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Price competition in finite markets with a rare good and private consumer valuations*

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Abstract

We consider a market with two sellers, each having one unit of identical product, who compete for potential buyers with one-unit demand and a private valuation of this product. First, firms simultaneously post their prices. Then buyers observe these prices and choose a seller to submit a request for purchase depending on their private valuation of the product. This framework is between the directed search theory and the theory of competing mechanisms. We prove that the private information about the consumer's willingness to pay produces an equilibrium price dispersion and endogenous consumer separation.

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Keywords: imperfect competition, private valuations, price dispersion, directed search, market inefficiency.

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1 Introduction

This study clarifies the influence of private information about consumer valuations of a good on the equilibrium price distribution in a model of competition with the mechanism of a direct price announcement. The work bridges directed search theory (Burdett, Shi, and Wright 2001), (Julien, Kennes, and King 2000), (Wright et al. 2021) and the theory of competing mechanisms (McAfee 1993), (Peters and Severinov 1997), (Burguet and Sákovics 1999), (Albrechta, Gautierb, and Vroman 2012), (Virág 2010), accounting for the features of both approaches.

Directed search theory says that when a consumer chooses the firm for a purchase she minimizes not only the full price, but also the probability of being unserved in the case of competition among several consumers coming to a particular seller. This may happen if the quantity of the product is limited. Consumers may be ready to pay more if this increases the chance of making a purchase. The high price of a particular firm is a signal for prosperous consumers that there is a higher probability of being able to buy the product, but the consumer must overpay for this.

However, extracting the maximal revenue from consumers with high valuations is complicated by the mechanism of the direct price announcement without elements of bargaining. This mechanism is still one of the most popular way of market organization, but, in contrast to an auction, sellers cannot force consumers to reveal their true valuations. Thus, they must find a balance between a high price and the consequent the high marginal revenue, and the possibility of not attracting any buyers because of their over-estimation of consumer willingness to pay. Even when a seller accurately estimates the demand, i.e. the distribution of buyers' valuations and reserve prices, she still faces uncertainty about which exact buyers are currently in the market.

This factor is a potential source of market inefficiency which is a puzzling phenomenon in different markets. This study offers a new answer to the question of the appearance of equilibrium price dispersion in markets with the a finite number of homogeneous goods. The model can also be used to explain the coexistence of job vacancies and unemployed in the labor market, and superfluous food discarded by stores and starving people.

The paper presents a market which consists of two sellers with one unit of a homogeneous product, and n buyers with private valuations of the product; every buyer has a one-unit demand. At the first stage, firms publicly, simultaneously and independently set their prices. At the second stage, buyers choose which seller to visit in order to buy the product for the announced price, or whether to visit at all. If several buyers come to the same seller, she chooses one buyer to serve equiprobably.

We demonstrate that non-equal prices lead to endogenous separation among consumers: buyers with very low valuations leave the market, those with intermediate valuations try to buy at the cheapest firm, while those with high valuations are ready to pay more. This sorting is caused by the fact that the price itself is not a unique factor in the purchase, and the probability of being served is what makes prosperous consumers to pay extra for lower competition.

Although the setting is symmetric, we prove that a pure strategy price equilibrium must be asymmetric if it exists, even for the large number of buyers. This means that sellers split the market into two disjoint parts such that each firm specializes in either medium-value consumers, or high-value consumers. These parts are unequal, i.e. more expensive sellers earn slightly less than cheaper ones, nevertheless, they cannot undercut the opponent because this leads to a decrease in the threshold between the two groups of consumers.

To better understand the equilibrium characteristics, we analyze a numerical solution for the special case of valuation distribution, in particular, for the class of power functions. We show that the price difference behaves non-monotonically with the growth of the number of buyers, which is driven by the different influence of the low and high prices. With demand side growth, the market becomes more efficient, which means that the probability of a seller making a transaction tends to 1. Yet it is always less than that for a market with perfect information about the buyer valuations.

The main contribution of the paper is that it clarifies how private information about consumer willingness to pay affects the market equilibrium and produces price dispersion and third-degree price discrimination. This type of incomplete information is widespread since, even if market analysts can make an accurate estimation of the demand function, the anonymity of buyers in a market with direct pricing protects them from revealing their true valuations.

It is to be noted that the incomplete information in this paper is on the consumer side, which differs from the approach with incomplete information about offers. That line has been developed in many studies, starting from (Baye and Morgan 2001), and extends the idea of different access to information about sellers in the market in the presence of an information gatekeeper. However, in our setting the market is free from any institutional limitations, and the key problem lies in the coordination failure stemming from the limited availability of the product and the uncertainty of buyer behavior.

In the next section, we briefly describe the closest related models from directed search theory and competing mechanisms theory in order to determine more accurately the exact place of this paper in the and to convince the reader that the intuition about possibility of price dispersion is ambivalent. In Section 3, we introduce the formal model, solve the consumers' second-stage subgame, and analyze the equilibrium pricing in the sellers' game to deduce that prices separate. Section 4 is devoted to a numerical analysis of the special case of a power distribution of valuations and the limited properties of the equilibrium and other market characteristics. Finally, Section 5 concludes with a discussion of the features and potential extensions.

2 Related Models

2.1 Directed search: price posting, common values. Price unification

Directed search theory was initially developed in a series of articles by Michael Peters (Peters 1984, 1991, 1997, 2000). The modification closest to our setting was introduced in (Burdett, Shi, and Wright 2001) and (Julien, Kennes, and King 2000). The baseline model presents a market with 2 buyers and 2 sellers. Each seller can produce one unit of an indivisible product, and each buyer has a one-unit demand. The value of the product is c for a seller and $u > c$ for a buyer. At the first stage, firms simultaneously and independently propose their prices $\mathbf{p} = (p_1, p_2)$. At the second stage, buyers observe these prices and choose the seller to submit their request for purchase. If two requests are sent to the same seller she randomly chooses one buyer to serve. The payoffs of the buyers and sellers who make a trade are $p - c$ and $u - p$, respectively. The authors demonstrate that the buyers' subgame has several regimes of equilibria dependent on the first stage prices, and some of them involve buyer coordination. This leads to a multiple coordinated equilibria in the sellers' game. But the non-coordinated equilibrium is unique and predicts the same prices $p_1 = p_2 = (u + c)/2$, while buyers pick sellers at random.

A market without coordination is inefficient in the sense that the expected number of deals is $3/2$, and the buyer's individual probability to be served is $3/4$. Because of the symmetry, complete information, and anonymity, the firms do not have a mechanism to redistribute the buyers more efficiently. One more interesting result (Julien, Kennes, and King 2000) is that if direct price announcement is replaced by a second price auction with a reserve price, then the optimal reserve prices should be the same, $(u + c)/2$, while the expected revenues of buyers are greater.

The extension to a larger market is developed in (Coles and Eeckhout 2003). By analogy with the 2 by 2 case, they show that in a market with n_b buyers and n_s sellers, in a unique non-coordinated equilibrium, all prices are equal and every buyer visits each seller with probability $1/n_s$. The equilibrium price p is the weighted average of u and c such that the weight of u is the probability a seller gets at least 2 buyers, and the weight of c is the probability she gets just 1. The results are in agreement with (Burdett, Shi, and Wright 2001). As in the 2 by 2 game, in equilibrium, buyers visit sellers at random, but the fact that the search can be directed disciplines prices.

The papers above demonstrate that if we take a model similar to ours, but make the private values common and symmetric, then, under the very natural assumption of a lack of coordination among buyers, we will observe the total randomization and complete anonymity of the market with unique prices for all sellers. We do not observe any price dispersion or any buyer specification of more preferred sellers.

2.2 Competing mechanisms: reserve price posting, private values. Price dispersion

The idea that buyers can choose a seller using public information about her sales mechanism is developed in the theory of competing mechanisms. It holds that sellers are able to propose a direct mechanism, for instance, second price auctions with a reserve price, and buyers decide which auction to attend.

The papers of (McAfee 1993), (Peters and Severinov 1997), (Albrechta, Gautierb, and Vroman 2012) consider large finite markets and focus on the symmetric case which is motivated by the anonymity of buyers and sellers. Using different technical assumptions, all papers deduce the main result that, in a limit equilibrium, prices equal to sellers' marginal costs rise.

But the result is inverted when we come to a finite market. The model with 2 sellers and n buyers with an auction mechanism is analyzed in (Burguet and Sákovics 1999), and the authors face technical problems which were previously avoided for the infinite market. The only difference between this setting and ours is the mechanism of sales under which sellers post not prices but reserve prices in second price auctions.

The main characteristics of equilibrium pricing in that framework are the following. Firstly, the authors demonstrate that there is no equilibrium in pure strategies in which firms set the same price. Secondly, they prove that a mixed strategy equilibrium exists but the marginal cost is never in its support. These results are in contrast to all the models above, so that the finiteness of the market is crucial. Thus, this paper provides arguments in favor of the possibility of an equilibrium price dispersion in our model with a direct price announcement.

One more related paper (Virág 2010) continues the analysis of (Peters and Severinov 1997) and (Burguet and Sákovics 1999) and clarifies the existence conditions for a large market framework. They prove that the necessary and sufficient conditions for existing a symmetric pure strategy equilibrium are weak enough, and at the limit, the equilibrium reserve price tends to the marginal cost.

