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Formal Expansions in Stochastic Model for Wave Turbulence 1: Kinetic Limit

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Abstract: We consider the damped/driven (modified) cubic NLS equation on a large torus with a properly scaled forcing and dissipation, and decompose its solutions to formal series in the amplitude. We study the second order truncation of this series and prove that when the amplitude goes to zero and the torus' size goes to infinity the energy spectrum of the truncated solutions becomes close to a solution of the damped/driven wave kinetic equation. Next we discuss higher order truncations of the series.

Contents

1. Introduction

1.1. The setting. The wave turbulence (WT) was developed in 1960's as a heuristic tool to study small-amplitude oscillations in nonlinear Hamiltonian PDEs and Hamiltonian systems on lattices. We start with recalling basic concepts of the theory in application to the cubic non-linear Schrödinger equation (NLS).

Classical setting. Consider the cubic NLS equation

$$\frac{\partial}{\partial t}u + i\Delta u - i\lambda\,|u|^2 u = 0, \qquad x \in \mathbb{T}^d_L = \mathbb{R}^d/(L\mathbb{Z}^d), \tag{1.1}$$

where $\Delta = (2\pi)^{-2}\sum_{j=1}^d(\partial^2/\partial x_j^2)$, $d \geq 2$, $L \geq 1$ and $0 < \lambda \leq 1$. Denote by H the space $L_2(\mathbb{T}^d_L;\mathbb{C})$, given the L_2-norm with respect to the normalised Lebesgue measure:

$$\|u\|^2 = \|u\|^2_{L_2(\mathbb{T}^d_L)} = \langle u, u\rangle, \quad \langle u, v\rangle = L^{-d}\Re\int_{\mathbb{T}^d_L} u\bar{v}\,dx.$$

The NLS equation is a hamiltonian system in H with two integrals of motion—the Hamiltonian and $\|u\|^2$. The equation with the Hamiltonian $\lambda\|u\|^4$ is $\frac{\partial}{\partial t}u - 2i\lambda\|u\|^2 u = 0$ and its flow commutes with that of NLS. We modify the NLS equation by subtracting $\lambda\|u\|^4$ from its Hamiltonian, thus arriving at the equation

$$\frac{\partial}{\partial t}u + i\Delta u - i\lambda\left(|u|^2 - 2\|u\|^2\right)u = 0, \qquad x \in \mathbb{T}^d_L. \tag{1.2}$$

This modification is used by mathematicians, working with hamiltonian PDEs, since it keeps the main features of the original equation, reducing some non-crucial technicalities. It is also used by physicists, studying WT; e.g. see [24], pp. 89–90.[1] Below we work with eq. (1.2) and write its solutions as functions $u(t,x) \in \mathbb{C}$ or as curves $u(t) \in H$.

The objective of WT is to study solutions of (1.1) and (1.2) when

$$\lambda \to 0 \text{ and } L \to \infty \tag{1.3}$$

on large time intervals.

We will write the Fourier series for an $u(x)$ as

$$u(x) = L^{-d/2}\sum\nolimits_{s\in\mathbb{Z}^d_L} v_s e^{2\pi i s\cdot x}, \qquad \mathbb{Z}^d_L = L^{-1}\mathbb{Z}^d, \tag{1.4}$$

where the vector of Fourier coefficients $v = \{v_s, s \in \mathbb{Z}^d_L\}$ is the Fourier transform of $u(x)$:

$$v = \hat{u} = \mathcal{F}(u), \qquad v_s = \hat{u}(s) = L^{-d/2}\int_{\mathbb{T}^d_L} u(x)e^{-2\pi i s\cdot x}\,dx. \tag{1.5}$$

Given a vector $v = \{v_s, s \in \mathbb{Z}^d_L\}$ we will regard the sum in (1.4) as its inverse Fourier transform $(\mathcal{F}^{-1}v)(x)$, which we will also write as $(\mathcal{F}^{-1}v_s)(x)$.[2] I.e.,

$$u(x) = (\mathcal{F}^{-1}v_s)(x) = (\mathcal{F}^{-1}v)(x).$$

Then $\|u\|^2 = \sum\!\!\!\!\!\!\bigcirc_s |v_s|^2$, where for a complex sequence $(w_s, s \in \mathbb{Z}^d_L)$ we denote

$$\sum\!\!\!\!\!\!\bigcirc_{s\in\mathbb{Z}^d_L} w_s = L^{-d}\sum\nolimits_{s\in\mathbb{Z}^d_L} w_s$$

[1] Note in addition that if $u(t,x)$ satisfies eq. (1.2), then $u' = e^{2it\lambda\|u\|^2}u$ is a solution of (1.1).

[2] The symmetric form of the Fourier transform which we use—with the same scaling factor $L^{-d/2}$ for the direct and the inverse transformations—is convenient for the heavy calculation below in the paper.

(this equals to the integral over $\mathbb{R}^d$ of w_s, extended to a function, constant on the cells of the mesh in $\mathbb{R}^d$ of size L^{-1}). By h we will denote the Hilbert space $h = L_2(\mathbb{Z}_L^d; \mathbb{C})$, given the norm

$$\|w\|^2 = \sum |w_s|^2.$$

Abusing a bit notation we denote by the same symbol the norms in the spaces H and h. This is justified by the fact that the Fourier transform (1.5) defines an isometry $\mathcal{F} : H \to h$.

Equations (1.1), (1.2) and other NLS equations on the torus $\mathbb{T}_L^d$ with fixed L are intensively studied by mathematicians, e.g. see the book [2] and references in it. The limit $\lambda \to 0$ with L fixed was rigorously treated in a number of publications, e.g. see [13]. But there are just a few mathematical works, addressing the limit (1.3). In the paper [11] $d = 2$ and the limit (1.3) is taken in a specific regime, when $L \to \infty$ much slower than λ^{-1}. The elegant description of the limit, obtained there, is far from the prediction of WT, and rather should be regarded as a kind of averaging. In recent paper [4] the authors study eq. (1.1) with random initial data $u(0, x) = u_0(x)$ such that the phases $\{\arg v_{0s}, s \in \mathbb{Z}_L^d\}$ of components of the vector $v_0 = \mathcal{F}(u_0)$ are independent uniformly distributed random variables. In the notation of our work they prove that under the limit (1.3), if L goes to infinity slower than λ^{-1} but not too slow, then for the values of time of order $\lambda^{-1}L^{-\delta}$, $\delta > 0$, the energy spectrum $n_s(\tau)$ approximately satisfies the WKE, linearised on $u_0(x)$ and scaled by the factor λ. The authors of [23] start with eq. (1.1), replace there the space-domain $\mathbb{T}_L^d$ by the discrete torus $\mathbb{Z}^d/(L\mathbb{Z}^d)$, modify the discrete Laplacian on $\mathbb{Z}^d/(L\mathbb{Z}^d)$ to a suitable operator, diagonal in the Fourier basis, and study the obtained equation, assuming that the distribution of the initial data $u(0, x)$ is given by the Gibbs measure of the equation. See [10] for a related result.

From other hand, there are plenty of physical works on equations (1.1) and (1.2) under the limit (1.3); many references may be found in [24,25,27]. These papers contain some different (but consistent) approaches to the limit. Non of them was ever rigorously justified, despite the strong interest in physical and mathematical communities to the questions, addressed by these works.

Zakharov–L'vov setting. When studying eq. (1.2), members of the WT community talk about "pumping the energy to low modes and dissipating it in high modes". To make this rigorous, Zakharov–L'vov [26] (also see [3], Section 1.2) suggested to consider the NLS equation, dumped by a (hyper) viscosity and driven by a random force:

$$\frac{\partial}{\partial t}u + i\Delta u - i\lambda(|u|^2 - 2\|u\|^2)u = -\nu\,\mathfrak{A}(u) + \sqrt{\nu}\,\frac{\partial}{\partial t}\eta^\omega(t, x),$$
$$\eta^\omega(t, x) = \mathcal{F}^{-1}\big(b(s)\beta_s^\omega(t)\big)(x) = L^{-d/2}\sum\nolimits_{s\in\mathbb{Z}_L^d} b(s)\beta_s^\omega(t)e^{2\pi i s\cdot x}. \tag{1.6}$$

Here $0 < \nu \le 1/2$, $\{\beta_s(t), s \in \mathbb{Z}_L^d\}$ are standard independent complex Wiener processes,[3] b is a positive Schwartz function on $\mathbb{R}^d \supset \mathbb{Z}_L^d$ and $\mathfrak{A}$ is the dissipative linear operator

$$\mathfrak{A}(u(x)) = L^{-d/2}\sum\nolimits_{s\in\mathbb{Z}_L^d} \gamma_s v_s e^{2\pi i s\cdot x}, \qquad v = \mathcal{F}(u), \quad \gamma_s = \gamma^0(|s|^2),$$

[3] i.e. $\beta_s = \beta_s^1 + i\beta_s^2$, where $\{\beta_s^j, s \in \mathbb{Z}_L^d, j = 1, 2\}$ are standard independent real Wiener processes.

where $\gamma^0(y)$ is a smooth increasing function of $y \in \mathbb{R}$ that has at most a polynomial growth at infinity together with its derivatives of all orders, such that $\gamma^0 \geq 1$ and

$$c(1+y)^{r_*} \leq \gamma^0(y), \quad |\partial^k \gamma^0(y)| \leq C(1+y)^{r_*-k} \quad \text{for } 0 \leq k \leq 3, \ \forall\, y \geq 0. \tag{1.7}$$

The exponent $r_* \geq 0$ and c, C are positive constants.[4]

It is convenient to pass to the slow time $\tau = \nu t$ and re-write eq. (1.6) as

$$\begin{aligned} &\dot{u} + i\nu^{-1}\Delta u + \mathfrak{A}(u) = i\rho\left(|u|^2 - 2\|u\|^2\right)u + \dot{\eta}^\omega(\tau, x), \\ &\eta^\omega(\tau, x) = L^{-d/2}\sum\nolimits_{s\in\mathbb{Z}^d_L} b(s)\beta^\omega_s(\tau)e^{2\pi i s\cdot x}, \end{aligned} \tag{1.8}$$

where $\rho = \lambda\nu^{-1}$. Here and below the upper-dot stands for $\partial/\partial\tau$ and $\{\beta_s(\tau)\}$ is another set of standard independent complex Wiener processes.

Solutions $u(t)$ and $u(\tau)$ are random processes in the space H. If $r_* \gg 1$, then equations (1.6) and (1.8) are well posed. In the context of equation (1.8), the objective of WT is to study its solutions when

$$\nu \to 0, \quad L \to \infty. \tag{1.9}$$

Below we show that the main characteristics of the solutions have non-trivial behaviour under the limit above if $\rho \sim \nu^{-1/2}$. We will choose

$$\rho = \nu^{-1/2}\varepsilon^{1/2}, \tag{1.10}$$

and will examine the limiting characteristics of the solutions when ε is a fixed small positive constant ε. We will show that $\sqrt{\varepsilon}$ measures the properly scaled amplitude of the solutions, so indeed it should be small for the methodology of WT to apply. The parameter $\lambda = \nu\rho = \nu^{1/2}\varepsilon^{1/2}$ will be used only in order to discuss the obtained results in the original scaling (1.6). Note that as $\lambda \sim \sqrt{\nu}$, then we study eq. (1.6)=(1.8) in the regime when it is a perturbation of the original NLS equation (1.2) by dissipative and stochastic terms much smaller then the terms of (1.2). We will examine solutions of (1.8) on the time-scale $\tau \sim 1$. For the original fast time t in eq. (1.6) it corresponds to the scale $t \sim \nu^{-1} \sim \lambda^{-2}$ =*(size of the nonlinearity)*$^{-2}$. This is exactly the time-scale usually considered in WT, see in [23,24,27].

Applying Ito's formula to a solution u of (1.8) and denoting $B = \sum_s b(s)^2$ we arrive at the balance of energy relation:

$$\mathbb{E}\|u(\tau)\|^2 + 2\mathbb{E}\int_0^\tau \|\mathfrak{A}(u(s))\|^2\, ds = \mathbb{E}\|u(0)\|^2 + 2B\tau. \tag{1.11}$$

The quantity $\mathbb{E}\|u(\tau)\|^2$—the "averaged energy per volume" of a solution u—should be of order one, see [24,25,27]. This agrees well with (1.11) since there $B \sim \int b^2\, dx \sim 1$ as $L \to \infty$, and we immediately get from (1.11) that

$$\mathbb{E}\|u(\tau)\|^2 \leq B + (\mathbb{E}\|u(0)\|^2 - B)e^{-2\tau},$$

uniformly in ν, ρ and L.

Using (1.4) and the relations

$$\widehat{u_1 u_2}(s) = L^{-d/2}\sum\nolimits_{s_1} \hat{u}_1(s_1)\hat{u}_2(s - s_1), \qquad \widehat{\bar{u}}(s) = \bar{\hat{u}}_{-s},$$

[4] For example, if $\gamma_s = (1+|s|^2)^{r_*}$, then $\mathfrak{A} = (1-\Delta)^{r_*}$.

we write (1.8) as the system of equations

$$\dot{v}_s - i\nu^{-1}|s|^2 v_s + \gamma_s v_s = i\rho L^{-d}\Big(\sum\nolimits_{s_1,s_2} \delta'^{12}_{3s} v_{s_1} v_{s_2} \bar{v}_{s_3} - |v_s|^2 v_s\Big) + b(s)\dot{\beta}_s, \quad (1.12)$$

$s \in \mathbb{Z}^d_L$, where

$$\delta'^{12}_{3s} = \begin{cases} 1, & \text{if } s_1 + s_2 = s_3 + s \text{ and } \{s_1, s_2\} \neq \{s_3, s\}, \\ 0, & \text{otherwise.} \end{cases}$$

Note that if $\{s_1, s_2\} \cap \{s_3, s\} \neq \emptyset$, then $\delta'^{12}_{3s} = 0$.

In view of the factor δ'^{12}_{3s}, in the double sum in (1.12) the index s_3 is a function of s_1, s_2, s.

Now using the interaction representation

$$v_s = \exp(i\nu^{-1}\tau|s|^2)\, \mathfrak{a}_s, \quad s \in \mathbb{Z}^d_L, \quad (1.13)$$

we re-write (1.12) as

$$\begin{aligned} &\dot{\mathfrak{a}}_s + \gamma_s \mathfrak{a}_s = i\rho\Big(\mathcal{Y}_s(\mathfrak{a}, \nu^{-1}\tau) - L^{-d}|\mathfrak{a}_s|^2 \mathfrak{a}_s\Big) + b(s)\dot{\beta}_s, \quad s \in \mathbb{Z}^d_L, \\ &\mathcal{Y}_s(\mathfrak{a}; t) = L^{-d}\sum\nolimits_{s_1}\sum\nolimits_{s_2} \delta'^{12}_{3s} \mathfrak{a}_{s_1}\mathfrak{a}_{s_2}\bar{\mathfrak{a}}_{s_3} e^{it\omega^{12}_{3s}}. \end{aligned} \quad (1.14)$$

Here $\{\beta_s\}$ is an another set of standard independent complex Wiener processes, and

$$\omega^{12}_{3s} = |s_1|^2 + |s_2|^2 - |s_3|^2 - |s|^2 = 2(s_1 - s)\cdot(s - s_2) \quad (1.15)$$

(the second equality holds since $s_3 = s_1 + s_2 - s$ due to the factor δ'^{12}_{3s}).

Despite stochastic models of WT (including (1.6)) are popular in physics, it seems that the only their mathematical studies were performed by the authors of this work and their collaborators, and in paper [10]. In the same time the models, given by Hamiltonian chains of equations with stochastic perturbations, have received significant attention from the mathematical community, e.g. see [6,9,18] and references in these works. The equations in the corresponding works are related to the Zakharov–L'vov model, written as the perturbed chain of hamiltonian equations (1.12), but differ from it significantly since there, in difference with (1.12), the interaction between the equations is local. This leads to rather different results, obtained by tools, rather different from those in our paper.

1.2. The results. The *energy spectrum* of a solution $u(\tau)$ of eq. (1.8) is the function

$$\mathbb{Z}^d_L \ni s \mapsto \mathcal{N}_s(\tau) = \mathcal{N}_s(\tau; \nu, L) = \mathbb{E}|v_s(\tau)|^2 = \mathbb{E}|\mathfrak{a}_s(\tau)|^2. \quad (1.16)$$

Traditionally in the center of attention of the WT community is the limiting behaviour of the energy spectrum $\mathcal{N}_s$, as well as of correlations of solutions $\mathfrak{a}_s(\tau)$ and $v_s(\tau)$ under the limit (1.9). Exact meaning of the latter is unclear since no relation between the small parameters ν and L^{-1} is postulated by the theory. In [13,22] it was proved that for ρ and L fixed, eq. (1.8) has a limit as $\nu \to 0$, called the *limit of discrete turbulence,* see [16,24] and Appendix 12.1. Next it was demonstrated in [21] on the physical level of rigour that if we scale ρ as $\tilde{\varepsilon}\sqrt{L}$, $\tilde{\varepsilon} \ll 1$, then the iterated limit $L \to \infty$ leads to a kinetic equation for the energy spectrum. Attempts to justify this rigorously so far failed. Instead in this work we specify the limit (1.3) as follows:

$$\nu \to 0 \text{ and } L \geq \nu^{-2-\epsilon} \text{ for some } \epsilon > 0, \\ \text{or first } L \to \infty \text{ and next } \nu \to 0 \tag{1.17}$$

(the second option formally corresponds to the first one with $\epsilon = \infty$). To present the results it is convenient for the moment to regard ρ as an independent parameter, however later we will choose it to be of the form (1.10) with a fixed small positive constant ε.

Accordingly to (1.17), everywhere in the introduction we assume that

$$L \geq \nu^{-2-\epsilon} \geq 1, \quad \epsilon > 0. \tag{1.18}$$

Let us supplement equation (1.8)=(1.14) with the initial condition

$$u(-T) = 0, \tag{1.19}$$

for some $0 < T \leq +\infty$, and in the spirit of WT decompose a solution of (1.14), (1.19) to formal series in ρ:

$$\mathfrak{a} = \mathfrak{a}^{(0)} + \rho\mathfrak{a}^{(1)} + \dots. \tag{1.20}$$

Substituting the series in the equation we get that $\mathfrak{a}^{(0)}$ satisfies the linear equation

$$\dot{\mathfrak{a}}_s^{(0)} + \gamma_s \mathfrak{a}_s^{(0)} = b(s)\dot{\beta}_s, \quad s \in \mathbb{Z}_L^d,$$

so this is the Gaussian process

$$\mathfrak{a}_s^{(0)}(\tau) = b(s) \int_{-T}^{\tau} e^{-\gamma_s(\tau - l)} d\beta_s(l), \tag{1.21}$$

while $\mathfrak{a}^{(1)}$ satisfies

$$\dot{\mathfrak{a}}_s^{(1)}(\tau) + \gamma_s \mathfrak{a}_s^{(1)}(\tau) = i\mathcal{Y}_s(\mathfrak{a}^{(0)}(\tau), \nu^{-1}\tau) - iL^{-d}|\mathfrak{a}_s^{(0)}(\tau)|^2 \mathfrak{a}_s^{(0)}(\tau), \qquad \tau > -T,$$

so

$$\mathfrak{a}_s^{(1)}(\tau) = iL^{-d} \int_{-T}^{\tau} e^{-\gamma_s(\tau - l)} \Big(\sum_{s_1, s_2} \delta'^{12}_{3s} (\mathfrak{a}_{s_1}^{(0)} \mathfrak{a}_{s_2}^{(0)} \bar{\mathfrak{a}}_{s_3}^{(0)})(l) e^{i\nu^{-1} l \omega^{12}_{3s}} \\ - |\mathfrak{a}_s^{(0)}(l)|^2 \mathfrak{a}_s^{(0)}(l) \Big) \, dl \tag{1.22}$$

is a Wiener chaos of third order (see [15]). Similarly, for $n \geq 1$

$$\mathfrak{a}_s^{(n)}(\tau) = iL^{-d} \int_{-T}^{\tau} e^{-\gamma_s(\tau - l)} \sum_{n_1 + n_2 + n_3 = n-1} \\ \Big(\sum_{s_1, s_2} \delta'^{12}_{3s} \big(\mathfrak{a}_{s_1}^{(n_1)} \mathfrak{a}_{s_2}^{(n_2)} \bar{\mathfrak{a}}_{s_3}^{(n_3)}\big)(l) e^{i\nu^{-1} l \omega^{12}_{3s}} - \big(\mathfrak{a}_s^{(n_1)} \mathfrak{a}_s^{(n_2)} \bar{\mathfrak{a}}_s^{(n_3)}\big)(l) \Big) \, dl \tag{1.23}$$

is a Wiener chaos of order $2n + 1$.

Quasisolutions. It is traditional in WT to retain the quadratic in ρ part of the decomposition (1.20) and analyse it, postulating that it well approximates small amplitude solutions. Thus motivated we start our analysis with the quadratic truncations of the series (1.20), which we call the *quasisolutions* and denote $\mathcal{A}(\tau)$. So

$$\mathcal{A}(\tau) = (\mathcal{A}_s(\tau), s \in \mathbb{Z}_L^d), \qquad \mathcal{A}_s(\tau) = \mathfrak{a}_s^{(0)}(\tau) + \rho\mathfrak{a}_s^{(1)}(\tau) + \rho^2\mathfrak{a}_s^{(2)}(\tau), \tag{1.24}$$

where $\mathfrak{a}^{(0)}$, $\mathfrak{a}^{(1)}$ and $\mathfrak{a}^{(2)}$ were defined above. The energy spectrum of a quasisolution $\mathcal{A}(\tau)$ is

$$\mathfrak{n}_s(\tau) = \mathbb{E}|\mathcal{A}_s(\tau)|^2, \quad s \in \mathbb{Z}_L^d, \tag{1.25}$$

where

$$\mathfrak{n}_s = \mathfrak{n}_s^{(0)} + \rho\,\mathfrak{n}_s^{(1)} + \rho^2\mathfrak{n}_s^{(2)} + \rho^3\mathfrak{n}_s^{(3)} + \rho^4\mathfrak{n}_s^{(4)}, \quad s \in \mathbb{Z}_L^d, \tag{1.26}$$

with

$$\mathfrak{n}_s^{(k)} = \sum_{k_1+k_2=k,\, k_1,k_2\le 2} \mathbb{E}\mathfrak{a}_s^{(k_1)}\bar{\mathfrak{a}}_s^{(k_2)}, \quad 0 \le k \le 4. \tag{1.27}$$

In particular, $\mathfrak{n}_s^{(0)} = \mathbb{E}|\mathfrak{a}_s^{(0)}|^2$ and we easily derive from (1.21) that

$$\mathfrak{n}_s^{(0)}(\tau) = \mathbb{E}|\mathfrak{a}_s^{(0)}(\tau)|^2 = B(s)\big(1 - e^{-2\gamma_s(T+\tau)}\big), \quad B(s) = \frac{b(s)^2}{\gamma_s}. \tag{1.28}$$

So $\mathfrak{n}_s^{(0)}$ extends to a Schwartz function of $s \in \mathbb{R}^d$, uniformly in $\tau \ge -T$. Also it is not hard to see that $\mathfrak{n}_s^{(1)} = 0$, see in Section 2. The coefficients $\mathfrak{n}_s^{(k)}$ with $k \ge 2$ are more complicated.

The processes $\mathfrak{a}_s^{(r)}(\tau)$ and the functions $\mathfrak{n}_s^{(r)}(\tau)$, $r \ge 0$, depend on ν and L. When it will be needed to indicate this dependence, we will write them as $\mathfrak{a}_s^{(r)}(\tau) = \mathfrak{a}_s^{(r)}(\tau; \nu, L)$, etc. The dependence of the objects on T will not be indicated.

It was explained on the heuristic and half-heuristic level in many physical works concerning various models of WT that the term $\rho^2\mathfrak{n}^{(2)}(\tau)$ is the crucial non-trivial component of the energy spectrum, while the terms $\rho^3\mathfrak{n}^{(3)}(\tau)$ and $\rho^4\mathfrak{n}^{(4)}(\tau)$ make its perturbations. Our results justify this insight on the energy spectra of quasisolutions. Firstly we consider $\mathfrak{n}_s^{(2)} = \mathbb{E}|\mathfrak{a}_s^{(1)}|^2 + 2\Re\mathbb{E}\mathfrak{a}_s^{(2)}\bar{\mathfrak{a}}_s^{(0)}$. The two terms on the right are similar. Consider the first one, $\mathbb{E}|\mathfrak{a}_s^{(1)}|^2$. The theorem below describes its asymptotic behaviour under the limit (1.17), where for simplicity we assume that $T = \infty$.

We set $B(s_1, s_2, s_3) := B(s_1)B(s_2)B(s_3)$ and denote by $C^\#(s)$ various positive continuous functions of s which decay as $|s| \to \infty$ faster than any negative degree of $|s|$.

$$\begin{aligned}&\text{The constants } C, C_1 \text{ etc and the functions } C^\#(s)\\ &\text{never depend on } \nu, L, \rho, \varepsilon \text{ and on the times } T, \tau,\end{aligned} \tag{1.29}$$

unless the dependence is indicated.

Theorem A. *Let in* (1.19) $T = \infty$. *Then for any* τ *and any* $s \in \mathbb{Z}_L^d$,

$$\begin{aligned}\Big|\mathbb{E}|\mathfrak{a}_s^{(1)}(\tau)|^2 - \frac{\pi\nu}{\gamma_s}\int_{\Sigma_s} \frac{B(s_1, s_2, s_1+s_2-s)}{\sqrt{|s_1-s|^2+|s_2-s|^2}}\, ds_1 ds_2|_{\Sigma_s}\Big|&\\ \le \big(\nu^2 + L^{-2}\nu^{-2}\big)C^\#(s).&\end{aligned} \tag{1.30}$$

Here Σ_s *is the quadric* $\{(s_1, s_2) : (s_1 - s)\cdot(s_2 - s) = 0\}$ *and* $ds_1 ds_2|_{\Sigma_s}$ *is the volume element on it, corresponding to the Euclidean structure on* $\mathbb{R}^{2d}$. *If* $d = 2$, *then the term* $C^\#(s)\nu^2$ *in the r.h.s. of* (1.30) *should be replaced by* $C^\#(s; \aleph)\,\nu^{2-\aleph}$, *where* $\aleph$ *is arbitrary positive number.*

See Theorems 3.1, 3.3 and Corollary 3.4. Due to (1.15), $\Sigma_s = \{(s_1, s_2, s_3) : s_1+s_2 = s_3 + s \text{ and } \omega^{12}_{3s} = 0\}$. This quadric is the set of resonances for eq. (2.1).

Denote $F_s(s_1, s_2) := (s_1 - s) \cdot (s_2 - s) = -\frac{1}{2}\omega^{12}_{3s}$. Then $|\partial F_s|$ equals the divisor of the integrand in (1.30). So the integral in (1.30) is exactly what physicists call the integral of B over the delta-function of F_s and denote $\int B\delta(F_s)$, see [27], p. 67. For rigorous mathematical treatment of this object see [12], Section III.1.3, or [17], pp. 36-37. As $\delta(F_s) = \delta\big(-\frac{1}{2}\,\omega^{12}_{3s}\big) = -2\delta(\omega^{12}_{3s})$, where $s_3 := s_1 + s_2 - s$, then neglecting the minus-sign (as physicists do) we may write the integral from (1.30) as

$$\nu\frac{2\pi}{\gamma_s}\int B(s_1, s_2, s_3)\,\delta(\omega^{12}_{3s})\delta^{12}_{3s}\,ds_1ds_2ds_3 \tag{1.31}$$

(since $\int \ldots\ \delta^{12}_{3s}\,ds_1ds_2ds_3 = \int \cdots |_{s_3=s_1+s_2-s}\ ds_1ds_2$).

Theorem A and its variations play important role in our work since the terms, quadratic in $\mathfrak{a}^{(1)}$ and in its increments, as well as quadratic terms, linear in $\mathfrak{a}^{(0)}$, $\mathfrak{a}^{(2)}$ and in their increments, play a leading role in the analysis of the energy spectrum $\mathfrak{n}_s(\tau)$. It turns out that their asymptotic behaviour under the limit (1.17) is described by integrals, similar to (1.30). These results imply that

$$\mathfrak{n}^{(2)}_s \sim \nu, \tag{1.32}$$

(recall (1.18)). The terms $\mathfrak{n}^{(3)}_s$ and $\mathfrak{n}^{(4)}_s$ and their variations which appear in our analysis are smaller:

$$|\mathfrak{n}^{(3)}_s|,\ |\mathfrak{n}^{(4)}_s| \le C^{\#}(s)\nu^2. \tag{1.33}$$

If $d = 2$ the estimate for $\mathfrak{n}^{(3)}_s$ is slightly weaker: $|\mathfrak{n}^{(3)}_s| \le C^{\#}(s)\nu^2 \ln \nu^{-1}$. Upper bounds (1.33) may be obtained by annoying but rather straightforward analysis of certain integrals with oscillating exponents, based on results from Sections 5 and 10, but instead we get them from a much deeper result, presented below in Theorem D.

Estimates (1.32) and (1.33) justify the scaling (1.10) since for such a choice of ρ

$$\mathfrak{n}_s = \mathfrak{n}^{(0)}_s + \varepsilon\widetilde{\mathfrak{n}}^{(2)}_s + O(\varepsilon^2) \tag{1.34}$$

(if $\nu \ll 1$), where $\widetilde{\mathfrak{n}}^{(2)}_s = \nu^{-1}\mathfrak{n}^{(2)}_s \sim 1$. Thus, the principal non-trivial component of $\mathfrak{n}_s$ is given by $\rho^2\mathfrak{n}^{(2)}_s = \varepsilon\widetilde{\mathfrak{n}}^{(2)}_s \sim \varepsilon$, while the other terms in (1.26)=(1.34) are smaller, of the size $O(\varepsilon^2)$. This also shows that the small parameter $\sqrt{\varepsilon}$ measures the properly scaled amplitude of the oscillations.

Since equation (1.14) for the processes $\mathfrak{a}_s(\tau)$ fast oscillates in time, then the task to describe the behaviour of $\mathfrak{n}_s(\tau)$ has obvious similarities with the problem, treated by the Krylov–Bogolyubov averaging (see in [1]). Accordingly it is natural to try to study the required limiting behaviour following the suit of the Krylov–Bogolyubov theory. That is, by considering the increments $\mathfrak{n}_s(\tau+\theta) - \mathfrak{n}_s(\tau)$ with $\nu \ll \theta \ll 1$ and passing firstly to the limit as $\nu \to 0$ and next—to the limit $\theta \to 0$. That insight was exploited heuristically in many works on WT (e.g. see [24], Section 7), while in [13,22] it was rigorously applied to pass to the limit of discrete turbulence $\nu \to 0$ with L and ρ fixed. In this work we also argue as it is customary in the classical averaging and analyse the increments $\mathfrak{n}_s(\tau+\theta) - \mathfrak{n}_s(\tau)$, using the asymptotical results like Theorem A and estimates like (1.33). This analysis shows that due to the important role, played by the integrals like (1.30), the leading nonlinear contribution to the increments is described by the cubic wave kinetic integral operator K, sending a function $y(s)$, $s \in \mathbb{R}^d$, to the function

$$K_s(y(\cdot)) = 2\pi \int_{\Sigma_s} \frac{ds_1\, ds_2\, |_{\Sigma_s}\;\; y_1 y_2 y_3 y_s}{\sqrt{|s_1 - s|^2 + |s_2 - s|^2}} \left(\frac{1}{y_s} + \frac{1}{y_3} - \frac{1}{y_1} - \frac{1}{y_2} \right),$$

where $y_s = y(s)$ and for $j = 1, 2, 3$ we denote $y_j = y(s_j)$ with $s_3 = s_1 + s_2 - s$. Using the notation (1.31), $K_s(y(\cdot))$ may be written as

$$4\pi \int y_1 y_2 y_3 y_s \left(\frac{1}{y_s} + \frac{1}{y_3} - \frac{1}{y_1} - \frac{1}{y_2} \right) \delta(\omega^{12}_{3s})\, \delta^{12}_{3s}\, ds_1 ds_2 ds_3.$$

This is exactly the wave kinetic integral which appears in physical works on WT to describe the 4-waves interaction, see [27], p. 71 and [24], p. 91.

The operator K is defined in terms of the measure $\mu_s = \frac{ds_1\, ds_2}{(|s_1 - s|^2 + |s_2 - s|^2)^{1/2}} |_{\Sigma_s}$, and our study of the operator is based on the following useful disintegration of μ_0 (the measure μ_s with $s = 0$), obtained in Theorem 3.6:

$$\mu_0(ds_1, ds_2) = |s_1|^{-1} ds_1\, d_{s_1^\perp} s_2,$$

where for a non-zero vector s_1 we denote by $d_{s_1^\perp} s$ the Lebesgue measure on the hyperspace $s_1^\perp = \{s : s \cdot s_1 = 0\}$. Based on this disintegration, in Section 4 we prove the result below, where for $r \in \mathbb{R}$ we denote by $\mathcal{C}_r(\mathbb{R}^d)$ the space of continuous complex functions on $\mathbb{R}^d$ with finite norm $|f|_r = \sup_x |f(x)| \langle x \rangle^r$.

