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On the Number of the Classes of Topological Conjugacy of Pixton Diffeomorphisms

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Abstract

For a wide class of dynamical systems known as Pixton diffeomorphisms the topological conjugacy class is completely defined by the Hopf knot equivalence class, i.e. the knot whose equivalence class under homotopy of the loops is a generator of the fundamental group $\pi_1(S^2 \times S^1)$. Moreover, any Hopf knot can be realized by a Pixton diffeomorphism. Nevertheless, the number of the classes of topological conjugacy of these diffeomorphisms is still unknown. This problem can be reduced to finding topological invariants of Hopf knots. In the present paper we describe a first order invariant for these knots. This result allows one to model countable families of pairwise non-equivalent Hopf knots and, therefore, infinite set of topologically non-conjugate Pixton diffeomorphisms.

Keywords Hopf knot · Pixton diffeomorphism · Homotopy invariant

Mathematics Subject Classification 37D15

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1 Introduction and Results

Invariants of oriented knots in 3-manifold M^3 equipped with a chosen equivalence class of the fundamental group $\pi_1(M^3)$ were studied from various points of view. The case $M^3 \cong S^2 \times S^1$ with the generator (positive) equivalence class whose knots are called *Hopf knots* is of special interest for the theory of dynamical systems. In the 3-dimensional dynamics the manifold $S^2 \times S^1$ appears in the natural way as the space of wandering orbits in the basin of a hyperbolic sink while the generator (positive) equivalence class of its fundamental group bears the comprehensive information on the dynamics of the system in this basin. In particular, a 1-dimensional saddle separatrix in the basin of this sink has a corresponding Hopf knot in $S^2 \times S^1$.

For the class of dynamical systems known as *Pixton diffeomorphisms* (see Sect. 5) this knot (up to a homeomorphism of $S^2 \times S^1$) completely defines the class of topological conjugacy of the Pixton diffeomorphism [1] and, moreover, any Hopf knot can be realized as some Pixton diffeomorphism. Thus we have the complete topological classification of Pixton diffeomorphisms. Nevertheless the problem of the cardinality of the set of topological conjugacy classes of these diffeomorphisms is still open[1] and it can be reduced to finding invariants of Hopf knots.

In the present paper we state the existence of an invariant of the first order for Hops knots. This allows one to model countable families of pairwise non-equivalent Hopf knots and, therefore, infinite set of topologically non-conjugate Pixton diffeomorphisms.

Now we give the precise definitions and formulate our results.

A knot on a manifold $S^2 \times S^1$ is a smooth embedding $\gamma : S^1 \to S^2 \times S^1$ or the image of this embedding $L = \gamma(S^1)$.

Two knots γ, γ' are said to be *smoothly homotopic*, if there exists a smooth map $\Gamma : S^1 \times [0, 1] \to S^2 \times S^1$ such that $\Gamma(s, 0) = \gamma(s)$ and $\Gamma(s, 1) = \gamma'(s)$ for every $s \in S^1$. If, additionally, $\Gamma|_{S^1 \times \{t\}}$ is an embedding for every $t \in [0, 1]$ then the knots are *isotopic*.

Any Hopf knot $L \subset S^2 \times S^1$ is smoothly homotopic to the standard Hopf knot $L_0 = \{x\} \times S^1$ (see for example [2]) but generally L is neither isotopic nor equivalent to L_0. Denote by $\bar{L} = p^{-1}(L)$ the non-compact arc which is the lifting of a Hopf knot L by means of a universal covering map $p : S^2 \times \mathbb{R} \to S^2 \times S^1$.

Lemma 1 (Criterion of equivalence of a Hopf knot to the standard one) *A Hopf knot L is equivalent to the standard one if and only if there is a 2-sphere Σ which is smoothly embedded in $S^2 \times \mathbb{R}$ and which intersects the arc $\bar{L} = p^{-1}(L)$ transversally at a unique point.*[2]

B. Mazur in [4] constructed the Hopf knot L_M (see Fig. 1) which is non-equivalent and non-isotopic to L_0. We call this knot the *Mazur knot*.

[1] C. Bonatti and V.Z. Grines knew that there exists a countable set of pairwise non-equivalent Hopf knots. For the first time this fact was mentioned by I.V. Itenberg (who discussed the subject with O.Ya. Viro and learned from him about Mazur knot) to V.Z. Grines and later it was confirmed by V.A. Vasilyev to E.V. Zhuzhoma during their meeting in Rennes.

[2] The idea of the proof follows from the results of Section 4.1 [3] formulated in terms of dynamical systems.

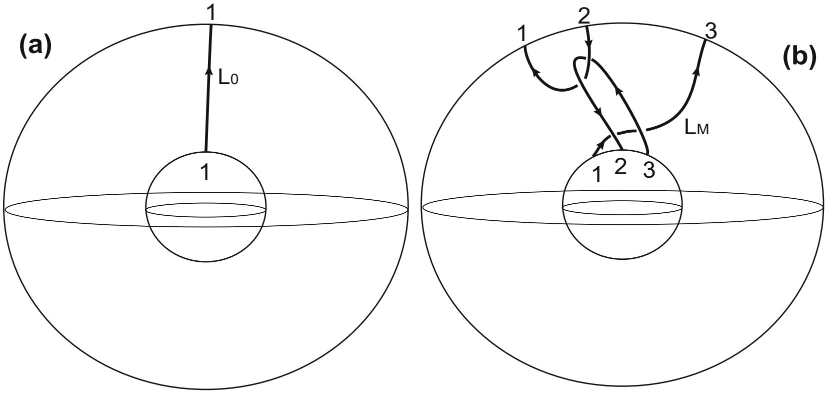


Fig. 1 The figure shows the fundamental domain of the cyclic covering over $S^2 \times S^1$. In order to get the Hopf knot in $S^2 \times S^1$ one glues the outer sphere with the inner one by identifying the points on the ray from the coordinate origin. In particular, the points with the same numbers are glued. Thus, two non-isotopic and non-equivalent Hopf knots L_0 and L_M are shown: **a** the standard Hopf knot L_0, **b** the Mazur knot L_M

An isotopic invariant of a knot L is said to be *of the first order in Vasilyev sense* if its change by a homotopy with one transversal self-intersection is defined by homotopy classes of the two loops created by this intersection.

