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On 3-Colouring Of Graphs with Short Faces and Bounded Maximum Vertex Degree

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Abstract—The vertex 3-colourability problem is to verify whether it is possible to split the vertex set of a given graph into three subsets of pairwise nonadjacent vertices or not. This problem is known to be NP-complete for planar graphs of the maximum face length at most 4 (and, even, additionally, of the maximum vertex degree at most 5), and it can be solved in linear time for planar triangulations. Additionally, the vertex 3-colourability problem is NP-complete for planar graphs of the maximum vertex degree at most 4, but it can be solved in constant time for graphs of the maximum vertex degree at most 3. It would be interesting to investigate classes of planar graphs with simultaneously bounded length of faces and the maximum vertex degree and to find the threshold, for which the complexity of the vertex 3-colourability problem is changed from polynomial-time solvability to NP-completeness. In this paper, we prove NP-completeness of the vertex 3-colourability problem for planar graphs of the maximum vertex degree at most 4, whose faces are of length no more than 5.

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1. INTRODUCTION

A *proper vertex colouring* of a graph G is a mapping $c : V(G) \longrightarrow N$, such that $c(v_1) \neq c(v_2)$, for all adjacent vertices $v_1, v_2 \in V(G)$. All the elements of $\{c(v) : v \in V(G)\}$ are said to be *colours*. A proper vertex colouring c of G is called a *k-colouring* if $c : V(G) \longrightarrow \{1, \ldots, k\}$. If a graph G has a k-colouring, then G is called *k-colourable*. The *vertex k-colourability problem* (briefly, the *k-VC problem*), for a given graph G, consists in determining whether the graph G is k-colourable or not. This problem is NP-complete for any $k \geq 3$.

The 3-VC problem is NP-complete in the class of planar graphs of the maximum vertex degree at most 4 [3]. Also, by the well-known Brooks' theorem [2], it can be solved in constant time for graphs of the maximum vertex degree at most 3. The 3-VC problem can be solved in linear time for planar graphs, all the faces of which (including external) are triangles [1]. Also, this problem (see [4]) remains NP-complete for planar graphs, all the faces of which (including external) are triangles or quadrangles (and, additionally, of the maximum vertex degree at most 5).

So, for planar graphs and the 3-VC problem, the complexity thresholds are known in terms of the maximum vertex degree and the maximum length of faces separately. We aim to find a similar threshold for the combination of these two parameters. In this paper, we prove that the 3-VC problem is NP-complete for the class of planar graphs simultaneously with the maximum face length at most 5 and of the maximum degree at most 4. This result leaves the only planar graphs of the maximum vertex degree

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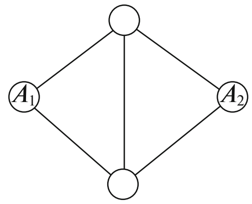

Fig. 1. The diamond.

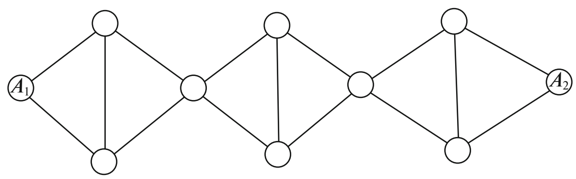

Fig. 2. The diamond chain.

at most 4, all which faces are triangles and quadrangles, as the case with an open complexity status of the 3-VC problem.

This paper is organized as follows. In Section 2, we prove NP-completeness of the 3-VC problem for planar graphs of the maximum vertex degree at most 5 and of the maximum face length at most 16 simultaneously. In Section 3, we prove the main result of this paper.

2. AUXILIARY RESULTS

Recall, it was shown in [4] that the 3-VC problem is NP-complete for planar graphs of the maximum vertex degree at most 5 with all the faces of length at most 4.

Let G be a k-colourable graph and $A \subseteq V(G)$, such that any k-colouring of G colours all the vertices from A in the same colour. Let H be an arbitrary graph and v be an arbitrary its vertex. We define the following replacement operation:

1. Firstly, we delete v together with all its incident edges.
2. After that we add G to the resulting graph as a separate connected component.
3. Lastly, we add some edges (their choice will be clear from a context) of the form au, where u is a neighbour of v in H and a is an element of A.

Denote the resulting graph by H'.

Lemma 1. *The graph H is k-colourable if and only if H' does.*

Proof. If H is k-colourable, then we can keep colours of its vertices for $V(H') \cap V(H)$, assign the colour of v for all the vertices in A, and extend the partial k-colouring of G to some its k-colouring.

