

UNDERLYING VARIETIES AND GROUP STRUCTURES

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ABSTRACT. Starting with exploration of the possibility to present the underlying variety of an affine algebraic group in the form of a product of some algebraic varieties, we then explore the naturally arising problem as to what extent the group variety of an algebraic group determines its group structure.

1. Introduction. This paper arose from a short preprint [16] and is its extensive extension. As starting point of [14] served the question of B. Kunyavsky [10] about the validity of the statement, which is formulated below as Corollary of Theorem 1. This statement concerns the possibility to present the underlying variety of a connected reductive algebraic group in the form of a product of some special algebraic varieties.

Sections 2, 3 make up the content of [16], where the possibility of such presentations is explored. For some of them, in Theorem 1 is proved their existence, and in Theorems 2–5, on the contrary, their non-existence.

Theorem 1 shows that there are non-isomorphic reductive groups whose underlying varieties are isomorphic. In Sections 4–10, we explore the problem, naturally arising in connection with this, as to what extent the underlying variety of an algebraic group determines its group structure. In Theorems 6–8, it is shown that some group properties (dimension of unipotent radical, reductivity, solvability, unipotency, toroidality in the sense of Rosenlicht) are equivalent to certain geometric properties of the underlying group variety. Theorem 8 generalizes to solvable groups M. Lazard’s theorem on unipotent groups. In Sections 6, 7, a method for constructing non-isomorphic connected semisimple groups, which are isomorphic as algebraic varieties, is found, and in Theorem 11 it is proved that for any connected reductive algebraic group R , the number of all, considered up to isomorphism, algebraic groups whose underlying varieties are isomorphic to that of R , is finite. Generally speaking, this number is greater than 1. It is proved in Theorem 12 that if the group R is simple, then it is equal to 1. The appendix

contains a finiteness theorem for reductive groups (Theorem 14), the proof of which provides an upper bound for the specified number.

My thanks go to T. Bandman, V. Gorbatsevich, and Yu. Zarhin who drew my attention to some related publications.

Conventions and notation

- k is an algebraically closed field, over which all algebraic varieties considered below are defined.
- Groups are considered in multiplicative notation. The unit element of a group G is denoted by e (which group is meant will be clear from the context).
- For groups G and H , the notation $G \simeq H$ means that they are isomorphic.
- $\mathcal{C}(G)$ is the center of a group G .
- $\mathcal{D}(G)$ is the derived group of a group G .
- $\langle g \rangle$ is the cyclic group generated by g .
- A torus means affine algebraic torus, and a homomorphism of algebraic groups means their algebraic homomorphism.
- $\mathcal{R}_u(G)$ is the unipotent radical of an affine algebraic group G .
- G° is the identity connected component of an algebraic group or a Lie group G .
- $\text{Lie}(G)$ is the Lie algebra of an algebraic group or a Lie group G .
- $\text{Hom}(G, H)$ and $\text{Aut}(G)$ are the groups of algebraic homomorphisms if G and H are algebraic groups. The character of such a group G is an element of the group $\text{Hom}(G, \mathbb{G}_m)$.
- \mathbb{A}_n is the n -dimensional coordinate affine space.
- \mathbb{A}_*^n is the product of n copies of the variety $\mathbb{A}^1 \setminus \{0\}$.
- Let $p := \text{char}(k)$ and $a \in \mathbb{Z}$. If $ap \neq 0$, then a' denotes quotient of dividing a by the greatest power of p that divides a . If $ap = 0$, then $a' := a$.

2. Reductive groups with isomorphic underlying varieties. In this section, we prove the existence of some presentations of underlying varieties of affine algebraic groups in the form of products of algebraic varieties, and also the existence of non-isomorphic reductive algebraic groups that are isomorphic as algebraic varieties.

Let G be a connected reductive algebraic groups. Then

$$D := \mathcal{D}(G) \quad \text{and} \quad Z := \mathcal{C}(G)^\circ$$

are respectively a connected semisimple algebraic group and a torus (see [5, Sect. 14.2, Prop. (2)]). The algebraic groups $D \times Z$ and G are not always isomorphic; the latter is equivalent to the equality $D \cap Z =$

$\{e\}$, which, in turn, is equivalent to the property that the isogeny of algebraic groups $D \times Z \rightarrow G$, $(d, z) \mapsto dz$, is their isomorphism.