Let $L \subset S^2 \times S^1$ be a Hopf knot and let $F_t : S^1 \to S^2 \times S^1$ be a smooth *generic* homotopy from $F_0(S^1) = L$ to L_0. This means that F_t has a finite number of the *singular values* $t_1, \dots, t_m$ for which F_{t_i} is not an embedding while each arc $C_i = F_{t_i}(S^1)$ has a unique point of self-intersection x_i, the self-intersection being transversal. Then one of the loops of the arc C_i realizes the even homology class $2s_i$, $s_i \in \mathbb{Z}$ while the other realizes the odd one. The standard rule (see Sect. 2) defining the sign of the singularity of the intersection ± 1 determines the monomial $\pm\tau^{s_i}$ corresponding to the self-intersection point x_i. The sum of the monomials for all singular points makes a Laurent polynomial $P_{F_t}(\tau)$ over $\mathbb{Z}$. We say the degrees s_i of the monomials τ^{s_i} of P_{F_t} to be *the exponents of the homotopy* F_t.

Let

$$I_{F_t}(\tau) = P_{F_t}(\tau) - P_{F_t}(0).$$

The main result of this paper is the following theorem.

Theorem 1 *For a Hopf knot $L \subset S^2 \times S^1$ the polynomial I_{F_t} is independent of the choice of a homotopy F_t between L and L_0 and $I_L = I_{F_t}$ is an isotopy invariant of the first order.*[3]

The following lemma shows that I_L is an equivalence invariant of Hopf knot L.

Lemma 2 *If two Hopf knots L, L' are equivalent then $I_L = I_{L'}$.*

[3] Notice that in [2] there is an invariant of the first order for the contractible knot $L \subset S^2 \times S^1$ (this case is included in the more general one). But Theorem 1 does not follow from these results.

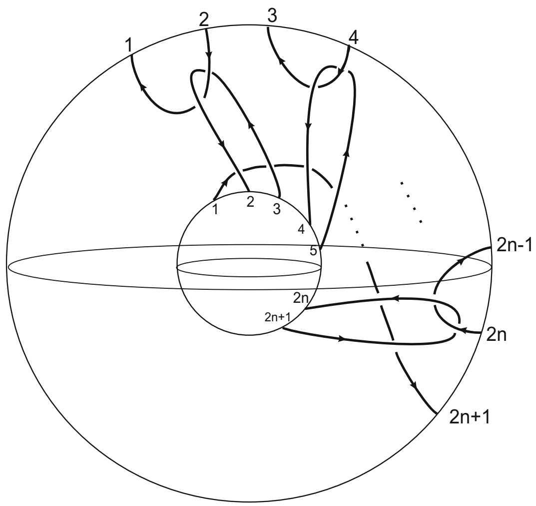


Fig. 2 The generalized Mazur knot $L_{M,n}$

These results allow us to point out a countable set of pairwise non-equivalent Hopf knots. The Mazur knot has exactly one pair of snagged arcs in the cover $S^2 \times [0, 1]$ obtained by cutting $S^2 \times S^1$ along the fiber $S^2 \times \{z\},\ z \in S^1$ (see Fig. 1). Therefore it can be generalized in the natural way by the generalized Mazur knot $L_{M,n},\ n \in \mathbb{N}$ with n pairs of snagged arcs in $S^2 \times [0, 1]$ (see Fig. 2). The direct calculation of the monomial $P_{F_t}(\tau)$ for the knot $L_{M,n}$ gives us the following result.

Proposition 1 *$I_{L_{M,n}}(\tau) = n\tau$ for the generalized Mazur knot $L_{M,n}$ and, therefore, the Hopf knots $\{L_{M,n},\ n \in \mathbb{N}\}$ are pairwise non-equivalent and non-isotopic.*

Consider another generalization $L^k_M,\ k \in \mathbb{N}$ of the Mazur knot when two snagged arcs turn k times around the generator (see Fig. 3). The direct calculation of the polynomial $P_{F_t}(\tau)$ for L^k_M produces the following result.

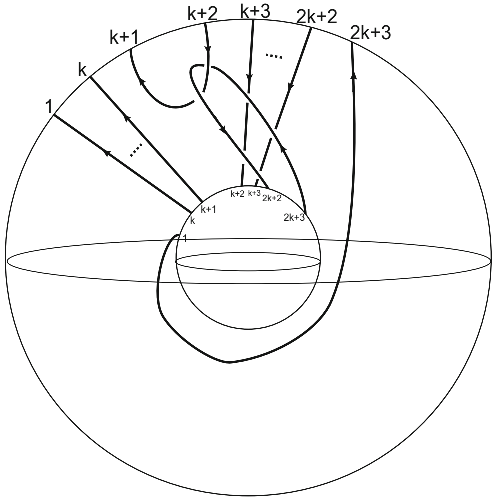
Proposition 2 *$I_{L^k_M}(\tau) = -\tau^{m+1}$ for even $k = 2m$ and $I_{L^k_M}(\tau) = -\tau^{-m}$ for odd $k = 2m-1$ for the generalized Mazur knot L^k_M. Therefore the Hopf knots $\{L^k_M,\ k \geq 0\}$ are pairwise non-equivalent and non-isotopic.*

There is one more way of getting different topology classes of Hopf knots. Let $\nu_l : S^2 \times S^1 \to S^2 \times S^1$ be a covering map of order $l \in \mathbb{N}$, that is $\nu_l(s, z) = (s, z^l)$. The *l-transfer of a Hopf knot* $L \subset S^2 \times S^1$ is the knot $\tilde{L} \subset S^2 \times S^1$ defined as the lift of L by ν_l. Notice that the transfer of a Hopf knot is a Hopf knot again.

A homotopy $\tilde{F}_t$ from $\tilde{L}$ is defined as a generic homotopy which is a small deformation of the homotopy that covers the homotopy F_t from L. The following lemma describes the connection between the polynomials $P_{F_t}(\tau)$ and $P_{\tilde{F}_t}(\tau)$ when the homotopy F_t has a unique point of self-intersection.