Theorem B. *For any $r > d$ the operator K defines a continuous 3-homogeneous mapping $K : \mathcal{C}_r(\mathbb{R}^d) \to \mathcal{C}_{r+1}(\mathbb{R}^d)$.*

See Theorem 4.1. Now consider the damped/driven wave kinetic equation (WKE):

$$\dot{m}(\tau, s) = -2\gamma_s m(\tau, s) + \varepsilon K(m(\tau, \cdot))(s) + 2b(s)^2, \quad m(-T) = 0. \tag{1.35}$$

In Theorem 4.4 we easily derive from Theorem B that for small ε this equation has a unique solution m. The latter can be written as $m = m^0(\tau, s) + \varepsilon m^1(\tau, s)$, where $m^0, m^1 \sim 1, m^0$ is a solution of the linear equation $(1.35)|_{\varepsilon=0}$ and equals $\mathfrak{n}^{(0)}$. Analysing the increments $\mathfrak{n}_s(\tau + \theta) - \mathfrak{n}_s(\tau)$ using the results, discussed above, in Section 5 we show that they have approximately the same form as the increments of solutions for eq. (1.35). Next in Section 7, arguing by analogy with the classical averaging theory, we get the stated below main result of this work (recall agreement (1.29)):

Theorem C (Main theorem). *The energy spectrum $\mathfrak{n}_s(\tau) = \mathfrak{n}_s(\tau; \nu, L)$ of the quasisolution $\mathcal{A}_s(\tau)$ of (1.14), (1.19) satisfies the estimate $\mathfrak{n}_s(\tau) \le C^\#(s)$ and is close to the solution $m(\tau, s)$ of WKE (1.35). Namely, under the scaling (1.10) for any r there exists $\varepsilon_r > 0$ such that for $0 < \varepsilon \le \varepsilon_r$ we have*

$$|\mathfrak{n}_\cdot(\tau) - m(\tau, \cdot)|_r \le C_r \varepsilon^2 \qquad \forall\, \tau \ge -T, \tag{1.36}$$

if $0 < \nu \le \nu_\varepsilon(r)$ for a suitable $\nu_\varepsilon(r) > 0$, and if L satisfies (1.18). Moreover, the limit $\mathfrak{n}_s(\tau; \nu, \infty)$ of $\mathfrak{n}_s(\tau; \nu, L)$ as $L \to \infty$ exists, is a Schwartz function of $s \in \mathbb{R}^d$ and also satisfies the estimate above for any r.

Since the energy spectrum $\mathfrak{n}_s$ is defined for $s \in \mathbb{Z}^d_L$ with finite L, then the norm in (1.36) is understood as $|f|_r = \sup_{s\in\mathbb{Z}^d_L} |f(s)|\langle s\rangle^r$.

For $\varepsilon = 0$ eq. (1.35) has the unique steady state m^0, $m^0_s = b^2_s/\gamma_s$, which is asymptotically stable. By the inverse function theorem, for $\varepsilon < \varepsilon'_r$ ($\varepsilon'_r > 0$), eq. (1.35) has a unique steady state $m^\varepsilon \in \mathcal{C}_r(\mathbb{R}^d)$, close to m^0. It also is asymptotically stable. Jointly with Theorem C this result describes the asymptotic in time behaviour of the energy spectrum $\mathfrak{n}_s$:

$$|\mathfrak{n}_\cdot(\tau) - m^\varepsilon_\cdot|_r \le |m^\varepsilon_\cdot|_r e^{-\tau-T} + C_r\varepsilon^2, \qquad \forall\, \tau \ge -T. \tag{1.37}$$

See in Section 7.

Due to Theorem A and some modifications of this result, the iterated limit $\lim_{\nu\to 0}\lim_{L\to\infty} \nu^{-1}\mathfrak{n}^{(2)}_s(\tau;\nu,L)$ exists and is non-zero. It is hard to doubt (however, we have not proved this yet) that a similar iterated limit also exists for $\nu^{-2}\mathfrak{n}^{(4)}_s$ (cf. estimate (1.33)). Then, in view of (1.33) for $\mathfrak{n}^{(3)}_s$, under the scaling (1.10) exists a limit $\mathfrak{n}_s(\tau;0,\infty) = \lim_{\nu\to 0}\lim_{L\to\infty}\mathfrak{n}_s(\tau;\nu,L)$. If so, then $\mathfrak{n}_s(\tau;0,\infty)$ also satisfies the assertion of Theorem C and obeys the time-asymptotic (1.37).

Theorems A–C are proved in Sections 2–7 and Sections 9–11, where Sections 9–11 contain a demonstration of Theorem A as well as of some lemmas, needed to prove Theorems B, C in Sections 2–7.

Section 8 presents the results of our second paper [8] on formal expansions (1.20) in series in ρ. There we decompose the spectrum $\mathcal{N}_s(\tau)$, defined in (1.16), in formal series,

$$\mathcal{N}_s(\tau) = \mathfrak{n}^0_s(\tau) + \rho\mathfrak{n}^1_s(\tau) + \rho^2\mathfrak{n}^2_s(\tau) + \dots. \tag{1.38}$$

where $\mathfrak{n}^0_s = \mathfrak{n}^{(0)}_s$ and for $k \ge 1$ $\mathfrak{n}^k_s$ is given by the formulas (1.27) with the restriction $k_1, k_2 \le 2$ being dropped, i.e. $\mathfrak{n}^k_s = \sum_{k_1+k_2=k}\mathbb{E}\mathfrak{a}^{(k_1)}_s\bar{\mathfrak{a}}^{(k_2)}_s$, so $\mathfrak{n}^j_s$ equals $\mathfrak{n}^{(j)}_s$ for $j \le 2$, but not for $j = 3, 4$. Still the estimates (1.33) remain true for $\mathfrak{n}^3_s$ and $\mathfrak{n}^4_s$. The bounds on $\mathfrak{n}^0_s, \dots, \mathfrak{n}^4_s$ suggest that $|\mathfrak{n}^k_s| \lesssim \nu^{k/2}$. The results of Section 8 put some light on this assumption. Namely, it is proved there that $\mathfrak{n}^k_s(\tau)$ may be approximated by a finite sum $\sum_{\mathcal{F}_k} I^k_s(\mathcal{F}_k)$ of integrals $I^k_s(\mathcal{F}_k)$, naturally parametrised by a certain class of Feynman diagrams $\mathcal{F}_k$. These integrals satisfy the following assertion, where $\lceil x\rceil$ stands for the smallest integer $\ge x$.

Theorem D. *For each k,*

(a) *every integral* $I^k_s(\mathcal{F}_k)$ *satisfies*

$$|I^k_s(\mathcal{F}_k)| \le C^\#(s;k)\max(\nu^{\lceil k/2\rceil}, \nu^d), \tag{1.39}$$

if L is so big that $L^{-2}\nu^{-2} \le \max(\nu^{\lceil k/2\rceil}, \nu^d)$. *If* $d = 2$ *and* $k = 3$ *then the maximum in the r.h.s. above should be multiplied by* $\ln\nu^{-1}$.

(b) *estimate* (1.39) *is sharp in the sense that if* $k > 2d$ *(so that* $\max(\nu^{\lceil k/2\rceil}, \nu^d) = \nu^d$*), then for some diagrams* $\mathcal{F}_k$ *we have*

$$|I^k_s(\mathcal{F}_k)| \sim C^\#(s;k)\nu^d \gg C^\#(s;k)\nu^{\lceil k/2\rceil}. \tag{1.40}$$

Theorem D is a new result on the integrals with fast oscillating quadratic exponents (see the integral in (8.9)). We hope that the theorem and its variations (cf. [7] and the last section of [8]) will find applications outside the framework of WT.

As $\mathfrak{n}_s^k$ is approximated by a finite sum of integrals $I_s^k(\mathcal{F}_k)$, it also obeys (1.39). Since $d \geq 2$, then (1.39) implies estimate (1.33), needed to prove Theorem C. Also, by (1.39)

$$|\mathfrak{n}_s^k| \lesssim \nu^{k/2} \quad \text{if} \quad k \leq 2d. \tag{1.41}$$

Validity of inequality $|\mathfrak{n}_s^k| \lesssim \nu^{k/2}$ for $k > 2d$ is a delicate issue. Assertion (1.40) implies that the inequality does not hold for all summands, forming $\mathfrak{n}_s^k$, but still it may hold for $\mathfrak{n}_s^k = \sum_{\mathcal{F}_k} I_s^k(\mathcal{F}_k)$ due to cancellations. And indeed, we prove that some cancellations do happen in the sub-sum over a certain subclass $\mathfrak{F}_k^B$ of diagrams $\mathcal{F}_k$ which give rise to the biggest integrals $I_s^k(\mathcal{F}_k)$. Namely, for any $\mathcal{F}_k \in \mathfrak{F}_k^B$ we have (1.40) but $\left|\sum_{\mathcal{F}_k \in \mathfrak{F}_k^B} I_s^k(\mathcal{F}_k)\right| \leq C^\#(s;k)\nu^{k-1} \leq C^\#(s;k)\nu^{k/2}$. This does not imply the validity of (1.41) for all k, which remains an open problem:

Problem 1.1. *Prove that for any* $k \in \mathbb{N}$

$$|\mathfrak{n}_s^k(\tau)| \leq C^\#(s;k)\nu^{k/2} \quad \forall s, \ \forall \tau \geq -T, \tag{1.42}$$

if L is sufficiently big in terms of ν^{-1}.

If the conjecture (1.42) is correct, then under the scaling (1.10), for any $M \geq 2$ the order M truncation of the series (1.38), namely $\mathcal{N}_{s,M}(\tau) = \sum_{0 \leq k \leq M} \rho^k \mathfrak{n}_s^k(\tau)$, also meets the assertion of Theorem C, i.e. satisfies the WKE with the accuracy ε^2. It is unclear for us if $\mathcal{N}_{s,M}$ satisfies the equation with better accuracy, i.e. if it better approximates a solution of (1.35) than $\mathfrak{n}_s(\tau)$. On the contrary, if (1.42) fails in the sense that for some k we have $\|\mathfrak{n}_\cdot^k\| \gtrsim C\nu^{k'}$ with $k' < k/2$, then under the scaling (1.10) the sum (1.38) is not a formal series in $\sqrt{\varepsilon}$, uniformly in ν and L.

1.3. Conclusions.

- If in eq. (1.8) ρ is chosen to be $\rho = \nu^{-1/2}\varepsilon^{1/2}$ with $0 < \varepsilon \leq 1$, then the energy spectra $\mathfrak{n}_s(\tau)$ of quasisolutions for the equation (i.e. of quadratic in ρ truncations of solutions u, decomposed in formal series in ρ) under the limit (1.17) satisfy the damped/driven wave kinetic equation (1.35) with accuracy ε^2. If we write the equation which we study using the original fast time t in the form (1.6), then the kinetic limit exists if there $\lambda \sim \sqrt{\nu}$. The time, needed to arrive at the kinetic regime is $t \sim \lambda^{-2}$.
- If (1.42) is true, then the energy spectra of higher order truncations for decompositions of solutions for (1.8) in series in ρ also satisfy (1.35) at least with the same accuracy ε^2.
- Similar, if the energy spectrum $\mathcal{N}_s(\tau) = \mathbb{E}|v_s|^2$ admits the second order truncated Taylor decomposition in ρ of the form

  $$\mathcal{N}_s(\tau) = \mathfrak{n}_s^{(0)} + \rho^2 \mathfrak{n}_s^{(2)} + O(\rho^3 \nu^{3/2}),$$

  then under the scaling (1.10) and the limit (1.17) the spectrum $\mathcal{N}_s(\tau)$ satisfies the WKE (1.35) with accuracy $\varepsilon^{3/2}$. At this point we recall that different NLS equations and their damped/driven versions appear in physics as models for small oscillations in various media, obtained by neglecting in the exact equations terms of higher order of smallness. So it is not impossible that the kinetic limit holds for the energy spectra of the second order jets in ρ of solutions $v(\tau)$ (which we call quasisolutions), but not for the solutions themselves since the former are closer to the physical reality.

- To prove our results we have developed in this work and in its second part [8] new analytic and combinatoric techniques. They apply to quasisolutions of equations (1.6) under the WT limit (1.9) and for this moment give no non-trivial information about the exact solutions. Still we believe that these techniques make a basis for further study of the damped/driven NLS equations under the WT limit, as well as of other stochastic models of WT. In particular, we hope that being applied to some other models they will give there stronger and "more final" results. To verify this belief is our next goal.

1.4. Notation. By $\mathbb{R}^n_+$ and $\mathbb{R}^n_-$ we denote, respectively, the sets $[0,\infty)^n$ and $(-\infty,0]^n$. For a vector v we denote by $|v|$ its Euclidean norm and by $v\cdot u$ its Euclidean scalar product with a vector u. We write $\langle v\rangle=(1+|v|^2)^{1/2}$. For a real number x, $\lceil x\rceil$ stands for the smallest integer $\ge x$. We denote by $\chi_d(\nu)$ the constant

$$\chi_d(\nu)=\begin{cases}1, & d\ge 3,\\ \ln\nu^{-1}, & d=2.\end{cases} \tag{1.43}$$

The exponent $\aleph_d$ is zero if $d\ge 3$ and is any positive number if $d=2$. For an integral $I=\int_{\mathbb{R}^N} f(z)\,dz$ and a domain $M\subset\mathbb{R}^N$, open or closed, we write

$$\langle I,M\rangle=\int_M f(z)\,dz. \tag{1.44}$$

Similar we write $\langle |I|,M\rangle=\int_M |f(z)|\,dz$.

We denote $\tilde{\sum}_{s_1,\dots,s_k\in\mathbb{Z}^d_L} := L^{-kd}\sum_{s_1,\dots,s_k\in\mathbb{Z}^d_L}$. Following the tradition of WT, we often abbreviate v_{s_j}, $\mathfrak{a}_{s_j}$, $\gamma_{s_j},\dots$ to v_j, $\mathfrak{a}_j$, $\gamma_j,\dots$, and abbreviate the sums $\sum_{s_1,\dots,s_k\in\mathbb{Z}^d_L}$ and $\tilde{\sum}_{s_1,\dots,s_k\in\mathbb{Z}^d_L}$ to $\sum_{1,\dots,k}$ and $\tilde{\sum}_{1,\dots,k}$. By δ^{12}_{3s} we denote the Kronecker delta of the relation $s_1+s_2=s_3+s$.

Finally, $C^\#(\cdot)$, $C^\#_1(\cdot),\dots$ stand for various non-negative continuous functions, fast decaying at infinity:

$$0\le C^\#(x)\le C_N\langle x\rangle^{-N}\quad \forall x, \tag{1.45}$$

for every N, with suitable constants C_N. By $C^\#(x;a)$ we denote a function $C^\#$, depending on a parameter a. Below we discuss some properties of the functions $C^\#$.

Functions $C^\#$. For any Schwartz function f, $|f(x)|$ may be written as $C^\#(x)$. If $\mathcal{L}:\mathbb{R}^n\to\mathbb{R}^n$ is a linear isomorphism, then the function $g(y)=C^\#(\mathcal{L}y)$ may be written as $C^\#_1(y)$. Next, for any $C^\#(x,y)$, where $(x,y)\in\mathbb{R}^{d_1+d_2}$, $d_1,d_2\ge 1$, there exist $C^\#_1(x)$ and $C^\#_2(y)$ such that

$$C^\#(x,y)\le C^\#_1(x)C^\#_2(y).$$

Indeed, consider $C^\#_0(t)=\sup_{|(x,y)|\ge t}C^\#(x,y)$, $t\ge 0$. This is a non-increasing continuous function, satisfying (1.45). Then

$$C^\#(x,y)\le\sqrt{C^\#_0\big(\tfrac{1}{\sqrt2}(|x|+|y|)\big)}\sqrt{C^\#_0\big(\tfrac{1}{\sqrt2}(|x|+|y|)\big)}\le C^\#_1(x)C^\#_2(y),$$

where $C^\#_j(z)=\big(C^\#_0(\tfrac{1}{\sqrt2}|z|)\big)^{1/2}$, $j=1,2$. Finally, for any $C^\#_1(x)$ and $C^\#_2(y)$ the function $C^\#_1(x)C^\#_2(y)$ may be written as $C^\#(x,y)$.

2. Formal Decomposition of Solutions in Series in ρ

2.1. Approximate a-equation. Despite that the term $L^{-d}|\mathfrak{a}_s|^2\mathfrak{a}_s$ is only a small perturbation in equation (1.14), it is rather inconvenient for our analysis. Dropping it we consider the following more convenient equation:

$$\dot{a}_s + \gamma_s a_s = i\rho\mathcal{Y}_s(a, \nu^{-1}\tau) + b(s)\dot{\beta}_s, \quad a_s(-T) = 0, \quad s \in \mathbb{Z}_L^d. \tag{2.1}$$

Similar to the process $\mathfrak{a}_s$, we decompose a_s to the formal series in ρ :

$$a = a^{(0)} + \rho a^{(1)} + \dots. \tag{2.2}$$

Here $a_s^{(0)}(\tau) = \mathfrak{a}_s^{(0)}(\tau)$ is the Gaussian process given by (1.21), while the processes $a_s^{(n)}(\tau)$ with $n \geq 1$, where

$$a_s^{(1)}(\tau) = iL^{-d}\int_{-T}^{\tau} e^{-\gamma_s(\tau-l)} \sum_{s_1,s_2} \delta'^{12}_{3s}(a_{s_1}^{(0)}a_{s_2}^{(0)}\bar{a}_{s_3}^{(0)})(l)e^{i\nu^{-1}l\omega^{12}_{3s}}\,dl \tag{2.3}$$

and for $n \geq 1$

$$a_s^{(n)}(\tau) = iL^{-d}\int_{-T}^{\tau} e^{-\gamma_s(\tau-l)} \times \sum_{n_1+n_2+n_3=n-1}\sum_{s_1,s_2} \delta'^{12}_{3s}\big(a_{s_1}^{(n_1)}a_{s_2}^{(n_2)}\bar{a}_{s_3}^{(n_3)}\big)(l)e^{i\nu^{-1}l\omega^{12}_{3s}}\,dl, \tag{2.4}$$

are Wiener chaoses of order $2n+1$. In Section 12.4 we prove that the processes $a^{(n)}(\tau)$ and $\mathfrak{a}^{(n)}(\tau)$ are L^{-d}-close:

Proposition 2.1. *For any $n \geq 0$,*

$$\mathbb{E}|a_s^{(n)}(\tau) - \mathfrak{a}_s^{(n)}(\tau)|^2 \leq L^{-2d}C^\#(s;n), \qquad s \in \mathbb{Z}_L^d, \tag{2.5}$$

uniformly in $\tau \geq -T$ and ν.

Since $\mathbb{E}|a_s^{(n)}(\tau)|^2$, $\mathbb{E}|\mathfrak{a}_s^{(n)}(\tau)|^2 \leq C^\#(s;n)$ for any τ, n (this follows from Theorem 5.1 jointly with (2.5)), Proposition 2.1 and the Cauchy inequality imply

Corollary 2.2. *For any $m, n \geq 0$,*

$$\big|\mathbb{E}a_s^{(n)}(\tau_1)\bar{a}_s^{(m)}(\tau_2) - \mathbb{E}\mathfrak{a}_s^{(n)}(\tau_1)\bar{\mathfrak{a}}_s^{(m)}(\tau_2)\big| \leq L^{-d}C^\#(s;m,n), \qquad s \in \mathbb{Z}_L^d, \tag{2.6}$$

uniformly in τ_1, τ_2 and ν.

We define a quasisolution $A(\tau)$ of equation (2.1) as in definition (1.24), where the processes $\mathfrak{a}^{(n)}$ are replaced by $a^{(n)}$, and define its energy spectrum as $n_s(\tau) = \mathbb{E}|A_s(\tau)|^2$, cf. (1.25). Due to Corollary 2.2 it suffices to prove the results, formulated in the introduction, replacing in their statements processes $\mathfrak{a}^{(k)}(\tau)$ and $\mathfrak{n}(\tau)$ with $a^{(k)}(\tau)$ and $n(\tau)$. Indeed, for Theorem A this assertion holds since $d \geq 2$. For Theorem C it follows from (1.17) as $|n_s - \mathfrak{n}_s| \leq \rho^4 L^{-d}C^\#(s) \leq \nu^{-2}L^{-d}C^\#(s)$, where we recall that ρ is given by (1.10). For other results the argument is similar.

Below we work with the processes $a^{(j)}(\tau)$ and $n(\tau)$, and never with $\mathfrak{a}^{(j)}(\tau)$ and $\mathfrak{n}(\tau)$.

2.2. Nonlinearity $\mathcal{Y}_s$. The cubic nonlinearity $\mathcal{Y}$ in (2.1) defines the 3-linear over real numbers mapping $(u, v, w) \mapsto \mathcal{Y}(u, v, w; t)$, where

$$\mathcal{Y}_s(u, v, w; t) = L^{-d} \sum_{s_1, s_2} \delta'^{12}_{3s}\, u_{s_1} v_{s_2} \bar{w}_{s_3} e^{it\omega^{12}_{3s}},$$

so $\mathcal{Y}_s(a; t) = \mathcal{Y}_s(a, a, a; t)$. Often it will be better to use the symmetrisation

$$\mathcal{Y}^{sym}_s(u, v, w; t) = \frac{L^{-d}}{3} \sum_{s_1, s_2} \delta'^{12}_{3s}\big(u_{s_1} v_{s_2} \bar{w}_{s_3} + v_{s_1} w_{s_2} \bar{u}_{s_3} + w_{s_1} u_{s_2} \bar{v}_{s_3}\big)\, e^{it\omega^{12}_{3s}}. \quad (2.7)$$

Clearly $\mathcal{Y}^{sym}(v, v, v) = \mathcal{Y}(v)$. Besides,

$$\mathcal{Y}_s(v^0 + \delta v^1 + \delta^2 v^2) = \mathcal{Y}_s(v^0) + 3\delta\, \mathcal{Y}^{sym}_s(v^0, v^0, v^1) + 3\delta^2\Big(\mathcal{Y}^{sym}_s(v^0, v^1, v^1) + \mathcal{Y}^{sym}_s(v^0, v^0, v^2)\Big) + O(\delta^3).$$

2.3. Correlations of first terms in the formal decomposition. Let us go back to the formal decomposition (2.2). Correlations of the processes $a^{(n)}_s(\tau)$ are important for what follows. Since $a^{(0)} = \mathfrak{a}^{(0)}$, then by (1.21) for any τ_1, τ_2,

$$\mathbb{E} a^{(0)}_s(\tau_1) a^{(0)}_{s'}(\tau_2) \equiv \mathbb{E} \bar{a}^{(0)}_s(\tau_1) \bar{a}^{(0)}_{s'}(\tau_2) \equiv 0,$$
$$\mathbb{E} a^{(0)}_s(\tau_1) \bar{a}^{(0)}_{s'}(\tau_2) = \delta^s_{s'} \frac{b(s)^2}{\gamma_s}\big(e^{-\gamma_s|\tau_1-\tau_2|} - e^{-\gamma_s(2T+\tau_1+\tau_2)}\big). \quad (2.8)$$

Indeed, the first relations are obvious. To prove the last we may assume that $-T \le \tau_1 \le \tau_2$. Then the l.h.s. vanishes if $s \ne s'$, while for $s = s'$ it equals

$$b(s)^2 \mathbb{E}\Big(\int_{-T}^{\tau_1} e^{-\gamma_s(\tau_1 - l_1)} d\beta_s(l_1)\Big)\Big(\int_{-T}^{\tau_1} e^{-\gamma_s(\tau_2 - l_2)} d\bar{\beta}_s(l_2) + \int_{\tau_1}^{\tau_2} e^{-\gamma_s(\tau_2 - l_2)} d\bar{\beta}_s(l_2)\Big).$$

The expectation of the second term vanishes, and that of the first equals $2\,b(s)^2 \int_{-T}^{\tau_1} e^{-\gamma_s(\tau_1 - l + \tau_2 - l)} dl$, which is the r.h.s. of (2.8) (recall that $d\beta \cdot d\bar{\beta} = 2dt$).

For the process $a^{(1)}$ we have

Lemma 2.3. *For any τ_1, τ_2 and s', s'',*

(i) $\mathbb{E} a^{(1)}_{s'}(\tau_1) a^{(1)}_{s''}(\tau_2) = 0$;
(ii) $\mathbb{E} a^{(1)}_{s'}(\tau_1) \bar{a}^{(1)}_{s''}(\tau_2) = 0$ *if* $s' \ne s''$;
(iii) $\mathbb{E} a^{(1)}_{s'}(\tau_1) a^{(0)}_{s''}(\tau_2) = \mathbb{E} a^{(1)}_{s'}(\tau_1) \bar{a}^{(0)}_{s''}(\tau_2) = 0$.

Proof. Let us verify ii). Due to (2.3), the expectation we examine is a sum over s_1, s_2 and s'_1, s'_2 of integrals of functions

$$\mathbb{E}\Big(\delta'^{12}_{3s'}(a^{(0)}_1 a^{(0)}_2 \bar{a}^{(0)}_3)(l)\, \delta'^{1'2'}_{3's''}(\bar{a}^{(0)}_{1'} \bar{a}^{(0)}_{2'} a^{(0)}_{3'})(l')\Big),$$

multiplied by some density-functions. Since each $a^{(0)}_j$ is a Gaussian process, then Wick's theorem applies. By (2.8) and due to the the factor δ'^{12}_{3s}, $a^{(0)}_1$ must be coupled with $\bar{a}^{(0)}_{1'}$ or with $\bar{a}^{(0)}_{2'}$, $a^{(0)}_2$—with $\bar{a}^{(0)}_{2'}$ or $\bar{a}^{(0)}_{1'}$, and $\bar{a}^{(0)}_3$—with $a^{(0)}_{3'}$. So $s' = s_1 + s_2 - s_3 = s'_1 + s'_2 - s'_3 = s''$ and ii) follows. The proof of i) and iii) is similar. □

By a similar argument it can be shown that $\mathbb{E}a_{s'}^{(m)}a_{s''}^{(n)} = 0$ for any m, n and s', s'', while $\mathbb{E}a_{s'}^{(m)}\bar{a}_{s''}^{(n)} = 0$ for any m, n and $s' \neq s''$. Moreover, $\Re\mathbb{E}\mathfrak{a}_s^{(1)}\bar{\mathfrak{a}}_s^{(0)} = 0$ (while Corollary 2.2 implies only that this expectation is of the size L^{-d}), so, as it is claimed in the introduction, $\mathfrak{n}_s^{(1)} = 0$. We do not prove and do not use these observations.

2.4. Second moments $\mathbb{E}a_s^{(1)}(\tau)\bar{a}_s^{(1)}(\tau)$. In the center of attention of WT are the limiting, as $L \to \infty$, $\nu \to 0$, correlations of solutions $a_s(\tau)$ (and $v_s(\tau)$). Accordingly we should analyse limiting correlations of the processes $a_s^{(n)}(\tau)$. To give an idea what we should expect there, let us consider the second moments of the process $a_s^{(1)}(\tau)$. The tools, needed for this analysis, will be systematically used later.

We have

$$\mathbb{E}|a_s^{(1)}(\tau)|^2 = L^{-2d} \int_{-T}^{\tau} dl_1 \int_{-T}^{\tau} dl_2\, e^{\gamma_s(l_1+l_2-2\tau)}$$
$$\times \sum_{1,2} \sum_{1',2'} \mathbb{E}\Big(\delta'^{12}_{3s}\delta'^{1'2'}_{3's}\,(a_1^{(0)}a_2^{(0)}\bar{a}_3^{(0)})(l_1)(\bar{a}_{1'}^{(0)}\bar{a}_{2'}^{(0)}a_{3'}^{(0)})(l_2)e^{i\nu^{-1}(l_1\omega^{12}_{3s}-l_2\omega^{1'2'}_{3's})}\Big).$$

By the Wick theorem

$$\sum_{s_{1'},s_{2'}} \mathbb{E}\Big(\delta'^{12}_{3s}(a_1^{(0)}a_2^{(0)}\bar{a}_3^{(0)})(l)\,\delta'^{1'2'}_{3's}(\bar{a}_{1'}^{(0)}\bar{a}_{2'}^{(0)}a_{3'}^{(0)})(l')\Big) = 2\prod_{j=1}^{3} \mathbb{E}a_j^{(0)}(l)\bar{a}_j^{(0)}(l'),$$

where we used that $\mathbb{E}\bar{a}_j^{(0)}(l)a_j^{(0)}(l') = \mathbb{E}a_j^{(0)}(l)\bar{a}_j^{(0)}(l')$. Then, recalling the notation $\sum\!\!\!\!\!\!\sum$ introduced in Section 1.4, by (2.8) we get

$$\mathbb{E}|a_s^{(1)}(\tau)|^2 = 2\sum\!\!\!\!\!\!\sum_{1,2}\delta'^{12}_{3s}\int_{-T}^{\tau} dl_1 \int_{-T}^{\tau} dl_2\, B_{123}$$
$$\times \prod_{j=1}^{3}\Big(e^{-\gamma_j|l_1-l_2|} - e^{-\gamma_j(2T+l_1+l_2)}\Big)e^{\gamma_s(l_1+l_2-2\tau)+i\nu^{-1}\omega^{12}_{3s}(l_1-l_2)},$$

where we denoted

$$B_{123} = B_1B_2B_3, \qquad B_r = b(s_r)^2/\gamma_{s_r} \quad \text{for } r = 1, 2, 3. \tag{2.9}$$

To simplify the computations, we first assume that $T = \infty$. In this case $\mathbb{E}|a_s^{(1)}(\tau)|^2$ does not depend on τ and equals Σ_s, where

$$\Sigma_s = 2\sum\!\!\!\!\!\!\sum_{1,2}\delta'^{12}_{3s}\int_{-\infty}^{0} dl_1 \int_{-\infty}^{0} dl_2\, B_{123}e^{-|l_1-l_2|(\gamma_1+\gamma_2+\gamma_3)+\gamma_s(l_1+l_2)+i\nu^{-1}\omega^{12}_{3s}(l_1-l_2)}. \tag{2.10}$$

Since for $a, b > 0$, $c \in \mathbb{R}$ we have

$$\int_{-\infty}^{0} dl_1 \int_{-\infty}^{0} dl_2 e^{-a|l_1-l_2|+b(l_1+l_2)+ic(l_1-l_2)} = \frac{a+b}{b\big((a+b)^2+c^2\big)},$$

then

$$\Sigma_s = \frac{2\nu^2}{\gamma_s}\sum\!\!\!\!\!\!\sum_{1,2}\delta'^{12}_{3s}B_{123}\,\frac{\gamma_1+\gamma_2+\gamma_3+\gamma_s}{(\omega^{12}_{3s})^2+\nu^2(\gamma_1+\gamma_2+\gamma_3+\gamma_s)^2}. \tag{2.11}$$

For the reason of equality (2.11), below we call expressions like those in the r.h.s. of (2.10) "sums", meaning that they become sums after the explicit integrating over dl_j.

3. Limiting Behaviour of Second Moments

In this section we study the asymptotic behaviour of the sum Σ_s (see (2.10)=(2.11)) as $L \to \infty$ and $\nu \to 0$, assuming that $L \gg \nu^{-1} \gg 1$. The latter inequality will be specified later.

3.1. Approximation of the sums Σ_s by integrals. Let us naturally extend $\gamma_s = \gamma^0(|s|^2)$ to a function on $\mathbb{R}^d$ and denote

$$\gamma_1 + \gamma_2 + \gamma_3 + \gamma_s =: \Gamma(s_1, s_2, s_3, s), \qquad s_j, s \in \mathbb{R}^d.$$

We also extend B_s, $s \in \mathbb{Z}^d_L$, to the function $B(s) = b(s)^2/\gamma_s$, $s \in \mathbb{R}^d$, and extend B_{123} to $B(s_1, s_2, s_3) := B(s_1)B(s_2)B(s_3)$, $s_j \in \mathbb{R}^d$. Recalling (1.15) we see that

$$\Sigma_s \text{ naturally extends to a function on } \mathbb{R}^d \ni s, \tag{3.1}$$

both in the form (2.10) and (2.11). Doing that we understand the relation $s_1+s_2 = s_3+s$, following from the factor δ'^{12}_{3s} as the rule "substitute $s_3 = s_1 + s_2 - s$". In this case in (2.10) and (2.11) the indices s_1, s_2 belong to $\mathbb{Z}^d_L$, while s and s_3 are vectors in $\mathbb{R}^d$. Here and in similar situations below we will keep denoting the extended functions by the same letters.