Theorem 1. *There is an injective homomorphism of algebraic groups $\iota: Z \hookrightarrow G$ such that the mapping*

$$\varphi: D \times Z \rightarrow G, (d, z) \mapsto d \cdot \iota(z),$$

is an isomorphism of algebraic varieties (but, generally speaking, not a homomorphism of algebraic groups).

Corollary. *The underlying varieties of (in general, non-isomorphic) algebraic groups $D \times Z$ and G are isomorphic.*

Remark 1. The existence of ι in the proof of Theorem 1 is established by an explicit construction.

Example ([15, Thm. 8, Proof]). Let $G = \mathrm{GL}_n$. Then $D = \mathrm{SL}_n$, $Z = \{\mathrm{diag}(t, \dots, t) \mid t \in k^\times\}$, and one can take

$$\mathrm{diag}(t, \dots, t) \mapsto \mathrm{diag}(t, 1, \dots, 1)$$

as ι . In this example, G and $D \times Z$ are nonisomorphic algebraic groups.

Proof of Theorem 1. Let T_D be a maximal torus of the group D , and let T_G be a maximal torus of the group G containing T_D . The torus T_D is a direct factor of the torus T_G : in the latter, there is a torus S such that the map $T_D \times S \rightarrow T_G$, $(t, s) \mapsto ts$, is an isomorphism of algebraic groups (see [5, 8.5, Cor.]). We shall show that the mapping

$$\psi: D \times S \rightarrow G, (d, s) \mapsto ds, \tag{1}$$

is an isomorphism of algebraic varieties.

We have (see [5, Sect. 14.2, Prop. (1),(3)]):

$$(a) \ Z \subseteq T_G, \quad (b) \ DZ = G, \quad (c) \ |D \cap Z| < \infty. \tag{2}$$

Let $g \in G$. In view of (2)(b), there are $d \in D$, $z \in Z$ such that $g = dz$, and in view of (2)(a) and the definition of S , there are $t \in T_D$, $s \in S$ such that $z = ts$. We have $dt \in D$ and $\psi(dt, s) = dts = g$. Therefore, the morphism ψ is surjective.

Consider in G a pair of mutually opposite Borel subgroups containing T_G . Their unipotent radicals U and U^- lie in D . Let $N_D(T_D)$ and $N_G(T_G)$ be the normalizers of tori T_D and T_G in the groups D and G respectively. Then $N_D(T_D) \subseteq N_G(T_G)$ in view of (2)(b). The homomorphism $N_D/T_D \rightarrow N_G/T_G$ induced by this embedding is an isomorphism of groups (see [5, IV.13]), by which we identify them and denote by W . For every $\sigma \in W$, fix a representative $n_\sigma \in N_D(T_D)$. The group $U \cap n_\sigma U^- n_\sigma^{-1}$ does not depend on the choice of this representative because T_D normalizes U^- ; we denote it by U'_σ .

It follows from the Bruhat decomposition that for each $g \in G$, there are uniquely defined $\sigma \in W$, $u \in U$, $u' \in U'_\sigma$ and $t_G \in T_G$ such that $g = u'n_\sigma u t_G$ (see [9, 28.4, Thm.]). In view of the definition of S , there are uniquely defined $t_D \in T_D$ and $s \in S$ such that $t_G = t_D s$, and since $u', n_\sigma, u, t_D \in D$, the condition $g \in D$ is equivalent to the condition $s = e$. It follows from this and the definition of the morphism ψ that the latter is injective.

Thus, ψ is a bijective morphism. Therefore, to prove that it is an isomorphism of algebraic varieties, it remains to prove its separability (see [5, Sect. 18.2, Thm.]). We have $\text{Lie}(G) = \text{Lie}(D) + \text{Lie}(T_G)$ (see [5, Sect. 13.18, Thm.]) and $\text{Lie}(T_G) = \text{Lie}(T_D) + \text{Lie}(S)$ (in view of the definition of S). Therefore,

$$\text{Lie}(G) = \text{Lie}(D) + \text{Lie}(S). \quad (3)$$

On the other hand, from (1) it is obvious that the restrictions of the morphism ψ to the subgroups $D \times \{e\}$ and $\{e\} \times S$ in $D \times S$, are isomorphisms respectively with the subgroups D and S in G . Since $\text{Lie}(D \times S) = \text{Lie}(D \times \{e\}) + \text{Lie}(\{e\} \times S)$, from (3) it follows that the differential of morphism ψ at the point (e, e) is surjective. Therefore (see [5, Sect. 17.3, Thm.]), the morphism ψ is separable.