If H' is k-colourable, then it induces a k-colouring of G. Hence, in H', all the elements of A receive the same colour. Deleting all the elements of $V(G) \setminus A$ from H' and contracting A into a vertex, we obtain a k-colouring of H. Thus, this Lemma holds. □

Consider the following three graphs, depicted on Figures 1, 2, and 3. We will call these graphs as the *diamond*, the *diamond chain*, and the *basic replacement graph*, correspondingly. The vertices A_1, A_2, and A_3 in these graphs form the set A.

It is easy to see that any 3-colouring of any of the mentioned three graphs colours all the vertices of A in the same colour. Moreover, there is only one 3-colouring of any of these graphs. Hence, they can serve as replacement graphs in the operation, described above. Clearly, the replacement operation for the diamond, the diamond chain, and the basic replacement graph keeps the planarity, i.e. H' is planar whenever H is planar.

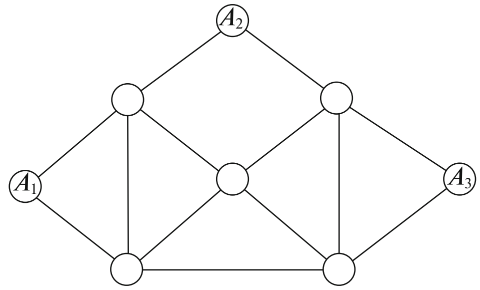

Fig. 3. The basic replacement graph.

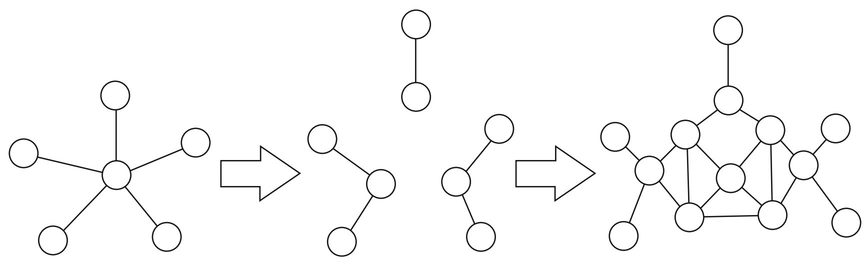

Fig. 4. The replacement of a degree 5 vertex.

Lemma 2. *The 3-VC problem is NP-complete for planar graphs of the maximum vertex degree at most 4, in which all the faces have length at most 16.*

Proof. We present a polynomial-time reduction for the NP-complete 3-VC problem in the class of planar graphs of the maximum vertex degree at most 5 with length of all the faces at most 4 (see [4]) to the same problem for planar graphs of the maximum vertex degree at most 4 and of all the face lengths at most 16. Hence, this Lemma will hold.

Consider an arbitrary planar graph of the maximum vertex degree at most 5 with length of all the faces at most 4. Firstly, let us remove all its degree 1 and 2 vertices. This operation keeps the 3-colourability. Secondly, we add $5-i$ pendant neighbours to any degree i vertex and, next, replace every vertex (its degree equals 5) with the basic replacement graph as it shown on Figure 4.

It can be done in polynomial time. Any of the vertices A_1, A_2, A_3 in all the copies of the basic replacement graph has at least one, but no more than two neighbours in the whole graph. The replacements keep the 3-colourability, remove all the degree 5 vertices, and add no new ones. All the faces were of length no more than 4 before. The transformation adds no more than 3 new edges to every vertex in each face. Hence, it gives at most 12 new edges to each face. This means that in the resulting graph length of all the faces is at most 16. So, this Lemma holds. □

The *n-sunlet* is a graph on $2n$ vertices, obtained by attaching n pendant edges to the n-cycle. The 5-sunlet is pictured on Figure 5.

Lemma 3. *A 3-colouring of leaves of the n-sunlet can be extended to a 3-colouring of the whole n-sunlet with the unique exceptional case, when n is odd and all the leaves have the same colour.*

Proof. Obviously, if all the leaves are of the same colour, then the n-sunlet is 3-colourable if and only if n is even.

We will prove that if there are at least two leaves with different colours in the n-sunlet, then it is 3-colourable. If there exist two leaves with different colours, then there are two leaves a and b with different colours, which are adjacent to consecutive vertices of the cycle in the n-sunlet. Let the colour of a is 1, and 2 is the colour of b. Assume that a and b are located counterclock-wisely (as shown on Figure 6). Let us colour the neighbour of a in the cycle in the colour 2.

