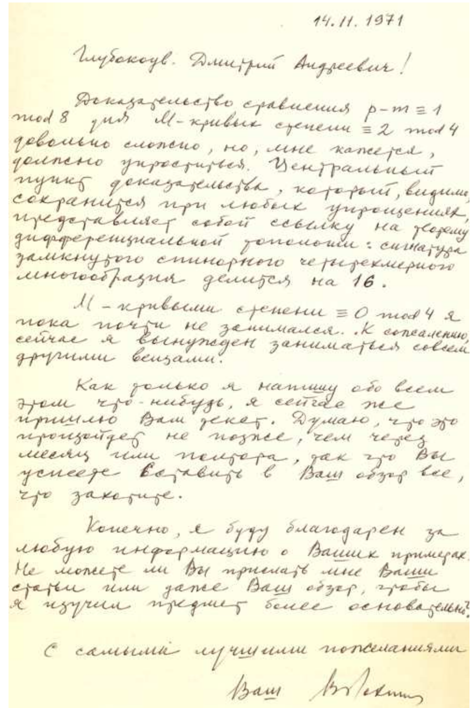
It can be seen from the table that Hilbert's conjecture about the unrealizability of the scheme $\frac{5}{1}5$ turned out to be incorrect. Gudkov wrote down the periodicity of the positions of realizable schemes in the top row of this table as a congruence $\mathcal{X}(B_+) \equiv k^2 \pmod 8$ with $k = 3$, where $\mathcal{X}(B_+)$ is the Euler characteristic of the oriented part of the complement to the curve in the projective plane.[1] Having verified that this congruence is true for all M-curves[2] of even degree $2k$ that he could construct, Gudkov put it forward as a conjectural necessary condition for the existence of an M-curve of degree $2k$. This Gudkov's conjecture, first published in [3] but even earlier popularized by him in conference talks and private conversations, was soon proved. First, V. I. Arnold [4] proved "half" of the congruence, modulo 4, and then V. A. Rokhlin proved it completely in [5] and [6].[3] Thanks to Rokhlin, the congruence is now called "Gudkov's congruence."



14.11.1971

Глубокоув. Дмитрий Андреевич!

Доказательство сравнения $p-m \equiv 1$ mod 8 для M-кривых степени $\equiv 2$ mod 4 довольно сложно, но, мне кажется, должно упроститься. Центральный пункт доказательства, который, видимо, сохранится при любых упрощениях, представляет собой ссылку на теорему дифференциальной топологии: сигнатура замкнутого спинорного четырехмерного многообразия делится на 16.

M-кривыми степени $\equiv 0$ mod 4 я пока почти не занимался. К сожалению, сейчас я вынужден заниматься совсем другими вещами.

Как только я напишу обо всем этом что-нибудь, я сейчас же пришлю Вам текст. Думаю, что это произойдет не позже, чем через месяц или полтора, так что Вы успеете вставить в Ваш обзор все, что захотите.

Конечно, я буду благодарен за любую информацию о Ваших примерах. Не можете ли Вы прислать мне Ваши статьи или даже Ваш обзор, чтобы я изучил предмет более основательно?

С самыми лучшими пожеланиями

Ваш

V. A. Rokhlin's letter dated November 14, 1971.

The described results of Gudkov, Arnold, and Rokhlin marked the beginning of a new stage in the development of the subject.[4] "Research on the topology of real algebraic varieties has merged into the general flow of research on differential topology" ([9, p. 5]). Also, they initiated a correspondence between Rokhlin and Gudkov. This correspondence took place in 1971–1982. In Gudkov's archive, there are 23 letters: 15 letters by Rokhlin and 8 letters by Gudkov.[5]

Of course, the correspondence mostly included an exchange of information about Hilbert's 16th problem, in particular, a discussion of the contents of the survey [9], on which Gudkov worked in 1970–1973. For example, on November 14, 1971, Rokhlin wrote:

"I almost have not dealt with M-curves of degree $\equiv 0 \mod 4$. Unfortunately, now I have to do completely different things. As soon as I write something about all this, I will send you the text immediately. I think that this will happen within a month or a month and a half, so you will have time to include anything you want into your survey. Of course, I would be grateful for any information about your examples. Could you send me your articles, or even your survey, so that I could study the subject more thoroughly?"

Here is V. A. Rokhlin's letter dated March 21, 1972, in full:

"Dear Dmitry Andreevich!

