EMBEDDINGS OF GROUPS $\mathrm{Aut}(F_n)$ INTO AUTOMORPHISM GROUPS OF ALGEBRAIC VARIETIES

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Abstract. For each integer $n > 0$, we construct a series of irreducible algebraic varieties X, for which the automorphism group $\mathrm{Aut}(X)$ contains as a subgroup the automorphism group $\mathrm{Aut}(F_n)$ of a free group F_n of rank n. For $n \geqslant 2$, such groups $\mathrm{Aut}(X)$ are nonamenable, and for $n \geqslant 3$, they are nonlinear and contain the braid group B_n. Some of these varieties X are affine, and among affine, some are rational and some are not, some are smooth and some are singular. The byproduct is that for $n \geqslant 3$, each Cremona group of rank $\geqslant 3n$ contains $\mathrm{Aut}(F_n)$ and the braid group B_n.

1. In the last decade, studying biregular and birational automorphism groups of algebraic varieties, in particular, their abstract group properties and their subgroups, has become a trend. In terms of popularity, probably the leaders among the groups in the focus of these studying are the Cremona groups.

Here we construct, for each integer $n > 0$, a series of irreducible algebraic varieties whose automorphism group contains as a subgroup the automorphism group $\mathrm{Aut}(F_n)$ of a free group F_n of rank n.

This finds an application to the linearity problem for automorphism groups of algebraic varieties considered in [8, Prop. 5.1], [9], [6]: the existence of such a subgroup implies that the automorphism groups of the constructed varieties are nonamenable for $n \geqslant 2$ and nonlinear for $n \geqslant 3$. In addition, since some important groups (for example, the braid group B_n for $n \geqslant 3$) are the subgroups of $\mathrm{Aut}(F_n)$, they get a realization in the form of subgroups of automorphism groups of the constructed algebraic varieties.

Among these varieties some are affine (some of them smooth and some with singularities), and among them some are rational and some are nonrational (and even are not stably rational).

As another application we obtain that for $n \geqslant 3$, each Cremona group of rank $\geqslant 3n$ contains the group $\mathrm{Aut}(F_n)$ and the braid group B_n. This bound for the rank of the Cremona group containing B_n is better than that following from [13, Thm. 4.6], where the linearity of the group B_n is proved.

2. In what follows, algebraic varieties are considered over an algebraically closed field k. With regard to algebraic geometry and algebraic groups, we follow conventions and notation from [2].

Groups are considered in multiplicative notation.

When saying that a group G contains a group H, we mean the existence of a monomorphism $\iota\colon H \hookrightarrow G$, by which H is identified with $\iota(H)$.

$\mathscr{C}(G)$ denotes the center of a group G.

$\mathrm{Sym}(X)$ denotes the symmetric group of a set X.

If X is an algebraic variety (respectively, a differentiable manifold), then the subgroup of $\mathrm{Sym}(X)$ consisting of all automorphisms (respectively, diffeomorphisms) of X is denoted by $\mathrm{Aut}(X)$. To avoid confusion, if X is an algebraic group or a real Lie group, $\mathrm{Aut}(\underline{X})$ denotes the automorphism group of its underlying algebraic variety (respectively, manifold), so that $\mathrm{Aut}(X)$ is a subgroup of $\mathrm{Aut}(\underline{X})$, whose elements are group automorphisms of X.

3. Consider an integer $n > 0$, an abstract group G with the identity element e, and the group

$$X := G^{\times n} := G \times \cdots \times G \quad (n \text{ factors}). \tag{1}$$

We fix in a free group F_n of rank n a free system of generators $f_1, \ldots, f_n$ and denote the unit element in $\mathrm{Aut}(F_n)$ by 1.

For any elements $w \in F_n$ and

$$x = (g_1, \ldots, g_n) \in X, \ \ g_j \in G, \tag{2}$$

denote by

$$w(x) = w(g_1, \ldots, g_n)$$

the element of G that is the image of element w under the homomorphism $F_n \to G$ that maps f_j to g_j for each j. In other words, $w(x)$ is obtained from the word w in $f_1, \ldots, f_n$ by substituting g_j in place of f_j for each j.

Any element $\sigma \in \mathrm{End}(F_n)$ is uniquely determined by the sequence of elements $\sigma(f_1), \ldots, \sigma(f_n)$, and any sequence of elements of F_n of length n is of this form for some σ. This sequence defines the following map (which, generally speaking, is not an endomorphism of the group X):

$$\sigma_X\colon X \to X, \quad x \mapsto (\sigma(f_1)(x), \ldots, \sigma(f_n)(x)). \tag{3}$$

Some properties of such maps are listed below.