Lemma 3 *Let the monomial $P_{F_t}(\tau)$ for a homotopy F_t from a Hopf knot L be $P_{F_t}(\tau) = \varepsilon \cdot \tau^s,\ \varepsilon \in \{-1, +1\}$. Let $l \in \mathbb{N}$ and let the number $q \in \mathbb{Z}$ be uniquely defined by*

Fig. 3 The generalized Mazur knot L_M^k



$(q-1)l < 2s \le ql$. Then for the homotopy $\tilde{F}_t$ from the l-transfer $\tilde{L}$ the polynomial $P_{\tilde{F}_t}(\tau)$ is

$$P_{\tilde{F}_t}(\tau) = \varepsilon \cdot l\tau^{\tilde{s}},$$

where $\tilde{s} = q/2$ if q is even and $\tilde{s} = (1-q)/2$ if q is odd.

Let F_t be a homotopy with a unique point of self-intersection from a generalized Mazur knot L_M^k to the standard Hopf knot L_0. Then from Lemma 3 and Proposition 2 for the 2-transfer it follows that $P_{\tilde{F}_t}(\tau) = 2\tau^{-p}$ if $k = 4p$ or $k = 4p-1$ and $P_{\tilde{F}_t}(\tau) = 2\tau^{p}$ if $k = 4p-2$ or $k = 4p-3$. In particular, $I_{\tilde{L}_M}(\tau) = 0$ for the 2-transfer of the Mazur knot $L_M = L_M^0$.

The following theorem shows that if the order of the covering map is great enough then the invariants of the first order do not distinguish the standard Hopf knot from the other knots.

Theorem 2 *Let l be greater than twice the absolute value of each exponent of the invariant I_L of a Hopf knot $L \subset S^2 \times S^1$. Then $I_{\tilde{L}}(\tau) = 0$ for the l-transfer $\tilde{L}$.*

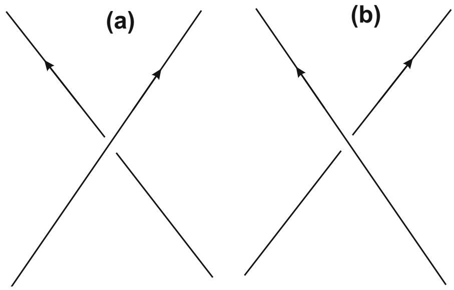
On the other hand using Lemma 1 one can prove the following fact.

Lemma 4 *If a Hopf knot is non-equivalent to the standard one then its transfer is non-equivalent to the standard one, as well.*

These results show that the complete classification up to equivalence of Hopf knots cannot be constructed on the invariants of the first order only.

From the point of view of the theory of dynamical systems if a knot L corresponds to a Pixton diffeomorphism f then its l-transfer $\tilde{L}$ corresponds to the diffeomorphism f^l. From Theorem 2 it follows that if the Pixton diffeomorphism f is realized for a knot L with the non-zero invariant I_L then f is not topologically conjugate to its power f^l for l starting from some $l_0 \in \mathbb{N}$ and greater.

Fig. 4 **a** underpass +1, **b** overpass −1



2 The Isotopy Invariant of the First Order for Hopf Knots (Proof of Theorem 1)

In this section we prove the existence of an isotopy invariant of the first order for Hops knots (Theorem 1). The proof follows from the auxiliary Lemma 5 of this section.

Let $L \subset S^2 \times S^1$ be a Hopf knot and let $F_t : S^1 \to S^2 \times S^1$ be a smooth general position homotopy from $F_0(S^1) = L$ which means that F_t has a finite number of the *singular values* $t_1, \dots, t_m$ for which F_{t_i} is not an embedding while each arc $C_i = F_{t_i}(S^1)$ has a unique point of self-intersection x_i and this self-intersection is transversal. Then one of the loops of the arc C_i realizes the even homology class $2s_i,\ s_i \in \mathbb{Z}$ while the other realizes the odd one.

Recall the standard rule defining the sign ± 1 of the singularity of the intersection. Consider an ordered frame of 3 vectors in the tangent space T_{x_i} of the manifold $S^2 \times S^1$. Let two vectors of the frame be tangent to the branches of the knot arc (arbitrary ordered) at the point x_i and let the third vector of the frame be transversal to the first two, its direction being the direction of transition from underpass to overpass at the point of the singular self-intersection when the homotopy parameter increases. Define the sign of the singularity to be positive if this frame is positively oriented in T_{x_i} and define it to be negative otherwise. The sign is independent of the order of the knot branches at the self-intersection and, therefore, it uniquely defines the sign of the monomial $\pm\tau^{s_i}$ in the polynomial $P_{F_t}(\tau)$ (see Fig. 4).

We say a homotopy F_t connecting the standard Hopf knot L_0 to itself to be *periodic*.

The proof of Theorem 1 follows from Lemma 5

Lemma 5 $I_{F_t}(\tau) = 0$ *for any periodic homotopy* F_t.

Proof Let $F_t : S^1 \to S^2 \times S^1$ be a periodic homotopy. Let F_t be defined by

$$F(z,t) = (s(z,t), w(z,t)),$$

where $z \in S^1, t \in [0,1], s(z,t) \in S^2, w(z,t) \in S^1$. Let the smooth map $\Psi_F : S^1 \times S^1 \to S^2 \times S^1$ be defined by

$$\Psi_F(z, t \pmod 1) = (s(z,t), w(z,t)).$$

Then Ψ_F is in one-to-one correspondence with $F(z,t)$.

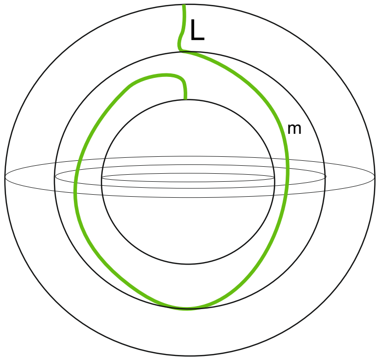


Fig. 5 Knot L

Consider the space of all maps Ψ_F constructed for periodic homotopies F_t. From [5] it follows that a connected component of this space is defined by two invariants $\theta_1(\Psi_F) \in \pi_0(Map\,(S^1 \times S^1; S^1))$ and $\theta_2(\Psi_F) \in \pi_0(Map\,(S^1 \times S^1; S^2))$. From Brushlinsky's theorem [6] it follows that $\pi_0(Map\,(S^1 \times S^1; S^1)) = H^1(S^1 \times S^1; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$. From Hopf-Whitney theorem ([7], Theorem 5, Chapter 1) one has $\pi_0(Map\,(S^1 \times S^1; S^2) = H^2(S^1 \times S^1; \mathbb{Z}) = \mathbb{Z}$.