I have found another, much more elementary, proof of your congruence $p - n \equiv k^2 \mod 8$. This proof clarifies the proof of Arnold's congruence $p - n \equiv k^2 \mod 4$ contained in the last section of my note (which you have).[6] It turned out that the congruence $\chi(F) - \sigma(X) \equiv 2\chi(F)$

[1] In Gudkov's original wording, the left-hand side of the congruence was written in other combinatorial terms.

[2] An M-curve is a curve with the maximum possible number of ovals according to Harnack.

[3] As A. Maren discovered later (see [7]), Rokhlin's first proof in [5] contained an error.

[4] For an overview of the history of the study of the topology of algebraic curves from Antiquity to the first years of the 21st century, see [8].

[5] Gudkov maintained a wide correspondence and kept copies and drafts of many letters. Most of Gudkov's correspondence with mathematicians (including the complete correspondence with Rokhlin) was published in [10].

[6] V. A. Rokhlin means the paper [5]. – G.P.

mod 4 *used there can be refined as* $\chi(F) - \sigma(X) \equiv 2\tau(F) \mod 8$, *where* $\tau(F)$ *is the residue modulo* 4 *that can be described by the location of the surface* F *in* X. *In particular, it can be calculated for Arnold's surfaces in* $\mathbb{C}P^2$, *which leads to the congruence* $p - n \equiv k^2 \mod 8$. *In this case, the proof of the congruence* $\chi(F) - \sigma(X) \equiv 2\tau(F)$ *[in contrast to the proof of the congruence containing the* Arf-*invariant from* §3 *of my note] is quite elementary. No covers are needed.*

Vladimir Abramovich Rokhlin

No doubt, this proof better expresses the topological essence of the matter than the first one. It has not been found immediately only because the general topological theorem underlying it [*i.e., the congruence* $\chi(F) - \sigma(X) \equiv 2\tau(F) \mod 8$] *was unknown. I will probably write this proof in detail soon. Of course, I will send it to you. I do not know whether you will still be able to take it into account in your survey.*

Please let me know if there are similar conjectures related to other situations,[7] *for example, to curves of odd degree, to surfaces, or to non-plane curves.*

With best wishes,

V. Rokhlin."

In a long reply, dated April 24, 1972, D. A. Gudkov lays out a number of his conjectures with detailed motivations. I cite two of them as examples.

"*Conjecture. All* M*-curves of odd degree* m *admissible by Harnack's theorem and having "sufficiently few" nests* (*from considerations of intersection with a line, a second-order curve, etc.*) *do exist.*"

"*The conjecture on surfaces* F *of degree* 4 *not homotopic to zero in* $\mathbb{R}P^3$.

There are only three surfaces for which the sum of the Betti numbers $\sigma(F)$ *is* 24, *and this sum is the largest possible* (M*-surfaces*):

$$F : R^1_2 + 9R^0_0; \quad R^1_6 + 5R^0_0; \quad R^1_{10} + R^0_0.$$

($R^1_k + lR^0_0$ *consists of a piece* R^1_k *of the type of a single-sheeted hyperboloid with* $k-1$ *handles, the genre of the piece is* k; R^0_0 *is an ovaloid;* lR^0_0 *is* l *ovaloids in one of the regions into which* R^1_k *divides* $\mathbb{R}P^3$. *A surface of type* $R^1_{10} + R^0_0$ *was constructed by Hilbert in* 1907; *a surface of type* $R^1_6 + 5R^0_0$ *was constructed by Utkin recently; a surface of type* $R^1_2 + 9R^0_0$ *has not yet been constructed*)."

The conjecture about surfaces of degree 4 was confirmed: in 1976, V. M. Kharlamov [11] proved the existence of the missing surface. The conjecture about curves of odd degree turned out to be false: in 1980, O. Ya. Viro [12] completed the classification of nonsingular curves of degree 7 and proved that a curve of degree 7 containing 14 ovals inside one is not possible. I would like to note that Dmitry Andreevich was not at all upset when his conjectures were refuted. He once told me: "I deliberately put forward a lot of conjectures, although I am totally unsure about some of them: my goal is to attract people to this topic."

Dmitry Andreevich Gudkov

In the correspondence, various organizational problems were also discussed. For example, several letters dated the end of 1972 and the beginning of 1973 were devoted to resolving the situation with simultaneous and independent papers by D. A. Gudkov and A. D. Krakhnov in Gorky and V. M. Kharlamov in Leningrad

[7]Note that exactly the same question – "Do you have other conjectures?" – was asked by V. I. Arnold in his letter to D. A. Gudkov dated March 27, 1971. – G.P.

which contained intersecting results. Eventually, the corresponding articles [13] and [14] were published in the same issue of a journal.