Since ψ is an isomorphism, from (1) it follows that $\dim(G) = \dim(D) + \dim(S)$. On the other hand, from (2)(b),(c) it follows that $\dim(G) = \dim(D) + \dim(Z)$. Therefore, Z and S are equidimensional and hence isomorphic tori. Consequently, as ι one can take the composition of any tori isomorphism $Z \rightarrow S$ with the identity embedding $S \hookrightarrow G$. \square

3. Properties of factors. In contrast to the previous section, this one, on the contrary, concerns the non-existence of some presentations the underlying variety of an affine algebraic group as a product of algebraic varieties.

Theorem 2. *An algebraic variety on which there is a nonconstant invertible regular function, cannot be a direct factor of the underlying variety of a connected semisimple algebraic group.*

Proof. If the statement of Theorem 2 were not true, then the existence of the non-constant invertible regular function specified in it would imply the existence of such a function on a connected semisimple algebraic group. Dividing this function by its value at the unit element, we would then get, according to [17, Thm. 3], a non-trivial character of this group, which contradicts the absence of nontrivial characters of connected semisimple groups. \square

Theorem 3. *An algebraic curve cannot be a direct factor of the underlying variety of a connected semisimple algebraic group.*

Proof. Suppose an algebraic curve X is a direct factor of the underlying variety of a connected semisimple algebraic group G . Then X is irreducible, non-singular, affine, and there is a surjective morphism $\pi: G \rightarrow X$. Due to rationality of the underlying variety of G (see [4, 14.14]), the existence of π implies unirationality, and hence, by Luroth's theorem, rationality of X . Therefore, X is isomorphic to an open subset U of \mathbb{A}^1 . The case $U = \mathbb{A}^1$ is impossible due to Theorem 4. If $U \neq \mathbb{A}^1$, then there is a non-constant invertible regular function on X , which is impossible in view of Theorem 2. \square

Below, unless otherwise stated, we assume that $k = \mathbb{C}$. By the Lefschetz principle, Theorems 5, 10, 11, 12 proved below are valid for any field k of characteristics zero. Topological terms refer to classical topology, and homology and cohomology are singular.

Every complex reductive algebraic group G has a compact real form, any two such forms are conjugate, and if \mathbf{G} is one of them, then the topological manifold G is homeomorphic to the product of \mathbf{G} and a Euclidean space; see [14, Chap. 5, §2, Thms. 2, 8, 9]. Therefore, G and \mathbf{G} have the same homology and cohomology. This is used below without further explanation.

Theorem 4. *If a d -dimensional algebraic variety X is a direct factor of the underlying variety of a connected reductive algebraic group, then*

$$H_d(X, \mathbb{Z}) \simeq \mathbb{Z} \quad \text{and} \quad H_i(X, \mathbb{Z}) = 0 \quad \text{for } i > d.$$

Proof. Suppose there is a connected reductive algebraic group G and an algebraic variety Y such that the underlying variety of G is isomorphic to $X \times Y$. Let $n := \dim(G)$; then $\dim(Y) = n - d$. The algebraic varieties X and Y are irreducible non-singular and affine. Therefore (see [12, Thm. 7.1]),

$$H_i(X, \mathbb{Z}) = 0 \quad \text{for } i > d, \quad H_j(Y, \mathbb{Z}) = 0 \quad \text{for } j > n - d. \quad (4)$$

By the universal coefficient theorem, for any algebraic variety V and every i , we have

$$H_i(V, \mathbb{Q}) \simeq H_i(V, \mathbb{Z}) \otimes \mathbb{Q}, \quad (5)$$

and by the Künneth formula,

$$H_n(G, \mathbb{Q}) \simeq H_n(X \times Y, \mathbb{Q}) \simeq \bigoplus_{i+j=n} H_i(X, \mathbb{Q}) \otimes H_j(Y, \mathbb{Q}). \quad (6)$$