In our case for Hopf knots the values of the invariant $\theta_1(\Psi_F)$ are in $1 \oplus \mathbb{Z} \subset H^1(S^1 \times S^1; \mathbb{Z})$. Since every homotopy $F(z, t)$ can be modified and left-composed with the homotopy of $-\theta_1$ rotations on t-family of z-circles for $t \in [0, 1]$ without loss of generality we have $\theta_1(\Psi_F) = 1 \oplus 0$.

The proof consists of two steps: 1) construction of a periodic isotopy F'_t for which $\theta_2(\Psi_{F'}) = \theta_2(\Psi_F) = \theta_2$ and $I_{F'_t}(\tau) = 0$; 2) the proof that $I_{F_t}(\tau) = I_{F'_t}(\tau)$.

Step 1. Consider the periodic isotopy $J : S^1 \times S^1 \to S^2 \times S^1$ defined by $J(z, t) = (s_0, z)$. If $\theta_2 = 0$ then let $F'_t = J_t$ and $I_{F'_t}(\tau) = 0$. Otherwise consider the Hopf knot L which coincides with L_0 outside $S^2 \times (\frac{1}{4}, \frac{3}{4})$ and whose intersection with the sphere $S^2 \times \{\frac{1}{2}\}$ is the segment m of the meridian directed from the south pole to the north pole (see Fig. 5). The isotopy $F'(z, t)$ is constructed in the following way:

- $F'(z, t)$ coincides with $J(z, t)$ on $([0, \frac{1}{4}] \cup [\frac{3}{4}, 1]) \times [0, 1]$,
- $F'(z, t),\ t \in [0, \frac{1}{4}]$ is the isotopy from L_0 to L such that $F'([\frac{3}{8}, \frac{5}{8}] \times \{\frac{1}{4}\}) = m$ and $F'(z, t),\ t \in [\frac{3}{4}, 1]$ is the reverse isotopy,
- $F'(z, t)$ on $[\frac{3}{8}, \frac{5}{8}] \times [\frac{1}{4}, \frac{3}{4}]$ coincides with the composition of the map $F'(z, \frac{1}{4})$ and t-isotopy of θ_2 rotations of the meridian m around north pole - south pole axis,
- $F'(z, t)$ on $\left([\frac{1}{4}, \frac{3}{8}] \cup [\frac{5}{8}, \frac{3}{4}]\right) \times [\frac{1}{4}, \frac{3}{4}]$ coincides with $F'(z, \frac{1}{4})$.

Smooth F'_t to a smooth isotopy in the standard way. By construction $I_{F'_t}(\tau) = 0$, $\theta_2(\Psi_{F'}) = \theta_2$.

Step 2. Let $\Xi(z, t, r) : S^1 \times S^1 \times [0, 1] \to S^2 \times S^1,\ r \in [0, 1]$ be an r-homotopy connecting the map $\Xi(z, t, 0) = \Psi_{F'}(z, t)$ and $\Xi(z, t, 1) = \Psi_F(z, t)$. Consider its lift

$$\tilde{\Xi} : S^1 \times [0, 1] \times [0, 1] \to S^2 \times S^1 \times [0, 1] \times [0, 1]$$

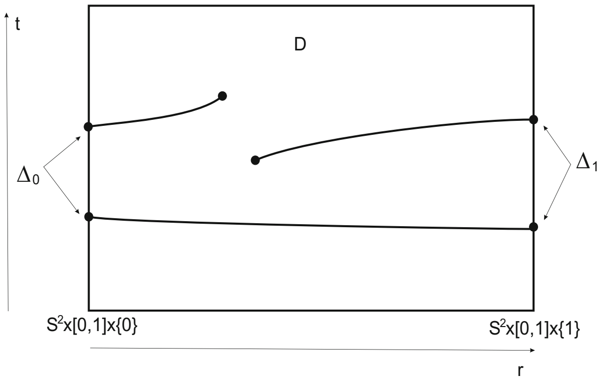


Fig. 6 The manifold Δ

defined by $\tilde{\Xi}(z, t, r) = (\Xi(z, t, r), t, r)$. Without loss of generality let the self-intersections of $\tilde{\Xi}(S^1 \times [0, 1] \times [0, 1])$ be transversal in $S^2 \times S^1 \times [0, 1] \times [0, 1]$ and let the self-intersection curves $\tilde{\Xi}(S^1 \times [0, 1] \times [0, 1])$ in $S^2 \times S^1 \times [0, 1] \times [0, 1]$ be in general position with respect to the projection to the segment $[0, 1](r)$.

Denote by Δ_0, Δ_1 the respective sets of points of self-intersection of $\tilde{\Xi}(S^1 \times [0, 1] \times \{0\})$, $\tilde{\Xi}(S^1 \times [0, 1] \times \{1\})$ and denote by Δ the set of self-intersection curves of $\tilde{\Xi}(S^1 \times [0, 1] \times [0, 1])$. The sets Δ_0, Δ_1 are 0-dimensional oriented manifolds (finite sets of points equipped with signs). The orientation on the manifold Δ is defined by its normal bundle in $S^2 \times S^1 \times [0, 1] \times [0, 1]$ represented as the (unordered) Whitney sum of the two oriented 2-bundles, the orientation of this 4-bundle being independent of the order of the 2-bundles. The manifold Δ consists of segments and circles while $(\Delta_0 \cup \Delta_1) \subset \partial\Delta$. Let $D = \partial\Delta \backslash (\Delta_0 \cup \Delta_1)$ (see Fig. 6).

Consider the knot $\tilde{\Xi}(S^1, t_0, r_0)$ containing the self-intersection point $x \in \Delta$. At the point x the even loop on the knot $\tilde{\Xi}(S^1, t_0, w_0)$ is well defined, its homotopy type being $2s \in \pi_1(S^1)$. Since a homotopy is continuous the value s is constant on the components of Δ. If a component contains a boundary point of D then $s = 0$ because in a neighborhood of a boundary point of D for s-parameter deformations of t-families of knots there can only be a self-intersection point for which the even loop is small and, therefore, it represents the zero element in $\pi_1(S^1)$.