Proposition 1. *Let σ and τ be any elements of $\mathrm{End}(F_n)$. Then:*

(a) $(\sigma\circ\tau)_X=\tau_X\circ\sigma_X$.
(b) $1_X=\mathrm{id}$.
(c) $\sigma_X(S^{\times n})\subseteq S^{\times n}$ *for any subgroup S of the group G.*
(d) *The following properties of element* (2) *are equivalent:*
  (a) $\sigma_X(x)=x$ *for each* $\sigma\in\mathrm{Aut}(F_n)$;
  (b) *if $n>1$, then $g_1=\ldots=g_n=e$, and if $n=1$, then $g_1^2=e$.*
(e) *Let $\gamma\colon G\to H$ be a group homomorphism, and $Y:=H^{\times n}$. Then the map*

$$\gamma_n\colon X\to Y,\quad (g_1,\ldots,g_n)\mapsto(\gamma(g_1),\ldots,\gamma(g_n))$$

is $\mathrm{End}(F_n)$-equivariant, i.e., $\gamma_n\circ\sigma_X=\sigma_Y\circ\gamma_n$.
(f) $\sigma_X(xz)=\sigma_X(x)\sigma_X(z)$ *for all $x\in X$, $z\in\mathscr{C}(X)$. In particular, the restriction of σ_X to the group $\mathscr{C}(X)$ is its endomorphism.*
(g) σ_X *commutes with the diagonal action of G on X by conjugation.*
(h) *If G is an algebraic group (respectively, a real Lie group), then σ_X is a morphism (respectively, a differentiable mapping).*

Proof. Statement (d) follows from the fact that if $n=1$, then the only element of the group $\mathrm{Aut}(F_n)$ not equal to 1 maps f_1 to f_1^{-1}, and if $n\geqslant 2$, then for any $i,j\in\{1,\ldots,n\}$, $i\neq j$, the element $\sigma_{ij}\in\mathrm{End}(F_n)$ defined by formula

$$\sigma_{ij}(f_l)=\begin{cases} f_l & \text{if } l\neq i,\\ f_if_j & \text{if } l=i,\end{cases}$$

lies in $\mathrm{Aut}(F_n)$.

The rest of the statements follow directly from the definitions and the fact that each element of the group F_n is written as a non-commutative Laurent monomial in $f_1,\ldots,f_n$, i.e.,

$$f_{i_1}^{\varepsilon_1}\cdots f_{i_s}^{\varepsilon_s},\quad \text{where } \varepsilon_j\in\mathbb{Z} \tag{4}$$

(uniquely defined if monomial (4) is reduced). □

4. It follows from statements (a), (b), (g) of Proposition 1 that if $\sigma\in\mathrm{Aut}(F_n)$, then $\sigma_X\in\mathrm{Sym}(X)$ with $(\sigma_X)^{-1}=(\sigma^{-1})_X$, and the map

$$\mathrm{Aut}(F_n)\to\mathrm{Sym}(X),\quad \sigma\mapsto(\sigma^{-1})_X, \tag{5}$$

is a group homomorphism. Moreover, if G is an algebraic group or a real Lie group, then $\sigma_X\in\mathrm{Aut}(\underline{X})$ and we obtain a group homomorphism

$$\mathrm{Aut}(F_n)\to\mathrm{Aut}(\underline{X}),\quad \sigma\mapsto(\sigma^{-1})_X. \tag{6}$$

Theorem 1.

(a) *If the group G is solvable and $n \geqslant 3$, then homomorphism* (5) *is not an embedding.*

(b) *Let the group G be nonsolvable and any of the following properties hold:*

(b$_1$) *G is a connected algebraic group, and the field k is uncountable if* $\operatorname{char}(k) > 0$.

(b$_2$) *G is a connected real Lie group.*

Then homomorphism (6) *is an embedding.*

Proof. (a) Let $n \geqslant 3$ and let the group G be solvable, i.e., the descending series of its successive commutator subgroups $G = \mathscr{D}^0(G) \supseteq \ldots \supseteq \mathscr{D}^i(G) \supseteq \cdots$ terminates at some step with a number s:

$$\mathscr{D}^s(G) = \{e\}. \tag{7}$$

Setting $[a, b] := aba^{-1}b^{-1}$, we define inductively the elements $\theta_i \in F_n$ as follows:

$$\begin{aligned} &\theta_1 := [f_1, f_2],\\ &\theta_i := [\theta_{i-1}, f_d], \quad \text{where } d = \begin{cases} 1 & \text{if } i \text{ is even,}\\ 2 & \text{if } i \text{ is odd,}\end{cases} \quad \text{for } i \geqslant 2. \end{aligned} \tag{8}$$

From (8) it follows that θ_i for any i is a nonempty reduced word in f_1 and f_2. Consider the elements $\sigma, \tau \in \operatorname{End}(F_n)$ given by the equalities

$$\sigma(f_i) := \begin{cases} f_i & \text{if } i < n,\\ f_n\theta_s & \text{if } i = n,\end{cases} \qquad \tau(f_i) := \begin{cases} f_i & \text{if } i < n,\\ f_n\theta_s^{-1} & \text{if } i = n.\end{cases} \tag{9}$$

Since $n \geqslant 3$, and θ_s is a word only in f_1, f_2, it follows from (9) that $\sigma \circ \tau = \tau \circ \sigma = 1$. Hence σ is a nonidentity element of the group $\operatorname{Aut}(F_n)$ with $\sigma^{-1} = \tau$.