This short description of the extensive related literature aims to demonstrate that we know much more about large markets with the great number of buyers and sellers and that the analysis of these markets is well developed because of the possibility to reduce, or even ignore, any strategic interaction on one side of the market. In the case of small markets, however, our knowledge about the price distribution is imperfect. The two principal models, one with common valuations and direct pricing, and other with private values and an auction, demonstrate contradictory predictions. The problem here is what happens in the equilibrium for the in-between model, with direct price announcements and private buyer valuations. An alternative way to ask this question is what exactly makes prices disperse in (Burguet and Sákovics 1999) in comparison with (Julien, Kennes, and King 2000) and (Coles and Eeckhout 2003). Is it incomplete information about buyers or the more complicated mechanism of sales? This paper sheds light on this problem and deepens our understanding of the micro-process of sorting prices and sharing the market. We argue that it is incomplete information that produces the price dispersion effect.

3 The Finite Market Model and Its Solution

3.1 Problem statement

Consider a finite market with n buyers and 2 sellers. Let $I = \{1, 2, \ldots, n\}$ be the set of buyers and $J = \{1, 2\}$ be the set of sellers. Each seller has one unit of a homogeneous indivisible good which she aims to sell for any positive price, such that an individual seller's valuation of the product is normalized to 0. Every buyer is interested in purchasing one unit of the product if the price does not exceed her maximal valuation, i.e. the reserve price. This valuation is private for the buyer, while sellers and other buyers treat it as a random variable with a continuous distribution function $F(x)$ on $[0, 1]$. Buyers' valuations are made independently.

The game is played in two stages. First, sellers simultaneously and independently announce their prices for the product: $\mathbf{p} = (p_1, p_2) \in [0, 1] \times [0, 1]$. Secondly, buyers observe prices and choose one seller to submit a purchase request to at her price. If both prices are too high for a particular buyer, she has the option not to buy at all. If a seller gets exactly one request, she immediately sells the product to this buyer for the announced price. If a seller gets several requests, she believes that all buyers are indistinguishable and choose one of them with equal probabilities. There is no any mechanism for a seller to reveal buyers' valuations, in contrast to a market with auctions, so buyers keep themselves anonymous. Every buyer maximizes her expected profit from the purchase, which is the valuation minus the price of a successful deal.

3.2 The buyers' subgame

3.2.1 Equilibrium concept

First, let us order the price vector $\mathbf{p}$ using the permutation $\sigma(\cdot)$.

$$\tilde{p} = \sigma(\mathbf{p}) : 0 \leqslant \tilde{p}_1 \leqslant \tilde{p}_2 \leqslant 1$$

Define the set of possible buyer actions as follows

$$s = \begin{cases} A: \text{send a request to seller } \tilde{1}; \\ B: \text{send a request to seller } \tilde{2}; \\ C: \text{do not send any request (leave the market).} \end{cases}$$

Each buyer's optimal strategy s^* necessarily depends on her valuation, other buyers' strategies, and the price vector $\tilde{p}$. Standard domination considerations yield:

- If $0 \leqslant v < \tilde{p}_1$, then $s^* = C$;
- If $\tilde{p}_1 \leqslant v < \tilde{p}_2$, then $s^* = A$;
- If $\tilde{p}_2 \leqslant v_i \leqslant 1$, then a buyer with high valuation may prefer strategy B. We will focus on a symmetric Bayesian-Nash equilibrium with the property of monotonicity in valuations immediately. This means that there exists a threshold $k \in [\tilde{p}_2, 1]$, the same for all buyers, such that buyers with values less than k use strategy A, while those with valuations greater or equal to k play strategy B. Note the similarity of the logic with (Burguet and Sákovics 1999).

Consider a fixed buyer with valuation v who thinks about going to a fixed seller who set price p and this buyer knows that each of the rest $n-1$ buyers in the market could come to this seller with probability x. Then her expected gain from this action is the following:

$$\textit{Expected gain} = (v-p) \times \textit{Probability of being served.} \tag{1}$$

Note, that here we use the property that the probability of being served does not depend on buyers' valuations (sellers cannot distinguish among buyers). Let us now compute the probability of being served in the situation described above.

$$\begin{aligned} &\textit{Probability of being served} = \\ &= \sum_{i=0}^{n-1} \underbrace{C_{n-1}^{i} \times x^{i} \times (1-x)^{n-1-i}}_{\text{Prob}(i \text{ that other buyers } n-1 \text{ come to the seller})} \times \\ &\times \underbrace{\frac{1}{i+1}}_{\text{Prob (buyer will be served} \mid i \text{ other buyers came to the seller)}} \end{aligned} \tag{2}$$

In the following proposition we transform and simplify this expression.

Proposition 1. *Let us introduce the function $z(x,n)$:*

$$z(x,n) = \sum_{s=0}^{n} C_n^s \times x^s \times (1-x)^{n-s} \times \frac{1}{s+1}, x \in \mathbb{R}, n \in \mathbb{N}$$

After some transformations it could be rewritten as:

$$z(x,n) = \begin{cases} \frac{1-(1-x)^{n+1}}{x(n+1)}, \textit{if } x \neq 0, \\ 1, \textit{if } x = 0. \end{cases}$$

Note that for any fixed $n \in \mathbb{N}$

$$\lim_{x \to 0} \frac{1-(1-x)^{n+1}}{x\,(n+1)} = 1.$$

To prove this proposition we use following lemmas.

Lemma 1. *Let* $f(x,n)$*:*

$$f(x,n) = \sum_{s=0}^{n} C_n^s \times \frac{1}{s+1} \times x^s \times (1-x)^{n-s}, x \in \mathbb{R}, n \in \mathbb{N}$$

Its derivative has the form

$$\frac{\partial f(x,n)}{\partial x} = \frac{1}{1-x} \times \left(\frac{1}{x} - \frac{f(x,n)}{x} - nf(x,n) \right)$$

Lemma 2. *The function*

$$z(x,n) = \begin{cases} \frac{1-(1-x)^{n+1}}{x(n+1)}, \textit{if } x \neq 0, \\ 1, \textit{if } x = 0. \end{cases}$$

is the solution of the differential equation

$$\frac{\partial f(x)}{\partial x} = \frac{1}{1-x} \times \left(\frac{1}{x} - \frac{f(x)}{x} - nf(x) \right)$$

with the boundary condition $f(0) \to 1$.

Proofs of these lemmas are in Appendix.

Recall, that we have defined function $z(x, n-1)$ as the probability for a given buyer to be served if she comes to a seller and each of the remaining $n-1$ buyers can independently come to this seller with a probability x. If we look at the function $z(x, n-1)$ after transformations, there is an interesting way to interpret it. The expression in the nominator $1-(1-x)^n$ is the probability that at least one buyer comes to the seller when each of the n buyers can come to this seller with probability x. The expression in the denominator $n \times x$ is the expected number of buyers coming to the seller when each buyer can do this independently with probability x. Note, that we start the derivation of the function $z(x, n-1)$ from the point of view of one fixed buyer (this buyer comes to the seller with probability 1 and each of the remaining buyers can do this with probability x) and now we come to the interpretation in which all (including previously fixed) buyers can come to the seller with probability x.

Hereinafter for simplicity we will write

$$z(x,n) = \frac{1-(1-x)^{n+1}}{x\,(n+1)}, n \in \mathbb{N}$$

instead of

$$z(x,n) = \begin{cases} \frac{1-(1-x)^{n+1}}{x(n+1)}, \text{if } x \neq 0, \\ 1, \text{if } x = 0; \end{cases}$$

because, as we noted above, $\lim_{x\to 0} \frac{1-(1-x)^{n+1}}{x(n+1)} = 1$.

The following lemma gives us some important properties of the function $z(x,n)$.

Lemma 3. *The function*

$$z(x,n) = \frac{1-(1-x)^{n+1}}{x\,(n+1)}, n \in \mathbb{N}$$

is strictly decreasing and strictly concave in x in $[0,1]$.

$z(0,n) = 1,\ z(1,n) = \frac{1}{n+1},\ z'(0,n) = -\frac{n}{2},\ z'(1,n) = -\frac{1}{n+1}.$

The proof is in Appendix.

3.2.2 Equilibrium in the buyers' subgame

Let us compute the expected profits from using strategies A and B of buyer $i \in I$ with value $v_i \in [\tilde{p}_2, 1]$ if other buyers apply the threshold strategy determined above.

$$\mathbb{E}[gain_i(s_i = A, s_{-i} = s^*)] = (v_i - \tilde{p}_1) \times z\,(F(k) - F(\tilde{p}_1), n-1) \quad (3)$$

$$\mathbb{E}[gain_i(s_i = B, s_{-i} = s^*)] = (v_i - \tilde{p}_2) \times z\,(1 - F(k), n-1) \quad (4)$$

Because the number of buyers in the market is fixed, hereinafter for simplicity we will use $z(x)$ instead of $z(x, n-1)$.