Hence, the manifold Δ defines the cobordism from Δ_0 to Δ_1 modulo points from D. Therefore, $I_{F_t}(\tau) = I_{F'_t}(\tau)$. □

3 Transfer Invariants for Hopf Knots

Recall that an *l-transfer of a Hopf knot* $L \subset S^2 \times S^1$ is the Hopf knot $\tilde{L} \subset S^2 \times S^1$ defined as the lift of L by a l-fold covering $\nu_l : S^2 \times S^1 \to S^2 \times S^1$. A homotopy $\tilde{F}_t$ from $\tilde{L}$ is defined to be a generic homotopy which is a small deformation of the homotopy that covers the homotopy F_t from L.

In this section we prove Lemma 3 and Theorem 2. Lemma 3 describes the relation between the polynomials $P_{F_t}(\tau)$ and $P_{\tilde{F}_t}(\tau)$ when the homotopy F_t has a unique point of self-intersection.

Lemma 3 *Let the monomial $P_{F_t}(\tau)$ for a homotopy F_t from a Hopf knot L be $P_{F_t}(\tau) = \varepsilon \cdot \tau^s$, $\varepsilon \in \{-1, +1\}$. Let $l \in \mathbb{N}$ and let the number $q \in \mathbb{Z}$ be uniquely defined by $(q-1)l < 2s \leq ql$. Then for the homotopy $\tilde{F}_t$ from the l-transfer $\tilde{L}$ the polynomial $P_{\tilde{F}_t}(\tau)$ is*

$$P_{\tilde{F}_t}(\tau) = \varepsilon \cdot l\tau^{\tilde{s}},$$

where $\tilde{s} = q/2$ if q is even and $\tilde{s} = (1-q)/2$ if q is odd.

Proof Let $L \subset S^2 \times S^1$ be a Hopf knot and let $F_t : S^1 \to S^2 \times S^1$ be a smooth generic homotopy from $F_0(S^1) = L$ with the unique self-intersection point x_0 on the arc $C = F_{t_0}(S^1)$, $t_0 \in (0, 1)$. The point x_0 produces two loops $J^e_{x_0}$, $J^o_{x_0}$ one of which realizes the even homology class $2s$, $s \in \mathbb{Z}$ while the other realizes the odd one $(1-2s)$ and $P_{F_t}(\tau) = \varepsilon \cdot \tau^s$ where $\varepsilon \in \{-1, +1\}$.

All the singular self-intersection points of $\tilde{F}$ are cycle ordered with respect to the positive unit shift of the cover by the generator of the cover; denote these points by $\tilde{x}^1_0, \ldots, \tilde{x}^l_0$ and denote by $t^1_0, \ldots, t^l_0 \in (0, 1)$ the corresponding time values of the homotopy $\tilde{F}_t$. Therefore the monomials corresponding to these points and the monomial τ^s have the same sign and the exponents at all the points are the same. Hence, $P_{\tilde{F}_t}(\tau) = \varepsilon \cdot l\tau^{\tilde{s}}$,.

Now we calculate $q = 2\tilde{s}$ as the class of the even loop produced by the unique self-intersection point $\tilde{x}^1_0$ on the arc $\tilde{C} = \tilde{F}_{t^1_0}(S^1)$, $t^1_0 \in (0, 1)$.

Consider the diffeomorphism $a : S^2 \times \mathbb{R} \to S^2 \times \mathbb{R}$ defined by $a(s, r) = (s, r+1)$. Then $S^2 \times S^1 = (S^2 \times \mathbb{R})/a$ is the orbit space of a on $S^2 \times \mathbb{R}$ and $p : S^2 \times \mathbb{R} \to S^2 \times S^1$ is the universal cover. Consider the action of the diffeomorphism a^l on $S^2 \times \mathbb{R}$, consider the orbit space $S^2 \times S^1 = (S^2 \times \mathbb{R})/a^l$ of the action of a^l on $S^2 \times \mathbb{R}$ and the universal cover $p_l : S^2 \times \mathbb{R} \to S^2 \times S^1$. Then $\nu_l = pp_l^{-1} : S^2 \times S^1 \to S^2 \times S^1$.

Pick a point $\alpha \in p_l^{-1}(x_0)$ and consider the lifting $\bar{C}$ of the loop $\tilde{C}$ by the covering p_l with the starting point α. Since the class of $\tilde{C}$ is 1 the end point of $\bar{C}$ coincides with $\beta = a^l(\alpha)$. Since the arc $\tilde{C}$ has the unique self-intersection point there is the unique interior point of $a^{ql}(\alpha)$, $q \in \mathbb{Z}$ on the arc $\bar{C}$. Denote this point by γ.

The point γ divides the arc $\bar{C}$ into two parts $\bar{J}_\alpha \ni \alpha$, $\bar{J}_\beta \ni \beta$. The loops $\tilde{J}_\alpha = p_l(\bar{J}_\alpha)$, $\tilde{J}_\beta = p_l(\bar{J}_\beta)$ produced by the self-intersection point $\tilde{x}^1_0$ of $\tilde{C}$ are of classes q, $1-q$, respectively. To be able to express q by s notice that a part of the arc $\bar{J}_\alpha$ coincides with the lift $\bar{J}^e_{x_0}$ of the loop $J^e_{x_0}$ by the cover p with the starting point α. Since the class of the loop $J^e_{x_0}$ is $2s$ the end point of the arc $\bar{J}^e_{x_0}$ coincides with the point $\delta = a^{2s}(\bar{x}^1_0)$ and the interior points of the arc $\bar{J}^e_{x_0}$ do not intersect the set $\{a^n(\bar{x}^1_0),\ n \in \mathbb{Z}\}$.