Consider now an element (2) from X. In view of (9), we have

$$\sigma(f_i)(x) = f_i(x) = g_i \quad \text{for } i < n \tag{10}$$

and $\sigma(f_n)(x) = (f_n\theta_s)(x)$. It follows from (8) that $\theta_s(x) \in \mathscr{D}^s(G)$. In view of (7), this yields $\theta_s(x) = e$. Therefore,

$$\sigma(f_n)(x) = f_n(x) = g_n. \tag{11}$$

Thus, in view of (2), (3), (10), (11), we have $\sigma_X = \mathrm{id}$. Hence, σ is a nonidentity element of the kernel of homomorphism (6). This proves statement (a).

(b) If $\operatorname{char}(k) = 0$, then by the Lefschetz principle we can assume $k = \mathbb{C}$, and then G in case (b$_1$) is a real Lie group and therefore it suffices to give a proof only for case (b$_2$). If no restrictions are imposed on char(k), then (b$_1$) admits some reduction, after which the proof for (b$_1$) and (b$_2$) follows the same line.

Namely, if G is a connected algebraic group, by Chevalley's theorem it contains a largest connected affine normal subgroup G_{aff}, and the group G/G_{aff} is an abelian variety. If G is nonsolvable, then since G/G_{aff} is abelian (hence solvable), G_{aff} is nonsolvable. Therefore, G_{aff} does not coincide with its radical $\mathrm{Rad}(G_{\mathrm{aff}})$ and hence $G_{\mathrm{aff}}/\mathrm{Rad}(G_{\mathrm{aff}})$ is a nontrivial connected semisimple algebraic group. In view of Proposition 1(c), this reduces the proof of statement (b) to the case when G in (b$_1$) is a nontrivial connected semisimple algebraic group. We will therefore assume now that in (b$_1$) this additional condition is satisfied. Then the group G in statement (b) contains F_m for any m: for case (b$_1$), this is proved in [4, Thm. 1.1], [5, App. D], and for case (b$_2$), in [11, Thm.]. We can therefore find in G a free subgroup S of rank n.

Arguing by contradiction, suppose now that homomorphism (6) is not an embedding, and let $\sigma \in \mathrm{Aut}(F_n)$ be a nonidentity element of its kernel. Since $\sigma \neq 1$, there is a number i such that the reduced word $\sigma(f_i)$ in $f_1, \ldots, f_n$ is different from f_i. Therefore, $w := \sigma(f_i)f_i^{-1}$ is a nonidentity element of the group F_n. Consider element (2), in which $g_1, \ldots, g_n$ is a free generating system of the group S. Since σ lies in the kernel of homomorphism (6), we have $\sigma_X(x) = x$, hence (3) yields $\sigma(f_i)(x) = g_i = f_i(x)$, and therefore $w(x) = e$. Thus, w is a nontrivial relation between the elements $g_1, \ldots, g_n$ despite the fact that $g_1, \ldots, g_n$ is a free system of generators of the group S. This proves statement (b). □

5. Corollaries 1 and 2 formulated below follow from Theorem 1 in view of the following Proposition 2:

Proposition 2. *Let H be a group containing* $\mathrm{Aut}(F_n)$. *Then*

(a) *The group H contains* $\mathrm{Aut}(F_s)$ *for any $s \leqslant n$.*
(b) *If $n \geqslant 2$, then the group H contains F_m for any m and is nonamenable.*
(c) *If $n \geqslant 3$, then the group H is nonlinear.*
(d) *If $n \geqslant 3$, then the group H contains the braid group B_n.*

Proof. Clearly the group $\mathrm{Aut}(F_n)$ contains $\mathrm{Aut}(F_s)$ for any $s \leqslant n$. This gives (a).

The group $\mathscr{C}(F_n)$ is trivial for $n \geqslant 2$ (see [14, Chap. I, Prop. 2.19]). Therefore, the group $\mathrm{Int}(F_n)$ is isomorphic to F_n, and hence (see [14, Chap. I, Prop. 3.1]) contains F_m for every m. This gives (b).

The group $\mathrm{Aut}(F_n)$ is nonlinear for $n \geqslant 3$ (see [12]). This gives (c).