Buyer i will choose strategy A, if

$$\mathbb{E}[gain_i(s_i = A, s_{-i} = s^*)] > \mathbb{E}[gain_i(s_i = B, s_{-i} = s^*)].$$

She may also prefer A if

$$\mathbb{E}[gain_i(s_i = A, s_{-i} = s^*)] = \mathbb{E}[gain_i(s_i = B, s_{-i} = s^*)].$$

We represent the difference of expected profits as a function of v_i

$$\phi(v_i) = v_i \times (z\,(F(k) - F(\tilde{p}_1)) - z\,(1 - F(k))) + \\ + \tilde{p}_2 \times z\,(1 - F(k)) - \tilde{p}_1 \times z\,(F(k) - F(\tilde{p}_1))\,. \quad (5)$$

In order for buyer i to use the threshold strategy above, function $\phi(v_i)$ must either strictly decrease in v_i, or be constant in v_i in the case of $\phi(v_i) = 0$.

$$\frac{\partial \phi(v_i)}{\partial v_i} \leqslant 0 \wedge \frac{\partial \phi(v_i)}{\partial v_i} = 0 \iff \phi(v_i) = 0 \tag{6}$$

The intuition behind this is the following: with an increase of the buyer's valuation, the product becomes more attractive for the buyer with respect to its price, hence, incentives to go to a seller with a higher price to increase the probability of a successful purchase increase. After the valuation exceeds some critical level, the buyer would prefer to overpay for the product. Taking the derivative leads to

$$\frac{\partial \phi(v_i)}{\partial v_i} = z\left(F(k) - F(\tilde{p}_1)\right) - z\left(1 - F(k)\right) < 0 \iff 2F(k) > F(\tilde{p}_1) + 1, \tag{7}$$

here we use the monotonicity of function $z(x)$.

Consider separately the boundary case:

$$\frac{\partial \phi(v_i)}{\partial v_i} = z\left(F(k) - F(\tilde{p}_1)\right) - z\left(1 - F(k)\right) = 0 \iff 2F(k) = F(\tilde{p}_1) + 1, \tag{8}$$

If $2F(k) = F(\tilde{p}_1) + 1$, function $\phi(v_i)$ has the form

$$\phi(v_i) = v_i \times 0 + z(1 - F(k)) \times (\tilde{p}_2 - \tilde{p}_1) = z(1 - F(k)) \times (\tilde{p}_2 - \tilde{p}_1). \tag{9}$$

Let us claim now $\phi(v_i) = 0$, which corresponds the situation when the relative difference of the attractiveness of strategies A and B does not depend on the valuations. At the same time, the threshold strategy that we are looking for requires that some of the buyers must prefer A $(i \in I : 0 \leqslant v_i < k)$, while buyers with higher valuations $(i \in I : k \leqslant v_i \leqslant 1)$ prefer B. Therefore, the profit of both strategies should coincide. It is easy to obtain that $z(1) = \frac{1}{n}$, which provides $\tilde{p}_2 = \tilde{p}_1$.

Substituting $\tilde{p}_2 = \tilde{p}_1$ into 5 leads to

$$\phi(v_i) = (v_i - p) \times \left(z\left(F(k) - F(p)\right) - z\left(1 - F(k)\right)\right). \tag{10}$$

If one claims that there are two groups of consumers with $v \geqslant p$, those who use strategy A and those who use B, then it is necessarily needed that $2F(k) = F(p) + 1$.

Therefore, it follows from $2F(k) = F(\tilde{p}_1) + 1$ that $\tilde{p}_2 = \tilde{p}_1$. At the same time, it follows from $\tilde{p}_2 = \tilde{p}_1 = p$ that $2F(k) = F(p) + 1$.

After the discussion of the boundary case $\left(\frac{\partial \phi(v_i)}{\partial v_i} = 0\right)$, let us turn back to the interior one $\left(\frac{\partial \phi(v_i)}{\partial v_i} > 0\right)$. Given $\tilde{p}$, we determine the valuation such that the buyer with this valuation is indifferent between A and B. Equalize 5 to zero and express v_i.

$$v_i^* = \frac{\tilde{p}_2 \times z\left(1 - F(k)\right) - \tilde{p}_1 \times z\left(F(k) - F(\tilde{p}_1)\right)}{z\left(1 - F(k)\right) - z\left(F(k) - F(\tilde{p}_1)\right)} \tag{11}$$

Because of the symmetry, buyer i's optimal strategy coincides with other buyers strategies. Thus, threshold level k must be a solution of the equation

$$x = \frac{\tilde{p}_2 \times z\left(1 - F(x)\right) - \tilde{p}_1 \times z\left(F(x) - F(\tilde{p}_1)\right)}{z\left(1 - F(x)\right) - z\left(F(x) - F(\tilde{p}_1)\right)}, \tag{12}$$

if it exists in the interval $[\tilde{p}_2, 1]$. Otherwise $k = 1$.

Rewrite 12 in the form

$$\frac{x - \tilde{p}_2}{x - \tilde{p}_1} = \frac{z(F(x) - F(\tilde{p}_1))}{z(1 - F(x))} \tag{13}$$

Lemma 4. *Equation 13 has a unique solution on the interval* $[\tilde{p}_2, 1]$ *iff*

$$\tilde{p}_2 \leqslant 1 - \frac{1 - \tilde{p}_1}{1 - F(\tilde{p}_1)} \times \frac{1}{n} \times \left(1 - ((F(\tilde{p}_1))^n\right).$$

The proof is in Appendix.

Note that $\frac{k - \tilde{p}_2}{k - \tilde{p}_1} \leqslant 1$, that is why, using the monotonicity of function $z(x)$, we can conclude that $F(k) - F(\tilde{p}_1) \geqslant 1 - F(k)$, so the inequality 6 is satisfied.

The calculation of threshold k in the explicit form is a complicated computational task for a given n and other fixed combinations of parameters. Hence, we prove the following theorem describing the equilibrium buyer behavior.

Theorem 1. *Under given price vector* $\tilde{p} = (\tilde{p}_1, \tilde{p}_2)$*, there exists a Bayesian-Nash equilibrium in the buyers' subgame in which the following strategies are chosen:*

- *If* $0 \leqslant v_i < \tilde{p}_1$*, then* $s_i^* = C$*,* $i \in I$*;*
- *If* $\tilde{p}_1 \leqslant v_i < k$*, then* $s_i^* = A$*,* $i \in I$*;*
- *If* $k \leqslant v_i \leqslant 1$*, then* $s_i^* = B$*,* $i \in I$*,*

where k *is either the root of equation 13 if it exists in the interval* $[\tilde{p}_2, 1]$*, or equals* 1*. Lemma 4 presents the necessary and sufficient conditions for the existence of the solution of equation 13 in the given interval.*

We call k internal if it is the root of equation 13 in the interval $[\tilde{p}_2, 1]$. In the other case, we take k equal to 1 and call it boundary. Before we come to the sellers' subgame, let us discuss the main properties of the internal critical value k.

If the higher price $\tilde{p}_2$ becomes a little higher, then a buyer who, before the increase, was indifferent between options A and B, $v = k$, now strictly prefers option A. Changes in $\tilde{p}_2$ has no direct impact on k, so the probability of being served by seller $\tilde{2}$, $1 - F(k)$, remains the same, but the possible gain, $k - \tilde{p}_2$, becomes lower. At the same time, both the probability of being served by seller $\tilde{1}$, $F(k) - F(\tilde{p}_1)$, and the possible gain, $k - \tilde{p}_1$, are the same as before. So option A after the changes becomes relatively more attractive than option B. That is why the new critical value k will be higher than the previous one.

If the lower price $\tilde{p}_1$ becomes higher, then there are two opposite effects. On the one hand, some buyers with relatively low reserve prices, who went to seller $\tilde{1}$ before, now leave the market, so the probability of being served by seller $\tilde{1}$ for those buyers remaining in the market becomes higher (this effect pushes k up). On the other hand, the possible gain, $k - \tilde{p}_1$, becomes lower (this effect pushes $\overline{k}$ down). We cannot say which effect is stronger, but we can prove the following result.

Proposition 2. *Let k be the root of the equation*

$$\frac{x - \tilde{p}_2}{x - \tilde{p}_1} = g(x, \tilde{p}_1),$$

inside $[\tilde{p}_2, 1]$, where $g(x, \tilde{p}_1) = \frac{z(F(x) - F(\tilde{p}_1))}{z(1 - F(x))}$. Then the partial derivatives of k with respect to prices have the form

$$\frac{\partial k}{\partial \tilde{p}_1} = \frac{-g(k, \tilde{p}_1) + g'_{\tilde{p}_1}(k, \tilde{p}_1) \times (k - \tilde{p}_1)}{1 - g(k, \tilde{p}_1) - g'_x(k, \tilde{p}_1) \times (k - \tilde{p}_1)} < 1 \tag{14}$$

$$\frac{\partial k}{\partial \tilde{p}_2} = \frac{1}{1 - g(k, \tilde{p}_1) - g'_x(k, \tilde{p}_1) \times (k - \tilde{p}_1)} > 0. \tag{15}$$

The proof is in Appendix.