Considering (2.11) with $s \in \mathbb{R}^d$ we may replace there the sum $\sum_{1,2}$ by the integral over $\mathbb{R}^d \times \mathbb{R}^d$, thus getting the function I_s, defined as

$$\begin{aligned} I_s &= \frac{\nu^2}{2\gamma_s} \int_{\mathbb{R}^d\times\mathbb{R}^d} ds_1\, ds_2\, \frac{\delta^{12}_{3s}\, B(s_1, s_2, s_3)\Gamma(s_1, s_2, s_3, s)}{((s_1 - s)\cdot(s_2 - s))^2 + (\frac{1}{2}\,\nu\Gamma(s_1, s_2, s_3, s))^2} \\ &=: \nu^2 \int_{\mathbb{R}^d\times\mathbb{R}^d} ds_1\, ds_2\, \frac{\delta^{12}_{3s}\, F_s(s_1, s_2)}{((s_1 - s)\cdot(s_2 - s))^2 + (\frac{1}{2}\,\nu\Gamma(s_1, s_2, s_3, s))^2}. \end{aligned} \tag{3.2}$$

Here δ^{12}_{3s} is the Kronecker delta of the relation $s_1 + s_2 = s_3 + s$, so the factor δ^{12}_{3s} in the integrands means that there $s_3 := s_1 + s_2 - s$. By F_s we denoted the positive Schwartz function on $\mathbb{R}^{3d}$

$$F_s(s_1, s_2) = \Gamma(s_1, s_2, s_3, s)B(s_1, s_2, s_3)/2\gamma_s, \quad s_3 := s_1 + s_2 - s. \tag{3.3}$$

Reverting the transformation, used to get (2.11) from (2.10) we find that

$$\begin{aligned} I_s = 2 \int_{\mathbb{R}^d\times\mathbb{R}^d} ds_1 ds_2 \int_{-T}^{0} dl_1 \int_{-T}^{0} dl_2\, \delta^{12}_{3s}\, B(s_1, s_2, s_3) \\ \times e^{-|l_1-l_2|(\gamma_1+\gamma_2+\gamma_3)+\gamma_s(l_1+l_2)+i\nu^{-1}\omega^{12}_{3s}(l_1-l_2)}. \end{aligned} \tag{3.4}$$

Our goal in this section is to estimate the difference between the sum Σ_s and the integral I_s, while in the next section we will study asymptotical behaviour of the latter as $\nu \to 0$. Moreover, we consider a bit more general sums, needed in the work [8].

We will study sums with the summation index $(s_1, \ldots, s_k) =: z \in \mathbb{Z}^{kd}_L$, $k \geq 1$, and integrals with the integrating variable $z = (s_1, \ldots, s_k) \in \mathbb{R}^{kd}$. For $s \in \mathbb{R}^d$ let us consider the union $D_s = \cup_{j=1}^{p} D_s^j$ of $p \geq 0$ affine subspaces D_s^j in $\mathbb{R}^{kd}$ (if $p = 0$ then D_s is empty),

$$D_s^j = \left\{z = (s_1, \dots s_k) \in \mathbb{R}^{kd} \,:\, c_0^j s + \sum_{i=1}^{k} c_i^j s_i = 0 \in \mathbb{R}^d\right\},$$

where c_i^j are some real numbers. We assume that the k-vectors $(c_1^j, \dots, c_k^j) \neq 0 \in \mathbb{R}^k$ for all j, so the subspaces D_s^j have dimension $d(k-1)$ for every s. Consider the sum/integral

$$S_s = \int_{\mathbb{R}^m} \sum\nolimits_{z \in \mathbb{Z}_L^{kd} \setminus D_s} G_s(z, \theta; \nu)\, d\theta, \qquad m \in \mathbb{N} \cup \{0\}, \quad s \in \mathbb{R}^d,$$

(if $m = 0$, then there is no integration over $\mathbb{R}^m$), where G_s is a measurable function of $(z, \theta, \nu) \in \mathbb{R}^{kd} \times \mathbb{R}^m \times (0, 1/2]$, C^2-smooth in z and satisfying

$$\left|\partial_z^\alpha G_s(z, \theta; \nu)\right| \leq \nu^{-|\alpha|} C^\#(s) C^\#(z) C^\#(\theta) \qquad \text{if} \quad 0 \leq |\alpha| \leq 2, \tag{3.5}$$

for all values of the arguments. Our goal is to compare S_s with the integral

$$J_s = \int_{\mathbb{R}^m} \int_{\mathbb{R}^{kd}} G_s(z, \theta; \nu)\, dz d\theta, \qquad s \in \mathbb{R}^d.$$

Theorem 3.1. *Under the assumption* (3.5)*,*

$$|S_s - J_s| \leq C^\#(s) \nu^{-2} L^{-2}, \qquad s \in \mathbb{R}^d. \tag{3.6}$$

Theorem 3.1 applies to the sum Σ_s where, due to the factor δ', we take the summation over $s_1, s_2 \neq s$. Indeed, in this case $D_s = D_s^1 \cup D_s^2$ with $D_s^j = \{(s_1, s_2) \,:\, s - s_j = 0\}$.

Proof. Denote by $\hat{S}_s$ the sum S_s, where D_s is replaced by the empty set. Firstly we claim that

$$|S_s - \hat{S}_s| \leq C^\#(s) L^{-d} \leq C^\#(s) L^{-2}, \tag{3.7}$$

where in the last inequality we used that $d \geq 2$. Indeed, due to (3.5) with $\alpha = 0$, $|S_s - \hat{S}_s| \leq C^\#(s) L^{-kd} \sum_{z \in D_s \cap \mathbb{Z}_L^{kd}} C^\#(z)$, and the claim follows from the fact that the affine subspaces D_s^j have dimension $d(k-1)$.

Now we consider a mesh in $\mathbb{R}^d$, formed by cubes of size L^{-1}, centred at the points of the lattice $\mathbb{Z}_L^d$. For any $l \in \mathbb{Z}_L^d$ denote by $m(l)$ the cell of the mesh with the centre in l, and consider the measurable mapping

$$\Pi : \mathbb{R}^d \mapsto \mathbb{Z}_L^d, \quad \Pi(x) = \begin{cases} l, & \text{if } x \in \text{ interior of } m(l), \ l \in \mathbb{Z}_L^d, \\ 0, & \text{if } x \in \partial m(l') \text{ for some } l' \in \mathbb{Z}_L^d. \end{cases}$$

Let $\Pi^k = \Pi \times \ldots \times \Pi$, where the product is taken k times. Then

$$\hat{S}_s = \int_{\mathbb{R}^m} \int_{\mathbb{R}^{kd}} G_s(z, \theta; \nu) \circ \left(\Pi^k \times \mathrm{id}\right) dz d\theta.$$

Setting $G_s^\Delta = G_s - G_s \circ (\Pi^k \times \mathrm{id})$, we see that

$$J_s - \hat{S}_s = \int_{\mathbb{R}^m} \int_{\mathbb{R}^{kd}} G_s^\Delta(z, \theta; \nu)\, dz d\theta.$$

For any $\mathbf{l} = (l_1, \dots, l_k) \in \mathbb{Z}_L^{dk}$ let us denote $\mathbf{m(l)} = m(l_1) \times \dots \times m(l_k)$, restrict G_s^Δ to the cell $\mathbf{m(l)}$ and write it as

$$G_s^\Delta(z,\theta) = \partial_z G_s(\mathbf{l},\theta)\cdot(z-\mathbf{l}) + T_s(z,\theta), \quad z \in \mathbf{m(l)}.$$

Here $|T_s(z,\theta)| \le CL^{-2}\sup_{\xi\in\mathbf{m(l)}} |\partial_\xi^2 G_s(\xi,\theta)|$ for $z \in \mathbf{m(l)}$. Then

$$\left|\int_{\mathbf{m(l)}} G_s^\Delta(z,\theta)dz\right| \le L^{-kd}L^{-2}\nu^{-2}C^\#(s)C^\#(\mathbf{l})C^\#(\theta),$$

and accordingly

$$|J_s - \hat{S}_s| \le \nu^{-2}L^{-2}C^\#(s)\int_{\mathbb{R}^m}\int_{\mathbb{R}^{kd}} C^\#(z)C^\#(\theta)\,dzd\theta \le C_1^\#(s)\nu^{-2}L^{-2}. \qquad (3.8)$$

The theorem is proved. □

Remark 3.2. Theorem 3.1 remains true for $d = 1$ if we replace (3.6) by the weaker estimate

$$|S_s - J_s| \le C^\#(s)(\nu^{-2}L^{-2} + L^{-1}), \qquad s \in \mathbb{R}^d.$$

Indeed, in the proof of the theorem relation $d \ge 2$ was used only once, to get the second inequality in (3.7).

3.2. Limiting behaviour of the integrals I_s. Here we study the integral I_s, written in the form (3.2), when $\nu \to 0$. With the notation $\Gamma_s(s_1,s_2) = \frac{1}{2}\Gamma(s_1,s_2,s_1+s_2-s,s)$ the integral takes the form

$$I_s = \nu^2\int_{\mathbb{R}^{2d}} ds_1\,ds_2\ \frac{F_s(s_1,s_2)}{((s_1-s)\cdot(s_2-s))^2 + (\nu\Gamma_s(s_1,s_2))^2}.$$

We study its asymptotical behaviour when $\nu \to 0$ in an abstract setting and do not use the explicit forms of the functions F_s and Γ_s. Instead we assume that they are C^2-smooth and C^3-smooth real functions, correspondingly, satisfying certain restrictions on their behaviour at infinity. Namely, it suffices to assume that

$$|\partial^\alpha_{s_1,s_2,s} F_s(s_1,s_2)| \le C_1^\#(s,s_1,s_2) \qquad \forall\, s_1,s_2,s \in \mathbb{R}^d, \quad \forall\, |\alpha| \le 2, \qquad (3.9)$$

and, for some real numbers $r_1 \ge 0$ and $K > 0$,

$$|\Gamma_s(s_1,s_2)| \ge K^{-1}, \quad |\partial^\alpha_{s_1,s_2,s}\Gamma_s(s_1,s_2)| \le K\langle(s_1,s_2,s)\rangle^{r_1-|\alpha|} \qquad (3.10)$$

for all s_1, s_2, s and all $|\alpha| \le 3$.

Note that function F_s from (3.3) satisfies (3.9), while function $\Gamma_s = \frac{1}{2}\Gamma(s_1,s_2,s_1+s_2-s,s)$ satisfies (3.10) with $r_1 = \max(2r_*, 3)$, as well as the functions $\Gamma_s(s_1,s_2) = \gamma^0(|s_i|^2)$, $i = 1,2,3,4$, where $s_3 = s_1+s_2-s$ and $s_4 = s$.

In the theorem below we denote by ${}^s\Sigma$ the quadric

$${}^s\Sigma = \{(s_1,s_2) : \omega^{12}_{3s} = 0\}, \qquad s_3 = s_1+s_2-s. \qquad (3.11)$$

The latter has a singularity at the point (s,s), and we denote by ${}^s\Sigma_*$ its smooth part ${}^s\Sigma_* = {}^s\Sigma \setminus \{(s,s)\}$.

Theorem 3.3. *As $\nu \to 0$, the integral I_s, $s \in \mathbb{R}^d$, may be written as*

$$I_s = \nu I_s^0 + \nu^2 I_s^\Delta, \tag{3.12}$$

$$I_s^0 = \pi \int_{^s\Sigma_*} \frac{F_s(s_1, s_2)}{\sqrt{|s-s_1|^2 + |s-s_2|^2}\, \Gamma_s(s_1, s_2)}\, ds_1\, ds_2\, |_{^s\Sigma_*}\,. \tag{3.13}$$

Here $ds_1 ds_2|_{^s\Sigma_*}$ is the volume element on $^s\Sigma_*$, induced from $\mathbb{R}^{2d}$, and the integral for I_s^0 converges absolutely. The functions I_s^0 and I_s^Δ satisfy the estimates

$$|I_s^0| \le C^\#(s), \qquad |I_s^\Delta| \le C^\#(s; \aleph_d)\, \nu^{-\aleph_d},$$

where $\aleph_d = 0$ if $d \ge 3$, while for $d = 2$, $\aleph_d$ is any positive number.

The theorem is proved in Section 9. If F_s and Γ_s do not depend on s, then the theorem holds under related (but milder) restrictions on F and Γ, and in that case $|I_s^\Delta| \le C\chi_d(\nu)$, where χ_d is defined in (1.43), see [19].

Theorem 3.3 implies that

$$|I_s| \le \nu C^\#(s). \tag{3.14}$$

In Appendix 12.3 we show that this inequality may be obtained easier and under weaker restrictions on the functions F_s and Γ_s. This observation is important since later in the text we use various generalisations of inequality (3.14) in situations, where analogies of the asymptotic expansion (3.12) are not known for us.

Applying Theorems 3.1 and 3.3 to the sum $\mathbb{E}|a_s^{(1)}(\tau)|^2 = \Sigma_s$ in (2.11) and recalling that Σ_s was extended to a function on $\mathbb{R}^d$, we get

Corollary 3.4. *Assume $T = \infty$. Then for $s \in \mathbb{R}^d$,*

$$\Big| \mathbb{E}|a_s^{(1)}(\tau)|^2 - \nu \frac{\pi}{\gamma_s} \int_{^s\Sigma_*} \frac{B(s_1, s_2, s_1 + s_2 - s)}{\sqrt{|s_1 - s|^2 + |s_2 - s|^2}}\, ds_1 ds_2|_{^s\Sigma_*} \Big|$$
$$\le C^\#(s; \aleph_d)(L^{-2}\nu^{-2} + \nu^{2-\aleph_d}).$$

It is convenient to pass in (3.13) from the variables (s_1, s_2, s) to

$$(x, y, s) = (s_1 - s, s_2 - s, s), \qquad (x, y) =: z. \tag{3.15}$$

Then the quadric $^s\Sigma$ becomes

$$\Sigma = \{z : x \cdot y = 0\} \subset \mathbb{R}_x^d \times \mathbb{R}_y^d = \mathbb{R}_z^{2d}.$$

The locus of Σ is the point $(0, 0)$, and the regular part is $\Sigma_* = \{(x, y) \ne 0 : x \cdot y = 0\}$. Now we write the integral I_s^0 as

$$\int_{\Sigma_*} f(z)|z|^{-1}\, dz\, |_{\Sigma_*}, \tag{3.16}$$

where $f = \pi F_s / \Gamma_s$.

Integrals of the form (3.16) are important for what follows. In next section we discuss some their properties.

3.3. Integrals (3.16). Let us extend the measure $|z|^{-1}\,dz\,|_{\Sigma_*}$ to a measure μ^Σ on Σ, where $\mu^\Sigma(\{0\}) = 0$, and next extend μ^Σ to a Borel measure on $\mathbb{R}^{2d}$, supported by Σ, keeping for the latter the same name. Then the integrals (3.16) may be written as $\int_{\Sigma_*} f(z)\,\mu^\Sigma(dz)$, or as $\int_\Sigma f(z)\,\mu^\Sigma(dz)$, or as $\int_{\mathbb{R}^{2d}} f(z)\,\mu^\Sigma(dz)$.

For any real number r let $\mathcal{C}_r(\mathbb{R}^{2d})$ be the space of continuous complex functions on $\mathbb{R}^{2d}$ with the finite norm

$$|f|_r = \sup_z |f(z)|\langle z\rangle^r. \tag{3.17}$$

Proposition 3.5. *Integral* (3.16) *as a function of f defines a continuous linear functional on the space $\mathcal{C}_r(\mathbb{R}^{2d})$ if $r > 2d-1$.*

The proposition is proved in Section 9.9. To study further the measure μ^Σ we consider the projection

$$\Pi : \Sigma_* \to \mathbb{R}^d_x, \quad z = (x, y) \mapsto x. \tag{3.18}$$

It defines a fibering of Σ_*, where the fiber $\Pi^{-1}0 = \{0\} \times \{\mathbb{R}^d_y \setminus \{0\}\}$ is singular, while for any non-zero x the fiber $\Pi^{-1}x$ equals $\{x\} \times x^\perp$, where $x^\perp$ is the orthogonal complement to x in $\mathbb{R}^d_y$. So the restriction of Π to the domain $\Sigma^x = \Sigma \setminus (\{0\} \times \mathbb{R}^d_y)$ is a smooth euclidean vector bundle over $\mathbb{R}^d \setminus \{0\}$.

Let us us abbreviate to μ the volume element $dz\,|_{\Sigma_*}$. Since the measure $\mu^\Sigma\,|_{\Sigma_*}$ is absolutely continuous with respect to μ, then $\mu^\Sigma(\Sigma_* \setminus \Sigma^x) = \mu(\Sigma_* \setminus \Sigma^x) = 0$; so to calculate the integrals (3.16) it suffices to know the restriction of μ^Σ to Σ^x. The result below shows how to disintegrate the measures $\mu\,|_{\Sigma^x}$ and $\mu^\Sigma\,|_{\Sigma^x}$ with respect to Π, and allows to integrate explicitly over them.

Theorem 3.6. *The measures $\mu\,|_{\Sigma^x}$ and $\mu^\Sigma\,|_{\Sigma^x}$ disintegrate as follows:*

$$\mu(dz) = |x|^{-1} dx |z|\, d_{x^\perp} y, \tag{3.19}$$

$$\mu^\Sigma(dz) = |x|^{-1} dx\, d_{x^\perp} y, \tag{3.20}$$

where $d_{x^\perp} y$ is the volume element on the space $x^\perp$ (the orthogonal complement to x in $\mathbb{R}^d_y$).

We recall that equality (3.19) means that for any continuous function f on Σ^x with compact support

$$\int_{\Sigma^x} f(z)\mu(dz) = \int_{\mathbb{R}^d\setminus\{0\}} |x|^{-1} dx \int_{x^\perp} f(z)\,|z|\,d_{x^\perp} y. \tag{3.21}$$

Proof. It suffices to verify (3.21) for all continuous functions f, supported by a compact set K, for every $K \Subset (\mathbb{R}^d \setminus \{0\}) \times \mathbb{R}^d$. For $x' \in \mathbb{R}^d \setminus \{0\}$ and $m \in \mathbb{N}$ we denote $r' = |x'| > 0$ and set $U_{x'} = \{x : |x - x'| < \frac{1}{2} r'\}$ and $U^m = \{y : |y| < m\}$. Since any K as above can be covered by a finite system of domains $U_{x'} \times U^m$, it suffices to prove (3.21) for any set $U_{x'} \times U^m =: U$ and any $f \in C_0(U)$, where $C_0(U)$ is the space of continuous compactly supported functions on U.

Now we construct explicitly a trivialisation of the linear bundle Π over $U_{x'}$. To do this we fix in $\mathbb{R}^d$ a coordinate system such that

$$x_1 \ge \kappa > 0 \quad \text{for any} \quad x = (x_1, x_2, \ldots, x_d) =: (x_1, \bar{x}) \in U_{x'}. \tag{3.22}$$

We denote $Q = \Pi^{-1}U_{x'} \subset \Sigma^x$ and construct a linear in the second argument $\bar\eta$ coordinate mapping $\Phi : U_{x'} \times \mathbb{R}^{d-1} \to Q$ of the form

$$\Phi(x, \bar\eta) = \big(x, \Phi_x(\bar\eta)\big), \qquad \Phi_x(\bar\eta) = (\varphi(x, \bar\eta), \bar\eta).$$

The function φ should be such that $\Phi_x(\bar\eta) \in x^\perp$. That is, it should satisfy $x \cdot \Phi_x(\bar\eta) = x_1\varphi + \bar x \cdot \bar\eta = 0$. From here we find that $\varphi = -\frac{\bar x \cdot \bar\eta}{x_1}$. Thus obtained mapping Φ_x is linear in $\bar\eta$, and the image of Φ is the set Q. In the coordinates $(x, \bar\eta) \in U_{x'} \times \mathbb{R}^{d-1}$ the hypersurface Σ^x is embedded in $\mathbb{R}^{2d}$ as a graph of the function $\varphi(x, \bar\eta)$. Accordingly in these coordinates the volume element μ on Σ^x reeds $\mu = \bar p(x, \bar\eta)dx\, d\bar\eta$, where[5]

$$\bar p(x, \bar\eta) = \big(1 + |\partial\varphi(x, \bar\eta)|^2\big)^{1/2} = \Big(1 + x_1^{-2}\big((x_1^{-1}\bar x \cdot \bar\eta)^2 + |\bar\eta|^2 + |\bar x|^2\big)\Big)^{1/2}.$$

So

$$\int_{U\subset Q} f(z)\mu(dz) = \int_{U_{x'}} \Big(\int_{\mathbb{R}^{d-1}} f(x, \Phi_x(\bar\eta))\, \bar p(x, \bar\eta)\, d\bar\eta\Big)dx.$$

Passing from the variable $\bar\eta$ to $y = \Phi_x(\bar\eta) \in x^\perp$ we write the measure $\bar p(x, \bar\eta)d\bar\eta$ as $p(z)d_{x^\perp}y$ with

$$p(z) = p(x, y) = \bar p(x, \Phi_x^{-1}y)|\det \Phi_x|^{-1}.$$

Then

$$\int_U f(z)\mu(dz) = \int_{U_{x'}} \Big(\int_{U^m\cap x^\perp} f(z)p(z)d_{x^\perp}y\Big)dx. \tag{3.23}$$

The smooth function p in the integral above is defined on $U \cap \Sigma^x$ in a unique way and does not depend on the trivialisation of Π over $U_{x'}$, used to obtain it. Indeed, if $p_1(z)$ is another smooth function on $U \cap \Sigma^x$ such that (3.23) holds with $p := p_1$, then

$$\int dx \int_{x^\perp} f(z)(p(z) - p_1(z))d_{x^\perp}y = 0 \qquad \forall f \in C_0(U),$$

which obviously implies that $p = p_1$. To establish (3.21) it remains to verify that in (3.23)

$$p(x_*, y_*) = |x_*|^{-1}|z_*| \qquad \forall z_* = (x_*, y_*) \in U \cap \Sigma^x. \tag{3.24}$$

To prove this equality let us choose in $\mathbb{R}^d$ euclidean coordinates with the first basis vector $e_1 = x_*/R_*$, $R_* = |x_*| \ge \frac12 r'$. In these coordinates condition (3.22) holds, $x_* = (R_*, 0)$ and $y_* = (0, \bar\eta_*)$, $\bar\eta_* \in \mathbb{R}^{d-1}$. Then

$$\bar p(x_*, \Phi_{x_*}^{-1}y_*) = \bar p(x_*, \bar\eta_*) = R_*^{-1}(R_*^2 + |\bar\eta_*|^2)^{1/2} = |x_*|^{-1}|z_*|,$$

and (3.24) follows since $\det \Phi_{x_*} = 1$. This proves (3.19). Relation (3.20) follows from (3.19) and the definition of the measure μ^Σ. □

[5] Indeed, denoting $\xi = (x, \bar\eta)$ we see that the first fundamental form I^ξ of Σ^x is given by $I^\xi_{ij} = (\delta_{i,j} + \theta_i\theta_j)$, where $\theta = \partial_\xi\varphi \in \mathbb{R}^{2d-1}$. So for $X \in \mathbb{R}^{2d-1}$,

$$I^\xi(X, X) = \sum X_j^2 + \sum_{i,j} X_i\theta_i X_j\theta_j = \sum X_j^2 + (X \cdot \theta)^2.$$

Choosing in $\mathbb{R}^{2d-1}$ a coordinate system with the first basis vector $\theta/|\theta|$ we find that $I^\xi(X, X) = X_1^2(1 + |\theta|^2) + \sum_{j\ge2} X_j^2$. So $\det I^\xi = 1 + |\partial_\xi\varphi|^2$, which implies the formula for the density $\bar p$.

Considering the projection $(x, y) \mapsto y$ instead of (3.18) we see that the measure μ^{Σ}, restricted to the domain $\Sigma^y = \{(x, y) \in \Sigma : y \neq 0\}$, disintegrates as

$$\mu^{\Sigma} \mid_{\Sigma^y} = dy\, |y|^{-1} d_{y^\perp} x, \quad y \in \mathbb{R}^d \setminus \{0\}. \tag{3.25}$$

Example 3.7. Let us calculate the μ^{Σ}-volumes of the balls $B_R^{2d} = \{|z| \leq R\}$. Denoting by A_n (by V_n) the area of the unit sphere (the volume of the unit ball) in $\mathbb{R}^n$, we have

$$\mu^{\Sigma}(B_R^{2d}) = \int_{|x| \leq R} \frac{1}{|x|} \int_{|y| \leq \sqrt{R^2 - |x|^2}} 1\, dy = A_d V_{d-1} \int_0^R r^{d-2} (R^2 - r^2)^{(d-1)/2} dr.$$

If $d = 3$ this equals $A_3 V_2 \frac{1}{4} R^4 = \pi^2 R^4$. If $d = 2$, this equals $A_2 V_1 \int_0^R \sqrt{R^2 - r^2} dr = \pi^2 R^2$.

4. Wave Kinetic Integrals and Equations

4.1. Wave kinetic integrals. For a complex function $v(s)$, $s \in \mathbb{R}^d$, and $s_1, s_2, s_3, s_4 \in \mathbb{R}^d$ with $s_4 = s$ we denote $v_j = v(s_j)$, $j = 1, 2, 3, 4$. In this section we study the wave kinetic integral $\big(Kv\big)(s)$, defined as follows:

$$(Kv)(s) = 2\pi \int_{{}^s\Sigma_*} \frac{ds_1\, ds_2 \mid_{{}^s\Sigma_*} \delta_{34}^{12}\, v_1 v_2 v_3 v_4}{\sqrt{|s_1 - s|^2 + |s_2 - s|^2}} \left(\frac{1}{v_4} + \frac{1}{v_3} - \frac{1}{v_2} - \frac{1}{v_1} \right), \tag{4.1}$$

where ${}^s\Sigma_*$ is the quadric (3.11) without the singular point (s, s), and $s_4 = s$. Passing to the variable $z = (x, y) = (s_1 - s, s_2 - s)$ we write K as an integral over Σ_* with respect to the measure $\mu^{\Sigma} = |z|^{-1} dz \mid_{\Sigma_*}$ (see (3.16)):

$$\begin{aligned}(Kv)(s) &= 2\pi \int_{\Sigma_*} d\mu^{\Sigma}(z) \delta_{34}^{12} \big(v_1 v_2 v_3 + v_1 v_2 v_4 - v_1 v_3 v_4 - v_2 v_3 v_4\big)(z) \\ &=: K_4(s) + K_3(s) + K_2(s) + K_1(s), \end{aligned} \tag{4.2}$$

where now v_1, v_2, v_3 should be written as functions of z and $s_4 = s$ (note the minus-signs for K_2 and K_1).

Now evoking Theorem 3.6 we will show that the wave kinetic integral K defines 1-smoothing continuous operators in the complex spaces $\mathcal{C}_r(\mathbb{R}^d)$ (see (3.17)) with r not too small.

Theorem 4.1. *If $v \in \mathcal{C}_r(\mathbb{R}^d)$ with $r > d$, then $K(v) \in \mathcal{C}_{r+1}(\mathbb{R}^d)$ and*

$$|K(v)|_{r+1} \leq C_r |v|_r^3, \tag{4.3}$$

where C_r is an absolute constant.

That is, the kinetic integral defines a continuous complex-homogeneous mapping of third degree $K : \mathcal{C}_r(\mathbb{R}^d) \to \mathcal{C}_{r+1}(\mathbb{R}^d)$ if $r > d$. We will derive the theorem's assertion from an auxiliary lemma, stated below and proved later in Section 11.1.

For $l = 1, \ldots, 4$ let $J_l(u^1, u^2, u^3, u^4)$ be the complex poly-linear operator of the third order, which does not depend on u^l and sends the quadruple of complex functions

$(u^1(s), \dots, u^4(s))$, $s \in \mathbb{R}^d$, to the function $U_l(s)$, equal to the integral which defines $K_l(s)$ without the factor 2π (see (4.2)), where we substitute

$$v_1 := u^1(x+s), \quad v_2 := u^2(y+s), \quad v_3 := u^3(x+y+s), \quad v_4 := u^4(s), \qquad (4.4)$$

in accordance to the relation between the coordinates (s_1, s_2, s_3, s_4) and (x, y). That is,

$$J_l(u^1, u^2, u^3, u^4)(s) = \int_{\Sigma_*} d\mu^\Sigma(z) \prod_{\substack{1 \le i \le 4 \\ i \ne l}} u^i, \quad l = 1, \dots, 4, \qquad (4.5)$$

where $u^1 \dots, u^4$ depend on the argument $z = (x, y)$ as in (4.4). Then for $l = 1, \dots, 4$

$$(K_l v)(s) = 2\pi \sigma_l J_l(v, v, v, v)(s), \qquad (4.6)$$

where $\sigma_1, \sigma_2 := -1$ and $\sigma_3, \sigma_4 := 1$. Theorem 4.1 is an easy consequence of the following assertion:

Lemma 4.2. *Let $u^1, \dots, u^4 \in \mathcal{C}_r(\mathbb{R}^d)$ where $r > d$. Then for $s \in \mathbb{R}^d$ and $1 \le l \le 4$ the integral, defining $J_l(u^1, u^2, u^3, u^4)(s) =: J_l(s)$ converges absolutely and satisfies*

$$|J_l|_{r+1} \le C_r \prod_{j \ne l} |u^j|_r. \qquad (4.7)$$

The lemma is proved in Section 11.1. To derive from it Theorem 4.1 we note that if the four functions $J_l(s)$ are proved to be continuous, then the theorem's assertion would follow from (4.6) and (4.7). To establish the continuity of—say—function J_4, we note that for any d-vector ξ, $J_4(s+\xi)$ equals $J_4(u^1_\xi, \dots, u^4_\xi)(s)$, where $u^j_\xi(\eta) = u^j(\eta + \xi)$. So

$$|J_4(s) - J_4(s+\xi)| \le \int_{\Sigma_*} d\mu^\Sigma(z) |u^1 u^2 u^3 - u^1_\xi u^2_\xi u^3_\xi|.$$

If $|\xi| \le 1$, then $|u^j_\xi|_r \le 2^r |u^j|_r$ for each j. The integrand is bounded by the sum of three terms, vanishing with ξ, where the first one is $|(u^1 - u^1_\xi) u^2 u^3|$. Since $|u^j - u^j_\xi|(z) \le |u^j|_r (1 + 2^r) \langle z \rangle^{-r}$ uniformly in $|\xi| \le 1$, then (4.7) and Lebesgue's theorem imply that $|J_4(s) - J_4(s+\xi)| \to 0$ as $\xi \to 0$. So J_4 is a continuous function. For the same reason all other functions J_j are continuous, and we have completed the derivation of Theorem 4.1 from the lemma.

The representation (4.6) together with (4.7) imply an estimate for increments of K_l:

Corollary 4.3. *If $v^1, v^2 \in \mathcal{C}_r(\mathbb{R}^d)$ are such that $|v^1|_r, |v^2|_r \le R$, where $r > d$, then*

$$|K_l(v^1) - K_l(v^2)|_{r+1} \le C_r R^2 |v^1 - v^2|_r, \quad l = 1, \dots, 4. \qquad (4.8)$$

4.2. Wave kinetic equations. Now we pass to the main topic of this section—the wave kinetic equation:

$$\dot{u}(\tau,s) = -\mathcal{L}u + \varepsilon K(u) + f(\tau,s), \qquad s \in \mathbb{R}^d, \tag{4.9}$$
$$u(0,s) = u_0(s), \tag{4.10}$$

where $0 < \varepsilon \le 1$, $K(u)(\tau,s) = K(u(\tau,\cdot))(s)$ is the wave kinetic integral (4.1) and $\mathcal{L}$ is the linear operator

$$(\mathcal{L}u)(s) = 2\gamma_s u(s), \quad s \in \mathbb{R}^d. \tag{4.11}$$

This operator defines in the spaces $\mathcal{C}_r$ semigroups of contractions:

$$\|\exp(-t\mathcal{L})\|_{\mathcal{C}_r(\mathbb{R}^d),\mathcal{C}_r(\mathbb{R}^d)} \le \exp(-2t), \qquad \forall t \ge 0,\ \forall r. \tag{4.12}$$

We denote by X_r the space of continuous curves $u : [0,\infty) \to \mathcal{C}_r(\mathbb{R}^d)$, given the uniform norm $\|u\|_r = \sup_{t\ge 0} |u(t)|_r$.