Therefore, from (4) it follows that

$$H_n(G, \mathbb{Q}) \simeq H_d(X, \mathbb{Q}) \otimes H_{n-d}(Y, \mathbb{Q}). \quad (7)$$

Consider a compact real form \mathbf{G} of the group G . Since \mathbf{G} is a closed connected orientable n -dimensional topological manifold, $H_n(\mathbf{G}, \mathbb{Q}) \simeq \mathbb{Q}$. Hence, $H_n(G, \mathbb{Q}) \simeq \mathbb{Q}$. From this and (24) it follows that $H_d(X, \mathbb{Q}) \simeq \mathbb{Q}$. In turn, in view of (5), the latter implies $H_d(X, \mathbb{Z}) \simeq \mathbb{Z}$, because $H_d(X, \mathbb{Z})$ is a finitely generated (see [6, Sect. 1.3]) torsion-free abelian group (see [1, Thm. 1]). \square

Corollary. *A contractible algebraic variety (in particular, \mathbb{A}^d) of positive dimension cannot be a direct factor of the underlying variety of a connected reductive algebraic group.*

Theorem 5. *An algebraic surface cannot be a direct factor of the underlying variety of a connected semisimple algebraic group.*

Proof. Suppose there are a connected semisimple algebraic group G and the algebraic varieties X and Y such that X is a surface, and $X \times Y$ is isomorphic to the algebraic variety G . We keep the notation of the proof of Theorem 4. Since G is semisimple, \mathbf{G} is semisimple too. Hence, $H^1(\mathbf{G}, \mathbb{Q}) = H^2(\mathbf{G}, \mathbb{Q}) = 0$ (see [13, §9, Thm. 4, Cor. 1]). Insofar as the \mathbb{Q} -vector spaces $H^i(\mathbf{G}, \mathbb{Q})$ and $H_i(\mathbf{G}, \mathbb{Q})$ are dual to each other, this gives

$$H_1(G, \mathbb{Q}) = H_2(G, \mathbb{Q}) = 0. \quad (8)$$

Since G is connected, X and Y are connected too. Therefore,

$$H_0(X, \mathbb{Q}) = H_0(Y, \mathbb{Q}) = \mathbb{Q}. \quad (9)$$

From (6), (8), and (9) it follows that $H_2(X, \mathbb{Q}) = 0$. In view of (5), this contradicts Theorem 4, which completes the proof. \square

4. Group properties determined by properties of underlying variety. Theorem 1 naturally leads to the question of to what extent the underlying variety of an algebraic group determines its group structure.

Explicitly or implicitly, this question has long been considered in the literature.

For example, M. Lazar proved in [11] that if the underlying variety of an algebraic group is isomorphic to an affine space, then this group is unipotent (for a short proof, see Remark 2 below).

By Chevalley's theorem, every connected algebraic group G contains the largest connected affine normal subgroup G_{aff} , and the group G/G_{aff} is an abelian variety. M. Rosenlicht in [17] considered G , which he called *toroidal*, such that G_{aff} is a torus; this property is equivalent to the absence connected one-dimensional unipotent subgroups. Theorem 6 shows that it can be equivalently reformulated in terms of

geometric properties of the underlying variety of G (the proof does not use restrictions on the characteristic of k):

Theorem 6. *The following properties of a connected algebraic group G are equivalent:*

- (a) G is toroidal;
- (b) G does not contain subvarieties isomorphic to \mathbb{A}^1 .

Proof. Let $\pi: G \rightarrow G/G_{\text{aff}}$ be the natural epimorphism, and let X be a subvariety of G isomorphic to \mathbb{A}^1 . By shifting it by an appropriate element of G , we can assume that $e \in X$. Since X is isomorphic to the underlying variety of the group \mathbb{G}_a , we can endow X with a structure of an algebraic group isomorphic to \mathbb{G}_a with the unit element e . Then $\pi|_X: X \rightarrow G/G_{\text{aff}}$ is a homomorphism of algebraic groups in view of [17, Thm. 3]. Since X is an affine and G/G_{aff} is a complete algebraic variety, this yields $X \subseteq G_{\text{aff}}$. Therefore, it comes down to proving equivalence of the following properties:

- (a') G_{aff} is a torus;
- (b') G_{aff} does not contain subvarieties isomorphic to \mathbb{A}^1 .