After that we clock-wisely colour vertices of the cycle. At every but the last step, the new uncoloured vertex will be adjacent to two coloured ones. Hence, there is a feasible colour for it. There is a feasible colour for the last vertex, since it will be adjacent to vertices of no more than two colours, one of them is the colour 2. □

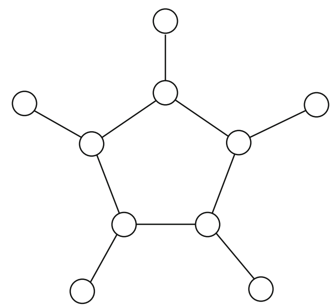

Fig. 5. The 5-sunlet.

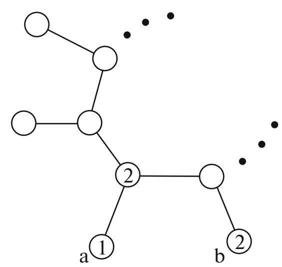

Fig. 6. Sunlet colouring.

3. THE MAIN RESULT

The next statement is the main result of the whole paper.

Theorem 1. *The 3-VC problem is NP-complete in the class of planar graphs of the maximum vertex degree at most 4 and of the maximum face length at most 5 simultaneously.*

Proof. To obtain a polynomial-time reduction to the 3-VC problem for the required graphs, we will conduct a series of graph transformations, participating in the proof of Lemma 2. Each transformation will have the following properties:

1. It will keep the 3-colourability,
2. It will keep the planarity,
3. It will result a graph of the maximum vertex degree at most 4,
4. It can be done in polynomial time.

Let us consider a planar graph G of the maximum vertex degree at most 4 and of the maximum face length at most 16 simultaneously. The 3-VC problem is NP-complete for these graphs, by Lemma 2. As in the proof of Lemma 2, we delete all the degree 1 and 2 vertices of G, next, add pendant vertices to receive a graph with all the vertex degrees equal 5, next, we replace each vertex with the basic replacement graph. Finally, we replace each vertex in the resulting graph with the diamond chain. We do these replacements in such a way that every instance of the basic replacement graph is transformed to the following subgraph:

From now on, we will call the *end vertices* degree 2 vertices in the diamond chain, the *exterior vertices* its degree 3 vertices, and the *interior vertices* its degree 4 vertices. For the next step, we will need to specify a 3-colouring of the graph a bit. In the diamond chain graph, its end and interior vertices have the same colour. However, we are free to choose how to colour the exterior vertices in the diamond chain. We will alternate colours of the exterior vertices from the same side of the diamond chain in the 3-colouring, like shown on Figure 8.

Denote by G' the resulting graph from G with respect to the replacements above. Clearly, G' is 3-colourable if and only if G is 3-colourable. We call an *outer cycle* any cycle, surrounding a face of G'

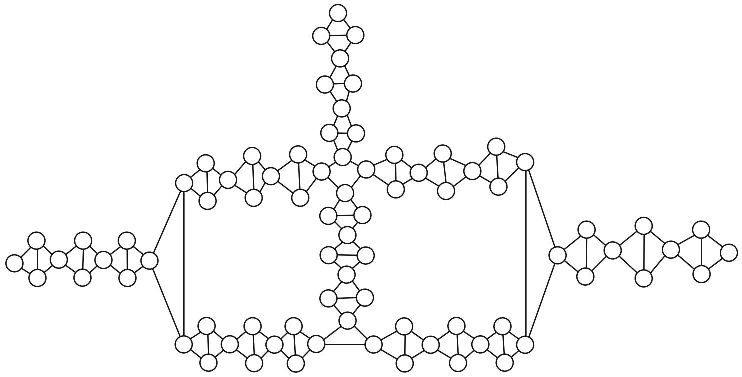

Fig. 7. The basic replacement graph with every vertex, replaced with the diamond chain.

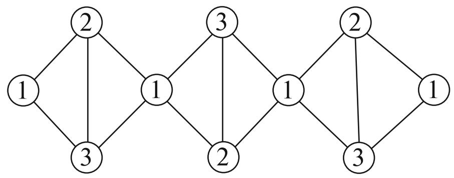

Fig. 8. The coloured diamond chain.

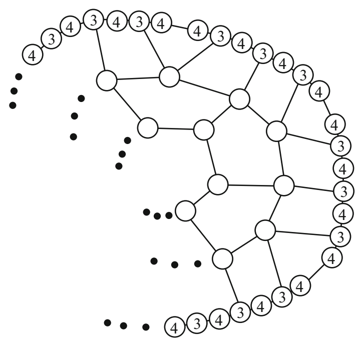

Fig. 9. Adding a middle cycle.

of length at least 5. All vertices of any outer cycle have degree 3 or 4 in G'. Inside any outer cycle, we create a cycle with the number of vertices, twice the quantity of the diamond chains in the outer cycle. We will call them *middle cycles*. The exterior vertices of each diamond chain are connected to the corresponding vertices in the middle cycle (see Figure 9, where the numbers are vertex degrees in G'), such that the rightmost exterior vertex of any diamond chain and the leftmost exterior vertex of the next diamond chain, going clockwise, are connected to the same vertex of the middle cycle.