Theorem 4.4. *If $r > d$, then*

(1) *for any $u_0 \in \mathcal{C}_r(\mathbb{R}^d)$, $f \in X_r$ and any ε the problem* (4.9), (4.10) *has at most one solution in X_r.*

(2) *If*

$$|u_0|_r \le C_*, \ \ \|f\|_r \le C_* \tag{4.13}$$

for some constant C_*, then there exist positive constants $\varepsilon_* = \varepsilon_*(C_*, r)$ and $R = R(C_*, r)$ such that if $0 < \varepsilon \le \varepsilon_*$, then the problem (4.9), (4.10) *has a unique solution $u \in X_r$, and $\|u\|_r \le R$. Moreover, if (u_{01}, f_1) and (u_{02}, f_2) are two sets of initial data, satisfying* (4.13)*, and u^1, u^2 are the corresponding solutions, then*

$$\|u^1 - u^2\|_r \le C(r)(|u_{01} - u_{02}|_r + \|f_1 - f_2\|_r). \tag{4.14}$$

The first assertion is obvious in view of the contraction property (4.12), since the mapping K is locally Lipschitz by Corollary 4.3, cf. the proof of Proposition 4.6 below. The second result follows elementary from Theorem 4.1. Details are given in Section 11.2.

Let $u^0(\tau,s)$ solves (4.9), (4.10) with $\varepsilon := 0$. Writing the solution u, constructed in Theorem 4.4, as $u^0 + \varepsilon v$ we get for v the equation $\dot{v} = -\mathcal{L}v + K(u^0 + \varepsilon v)$, $v(0) = 0$. So $\|v\|_r \le C(C_*, r)$ and Corollary 4.3 implies that $K(u^0 + \varepsilon v) = K(u^0) + O(\varepsilon)$. Accordingly, $v = u^1 + O(\varepsilon)$, where

$$\dot{u}^1 = -\mathcal{L}u^1 + K(u^0), \qquad u^1(0) = 0.$$

We have seen that the solution $u(\tau,s)$, built in Theorem 4.4, may be written as

$$u(\tau,s) = u^0(\tau,s) + \varepsilon u^1(\tau,s) + O(\varepsilon^2), \tag{4.15}$$

where u^0 and u^1 are defined above.

Let us fix any $r_0 > d$ and denote $\varepsilon(f) = \varepsilon_*\big(\|f\|_{r_0}, r_0\big)$.

Corollary 4.5. *Let $u_0 = 0$, $f \in X_r$ $\forall r$ and $0 \le \varepsilon \le \varepsilon(f)$. Then the problem* (4.9), (4.10) *has a unique solution u such that $u \in X_r$ $\forall r$. Its norms $\|u\|_r$ are bounded by constants, depending only on r and $\|f\|_r$.*

Proof. By the theorem the problem has a unique solution $u \in X_r$ with $r = r_0$. By Corollary 4.3, $K(u) \in X_{r+1}$ and we get from the equation (4.9) that also $u \in X_{r+1}$. Iterating this argument we see that $u \in \cap X_r$. The second assertion follows from the theorem. □

Now let us assume that in (4.9) the function f does not depend on τ, so $f(\tau, s) = f_s$, where

$$f \in \mathcal{C}_r, \quad r > d. \tag{4.16}$$

Then for $\varepsilon = 0$ the only steady state of (4.9), i.e. a solution of the equation $-\mathcal{L}u + f = 0$, is $u^0 = \mathcal{L}^{-1} f$; it is asymptotically stable. By the implicit function theorem, there exists ε_r' such that for $0 < \varepsilon \le \varepsilon_r'$ equation (4.9) has a unique steady state u^ε, close to u^0, $-\mathcal{L}u^\varepsilon + \varepsilon K(u^\varepsilon) + f = 0$, and $|u^\varepsilon - u^0|_r \le C_r \varepsilon$.

Proposition 4.6. *Let f and r be as in (4.16) and (4.13) holds for some C_*. Let $\varepsilon \le \varepsilon_*(C_*, r)$ and $u(\tau)$ be a solution of (4.9), (4.10). Then there exists $\varepsilon' = \varepsilon'(C_*, r)$ such that if $0 < \varepsilon \le \min(\varepsilon_*, \varepsilon')$, then*

$$|u(\tau) - u^\varepsilon|_r \le |u_0 - u^\varepsilon|_r e^{-\tau} \quad \textit{for} \quad \tau \ge 0.$$

Proof. Denoting $w(\tau) = u(\tau) - u^\varepsilon$, we find

$$w(\tau) = e^{-\mathcal{L}\tau} w(0) + \varepsilon \int_0^\tau e^{-\mathcal{L}(\tau - t)}\big(K(u(t)) - K(u^\varepsilon)\big)dt.$$

Since $|u^0|_r \le |f|_r \le C_*$, then we may assume that $|u^\varepsilon|_r \le 2C_*$ since ε is sufficiently small. Moreover, by Theorem 4.4, $|u(\tau)|_r \le R(C_*, r)$ for all $\tau \ge 0$. In particular, this implies that $|w(0)|_r \le C(C_*, r)$. Then, in view of Corollary 4.3 and (4.12), we find

$$|w(\tau)|_r \le e^{-2\tau}|w(0)|_r + \varepsilon C_r R^2 \int_0^\tau e^{-2(\tau - t)}|w(t)|_r dt.$$

This relation and Gronwall's lemma, applied to the the function $e^{2t}|w(t)|_r$, imply that

$$|w(\tau)|_r \le |w(0)|_r e^{-\tau(2 - \varepsilon C_r R^2)}.$$

Choosing ε so small that $\varepsilon C_r R(C_*, r)^2 \le 1$ we obtain the desired estimate. □

5. Quasisolutions

In this section we start to study quasisolutions $A(\tau) = A(\tau; \nu, L)$ of eq. (2.1) (where $r_* > 0$), which are second order truncations of series (2.2):

$$A(\tau) = (A_s(\tau), s \in \mathbb{Z}_L^d), \qquad A_s(\tau) = a_s^{(0)}(\tau) + \rho a_s^{(1)}(\tau) + \rho^2 a_s^{(2)}(\tau). \tag{5.1}$$

Our main goal is to examine the energy spectrum of A,

$$n_s(\tau) = n_s(\tau; \nu, L) = \mathbb{E}|A_s(\tau)|^2, \quad s \in \mathbb{Z}_L^d,$$

when L is large and ν is small and to show that $n_s(\tau)$ approximately satisfies the wave kinetic equation (WKE) (1.35). The energy spectrum n_s is a polynomial in ρ of degree four,

$$n_s = n_s^{(0)} + \rho\, n_s^{(1)} + \rho^2 n_s^{(2)} + \rho^3 n_s^{(3)} + \rho^4 n_s^{(4)}, \quad s \in \mathbb{Z}_L^d, \tag{5.2}$$

where $n_s^{(k)}(\tau) = n_s^{(k)}(\tau; \nu, L)$,

$$n_s^{(k)} = \sum_{k_1 + k_2 = k,\, k_1, k_2 \le 2} \mathbb{E} a_s^{(k_1)} \bar{a}_s^{(k_2)}, \quad 0 \le k \le 4.$$

The term $n_s^{(0)}$ is given by (1.28) while by Lemma 2.3,

$$n_s^{(1)} = 2\Re\mathbb{E}\bar{a}_s^{(0)}a_s^{(1)} = 0. \tag{5.3}$$

Writing explicitly $n_s^{(i)}$ with $2 \le i \le 4$, we find that

$$n_s^{(2)} = \mathbb{E}\big(|a_s^{(1)}|^2 + 2\Re\bar{a}_s^{(0)}a_s^{(2)}\big), \quad n_s^{(3)} = 2\Re\mathbb{E}\bar{a}_s^{(1)}a_s^{(2)}, \quad n_s^{(4)} = \mathbb{E}|a_s^{(2)}|^2. \tag{5.4}$$

We decompose

$$n_s = n_s^{\le 2} + n_s^{\ge 3},$$

where

$$n_s^{\le 2} = n_s^{(0)} + \rho^2 n_s^{(2)} \quad \text{and} \quad n_s^{\ge 3} = \rho^3 n_s^{(3)} + \rho^4 n_s^{(4)}.$$

Let us extend $n_s^{(0)}$ to the Schwartz function on $s \in \mathbb{R}^d$ given by (1.28). Iterating formula (2.4) we write $a_s^{(n)}(\tau)$, $n \in \mathbb{N}$, as an iterative integral of polynomials of $a_{s'}^{(0)}(\tau')$, $s' \in \mathbb{Z}_L^d$, $\tau' \le \tau$. Since each $a_{s'}^{(0)}(\tau')$ is a Gaussian random variable (1.21), the Wick formula applies to every term $\mathbb{E}a_s^{(k_1)}\bar{a}_s^{(k_2)}$ and we see that

$$\begin{aligned}&\text{for any } 0 \le k_1, k_2 \le 2 \text{ and any } \nu, L \text{ the second moment } \mathbb{E}a_s^{(k_1)}\bar{a}_s^{(k_2)}\\ &\text{naturally extends to a Schwartz function of } s \in \mathbb{R}^d,\end{aligned} \tag{5.5}$$

cf. (3.1).

The function $n_s^{(0)}$ is of order one, and $n_s^{(1)} \equiv 0$ by (5.3). Consider the function $\mathbb{R}^d \ni s \mapsto n_s^{(2)}(\tau)$. It is made by two terms (see (5.4)). By Corollary 3.4 the first may be written as

$$\mathbb{E}|a_s^{(1)}(\tau; \nu, L)|^2 = \nu\Phi_1(s, \tau) + O(\nu^{2-\aleph_d})C^\#(s; \aleph_d) \quad \text{if} \quad L \ge \nu^{-2},$$

where Φ_1 is a Schwartz function of s, independent from L and ν. This relation was proved for $T = \infty$, but it remains true for any finite T due to a similar argument. Similarly if $L \ge \nu^{-2}$, then $\mathbb{E}\Re\big(\bar{a}_s^{(0)}a_s^{(2)}(\tau; \nu, L)\big) = \nu\Phi_2(s, \tau) + O(\nu^{2-\aleph_d})C^\#(s; \aleph_d)$, so

$$n_s^{(2)}(\tau; \nu, L) = \nu\Phi(s, \tau) + O(\nu^{2-\aleph_d})C^\#(s; \aleph_d) \quad \text{if} \quad L \ge \nu^{-2}, \tag{5.6}$$

where $s \mapsto \Phi(s, \tau)$ is a Schwartz function (we are not giving a complete proof of (5.6) since this relation is used only for motivation and discussion). To understand the limiting behaviour of $n_s^{(3)}$ and $n_s^{(4)}$ we will use another result, proved in [8], where $a_s^{(i)}(\tau)$ denote the terms of the series (2.2). Recall that the function χ_d is defined in (1.43) and $\lceil\cdot\rceil$—in Notation.

Theorem 5.1. *For any $k_1, k_2 \ge 0$ and $k := k_1 + k_2$, we have*

$$\big|\mathbb{E}a_s^{(k_1)}(\tau_1)\bar{a}_s^{(k_2)}(\tau_2)\big| \le C^\#(s; k)\big(\nu^{-2}L^{-2} + \max(\nu^{\lceil k/2\rceil}, \nu^d)\,\chi_d^k(\nu)\big) \tag{5.7}$$

for any $s \in \mathbb{Z}_L^d$, uniformly in $\tau_1, \tau_2 \ge -T$, where $\chi_d^k(\nu) = \chi_d(\nu)$ if $k = 3$ and $\chi_d^k(\nu) \equiv 1$ otherwise. The second moment $\mathbb{E}a_s^{(k_1)}(\tau_1)\bar{a}_s^{(k_2)}(\tau_2)$ extends to a Schwartz function of $s \in \mathbb{R}^d \supset \mathbb{Z}_L^d$ which satisfies the same estimate (5.7).

Note that for $k \le 2$ the theorem's assertion follows from the preceding discussion since $d \ge 2$ (and recall that for $k = 1$ the l.h.s. of (5.7) vanishes by Lemma 2.3). A short direct proof of (5.7) with $k_1 = k_2 = 1$ and $T = \infty$ is given in Addendum 12.3. In Section 8 we discuss a strategy of the theorem's proof for any k, given in [8]. By Theorem 5.1, for any $s \in \mathbb{R}^d$

$$|n_s^{(3)}| \le C^\#(s)\nu^2\chi_d(\nu), \quad |n_s^{(4)}| \le C^\#(s)\nu^2 \quad \text{if} \quad L \ge \nu^{-2}. \tag{5.8}$$

Choosing the parameter ρ in the form (1.10), we will examine the energy spectrum n_s under the limit (1.17). Due to the discussion above, under this limit

$$n_s^{(0)} = B(s)\big(1 - e^{-2\gamma_s(T+\tau)}\big), \quad \rho n_s^{(1)} \equiv 0, \quad \rho^2 n_s^{(2)} \to \varepsilon\Phi(s,\tau), \quad \rho^3 n_s^{(3)} \to 0.$$

Concerning the term $\rho^4 n_s^{(4)}$, our results do not allow to find its asymptotic under the limit, but only imply that $|\rho^4 n_s^{(4)}| \le \varepsilon^2 C^\#(s)$. Accordingly our goal is to examine the energy spectrum n_s under the limit (1.17) and the scaling (1.10) with precision $\varepsilon^2 C^\#(s)$, regarding the constant $\varepsilon \le 1$ (which measures the size of solutions for (1.8) under the proper scaling) as a fixed small parameter.

5.1. Increments of the energy spectra $n_s^{\le 2}$ and the reminder $n_s^{\ge 3}$. We will show that the process $n_s^{\le 2}$ approximately satisfies the WKE (1.35), while the reminder $n_s^{\ge 3}$ is small. This will imply that $n_s(\tau)$ is an approximate solution of the WKE. We always assume (1.10) and that $L \ge 1$, $0 < \nu \le 1/2$.

Now for $u \in \mathcal{C}_r(\mathbb{R}^d)$, $r > d$, and for $\tau \in (0, 1]$, we consider the kinetic integral $K^\tau(u) = \big((K^\tau u)(s), s \in \mathbb{R}^d\big)$:

$$K^\tau(u) = \int_0^\tau e^{-t\mathcal{L}} K(u)\, dt, \tag{5.9}$$

where the operator $K = K_1 + \cdots + K_4$ is defined in Section 4 and the linear operator $\mathcal{L}$ is introduced in (4.11). That is,

$$(K^\tau u)(s) = \frac{1 - e^{-2\gamma_s\tau}}{2\gamma_s}(Ku)(s) = \frac{1 - e^{-2\gamma_s\tau}}{2\gamma_s}\sum_{j=1}^4 (K_j u)(s). \tag{5.10}$$

The result below is the main step in establishing the wave kinetic limit. There, using (5.5), we regard $n_s^{\le 2}(\tau)$ as a Schwartz function of $s \in \mathbb{R}^d$.

Theorem 5.2. *For any* $0 < \tau \le 1$ *we have*

$$n^{\le 2}(\tau) = e^{-\tau\mathcal{L}} n^{\le 2}(0) + 2\int_0^\tau e^{-t\mathcal{L}} b^2\, dt + \varepsilon K^\tau(n^{\le 2}(0)) + \mathcal{R}, \tag{5.11}$$

where $b^2 = \{b^2(s), s \in \mathbb{R}^d\}$ *and the reminder* $\mathcal{R}(\tau, s)$ *satisfies*

$$|\mathcal{R}(\tau)|_r \le C_{r,\aleph_d}\, \varepsilon\big(\nu^{1-\aleph_d} + \nu^{-3}L^{-2} + \tau^2 + \varepsilon\tau\big), \quad \forall r, \tag{5.12}$$

and $\aleph_d$ *is defined as in Theorem 3.3.*

Proof of Theorem 5.2 is given in Section 5.2. Since for any $\tau' \geq -T$ the process $\tau \to (A_s(\tau'+\tau), s \in \mathbb{Z}_L^d)$, is a quasisolution of the problem (2.1), (1.19) with $T := T+\tau'$ and $\beta_s(\tau) := \beta_s(\tau+\tau')$, then the theorem applies to study the increments of $n^{\leq 2}$ from τ' to $\tau'+\tau$, for any $\tau' \geq -T$. That is, (5.11) remains true if we replace $n^{\leq 2}(0)$ by $n^{\leq 2}(\tau')$ and $n^{\leq 2}(\tau)$ by $n^{\leq 2}(\tau'+\tau)$.

We also need an estimate on the reminder $n_s^{\geq 3}$. It is a part of the assertion below, which is an immediate consequence of (5.7)-(5.8) since $d \geq 2$:

Proposition 5.3. *If $L \geq \nu^{-2}$, then for $k = 0, 1, 2, 4$ we have*

$$\left|n_s^{(k)}(\tau)\right| \leq C^{\#}(s)\nu^{\lceil k/2 \rceil}, \tag{5.13}$$

and $\left|n_s^{(3)}(\tau)\right| \leq C^{\#}(s)\nu^2\chi_d(\nu)$, uniformly in $\tau \geq -T$. So

$$\left|n_s^{\leq 2}(\tau)\right| \leq C^{\#}(s) \tag{5.14}$$

and if $\nu(\chi_d(\nu))^{1/2} \leq \varepsilon$, then

$$\left|n_s^{\geq 3}(\tau)\right| \leq C^{\#}(s)\varepsilon^2. \tag{5.15}$$

In accordance with (1.17) we will study the energy spectrum $n_s(\tau)$ under the two limiting regimes: $L \gg \nu^{-1}$ *when* $\nu \to 0$; or *first* $L \to \infty$ *and then* $\nu \to 0$. To treat the latter we will need the following result.

Proposition 5.4. *For any $\nu \in (0, 1/2]$, $\tau_1, \tau_2 \geq -T$, $k_1, k_2 \geq 0$ and $s \in \mathbb{R}^d$ the moment $\mathbb{E}a_s^{(k_1)}(\tau_1; \nu, L)\bar{a}_s^{(k_2)}(\tau_2; \nu, L)$ admits a finite limit as $L \to \infty$. The limit is a Schwartz function of s.*

In particular, this result implies that any $n_s^{(k)}(\tau; \nu, L)$ converges, as $L \to \infty$, to a Schwartz function $n_s^{(k)}(\tau; \nu, \infty)$ of $s \in \mathbb{R}^d$. The proposition can be obtained directly by iterating the Duhamel formula (2.4) and using Theorem 3.1 to replace the corresponding sum by an L-independent integral (cf. Section 3.1, where the moments with $k_1 = k_2 = 1$ are approximated by integrals I_s). We do not give here a proof since it follows from a stronger result in [8], discussed in Section 8 (see there (8.7) and (8.9)).

5.2. *Proof of Theorem 5.2.* It is convenient to decompose the processes $a_s^{(i)}(\tau)$, $\tau \geq 0$, as

$$a_s^{(i)}(\tau) = c_s^{(i)}(\tau) + \Delta a_s^{(i)}(\tau), \qquad i = 0, 1, 2, \quad s \in \mathbb{Z}_L^d, \tag{5.16}$$

where

$$c_s^{(i)}(\tau) = e^{-\gamma_s \tau} a_s^{(i)}(0)$$

and $\Delta a_s^{(i)}$ is defined via (5.16). That is, $c_\cdot(\tau) := c_\cdot^{(0)}(\tau) + \rho c_\cdot^{(1)}(\tau) + \rho^2 c_\cdot^{(2)}(\tau)$ with $\tau \geq 0$ is a solution of the linear equation $(2.1)_{\rho=0, b(s)\equiv 0}$, equal $A_\cdot(0)$ at $\tau = 0$, and $\Delta a_\cdot(\tau)$ equals $A_\cdot(\tau) - c_\cdot(\tau)$. By (5.5), for $0 \leq i, j \leq 2$ the functions

$$\mathbb{E}c_s^{(i)}\bar{c}_s^{(j)}, \quad \mathbb{E}c_s^{(i)}\Delta\bar{a}_s^{(j)}, \quad \mathbb{E}\Delta a_s^{(i)}\Delta\bar{a}_s^{(j)}$$
$$\text{naturally extend to Schwartz functions of } s \in \mathbb{R}^d. \tag{5.17}$$

Due to (5.3) and (5.4),

$$e^{-2\gamma_s\tau} n_s^{\le 2}(0) = \mathbb{E}|c_s^{(0)}(\tau)|^2 + \rho^2 \mathbb{E}\big(|c_s^{(1)}(\tau)|^2 + 2\Re \bar c_s^{(0)}(\tau) c_s^{(2)}(\tau)\big), \quad \forall s \in \mathbb{R}^d.$$

Also,

$$\begin{aligned} n_s^{\le 2}(\tau) - e^{-2\gamma_s\tau} n_s^{\le 2}(0) =& \mathbb{E}\Big(|a_s^{(0)}(\tau)|^2 - |c_s^{(0)}(\tau)|^2 \\ &+ \rho^2\big(|a_s^{(1)}(\tau)|^2 - |c_s^{(1)}(\tau)|^2 + 2\Re\big(a_s^{(2)}\bar a_s^{(0)}(\tau) - c_s^{(2)}\bar c_s^{(0)}(\tau)\big)\Big). \end{aligned} \tag{5.18}$$

Writing explicitly processes $\Delta a_s^{(i)}(\tau)$, $s \in \mathbb{Z}_L^d$, from eq. (2.1), we find

$$\begin{aligned} \Delta a_s^{(0)}(\tau) &= b(s) \int_0^\tau e^{-\gamma_s(\tau - l)}\, d\beta_s(l), \\ \Delta a_s^{(1)}(\tau) &= i \int_0^\tau e^{-\gamma_s(\tau-l)} \mathcal{Y}_s(a^{(0)}, \nu^{-1} l)\, dl, \\ \Delta a_s^{(2)}(\tau) &= i \int_0^\tau e^{-\gamma_s(\tau-l)} 3\mathcal{Y}_s^{sym}(a^{(0)}, a^{(0)}, a^{(1)}; \nu^{-1} l)\, dl, \end{aligned} \tag{5.19}$$

where $a^{(0)} = a^{(0)}(l)$ and we recall that $\mathcal{Y}_s^{sym}$ is defined at the beginnig of Section 2. Let us note that to get explicit formulas for $c_s^{(i)}(\tau)$, $i = 0, 1, 2$, it suffices to replace in the r.h.s.'s of the relations in (5.19) the range of integrating from $[0, \tau]$ to $[-T, 0]$. For example, $c_s^{(0)}(\tau) = e^{-\gamma_s\tau} a_s^{(0)}(0) = b(s) \int_{-T}^0 e^{-\gamma_s(\tau-l)}\, d\beta_s(l)$.

Using that $\mathbb{E} c_s^{(i)} \Delta \bar a_s^{(0)} = \mathbb{E} c_s^{(i)} \mathbb{E} \Delta \bar a_s^{(0)} = 0$ for any i and s, we obtain

$$\mathbb{E}\big(a_s^{(2)} \bar a_s^{(0)}(\tau) - c_s^{(2)} \bar c_s^{(0)}(\tau)\big) = \mathbb{E}\Delta a_s^{(2)} \bar a_s^{(0)}(\tau), \tag{5.20}$$

and from (5.16) we get that

$$\begin{aligned} |a_s^{(1)}(\tau)|^2 - |c_s^{(1)}(\tau)|^2 &= |\Delta a_s^{(1)}(\tau)|^2 + 2\Re \Delta a_s^{(1)} \bar c_s^{(1)}(\tau), \\ \mathbb{E}|a_s^{(0)}(\tau)|^2 - \mathbb{E}|c_s^{(0)}(\tau)|^2 &= \mathbb{E}|\Delta a_s^{(0)}(\tau)|^2. \end{aligned} \tag{5.21}$$

Then, inserting (5.20) and (5.21) into (5.18), we find

$$n_s^{\le 2}(\tau) - e^{-2\gamma_s\tau} n_s^{\le 2}(0) = \mathbb{E}\big|\Delta a_s^{(0)}(\tau)\big|^2 + \rho^2 Q_s(\tau), \quad s \in \mathbb{R}^d,$$

where

$$Q_s(\tau) := \mathbb{E}|\Delta a_s^{(1)}(\tau)|^2 + 2\Re\mathbb{E}\big(\Delta a_s^{(1)}(\tau) \bar c_s^{(1)}(\tau) + \Delta a_s^{(2)}(\tau) \bar a_s^{(0)}(\tau)\big), \tag{5.22}$$

and we recall (5.17). Since

$$\mathbb{E}\big|\Delta a_s^{(0)}(\tau)\big|^2 = \frac{b(s)^2}{\gamma_s}(1 - e^{-2\gamma_s\tau}) = 2\int_0^\tau e^{-t\mathcal{L}} b^2(s)\, dt,$$

then

$$n^{\le 2}(\tau) - e^{-t\mathcal{L}} n^{\le 2}(0) = 2\int_0^\tau e^{-t\mathcal{L}} b^2\, dt + \rho^2 Q(\tau),$$

for $n^{\le 2} = (n_s^{\le 2}, s \in \mathbb{R}^d)$. So the desired formula (5.11) is an immediate consequence of the assertion below:

Proposition 5.5. *We have*

$$\rho^2 Q_s(\tau) = \varepsilon K^\tau(n^{\le 2}(0))(s) + \mathcal{R}(\tau, s), \quad s \in \mathbb{R}^d,$$

where the reminder $\mathcal{R}$ satisfies (5.12).

Proof. Below we abbreviate $n_s^{(0)}(0)$ to $n_s^{(0)}$.

Since $\rho^2\nu = \varepsilon$, then we should show that for any r,

$$\big|Q(\tau) - \nu K^\tau(n^{\le 2}(0))\big|_r \le C_{r,\aleph_d}(\nu^{2-\aleph_d} + \nu^{-2}L^{-2} + \nu\tau^2 + \varepsilon\nu\tau). \tag{5.23}$$

To this end, iterating formula (2.4), we will express the processes $\Delta a_s^{(2)}$, $\Delta a_s^{(1)}$ and $c_k^{(1)}$, entering the definition (5.22) of Q_s, through the processes $a_k^{(0)}$. Then, applying the Wick formula, we will see that Q_s depends on the quasisolution A_s only through the correlations of the form $\mathbb{E}a_k^{(0)}(l)\bar{a}_k^{(0)}(l')$. We will show that the main input to Q_s comes from those terms which depend only on the correlations with the times l, l' satisfying $0 \le l, l' \le \tau$. Then, approximating these correlations by their values at $l = l' = 0$, we will see that $Q_s(\tau)$ is close to a sum $\mathcal{Z}_s$ from Proposition 5.6 below, which depends only on τ and the energy spectrum $n_k^{(0)}(0) = \mathbb{E}|a_k^{(0)}(0)|^2$. Next we will approximate the sum $\mathcal{Z}_s$ by its asymptotic as $\nu \to 0$ and $L \to \infty$, which is given by the kinetic integral $\nu K^\tau(n^{(0)}(0))$. Finally, replacing $\nu K^\tau(n^{(0)}(0))$ with $\nu K^\tau(n^{\le 2}(0))$ and estimating the difference of the two kinetic integrals we will get (5.23).

We will derive (5.23) from the following result:

Proposition 5.6. *We have*

$$\big|Q_s(\tau) - \mathcal{Z}_s\big| \le C^\#(s)(\nu^2\chi_d(\nu) + \nu^{-2}L^{-2} + \nu\tau^2), \quad s \in \mathbb{R}^d, \tag{5.24}$$

where

$$\begin{aligned}\mathcal{Z}_s := 2\sum\nolimits_{1,2}\delta'^{12}_{3s}\big(&\mathcal{Z}^4 n_1^{(0)}n_2^{(0)}n_3^{(0)} + \mathcal{Z}^3 n_1^{(0)}n_2^{(0)}n_s^{(0)} \\ &- 2\mathcal{Z}^1 n_2^{(0)}n_3^{(0)}n_s^{(0)}\big) =: 2S_s^1 + 2S_s^2 - 4S_s^3,\end{aligned} \tag{5.25}$$

and the terms $\mathcal{Z}^i = \mathcal{Z}^i(s_1, s_2, s_3, s, \tau)$ have the following form:

$$\mathcal{Z}^4 = \frac{\big|e^{i\nu^{-1}\omega^{12}_{3s}\tau} - e^{-\gamma_s\tau}\big|^2}{\gamma_s^2 + (\nu^{-1}\omega^{12}_{3s})^2}, \quad \mathcal{Z}^j = 2\,\frac{1 - e^{-\gamma_s\tau}}{\gamma_s}\,\frac{\gamma_j}{\gamma_j^2 + (\nu^{-1}\omega^{12}_{3s})^2} \quad \textit{for } j = 1, 2, 3. \tag{5.26}$$

The proposition is proved in the next section.

To deduce the desired estimate (5.23) from (5.24) we will approximate the sums S_s^j ($j = 1, 2, 3$) in (5.25) by their asymptotic as $\nu \to 0$ and $L \to \infty$:

The sum S_s^1. It has the form (10.2) with $F_s(s_1, s_2) = n_{s_1}^{(0)} n_{s_2}^{(0)} n_{s_1+s_2-s}^{(0)}$. So by (10.4) and Theorem 10.2,

$$\Big|S_s^1 - \nu\frac{1 - e^{-2\gamma_s\tau}}{4\gamma_s}(K_4 n^{(0)})(s)\Big| \le C^\#(s;\aleph_d)(\nu^{2-\aleph_d} + \nu^{-2}L^{-2}) \tag{5.27}$$

for all $s \in \mathbb{R}^d$, where we recall that the integral K_4 is defined in (4.2).

The sum S_s^2. Let us set $F_s(s_1,s_2) := \gamma_3 n_{s_1}^{(0)} n_{s_2}^{(0)} n_s^{(0)}$ and $\Gamma_s(s_1,s_2) = \gamma_3/2$. Then the sum takes the form

$$S_s^2 = \frac{1-e^{-\gamma_s\tau}}{2\gamma_s}\nu^2 \sum\nolimits_{s_1,s_2} \delta'^{12}_{3s} \frac{F_s(s_1,s_2)}{(\nu\Gamma_s)^2 + (\omega^{12}_{3s}/2)^2}.$$

Applying Theorems 3.1 and 3.3 we get

$$\Big| S_s^2 - \nu\frac{1-e^{-\gamma_s\tau}}{2\gamma_s}(K_3 n^{(0)})(s)\Big| \le C^\#(s;\aleph_d)(\nu^{2-\aleph_d} + \nu^{-2}L^{-2}). \tag{5.28}$$

The sum S_s^3. Let us note that the function $\mathcal{Z}^1$ equals to $\mathcal{Z}^3$, if we there replace γ_3 by γ_1. Then, repeating the argument used to analyse the sum S_s^2, we get

$$\Big| - S_s^3 - \nu\frac{1-e^{-\gamma_s\tau}}{2\gamma_s}(K_1 n^{(0)})(s)\Big| \le C^\#(s;\aleph_d)(\nu^{2-\aleph_d} + \nu^{-2}L^{-2}). \tag{5.29}$$

Using the symmetry of the integral K_1 with respect to the transformation $(s_1,s_2)\mapsto (s_2,s_1)$ in its integrand, we see that $K_1(n^{(0)}) = K_2(n^{(0)})$.

In (5.27) the sum S_s^1 is approximated by the integral $\frac{\nu}{2}K_4(n^{(0)})$, multiplied by the factor $\frac{1-e^{-2\gamma_s\tau}}{2\gamma_s}$ which also arises in (5.10), while for the sums S_s^2 and S_s^3 the corresponding factors are slightly different, see (5.28) and (5.29). To handle this difficulty we consider $K_j(n^{(0)}) =: \eta$, where j is 1 or 3. By (1.28) and Lemma 4.2, $|\eta|_r \le C_r$ for all r. Denote

$$\xi_s = \Big(\frac{1-e^{-\gamma_s\tau}}{\gamma_s} - \frac{1-e^{-2\gamma_s\tau}}{2\gamma_s}\Big)\eta_s, \quad s\in\mathbb{R}^d.$$

Since $\frac{1-e^{-\gamma_s\tau}}{\gamma_s} - \frac{1-e^{-2\gamma_s\tau}}{2\gamma_s} = \frac{(1-e^{-\gamma_s\tau})^2}{2\gamma_s} \le \tau^2\gamma_s$, we find

$$|\xi|_r \le \tau^2|\eta|_{r+2r_*} \le \tau^2 C_r.$$

This estimate allows to replace in (5.28) and (5.29) $\nu\frac{1-e^{-\gamma_s\tau}}{2\gamma_s}K_j(n^{(0)})$ by $\nu\frac{1-e^{-2\gamma_s\tau}}{4\gamma_s}$ $K_j(n^{(0)})$ with accuracy $\nu\tau^2$. So recalling the definition of K^τ in (5.10), combining (5.27), (5.28), (5.29) and using that $K_1(n^{(0)}) = K_2(n^{(0)})$, for any r we get

$$\big|\mathcal{Z}_s - \nu(K^\tau n^{(0)})(s)\big|_r \le C_{r,\aleph_d}(\nu^{2-\aleph_d} + \nu^{-2}L^{-2}) + C_r\nu\tau^2. \tag{5.30}$$

Finally, since by Proposition 5.3 together with (5.3) $|n_s^{\le 2}(\tau) - n_s^{(0)}(\tau)| = \rho^2|n_s^{(2)}(\tau)| \le C^\#(s)\varepsilon$, then $|K^\tau(n^{\le 2}(0)) - K^\tau(n^{(0)}(0))|_{r+1} \le C_r\varepsilon\tau$ in view of Corollary 4.3 and (5.9). This inequality jointly with estimates (5.30) and (5.24) imply the desired relation (5.23). □

6. Proof of Proposition 5.6

To conclude the proof of Theorem 5.2, it remains to establish Proposition 5.6. The proof of the proposition is somewhat cumbersome since we have to consider a number of different terms and different cases. During the proof we will often skip the upper index (0), so by writing a and a_s we will mean $a^{(0)}$ and $a_s^{(0)}$.