3.3 The sellers' competition

The buyers' optimal behavior at the second stage affects the price choice at the first one. We now start from an analysis of the sellers' best response functions. Let seller 1 observe, or perfectly predict, the posted price p_2 of seller 2. When finding the optimal reaction, seller 1 should compare her expected revenues in the two cases: when she posts a price lower than her competitor $0 \leqslant p_1 \leqslant p_2$, and when she posts a price higher than her competitor $p_2 \leqslant p_1 \leqslant 1$. Note that two different scenarios are possible. In the first case, price p_2 is such that for any $p_1 \leqslant p_2$ threshold $\overline{k}$ is an interior root, i.e. $\overline{k}$ is sensitive to price changes. In the second case, seller 1 is able to set such a price $p_1 \leqslant p_2$, that $\overline{k} = 1$ and becomes nonsensitive to ε-changes in prices. Lemma 4 states that equation 13 has an interior solution in the interval $[\tilde{p}_2, 1]$ iff $\tilde{p}_2 \leqslant 1 - \frac{1 - \tilde{p}_1}{1 - F(\tilde{p}_1)} \times \frac{1}{n} \times (1 - ((F(\tilde{p}_1))^n)$. Determine $b^*(\tilde{p}_2)$ as a minimum of the right part of this inequality under $0 \leqslant \tilde{p}_1 \leqslant \tilde{p}_2$. In particular,

$$b^*(y) = \min_{0 \leqslant x \leqslant y} \left(1 - \frac{1 - x}{1 - F(x)} \times \frac{1}{n} \times (1 - ((F(x))^n)\right). \tag{16}$$

Obviously, if $\tilde{p}_2 \leqslant b^*(\tilde{p}_2)$ then it is impossible for seller 2 to undercut her competitor as she will lose all potential buyers ($\overline{k} = 1$). Otherwise, there exists a price $\tilde{p}_1$ that no buyer will come to the seller with the highest price with probability 1. Define $A(\tilde{p}_2)$ as the set of all values $\tilde{p}_1$ such that for a given $(\tilde{p}_1, \tilde{p}_2)$ the corresponding threshold $\overline{k} = 1$ is on the boundary. Formally,

$$A(y) = \left\{ a : \left(y > 1 - \frac{1-a}{1-F(a)} \times \frac{1}{n} \times (1 - ((F(a))^n) \right) \& (0 \leqslant a \leqslant y) \right\}. \tag{17}$$

Introduce the function

$$\gamma(x) = 1 - \frac{1-x}{1-F(x)} \times \frac{1}{n} \times (1 - ((F(x))^n). \tag{18}$$

Summarizing the considerations above, we can write the best response function of seller 1 to the arbitrary price of her competitor:

$$p_1^* = \arg\max_{p_1} \begin{cases} p_2 : \begin{cases} 0 \leqslant p_2 \leqslant 1 \\ p_2 \leqslant b^*(p_2) \end{cases} & : \begin{cases} p_1 \times \left(1 - \left(F(p_1) + 1 - F(\overline{k})\right)^n\right), \ p_1 \in [0, p_2] \\ p_1 \times \left(1 - \left(F(\overline{k})\right)^n\right), \ p_1 \in [p_2, 1], \ p_2 \notin A(p_1) \\ 0, \ p_1 \in [p_2, 1], \ p_2 \in A(p_1) \end{cases} \\ p_2 : \begin{cases} 0 \leqslant p_2 \leqslant 1 \\ p_2 > b^*(p_2) \end{cases} & : \begin{cases} p_1 \times (1 - (F(p_1))^n), \ p_1 \in A(p_2) \\ p_1 \times \left(1 - \left(F(p_1) + 1 - F(\overline{k})\right)^n\right), \ p_1 \in [0, p_2], \\ \qquad p_1 \notin A(p_2) \\ p_1 \times \left(1 - \left(F(\overline{k})\right)^n\right), \ p_1 \in [p_2, 1], \ p_2 \notin A(p_1) \\ 0, \ p_1 \in [p_2, 1], \ p_2 \in A(p_1). \end{cases} \end{cases} \tag{19}$$

Because of the bulky form of the reaction function, further equilibrium analysis is complicated. The problem is that when the elimination of a competitor from the market is possible, the structure of set $A(\tilde{p}_2)$ is unclear. Though the main result on the price distribution will be proved for the general case, one can see that there is a large class of private value distributions $F(x)$ for which the problem can be simplified significantly.

Assume now that $\gamma(x)$ is an increasing function in $[0, 1]$. Then, regardless of $\tilde{p}_2$, function $b^*(\tilde{p}_2)$ reaches its minimal value in $[0, \tilde{p}_2]$ at the left point of the interval, and this minimum equals $\left(1 - \frac{1}{n}\right)$. Moreover, because of the monotonicity of $\gamma(x)$, it is easy to clarify the structure of $A(\tilde{p}_2)$. Note that $\gamma(0) = \frac{n-1}{n}$ is less than any $\tilde{p}_2 \in \left(\frac{n-1}{n}, 1\right]$. This means that $\tilde{p}_1 = 0$ belongs to $A(\tilde{p}_2)$ for any $\tilde{p}_2 \in \left(\frac{n-1}{n}, 1\right]$. If $\tilde{p}_2 \in \left[0, \frac{n-1}{n}\right]$, then $A(\tilde{p}_2)$ is empty. Now let us show that $\tilde{p}_1 = \tilde{p}_2$ is not in $A(\tilde{p}_2)$.

Lemma 5. *The element $\tilde{p}_1 = \tilde{p}_2$ does not belong to the set $A(\tilde{p}_2)$ for any $\tilde{p}_2 \in [0, 1]$, where*

$$A(\tilde{p}_2) = \left\{ a : \left(\tilde{p}_2 > 1 - \frac{1-a}{1-F(a)} \times \frac{1}{n} \times (1 - ((F(a))^n) \right) \& (0 \leqslant a \leqslant \tilde{p}_2) \right\}.$$

Because of the monotonicity and continuity of $\gamma(a)$, this also holds for the elements with $\tilde{p}_1 = \tilde{p}_2 - \varepsilon$, where $\varepsilon \to 0+$.

The proof is in Appendix.

Thus, there must be a threshold level $\tilde{a} \in (0, \tilde{p}_2)$ such that all $\tilde{p}_1$ from $[0, \tilde{a}]$ belong to $A(\tilde{p}_2)$, while all $\tilde{p}_1$ from $(\tilde{a}, \tilde{p}_2]$ do not. The exact value of $\tilde{a}$ depends on $\tilde{p}_2$.

In the case of a monotonously increasing $\gamma(x)$ in $[0, 1]$, the best response function of seller 1 has the form:

$$p_1^* = \underset{p_1}{\operatorname{argmax}} \begin{cases} p_2 \in \left[0, \frac{n-1}{n}\right] : \begin{cases} p_1 \times \left(1 - \left(F(p_1) + 1 - F(\bar{k})\right)^n\right), \; p_1 \in [0, p_2] \\ p_1 \times \left(1 - \left(F(\bar{k})\right)^n\right), \; p_1 \in [p_2, \gamma(p_2)] \\ 0, \; p_1 \in [\gamma(p_2), 1] \end{cases} \\ p_2 \in \left[\frac{n-1}{n}, 1\right] : \begin{cases} p_1 \times \left(1 - \left(F(p_1)\right)^n\right), \; p_1 \in [0, \tilde{a}(p_2)] \\ p_1 \times \left(1 - \left(F(p_1) + 1 - F(\bar{k})\right)^n\right), \; p_1 \in (\tilde{a}(p_2), p_2] \\ p_1 \times \left(1 - \left(F(\bar{k})\right)^n\right), \; p_1 \in [p_2, \gamma(p_2)] \\ 0, \; p_1 \in [\gamma(p_2), 1]\,. \end{cases} \end{cases} \tag{20}$$

A related question is to clarify how restrictive the assumption on a monotonous increasing $\gamma(x)$ is. The answer is given by the following statement.

Proposition 3. *For every convex cumulative distribution function of private consumer values ($F^{''}(x) \geqslant 0$) and an arbitrary finite n, function $\gamma(x)$ monotonously increases in $[0, 1]$. For every strictly concave function $F^{''}(x) < 0$ and finite n, function $\gamma(x)$ monotonously decreases in $[0, 1]$. In general, for $\gamma(x)$ to monotonously increase in $[0, 1]$ for every n, it is sufficient that for all $x^* \in [0, 1]$ such that $F^{''}(x^*) = 0$, and for x^* equals 0 and 1, the following condition holds*

$$1 - F(x^*) - F'(x^*)(1 - x^*) \geqslant 0 \tag{21}$$

The proof is in Appendix.

It is to be noted that the uniform distribution function of private values satisfies Proposition 3.

Before concluding the equilibrium price structure, we introduce one more required lemma.

Lemma 6. *Let k be the root of the equation*

$$\frac{x - \tilde{p}_2}{x - \tilde{p}_1} = \frac{z\left(F(x) - F(\tilde{p}_1)\right)}{z\left(1 - F(x)\right)}$$

inside $[p_h, 1]$. Then the following relation holds

$$\left.\frac{\partial k}{\partial \tilde{p}_1}\right|_{\tilde{p}_1 = \tilde{p}_2 = p} + \left.\frac{\partial k}{\partial \tilde{p}_2}\right|_{\tilde{p}_1 = \tilde{p}_2 = p} = \frac{F'(p)}{2F'(k)}.$$

The proof is in Appendix.

Let us now formulate and prove the main theorem of the paper.