By G'', we denote the resulting graph from G'. Clearly, G' is 3-colourable whenever G'' is 3-colourable. Let us show that the opposite fact is true. Consider a 3-colouring of G'. Firstly, we colour all the degree 4 vertices in the middle cycles in arbitrary feasible colours. Each of them has exactly two already coloured neighbours. If some degree 3 vertex v in a middle cycle has no feasible colour, then all its 3 neighbours must be coloured in all the 3 colours. Each of these neighbours is of of the different color with the rightmost and the leftmost exterior vertices from the corresponding diamond chain. The neighbors from the middle cycle are adjacent to it and the neighbor from the outer cycle has a different color because the colors of exterior vertices alternate in a diamond chain. The rightmost and the leftmost exterior vertices from the corresponding diamond chain are of the same colour, which means, that their the neighbours of v cannot be of this colour. Hence, we can colour v in a feasible colour.

Next, we put the n-cycle inside any middle $2n$-cycle and call it an *inner cycle*. We add edges between degree 3 vertices of the middle $2n$-cycle and all the vertices of the inner n-cycle. Basically, we insert the

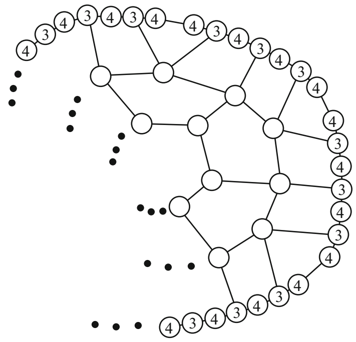

Fig. 10. Adding an inner cycle.

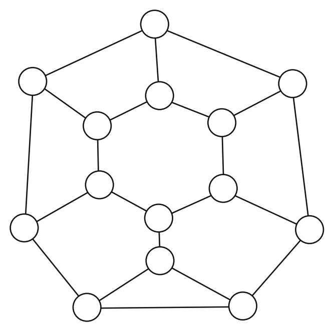

Fig. 11. One step of the final transformation.

n-sunlet inside the graph G'', see Figure 10. As the maximum face length of G is at most 16, then $n \leq 16$.

By G''', we denote the resulting graph from G''. Let us show that there exists a 3-colouring of G'', such that not all leaves of any corresponding n-sunlet of G''' receive the same colour. By Lemma 3, it would imply the 3-colourability of G'''. Suppose that in any 3-colouring of G'' all the n-sunlet leaves have the same colour 1. Let us take an arbitrary n-sunlet leaf v. It has two neighbours u_1 and u_2 in the middle cycle. The neighbours of u_1 and u_2 in the corresponding diamond chain have the same colour. Suppose that this colour is different from the colour 1, e.g. they are of the colour 2. Then, u_1 and u_2 have the colour 3, and v can be recoloured in the colour 2, keeping the feasibility of a colouring. Hence, the neighbours of u_1 and u_2 in the corresponding diamond chain are of the colour 1. It is true for all the diamond chains. Every vertex from the middle cycle, which is not an n-sunlet leaf, is then only adjacent to vertices of the colour 1.

Let us take any vertex v from the middle circle, which is an n-sunlet leaf. It is connected to the central exterior vertex u of some diamond chain from the outer cycle and two vertices u_1 and u_2 from the middle cycle, which are not n-sunlet leaves. We recolour u_1 and u_2 in the colour of u. Since the colour of u is either 2 or 3, this recolouring keeps the feasibility of the 3-colouring. After that u_1 and u_2 have the same colour and, hence, we can recolour v from the colour 1 to a colour in $\{2, 3\}$. So, G''' is 3-colourable if and only if G'' is 3-colourable.

Finally, we will contract by one length of faces, constituted by the inner cycles. At each step, we will break the face, bounded by an inner cycle, like shown on Figure 11, and redefine the inner cycle. By Lemma 3, the new graph is 3-colourable if and only if G''' does, since all the neighbours of the new inner cycle cannot have the same colour. This way, using no more, than 11 iterations, we will insure that all the faces are of length at most 5.

Thus, the 3-VC problem for planar graphs of the maximum vertex degree at most 4 and of the maximum face length at most 16, which is NP-complete, can be polynomially reduced to the same problem for planar graphs of the maximum vertex degree at most 4 and the maximum face length at most 5. So, this Theorem is true. □

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