There was also an exchange of bibliographic information. For example, on April 23, 1973, V. A. Rokhlin wrote:

“Dear Dmitry Andreevich, being in a hurry, I have written down neither the surveys you mentioned, nor the lady’s 1906 *paper with Petrovsky’s inequalities. Please send the coordinates!”*

And soon he received a reply:

“Dear Vladimir Abramovich! Here is the bibliography. Ragsdale V. (*the lady*) *“On the Arrangement of the Real Branches of Plane Algebraic Curves” Amer. Jour. of Math.* 28, *pp.* 377–404, 1906*”*[8] followed by more than ten references to articles on algebraic curves published in difficult-to-access sources. On January 15, 1974, Rokhlin wrote:

“I searched the writings you sent me for information on curves of degrees $8, 10, 12$*, but did not found much. I am now interested not in the construction techniques, but in the fact of existence. Is there any more or less complete table of the constructed curves?”*

Dmitry Andreevich asked me to compile such a table and then sent it to V. A. Rokhlin:

“I am sending you the tables of M*-curves* (*up to degree* 12) *constructed by known methods. The tables were compiled by G. Polotovskiy. In my opinion, it would be worthwhile to publish a survey on Bruzotti’s methods with an appendix containing the table of curves, but I do not know where.”*[9]

For his part, V. A. Rokhlin wrote to D. A. Gudkov about literature on topology:

“The best educational text on characteristic classes is Milnor’s lectures (*Mathematics,* 3 : 4 *and* 9 : 4). *The Arf-invariant, defined in my note, is calculated directly by definition. Other Arf-invariants used in topology can be found, for example, in Kerver and Milnor* (*Ann. Math.* 77*, No.* 3) *and Pontryagin* (*Trudy* MIAN, XLV*, the last section*). *However, do not expect much from learning topology from books and articles”* (January 21, 1972).

It is worth noting that the correspondence repeatedly addresses issues related to teaching mathematics. One of the reasons is Gudkov’s desire to set up teaching topology at the Gorky University[10]. Understanding well Rokhlin’s comment about learning topology from books, he wrote on January 7, 1973:

“The fact is that in Gorky there is no true culture in many areas of mathematics, in particular, in algebraic geometry and topology. Many mathematicians here feel this deficiency $< \cdots >$*. It would be good to obtain from you also* $< \cdots >$ *a specialist in algebraic topology* (*topology of manifolds, etc.*) *for a permanent position at the Gorky University. Of course, provided that this is the wish of the specialist himself. It would be best if he stayed in touch with you. This could be a very good undergraduate planning to enroll in the graduate program under your supervision, or a graduate student, or an already working mathematician. For my part, I would make every effort to implement such a plan.”*

No suitable specialist was found, and D. A. Gudkov himself began to teach an elective course in topology based on student notes of V. A. Rokhlin’s lectures. At the beginning of 1975, Gudkov sent a “report” to Rokhlin:

[8] A few years later, O. Ya. Viro [10], and then his student I. V. Itenberg [15], obtained counterexamples to Ragsdale’s conjecture from this paper.

[9] Such a paper [16] was published.

[10] Until mid-1970s, topology was not included in the curricula of Soviet universities, but V. A. Rokhlin taught a topology course at the Leningrad University since 1965 [17].

"I have taught your course, which I call a beginner's course, up to the third chapter, that is, I have given 23 lectures. $<\cdots>$ *The two remaining chapters,* 4. *Smooth manifolds and* 5. *Riemannian spaces, will be discussed in the next term. I cannot give more than one two-hour lecture a week, because the audience is very busy and I am afraid to abuse their attention* (*they may run away*). *At the listeners' request, I have written detailed notes of the lectures of the first chapter and will write notes of the others. I have an idea to make a rotaprint edition of these notes. In this regard, I have some thoughts which I would like to discuss with you:* 1) *In the preface, I mention that my course is based on student notes of your lectures.* 2) *Perhaps you will agree to be added as a coauthor, with an indication that all errors are completely my responsibility.* 3) *Perhaps you would like to look at these notes?* 4) *If you have objections, I will not publish these lectures."*