The group $\mathrm{Aut}(F_n)$ contains B_n for $n \geqslant 3$ (see [15, Chap. 3, 3.7]). This gives (d). □

Corollary 1. *Let the group G be nonsolvable and let it be either a connected algebraic group with uncountable field k for* $\mathrm{char}(k) > 0$*, or a connected real Lie group. Then for any integers $n > 0$ and $0 < s \leqslant n$, the group* $\mathrm{Aut}(\underline{G^{\times n}})$ *contains* $\mathrm{Aut}(F_s)$*, and for $n \geqslant 3$, it is nonlinear and contains the braid group B_n.*

Remark 1. The action G on itself by left or right translations defines an embedding of G into $\mathrm{Sym}(G)$. Therefore, $\mathrm{Sym}(G^{\times n})$ contains $G^{\times n}$. If G is a connected algebraic group or a real Lie group, then this construction shows that $\mathrm{Aut}(\underline{G^{\times n}})$ contains the group $G^{\times n}$ acting on $G^{\times n}$ simply transitively. If, moreover, G is nonsolvable, then, as noted in the proof of Theorem 1, it contains F_m for any m and, therefore, is nonamenable. Hence the same is true for the group $\mathrm{Aut}(\underline{G^{\times n}})$.

Example 1. If we take $G = \mathrm{SL}_2(k)$, where the field k is uncountable for $\mathrm{char}(k) > 0$, then the underlying variety of the group $G^{\times n}$ is isomorphic to $Q \times \cdots \times Q$ (n factors), where Q is the affine quadric in $\mathbb{A}^4$ defined by the equation $x_1x_2 + x_3x_4 = 1$. Thus, according to Corollary 1, the automorphism group of this irreducible smooth affine algebraic variety contains $\mathrm{Aut}(F_s)$ for any $s \leqslant n$, and if $n \geqslant 3$, then it is nonlinear and contains the braid group B_n. The same is true if, for $\mathrm{char}(k) \neq 2$, one replaces Q with Q/Z, where Z is the group of order two generated by the involution $(a, b, c, d) \mapsto (-a, -b, -c, -d)$: this follows from Corollary 1 for $G = \mathrm{PSL}_2(k)$.

Example 2. The following construction [18, Sect. 2, Example 4] gives an embedding of the group $\mathrm{Aut}(F_n)$ into the automorphism group of an affine open subset of the affine space $\mathbb{A}^m$ for some m, and hence into the Cremona group Cr_n of rank m. In principle, it allows one to describe the birational transformations of the space $\mathbb{A}^m$ that lie in the image of $\mathrm{Aut}(F_n)$ by means of explicit formulas.

Namely, consider a d-dimensional associative k-algebra A with an identity element. Having fixed its basis, we identify the set A with $\mathbb{A}^d$. The group A^* of invertible elements of the algebra A is a connected affine algebraic group whose underlying variety is an affine open subset of $\mathbb{A}^d$. If A^* is nonsolvable, then formulas (3), (6) for $G = A^*$ define an embedding of $\mathrm{Aut}(F_n)$ into the Cremona group of rank nd. For example, this is the case for

$$A = \mathrm{Mat}_{s\times s}(k), \quad s \geqslant 2:$$

in this case, A^* is the nonsolvable group GL_s, and explicit description of birational transformations σ_X in coordinates is reduced to multiplying functional matrices and their inverses.

Corollary 2. *For any integers $n > 0$, $r \geqslant 3n$, and $0 < s \leqslant n$, the Cremona group Cr_r of rank r contains $\mathrm{Aut}(F_s)$, and for $n \geqslant 3$, contains the braid group B_n.*

Proof. Let G be the group $\mathrm{SL}_2(k)$. Its underlying variety is three-dimensional and rational. Therefore, Cr_{3n} contains $\mathrm{Aut}(\underline{G^{\times n}})$. The statement now follows from Corollary 1 and the fact that Cr_{r+1} contains Cr_r for any r. □

6. Since Corollary 2 concerns subgroups of the Cremona group, it is appropriate to complement it here with the following remark on Cantat's question about these subgroups.

Remark 2. In [9] are given examples of finitely generated (and even finitely presented) groups that do not admit embeddings into any Cremona group, which gives an answer to Cantat's question about the existence of such groups (see also [7]). These examples are based on Theorem 1.2 of [10], according to which the word problem is solvable in every finitely generated subgroup of any Cremona group. However, the answer to the above question — and even in a stronger form, with the addition of the condition of simplicity of the subgroup — can be obtained without using this theorem.