(a') \Rightarrow (b'): Let the subvariety X of the torus G_{aff} be isomorphic to \mathbb{A}^1 . The algebra of regular functions on G_{aff} is generated by invertible functions. This means that this is also the case for the algebra of regular functions on X . This contradicts the fact that there are no non-constant invertible regular functions on \mathbb{A}^1 .

(b') \Rightarrow (a'): If (b') holds, then G_{aff} is reductive, since the variety $\mathcal{R}_u(G_{\text{aff}})$ is isomorphic to \mathbb{A}^d (see [7, p. 5-02, Cor.]), which for $d > 0$ contains affine lines. In addition, $\mathcal{Z}(G_{\text{aff}}) = \{e\}$, because root subgroups in a semisimple group are isomorphic to \mathbb{G}_a . Hence, G_{aff} is a torus. \square

Below, in Theorem 7, its Corollary, and Theorem 8 several group properties of connected affine algebraic groups are pointed out, which are determined by the properties of underlying varieties. In the formulations the following notation is used.

Let X be an irreducible algebraic variety. The multiplicative group $k[X]^\times$ of invertible regular functions on X contains the subgroup of nonzero constants k^\times , and $k[X]^\times/k^\times$ is a finitely generated free abelian group (see [17, Thm. 1]). Let us denote

$$\text{units}(X) := \text{rank}(k[X]^\times/k^\times).$$

According to [17, Thms. 2, 3], this invariant has the following properties:

- (i) If X and Y are irreducible algebraic varieties, then

$$\text{units}(X \times Y) = \text{units}(X) + \text{units}(Y). \quad (10)$$

(ii) If G is a connected algebraic group, then

$$\text{units}(G) = \text{rank}(\text{Hom}(G, \mathbb{G}_m)). \quad (11)$$

In what follows, we use the following notation:

$$\text{mh}(X) := \max\{d \in \mathbb{Z}_{\geq 0} \mid H_d(G, \mathbb{Q}) \neq 0\}. \quad (12)$$

If X is a non-singular affine algebraic variety, then, according to [12, Thm. 7.1],

$$\text{mh}(X) \leq \dim(X).$$

Theorem 7. *If G is a connected affine algebraic group, then*

$$\dim(\mathcal{R}_u(G)) = \dim(G) - \text{mh}(G). \quad (13)$$

Proof. Since the underlying variety of $\mathcal{R}_u(G)$ is isomorphic to an affine space, the topological manifolds G and $R := G/\mathcal{R}_u(G)$ are homotopy equivalent. Therefore, $H_i(G, \mathbb{Q}) \simeq H_i(R, \mathbb{Q})$ for every i and, therefore,

$$\text{mh}(G) = \text{mh}(R). \quad (14)$$

Since R is reductive, it follows from (5) and Theorem 4 that

$$\text{mh}(R) = \dim(R). \quad (15)$$

In view of $\dim(R) = \dim(G) - \dim(\mathcal{R}_u(G))$, equalities (14) and (15) imply (13). \square

Corollary. *The following properties of a connected affine algebraic group G are equivalent:*

- (a) G is reductive;
- (b) $\text{mh}(G) = \dim(G)$.

The previous corollary shows that the property of a connected affine algebraic group to be reductive is expressed in terms the geometric property of its underlying variety. The following Theorem 8, generalizing M. Lazard's theorem [11], shows that the same is true for the property of a group to be solvable.

Theorem 8. *The following properties of a connected affine algebraic group S are equivalent:*

- (a) S is solvable;
- (b) $\text{mh}(S) = \text{units}(S)$;
- (c) *there are non-negative integers t and r such that the underlying variety of S is isomorphic to $\mathbb{A}_*^t \times \mathbb{A}^r$; wherein the equality $t = \text{units}(S)$ automatically holds.*

The group S is unipotent (respectively, a torus) if and only if its underlying variety is isomorphic to \mathbb{A}^r (respectively, \mathbb{A}_^t).*