Theorem 2. *For all finite $n \geqslant 2$, in the sellers' subgame, there is no pure strategy equilibrium with equal prices $p_1 = p_2 = p \in [0, 1]$.*

Proof. Assume that seller 1 observes some price $p_2 \in (0, 1)$ proposed by seller 2. For $p_1 = p_2$ be the best response of seller 1, it is necessary but not sufficient that an ε-deviation upwards and downwards from p_2 will lead to loses for seller 1. In other words, $p_1 = p_2$ must be the local maximum of the piecewise given, best response function of seller 1. The non-profitability of deviation by ε downwards is provided by

$$\left.\frac{\partial \pi_1}{\partial \tilde{p}_1}\right|_{p_1=p_2} > 0, \tag{22}$$

where

$$\pi_1 = p_1 \times (1 - (F(p_1) + 1 - F(k))^n) \, .$$

Here we apply Lemma 5: as $\tilde{p}_1 \to \tilde{p}_2$, the profit function has exactly this form.

The non-profitability of deviation by ε upwards is provided by

$$\left.\frac{\partial \pi_1}{\partial \tilde{p}_2}\right|_{p_1=p_2} < 0, \tag{23}$$

where

$$\pi_1 = p_1 \times (1 - (F(k)^n) \, .$$

Recall that $F(k) = \frac{1+F(p)}{2}$ when $p_1 = p_2 = p$, and replace this in the partial derivatives.

$$\begin{aligned}\left.\frac{\partial \pi_1}{\partial p_1}\right|_{p_1 \in [p_2-\varepsilon, p_2]} &= 1 - (F(p_1) + 1 - F(k))^n - \\ &- n \times p_1 \times (F(p_1) + 1 - F(k))^{n-1} \times \left(f(p_1) - f(k) \times \frac{\partial k}{\partial p_1}\right) = \\ = 1 - (F(p_1) + 1 - F(k))^{n-1} &\times \left(1 + F(p_1) - F(k) + n \times p_1 \times \left(f(p_1) - f(k) \times \frac{\partial k}{\partial \tilde{p}_1}\right)\right)\end{aligned} \tag{24}$$

$$\left.\frac{\partial \pi_1}{\partial \tilde{p}_1}\right|_{p_1=p_2=p} = 1 - \left(\frac{1+F(p)}{2}\right)^{n-1} \times \left(\frac{1+F(p)}{2} + n \times p \times \left(f(p) - f(k) \times \left.\frac{\partial k}{\partial \tilde{p}_1}\right|_{p_1=p_2=p}\right)\right)$$

$$\begin{aligned}&\left.\frac{\partial \pi_1}{\partial \tilde{p}_1}\right|_{p_1=p_2=p} > 0 \iff \\ &\iff \left(\frac{1+F(p)}{2}\right)^{n-1} \times \left(\frac{1+F(p)}{2} + n \times p \times \left(f(p) - f(k) \times \left.\frac{\partial k}{\partial \tilde{p}_1}\right|_{p_1=p_2=p}\right)\right) < 1\end{aligned} \tag{25}$$

$$\frac{\partial \pi_1}{\partial \tilde{p}_2} = 1 - (F(k))^n - n \times p_1 \times (F(k))^{n-1} \times f(k) \times \frac{\partial k}{\partial p_1} =$$
$$= 1 - (F(k))^{n-1} \times \left(F(k) + n \times p_1 \times f(k) \times \frac{\partial k}{\partial \tilde{p}_2} \right) \quad (26)$$

$$\left.\frac{\partial \pi_1}{\partial \tilde{p}_2}\right|_{p_1=p_2=p} = 1 - \left(\frac{1+F(p)}{2}\right)^{n-1} \times \left(\frac{1+F(p)}{2} + n \times p \times f(k) \times \left.\frac{\partial k}{\partial \tilde{p}_2}\right|_{p_1=p_2=p}\right)$$

$$\left.\frac{\partial \pi_1}{\partial \tilde{p}_2}\right|_{p_1=p_2=p} < 0 \iff$$
$$\iff \left(\frac{1+F(p)}{2}\right)^{n-1} \times \left(\frac{1+F(p)}{2} + n \times p \times f(k) \times \left.\frac{\partial k}{\partial \tilde{p}_2}\right|_{p_1=p_2=p}\right) > 1. \quad (27)$$

When conditions 25 and 27 hold simultaneously, this leads to

$$\left.\frac{\partial k}{\partial \tilde{p}_1}\right|_{p_1=p_2=p} + \left.\frac{\partial k}{\partial \tilde{p}_2}\right|_{p_1=p_2=p} > \frac{f(p)}{f(k)}. \quad (28)$$

It is easy to see that, when $p_1 = p_2$, $n \geqslant 2$, the conditions from Lemma 4 hold, which means that equation 13 has an interior solution in $[\tilde{p}_2, 1]$, i.e. partial derivatives from 28 exist.

Applying Lemma 6 yields

$$\left.\frac{\partial k}{\partial \tilde{p}_1}\right|_{p_1=p_2=p} + \left.\frac{\partial k}{\partial \tilde{p}_2}\right|_{p_1=p_2=p} = \frac{1}{2} \times \frac{f(p)}{f(k)} < \frac{f(p)}{f(k)}.$$

Therefore, for any $n \in N$, $n \geqslant 2$, there is no interior equilibrium ($p \in (0,1)$) with equal prices.

We stress that $p_1 = p_2 = 1$ and $p_1 = p_2 = 0$ are not equilibria. In the first case, a small decrease in the price is profitable since it increases the probability of sales from zero. In the second case, a price increase is profitable since it increases the gain from zero for a successful sale. □

Note that this result is in agreement with the similar theorem from Burguet and Sákovics 1999 for the case of quasi-effective mechanisms, and in particular, for the second price auction with a reserve price.

3.4 The existence problem

We have proved the absence of a symmetric pure strategy equilibrium in the buyers' game. However, the problem of the existence of an *asymmetric* equilibrium remains open. In the general case, this analysis is complicated by the unclear structure of the set A and the behavior of the function $b^*(\cdot)$, and, as a result, the potential jumps along the branches of the profit function for a seller who is able to exclude the opponent from the market (generate $k = 1$). That is why we restrict further consideration to the case of a monotonously increasing function $\gamma(x)$.

The best response function of seller 1 has the form 20. Assume that prices (p_1^*, p_2^*) form an equilibrium. It is easy to see that, in equilibrium, the threshold k must be smaller than 1, since otherwise the seller with the highest price has a profitable deviation, i.e. to the level $p_1 = p_2$, and thus obtain the expected positive profit. Therefore, if an equilibrium exists, then it must hold $\tilde{p}_2 \in [\tilde{p}_1, \gamma(\tilde{p}_1)]$ and

$$\begin{cases} \tilde{p}_1 \in [0, \tilde{p}_2], \text{if } \tilde{p}_2 \in \left[0, \frac{n-1}{n}\right], \\ \tilde{p}_1 \in [\tilde{a}(\tilde{p}_2), \tilde{p}_2], \text{if } \tilde{p}_2 \in \left[\frac{n-1}{n}, 1\right]. \end{cases}$$

This means that sellers are located on the following branches of the profit function:

$$\begin{cases} \tilde{\pi}_2 = \tilde{p}_2 \times (1 - (F(k))^n), \\ \tilde{\pi}_1 = \tilde{p}_1 \times (1 - (F(\tilde{p}_1) + 1 - F(k))^n). \end{cases}$$

If there is an equilibrium $(\tilde{p}_1^*, \tilde{p}_2^*)$ such that $\tilde{p}_2^* \leqslant \frac{n-1}{n}$, then these prices solve the following optimization problems:

$$\begin{cases} \tilde{\pi}_2 = \tilde{p}_2 \times (1 - (F(k))^n) \longrightarrow \max\limits_{\tilde{p}_2 \in [\tilde{p}_1^*, \gamma(\tilde{p}_1^*)]} \\ \tilde{\pi}_1 = \tilde{p}_1 \times (1 - (F(\tilde{p}_1) + 1 - F(k))^n) \longrightarrow \max\limits_{\tilde{p}_1 \in [0, \tilde{p}_2^*]} \end{cases} \tag{29}$$

If one seeks an equilibrium with $\tilde{p}_2^* \geqslant \frac{n-1}{n}$, the corresponding prices must be the solutions, if they exist, of the following problems:

$$\begin{cases} \tilde{\pi}_2 = \tilde{p}_2 \times (1 - (F(k))^n) \longrightarrow \max\limits_{\tilde{p}_2 \in [\tilde{p}_1^*, \gamma(\tilde{p}_1^*)]} \\ \tilde{\pi}_1 = \tilde{p}_1 \times (1 - (F(\tilde{p}_1) + 1 - F(k))^n) \longrightarrow \max\limits_{\tilde{p}_1 \in [\tilde{a}(\tilde{p}_2^*), \tilde{p}_2^*]}. \end{cases} \tag{30}$$

The solutions of the corresponding problems are necessarily the critical points of the profit functions in the given intervals. First order conditions, together with the equation for the threshold level k, provide a necessary condition for an equilibrium to exist.