Namely, according to [20, p. 188, Example 6], Richard Thompson's group V is an example of a non-Jordan finitely presented group. Since any Cremona group is Jordan (see [1, Cor. 1.5]), the group V cannot be embedded in it. Moreover, in addition to this property, V is simple, and therefore any homomorphism of the group V into a Cremona group is trivial (unlike [10], this proves Corollary 1.4 of [10] without usage of the amplification, obtained in [16], of the Boone–Novikov construction).

7. Let G be a connected (not necessarily affine) algebraic group. Below we show that the variety $X = G^{\times n}$ is only one of the "extreme" cases in a series of algebraic varieties related to X, on which there is a natural action of the group $\mathrm{Aut}(F_n)$. For some of them (but not for all), this action is faithful (i.e., its kernel is trivial), which gives new examples of algebraic varieties, the automorphism group of which contains $\mathrm{Aut}(F_n)$, and therefore, by virtue of Proposition 2, for $n \geqslant 2$, is non-amenable, and for $n \geqslant 3$, is nonlinear and contains the braid group B_n. Unlike X, they are no longer presented in the form of a Cartesian power of an algebraic variety, not necessarily smooth, not endowed with a simply transitive action of the group $G^{\times n}$, and among those of them that are affine, there are nonrational (and even not stably rational).

To obtain such varieties one can, first, replace X with an appropriate $\mathrm{Aut}(F_n)$-invariant open subset U of X (in general, U is not affine, even if G is affine). Such subsets do exist. For example, the set of points whose G-orbits have maximal dimension with respect to the diagonal action of G on X by conjugation is open in X, and from Proposition 1(g) it follows that it is $\mathrm{Aut}(F_n)$-invariant. If the action of $\mathrm{Aut}(F_n)$ on X is faithful, then the openness condition of U ensures faithfulness of the action of $\mathrm{Aut}(F_n)$ on U.

Next, the transition from X to U can be complemented with the following construction. Consider a closed subgroup S in G and its diagonal action on X by conjugation. Suppose that X contains an open subset U that is invariant under both $\mathrm{Aut}(F_n)$ and S and admits a geometric or categorical quotient under the action of S. Then, in view of Proposition 1(g), the action of $\mathrm{Aut}(F_n)$ on U descends to the quotient variety. The results obtained below show that under certain conditions this action is faithful.

Below we explore two cases, in which $U = G$. In the first, the group G is affine, and the group S is reductive, while in the second, no restrictions are imposed on the group G, and the group S is finite. In the corresponding criteria for an element $\sigma \in \mathrm{Aut}(F_n)$ to belong to the kernel of the action of the group $\mathrm{Aut}(F_n)$ on the quotient variety, we use the following notation:

If $w \in F_n$, $a \in G$, and $i \in \{1, \ldots, n\}$, we put

$$X_{w,a,i} := \{x = (g_1, \ldots, g_n) \in X \mid w(x)a^{-1}g_i^{-1}a = e\}. \tag{12}$$

We have $(e, \ldots, e) \in X_{w,a,i}$, therefore, $X_{w,a,i} \neq \varnothing$. Being the fiber of the morphism

$$X \to G, \quad x = (g_1, \ldots, g_n) \mapsto w(x)a^{-1}g_i^{-1}a$$

over the point e, the set $X_{w,a,i}$ is closed in X.

8. Turning to the first case, we assume that G is a connected affine algebraic group, and S is its reductive closed subgroup.

Consider the diagonal action of the group S on the affine algebraic variety $X = G^{\times n}$ by conjugation. Then the k-algebra $k[X]^S$ is finitely generated, and if $X/\!\!/S$ is the affine algebraic variety with $k[X/\!\!/S] = k[X]^S$, and

$$\pi\colon X \to X/\!\!/S,$$

is the morphism determined by the identity embedding $k[X]^S \hookrightarrow k[X]$, then the pair $(\pi, X/\!\!/S)$ is the categorical quotient for the action of S on

X. In view of Proposition 1(g), any comorphism σ_X^* preserves the algebra $k[X]^S$. Therefore, it indices an automorphism $\sigma_{X/\!\!/S} \in \operatorname{Aut}(X/\!\!/S)$, for which

$$\pi \circ \sigma_X = \sigma_{X/\!\!/S} \circ \pi. \tag{13}$$

It arises a homomorphism

$$\operatorname{Aut}(F_n) \to \operatorname{Aut}(X/\!\!/S), \quad \sigma \mapsto (\sigma^{-1})_{X/\!\!/S}. \tag{14}$$

Its kernel is described as follows:

Lemma 1. *Let G be a connected affine algebraic group, and let S be its reductive closed subgroup. The following properties of an element $\sigma \in \operatorname{Aut}(F_n)$ are equivalent:*

(a) *σ lies in the kernel of homomorphism* (14).
(b) *Every closed S-orbit lies in the set*

$$\textstyle\bigcup_{s\in S} \Big(\bigcap_{i=1}^{n} X_{\sigma(f_i),s,i}\Big). \tag{15}$$