If $\delta = \gamma$ then $2s = ql$. Otherwise the arc $\bar{J}_\alpha$ from the point δ coincides with the lifts of the knot L by the covering p with the starting points $\delta, a(\delta), \ldots, a^{i-1}(\delta)$, $i \in \{1, \ldots, l-1\}$ until $2s + i$ is a multiple of l. Then $ql = 2s + i$ and the number q is uniquely defined by $(q-1)l < 2s \le ql$. If q is even then $\tilde{s} = q/2$ and if q is odd then $\tilde{s} = (1-q)/2$. □

Theorem 2 *Let l be greater than twice the absolute value of each exponent of the invariant I_L of a Hopf knot $L \subset S^2 \times S^1$. Then $I_{\tilde{L}} = 0$ for the l-transfer $\tilde{L}$.*

Proof Let F_t be a homotopy from the knot L to the standard one and let $s_1, \ldots, s_\mu$ be the exponents of the polynomial $P_{F_t}(\tau)$. From Lemma 3 it follows that the number of the exponents $\tilde{s}_1, \ldots, \tilde{s}_\mu$ of the polynomial $P_{\tilde{F}_t}(\tau)$ is the same and $2\tilde{s}_i = q_i$ if q_i is even while $2\tilde{s}_i = 1 - q_i$ if q_i is odd; here q_i is defined by $(q_i - 1)l < 2s_i \le q_i l$.

If $l > |2s_i|$ then $0 \cdot l < 2s_i \le 1 \cdot l$ for positive s_i and $-1 \cdot l < 2s_i \le 0 \cdot l$ for negative $s_i \le 0$. This means $q_i = 1$, $\tilde{s}_i = 0$ for positive s_i and $q_i = 0$, $\tilde{s}_i = 0$ for $s_i \le 0$. Thus, all the exponents $\tilde{s}_i$ are zeros and, therefore, $I_{\tilde{L}}(\tau) = 0$. □

4 Equivalence of Hopf Knots

In this section we prove Lemmas 1, 2 and 4.

Recall that when proving Lemma 3 we introduced the diffeomorphism $a : S^2 \times \mathbb{R} \to S^2 \times \mathbb{R}$ defined by $a(s, r) = (s, r+1)$. Then $S^2 \times S^1 = (S^2 \times \mathbb{R})/a$ is the orbit space of a action on $S^2 \times \mathbb{R}$ and $p : S^2 \times \mathbb{R} \to S^2 \times S^1$ is a universal cover. For the Hopf knot L we defined $\bar{L} = p^{-1}(L)$.

Lemma 1 (Criterion of equivalence of a Hopf knot to the standard one) *A Hopf knot L is equivalent to the standard one if and only if there is a 2-sphere Σ which is smoothly embedded in $S^2 \times \mathbb{R}$ and which intersects the arc $\bar{L}$ transversally at a unique point.*

Proof ($\Rightarrow$) Let a Hopf knot L be equivalent to the standard one L_0. Then there is a diffeomorphism $h : S^2 \times S^1 \to S^2 \times S^1$ for which $h(L^0) = L$. Thus, the connected component Σ of the set $p^{-1}(h(S^2 \times \{0\}))$ is the desired sphere.

($\Leftarrow$) Let Σ be a 2-sphere smoothly embedded in $S^2 \times \mathbb{R}$ and such that

($*$) the intersection $\Sigma \cap \bar{L}$ is a unique point.

Step 1. Now we modify the sphere Σ to the sphere Σ' with the property ($*$) and for which $a(\Sigma') \cap \Sigma' = \emptyset$.

Let m be the least natural for which $a^k(\Sigma) \cap \Sigma = \emptyset$ for all $k > m$. Without loss of generality assume Σ to be transversal to its images $a(\Sigma), \ldots, a^m(\Sigma)$. If this is not the case then this can be achieved by the following construction.

For every $x \in \Sigma$ pick a compact neighborhood K_x of the point x in $\mathbb{R}^3$ such that $a(K_x) \cap K_x = \emptyset$. This neighborhood exists because there are no fixed points of the diffeomorphism a on the sphere Σ. Pick a finite subcover $K_1, \ldots, K_p$ of Σ from the cover $\{K_x,\ x \in \Sigma\}$. Approximate Σ by a smooth sphere for which $a(K_1)$ is transversal to Σ. Further approximations allow one to obtain transversality along $K_1 \cup K_2$ and so on up to $K_1 \cup \cdots \cup K_p$. Then approximate in the same way the resulting sphere so that it is transversal to its images by a and a^2. After m steps one gets the sphere (denote it again by Σ) transversal to all its images by $a, \ldots, a^m$. All the approximations can be done so that the property ($*$) is preserved.

Consider two cases: (I) $m = 1$, i.e. $a(\Sigma) \cap \Sigma \ne \emptyset$, $a^k(\Sigma) \cap \Sigma = \emptyset$ for all $k > 1$ and (II) $m > 1$, i.e. $a^k(\Sigma) \cap \Sigma \ne \emptyset$ for $k = 1, \ldots, m$, $a^k(\Sigma) \cap \Sigma = \emptyset$ for all $k > m$.

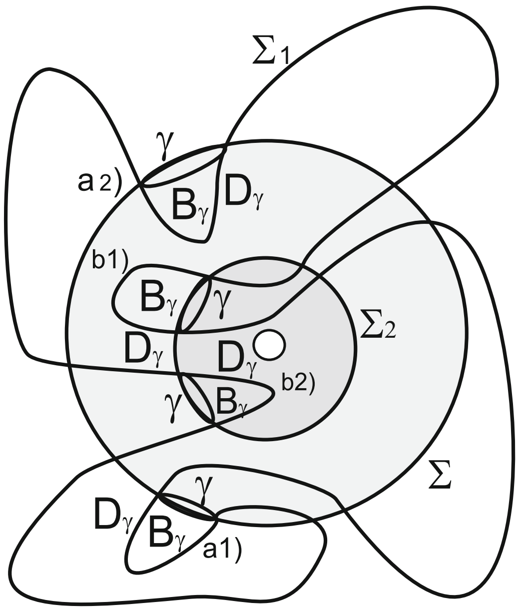


Fig. 7 Illustration to the proof of Lemma 1

Case (I). Now we modify the sphere Σ to the sphere S with the property (*) and for which

(i) $a^k(S) \cap S = \emptyset$ for all $k > 1$;

(ii) the number of curves in the intersection $a(S) \cap S$ is less than the number of curves in the intersection $a(\Sigma) \cap \Sigma$.