Proposition 4. *The asymmetric equilibrium in the sellers' subgame with $n \geqslant 2$ buyers* *exists, only if the following system is solvable:*

$$\begin{cases} \left.\frac{\partial \tilde{\pi}_1}{\partial \tilde{p}_1}\right|_{\tilde{p}_1^*} = 0 \\ \left.\frac{\partial \tilde{\pi}_2}{\partial \tilde{p}_2}\right|_{\tilde{p}_2^*} = 0 \\ \frac{k-\tilde{p}_2^*}{k-\tilde{p}_1^*} = \frac{z(F(k)-F(\tilde{p}_1))}{z(1-F(k))} \\ \tilde{p}_2^* \in (\tilde{p}_1^*, \gamma(\tilde{p}_1^*)] \\ \tilde{p}_1^* \in [\tilde{a}(\tilde{p}_2^*), \tilde{p}_2^*), \textit{ if } \tilde{p}_2^* \geqslant \frac{n-1}{n} \\ \tilde{p}_1^* \in [0, \tilde{p}_2^*), \textit{ if } \tilde{p}_2^* \leqslant \frac{n-1}{n} \end{cases} \tag{31}$$

where

$$\frac{\partial \tilde{\pi}_1}{\partial \tilde{p}_1} = 1 - (F(\tilde{p}_1) + 1 - F(k)))^{n-1} \times$$
$$\times \left(1 + F(\tilde{p}_1) - F(k) + n \times \tilde{p}_1 \times \left(f(\tilde{p}_1) - f(k) \times \frac{\partial k}{\partial \tilde{p}_1}\right)\right)$$

$$\frac{\partial \tilde{\pi}_2}{\partial \tilde{p}_2} = 1 - (F(k))^{n-1} \times \left(F(k) + n \times \tilde{p}_2 \times f(k) \times \frac{\partial k}{\partial \tilde{p}_2}\right).$$

Sufficient conditions must guarantee that the critical point obtained from system 31 provides the maximum of the best response function on the given branches and that there are no profitable deviations to other branches of the profit functions.

An analytical solution or a significant simplification of system 31 are not possible. Thus, the only way to check the solvability is to obtain numerical solutions for given cumulative distribution function of private valuations $F(\cdot)$ and different n.

4 The Special Case $F(x) = x^a$

We restrict ourselves to the family of power functions

$$F(x, a) = \begin{cases} 1, x \geqslant 1 \\ x^a, 0 < x < 1 \\ 0, x \leqslant 0 \end{cases}$$

with $a \geqslant 1$. Note that the uniform distribution of private valuations is a special case ($a = 1$). When a grows, the curve becomes more concave, and at the limit, as $a \to \infty$, it converges to the Dirac delta function with the whole mass at point 1. Such $F(x)$ are concave functions in $[0, 1]$ if $a > 1$ and, as Proposition 3 states, function $\gamma(x)$ is increasing.

For this class of functions, we have shown that, for all pairs (a, n) under consideration, system 31 has a solution. Below we present some properties of these solutions dependent on the parameters of the market.

In the numerical analysis, we treat all the values of the parameters in the intervals $2 \leqslant n \leqslant 10000$ and $1 \leqslant a \leqslant 60$. Figure 1 shows the results for $a \in \{1, 2, 4, 10, 25\}$ and $2 \leqslant n \leqslant 100$ for clarity, but they represents all main tendencies completely.

Figure 1 demonstrates the equilibrium dynamics of the highest price as the market grows.

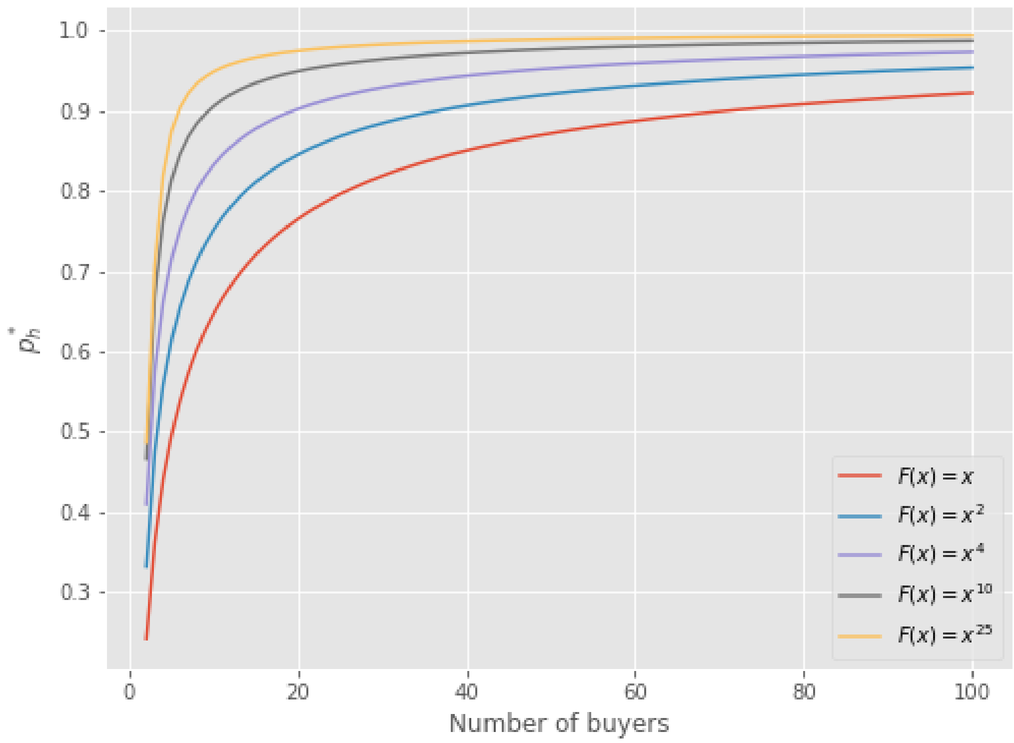

Figure 1: The highest price p_h as a function of n for different values a

First, the highest price increases in n. This is obvious since the greater the number of buyers in the market, the higher the probability that at least one buyer has a valuation greater than the given threshold level. This implies that a seller can raise the price which leads to a rise in k. Immediately, this explains why the behavior of k is similar to $\tilde{p}_2^*$.

Secondly, in equilibrium, the highest price increases with a for a given n. This is due to the increasing the probability of attracting at least one buyer with a valuation greater than the fixed level.

Thirdly, under fixed a, the sequence of $\tilde{p}_2^*(n)$ monotonously increases and tends to 1 as $n \to \infty$. All the described properties also hold for the lowest equilibrium price $(\tilde{p}_1^*)$, so we omit its plot.

The most interesting behavior is demonstrated by the equilibrium price range, which is one of the common measures of price dispersion. The range is given in Figure 2.

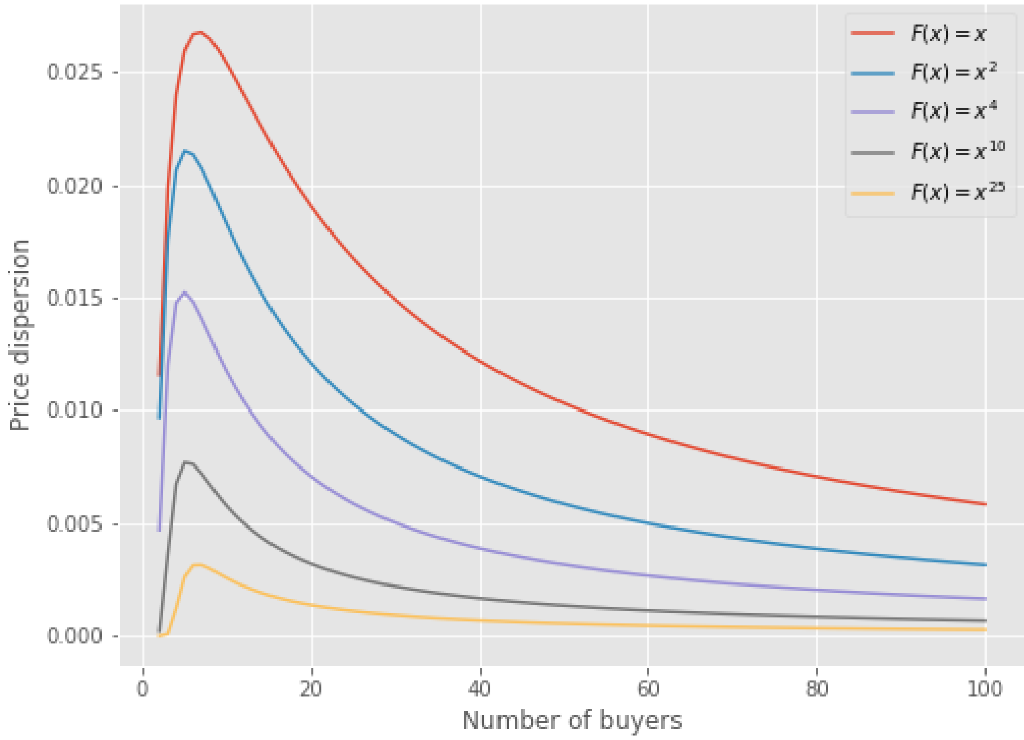

Figure 2: The equilibrium price range $\tilde{p}_2 - \tilde{p}_1$ as a function of n for different values a

Surprisingly, the form of the curve is bell-shaped. This is due to the difference in the growth rate for low and high equilibrium prices. For a relatively small number of buyers in the market, adding one more buyer significantly increases the probability of a more expensive seller making a sale, which leads to the rapid growth of the higher price. At the same time, the cheapest seller has already attracted enough buyers, such that a small expansion in the potential demand does not shift her price as quickly. When the number of buyers is large, this effect disappears and both prices increase equally slowly.