Proof. The morphism π is surjective, its fibers are S-invariant, and for every point $b \in X/\!\!/S$, the fiber $\pi^{-1}(b)$ contains a unique closed S-orbit $\mathcal{O}_b$ (see [17, §2 and Append. 1B]). It follows from (13) that the restriction of the morphism σ_X to the fiber $\pi^{-1}(b)$ is its S-equivariant isomorphism with the fiber $\pi^{-1}(\sigma_{X/\!\!/S}(b))$. In view of the uniqueness of closed orbits in the fibers, this means that $\sigma_X(\mathcal{O}_b) = \mathcal{O}_{\sigma_{X/\!\!/S}(b)}$. Therefore, $\sigma_{X/\!\!/S}(b) = b$ if and only if $\sigma_X(\mathcal{O}_b) = \mathcal{O}_b$. In view of (3) and definition (12), this implies the equivalence of (a) and (b). □

9. In the situation under consideration two extreme cases occur.

The first is the case of $S = \{e\}$. It is considered in Theorem 1, which shows that, depending on G, S, and k, both possibilities are realized for homomorphism (14): in one, (14) is an embedding, and in the other, it is not.

The second is the case of $S = G$ (so that G is reductive). It is considered in the following theorem:

Theorem 2. *Let G be a connected reductive algebraic group, and let $S = G$.*

(a) *If $n \geqslant 2$, then homomorphism* (14) *is not an embedding.*
(b) *If $n = 1$, then the following properties are equivalent:*
    (b_1) *Homomorphism* (14) *is an embedding.*
    (b_2) *The group G contains a connected simple normal subgroup either of the type D_ℓ, where ℓ is odd, or of the type E_6.*

Proof. Let $n > 1$ and let $\sigma, \tau \in \mathrm{End}(F_n)$ are given by the formulas

$$\sigma(f_i) = \begin{cases} f_1 & \text{if } i = 1, \\ f_1 f_i f_1^{-1} & \text{if } i > 1, \end{cases} \quad \tau(f_i) = \begin{cases} f_1 & \text{if } i = 1, \\ f_1^{-1} f_i f_1 & \text{if } i > 1. \end{cases} \tag{16}$$

Then $\sigma \circ \tau = \tau \circ \sigma = 1$. Therefore, $\sigma \in \mathrm{Aut}(F_n)$, and $\tau = \sigma^{-1}$.

Now let $x \in X$ be a point (2). Then it follows from (3), (16) that

$$\sigma_X(x) = (g_1, g_1 g_2 g_1^{-1}, \ldots, g_1 g_n g_1^{-1}) = g_1 \cdot x,$$

so x and $\sigma_X(x)$ lie in the same S-orbit. Therefore, the point $\pi(x)$ is fixed with respect to $\sigma_{X/\!\!/S}$. Since π is surjective, this shows that σ lies in the kernel of homomorphism (14). In view of $\sigma \neq 1$, this proves (a).

Now let $n = 1$, so that $X = G$. Then $\mathrm{Aut}(F_n)$ is the group of order two, and if $\sigma \in \mathrm{Aut}(F_n)$, $\sigma \neq 1$, then $\sigma(f_1) = f_1^{-1}$, so that $\sigma_X(g) = g^{-1}$ for any $g \in G$. Every fiber of the morphism π contains a unique orbit consisting of semisimple elements, and it is the unique closed orbit in this fiber (see [24]).

From this and Lemma 1 the equivalence of the following properties follows:

(i) σ lies in the kernel of homomorphism (14);
(ii) g and g^{-1} are conjugate for any semisimple element $g \in G$.

Since the intersection of any semisimple conjugacy class with a fixed maximal torus of the group G is nonempty (see [2, Thm. 11.10]) and is an orbit of the normalizer of this torus (see [24, 6.1] or [23, 1.1.1]), property (ii) is equivalent to the fact that the Weyl group of the group G contains -1. This, in turn, is equivalent to the fact that -1 is contained in the Weyl group of every nontrivial connected simple normal subgroup of G. As is known (see [3, Tabl. I–IX]), the Weyl group of a nontrivial connected simple algebraic group does not contain -1 exactly if it is either of the type D_ℓ, where ℓ is odd, or of the type E_6. This completes the proof. □

10. Now we consider the second case, when G is any (not necessarily affine) connected algebraic group, and S is its finite subgroup.