We recall that Q_s is given by formula (5.22) and first consider the term $\mathbb{E}\Delta a_s^{(2)}(\tau)\bar{a}_s(\tau)$. Inserting the identity $a^{(1)}(l) = c^{(1)}(l) + \Delta a^{(1)}(l)$ into formula (5.19) for $\Delta a_s^{(2)}$, we obtain

$$\mathbb{E}\Delta a_s^{(2)}(\tau)\bar{a}_s(\tau) = N_s + \widetilde{N}_s,$$

where

$$N_s := i\,\mathbb{E}\Big(\bar{a}_s(\tau)\int_0^\tau e^{-\gamma_s(\tau-l)}3\mathcal{Y}_s^{sym}(a, a, \Delta a^{(1)}; \nu^{-1}l)\,dl\Big) \tag{6.1}$$

and

$$\widetilde{N}_s := i\,\mathbb{E}\Big(\bar{a}_s(\tau)\int_0^\tau e^{-\gamma_s(\tau-l)}3\mathcal{Y}_s^{sym}(a, a, c^{(1)}; \nu^{-1}l)\,dl\Big).$$

Thus,

$$Q_s = \mathbb{E}|\Delta a_s^{(1)}(\tau)|^2 + 2\Re N_s + 2\Re\mathbb{E}\Delta a_s^{(1)}(\tau)\bar{c}_s^{(1)}(\tau) + 2\Re\widetilde{N}_s, \quad s \in \mathbb{R}^d.$$

So we have to analyse the four terms in the r.h.s. above.

6.1. The first term of Q_s. First we will show that the term $\mathbb{E}|\Delta a_s^{(1)}(\tau)|^2$ can be approximated by the first sum $2S_1$ from (5.25). Indeed, due to (5.19), we have

$$\mathbb{E}|\Delta a_s^{(1)}(\tau)|^2 = \mathbb{E}\int_0^\tau dl\int_0^\tau dl'\, e^{-\gamma_s(2\tau-l-l')}\mathcal{Y}_s(a, \nu^{-1}l)\overline{\mathcal{Y}_s(a, \nu^{-1}l')}. \tag{6.2}$$

Writing the functions $\mathcal{Y}_s$ explicitly and applying the Wick theorem, in view of (2.8) we find

$$\begin{aligned}\mathbb{E}|\Delta a_s^{(1)}(\tau)|^2 = 2\sum\nolimits_{1,2}\delta'^{12}_{3s}\int_0^\tau dl\int_0^\tau dl'\, e^{-\gamma_s(2\tau-l-l')+i\nu^{-1}\omega^{12}_{3s}(l-l')}\\ \mathbb{E}a_1(l)\bar{a}_1(l')\,\mathbb{E}a_2(l)\bar{a}_2(l')\,\mathbb{E}\bar{a}_3(l)a_3(l').\end{aligned} \tag{6.3}$$

Denoting $g_{s123}(l, l', \tau) = -\gamma_s(2\tau - l - l') + i\nu^{-1}\omega^{12}_{3s}(l - l')$ and computing the time integrals of the exponent above, we obtain

$$\int_0^\tau dl\int_0^\tau dl'\, e^{g_{s123}(l,l',\tau)} = \mathcal{Z}^4,$$

where $\mathcal{Z}^4$ is defined in (5.26). Together with the sum in (6.3) we consider a sum obtained from the latter, without factor 2, by replacing the processes $a_k(l)$, $a_k(l')$ by their value at zero $a_k(0)$:

$$\sum\nolimits_{1,2}\delta'^{12}_{3s}\mathcal{Z}^4\,\mathbb{E}|a_1(0)|^2\mathbb{E}|a_2(0)|^2\mathbb{E}|a_3(0)|^2 = \sum\nolimits_{1,2}\delta'^{12}_{3s}\mathcal{Z}^4\, n_1^{(0)}n_2^{(0)}n_3^{(0)} = S_s^1. \tag{6.4}$$

Our goal in this section is to show that

$$\big|\mathbb{E}|\Delta a_s^{(1)}(\tau)|^2 - 2S_s^1\big| \le C^\#(s)(\nu^{-2}L^{-2} + \tau^2\nu). \tag{6.5}$$

Due to (6.3) and (2.8),

$$\mathbb{E}|\Delta a_s^{(1)}(\tau)|^2 = 2\sum\nolimits_{1,2}\delta'^{12}_{3s}\int_0^\tau dl\int_0^\tau dl'\, e^{g_{s123}(l,l',\tau)}B_{123}h_{123}(l, l'), \tag{6.6}$$

where B_{123} is the function defined in (2.9), extended from $(\mathbb{Z}_L^d)^3$ to $(\mathbb{R}^d)^3$, and $h_{123}(l,l') = \prod_{j=1}^3 \left(e^{-\gamma_j|l-l'|} - e^{-\gamma_j(2T+l+l')}\right)$ (also viewed as a function on $(\mathbb{R}^d)^3$). Let f_{123} denotes the increment of the function h_{123}, that is

$$f_{123}(l,l') = h_{123}(l,l') - h_{123}(0,0).$$

It is straightforward to see that

$$|\partial^\alpha_{s_1,s_2,s_3} f_{123}(l,l')| \le C_{|\alpha|}\langle(s_1,s_2,s_3)\rangle^{m_{|\alpha|}}\,\tau \qquad \forall|\alpha|\ge 0, \tag{6.7}$$

for appropriate constants $C_k, m_k > 0$, uniformly in $0 \le l, l' \le \tau$. Since $n_1^{(0)}n_2^{(0)}n_3^{(0)} = B_{123}h_{123}(0,0)$, from (6.6) and (6.4) we see that $\mathbb{E}|\Delta a_s^{(1)}|^2 - 2S_s^1 = 2S_s^\Delta$ with

$$S_s^\Delta = \sum\nolimits_{1,2} \delta'^{12}_{3s} \int_0^\tau dl \int_0^\tau dl'\, e^{g_{s123}(l,l',\tau)} B_{123} f_{123}(l,l').$$

Now in the expression above we replace $\sum_{1,2}$ by $\int ds_1\,ds_2$ and denote the obtained integral by I_s^Δ,

$$\begin{aligned} I_s^\Delta = \int_0^\tau dl \int_0^\tau dl' \int_{\mathbb{R}^{2d}} ds_1\,ds_2\, e^{g_{s123}(l,l',\tau)} \\ B_{123} f_{123}(l,l'), \qquad s_3 = s_1+s_2-s. \end{aligned} \tag{6.8}$$

Since B_{123} with $s_3 = s - s_1 - s_2$ is a Schwartz function of s, s_1, s_2 and the function f_{123} satisfies (6.7), then Theorem 3.1 applies and we find

$$|S_s^\Delta - I_s^\Delta| \le C^\#(s)\nu^{-2}L^{-2}. \tag{6.9}$$

To establish (6.5) it remains to prove that $|I_s^\Delta| \le C^\#(s)\tau^2\nu$. To do that we divide the external integral (over $dldl'$) in (6.8) to two integrals:

Integral over $|l-l'| \le \nu$. In view of (6.7) with $\alpha = 0$, in this case the internal integral (over ds_1ds_2) is bounded by $C^\#(s)\tau$, so I_s^Δ is bounded by $C_1^\#(s)\tau^2\nu$.

Integral over $|l-l'| \ge \nu$. If $\tau < \nu$ than this integral vanishes, so we assume that $\tau \ge \nu$. Since $\omega_{3s}^{12} = 2(s_1-s)\cdot(s-s_2)$ is a non-degenerate quadratic form (with respect to the variable $z = (s_1-s, s_2-s) \in \mathbb{R}^{2d}$), for any l, l' from the considered domain the integral over $ds_1ds_2 = dz$ in (6.8) has the form (12.2) with $\nu := \nu|l-l'|^{-1} \le 1$ and $n = 2d$. In view of (6.7), estimate (12.4) together with (12.5) implies that this integral is bounded by $C^\#(s)\tau\nu^d|l-l'|^{-d}$. So

$$\begin{aligned} |I_s^\Delta| &\le C^\#(s)\tau\nu^d \int_0^\tau dl \int_0^\tau dl'\, |l-l'|^{-d}\chi_{\{|l-l'|\ge\nu\}} \\ &\le C_1^\#(s)\tau^2\nu^d \int_\nu^\tau x^{-d}\,dx \le C_2^\#(s)\tau^2\nu. \end{aligned}$$

We saw that $|I_s^\Delta| \le C^\#(s)\tau^2\nu$. This relation and (6.9) imply (6.5).

6.2. The second term of Q_s. To study the term $2\Re N_s$ we use the same strategy as above. Namely, expressing in (6.1) the function $3\mathcal{Y}_s^{sym}$ via $\mathcal{Y}_s$, we write N_s as $N_s = N_s^1 + 2N_s^2$, $s \in \mathbb{R}^d$, where

$$N_s^1 = i\,\mathbb{E}\Big(\bar a_s(\tau)\int_0^\tau e^{-\gamma_s(\tau-l)}\mathcal{Y}_s(a,a,\Delta a^{(1)};\nu^{-1}l)\,dl\Big),$$
$$N_s^2 = i\,\mathbb{E}\Big(\bar a_s(\tau)\int_0^\tau e^{-\gamma_s(\tau-l)}\mathcal{Y}_s(\Delta a^{(1)},a,a;\nu^{-1}l)\,dl\Big).$$

We will show that the terms $2\Re N_s^1$ and $4\Re N_s^2$ can be approximated by the second and the third sums from (5.25).

Term N_s^1. Let us start with the term N_s^1: writing explicitly the function $\mathcal{Y}_s$ and then $\Delta\bar a_3^{(1)}$ we get

$$\begin{aligned} N_s^1 &= i\,L^{-d}\sum_{1,2}\delta'^{12}_{3s}\int_0^\tau dl\, e^{-\gamma_s(\tau-l)+i\nu^{-1}\omega^{12}_{3s}l}\,\mathbb{E}\big(a_1(l)a_2(l)\Delta\bar a_3^{(1)}(l)\bar a_s(\tau)\big)\\ &= L^{-2d}\sum_{1,2}\sum_{1',2'}\delta'^{12}_{3s}\delta'^{1'2'}_{3'3}\int_0^\tau dl\int_0^l dl'\, e^{-\gamma_s(\tau-l)+i\nu^{-1}\omega^{12}_{3s}l}e^{-\gamma_3(l-l')-i\nu^{-1}\omega^{1'2'}_{3'3}l'}\\ &\quad\times\mathbb{E}\big(a_1(l)a_2(l)\bar a_{1'}(l')\bar a_{2'}(l')a_{3'}(l')\bar a_s(\tau)\big). \end{aligned} \tag{6.10}$$

By the Wick theorem, we need to take the summation only over $s_{1'}, s_{2'}, s_{3'}$ satisfying $s_{1'} = s_1$, $s_{2'} = s_2$, $s_{3'} = s$ or $s_{1'} = s_2$, $s_{2'} = s_1$, $s_{3'} = s$. Since in the both cases we get $\delta'^{1'2'}_{3'3} = \delta'^{12}_{3s}$ and $\omega^{1'2'}_{3'3} = \omega^{12}_{3s}$, we find

$$\begin{aligned} N_s^1 = 2{\textstyle\sum_{1,2}}\delta'^{12}_{3s}\int_0^\tau dl\int_0^l dl'\, e^{-\gamma_s(\tau-l)-\gamma_3(l-l')+i\nu^{-1}\omega^{12}_{3s}(l-l')}\\ \times\,\mathbb{E}a_1(l)\bar a_1(l')\,\mathbb{E}a_2(l)\bar a_2(l')\,\mathbb{E}a_s(l')\bar a_s(\tau). \end{aligned} \tag{6.11}$$

Replacing in (6.11) the processes $a_k^{(0)}(l)$, $a_k^{(0)}(l')$ and $a_k^{(0)}(\tau)$ by their value at zero, we get instead of N_s^1 the sum

$$\hat N_s^1 := 2{\textstyle\sum_{1,2}}\delta'^{12}_{3s}T_{12}\,n_1^{(0)}n_2^{(0)}n_s^{(0)},$$

where T_{12} denotes the integral of the exponent above

$$T_{12} := \int_0^\tau dl\int_0^l dl'\, e^{-\gamma_s(\tau-l)-\gamma_3(l-l')+i\nu^{-1}\omega^{12}_{3s}(l-l')}. \tag{6.12}$$

Arguing as in Section 6.1 we find

$$|N_s^1 - \hat N_s^1| \le C^\#(s)(\nu^{-2}L^{-2} + \tau^2\nu). \tag{6.13}$$

The term $2\Re\hat N_s^1$ is not equal to the second sum from (5.25) yet. To extract the latter from the former we write the integral over dl' in (6.12) as $\int_0^l = \int_{-\infty}^l - \int_{-\infty}^0$. We get

$$T_{12} = \mathcal{T}_{12} + \mathcal{T}^r_{12}, \tag{6.14}$$

where

$$\mathcal{T}_{12} = e^{-\gamma_s \tau} \int_0^\tau dl\, e^{l(\gamma_s - \gamma_3 + i\nu^{-1}\omega_{3s}^{12})} \int_{-\infty}^{l} dl' e^{l'(\gamma_3 - i\nu^{-1}\omega_{3s}^{12})} = \frac{1 - e^{-\gamma_s \tau}}{(\gamma_3 - i\nu^{-1}\omega_{3s}^{12})\gamma_s}$$

and

$$\mathcal{T}_{12}^r = -\int_0^\tau dl \int_{-\infty}^0 dl'\, e^{-\gamma_s(\tau-l) - \gamma_3(l-l') + i\nu^{-1}\omega_{3s}^{12}(l-l')}. \tag{6.15}$$

Computing the real part of $\mathcal{T}_{12}$, we find that

$$2\Re\mathcal{T}_{12} = 2\,\frac{1 - e^{-\gamma_s \tau}}{\gamma_s}\, \frac{\gamma_3}{\gamma_3^2 + (\nu^{-1}\omega_{3s}^{12})^2} = \mathcal{Z}^3, \tag{6.16}$$

where $\mathcal{Z}^3$ is defined in (5.26). On the other hand,

$$\left|\hat{N}_s^1 - 2\sum\nolimits_{1,2} \delta_{3s}^{\prime 12} \mathcal{T}_{12}\, n_1^{(0)} n_2^{(0)} n_s^{(0)}\right| = 2\left|\sum\nolimits_{1,2} \delta_{3s}^{\prime 12} \mathcal{T}_{12}^r\, n_1^{(0)} n_2^{(0)} n_s^{(0)}\right| \tag{6.17}$$

by (6.15) equals to

$$2\left|\sum\nolimits_{1,2} \delta_{3s}^{\prime 12} \int_0^\tau dl \int_{-\infty}^0 dl'\, e^{l' + i\nu^{-1}\omega_{3s}^{12}(l-l')} F_s(s_1, s_2, l, l', \tau)\right| \tag{6.18}$$

with $F_s = e^{-\gamma_s(\tau-l) - \gamma_3 l + (\gamma_3 - 1)l'} n_1^{(0)} n_2^{(0)} n_s^{(0)}$. Since by (1.28) $n_r^{(0)} = n_r^{(0)}(0)$ is a Schwartz function of $r \in \mathbb{R}^d$ and $\gamma_3 - 1 \geq 0$, then Theorem 10.3 applies and implies that (6.18) is bounded by $C^\#(s)\big(\nu^2 \chi_d(\nu) + \nu^{-2} L^{-2}\big)$, so the l.h.s. of (6.17) also is.

Thus, due to (6.13) and (6.16) we arrive at the relation

$$\left|2\Re N_s^1 - 2\sum\nolimits_{1,2} \delta_{3s}^{\prime 12} \mathcal{Z}^3\, n_1^{(0)} n_2^{(0)} n_s^{(0)}\right| \leq C^\#(s)(\nu^2 \chi_d(\nu) + \nu^{-2} L^{-2} + \tau^2 \nu). \tag{6.19}$$

Term N_s^2. Finally, we study the term N_s^2 by literally repeating the argument we have applied to N_s^1. We find that

$$\begin{aligned} N_s^2 &= i\,L^{-d} \sum_{1,2} \delta_{3s}^{\prime 12} \int_0^\tau dl\, e^{-\gamma_s(\tau-l) + i\nu^{-1}\omega_{3s}^{12} l}\, \mathbb{E}\Delta\big(a_1^{(1)}(l) a_2(l) \bar{a}_3(l) \bar{a}_s(\tau)\big) \\ &= -L^{-2d} \sum_{1,2} \sum_{1',2'} \delta_{3s}^{\prime 12} \delta_{3'1}^{\prime 1'2'} \int_0^\tau dl \int_0^l dl'\, e^{-\gamma_s(\tau-l) + i\nu^{-1}\omega_{3s}^{12} l} e^{-\gamma_1(l-l') + i\nu^{-1}\omega_{3'1}^{1'2'} l'} \\ &\quad \times \mathbb{E}\big(a_{1'}(l') a_{2'}(l') \bar{a}_{3'}(l') a_2(l) \bar{a}_3(l) \bar{a}_s(\tau)\big). \end{aligned} \tag{6.20}$$

By the Wick theorem we should take summation either under the condition $s_{1'} = s_3$, $s_{2'} = s$, $s_{3'} = s_2$ or $s_{1'} = s$, $s_{2'} = s_3$, $s_{3'} = s_2$. Since in both cases $\delta_{3'1}^{\prime 1'2'} = \delta_{3s}^{\prime 12}$ and $\omega_{3'1}^{1'2'} = -\omega_{3s}^{12}$, then

$$\begin{aligned} N_s^2 = &-2\sum\nolimits_{1,2} \delta_{3s}^{\prime 12} \int_0^\tau dl \int_0^l dl'\, e^{-\gamma_s(\tau-l) - \gamma_1(l-l') + i\nu^{-1}\omega_{3s}^{12}(l-l')} \\ &\times \mathbb{E} a_2(l) \bar{a}_2(l')\, \mathbb{E} a_3(l') \bar{a}_3(l)\, \mathbb{E} a_s(l') \bar{a}_s(\tau). \end{aligned} \tag{6.21}$$

We set

$$M_{12} := \int_0^\tau dl \int_0^l dl' \, e^{-\gamma_s(\tau-l)-\gamma_1(l-l')+i\nu^{-1}\omega^{12}_{3s}(l-l')}$$

and note that M_{12} equals to T_{12}, defined in (6.12), if replace γ_3 by γ_1. Then, as in (6.14)–(6.16) we get $M_{12} = \mathcal{M}_{12} + \mathcal{M}^r_{12}$, where $2\Re\mathcal{M}_{12} = \mathcal{Z}^1$. Arguing again as in Section 6.1, we replace in (6.21) the processes $a_k^{(0)}(l)$, $a_k^{(0)}(l')$ and $a_k^{(0)}(\tau)$ by their value at zero $a_k^{(0)}(0)$. Next, using Theorem 10.3 we show that the input to the resulting sum of the term corresponding to $\mathcal{M}^r_{12}$ is small. Finally, similarly to (6.19), we get

$$\Big|4\Re N_s^2 + 4\sum\nolimits_{1,2} \delta'^{12}_{3s} \mathcal{Z}^1 \, n_2^{(0)} n_3^{(0)} n_s^{(0)}\Big| \le C^\#(s)(\nu^2\chi_d(\nu) + \nu^{-2}L^{-2} + \tau^2\nu),$$

where the sign "+" in the l.h.s. is due to the sign "−" in (6.21) (cf. (6.11)).

6.3. The last two terms of Q_s. In Sections 6.1 and 6.2 we have seen that the first two terms of Q_s approximate the sum $\mathcal{Z}_s$. So to get assertion of the proposition it suffices to show that

$$|\mathbb{E}\Delta a_s^{(1)}(\tau)\bar c_s^{(1)}(\tau)|, \ |\widetilde N_s| \le C^\#(s)\big(\nu^2\chi_d(\nu) + \nu^{-2}L^{-2}\big). \tag{6.22}$$

We have

$$\mathbb{E}\Delta a_s^{(1)}\bar c_s^{(1)}(\tau) = \mathbb{E}\int_0^\tau e^{-\gamma_s(\tau-l)}\mathcal{Y}_s(a,\nu^{-1}l)\,dl \int_{-T}^0 e^{-\gamma_s(\tau-l')}\overline{\mathcal{Y}_s(a,\nu^{-1}l')}\,dl'.$$

This expression coincides with (6.2) in which the integral $\int_0^\tau dl'$ is replaced by $\int_{-T}^0 dl'$. Then, $\mathbb{E}\Delta a_s^{(1)}\bar c_s^{(1)}(\tau)$ has the form (6.3) where the same replacement is done. Since the correlations $\mathbb{E}a_j(l)\bar a_j(l')$ are given by (2.8), Theorem 10.3 applies (see a discussion after its formulation) and we get the first inequality from (6.22).

Expressing the function $\mathcal{Y}_s^{sym}$ through $\mathcal{Y}_s$, for the term $\widetilde N_s$ we find $\widetilde N_s = \widetilde N_s^1 + 2\widetilde N_s^2$, where

$$\widetilde N_s^1 = i\mathbb{E}\Big(\bar a_s(\tau)\int_0^\tau e^{-\gamma_s(\tau-l)}\mathcal{Y}_s(a,a,c^{(1)};\nu^{-1}l)\,dl\Big)$$

and

$$\widetilde N_s^2 = i\mathbb{E}\Big(\bar a_s(\tau)\int_0^\tau e^{-\gamma_s(\tau-l)}\mathcal{Y}_s(c^{(1)},a,a;\nu^{-1}l)\,dl\Big).$$

Expressing $c^{(1)}$ through $a^{(0)}$, we see that the terms $\widetilde N_s^1$ and $\widetilde N_s^2$ have the forms (6.10) and (6.20) correspondingly, where the integral $\int_0^l dl'$ is replaced by $\int_{-T}^0 dl'$. Then the second inequality from (6.22) again follows from Theorem 10.3. □

7. Energy Spectra of Quasisolutions and Wave Kinetic Equation

Everywhere in this section in addition to (1.10) we assume that

$$L \geq \nu^{-2-\epsilon}, \qquad \epsilon > 0. \tag{7.1}$$

Let us consider equation (4.9) with $f(s) = 2b(s)^2$ for $\tau \geq -T$:

$$\dot{\mathfrak{z}}(\tau, s) = -\mathcal{L}\mathfrak{z} + \varepsilon K(\mathfrak{z}) + 2b(s)^2, \quad s \in \mathbb{R}^d, \tag{7.2}$$

with the initial condition

$$\mathfrak{z}(-T) = 0. \tag{7.3}$$

Fix any $r_0 > d$. Denoting $\varepsilon(b) = \varepsilon_*\big(2\|b^2(s)\|_{r_0}, r_0\big) > 0$, where ε_* is the constant from Theorem 4.4, we get from Corollary 4.5 that for $0 \leq \varepsilon \leq \varepsilon(b)$ a solution $\mathfrak{z}$ of (7.2), (7.3) exists, is unique and $\mathfrak{z} \in X_r$ for each r, so that

$$|\mathfrak{z}(\tau)|_r \leq C_r, \quad \forall r, \tag{7.4}$$

uniformly in τ. Our goal is to compare $n_s^{\leq 2}(\tau)$ (extended to a function on $\mathbb{R}^d = \{s\}$) with $\mathfrak{z}$ in the spaces $\mathcal{C}_r(\mathbb{R}^d)$, and we recall that, due to (5.14),

$$|n^{\leq 2}(\tau)|_r \leq C_r \quad \forall r, \tag{7.5}$$

uniformly in ν, L, τ. The constants C_r below vary from formula to formula and we often skip the dependence on r writing simply C.

Since both curves $n^{\leq 2}$ and $\mathfrak{z}$ vanish at $-T$, then their difference $w = n^{\leq 2} - \mathfrak{z}$ also does. Let us estimate the increments of w.

Proposition 7.1. *If ρ, ν, L satisfy (1.10), (7.1), $r > d$ and $\varepsilon \leq C_{1r}^{-1} \leq \varepsilon(b)$, then*

$$|w(\tau' + \tau)|_r \leq (1 - \tau/2)|w(\tau')|_r + C_{2r}\tau W \quad \forall \tau' \geq -T,\ 0 \leq \tau \leq 1/2, \tag{7.6}$$

where $W = \varepsilon\big(\tau + \varepsilon + \tau^{-1}\nu^{1-\aleph_d} + \tau^{-1}\nu^{-3}L^{-2}\big)$ and $\aleph_d$ is defined as in Theorem 3.3. The constants C_{1r}, C_{2r} do not depend on τ, τ', T and ν, L, ε, but C_{2r} depends on $\aleph_d$.

Proof. The calculation below does not depend on $\tau' \geq -T$ and to simplify presentation we take $\tau' = 0$. Then

$$\mathfrak{z}(\tau) = e^{-\tau\mathcal{L}}\mathfrak{z}(0) + 2\int_0^\tau e^{-t\mathcal{L}} b^2\, dt + \varepsilon \int_0^\tau e^{-(\tau-t)\mathcal{L}} K(\mathfrak{z}(t))\, dt.$$

From here and (5.11) we have

$$w(\tau) = e^{-\tau\mathcal{L}} w(0) + \varepsilon\Delta + \mathcal{R}, \tag{7.7}$$

where $\Delta = \big[K^\tau(n^{\leq 2}(0)) - \int_0^\tau e^{-(\tau-t)\mathcal{L}} K(\mathfrak{z}(t))\, dt\big]$. Now we will estimate Δ. By (5.9), it is made by the sum of four terms

$$\int_0^\tau e^{-t\mathcal{L}} K_j(n^{\leq 2}(0))\, dt - \int_0^\tau e^{-(\tau-t)\mathcal{L}} K_j(\mathfrak{z}(t))\, dt, \quad 1 \leq j \leq 4.$$

Their estimating is very similar. Consider, for example, the term with $j = 1$ and write it as

$$\int_0^\tau e^{-(\tau-t)\mathcal{L}}\big(K_1(n^{\leq 2}(0)) - K_1(\mathfrak{z}(0))\big)\, dt$$

$$+\int_0^\tau e^{-(\tau-t)\mathcal{L}}\big(K_1(\mathfrak{z}(0)) - K_1(\mathfrak{z}(t))\big)\,dt =: \Delta^1 + \Delta^2.$$

In view of Corollary 4.3 and (4.12), $|\Delta^1|_r \le C\tau|w(0)|_r$, where C depends on the norms $|n^{\le 2}(0)|_r$ and $|\mathfrak{z}(0)|_r$. By (7.5) and (7.4) the two norms are bounded by constants, so the constant $C = C_r$ is absolute. Consider Δ^2. From (7.4) and the equation on $\mathfrak{z}$ we have that $|\dot{\mathfrak{z}}|_r \le C_r$, so $|\mathfrak{z}(0) - \mathfrak{z}(\tau)|_r \le C_r\tau$. Hence, by Corollary 4.3 and (4.12), $|\Delta^2|_r \le C_r\tau^2$.

We have seen that

$$|\Delta|_r \le C'_{1r}\tau(\tau + |w(0)|_r). \tag{7.8}$$

Since $0 \le \tau \le 1/2$, then by (4.12) we have $|e^{-\tau\mathcal{L}}w(0)|_r \le (1-\tau)|w(0)|_r$. So using the bound (5.12) for $|\mathcal{R}|_r$, by (7.7) we get

$$|w(\tau)|_r \le (1-\tau)|w(0)|_r + C'_{1r}\varepsilon\tau|w(0)|_r + C_{2r}\tau W$$

with W as in the proposition. This relation implies the assertion with $C_{1r}^{-1} = \min((2C'_{1r})^{-1}, \varepsilon(b))$. $\square$

Denote $\tilde{w}(\tau) = e^{-\tau\mathcal{L}}w(0)$. Then $\frac{d}{d\tau}\tilde{w}(\tau) = -\mathcal{L}\tilde{w}(\tau)$, so $|\tilde{w}(\tau) - w(0)|_r \le \int_0^\tau |\mathcal{L}\tilde{w}(t)|_r\,dt \le C\tau$, where in the latter inequality we used (7.4) and (7.5). Note also that, due to (5.12), $|\mathcal{R}|_r \le C\tau$ provided that (7.1) holds and $\nu^{1-\aleph_d} \le C_1\tau$. These relations, equality (7.7) and estimate (7.8) on the term Δ imply

Corollary 7.2. *Assume that $\nu^{1-\aleph_d} \le C\tau$ for some constant $C > 0$. Then, under the assumptions of Proposition 7.1,*

$$|w(\tau'+\tau) - w(\tau')|_r \le C_{r,\aleph_d}\tau, \quad \forall\,\tau' \ge -T. \tag{7.9}$$

Now we state and prove the main result of our work. Let us again fix any $r_0 > d$ and set

$$\varepsilon_r = C^{-1}_{1\,\max(r,r_0)},$$

where C_{1r} is the constant from Proposition 7.1. We recall that the energy spectrum $n_s(\tau) = n_s(\tau;\nu,L)$ of a quasisolution $A(\tau)$ naturally extends to a Schwartz function of s, see (5.5).

Theorem 7.3. *Let ν and L satisfy (7.1), let $A(\tau)$ be the quasisolution (5.1), corresponding to $\rho^2 = \varepsilon\nu^{-1}$, let $n_s(\tau) = \mathbb{E}|A_s(\tau)|^2$ be its energy spectrum and $\mathfrak{z}(\tau, s)$ be the (unique) solution of (7.2), (7.3). Then for any r and for $0 < \varepsilon \le \varepsilon_r$ there exists $\nu_\varepsilon(r) > 0$ such that for $\nu \le \nu_\varepsilon(r)$ we have*

$$|n(\tau) - \mathfrak{z}(\tau)|_r \le C_r\varepsilon^2 \quad \forall\,\tau \ge -T, \quad \forall\, r. \tag{7.10}$$

The constant C_r does not depend on τ and T.

Note that $n(\tau)$ has the form (5.2), where $n^{(1)} = 0$ and the first nontrivial term $\rho^2 n^{(2)}$ is of order ε, which is much bigger than the r.h.s. of (7.10) if $\varepsilon \ll 1$. Similarly, $\mathfrak{z}(\tau)$ has the form (4.15), where the first nontrivial term εu^1 is also of the size ε.

Proof. Since $|\cdot|_{r'} \le |\cdot|_r$ for $r' \le r$, then estimate (7.10) for $r < r_0$ follows from (7.10) with $r = r_0$. Assume now that $r \ge r_0$. Since $w(t) = n^{\le 2}(t) - \mathfrak{z}(t)$, then in view of (5.15) it suffices to establish that

$$|w(\tau')|_r \le C\varepsilon^2 \quad \forall\, \tau' \ge -T \tag{7.11}$$

(we assume $\nu_\varepsilon \ll 1$). Let us fix some $0 < \aleph_d \ll 1$ and any time-step τ, satisfying

$$C_0 \varepsilon^{-1} \nu^{1-\aleph_d} \le \tau \le C_0^{-1} \varepsilon^2, \tag{7.12}$$

for a sufficiently large constant $C_0 > 0$. We claim that

$$w_n := |w(-T + n\tau)|_r \le C C_{2r} \varepsilon^2 \quad \forall n \in \mathbb{N} \cup \{0\}, \tag{7.13}$$

where $C_{2r} = C_{2r,\aleph_d}$ is the constant from Proposition 7.1. Indeed, let us fix any $N \in \mathbb{N}$ and let $w_n, n = 0, \dots, N$, attains its maximum at a point n, which we write as $n = k_0 + 1$. If $k_0 + 1 = 0$, then $w_k \equiv 0$. Otherwise, in view of (7.6),

$$w_{k_0} \le w_{k_0+1} \le (1 - \tau/2) w_{k_0} + C_{2r} \tau W;$$

so $w_{k_0} \le 2C_{2r} W$. From here $\max_{0 \le k \le N} |w(-T + n\tau)|_r = w_{k_0+1} \le 3C_{2r} W$ for any N, and (7.13) follows in view of (7.12) and (7.1).