Let $\Sigma_1 = a(\Sigma)$ and $\Sigma_2 = a^2(\Sigma)$. Denote by K_+ the closure of the connected component of $S^2 \times \mathbb{R} \setminus \Sigma$ containing Σ_2. Let $K_- = cl\,(S^2 \times \mathbb{R} \setminus K_+)$

Let γ be a curve from the intersection $\Sigma_1 \cap (\Sigma \cup \Sigma_2)$. We say the curve γ to be *innermost* on Σ_1 if it bounds the disk $D_\gamma \subset \Sigma_1$ whose interior $int\ D_\gamma$ contains no curves of the set $\Sigma_1 \cap (\Sigma \cup \Sigma_2)$. Consider the innermost curve γ. Two cases are possible: a) $\gamma \subset \Sigma$ and b) $\gamma \subset \Sigma_2$.

In the case a) consider two subcases: a1) $D_\gamma \subset K_-$ and a2) $D_\gamma \subset K_+$.

In the subcase a1) the disk D_γ divides the domain K_- into two parts, the closure of one of them being the 3-ball (denote it by B_γ). By construction $int\ D_\gamma \cap a^k(\Sigma) = \emptyset$ for each $k > 1$. Moreover, the disks D_γ and $d_\gamma = B_\gamma \cap \Sigma$ have the same number (0 or 1) of points of intersection with the curve $\bar{L}$. Then the desired sphere S is obtained by smoothing of the sphere $(\Sigma \setminus d_\gamma) \cup D_\gamma$ (see Fig. 7).

In the subcase a2) one constructs the desired sphere S by arguing in the same way with the domain K_+.

Consider two subcases of the case b): b1) $D_\gamma \subset a^2(K_-)$ and b2) $D_\gamma \subset a^2(K_+)$.

In the subcase b1) the disk D_γ divides the domain $a^2(K_-)$ into two parts, the closure of one of them being the 3-ball (denote it by B_γ). By construction $int\ D_\gamma \cap a^{-k}(\Sigma_2) = \emptyset$ for each $k > 1$. Moreover, the disks D_γ and $d_\gamma = B_\gamma \cap \Sigma_2$ have the same number (0 or 1) of points of intersection with the curve $\bar{L}$. Let the sphere S_2 be a smoothing of the sphere $(\Sigma_2 \setminus d_\gamma) \cup D_\gamma$. Then $S = a^{-2}(S^2)$ is the desired sphere.

In the subcase b2) one constructs the desired sphere S by arguing in the same way with the domain $a^2(K_+)$.

One repeats this construction for each innermost curve and one gets the desired sphere Σ'.

Consider case (II). Let $\tilde{m} = \left[\frac{m+1}{2}\right]$. Notice that $\tilde{m} < m$ for $m \geq 2$. From (II) $a^{k\tilde{m}}(\Sigma) \cap \Sigma = \emptyset$ for all $k > 1$. By the same technique one constructs the sphere $\tilde{\Sigma}'$ intersecting the arc $\bar{L}$ at a unique point and $a^{\tilde{m}}(\tilde{\Sigma}') \cap \tilde{\Sigma}' = \emptyset$. Thus the number m is reduced to $\tilde{m}$. One continues this process and one reduces m to 1 and constructs the desired sphere Σ' in the same way as in the case (I).

Step 2. Construction of the diffeomorphism $h : S^2 \times S^1 \to S^2 \times S^1$ for which $h(L_0) = L$.

Let $N(L)$ be a tubular neighborhood of the knot L in $S^2 \times S^1$ and let $N(\bar{L}) = p^{-1}(N(L))$. Without loss of generality assume that $\Sigma' \cap N(\bar{L})$ consists of one 2-disk d. Now we show that $G = S^2 \times S^1 \backslash N(L)$ is the solid torus.

Denote by $K \subset S^2 \times \mathbb{R}$ the 3-annulus bounded by the spheres Σ' and $a(\Sigma')$ and let $\bar{G} = K \backslash int\, N(\bar{L})$. Then the boundary of the domain $\bar{G}$ is the 2-sphere. From the generalized Schoenflies theorem it follows that $\bar{G}$ is a 3-ball. The set G is obtained from the 3-ball $\bar{G}$ by gluing its boundary 2-disks by the diffeomorphism a and, therefore, it is the solid torus.

Pick a tubular neighborhood $N(L_0)$ of the knot L_0 and a diffeomorphism $h_N : cl\, N(L) \to cl\, N(L_0)$ such that $h_N(L) = L_0$. Then h_N maps a meridian of the solid torus $cl\, N(L)$ to a meridian of the solid torus $cl\, N(L_0)$.

Let $G_0 = S^2 \times S^1 \backslash N(L_0)$. Since $S^2 \times S^1$ is the result of gluing of two solid tori by the diffeomorphism mapping a meridian to a meridian the diffeomorphism $h_N : \partial G \to \partial G_0$ maps a meridian of the solid torus G to a meridian of the solid torus G_0 and, therefore, it can be extended to the desired diffeomorphism $h : S^2 \times S^1 \to S^2 \times S^1$ which realizes equivalence of the knots L and L_0. □

Lemma 2 *If two Hopf knots L, L' are equivalent then $I_L = I_{L'}$.*

Proof Let two Hopf knots L, L' be equivalent, i.e. there is a diffeomorphism $h : S^2 \times S^1 \to S^2 \times S^1$ such that $L' = h(L)$. Let F_t be a homotopy from L to L_0. Then $h(F_t)$ is a homotopy from L' to $h(L_0)$. Thus, to prove lemma it suffices to prove that there is an isotopy G_t from the knot L_0 to the knot $h(L_0)$.