According to [22, Prop. 19 and Example 2) on p. 50], in this case there exist an algebraic variety X/S and a morphism

$$\rho\colon X \to X/S \tag{17}$$

such that the pair $(\rho, X/S)$ is the geometric quotient for the action of S on X. In particular,

$$\rho^*(k(X/S)) = k(X)^S. \tag{18}$$

For each $\sigma \in \mathrm{Aut}(F_n)$, from Proposition 1(g) and the properties of geometric quotient it follows the existence of $\sigma_{X/S} \in \mathrm{Aut}(X/S)$ such that

$$\rho \circ \sigma_X = \sigma_{X/S} \circ \rho.$$

It arises a homomorphism

$$\mathrm{Aut}(F_n) \to \mathrm{Aut}(X/S), \quad \sigma \mapsto (\sigma^{-1})_{X/S}. \tag{19}$$

Its kernel is described as follows:

Lemma 2. *Let G be a connected algebraic group, and let S be its finite subgroup. The following properties of an element $\sigma \in \mathrm{Aut}(F_n)$ are equivalent:*

(a) *σ lies in the kernel of homomorphism* (19)*.*

(b) *There exists an element $s \in S$ such that*

$$X = X_{\sigma(f_1),s,1} = \ldots = X_{\sigma(f_n),s,n}. \tag{20}$$

If the group G is nonsolvable, the field k is uncountable if $\mathrm{char}(k) > 0$*, and* (b) *holds, then the following conditions are equivalent:*

(b_1) *$\sigma = 1$;*

(b_2) *$s \in \mathscr{C}(G)$.*

Proof. Since the group S is finite, every S-orbit is closed in X. Since (17) is the geometric factor, the morphism ρ is surjective and each fiber is an S-orbit. The argument analogous to that used in the proof of Lemma 1 shows that condition (a) is equivalent to the condition that each S-orbit lies in set (15), that is, the condition that set (15) coincides with the whole set X.

Since the algebraic variety X is irreducible and set (15) is the union of a finite (because S is finite) collection of closed sets of the form $\bigcap_{i=1}^n X_{\sigma(f_i),s,i}$, the variety X must coincide with one of them. Finally, the equality $X = \bigcap_{i=1}^n X_{\sigma(f_i),s,i}$ is obviously equivalent to system of equalities (20). This proves the equivalence of (a) and (b).

Now let the group G be nonsolvable, let the field k be uncountable if $\mathrm{char}(k) > 0$, and let (b) holds. Consider an arbitrary point $x = (g_1, \ldots, g_n) \in X$.

If (b_1) holds, then for each $i \in \{1, \ldots, n\}$, we have the equality $\sigma(f_i) = f_i$, and therefore, in view of (12) and (20), the equality $g_i = s g_i s^{-1}$. Since g_i may be any element of G, this means that (b_2) holds.

Conversely, if (b_2) holds, then for any i, it follows from (12) and (20) that $\sigma(f_i)(x) = g_i$, i.e., σ lies in the kernel of homomorphism (6), which, in view of Theorem 1, is trivial. Hence, (b_1) is fulfilled. This proves the equivalence of (b_1) and (b_2). $\square$

11. Now we explore as to when in the considered situation homomorphism (19) is an embedding. Since the action of the group S on G by conjugation is trivial if and only if $S \subseteq \mathscr{C}(G)$, in what follows we can (and shall) assume that the group S does not lie in $\mathscr{C}(G)$ (i.e., is noncentral).

Theorem 3. *Let G be a nonsolvable connected algebraic group, and let the field k be uncountable if $\operatorname{char}(k) > 0$. Let S be a noncentral finite subgroup of the group G, and let $X := G^{\times n}$. Then homomorphism (19) is an embedding, so that the group $\operatorname{Aut}(X/S)$ contains $\operatorname{Aut}(F_n)$. For $n \geqslant 2$, the group $\operatorname{Aut}(X/S)$ is nonamenable, and for $n \geqslant 3$, is nonlinear and contains the braid group B_n.*

Proof. Arguing by contradiction, assume that the kernel of homomorphism (19) contains an element $\sigma \in \operatorname{Aut}(F_n)$, $\sigma \neq 1$. Then by Lemma 2, there is an element

$$s \in S \setminus \mathscr{C}(G), \tag{21}$$

such that equalities (20) hold. In view of (12), this means that for each $i \in \{1, \ldots, n\}$, the following group identity holds:

$$\sigma(f_i)(g_1, \ldots, g_n) = s g_i s^{-1} \quad \text{for any } g_1, \ldots, g_n \in G. \tag{22}$$

In particular, for any $g \in G$, the equality obtained by substituting $g_1 = \ldots = g_n = g$ in (22) holds. Since $\sigma(f_i)$ has form (4), this means the existence of an integer d such that the following group identity holds:

$$g^d = sgs^{-1} \quad \text{for each } g \in G. \tag{23}$$

Notice that

$$d \neq 1 \quad \text{and} \quad d \neq -1. \tag{24}$$

Indeed, in view of (23), if $d = 1$, then $s \in \mathscr{C}(G)$, which contradicts (21). If $d = -1$, then for any $g, h \in G$ the following equality holds

$$h^{-1}g^{-1} = (gh)^{-1} \overset{(23)}{=} s(gh)s^{-1} = sgs^{-1}shs^{-1} \overset{(23)}{=} g^{-1}h^{-1},$$

which mean commutativity of the group G and contradicts its nonsolvability.