Since for any $\tau' > -T$ we can find an n such that $\tau' \in [-T + n\tau, -T + (n+1)\tau]$, then (7.11) follows from (7.13) and (7.9). □

By Proposition 5.4, when $L \to \infty$ and ν stays fixed, the energy spectrum $n_s(\tau; \nu, L)$ admits a limit $n_s(\tau; \nu, \infty)$, which is a Schwartz function of $s \in \mathbb{R}^d$. Since estimate (7.10) is uniform in ν, L, we immediately get

Corollary 7.4. *For $0 < \varepsilon \le \varepsilon_r$, $\rho = \varepsilon^{1/2}\nu^{-1/2}$ and $\nu \le \nu_\varepsilon(r)$ ($\nu_\varepsilon(r)$ as in Theorem 7.3), the limiting energy spectrum $n_s(\tau; \nu, \infty)$ satisfies estimate (7.10).*

Jointly with Proposition 4.6, Theorem 7.3 implies exponential convergence of $n(\tau)$ to an equilibrium, modulo ε^2:

Corollary 7.5. *For $r > d$ there exists $\varepsilon' = \varepsilon'(r, |b^2|_r)$ such that if $0 < \varepsilon \le \min(\varepsilon', C_{1r}^{-1})$, then eq. (7.2) has a unique steady state $\mathfrak{z}^\varepsilon$ close to $\mathfrak{z}^0 = 2\mathcal{L}^{-1} b^2$, and*

$$|n(\tau) - \mathfrak{z}^\varepsilon|_r \le e^{-\tau - T} |\mathfrak{z}^\varepsilon|_r + C_r \varepsilon^2, \qquad \forall \tau \ge -T.$$

In view of Corollary 2.2, Theorem C from the introduction follows from Theorem 7.3 and Corollary 7.4. Similarly, the asymptotic (1.37) follows from Corollary 7.5.

By (4.15), the solution $\mathfrak{z}$ of (7.2), (7.3) may be written as $\mathfrak{z} = \mathfrak{z}^0 + \varepsilon \mathfrak{z}^1 + O(\varepsilon^2)$, where

$$\begin{aligned} \dot{\mathfrak{z}}^0 &= -\mathcal{L}\mathfrak{z}^0 + 2b(s)^2, \qquad \mathfrak{z}^0(-T) = 0, \\ \dot{\mathfrak{z}}^1 &= -\mathcal{L}\mathfrak{z}^1 + K(\mathfrak{z}^0), \qquad \mathfrak{z}^1(-T) = 0. \end{aligned}$$

From the first equation we see that $\mathfrak{z}^0(s) = n_s^{(0)}$ (see (1.28)), and from the second—that

$$\mathfrak{z}^1(\tau) = \int_{-T}^{\tau} e^{-(\tau - l)\mathcal{L}} K(n_s^{(0)}(l))\, dl.$$

Since, by (5.15), $n(\tau) = n^{(0)}(\tau) + \varepsilon(\nu^{-1} n^{(2)}(\tau)) + O(\varepsilon^2)$, then (7.10) implies that

$$n^{(2)}(\tau) = \nu \int_{-T}^{\tau} e^{-(\tau - l)\mathcal{L}} K(n_s^{(0)}(l))\, dl + O(\nu\varepsilon), \tag{7.14}$$

where $|O(\nu\varepsilon)|_r \le \nu\varepsilon C_r$ for every r. Other way round, now, after the exact form of the operator K in (7.2) is established, the validity of the presentation (7.14) (obtained by some direct calculation), jointly with estimate (5.15) would imply Theorem 7.3.

8. Energy Spectra of Solutions (2.2)

Results of Sections 5 and 7 concerning the quasisolutions suggest a natural question if they extend to higher order truncations of the complete decomposition (2.2) in series in ρ. In this section we discuss the corresponding positive and negative results, obtained in [8]. For a complex number z we denote by z^* either z or $\bar{z}$.

Firstly let us return to Section 2. Iterating the Duhamel integral in the r.h.s. of (2.4) and expressing there iteratively $a^{(n_j)}(l)$ with $1 \le n_j < n$ via integrals (2.4) with $n := n_j$, we will eventually represent each $a_s^{(n)}(\tau)$ as a sum of iterated integrals of the form

$$J_s(\mathcal{T}) = J_s(\tau; n, \mathcal{T}) = \int \dots \int dl_1 \dots dl_n L^{-nd} \sum_{s_1,\dots,s_{3n}} (\dots). \tag{8.1}$$

The zone of integrating in $l = (l_1, \dots, l_n)$ is a convex polyhedron in the cube $[-T, \tau]^n$, and the summation is taken over the vectors $(s_1, \dots, s_{3n}) \in \left(\mathbb{Z}_L^d\right)^{3n}$ subject to certain linear relations which follow from the factor δ'^{12}_{3s} in the definition (2.1) of $\mathcal{Y}_s$. The summand in brackets in (8.1) is a monomial of exponents $e^{-\gamma_{s'}(l_k - l_j)}$, $e^{\pm i\nu^{-1}\omega^{s'_1 s'_2}_{s'_3 s'_4}}$ and the processes $\left(a_{s''}^{(0)}\right)^*(l_r)$, which has degree $2n+1$ with respect to the processes. Each integral $J_s(\tau; n, \mathcal{T})$ corresponds to an oriented rooted tree $\mathcal{T}$ from a class $\Gamma(n)$ of trees with the root at $a_s^{(n)}(\tau)$, with random variables $\left(a_{s''}^{(0)}\right)^*(l_r)$ at its leaves and with vertices labelled by $\left(a_{s'}^{(n')}\right)^*(l'_r)$ $(1 \le n' < n)$. To any vertex $\left(a_{s'}^{(n')}\right)^*(l'_r)$ enters one edge of the tree and three edges outgo from it to the vertices or leaves, corresponding to some three specific terms $\left(a^{(n_1)}\right)^*$, $\left(a^{(n_2)}\right)^*$, $\left(a^{(n_3)}\right)^*$ in the decomposition (2.4) of $\left(a_{s'}^{(n')}\right)^*(l'_r)$. So

$$a_s^{(n)}(\tau) = \sum_{\mathcal{T} \in \Gamma(n)} J_s(\tau; n, \mathcal{T}). \tag{8.2}$$

Now let us consider the formal series

$$N_s(\tau) = n_s^0(\tau) + \rho n_s^1(\tau) + \rho^2 n_s^2(\tau) + \dots \tag{8.3}$$

for the energy spectrum $N_s = \mathbb{E}|a_s|^2$ of a solution $a_s(\tau)$, when the latter is written as the formal series (2.2). There $n_s^0 \sim 1$, $n_s^1 = 0$ and n_s^2 are the same as $n_s^{(0)}$, $n_s^{(1)}$ and $n_s^{(2)}$ in the decomposition (5.2) of quasisolutions, but n_s^3 and n_s^4 are different. This small ambiguity should not cause a problem, and we will see below that the new n_s^3 and n_s^4 still meet the estimates (5.13). Let us consider any $n_s^k(\tau)$. It equals

$$n_s^k(\tau) = \mathbb{E} \sum_{k_1 + k_2 = k} a_s^{(k_1)}(\tau) \bar{a}_s^{(k_2)}(\tau). \tag{8.4}$$

We analyse each expectation $\mathbb{E} a_s^{(k_1)} \bar{a}_s^{(k_2)}$ separately. Due to (8.2),

$$\mathbb{E} a_s^{(k_1)}(\tau) \bar{a}_s^{(k_2)}(\tau) = \sum_{\mathcal{T}_1 \in \Gamma(k_1), \mathcal{T}_2 \in \Gamma(k_2)} \mathbb{E} J_s(\tau; k_1, \mathcal{T}_1) \overline{J_s(\tau; k_2, \mathcal{T}_2)}, \tag{8.5}$$

with

$$\mathbb{E} J_s(\tau; k_1, \mathcal{T}_1) \overline{J_s(\tau; k_2, \mathcal{T}_2)} = \int \dots \int dl_1 \dots dl_k L^{-kd} \sum_{s_1,\dots,s_{3k}} \mathbb{E}(\dots), \tag{8.6}$$

where $k = k_1 + k_2$ and under the expectation sign stands a product of the terms in the brackets in the presentations (8.1) for $J_s(\tau; k_1, \mathcal{T}_1)$ and $\overline{J_s(\tau; k_2, \mathcal{T}_2)}$.

Since $a_s^{(0)}(l)$ are Gaussian random variables whose correlations are given by (2.8), then by the Wick theorem (see [15]) each expectation (8.6) is a finite sum over different Wick pairings between the conjugated and non-conjugated variables $a_{s'}^{(0)}(l'_r)$ and $\bar{a}_{s''}^{(0)}(l''_r)$, labelling the leaves of $\mathcal{T}_1 \cup \overline{\mathcal{T}_2}$. By (2.8), for each Wick pairing, in the sum $\sum_{s_1,\dots,s_{3k}}$ from (8.6) only those summands do not vanish for which indices s' and s'' of the Wick paired variables are equal, $s' = s''$. So we take the sum only over vectors $s_1, \dots, s_{3k}$ satisfying this restriction. The Wick pairings in (8.6) can be parametrised by Feynman diagrams $\mathcal{F}$ from a class $\mathfrak{F}(k_1, k_2)$ of diagrams, obtained from the union $\mathcal{T}_1 \cup \overline{\mathcal{T}}_2$ of various trees $\mathcal{T}_1 \in \Gamma(k_1)$ and $\overline{\mathcal{T}_2} \in \overline{\Gamma(k_2)}$ by all paring of conjugated leaves with non-conjugated leaves in $\mathcal{T}_1 \cup \overline{\mathcal{T}}_2$. We denote the summands, forming the r.h.s. of (8.6), by $I_s(\mathcal{F})$, where $\mathcal{F} \in \mathfrak{F}(k_1, k_2)$, and accordingly write (8.5) as

$$\mathbb{E} a_s^{(k_1)}(\tau) \bar{a}_s^{(k_2)}(\tau) = \sum_{\mathcal{F} \in \mathfrak{F}(k_1, k_2)} I_s(\mathcal{F}), \qquad I_s(\mathcal{F}) = I_s(\tau; k_1, k_2, \mathcal{F}). \tag{8.7}$$

Then, (8.4) takes the form

$$n_s(\tau) = \sum_{k_1+k_2=k} \sum_{\mathcal{F} \in \mathfrak{F}(k_1, k_2)} I_s(\mathcal{F}). \tag{8.8}$$

See [8] for a detailed explanation of the formulas (8.1)–(8.8).

Resolving all the restrictions on the indices $s_1, \dots, s_{3k}$ in (8.6) which follow from the rules, used to construct the trees and the diagrams, we find that among those indices exactly k are independent. Suitably parametrizing them by vectors $z_1, \dots, z_k \in \mathbb{Z}_L^d$ we write the sum in (8.6) as $\sum_{z_1,\dots,z_k \in \mathbb{Z}_L^d}$. Approximating the latter by an integral over $\mathbb{R}^{kd}$ using Theorem 3.1, we get for integrals $I_s(\mathcal{F})$ with $\mathcal{F} \in \mathfrak{F}(k_1, k_2)$ an explicit formula:

$$I_s(\mathcal{F}) = \int_{\mathbb{R}^k} dl \int_{\mathbb{R}^{kd}} dz\, F_{\mathcal{F}}(\tau, s, l, z) e^{i\nu^{-1} \sum_{i,j=1}^k \alpha_{ij}^{\mathcal{F}} (l_i - l_j) z_i \cdot z_j} + O(L^{-2}\nu^{-2}), \tag{8.9}$$

where $k = k_1 + k_2$, $z = (z_1, \dots, z_k)$, the function $F_{\mathcal{F}}$ is smooth in s, z and fast decays in s, z and l, while $\alpha^{\mathcal{F}} = (\alpha_{ij}^{\mathcal{F}})$ is a skew-symmetric matrix without zero lines and rows. Its rank is ≥ 2 and for any k it may be equal to 2. In particular, (8.7) together with (8.9) implies Proposition 5.4 since the integrals in (8.9) are independent from L and are Schwartz functions of s.

Now let us go back to the series (8.3). We know that $n_s^0 \sim 1$, $n_s^1 = 0$ and $n_s^2 \sim C^{\#}(s)\nu$. By a direct (but long) calculation, similar to that in Appendix 12.3, it is possible to verify that $|n_s^3| \leq C^{\#}(s)\nu^2 \chi_d(\nu) \leq C^{\#}(s)\nu^{3/2}$ and $|n_s^4| \leq C^{\#}(s)\nu^2$. This suggests to assume that

$$|n_s^k| \leq C^{\#}(s; k)\nu^{k/2} \quad \text{for any} \quad k. \tag{8.10}$$

If this is the case, then under the "kinetic" scaling $\rho = \nu^{-1/2}\varepsilon^{1/2}$ the series (8.3) becomes a formal series in $\sqrt{\varepsilon}$, uniformly in ν and L, and its truncation of any order $m \geq 2$ in ρ still is $O(\varepsilon^2)$-close to the solution $\mathfrak{z}(\tau)$ of the wave kinetic equation (7.2), (7.3). On the contrary, if this is not the case in the sense that $\|n_\cdot^k\| \geq C_k \nu^{k'}$ with $k' < k/2$ for some k, then (8.3) even is not a formal series, uniformly in ν, L under the kinetic scaling above.

Analysing formula (8.9), we obtain estimate (5.7) of Theorem 5.1 for every integral $I_s(\mathcal{F})$, which implies estimate (5.7) itself (see Theorem 1.2 in [8]). By (8.4), estimate (5.7) implies

Theorem 8.1. *For each k,*

$$|n_s^k| \le C^\#(s;k)\max(\nu^{\lceil k/2\rceil},\nu^d)\,\chi_d^k(\nu), \tag{8.11}$$

provided that L is so big that $C^\#(s;k)L^{-2}\nu^{-2}$, is smaller than the r.h.s. of (8.11).

Below in this section we always assume that L is as big as it is required in the theorem. The theorem implies estimate (8.10) only for $k \le 2d$. In particular, for $k \le 4$ since $d \ge 2$.

To get estimate (8.11), we establish it for each integral $I_s(\mathcal{F})$ separately. Our next result shows that estimate (8.10) can not be obtained by improving (8.11) for every integral $I_s(\mathcal{F})$, since inequality (8.11) is sharp for some of the integrals.

Let $\mathfrak{F}^B(k)$ be the set of Feynman diagrams in $\cup_{k_1+k_2=k}\mathfrak{F}(k_1,k_2)$ for which the matrix $\alpha^{\mathcal{F}}$ from (8.9) satisfies $\alpha_{ij}^{\mathcal{F}} = 0$ if $i \ne p$ and $j \ne p$ (so, only the p-th line and column of the matrix $\alpha^{\mathcal{F}}$ are non-zero). This set is not empty.

Proposition 8.2. *If $k > 2d$, then for any $\mathcal{F} \in \mathfrak{F}^B(k)$ the corresponding integrals $I_s(\mathcal{F})$ satisfy $I_s(\mathcal{F}) \sim C^\#(s;k)\nu^d$.*

Proposition 8.2 shows that the estimate (8.10) can be true for $k > 2d$ only if in the sum (8.8) the large terms cancel each other. And indeed, we observe strong cancellations among the integrals from the set $\mathfrak{F}^B(k)$:

Proposition 8.3. *For any k, $\left|\sum_{\mathcal{F}\in\mathfrak{F}^B(k)} I_s(\mathcal{F})\right| \le C^\#(s;k)\nu^{k-1}$.*

Since for $k \ge 2$ we have $\nu^{k-1} \le \nu^{k/2}$, the estimate in the proposition agrees with (8.10). In the proofs of Propositions 8.2 and 8.3 we use special structure of the set $\mathfrak{F}^B(k)$ and we do not have similar results for a larger set of diagrams, nevertheless we find it plausible that (8.10) is true. Namely, that the decomposition of integrals $I_s(\mathcal{F})$ in asymptotic sum in ν is such that a few main order terms of the decomposition of the sum (8.8) vanish due to a cancellation, so that (8.10) holds (see Problem 1.1). We understand the mechanism of this cancellation, but do not know if it goes till the order $\nu^{k/2}$.

9. Proof of Theorem 3.3

9.1. Vicinity of the point (s,s). Off the quadric ${}^s\Sigma$ the integral I_s is small—of order ν^2—and the main task is to examine it in the vicinity of ${}^s\Sigma$. First we will study I_s near the locus (s,s), and next—near the smooth part ${}^s\Sigma_*$.

Passing to the variables $(x,y,s) = (z,s)$ (see (3.15)) we write F_s and Γ_s as $F_s(z)$ and $\Gamma_s(z)$. The functions still satisfy (3.9) and (3.10) with (s_1,s_2) replaced by (x,y) since the map $(s_1,s_2,s) \mapsto (x,y,s)$ is a linear isomorphism.

Consider the domain

$$K_\delta = \{|x| \le \delta, |y| \le \delta\} \subset \mathbb{R}^d \times \mathbb{R}^d, \qquad 0 < \delta \le 1, \tag{9.1}$$

and the integral $\langle I_s, K_\delta\rangle$ (see *Notation*, formula (1.44)):

$$\langle I_s, K_\delta\rangle = \nu^2 \int_{K_\delta} \frac{F_s(z)\,dz}{(x\cdot y)^2 + (\nu\Gamma_s(z))^2}. \tag{9.2}$$

Obviously, everywhere in K_δ, $|F_s(z)| \le C^\#(s)$ and $\Gamma_s(z) \ge K^{-1}$. Then, since the volume of K_δ is bounded by $C\delta^{2d}$,

$$\left|\langle I_s, K_\delta\rangle\right| \le C^\#(s)\delta^{2d}. \tag{9.3}$$

Next we pass to the global study of integral I_s, written similar to (9.2) as

$$I_s = \nu^2 \int_{\mathbb{R}^{2d}} \frac{F_s(z)\,dz}{(x \cdot y)^2 + \big(\nu\Gamma_s(z)\big)^2}. \tag{9.4}$$

9.2. The manifold Σ_* and its vicinity. We denote by Σ the quadric $\Sigma = \{z = (x, y) : x \cdot y = 0\}$ and $\Sigma_* = \Sigma \setminus (0, 0)$. The set Σ_* is a smooth manifold of dimension $2d - 1$. Let $\xi \in \mathbb{R}^{2d-1}$ be a local coordinate on Σ_* with the coordinate mapping $\xi \mapsto z_\xi = (x_\xi, y_\xi) \in \Sigma_*$. Abusing notation we write $|\xi| = |(x_\xi, y_\xi)|$. The vector $N(\xi) = (y_\xi, x_\xi)$ is a normal to Σ_* at ξ of length $|\xi|$, and

$$\langle N(\xi), (x_\xi, y_\xi)\rangle = 2x_\xi \cdot y_\xi = 0.$$

For any $0 < R_1 < R_2$ we denote

$$\begin{aligned} S^{R_1} &= \{z \in \mathbb{R}^{2d} : |z| = R_1\}, & \Sigma^{R_1} &= \Sigma \cap S^{R_1}, \\ S^{R_2}_{R_1} &= \{z : R_1 < |z| < R_2\}, & \Sigma^{R_2}_{R_1} &= \Sigma \cap S^{R_2}_{R_1}, \end{aligned} \tag{9.5}$$

and for $t > 0$ denote by D_t the dilation operator

$$D_t : \mathbb{R}^{2d} \to \mathbb{R}^{2d}, \quad z \mapsto tz.$$

It preserves Σ_*, and we write $D_t\xi = t\xi$ for $\xi \in \Sigma_*$.

The following lemma, specifying the geometry of Σ_* and its vicinity, is proved in [19]:

Lemma 9.1. (1) *There exists $\theta_0^* > 0$ such that for any $0 < \theta_0 \le \theta_0^*$ a suitable neighbourhood $\Sigma^v = \Sigma^v(\theta_0)$ of Σ_* in $\mathbb{R}^{2d} \setminus \{0\}$ may be uniquely parametrised as*

$$\Sigma^v = \{\pi(\xi, \theta),\ \xi \in \Sigma_*,\ |\theta| < \theta_0\}, \tag{9.6}$$

where

$$\pi(\xi, \theta) = (x_\xi, y_\xi) + \theta N_\xi = (x_\xi, y_\xi) + \theta(y_\xi, x_\xi).$$

(2) *For any vector $\pi := \pi(\xi, \theta) \in \Sigma^v$ its length equals*

$$|\pi| = |\xi|\sqrt{1 + \theta^2}.$$

The distance from π to Σ equals $|\xi||\theta|$, and the shortest path from π to Σ is the segment $S := \{\pi(\xi, t\theta) : 0 \le t \le 1\}$.

(3) *If $z = (x, y) \in S^R$ is such that* $\operatorname{dist}(z, \Sigma) \le \frac{1}{2}R\theta_0$, *then $z = \pi(\xi, \theta) \in \Sigma^v$, where $|\xi| \le R \le |\xi|\sqrt{1 + \theta_0^2}$.*

(4) *If $(x, y) = \pi(\xi, \theta) \in \Sigma^v$, then*

$$x \cdot y = |\xi|^2\theta. \tag{9.7}$$

(5) *If $(x, y) \in S^R \cap (\Sigma^v)^c$, then $|x \cdot y| \ge c|\theta_0|^2 R^2$ for some $c > 0$.*

The coordinates (9.6) are known as the normal coordinates, and their existence follows easily from the implicit function theorem. The assertion 1) is a bit more precise than the general result since it specifies the width of the neighbourhood Σ^v.

Let as fix any $0 < \theta_0 \le \theta_0^*$, and consider the manifold $\Sigma^v(\theta_0)$. Below we provide it with additional structures and during the corresponding constructions decrease θ_0^*, if needed. Consider the set Σ^1 (see (9.5)). It equals

$$\Sigma^1 = \{(x, y) : x \cdot y = 0, |x|^2 + |y|^2 = 1\},$$

and is a smooth compact submanifold of $\mathbb{R}^{2d}$ of codimension 2. Let us cover it by some finite system of charts $\mathcal{N}_1, \dots, \mathcal{N}_{\tilde{n}}$, $\mathcal{N}_j = \{\eta^j = (\eta^j_1, \dots, \eta^j_{2d-2})\}$. Denote by $m(d\eta)$ the volume element on Σ^1, induced from $\mathbb{R}^{2d}$, and denote the coordinate maps as $\mathcal{N}_j \ni \eta^j \to (x_{\eta^j}, y_{\eta^j}) \in \Sigma^1$. We will write points of Σ^1 both as η and (x_η, y_η).

The mapping

$$\Sigma^1 \times \mathbb{R}^+ \to \Sigma_*, \quad (\eta, t) \to D_t(x_\eta, y_\eta) \in \Sigma^t,$$

is 1-1, and is a local diffeomorphism; so this is a global diffeomorphism. Accordingly we can cover Σ_* by the $\tilde{n}$ charts $\mathcal{N}_j \times \mathbb{R}_+$, with the coordinate maps $(\eta^j, t) \mapsto D_t(x_{\eta^j}, y_{\eta^j})$, $\eta^j \in \mathcal{N}_j$, $t > 0$. In these coordinates the volume element on Σ^t is $t^{2d-2} m(d\eta)$. Since $\partial/\partial t \in T_{\eta,t}\Sigma_*$ is a vector of unit length, perpendicular to Σ^t, then the volume element on Σ_* is

$$d_{\Sigma_*} = t^{2d-2} m(d\eta)\, dt. \tag{9.8}$$

The coordinates (η, t, θ) with $\eta \in \mathcal{N}_j, t > 0, |\theta| < \theta_0$, where $1 \le j \le \tilde{n}$, make coordinate systems on Σ^v. Since the vectors $\partial/\partial t$ and $t^{-1}\partial/\partial\theta$ form an orthonormal base of the orthogonal complement in $\mathbb{R}^{2d}$ to $T_{(\eta,t,\theta=0)}\Sigma^t$, then in the domain Σ^v the volume element $dz = dx\, dy$ may be written as

$$dz = t^{2d-1}\mu(\eta, t, \theta) m(d\eta) dt\, d\theta, \quad \mu(\eta, t, 0) = 1. \tag{9.9}$$

The transformation $D_r : (\eta, t, \theta) \mapsto (\eta, rt, \theta)$, $r > 0$, multiplies the form in the l.h.s. by r^{2d}, preserves $d\eta$ and $d\theta$, and multiplies dt by r. Hence, μ does not depend on t, and we have got

Lemma 9.2. *The coordinates (η^j, t, θ), where $\eta^j \in \mathcal{N}_j$, $t > 0$, $|\theta| < \theta_0$, and $1 \le j \le \tilde{n}$, define on Σ^v coordinate systems, jointly covering Σ^v. In these coordinates the dilations D_r, $r > 0$, reed as $D_r : (\eta, t, \theta) \mapsto (\eta, rt, \theta)$, and the volume element has the form (9.9), where μ does not depend on t.*

Consider the mapping

$$\Pi : \Sigma^v \to \Sigma_*, \qquad z = \pi(\xi, \theta) \mapsto \xi.$$

By Lemma 9.1.2, $|z| \le |\Pi(z)| \le 2|z|$.

For $0 < R_1 < R_2$ we will denote

$$\left(\Sigma^v\right)^{R_2}_{R_1} = \Sigma^v \cap \left(\Pi^{-1}\Sigma^{R_2}_{R_1}\right). \tag{9.10}$$

In coordinates (η^j, t, θ) this domain is $\{(\eta^j, t, \theta) : R_1 < t < R_2, |\theta| < \theta_0\}$.

Let us consider functions Γ and F in the variables (η, t, θ). Consider first $\Gamma_s(z)$ with $z = \pi(\xi, \theta) \in \Sigma^v$. Since $\pi(\xi, \theta) = z_\xi + \theta N_\xi$, then $\frac{\partial^k \Gamma_s(z)}{\partial \theta^k} = d_z^k \Gamma_s(z)(N(\xi), \dots, N(\xi))$. As $|N(\xi)| = |\xi| \le |z|$, then by (3.10),

$$|\Gamma_s| \ge K^{-1}, \quad \Big|\frac{\partial^k}{\partial \theta^k}\Gamma_s\Big| \le C_1 K \langle (z,s)\rangle^{r_1-k} |N_\xi|^k \le C_2 K \langle (z,s)\rangle^{r_1}, \tag{9.11}$$

for $k \le 3$. Similar, since F satisfies (3.9), then for $z = \pi(\eta, t, \theta)$ we have

$$F_s(\eta, t, \theta) \in C^2 \quad \text{and} \quad \Big|\frac{\partial^k}{\partial \theta^k} F_s\Big| \le C^\#(t,s), \quad k \le 2. \tag{9.12}$$

9.3. Integral over the complement to a neighbourhood of Σ. On $\mathbb{R}_+ \times \mathbb{R}^d$ let us define the function

$$\Theta = \Theta(t,s) = \langle (t,s)\rangle^{-r_1} \le 1$$

($r_1 \ge 0$ is the exponent in (3.10)), and consider a neighbourhood of Σ:

$$\Sigma^{nbh}(\theta_0) = \{\pi(\xi,\theta) : |\theta| \le \theta_{0m}\} \subset \Sigma^v(\theta_0), \quad \theta_{0m} = \theta_0 \Theta(t,s).$$

Consider the integral over its complement, $\Upsilon_s^m(\theta_0) = \langle |I_s|, \mathbb{R}^{2d} \setminus \Sigma^{nbh}(\theta_0)\rangle$. Using the polar coordinates in $\mathbb{R}^{2d}$, we have

$$\Upsilon_s^m(\theta_0) \le \langle |I_s|, \{|z| \le \nu^b\}\rangle + \nu^2 C_d \int_{\nu^b}^\infty dr\, r^{2d-1} \int_{S^1 \setminus \Sigma^{nbh}(\theta_0)} \frac{|F_s(z)|\, d_{S^1}}{(x\cdot y)^2 + (\nu \Gamma_s(z))^2},$$

where we choose $b = \frac{1}{d}$ and denote d_{S^1} is the normalised Lebesgue measure on S^1. By item 5) of Lemma 9.1 with θ_0 replaced by $\theta_0\Theta(t,s)$, the divisor of the integrand is $\ge C^{-2} r^4 \theta_0^4 \Theta^4$. So the internal integral is bounded by $C^\#(r,s) r^{-4} \langle (t,s)\rangle^{r_1}$. Due to this, (9.12) and (9.3) the r.h.s. is bounded by

$$C_1^\#(s)\nu^{2bd} + \nu^2 C^\#(s) \int_{\nu^b}^\infty C^\#(r) r^{2d-5}\, dr \le C_1^\#(s) \nu^2 \chi_d(\nu)$$

(we recall the notation (1.43)). Accordingly,

$$\Upsilon_s^m(\theta_0) \le C^\#(s) \nu^2 \chi_d(\nu). \tag{9.13}$$

9.4. Integral over the vicinity of Σ. Let Σ_s^{vm} be a neighbourhood of Σ_* such that

$$\Sigma^{nbh}(\tfrac12 \theta_0) \subset \Sigma_s^{vm} \subset \Sigma^{nbh}(2\theta_0). \tag{9.14}$$

Then in view of (9.13)

$$|\langle I_s, \Sigma^{nbh}(2\theta_0)\rangle - \langle I_s, \Sigma_s^{vm}\rangle| \le \langle |I_s|, \mathbb{R}^{2d} \setminus \Sigma^{nbh}(\tfrac12\theta_0)\rangle = \Upsilon_s^m\big(\tfrac{\theta_0}{2}\big) \le C_1^\#(s)\nu^2 \chi_d(\nu).$$

In view of this estimate and (9.13),

$$\Big| \langle I_s, \mathbb{R}^{2d} \rangle - \langle I_s, \Sigma_s^{vm} \rangle \Big| \le C_1^\#(s) \nu^2 \chi_d(\nu). \tag{9.15}$$

So to prove the theorem it suffices to calculate integrals $\langle I_s, \Sigma_s^{vm} \rangle$ over any domains Σ_s^{vm} as in (9.14). Below we will do this for domains Σ_s^{vm} of the form

$$\Sigma_s^{vm} = \{(\eta, t, \theta) : -\theta^-(t,s) < \theta < \theta^+(t,s)\},$$

with suitably defined functions $\theta_0^\pm(t,s) \in [\frac{1}{2}\theta_{0m}, 2\theta_{0m}]$.

We estimate the integrals $\langle I_s, \Sigma_s^{vm}(\theta_0) \rangle$ in 4 steps.

9.5. Step 1: disintegration of I_s. For any $0 < R_1 < R_2$ we define domains $(\Sigma^{nbh}(\theta_0))_{R_1}^{R_2} = \Sigma^{nbh}(\theta_0) \cap (\Pi^{-1}\Sigma_{R_1}^{R_2})$ and using (9.9) write integral $\langle I_s, \big(\Sigma^{nbh}(\theta_0)\big)_{R_1}^{R_2} \rangle$ as

$$\begin{aligned} &\nu^2 \int_{\Sigma^1} m(d\eta) \int_{R_1}^{R_2} dt\, t^{2d-1} \int_{-\theta_{0m}}^{\theta_{0m}} d\theta \, \frac{F_s(\eta, t, \theta)\mu(\eta, \theta)}{(x \cdot y)^2 + (\nu \Gamma_s(\eta, t, \theta))^2} \\ &= \nu^2 \int_{\Sigma^1} m(d\eta) \int_{R_1}^{R_2} J_s(\eta, t) t^{2d-1} dt \end{aligned}$$

(cf. (9.4)), where by (9.7)

$$J_s(\eta, t) = \int_{-\theta_{0m}}^{\theta_{0m}} d\theta \, \frac{F_s(\eta, t, \theta)\mu(\eta, \theta)}{t^4\theta^2 + (\nu \Gamma_s(\eta, t, \theta))^2}.$$

To study J_s let us write Γ_s as

$$\Gamma_s(\eta, t, \theta) = h_{\eta,t,s}(\theta) \Gamma_s(\eta, t, 0).$$

The function $h_{\eta,t,s}(\theta) =: h(\theta)$ is C^3-smooth and in view of (9.11) satisfies

$$h(0) = 1, \quad \Big| \frac{\partial^k}{\partial \theta^k} h(\theta) \Big| \le C\Theta^{-1} \quad \forall\, 1 \le k \le 3, \tag{9.16}$$

for all η, t, θ, s. Denoting $\varepsilon = \nu t^{-2} \Gamma_s(\eta, t, 0)$, we write J_s as

$$J_s = t^{-4} \int_{-\theta_{0m}}^{\theta_{0m}} \frac{F_s(\eta, t, \theta)\mu(\eta, \theta) h^{-2}(\theta)\, d\theta}{\theta^2 h^{-2}(\theta) + \varepsilon^2}. \tag{9.17}$$

9.6. Step 2: definition of domains Σ_s^{vm}. On the segment $\mathcal{I} = [-\theta_{0m}, \theta_{0m}] \subset [-\theta_0, \theta_0]$ consider the function h and the function

$$f = f_{\eta,t,s} : \mathcal{I} \ni \theta \mapsto \zeta = \theta/h(\theta),$$

By (9.16), $\frac{2}{3} \le h \le \frac{3}{2}$ on $\mathcal{I}$, if θ_0 is small enough. From here and (9.16), for $\theta \in \mathcal{I}$ we have $\frac{1}{2} \le f'(\theta) \le 2$ (if θ_0 is small), and

$$\left| \frac{\partial^k f}{\partial \theta^k} \right| \le C\Theta^{-(k-1)}(t,s) \quad \forall\, 1 \le k \le 3. \tag{9.18}$$

So f defines a C^3-diffeomorphism of $\mathcal{I}$ on $f(\mathcal{I})$, $\frac{1}{2}\mathcal{I} \subset f(\mathcal{I}) \subset 2\mathcal{I}$, such that $f'(0) = 1$ and f^{-1} also satisfies estimates (9.18) (with a modified constant C).