One more market characteristics concerns the efficiency of matching among buyers and sellers. This is the expected number of trades, or, equivalently, the average expected number of trades per seller. Simulations show that the average expected number of trades tends to 1 with market growth. This is in line with the model with common valuations equal to 1 (see Wright et al. 2021), where this parameter is given by $1 - 1/2^n \to 1$ as $n \to \infty$. Figure 3 illustrates the dynamics of the average expected number of trades per seller.

One can see that a market with common valuations is more efficient, which seems to be rather intuitive. Sellers lose some potential buyers when they are not fully informed

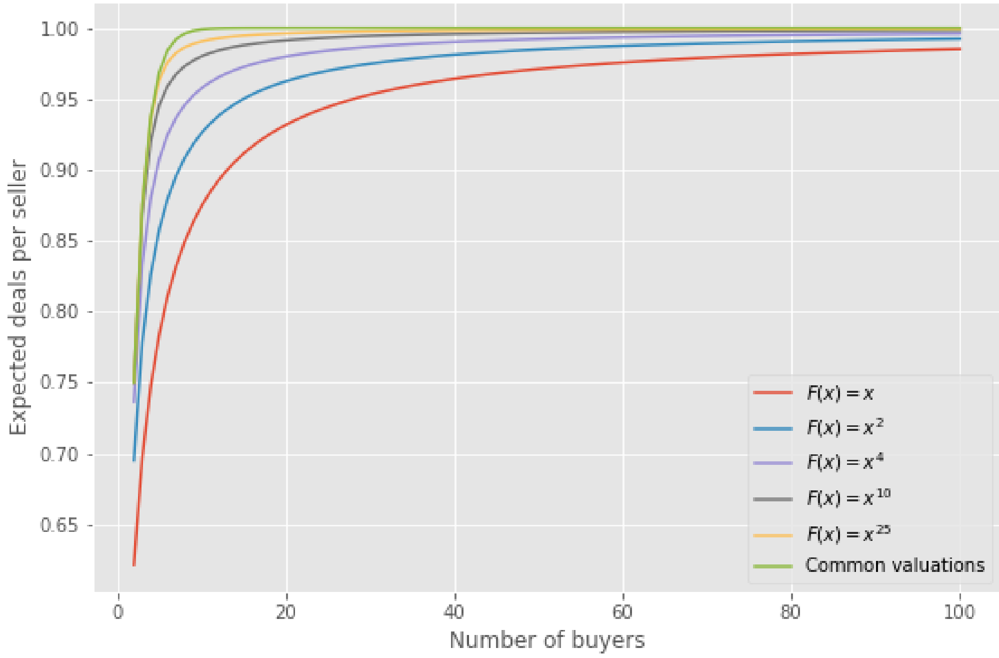

Figure 3: The equilibrium number of expected trades per seller as a function of n for different values a

about the actual buyers' willingness to pay.

5 Conclusion

In this paper, we clarify a mechanism that is able to produce a price dispersion in a finite market. Uncertainty about buyers' valuations inherently generates the expectation of heterogeneity among buyers. We have shown that the expectation of this difference in valuations forces sellers to split the market and choose a buyer specialization. The seller with a lower price attracts a larger pool of buyers, so she increases the probability of a trade, but with a lower margin. The seller with a higher price trades with a smaller but richer market segment, which makes her successful trades rarer but more profitable.

The paper fills the gap in our understanding of matching in a finite market with the simplest mechanism of price announcement. The lack of a pure strategy symmetric equilibrium means the presence of price dispersion, nevertheless, the pattern we obtained differs from a mixed strategy "symmetric" dispersion. It is explicitly non-equal pure prices. This is valuable since it explains not only the persistence price dispersion in some markets, but also the permanence of these prices, without frequent updates. This is in contrast to the previous search market models summarized in (Baye, Morgan, Scholten, et al. 2006) and the auction model by (Burguet and Sákovics 1999) where, in different frameworks, the price dispersion means the price realization from some mixed-strategy equilibrium distribution.

Technically, the key finding is the function of the probability of being served $z(x, n-1)$. Although the analysis is hard, we believe that it can be extended to larger markets with more than two sellers. The main result should remain the same, while more nontrivial features depending on the market structure may arise.

References

[1] James Albrechta, Pieter Gautierb, and Susan Vroman. "A note on Peters and Severinov, "Competition among sellers who offer auctions instead of prices"". In: *Journal of Economic Theory* 147 (2012), pp. 389–392.

[2] Michael R Baye and John Morgan. "Information gatekeepers on the internet and the competitiveness of homogeneous product markets". In: *American Economic Review* 91.3 (2001), pp. 454–474.

[3] Michael R Baye, John Morgan, Patrick Scholten, et al. "Information, search, and price dispersion". In: *Handbook on economics and information systems* 1 (2006), pp. 323–375.

[4] Kenneth Burdett, Shouyong Shi, and Randall Wright. "Pricing and matching with frictions". In: *Journal of Political Economy* 109.5 (2001), pp. 1060–1085.

[5] Roberto Burguet and József Sákovics. "Imperfect competition in auction designs". In: *International Economic Review* 40.1 (1999), pp. 231–247.

[6] Melvyn G Coles and Jan Eeckhout. "Indeterminacy and directed search". In: *Journal of Economic Theory* 111.2 (2003), pp. 265–276.

[7] Beno^ıt Julien, John Kennes, and Ian King. "Bidding for labor". In: *Review of Economic Dynamics* 3.4 (2000), pp. 619–649.

[8] R Preston McAfee. "Mechanism design by competing sellers". In: *Econometrica: Journal of the econometric society* (1993), pp. 1281–1312.

[9] Michael Peters. "Bertrand equilibrium with capacity constraints and restricted mobility". In: *Econometrica: Journal of the Econometric Society* (1984), pp. 1117–1127.

[10] Michael Peters. "Ex ante price offers in matching games non-steady states". In: *Econometrica: Journal of the Econometric Society* (1991), pp. 1425–1454.

[11] Michael Peters. "On the equivalence of Walrasian and non-Walrasian equilibria in contract markets: the case of complete contracts". In: *The Review of Economic Studies* 64.2 (1997), pp. 241–264.

[12] Michael Peters. “Limits of exact equilibria for capacity constrained sellers with costly search”. In: *Journal of Economic Theory* 95.2 (2000), pp. 139–168.

[13] Michael Peters and Sergei Severinov. “Competition among sellers who offer auctions instead of prices”. In: *Journal of Economic Theory* 75 (1997), pp. 141–179.

[14] Gábor Virág. “Competing auctions: finite markets and convergence”. In: *Theoretical Economics* 5.2 (2010), pp. 241–274.

[15] Randall Wright et al. “Directed Search and Competitive Search Equilibrium: A Guided Tour”. In: *Journal of Economic Literature* 59.1 (Mar. 2021), pp. 90–148.

Appendix

Proof of Lemma 1.

$$\frac{\partial f(x,n)}{\partial x} = \sum_{s=0}^{n} C_n^s \times \frac{1}{s+1} \times \left(s x^{s-1}(1-x)^{n-s} - (n-s)x^s(1-x)^{n-s-1} \right) =$$

$$= \frac{1}{x} \times \sum_{s=0}^{n} C_n^s \times \frac{s}{s+1} \times x^s(1-x)^{n-s} -$$

$$- \frac{1}{1-x} \times n \times \sum_{s=0}^{n} C_n^s \times \frac{1}{s+1} \times x^s \times (1-x)^{n-s} +$$

$$+ \frac{1}{1-x} \times \sum_{s=0}^{n} C_n^s \times \frac{s}{s+1} \times x^s(1-x)^{n-s} =$$

$$= \frac{1}{x(1-x)} \times (1 - f(x,n)) - \frac{1}{1-x} \times n \times f(x,n) = \frac{1}{1-x} \times \left(\frac{1}{x} - \frac{f(x,n)}{x} - n f(x,n) \right)$$

□

Proof of Lemma 2.

$$\begin{cases} f'(x) = \frac{1}{1-x} \times \left(\frac{1}{x} - \frac{f(x)}{x} - n \times f(x) \right) \\ f(0) \to 1 \end{cases}$$

$$\frac{df(x)}{dx} + \frac{f(x) \times (nx+1)}{x \times (1-x)} = \frac{1}{x \times (1-x)}$$

$$\frac{x}{(1-x)^{n+1}} \times \frac{df(x)}{dx} + \frac{f(x) \times (nx+1)}{(1-x)^{n+2}} = \frac{1}{(1-x)^{n+2}}$$

$$\frac{x}{(1-x)^{n+1}} \times \frac{df(x)}{dx} + \frac{d\left(\frac{x}{(1-x)^{n+1}} \right)}{dx} \times f(x) = \frac{1}{(1-x)^{n+2}}$$

$$\frac{d\left(\frac{x}{(1-x)^{n+1}} \times f(x) \right)}{dx} = \frac{1}{(1-x)^{n+2}}$$

$$\int \frac{d\left(\frac{x}{(1-x)^{n+1}} \times f(x) \right)}{dx} = \int \frac{1}{(1-x)^{n+2}}$$

$$\frac{x \times f(x)}{(1-x)^{n+1}} = \frac{1}{(n+1)(1-x)^{n+1}} + const$$

$$f(0) = 1 \to const = -\frac{1}{n+1}$$

$$f(x) = \frac{1 - (1-x)^{n+1}}{x \times (n+1)}$$

□

Proof of Lemma 3.

$$z(x,n) = \frac{1-(1-x)^{n+1}}{x \times (n+1)}$$

$$\frac{\partial z(x,n)}{\partial x} = \frac{nx(1-x)^n + (1-x)^n - 1}{(n+1) \times x^2}$$

The denominator of this expression is non-negative, the numerator is a decreasing function in x: $\frac{\partial (nx(1-x)^n + (1-x)^n - 1)}{\partial x} = -n(n+1)x(1-x)^{n-1} \leqslant 0$, which tends to the maximal value of the numerator in $[0,1]$ equal to $n \times 0 \times (1-0)^n + (1-0)^n - 1 = 0$. For all $x \in (0,1]$ the numerator is less than 0 and $\frac{\partial z(x,n)}{\partial x} < 0$.