Next, if r is a positive integer, then the following group identity holds:

$$s^r g s^{-r} = g^{d^r} \quad \text{for each } g \in G. \tag{25}$$

Indeed, (25) becomes (23) for $r = 1$. Arguing by induction, from $s^{r-1}gs^{-r+1} = g^{d^{r-1}}$ we obtain

$$s^r g s^{-r} = s(s^{r-1}gs^{-r+1})s^{-1} = sg^{d^{r-1}}s^{-1} \overset{(23)}{=} (g^{d^{r-1}})^d = g^{d^r},$$

as stated.

Since the group S is finite, the order of the element s in group identity (23) is finite. Let r in (25) be equal to this order. Then (25) becomes the group identity

$$e = g^{d^r-1} \quad \text{for each } g \in G. \tag{26}$$

Since, in view of (24), we have $d^r - 1 \neq 0$, it follows from group identity (26) that G is a torsion group. Being nonsolvable, the group G is not unipotent. Therefore, it contains a nonidentity semisimple element, and hence a nontrivial torus (see [2, Thms. 4.4, 11.10]). But any such torus contains an element of infinite order (see [2, Prop. 8.8]). The obtained contradiction completes the proof that homomorphism (14) is an embedding. The remaining statements of Theorem 3 follow from Proposition 2. □

12. There are affine rational algebraic varieties in the set, supplied by Theorem 3, of algebraic varieties X/S containing Aut(F_n) in their automorphism group: such is X itself if G is unsolvable and affine, because the underlying variety of any connected affine algebraic group is rational (see [2, Cor. 14.14]).

We will now show that in this set there are also affine nonrational (and even not stably rational) algebraic varieties.

We use the following known statement (see, e,g,, [19, Thm. 1]).

Lemma 3. *If the field of invariant rational functions of some faithful linear action of a finite group on a finite-dimensional vector space over k is stably rational over k, then the same property holds for any other such action of this group.*

Let p be a prime integer other than char(k). In [21] are found finite groups F of order p^9 and group embeddings

$$\iota\colon F \hookrightarrow \mathrm{GL}(V),$$

where V is a finite-dimensional vector space over k, such that the field of $\iota(F)$-invariant rational functions on V is not stably rational over k. In view of Lemma 3, replacing V and ι if needed, we can (and shall) assume that

$$\iota(F) \cap \mathscr{C}\big(\mathrm{GL}(V)\big) = \{\mathrm{id}_V\}. \tag{27}$$

Indeed, let L be a one-dimensional vector space over k. Since

$$\mathscr{C}\big(\mathrm{GL}(V \oplus L)\big) = \{c \cdot \mathrm{id}_{V\oplus L} \mid c \in k, c \neq 0\},$$

for the group embedding

$$\iota'\colon F \hookrightarrow \mathrm{GL}(V \oplus L), \quad f \mapsto \iota(f) \oplus \mathrm{id}_L,$$

we have $\iota'(F) \cap \mathscr{C}\big(\mathrm{GL}(V \oplus L)\big) = \{\mathrm{id}_{V\oplus L}\}$.

Now we put

$$G := \mathrm{GL}(\mathrm{V}), \quad S := \iota(F).$$

It follows from (27) that the diagonal action of the group S on the vector space $\mathrm{End}(\mathrm{V})^{\oplus n}$ by conjugation is a faithful linear action. Therefore, in view of Lemma 3, the field of rational S-invariant functions on $\mathrm{End}(\mathrm{V})^{\oplus n}$ is not stably rational over k. Since $X := G^{\times n}$ is an S-invariant open subset of $\mathrm{End}(V)^{\oplus n}$, this implies that the field $k(X)^S$ for the diagonal action of S on X by conjugation is not stably rational over k. This and (18) yield that the algebraic variety X/S is not stably rational. Since the group G is affine, we have $X/S = X/\!\!/S$ (see [22, Prop. 18 on p. 48]), so that the algebraic variety X/S is affine. Finally, in view of nonsolvability of the group G, it follows from Theorem 3 that if the field k is uncountable when $\mathrm{char}(k) > 0$, then the group $\mathrm{Aut}(X/S)$ contains $\mathrm{Aut}(F_n)$.

Remark 3. At present, the groups of orders p^6 and p^5 are known whose fields of rational invariants of faithful linear actions are not stably rational (see details and references in [19, Rem. on p. 414]). They can be taken as F in the construction described in this section.

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