Let us set $\zeta^+ = f(\theta_0\Theta)$ and $\zeta^- = -f(-\theta_0\Theta)$. Then $2^{-1}\theta_0\Theta \le \zeta^\pm \le 2\theta_0\Theta$, and passing in integral (9.17) from variable θ to $\zeta = f(\theta)$ we find that

$$J_s = t^{-4} \int_{-\zeta^-}^{\zeta^+} \frac{F_s(\eta,t,\theta)\mu(\eta,\theta)h^{-2}(\theta)(f^{-1}(\zeta))'\, d\zeta}{\zeta^2 + \varepsilon^2}.$$

Denoting the nominator of the integrand as $\Phi_s(\eta, t, \zeta)$ and using (9.12) we see that this is a C^2-smooth function, satisfying

$$|\frac{\partial^k}{\partial \theta^k}\Phi_s| \le C^{\#}(t,s) \quad \forall\, 0 \le k \le 2.$$

Moreover, since $h(0) = 1$ and $(f^{-1}(0))' = f'(0) = 1$, then in view of (9.9) we have that

$$\Phi_s(\eta,t,0) = F_s(\eta,t,0). \tag{9.19}$$

Now denote

$$\zeta_0 = \min(\zeta^+, \zeta^-) \in [\tfrac{1}{2}\theta_0\Theta, 2\theta_0\Theta], \tag{9.20}$$

set

$$\theta_0^-(\eta,t,s) = -f_{\eta,t,s}^{-1}(-\zeta_0), \quad \theta_0^+(\eta,t,s) = f_{\eta,t,s}^{-1}(\zeta_0),$$

and use these functions $\theta_0^\pm$ to define the domains Σ_s^{vm}. Then

$$\langle I_s, \left(\Sigma_s^{vm}\right)_{R_1}^{R_2}\rangle = \nu^2 \int_{\Sigma^1} m(d\eta) \int_{R_1}^{R_2} J_s^m(\eta,t)t^{2d-1}dt, \tag{9.21}$$

where

$$\begin{aligned} J_s^m(\eta,t) &= \int_{-\theta_0^-}^{\theta_0^+} d\theta\, \frac{F_s(\eta,t,\theta)\mu(\eta,\theta)}{t^4\theta^2 + (\nu\Gamma_s(\eta,t,\theta))^2} \\ &= t^{-4} \int_{-\zeta_0}^{\zeta_0} \frac{F_s(\eta,t,\theta)\mu(\eta,\theta)h^{-2}(\theta)(f^{-1}(\zeta))'\, d\zeta}{\zeta^2 + \varepsilon^2}. \end{aligned}$$

9.7. Step 3: calculating integrals $\langle I_s, \Sigma_s^{vm} \rangle$. Clearly,

$$|J_s^m| \le \frac{C^\#(t,s)}{t^4} \int_{-\zeta_0}^{\zeta_0} \frac{d\zeta}{\zeta^2+\varepsilon^2} \le \frac{C_1^\#(t,s)\theta_0\Theta}{\nu^2\Gamma_s^2}(\eta,t,0) \le \nu^{-2} C^\#(t,s)$$

(here and below the constants may depend on θ_0). So if $\alpha > 0$, then

$$\langle |I_s|, \left(\Sigma_s^{vm}\right)_{\nu^{-\alpha}}^\infty \rangle \le \nu^{-2}\nu^2 C^\#(s) \int_{\nu^{-\alpha}}^\infty t^{2d-1} C^\#(t)dt \le C_\alpha^\#(s)\nu^2. \tag{9.22}$$

Similar, if $|s| \ge \nu^{-\beta}$, $\beta > 0$, then

$$\langle |I_s|, \left(\Sigma_s^{vm}\right)_0^\infty \rangle \le C^\#(s) \int_0^\infty t^{2d-1} C^\#(t)dt \le C_\beta^\#(s)\nu^2. \tag{9.23}$$

To estimate $J_s^m(\eta,t)$ for t and s not very large, let us consider the integral J_s^{0m}, obtained from J_s^m by frosening Φ_s at $\zeta = 0$:

$$J_s^{0m} = t^{-4} \int_{-\zeta_0}^{\zeta_0} \frac{\Phi_s(\eta,t,0)\,d\zeta}{\zeta^2+\varepsilon^2} = 2t^{-4} F_s(\eta,t,0)\varepsilon^{-1} \tan^{-1}\frac{\zeta_0}{\varepsilon}$$

(we use (9.19)). As $0 < \frac{\pi}{2} - \tan^{-1}\frac{1}{\bar\varepsilon} < \bar\varepsilon$ for $0 < \bar\varepsilon \le \frac12$, then

$$0 < \pi\nu^{-1}t^{-2}(F_s/\Gamma_s)\,|_{\theta=0} - J_s^{0m} < \frac{2}{\zeta_0} t^{-4} F_s(\eta,t,0), \tag{9.24}$$

if $\nu t^{-2}\Gamma_s(\eta,t,0) \le \frac12\zeta_0$, which holds if

$$\nu \le C t^2 \Theta^2(t,s), \tag{9.25}$$

in view of (9.11) and (9.20). Now we estimate the difference between J_s^m and J_s^{0m}. We have:

$$(J_s^m - J_s^{0m})(\eta,t) = t^{-4} \int_{-\zeta_0}^{\zeta_0} \frac{\Phi_s(\eta,t,\zeta) - \Phi_s(\eta,t,0)}{\zeta^2+\varepsilon^2} d\zeta.$$

Since Φ_s is a smooth function and its C^2-norm is bounded by $C^\#(t,s)$, then

$$\Phi_s(\eta,t,\zeta) - \Phi_s(\eta,t,0) = A_s(\eta,t)\zeta + B_s(\eta,t,\zeta)\zeta^2,$$

where $|A_s|, |B_s| \le C^\#(t,s)$. From here

$$|J_s^m - J_s^{0m}| \le C_1^\#(t,s)t^{-4} \int_0^{\zeta_0} \frac{\zeta^2\,d\zeta}{\zeta^2+\varepsilon^2} \le C_2^\#(t,s)t^{-4}.$$

Denote

$$\mathcal{J}_s(\eta,t) = \pi t^{-2}(F_s/\Gamma_s)(\eta,t,0). \tag{9.26}$$

Then, jointly with (9.24), the last estimate tell us that

$$|J_s^m - \nu^{-1}\mathcal{J}_s(\eta,t)| \le C^\#(t,s)t^{-4} \quad \text{if (9.25) holds.} \tag{9.27}$$

9.8. Step 4: end of the proof. Let us write $\langle I_s, \Sigma_s^{vm}\rangle$ as

$$\langle I_s, \big(\Sigma_s^{vm}\big)_0^{\nu^b}\rangle + \langle I_s, \big(\Sigma_s^{vm}\big)_{\nu^b}^{\nu^{-a}}\rangle + \langle I_s, \big(\Sigma_s^{vm}\big)_{\nu^{-a}}^{\infty}\rangle.$$

We will analyse the three terms, choosing properly positive constants a, b.
1. By (9.22),

$$\Big|\langle I_s, (\Sigma_s^{vm})_{\nu^{-a}}^{\infty}\rangle\Big| \le C_a^{\#}(s)\nu^2.$$

Similar,

$$\Big|\nu\int_{\Sigma^1} m(d\eta)\int_{\nu^{-a}}^{\infty} dt\, t^{2d-1}\mathcal{J}_s(\eta,t)\Big| \le C^{\#}(s)\nu\int_{\nu^{-a}}^{\infty} t^{2d-3}C^{\#}(t)\,dt \le C_a^{\#}(s)\nu^2.$$

2. Since $(\Sigma_s^{vm})_0^{\delta} \subset K_{2\delta}$, estimate (9.3) with $\delta = \nu^b$ implies

$$\langle |I_s|, (\Sigma_s^{vm})_0^{\nu^b}\rangle \le \langle |I_s|, K_{2\nu^b}\rangle \le C^{\#}(s)\nu^{2bd}.$$

Besides,

$$\nu\int_{\Sigma^1} m(d\eta)\int_0^{\nu^b} dt\, t^{2d-1}\mathcal{J}_s(\eta,t) \le \nu C^{\#}(s)\int_0^{\nu^b} t^{2d-3}dt = C_1^{\#}(s)\nu^{1+2b(d-1)}.$$

3. Now consider

$$X_s := \Big|\langle I_s, (\Sigma_s^{vm})_{\nu^b}^{\nu^{-a}}\rangle - \nu\int_{\Sigma^1} m(d\eta)\int_{\nu^b}^{\nu^{-a}} dt\, t^{2d-1}\mathcal{J}_s(\eta,t)\Big|. \tag{9.28}$$

We claim that

$$X_s \le \nu^2\chi_d(\nu)C^{\#}(s), \tag{9.29}$$

where $\chi_d(\nu)$ was defined in (1.43). Indeed, if $|s| \ge \nu^{-a}$, then by (9.23) the modulus of the first term in the r.h.s. of (9.28) is $\le C_a^{\#}(s)\nu^2$. The second term also satisfies this estimate with some other $C_a^{\#}(s)$. So (9.29) is established if $|s| \ge \nu^{-a}$. By a similar (and even easier) argument the claimed estimate holds if $\nu_1 \le \nu \le 1$ for any fixed constant $\nu_1 > 0$.

Now let us consider the case $\nu \le \nu_1$, $|s| \le \nu^{-a}$. Then for $\nu^b \le t \le \nu^{-a}$ the r.h.s. of (9.25) is no smaller than $Y := C\nu^{2b}(1 + 2\nu^{-2a})^{-r_1}$. If

$$2b + 2ar_1 < 1, \tag{9.30}$$

then $Y \ge \nu$ if $\nu \le \nu_1$ and $\nu_1 > 0$ is small enough. Then assumption (9.25) holds, and (9.21) together with (9.27) implies (9.29):

$$\begin{aligned} X_s &\le \nu^2\int_{\Sigma^1} m(d\eta)\int_{\nu^b}^{\nu^{-a}} dt\, t^{2d-1}C^{\#}(t,s)t^{-4} \\ &\le \nu^2 C^{\#}(s)\int_{\nu^b}^{\nu^{-a}} dt\, C^{\#}(t)t^{2d-5} \le C_1^{\#}(s)\nu^2\chi_d(\nu). \end{aligned}$$

4. In the same time, in view of (9.26), for any $-\infty \le A < B \le \infty$ we have

$$\Big|\nu\int_{\Sigma^1} m(d\eta)\int_{\nu^B}^{\nu^A} dt\, t^{2d-1}\mathcal{J}_s(\eta,t)\Big| \le \nu\int_{\nu^B}^{\nu^A} dt\, t^{2d-1-2}C^{\#}(t,s) \le C_1^{\#}(s)\nu, \tag{9.31}$$

since $d \geq 2$.

Now we get from **1-4** that

$$\begin{aligned}
&\Big|\langle I_s, \Sigma_s^{vm}\rangle - \nu \int_{\Sigma^1} m(d\eta) \int_0^\infty dt\, t^{2d-1} \mathcal{J}_s(\eta, t)\Big| \\
&\leq \Big|\langle I_s, (\Sigma_s^{vm})_{\nu^{-a}}^{\infty}\rangle\Big| + \nu \int_{\Sigma^1} m(d\eta) \int_{\nu^{-a}}^\infty dt\, t^{2d-1} |\mathcal{J}_s(\eta, t)| \\
&\quad + \Big|\langle I_s, (\Sigma_s^{vm})_{\nu^b}^{\nu^{-a}}\rangle - \nu \int_{\Sigma^1} m(d\eta) \int_{\nu^b}^{\nu^{-a}} dt\, t^{2d-1} \mathcal{J}_s(\eta, t)\Big| \\
&\quad + \Big|\langle I_s, (\Sigma_s^{vm})_0^{\nu^b}\rangle\Big| + \nu \int_{\Sigma^1} m(d\eta) \int_0^{\nu^b} dt\, t^{2d-1} |\mathcal{J}_s(\eta, t)| \\
&\leq C^\#(s)\big(\nu^2 + \nu^2 \chi_d(\nu) + \nu^{2bd} + \nu^{1+2b(d-1)}\big) =: Z,
\end{aligned} \tag{9.32}$$

if condition (9.30) holds for some $a, b > 0$. If $d \geq 3$, we choose $b = \frac{1}{d} < 1$. Then (9.30) holds for some $a > 0$, and $Z \leq C^\#(s)\nu^2 \chi_d(\nu)$. If $d = 2$, then $\nu^{2bd} = \nu^{4b}$ and $\nu^{1+2b(d-1)} = \nu^{1+2b}$. We choose $b = 1/2 - \aleph/4$, $\aleph > 0$. Then again (9.30) holds for some $a(\aleph) > 0$, so $Z \leq C_\aleph^\#(s)\nu^{2-\aleph}$.

By (9.31) the improper integral $\int_{\Sigma^1} m(d\eta) \int_0^\infty dt\, t^{2d-1} \mathcal{J}_s$ converges absolutely and is bounded by $C_1^\#(s)$. In view of (9.8) and (9.26) it may be written as

$$\nu \int_{\Sigma^1} m(d\eta) \int_0^\infty dt\, t^{2d-1} \mathcal{J}_s(\eta, t) = \pi\nu \int_{\Sigma_*} \frac{F_s(z)}{\Gamma_s(z)\sqrt{|x|^2 + |y|^2}}\, dz\,|_{\Sigma_*}\ .$$

This result jointly with the estimates (9.32) and (9.15) imply the assertion of Theorem 3.3. □

9.9. Proof of Proposition 3.5. Denoting by B_r^R the spherical layer $\{r \leq |z| \leq R\}$, in view of (9.8) for $R \geq 0$ we have

$$\int_{B_R^{R+1}} \mu^\Sigma(dz) \leq C \int_R^{R+1} t^{2d-2} dt \leq C_1 (R+1)^{2d-2}.$$

So for any function $f \in \mathcal{C}_m(\mathbb{R}^{2d})$ with $m > 2d-1$ the integral $\int f(z)\, \mu^\Sigma(dz)$ is bounded by

$$|f|_m \sum_{R=0}^\infty \int_{B_R^{R+1}} \langle z\rangle^{-m}\, \mu^\Sigma(dz) \leq C_2 |f|_m \sum_{R=0}^\infty \frac{(R+1)^{2d-2}}{\langle R\rangle^m} = C_3 |f|_m.$$

This proves the proposition.

10. Oscillating Sums Under the Limit (1.17)

10.1. Correlations between increments of $a_s^{(1)}$. Correlations between the increments of $a_s^{(1)}$ and some similar quantities, treated in Section 6 lead to sums of the following form:

$$\Sigma_s^0(\tau) = \sum\nolimits_{1,2} F_s(s_1, s_2)\delta_{3s}'^{12} \int_0^\tau \int_0^\tau d\theta_1\, d\theta_2 e^{-\gamma_s\left(2\tau - \theta_1 - \theta_2\right) + i\nu^{-1}\omega_{3s}^{12}(\theta_1 - \theta_2)}, \tag{10.1}$$

where $s \in \mathbb{R}^d$ and $\tau \in (0, 1]$ is a parameter. Our goal is to study their asymptotical behaviour as $\nu \to 0$, $L \to \infty$, uniformly in τ. In (10.1) F_s is a real C^2-function on $\mathbb{R}^{2d}$, satisfying (3.9) (for example, F_s may be the function, defined in (3.3)). Integrating over $d\theta_1\, d\theta_2$ we find that

$$\Sigma_s^0(\tau) = \sum\nolimits_{1,2}' F_s(s_1, s_2)\delta_{3s}^{\prime 12} \frac{|e^{i\nu^{-1}\omega_{3s}^{12}\tau} - e^{-\gamma_s\tau}|^2}{\gamma_s^2 + (\nu^{-1}\omega_{3s}^{12})^2}. \tag{10.2}$$

Note that the nominator in the fraction above equals

$$|e^{i\nu^{-1}\omega_{3s}^{12}\tau} - e^{-\gamma_s\tau}|^2 = 1 + e^{-2\gamma_s\tau} - 2e^{-\gamma_s\tau}\cos(\nu^{-1}\tau\omega_{3s}^{12}). \tag{10.3}$$

We wish to study the asymptotical behaviour of the sum Σ_s^0 as $\nu \to 0,\ L \to \infty$ and, as before, will do this by comparing (10.1) with the integral

$$I_s^0 = \int\int ds_1\, ds_2 \int_0^\tau \int_0^\tau d\theta_1\, d\theta_2\, \delta_{3s}^{12} M_s(s_1, s_2, s_3; \theta_1, \theta_2),$$

where

$$M_s = F_s(s_1, s_2) e^{-\gamma_s\left(2\tau - \theta_1 - \theta_2\right) + i\nu^{-1}\omega_{3s}^{12}(\theta_1 - \theta_2)}.$$

By Theorem 3.1,

$$|I_s^0 - \Sigma_s^0| \le C^\#(s)\nu^{-2}L^{-2}. \tag{10.4}$$

So if $L \gg \nu^{-1}$, then to calculate the asymptotical behaviour of Σ_s^0 it suffices to calculate that of I_s^0. Integrating over $d\theta_1\, d\theta_2$ in the expression for I_s^0 we obtain (10.2) with $\sum_j'$ replaced by $\int ds_j$. In view of (10.3) we get that $I_s^0 = \nu^2 I_s^{0,1} + \nu^2 I_s^{0,2}$, where

$$I_s^{0,1} = (1 + e^{-2\gamma_s\tau}) \int\int ds_1\, ds_2 \frac{\delta_{3s}^{12} F_s(s_1, s_2)}{\nu^2\gamma_s^2 + (\omega_{3s}^{12})^2},$$

$$I_s^{0,2} = -2e^{-\gamma_s\tau} \int\int ds_1\, ds_2 \frac{\delta_{3s}^{12} F_s(s_1, s_2)\cos(\nu^{-1}\tau\omega_{3s}^{12})}{\nu^2\gamma_s^2 + (\omega_{3s}^{12})^2}.$$

Consider first $I_s^{0,1}$. Applying Theorem 3.3 with $\Gamma_s = \gamma_s/2$, we find

$$I_s^{0,1} = \frac{1 + e^{-2\gamma_s\tau}}{2\gamma_s}\pi\nu^{-1} \int_{^s\Sigma_*} \frac{F_s(z)}{|(s - s_1, s - s_2)|}\, dz\,|_{^s\Sigma_*} + O(1)\nu^{-\aleph_d} C^\#(s; \aleph_d), \tag{10.5}$$

where $z = (s_1, s_2)$.

It remains to study the asymptotical behaviour of $I_s^{0,2}$. It is described by the following result, proved in Section 10.3 (also see [20]):

Lemma 10.1. *For any $s \in \mathbb{R}^d$ and $\tau \in (0, 1]$,*

$$\left| I_s^{0,2} + \gamma_s^{-1}\nu^{-1}e^{-2\gamma_s\tau}\pi \int_{^s\Sigma_*} \frac{F_s(z)}{|(s - s_1, s - s_2)|}\, dz\,|_{^s\Sigma_*} \right| \le C^\#(s)\chi_d(\nu). \tag{10.6}$$

Relations (10.5) and (10.6) imply the main result of this section:

Theorem 10.2. *For any $s \in \mathbb{R}^d$ and $\tau \in (0, 1]$,*

$$\left| I_s^0 - \frac{1 - e^{-2\gamma_s\tau}}{2\gamma_s}\nu\pi \int_{^s\Sigma_*} \frac{F_s(z)}{|(s - s_1, s - s_2)|}\, dz\,|_{^s\Sigma_*} \right| \le C^\#(s; \aleph_d)\nu^{2-\aleph_d}.$$

Jointly with (10.4) this gives an asymptotic for the sum Σ_s^0.

10.2. Correlations between solutions and their increments. In this section we analyse sums similar to (10.1) in which the integral $\int_0^\tau d\theta_2$ is replaced by the integral $\int_{-\infty}^0 d\theta_2$. Sums of such form arise in Section 6, when studying the correlation

$$\mathbb{E}\Delta a_s^{(1)}(\tau)\bar{c}_s^{(1)}(\tau) = e^{-\gamma_s\tau}\mathbb{E}\big(a_s^{(1)}(\tau) - e^{-\gamma_s\tau}a_s^{(1)}(0)\big)\bar{a}_s^{(1)}(0)$$

and similar quantities. We will show that the considered sums are negligible. The reason for this is that for $\omega_{3s}^{12} \neq 0$ the "fast" frequency $\nu^{-1}\omega_{3s}^{12}(\theta_1 - \theta_2)$ is of the size $\lesssim 1$ only if $|\theta_1|, |\theta_2| \lesssim \nu$, so the Lebesgue measure of such resonant vectors (θ_1, θ_2) is only of order ν^2 (if $\tau \sim 1$). That is why such sums are much smaller that those of the form (10.1), where the measure of the resonant vectors (θ_1, θ_2) is of order $\nu \gg \nu^2$.

We consider the sum

$$S_s(\tau) = \sum\nolimits_{1,2}\delta'^{12}_{3s}\int_0^\tau dl\int_{-\infty}^0 dl'\, e^{\gamma l'}e^{i\nu^{-1}(l-l')\omega_{3s}^{12}}F_s(s_1, s_2, l, l', \tau), \quad s \in \mathbb{R}^d,$$

where $\gamma > 0$ and F_s is a real function, measurable in s_1, s_2, l, l', C^2-smooth in s_1, s_2, and such that

$$e^{\gamma l'}|\partial^\alpha_{s_1,s_2}F_s| \le C^\#(s)C^\#(s_1)C^\#(s_2) \qquad \forall\, 0 \le |\alpha| \le d+1, \tag{10.7}$$

uniformly in $l' \le 0,\ 0 \le \tau \le 1,\ 0 \le l \le \tau$.

Theorem 10.3. *Let* $0 \le \tau \le 1$. *Then under the assumption* (10.7) *the sum* S_s *meets the estimate*

$$|S_s(\tau)| \le C_1^\#(s)\big(\nu^2\chi_d(\nu) + \nu^{-2}L^{-2}\big),$$

where $C^\#(s)$ *depends only on* d, γ *and the function* $C_1^\#(s)$ *from* (10.7).

Note that since we assume no smoothness in l' for function F_s, then the theorem also applies to the sums

$$S_s^T(\tau) = \sum\nolimits_{1,2}\delta'^{12}_{3s}\int_0^\tau dl\int_{-T}^0 dl'\, e^{\gamma l'}e^{i\nu^{-1}(l-l')\omega_{3s}^{12}}F_s$$

since we may extend F_s by zero for $l' < -T$ and regard S_s^T as the sum S_s.

Proof. It is convenient to change the variable $\theta = -l'$, so that the sum S_s takes the form

$$S_s(\tau) = \sum\nolimits_{1,2}\delta'^{12}_{3s}\int_0^\infty d\theta\int_0^\tau dl\, e^{-\gamma\theta}e^{i\nu^{-1}(l+\theta)\omega_{3s}^{12}}F_s(s_1, s_2, \theta, l, \tau).$$

As usual, to estimate S_s we first approximate it by the integral

$$I_s(\tau) = \int_0^\infty d\theta\int_0^\tau dl\int\int ds_1 ds_2\,\delta_{3s}^{12}e^{-\gamma\theta}e^{i\nu^{-1}(l+\theta)\omega_{3s}^{12}}F_s.$$

Applying Theorem 3.1 we get that

$$|I_s(\tau) - S_s(\tau)| \le C^\#(s)\nu^{-2}L^{-2}. \tag{10.8}$$

To estimate I_s we write it as a sum of integrals over the domains $\{\nu^{-1}(l+\theta) \ge 1\} = \{l \ge \nu - \theta\}$ and $\{\nu^{-1}(l+\theta) \le 1\} = \{0 \le l \le \nu - \theta\}$.

Integral over $\{l \geq \nu - \theta\}$. Let us denote this integral I_s^1, and for any fixed $\theta \geq 0$ and $l \geq \nu - \theta$ consider the internal integral over $ds_1\, ds_2$:

$$I_s^1(l,\theta) = e^{-\gamma\theta} \int\int ds_1 ds_2\, \delta_{3s}^{12} e^{i\nu^{-1}(l+\theta)\omega_{3s}^{12}} F_s.$$

Since $\omega_{3s}^{12} = 2(s_1 - s)\cdot(s - s_2)$ is a non-degenerate quadratic form and $\nu^{-1}(l+\theta) \geq 1$ on the zone of integrating, then the integral $I_s^1(l,\theta)$ has the form (12.2) with $\nu := \nu(l+\theta)^{-1}$, $\varphi = e^{-\gamma\theta} F_s$ and $n = 2d$. So by (12.4), (12.5) and (10.7),

$$|I_s^1(l,\theta)| \leq C^\#(s)\nu^d (l+\theta)^{-d} e^{-\gamma\theta},$$

where $C^\#(s)$ is as in the theorem. Accordingly,

$$|I_s^1| \leq \mathfrak{C} \int_0^\infty d\theta e^{-\gamma\theta} \int_0^\tau dl\, (l+\theta)^{-d} \chi_{\{l \geq \nu-\theta\}}, \qquad \mathfrak{C} = C^\#(s)\nu^d.$$

We split the integrating zone $\{\theta \geq 0,\ 0 \leq l \leq \tau,\ l \geq \nu - \theta\}$ to the part where $\theta \geq \nu$ and its complement, and accordingly split the integral above as $I_s^{1,1} + I_s^{1,2}$, where

$$I_s^{1,1} = \mathfrak{C} \int_\nu^\infty d\theta e^{-\gamma\theta} \int_0^\tau dl\, (l+\theta)^{-d},$$
$$I_s^{1,2} = \mathfrak{C} \int_{(\nu-\tau)\vee 0}^\nu d\theta e^{-\gamma\theta} \int_{\nu-\theta}^\tau dl\, (l+\theta)^{-d}.$$

Consider first $I_s^{1,1}$. Computing the internal integral and replacing τ by ∞ we find

$$I_s^{1,1} \leq C\mathfrak{C} \int_\nu^\infty \frac{e^{-\gamma\theta}}{\theta^{d-1}}\, d\theta \leq \frac{C_1\mathfrak{C}}{\nu^{d-2}} \chi_d(\nu) = C_1^\#(s)\nu^2 \chi_d(\nu).$$

Now consider $I_s^{1,2}$. Replacing in the external integral $(\nu - \tau) \vee 0$ by 0 and in the internal one τ by ∞ we find

$$I_s^{1,2} \leq C\mathfrak{C} \int_0^\nu d\theta\, \nu^{-d+1} = C_1^\#(s)\nu^2.$$

Integral over $\{0 \leq l \leq \nu - \theta\}$. Denoting this integral as I_s^2, we find

$$|I_s^2| \leq \int_0^\nu d\theta \int_0^{(\nu-\theta)\wedge\tau} dl \int\int ds_1 ds_2\, |F_s| \leq C^\#(s)\nu^2,$$

since $\int\int ds_1 ds_2\, |F_s| \leq C^\#(s)$ and the area of integration over $d\theta dl$ is bounded by ν^2.

We have seen that $|I_s| \leq C^\#(s)\nu^2\chi_d(\nu)$. This and (10.8) jointly imply the assertion of the theorem. □

10.3. Proof of Lemma 10.1. Let us denote $\nu' = \frac{1}{2}\nu\gamma_s$. If $\nu' > 1$, then $|s| \geq C\nu^{-1/2r_*}$, so taking into account assumption (3.9) we see that the both summands in the l.h.s. of (10.6) are bounded by $C^\#(s)$ and the result follows. So we may assume that $\nu' \leq 1$.

Let us write $I_s^{0,2}$ as

$$I_s^{0,2} = -\frac{1}{2}\, e^{-\gamma_s\tau} \int\int ds_1\, ds_2\, \delta_{3s}^{12} \frac{F_s(s_1, s_2)\cos(\nu^{-1}\tau\omega_{3s}^{12})}{\nu'^{\,2} + (\omega_{3s}^{12}/2)^2}.$$

We will examine the integrals $\langle I_s^{0,2}, K_{2r}\rangle$, $r \ll 1$, $\langle I_s^{0,2}, \Sigma^\nu\rangle$ and $\langle I_s^{0,2}, \mathbb{R}^{2d} \setminus \Sigma^\nu\rangle$ (see Notation, (9.1) and Lemma 9.1), and will derive the lemma from this analysis. The constants below do not depend on τ, s and ν'.

Let us re-write $I_s^{0,2}$, using the variables $(x, y) = z$, see (3.15). Disregarding for a moment the pre-factor $-\frac{1}{2}e^{-\gamma_s\tau}$ we examine the integral

$$J_s = \int_{\mathbb{R}^{2d}} dz\, \frac{F_s(z)\cos\lambda x\cdot y}{(x\cdot y)^2 + \nu'^{\,2}}, \qquad \lambda = \nu'^{\,-1}\tau\gamma_s.$$

Step 1. Since $|\cos\lambda x\cdot y| \leq 1$, an upper bound for $\langle J_s,\ K_{2r}\rangle$, where $r \ll 1$, (see (9.1)) follows from (9.3):

$$\langle |J_s|, K_{2r}\ \rangle \leq C^\#(s)\nu'^{-2}r^{2d}. \tag{10.9}$$

Step 2. Integral over Σ^ν. We recall that $(\Sigma^\nu)_{R_1}^{R_2}$ is defined in (9.10). Passing to the variables (η, t, θ) (see Lemma 9.2) and using (9.7) we disintegrate $\langle J_s, (\Sigma^\nu)_r^\infty\rangle =: J_s^r$ as follows:

$$\begin{aligned} J_s^r &= \int_{\Sigma^1} m(d\eta) \int_r^\infty dt\, t^{2d-1} \int_{-\theta_0}^{\theta_0} d\theta\, \frac{F_s(\eta, t, \theta)\mu(\eta, \theta)\cos(\lambda x\cdot y)}{(t^2\theta)^2 + \nu'^{\,2}} \\ &= \int_{\Sigma^1} m(d\eta) \int_r^\infty dt\, t^{2d-1}\Upsilon_s(\eta, t), \end{aligned} \tag{10.10}$$

where

$$\Upsilon_s(\eta, t) = t^{-4}\int_{-\theta_0}^{\theta_0} \frac{F_s(\eta, t, \theta)\mu(\eta, \theta)\cos(\lambda t^2\theta)\, d\theta}{\theta^2 + \varepsilon^2}, \quad \varepsilon = \nu' t^{-2}.$$

To estimate Υ_s, consider first the integral Υ_s^0, obtained from Υ_s by frozening $F_s\mu$ at $\theta = 0$. Since $\mu(\eta, 0) = 1$, then Υ_s^0 equals

$$2t^{-4}F_s(\eta, t, 0)\int_0^{\theta_0} \frac{\cos(\lambda t^2\theta)\, d\,\theta}{\theta^2 + \varepsilon^2} = 2\nu'^{\,-1}t^{-2}F_s(\eta, t, 0)\int_0^{\theta_0/\varepsilon} \frac{\cos(\gamma_s\tau w)\, dw}{w^2 + 1}.$$

Consider the integral

$$2\int_0^{\theta_0/\varepsilon} \frac{\cos(\gamma_s\tau w)\, dw}{w^2 + 1} = 2\int_0^\infty \frac{\cos(\gamma_s\tau w)\, dw}{w^2 + 1} - 2\int_{\theta_0/\varepsilon}^\infty \frac{\cos(\gamma_s\tau w)\, dw}{w^2 + 1} =: I_1 - I_2.$$

Since

$$2\int_0^\infty \frac{\cos(\xi w)\, dw}{w^2 + 1} = \int_{-\infty}^\infty \frac{e^{i\xi w}\, dw}{w^2 + 1} = \pi e^{-|\xi|},$$

then $I_1 = \pi e^{-\gamma_s \tau}$. For I_2 we have an obvious estimate $|I_2| \le 2\varepsilon/\theta_0 = C_1 \nu' t^{-2}$. So

$$\Upsilon_s^0(\eta, t) = \nu'^{\,-1} \pi t^{-2} F_s(\eta, t, 0)(e^{-\gamma_s \tau} + \Delta_t), \qquad |\Delta_t| \le C\nu' t^{-2}. \tag{10.11}$$

Now we estimate the difference between Υ_s and Υ_s^0. Writing $(F_s\mu)(\eta, t, \theta) - (F_s\mu)(\eta, t, 0)$ as $A_s(\eta, t)\theta + B_s(\eta, t, \theta)\theta^2$, where $|A_s|, |B_s| \le C^\#(s, t)$, we have

$$\Upsilon_s - \Upsilon_s^0 = t^{-4} \int_{-\theta_0}^{\theta_0} \frac{(A_s\theta + B_s\theta^2)\cos(\lambda t^2 \theta)\, d\theta}{\theta^2 + \varepsilon^2}.$$

Since the first integrand is odd in θ, then its integral vanishes, and

$$|\Upsilon_s - \Upsilon_s^0| \le C^\#(s, t) t^{-4} \int_{-\theta_0}^{\theta_0} \frac{\theta^2\, d\theta}{\theta^2 + \varepsilon^2} \le 2C^\#(s, t) t^{-4} \theta_0.$$

So by (10.11)

$$\begin{aligned} &|\Upsilon_s(\eta, t) - \nu'^{\,-1} \pi t^{-2} F_s(\eta, t, 0) e^{-\gamma_s \tau}| \\ &\quad \le C^\#(s, t)\big(t^{-4} + \nu'^{\,-1} t^{-2}\, \nu' t^{-2}\big) = C_1^\#(s, t) t^{-4}. \end{aligned}$$

Then, by (10.10),

$$\begin{aligned} J_s^r &= \int_{\Sigma^1} m(d\eta) \int_r^\infty dt\, t^{2d-1} \big[\nu'^{\,-1} \pi t^{-2} e^{-\gamma_s \tau} F_s(\eta, t, 0) + O(C^\#(s, t) t^{-4})\big] \\ &= \int_{\Sigma^1} m(d\eta) \int_r^\infty dt\, \big[\nu'^{\,-1} \pi e^{-\gamma_s \tau} t^{2d-3} F_s(\eta, t, 0)\big] + O\big(C^\#(s) \chi_d(r)\big), \end{aligned}$$

since $\int_r^\infty t^{2d-1} C^\#(s, t) t^{-4}\, dt \le C^\#(s) \chi_d(r)$. Noting that

$$|\nu'^{\,-1} \pi e^{-\gamma_s \tau} \int_{\Sigma^1} m(d\eta) \int_0^r dt\, t^{2d-3} F_s(\eta, t, 0)| \le C^\#(s) \nu'^{\,-1} r^{2d-2},$$

we arrive at the inequality

$$\begin{aligned} &|J_s^r - \nu'^{\,-1} \pi e^{-\gamma_s \tau} \int_{\Sigma^1} m(d\eta) \int_0^\infty dt\, t^{2d-2} (t^{-1} F_s(\eta, t, 0))| \\ &\quad \le C^\#(s) \big(\nu'^{\,-1} r^{2d-2} + \chi_d(r)\big). \end{aligned} \tag{10.12}$$

Here in view of (9.8)

$$\int_{\Sigma^1} m(d\eta) \int_0^\infty dt\, t^{2d-2} (t^{-1} F_s(\eta, t, 0)) = \int_{\Sigma_*} F_s(z) |z|^{-1}. \tag{10.13}$$

Since $(\Sigma^v)_0^r \subset K_{2r}$ by Lemma 9.1.3, where θ_0 is assumed to be sufficiently small, then by (10.9),

$$|J_s^r - \langle J_s, \Sigma^v \rangle| \le C^\#(s) \nu'^{\,-2} r^{2d}. \tag{10.14}$$

Step 3. Final asymptotic. Consider the integral over the complement to Σ^v:

$$\big|\langle J_s, \mathbb{R}^{2d} \setminus \Sigma^v \rangle\big| \le \big\langle |J_s|, \{|z| \le r\} \big\rangle + C \int_r^\infty dt\, t^{2d-1} \int_{S^1 \setminus \Sigma^v} \frac{|F_s(z)|\, d_{S^1}}{\omega^2/4 + \nu'^{\,2}}.$$

By item 5) of Lemma 9.1, $|\omega(z)| \geq Ct^2$ in $S^t \setminus \Sigma^v$. Jointly with (10.9) it implies that

$$\left|\langle J_s, \mathbb{R}^{2d} \setminus \Sigma^v\rangle\right| \leq C^\#(s)\nu'^{\,-2} r^{2d} + C^\#(s)\int_r^\infty t^{2d-1}t^{-4}C^\#(t)\,dt \\ \leq C_1^\#(s)\big(\nu'^{\,-2}r^{2d} + \chi_d(r)\big). \tag{10.15}$$

Finally by (10.14), (10.15), (10.12) and (10.13) with $r = \sqrt{\nu}$, we have

$$\left| J_s - \nu'^{\,-1} e^{-\gamma_s \tau}\pi \int_{\Sigma_*} F_s(z)|z|^{-1}\right| \leq C^\#(s)\chi_d(\nu).$$

That is, $\left| I_s^{0,2} + \gamma_s^{-1}\nu^{-1}e^{-2\gamma_s\tau}\pi \int_{\Sigma_*} F_s(z)|z|^{-1}\right| \leq C^\#(s)\chi_d(\nu)$, and the lemma is proved.