The numerator is a decreasing function in x, the denominator is an increasing function in x. Therefore, $\frac{\partial z(x,n)}{\partial x}$ is a decreasing function in x. Thus, $\frac{\partial^2 z(x,n)}{\partial x^2} < 0$.

□

Proof of Lemma 4. Consider the functions $w(x) = \frac{x-\tilde{p}_2}{x-\tilde{p}_1}$ and $g(x) = \frac{z(F(x)-F(\tilde{p}_1))}{z(1-F(x))}$. Function $w(x)$ is continuous and increases monotonically on $x \in [\tilde{p}_2, 1]$. Function $g(x)$ is continuous and decreases monotonically on $x \in [\tilde{p}_2, 1]$, since its numerator decreases in x, while denominator increases in x on $[\tilde{p}_2, 1]$ (see Lemma 3).

$$w(\tilde{p}_2) = 0$$

$$w(1) = \frac{1-\tilde{p}_2}{1-\tilde{p}_1}$$

$$g(\tilde{p}_2) = \frac{z(F(\tilde{p}_2) - F(\tilde{p}_1))}{z(1-F(\tilde{p}_2))} > 0 = w(\tilde{p}_2)$$

$$g(1) = \frac{z(1-F(\tilde{p}_1))}{z(0)} = \frac{1-(F(\tilde{p}_1))^n}{n \times (1-F(\tilde{p}_1))}$$

For functions $w(x)$ and $g(x)$ to intersect on $[\tilde{p}_2, 1]$, it is necessary and sufficient that $w(1) \geqslant g(1)$. After minor simplifications this leads to the condition

$$\tilde{p}_2 \leqslant 1 - \frac{1-\tilde{p}_1}{1-F(\tilde{p}_1)} \times \frac{1}{n} \times (1 - ((F(\tilde{p}_1))^n)$$

□

Proof of Proposition 2. Let k be the root of the equation

$$\frac{x-\tilde{p}_2}{x-\tilde{p}_1} = g(x, \tilde{p}_1),$$

inside $[\tilde{p}_2, 1]$, where $g(x, \tilde{p}_1) = \frac{z(F(x)-F(\tilde{p}_1))}{z(1-F(x))}$.

Rewrite this equation in the form

$$k(\tilde{p}_1, \tilde{p}_2) - \tilde{p}_2 - k(\tilde{p}_1, \tilde{p}_2) \times g(k(\tilde{p}_1, \tilde{p}_2), \tilde{p}_1) + \tilde{p}_1 \times g(k(\tilde{p}_1, \tilde{p}_2), \tilde{p}_1) = 0$$

Taking derivatives with respect to prices, we have

$$\frac{\partial k}{\partial \tilde{p}_1} - \frac{\partial k}{\partial \tilde{p}_1} \times g(k,\tilde{p}_1) - k \times \left(g'_x(k,\tilde{p}_1) \times \frac{\partial k}{\partial p_1} + g'_{\tilde{p}_1}(k,\tilde{p}_1) \right) + \\ + g(k,\tilde{p}_1) + \tilde{p}_1 \times \left(g'_x(k,\tilde{p}_1) \times \frac{\partial k}{\partial \tilde{p}_1} + g'_{\tilde{p}_1}(k,\tilde{p}_1) \right) = 0$$

$$\frac{\partial k}{\partial \tilde{p}_1} = \frac{-g(k,\tilde{p}_1) + g'_{\tilde{p}_1}(k,\tilde{p}_1) \times (k - \tilde{p}_1)}{1 - g(k,\tilde{p}_1) - g'_x(k,\tilde{p}_1) \times (k - \tilde{p}_1)}$$

Note, that $g(k,\tilde{p}_1) \leqslant 1$, $g'_x(x,\tilde{p}_1) < 0$, $g'_{\tilde{p}_1}(x,\tilde{p}_1) > 0$ for x on $[0,1]$.

$$\frac{-g(k,\tilde{p}_1) + g'_{\tilde{p}_1}(k,\tilde{p}_1) \times (k - \tilde{p}_1)}{1 - g(k,\tilde{p}_1) - g'_x(k,\tilde{p}_1) \times (k - \tilde{p}_1)} \vee 1$$

$$-g(k,\tilde{p}_1) + g'_{\tilde{p}_1}(k,\tilde{p}_1) \times (k - \tilde{p}_1) \vee 1 - g(k,\tilde{p}_1) - g'_x(k,\tilde{p}_1) \times (k - \tilde{p}_1)$$

$$\left(g'_{\tilde{p}_1}(k,\tilde{p}_1) + g'_x(k,\tilde{p}_1) \right) \times (k - \tilde{p}_1) \vee 1$$

If we look at the structure of function g, we find that $|g'_{\tilde{p}_1}(x,\tilde{p}_1)| < |g'_x(x,\tilde{p}_1)|$ for all x in $[0,1]$. That is why the left side of the expression above is negative and less than 1.

$$\frac{\partial k}{\partial \tilde{p}_2} - 1 - \frac{\partial k}{\partial \tilde{p}_2} \times g(k,\tilde{p}_1) - k \times \left(g'_x(k,\tilde{p}_1) \times \frac{\partial k}{\partial p_2} \right) + \\ + \tilde{p}_1 \times \left(g'_x(k,\tilde{p}_1) \times \frac{\partial k}{\partial \tilde{p}_2} \right) = 0$$

$$\frac{\partial k}{\partial \tilde{p}_2} = \frac{1}{1 - g(k,\tilde{p}_1) - g'_x(k,\tilde{p}_1) \times (k - \tilde{p}_1)}$$

□

Proof of Lemma 5.

$$p > 1 - \frac{1-p}{1-F(p)} \times \frac{1}{n} \times (1 - ((F(p))^n)$$

$$(1-p) \times \left(1 - \frac{1}{1-F(p)} \times \frac{1}{n} \times (1 - ((F(p))^n) \right) < 0$$

$$1 < \frac{1}{1-F(p)} \times \frac{1}{n} \times (1 - ((F(p))^n)$$

$$(1 - F(p)) \times n < 1 - ((F(p))^n$$

The last inequality is wrong $\forall n \geqslant 2$.

□

Proof of Proposition 3.

$$\gamma(x) = 1 - \frac{1-x}{1-F(x)} \times \frac{1}{n} \times (1 - ((F(x))^n),$$

where $x \in [0,1]$.

$$\gamma'(x) = \frac{1}{n} \times \left[\frac{1-x}{1-F(x)} \times n \times (F(x))^{n-1} \times f(x) - (1-(F(x))^n) \times \frac{F(x) - 1 + f(x) \times (1-x)}{(1-F(x))^2} \right]$$

It is sufficient for $\gamma'(x)$ to be positive that $F(x) - 1 + f(x) \times (1-x) \leqslant 0 \; \forall x \in [0,1]$

$$\frac{1-F(x)}{1-x} \geqslant f(x)$$

From the mean value theorem we have

$$f(c) \geqslant f(x),$$

where $c \in [x, 1]$.

Obviously, the last inequality is true for $f' \geqslant 0$ and is false for $f' < 0$.

□

Proof of Lemma 6. Using results from proposition 2, we have

$$\frac{\partial k}{\partial \tilde{p}_1} + \frac{\partial k}{\partial \tilde{p}_2} = \frac{1 - g(k, \tilde{p}_1) + g'_{\tilde{p}_1}(k, \tilde{p}_1) \times (k - \tilde{p}_1)}{1 - g(k, \tilde{p}_1) - g'_x(k, \tilde{p}_1) \times (k - \tilde{p}_1)},$$

where $g(x, \tilde{p}_1) = \frac{z(F(x) - F(\tilde{p}_1))}{z(1-F(x))}$.

If $\tilde{p}_1 = \tilde{p}_2 = p$, then $g(k, \tilde{p}_1) = 1$. So

$$\left.\frac{\partial k}{\partial \tilde{p}_1}\right|_{\tilde{p}_1 = \tilde{p}_2 = p} + \left.\frac{\partial k}{\partial \tilde{p}_2}\right|_{\tilde{p}_1 = \tilde{p}_2 = p} = -\left.\frac{g'_{\tilde{p}_1}(k, \tilde{p}_1)}{g'_x(k, \tilde{p}_1)}\right|_{\tilde{p}_1 = \tilde{p}_2 = p}$$

Taking partial derivatives of g and using the fact that, under $\tilde{p}_1 = \tilde{p}_2 = p$, $F(x) - F(\tilde{p}_1) = 1 - F(x)$, we can observe the needed result.

□

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