V. A. Rokhlin answers on May 19, 1973:

"Forgive me for not having answered you at once: I have a great backlog of letters, reviews, etc., and I write slowly. $<\cdots>$ *I will try to answer your questions. Of course, I do not mind if you print your lectures* (*or their abstracts*). *As for the indication that they are mostly based on student notes of my lectures, I have nothing to say against this, since this is true. However, it is important to mention that at our university this is a required course in geometry, more exactly, its part taught in the third term.* (*I mean topology; Riemannian geometry is taught in the fourth term, and, by the way, not by me, but by Yu. A. Volkov, who wrote the corresponding part of our program*). *Of course, I cannot be a coauthor. I have no strength to look through these notes before publication, but if you want a critique from an absolutely competent person, you may ask, for example, Oleg Yanovich Viro, who in this academic year taught a required course in topology instead of me.* (*By the way, recently he has received an award of the Leningrad Mathematical Society, and defended his dissertation back in December.*) *Perhaps he will not refuse to come to Gorky for a week or ten days to look through your text and tell you about our main elective courses and notes of them* (*however, I have not spoken with him about this*)*."*

O. Ya. Viro did not refuse. Since then, he and V. M. Kharlamov repeatedly visited Gorky. Gudkov created a required course in topology at the Gorky University, and the role of Rokhlin, Viro, and Kharlamov in this cannot be overestimated. In the preface to a series of six manuals on "Beginnings of Topology" (1981–1983), Gudkov wrote: "A significant part of this manual was reviewed by O. Ya. Viro, who made helpful comments. I express my sincere gratitude to V. A. Rokhlin, V. M. Kharlamov, and O. Ya. Viro."

Unfortunately, the friendship between D. A. Gudkov and V. A. Rokhlin was mostly epistolary: poor health and busy schedule did not allow them to accept repeated mutual invitations, so Gudkov and Rokhlin met in person only three or four times. I know that in 1974, Gudkov gave a talk at Rokhlin's seminar in Leningrad. In the late 1970s, Rokhlin spoke about complex topological characteristics of real algebraic curves at the Moscow Mathematical Society, and Dmitry Andreevich went to Moscow to listen to the talk, and took me with him. Once (I guess, in the very early 1980s), Vladimir Abramovich and his wife A. A. Gurevich visited Gorky. Soon after that, Dmitry Andreevich told me with genuine pleasure: "Yesterday Natasha (Natalia Vassilievna, Dmitry Andreevich's wife. – G.P.) told me: "How come that you are visited by such wonderful people?"

In August 1982, Gudkov took part in the Leningrad International Topological Conference. I do not know whether he met Rokhlin during this visit. In his last letter to Gudkov, on February 2, 1980, Vladimir Abramovich wrote:

"I myself have not thought about real algebraic varieties for several years, but I follow what is happening."

However, here is what E. I. Gordon [18] wrote about this Gudkov's stay in Leningrad:

"I will tell about a serious confrontation between D.A. and the local party officials, which could have ended very badly. In 1983, during a reelection of the head of the Department of Geometry and Algebra, at first everything went smoothly and calmly, but suddenly all sorts of obstacles began to arise from the then rector A. G. Ugodchikov. The decision had been repeatedly postponed; some bizarre claims were made such as an insufficient introduction of computer technology (practically absent at the faculty at the time) in teaching and research, the lack of applied topics, etc. D.A. recalled that the troubles began one day at the University Council, when the rector, who had just returned from the regional party committee, for no reason and in the middle of a discussion of a completely different issue, suddenly attacked D.A. with great irritation. Since at the time D.A. had quite a peaceful relationship with the rector, he put forward the following version of what had happened. Not long before that, D.A. was in Leningrad at a topological conference. There he went to the party committee of the Leningrad University and "like a communist" began to demand that the outstanding mathematician Vladimir Abramovich Rokhlin, who had been shortly before forced to retire, return to work. Rokhlin's dismission was inspired by the highest party officials.[11] $< \cdots >$ That is why, the reaction of the party committee was furious. They yelled at D.A., who also answered sharply. Apparently, the party committee of the Leningrad State University called the Gorky regional committee and instructed them "to sort things out."

D. A. Gudkov's friendship and collaboration with V. A. Rokhlin, O. Ya. Viro, and V. M. Kharlamov spread rapidly, and Gudkov's students became friends with Viro, Kharlamov, and then with other Rokhlin's students and students of his students. This played a significant role in the intensive development of real algebraic geometry in the last decades of the 20th century.

Translated by the author.

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