11. Wave Kinetic Integrals and Equations: Proofs

11.1. Proof of Lemma 4.2. To prove the lemma we may assume that

$$|u^1|_r = \cdots = |u^4|_r = 1.$$

Consider first the integral J_4. By the above,

$$|J_4(s)| \leq \int_{\Sigma_*} d\mu^\Sigma(z)\langle x+s\rangle^{-r}\langle y+s\rangle^{-r}\langle x+y+s\rangle^{-r} =: \mathcal{J}_4(s), \quad z = (x, y).$$

We should verify that

$$\mathcal{J}_4(s) \leq C\langle s\rangle^{-r-1} \quad \forall s \in \mathbb{R}^d. \tag{11.1}$$

If $|s| \leq 2$, then it is not hard to check that

$$\langle x+s\rangle\langle y+s\rangle\langle x+y+s\rangle \geq C^{-1}\langle z\rangle^2$$

for a suitable constant C independent from s. So the integrand for $\mathcal{J}_4(s)$ is bounded by $C_1\langle z\rangle^{-2r}$. Since $r > d$, then by Proposition 3.5 $\mathcal{J}_4(s) \leq C$ if $|s| \leq 2$. This proves (11.1) if $|s| \leq 2$, and it remains to consider the case when

$$R := |s| \geq 2.$$

Assuming that a vector s as above is fixed, let us consider the sets

$$O_1 = \{\xi \in \mathbb{R}^d : |\xi| \leq \tfrac{9}{10}R\}, \quad O_2 = \{\xi : \tfrac{9}{10} \leq |\xi| \leq \tfrac{11}{10}R\},$$
$$O_3 = \{\xi : |\xi| \geq \tfrac{11}{10}R\}, \quad O_{ij} = O_i \cup O_j, \quad i, j = 1, 2, 3,$$

and define

$$\Sigma^{i,j} = \Sigma_* \cap (O_i \times O_j), \quad \Sigma^{12,j} = \Sigma_* \cap (O_{12} \times O_j), \quad \text{etc.}$$

Next we denote by $J_4^{i,j}(s)$ the part of the integral in (4.5) which comes from the integrating over $\Sigma^{i,j}$:

$$J_4^{i,j}(s) = \int_{\Sigma^{i,j}} d\mu^\Sigma(z)\big(u^1(x+s)u^2(y+s)u^3(x+y+s)\big),$$

and define similar integrals $J_4^{12,j}(s)$, etc. Then $J_4(s) = \sum_{i,j\in\{1,2,3\}} J_4^{i,j}(s)$ and $|J_4^{i,j}(s)| \le \mathcal{J}_4^{i,j}(s)$, where

$$\mathcal{J}_4^{i,j}(s) = \int_{\Sigma^{i,j}} d\mu^\Sigma(z)\langle x+s\rangle^{-r}\langle y+s\rangle^{-r}\langle x+y+s\rangle^{-r}.$$

Clearly

$$\mathcal{J}_4^{i,j}(s) = \mathcal{J}_4^{j,i}(s) \quad \forall\,(i,j), \tag{11.2}$$

and $\mathcal{J}_4^{12,j}(s) = \mathcal{J}_4^{1,j}(s) + \mathcal{J}_4^{2,j}(s)$, etc. To verify (11.1) it remains to check that

$$\mathcal{J}_4^{i,j}(s) \le CR^{-r-1} \quad \forall\,(i,j), \tag{11.3}$$

when $R = |s| \ge 2$. Applying Theorem 3.6 we find that

$$\mathcal{J}_4^{i,j}(s) = \int_{O_i} dx\,|x|^{-1}\langle x+s\rangle^{-r}\int_{x^\perp\cap O_j} d_{x^\perp}y\,\langle y+s\rangle^{-r}\langle x+y+s\rangle^{-r}. \tag{11.4}$$

By s_x (by s_y) we will denote the projection of s on the space $x^\perp$ (on $y^\perp$), and by $\hat s, \hat x, \hat y$—the vectors $s/R, x/R, y/R$; so $|\hat s| = 1$.

We will estimate the r.h.s. of (11.4) for each set $\{i, j\} \subset \{1, 2, 3\}$, using the following elementary inequalities, related to the domains O_j:

$$\langle y+s\rangle \ge |y+s| \ge \tfrac{1}{10}R \quad \forall\, y \in O_{13}, \tag{11.5}$$

$$\langle x+y+s\rangle \ge C^{-1}(|x|+|y|) \quad \forall\, z \in (O_{23}\times O_{23}) \cap \Sigma. \tag{11.6}$$

Proof of (11.6) uses that $|x+y| \ge (|x|+|y|)/\sqrt{2}$ for $x, y \in \Sigma$. We will also use the integral inequalities below, where $|s| = R \ge 2$ and $\hat\chi_l = \hat\chi_l(R)$ equals 1 if $l \ne 0$ and equals $\ln R$ if $l = 0$:

$$\int_{\mathbb{R}^d} \langle y+\xi\rangle^{-p} dy \le C \quad \forall \xi \in \mathbb{R}^d \quad \text{if } \ p > d, \tag{11.7}$$

$$\int_{|x|\le\frac{11}{10}R} |x|^{-1}\langle x+s\rangle^{-\hat r}dx \le C\hat\chi_{d-\hat r}R^{\max(-1,d-1-\hat r)}, \tag{11.8}$$

$$\int_{|x|\ge\frac{9}{10}R} |x|^{A}\langle x+s\rangle^{-\hat r}dx \le C\hat\chi_{d-\hat r}R^{\max(A,A+d-\hat r)} \tag{11.9}$$

if $A < \hat r - d$. We will use these relations with $d := d$ or $d := d-1$. The first inequality is obvious. To prove (11.8) note that there the integral equals

$$R^{d-1}\int_{|y|\le\frac{11}{10}} |y|^{-1}\langle R(y+\hat s)\rangle^{-\hat r}dy, \quad \hat s = s/R.$$

It is straightforward to see that this integral (with the pre-factor), taken over the ball $\{|y+\hat s| \le 1/10\}$ is $\le C\hat\chi_{d-\hat r}R^{\max(-1,d-1-\hat r)}$, while the integral over the ball's complement is $\le CR^{d-1-\hat r}$. It implies (11.8). Proof of (11.9) is similar.

Estimating the integral $\mathcal{J}_4^{23,23}(s)$. By (11.4) and (11.6) the integral is bounded by

$$C\int_{|x|\ge\frac{9}{10}R} dx\,|x|^{-1}\langle x+s\rangle^{-r}\int_{|y|\ge\frac{9}{10}R} d_{x^\perp}y\,\langle y+s\rangle^{-r}(|x|+|y|)^{-r}.$$

Since $\langle y+s\rangle \geq \langle y+s_x\rangle$ and $r>d$, the internal integral is less than

$$CR^{-r}\int_{|y|\geq \frac{9}{10}R} d_{x^\perp}y\,\langle y+s_x\rangle^{-r} \leq C_1R^{-r},$$

where we used (11.7) with $d:=d-1$. So

$$\mathcal{J}_4^{23,23}(s) \leq CR^{-r}\int_{|x|\geq \frac{9}{10}R} dx\,|x|^{-1}\langle x+s\rangle^{-r} \leq R^{-r-1}$$

by (11.9) with $A=-1$ and $\hat{r}=r$. This implies (11.3) if $i,j\in\{2,3\}$.
Estimating the integral $\mathcal{J}_4^{12,13}(s)$. By (11.4) and (11.5),

$$\mathcal{J}_4^{12,13}(s) \leq CR^{-r}\int_{|x|\leq \frac{11}{10}R} dx\,|x|^{-1}\langle x+s\rangle^{-r}\int d_{x^\perp}y\,\langle x+y+s\rangle^{-r}.$$

Since $\langle x+y+s\rangle \geq \langle y+s_x\rangle$, by (11.7) with $d:=d-1$ the internal integral is bounded by a constant. So in view of (11.8) with $\hat{r}=r$, we get $\mathcal{J}_4^{12,13}(s)\leq CR^{-r-1}$. This implies (11.3) if $i\in\{1,2\}$, $j\in\{1,3\}$.

Evoking the symmetry (11.2) we see that we have checked (11.3) for all (i,j), and thus have proved (4.7) for $l=4$.

Other three integrals are easier. Let us first consider $J_3(s)$,

$$J_3(s)=u^4(s)\int_{\Sigma_*} d\mu^\Sigma(z)u^1(x+s)u^2(y+s),$$

where $|u^j|_r=1$ for each j. Then

$$|J_3(s)| \leq \langle s\rangle^{-r}\int_{\Sigma_*} d\mu^\Sigma(z)\langle x+s\rangle^{-r}\langle y+s\rangle^{-r} =: \mathcal{J}_3(s).$$

If $|s|\leq 2$, then $\mathcal{J}_3(s)$ is bounded by

$$C\int_{\Sigma_*} d\mu^\Sigma(z)\langle x+s\rangle^{-r}\langle y+s\rangle^{-r} = C\int dx\,|x|^{-1}\langle x+s\rangle^{-r}\int_{x^\perp} d_{x^\perp}y\,\langle y+s\rangle^{-r}.$$

Applying (11.7) to the both internals, using that $r>d$ and the integrability of $|x|^{-1}$ over a neighbourhood of $0\in\mathbb{R}^d$, we see that $|\mathcal{J}_3(s)|\leq C$, which implies $|\mathcal{J}_3(s)|\leq C\langle s\rangle^{-r-1}$ for $|s|\leq 2$.

Now we pass to the case $R=|s|\geq 2$. Then

$$\mathcal{J}_3(s) \leq \langle s\rangle^{-r}\int dx\,|x|^{-1}\langle x+s\rangle^{-r}\int_{x^\perp} d_{x^\perp}y\,\langle y+s\rangle^{-r}.$$

Applying (11.7) we see that the internal integral is bounded by a constant, while by (11.8) and (11.9) with $A=-1$ and $\hat{r}=r$ the external is $\leq CR^{-1}$. This proves (4.7) for $l=3$.

The integrals J_1 and J_2 are very similar and we only consider J_1:

$$J_1(s)=u^4(s)\int_{\Sigma_*} d\mu^\Sigma(z)u^2(y+s)u^3(x+y+s).$$

Assume first $|s| \leq 2$. Using the disintegration of the measure $d\mu^{\Sigma}(z)$ in the form (3.25) we get

$$|J_1(s)| \leq C \int dy\, |y|^{-1} \langle y+s\rangle^{-r} \int_{y^{\perp}} d_{y^{\perp}}x\, \langle x+y+s\rangle^{-r}.$$

Since $\langle x+y+s\rangle^{-r} \leq \langle x+s_y\rangle^{-r}$ and $r > d$, then using (11.7) we see that $|J_1(s)| \leq C$ for $|s| \leq 2$.

If $R = |s| \geq 2$, then, in view of (3.25),

$$|J_1(s)| \leq \langle s\rangle^{-r} \int dy\, |y|^{-1} \langle y+s\rangle^{-r} \int_{y^{\perp}} d_{y^{\perp}}x\, \langle x+y+s\rangle^{-r}.$$

Again, using (11.7) we estimate the internal integral, using (11.8) and (11.9) estimate the external and get that $|J_1(s)| \leq CR^{-r-1}$.

Thus the lemma's assertion also holds for $l = 1, 2$, and Lemma 4.2 is proved.

11.2. Proof of Theorem 4.4. Let us denote by $\{S_t, t \geq 0\}$ the semigroup of the linear equation, i.e. $S_t = e^{-t\mathcal{L}}$. Obviously, $|S_t|_{\mathcal{C}_r,\mathcal{C}_r} \leq e^{-2t}$. The solution $u^0(t,x)$ of the linear problem $(4.9)_{\varepsilon=0}$, (4.10) is given by the Duhamel integral

$$u^0(t) = \mathfrak{C}(u_0, f)(t) := S_t u_0 + \int_0^t S_{t-s} f(s)\, ds,$$

so

$$\|u^0\|_r = \|\mathfrak{C}(u_0, f)\|_r \leq |u_0|_r + \frac{1}{2}\|f\|_r. \tag{11.10}$$

A solution u for the problem (4.9), (4.10) is a fixed point of the operator

$$u \mapsto \mathfrak{C}(u_0, f + \varepsilon K(u)) =: \mathfrak{B}(u).$$

Assuming (4.13) and using (4.3) we see from (11.10) that

$$\|\mathfrak{B}(u)\|_r \leq \frac{3}{2}C_* + \frac{\varepsilon}{2}C_r^K \|u\|_r^3,$$

where C_r^K is the constant from (4.3). So the operator $\mathfrak{B}$ preserves the ball

$$B_R = \{u \in X_r, \|u\|_r \leq R\}, \quad R = \frac{3}{2}C_* + 1,$$

if $\varepsilon \ll 1$. Using Corollary 4.3 we see that the operator $\mathfrak{B}$ defines a contraction of the ball B_R if $\varepsilon \ll 1$. Accordingly eq. (4.9) has a unique solution $u \in B_R$.

Finally, if (u_{01}, f_1) and (u_{02}, f_2) are two sets of initial data, satisfying (4.13), and u^1, u^2 are the corresponding solutions, then

$$\|u^1 - u^2\|_r \leq |u_{01} - u_{02}|_r + \frac{1}{2}\|f_1 - f_2\|_r + \frac{3}{2}\varepsilon\, C_r^K R^2 \|u^1 - u^2\|_r$$

for all $t \geq 0$, so

$$\|u^1 - u^2\|_r \leq (1 - \frac{3}{2}\varepsilon\, C_r^K R^2)^{-1}(|u_{01} - u_{02}|_r + \frac{1}{2}\|f_1 - f_2\|_r).$$

This implies (4.14) if $\varepsilon \ll 1$ and the theorem is proved.

12. Addenda

12.1. Discrete turbulence. In order to study the double limit (1.9) it is natural to examine first the limit $\nu \to 0$ (with L and ρ fixed), known as the *limit of discrete turbulence*, see [24]. To do this consider the following *effective equation*:

$$\dot{\mathfrak{a}}_s+\gamma_s\mathfrak{a}_s = i\rho L^{-d}\Big(\sum\nolimits_{s_1}\sum\nolimits_{s_2}\delta'^{12}_{3s}\delta(\omega)\mathfrak{a}_{s_1}\mathfrak{a}_{s_2}\bar{\mathfrak{a}}_{s_3}-\mathfrak{a}_s|\mathfrak{a}_s|^2\Big)+b(s)\dot{\beta}_s, \quad s\in\mathbb{Z}^d_L, \quad (12.1)$$

where $\delta(\omega)$ is the delta-function of $\omega=\omega^{12}_{3s}$ (equal 1 if $\omega^{12}_{3s}=0$ and equal 0 otherwise). The following result is proven in [13,22].

Theorem 12.1. *If r_* is sufficiently big in terms of d, then eq. (12.1) is well posed and mixing. When L and ρ are fixed and $\nu\to 0$, then*

(i) *solutions of (1.14) converge in distribution, on time intervals of order 1, to solutions of (12.1) with the same initial data at $\tau=0$;*
(ii) *the unique stationary measure $\mu_{\nu,L}$ of (1.14) weakly converges to the unique stationary measure of eq. (12.1).*

12.2. Estimates for integrals with fast oscillating exponents, given by quadratic forms. Let φ be a complex L_1-function on $\mathbb{R}^n$ such that its Fourier transform $\hat\varphi\in L_1(\mathbb{R}^n)$, where in this appendix we define $\hat\varphi(\xi)$ as $\int e^{-ix\cdot\xi}\varphi(x)\,dx$. Let Q be a symmetric non-degenerate real $n\times n$-matrix. Consider the integral

$$I(\nu)=\int_{\mathbb{R}^n} e^{i\nu^{-1}x\cdot Qx/2}\bar\varphi(x)\,dx, \quad 0<\nu\le 1. \qquad (12.2)$$

The Fourier transform of the function $e^{i\nu^{-1}x\cdot Qx/2}=: F_0$ is

$$\hat F_0=(2\pi\nu)^{n/2}\zeta\,|\det Q|^{-1/2}\,e^{-i\nu\xi\cdot Q^{-1}\xi/2},$$

where ζ is some complex number of unit norm (see [5,14]). Formally applying Parseval's identity we get

$$I(\nu)=(2\pi)^{-n}\int \hat F_0\bar{\hat\varphi}\,d\xi=\Big(\frac{\nu}{2\pi}\Big)^{n/2}\zeta\,|\det Q|^{-1/2}\int_{\mathbb{R}^n} e^{-i\nu\xi\cdot Q^{-1}\xi/2}\overline{\hat\varphi(\xi)}\,d\xi. \quad (12.3)$$

To justify the validity of (12.3) in out situation, we approximate F_0 by functions

$$F_\varepsilon=e^{i\nu^{-1}x\cdot(Q+i\varepsilon I)x/2}, \qquad \varepsilon>0.$$

These are Schwartz functions whose Fourier transforms

$$\hat F_\varepsilon=(2\pi\nu)^{n/2}(\det iQ-\varepsilon I)^{-1/2}\,e^{-i\nu\xi\cdot(Q+i\varepsilon I)^{-1}\xi/2}$$

converge to $\hat F_0(\xi)$ for each ξ, as $\varepsilon\to 0$ (see [5,14]). For every $\varepsilon>0$ Parseval's identity holds for F_0 replaced by F_ε. Passing there to the limit as $\varepsilon\to 0$ using Lebesgue's theorem we recover (12.3). So

$$|I(\nu)|\le\Big(\frac{\nu}{2\pi}\Big)^{n/2}|\det Q|^{-1/2}|\hat\varphi|_{L_1}. \qquad (12.4)$$

We recall that

$$|\hat\varphi|_{L_1}\le C_m\|\varphi\|_{H^m(\mathbb{R}^n)} \quad \text{for any } m>n/2, \qquad (12.5)$$

where $H^m(\mathbb{R}^n)$ denotes the Sobolev space on $\mathbb{R}^n$.

12.3. Direct proof of estimate (3.14). In this appendix we estimate directly integral I_s in (3.4) with $T = \infty$ (indirectly this integral in the form (3.2) was estimated in (3.14) via Theorem 3.3). Setting $x = \sqrt{2}(s_1 - s)$ and $y = \sqrt{2}(s_2 - s)$, in view of (1.15) we get $\omega^{12}_{3s} = -x \cdot y$. Then, denoting $z = (x, y)$, we obtain

$$I_s = \int_{\mathbb{R}^{2d}} dz \int_{\mathbb{R}^2_-} dl \; F_s(l, z) e^{i\nu^{-1} x\cdot y(l_2 - l_1)}, \qquad l = (l_1, l_2), \tag{12.6}$$

where $F_s(l, z)$ is the function

$$2B(s_1, s_2, s_3) e^{-|l_1 - l_2|(\gamma_1+\gamma_2+\gamma_3)+\gamma_s(l_1+l_2)}, \quad s_3 = s_1 + s_2 - s,$$

written in the coordinates l, z. Since B is a Schwartz function and $\gamma_s \geq 1$, for $l \in \mathbb{R}^2_-$ the function F_s satisfies the estimate

$$|\partial_{z^\alpha} F_s(l, z)| \leq C^\#_\alpha(s) C^\#_\alpha(z) e^{(l_1+l_2)}, \qquad \forall \alpha. \tag{12.7}$$

Let $\mathcal{N}$ be the subset of $\mathbb{R}^2_- = \{l = (l_1, l_2)\}$ where $|l_1 - l_2| \geq \nu$, and $\mathcal{N}^c$ be its complement. Then, bounding the complex exponent in (12.6) by one, we find

$$|\langle I_s, \mathcal{N}^c \rangle| \leq C^\#(s)\nu, \tag{12.8}$$

where we recall that the notation $\langle I_s, \mathcal{N}^c\rangle$ was introduced in (1.44).

To estimate the term $\langle I_s, \mathcal{N}\rangle$, we note that the integral over dz in (12.6) has the form (12.2) with $\nu := \nu|l_1 - l_2|^{-1}$ and $n = 2d$. Then, due to (12.4) and (12.5),

$$|\langle I_s, \mathcal{N}\rangle| \leq \frac{1}{(2\pi)^d} \int_{\mathcal{N}} \frac{\nu^d}{|l_1 - l_2|^d} \big|\hat{F}_s(l, \cdot)\big|_{L^1} \, dl \leq C^\#(s)\nu^d \int_{\mathcal{N}} \frac{e^{(l_1+l_2)}}{|l_1 - l_2|^d} \, dl,$$

where in the last inequality we used (12.7) and the definition of the set $\mathcal{N}$. Since $d \geq 2$, this implies $|\langle I_s, \mathcal{N}\rangle| \leq C^\#_1(s)\nu^d/\nu^{d-1} = C^\#_1(s)\nu$. Combining the obtained inequality with (12.8), we get the desired estimate (3.14).

12.4. Proof of Proposition 2.1. We start by explaining the scheme of the proof. Let us express the process $a^{(n)}_s$ through the processes $a^{(0)}_k$ by iterating formula (2.4) n times and compute the expectation $\mathbb{E}|a^{(n)}_s(\tau)|^2$ applying the Wick theorem. It can be shown that when $L \to \infty$ this expectation stays of size one. The reason for this is as follows: arguing by induction we see that an affine space of variables $k \in \mathbb{Z}^d_L$ which serves as the set of indices over which we take summation in the expression for $|a^{(n)}_s(\tau)|^2$ through $a^{(0)}_k$, $\bar{a}^{(0)}_k$ has dimension $4nd$ (indeed, for $n = 0$ this dimension obviously is zero, while for $n = 1$ due to (2.3) and the conjugated formula it is $4d$, etc.).

When computing the expectation $\mathbb{E}|a^{(n)}_s(\tau)|^2$ we make the Wick pairing of terms $a^{(0)}_k$, $\bar{a}^{(0)}_{k'}$ and it is possible to see that the dimension of the corresponding space of variables becomes $2nd$ (since in view of (2.8) for Wick-coupled terms $a^{(0)}_k$ and $\bar{a}^{(0)}_{k'}$ we should have $k = k'$).[6] At the same time, after n iterations of (2.4) we get a factor

[6] This is true for the most of Wick-pairings, while for some of them the affine space of variables may become empty because of the restrictions of the type $\{s_1, s_2\} \neq \{s_3, s\}$ imposed by δ'^{12}_{3s}.

L^{-2nd} in the formula for $\mathbb{E}|a_s^{(n)}(\tau)|^2$; this leads to the claim. For an example of such computation see Section 2.4. Similarly, $\mathbb{E}|\mathfrak{a}_s^{(n)}(\tau)|^2$ stays of order one when $L \to \infty$.

Now we express the difference $\Delta_s^n(\tau) := \mathfrak{a}_s^{(n)}(\tau) - a_s^{(n)}(\tau)$ through the processes $a_k^{(0)}$ and write $\Delta_s^n(\tau)$ as a finite sum $\Delta_s^n = \sum \Delta_s^{n,j}$. Each term $\Delta_s^{n,j}$ is obtained by iterating (2.4), where at at least one iteration $\delta_{3s}^{\prime 12}$ is replaced by $-\delta_{s_2}^{s_1}\delta_{s_3}^{s_2}\delta_s^{s_3}$ (this corresponds to the "diagonal" term $-\mathfrak{a}_s^{(n_1)}\mathfrak{a}_s^{(n_2)}\bar{\mathfrak{a}}_s^{(n_3)}$ in (1.23)).

This implies that dimension of the affine space of variables over which we take the summation when computing $\mathbb{E}|\Delta_s^{n,j}(\tau)|^2$ drops at least by $2d$ in comparison with that for $\mathbb{E}|a_s^{(n)}(\tau)|^2$ and $\mathbb{E}|\mathfrak{a}_s^{(n)}(\tau)|^2$. At the same time we still have the factor L^{-2nd} in the formula for $\mathbb{E}|\Delta_s^{n,j}|^2$. This implies the desired estimate $\mathbb{E}|\Delta_s^{n,j}|^2 \le C^\#(s)L^{-2d}$.

We give a complete proof only for the cases $n = 1, 2$ since in Theorem C which is the main result of this paper we only deal with $a^{(n)}$ and $\mathfrak{a}^{(n)}$ such that $n \le 2$. A general case can be considered similarly, by analyzing the dimensions of the affine spaces of variables over which we take the summation in the formula for Δ^n. In particular, no delicate cancellation argument is used. However, the proof is cumbersome due to the notational difficulty, arising when expressing Δ^n through $a^{(0)}$.

By (1.22) and (2.3),

$$\Delta_s^1(\tau) = -iL^{-d}\int_{-T}^{\tau} e^{-\gamma_s(\tau-l)}|a_s^{(0)}(l)|^2 a_s^{(0)}(l)\,dl, \tag{12.9}$$

so in view of (2.8) the Wick theorem implies $\mathbb{E}|\Delta_s^1(\tau)|^2 \le C^\#(s)L^{-2d}$. Next, $\Delta_s^2 = 2\Delta_s^{2,1} + \Delta_s^{2,2}$, where the term

$$\begin{aligned}\Delta_s^{2,1}(\tau) = iL^{-d}\int_{-T}^{\tau} e^{-\gamma_s(\tau-l)}\Big(&\sum_{s_1,s_2}\delta_{3s}^{\prime 12}(\Delta_{s_1}^1 a_{s_2}^{(0)}\bar{a}_{s_3}^{(0)})(l)e^{i\nu^{-1}l\omega_{3s}^{12}}\\ &-\mathfrak{a}_s^1(l)|a_s^{(0)}(l)|^2\Big)\,dl = \Delta_s^{2,1;1}(\tau) - \Delta_s^{2,1;2}(\tau)\end{aligned} \tag{12.10}$$

corresponds to the choice $n_1 = 1$ and $n_2 = n_3 = 0$ in (1.23) with $n = 2$. The term $\Delta_s^{2,2}$ corresponds to the choice $n_1 = n_2 = 0,\ n_3 = 1$ and has a similar form. Below we only discuss $\Delta^{2,1}$.

Because of the factor L^{-d} in (12.10) it is straightforward to see that $\mathbb{E}|\Delta_s^{2,1;2}(\tau)|^2 \le C^\#(s)L^{-2d}$. Let us study the term $\Delta_s^{2,1;1}$. By (12.10) joined with (12.9),

$$\begin{aligned}\mathbb{E}|\Delta_s^{2,1;1}(\tau)|^2 = L^{-4d}&\int_{-T}^{\tau}dl\int_{-T}^{l}dl_1\int_{-T}^{\tau}dl'\int_{-T}^{l'}dl_1'\,e^{-\gamma_s(2\tau-l-l')}\\ &\sum_{s_1,s_2,s_1',s_2'}\delta_{3s}^{\prime 12}\delta_{3's}^{\prime 1'2'}e^{-\gamma_1(l-l_1)-\gamma_{1'}(l'-l_1')}e^{i\nu^{-1}(l\omega_{3s}^{12}-l'\omega_{3's}^{1'2'})}\\ &\mathbb{E}\Big(a_1^{(0)}(l_1)|a_1^{(0)}(l_1)|^2\,a_2^{(0)}(l)\bar{a}_3^{(0)}(l)\,\bar{a}_{1'}^{(0)}(l_1')|a_{1'}^{(0)}(l_1')|^2\,\bar{a}_{2'}^{(0)}(l')a_{3'}^{(0)}(l')\Big).\end{aligned} \tag{12.11}$$

Next we apply the Wick theorem. Due to (2.8), non-conjugated variables $a_k^{(0)}$ can be coupled with conjugated variables $\bar{a}_{k'}^{(0)}$ only, and k should equals to k'. Consider e.g. the case when $a_1^{(0)}$ is coupled with $\bar{a}_1^{(0)}$, $a_1^{(0)}$ with $\bar{a}_{1'}^{(0)}$, $a_2^{(0)}$ with $\bar{a}_{2'}^{(0)}$, $a_{3'}^{(0)}$ with $\bar{a}_3^{(0)}$ and $a_{1'}^{(0)}$

with $\bar{a}^{(0)}_{1'}$ (we write $|a_k|^2$ as $a_k \bar{a}_k$). Then, due to (2.8), the corresponding to this Wick pairing term in the expression for $\mathbb{E}|\Delta_s^{2,1;1}(\tau)|^2$ does not exceed

$$CL^{-4d} \sum_{s_1, s_2} \delta'^{12}_{3s} \big(B(s_1)\big)^3 B(s_2) B(s_3) \le L^{-2d} C^{\#}(s),$$

where $B(s)$ is defined in (1.28). The other Wick pairings can be considered similarly—the dimension of the space of variables over which we take the summation always does not exceed $2d$, so the resulting estimate will be the same.

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