From [8] it follows that there are two possibilities for the diffeomorphism h: (1) h is isotopic to the identity diffeomorphism; (2) h is isotopic to the Gluck diffeomorphism g which is the rotation of horizontal spheres around the axis of the standard knot L_0 from 0 to 2π. In the case (1) the desired isotopy G_t is the isotopy from the identity to the diffeomorphism h. In the case (2) construct an isotopy H_t from the identity to the diffeomorphism gh^{-1}. Then $H_0(S^1) = h(L_0)$ and $H_1(S^1) = g(L_0)$. Since $g(L_0) = L_0$ the isotopy $G_t = H_{1-t}$ is the desired one. □

Lemma 4 *If a Hopf knot is non-equivalent to the standard one then its transfer is non-equivalent to the standard one, as well.*

Proof Suppose the contrary, i.e. the l-transfer $\tilde{L}$ of a Hopf knot L is equivalent to the standard one. Then by Lemma 1 there is a smooth 2-sphere $\Sigma \subset S^2 \times \mathbb{R}$ such that the

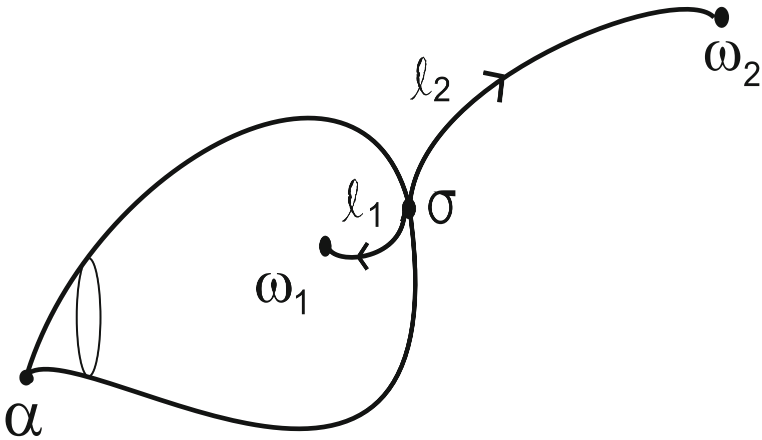


Fig. 8 Phase portrait of a diffeomorphism of $\mathcal{P}$

intersection $\Sigma \cap \bar{L}$ is the unique point. But this contradicts the fact that the knot L is not trivial. □

5 Pixton Diffeomorphisms

Denote by $\mathcal{P}$ the class of Morse-Smale diffeomorphisms $f : S^3 \to S^3$ whose non-wandering set consists of one fixed source α, one fixed saddle σ and two fixed sinks ω_1, ω_2. The phase portrait of such diffeomorphism is shown in Fig. 8. The original Pixton's example is of this class and we say the class $\mathcal{P}$ to be the *Pixton class*.

Recall the classification results from [1].

Let $f \in \mathcal{P}$. Denote by ℓ_1, ℓ_2 the unstable separatrices of the point σ. The closure $cl(\ell_i)$ $(i = 1, 2)$ of the 1-dimensional unstable separatrix of the point σ is homeomorphic to the simple compact arc and it is the union of this separatrix, the point σ and the sink ω_i (see Fig. 8).

Let $V_i = W^s_{\omega_i} \setminus \omega_i$ and $\hat{V}_i = V_i/f$ for $i = 1, 2$. The natural projection $p_{_i} : V_i \to \hat{V}_i$ is the covering and, therefore, it induces the epimorphism $\eta_{_i} : \pi_1(\hat{V}_i) \to \mathbb{Z}$. The manifold $\hat{V}_i$ is diffeomorphic to $S^2 \times S^1$ and the set $L_i = p_{_i}(\ell_i)$ is the $\eta_{_i}$-essential knot in the manifold $\hat{V}_i$ such that $\eta_{_i}(i_{L_{i*}}(\pi_1(L_i))) = \mathbb{Z}$, i.e. it is a Hopf knot.

Statement 1 *At least one of the knots L_1, L_2 is equivalent to the standard Hopf knot.*

If among the knots L_1, L_2 there is a non-equivalent to the standard one then denote it by $L(f)$; otherwise let $L(f) = L_1$. Figure 8 shows the phase portrait of a Pixton diffeomorphism whose both knots L_1, L_2 are equivalent to the standard one. Figure 9 shows the phase portrait of a Pixton diffeomorphism for which the knot L_1 is equivalent to the standard one while the knot $L(f) = L_2$ is the Mazur knot and the closure of the separatrix ℓ_2 is the wild Artin-Fox arc [9].

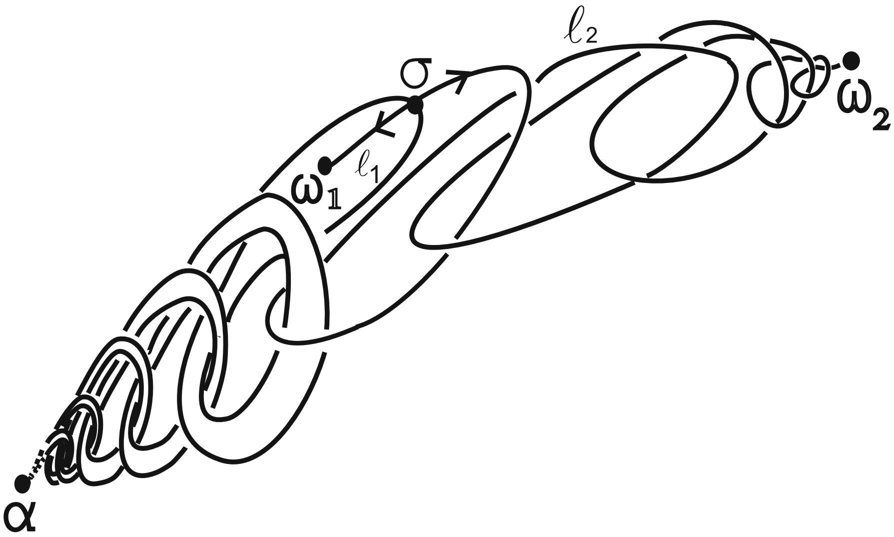


Fig. 9 The phase portrait of a Pixton diffeomorphism with the knot $L_2 = L_M$

Statement 2 *Two diffeomorphisms $f, f' \in \mathcal{P}$ are topologically conjugate if and only if the knots $L(f)$ and $L(f')$ are equivalent.*

Hence, the equivalence class of the knot $L(f)$ is the complete topological invariant for Pixton diffeomorphisms. Moreover, the realization theorem holds.

Statement 3 *For any Hopf knot $L \subset S^2 \times S^1$ there is a diffeomorphism $f \in \mathcal{P}$ such that the knots L and L_2 are equivalent.*

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