Proof. (a) \Leftrightarrow (b): Let $G := S/\mathcal{R}_u(S)$; it is a connected reductive algebraic group. We shall use the same notation as in the proof of Theorem 1. Solvability of S is equivalent to the equality $G = Z$, whence, since G and Z are connected, it follows that

$$S \text{ is solvable} \iff \dim(G) = \dim(Z). \quad (16)$$

In view of 7, we have

$$\dim(G) = \text{mh}(S). \quad (17)$$

The elements of $\text{Hom}(S, \mathbb{G}_m)$ (respectively, $\text{Hom}(G, \mathbb{G}_m)$) are trivial on $\mathcal{R}_u(S)$ (respectively, D). From this and (2)(b) it follows that

$$\begin{aligned} \text{Hom}(S, \mathbb{G}_m) &\simeq \text{Hom}(G, \mathbb{G}_m), \\ \text{Hom}(G, \mathbb{G}_m) &\simeq \text{Hom}(Z/(Z \cap D), \mathbb{G}_m). \end{aligned} \quad (18)$$

From (11), (18), and (2)(c) we get

$$\begin{aligned} \text{units}(S) &= \text{rank}(\text{Hom}(Z/(Z \cap D), \mathbb{G}_m)) \\ &= \dim(Z/(Z \cap D)) = \dim(Z). \end{aligned} \quad (19)$$

Matching (16), (17), and (19) completes the proof of the equivalence (a) \Leftrightarrow (b).

(a) \Rightarrow (c): This is proved in [7, p. 5-02, Cor.] for the field k of arbitrary characteristic.

(c) \Rightarrow (b): Let (c) be satisfied. From (10), (11) and the obvious equality $\text{units}(\mathbb{A}^r) = 0$ it follows that

$$\text{units}(\mathbb{A}^r \times \mathbb{A}_*^t) = t. \quad (20)$$

On the other hand, since the topological manifold \mathbb{A}^r is contractible, and \mathbb{A}_*^t is homotopy equivalent to the product of t circles, we have

$$H_j(\mathbb{A}_*^t \times \mathbb{A}^r, \mathbb{Q}) = \begin{cases} \mathbb{Q}, & \text{if } j = t, \\ 0 & \text{if } j > t; \end{cases}$$

from this and (12) we conclude that

$$\text{mh}(\mathbb{A}_*^t \times \mathbb{A}^r) = t. \quad (21)$$

Comparing (20) with (21) completes the proof of implication (c) \Rightarrow (b).

The group S is unipotent (respectively, is a torus), if and only if it is solvable (i.e., according to (c), its underlying variety is isomorphic to $\mathbb{A}_*^t \times \mathbb{A}^r$) and, by Theorem 7, the equality $\text{mh}(S) = 0$ (respectively, $\text{mh}(S) = \dim(S)$) holds. The last statement of the theorem now follows from (21). \square

Remark 2. The last statement of Theorem 8 is true for the field k of any characteristic. For a torus, this follows from Corollary 1 of Theorem 9. For unipotent groups, it is obtained by concatenating [7, p. 5-02, Cor.] and [11]. Here is a short proof of M. Lazard's theorem [11], suitable for the field k of any characteristic.

Proof of M. Lazard's theorem. Let the underlying variety of G be isomorphic to \mathbb{A}^r . If G is not unipotent, then G contains a non-identity semisimple element, and therefore, also a nonidentity torus (see [5, Thms. 4.4(1), 11.10]). The action of this torus on G by left translations has no fixed points. This contradicts the fact that every algebraic torus action on \mathbb{A}^r has a fixed point, see [2, Thm. 1]. \square

5. Different group structures on the same variety. As is known, there are infinitely many pairwise non-isomorphic unipotent groups of a fixed sufficiently large dimension; their underlying varieties, however, are all isomorphic (see Theorem 8). On the other hand, underlying varieties of toroidal groups uniquely determine their group structure:

Theorem 9. *Let G_1 and G_2 be algebraic groups, one of which is toroidal. The following properties are equivalent:*

- (a) *the underlying varieties of G_1 and G_2 are isomorphic;*
- (b) *G_1 and G_2 are isomorphic algebraic groups.*

Proof. Let G_1 be toroidal. Then, according to [17, Thm. 3], the composition of an isomorphism of the underlying variety of G_2 to that of G_1 with a suitable left translation of G_1 is an isomorphism of algebraic groups, which proves (a) \Rightarrow (b). \square

Corollary 1. *Tori are isomorphic if and only if their underlying varieties are isomorphic.*

Corollary 2. *Abelian varieties are isomorphic if and only if their underlying varieties are isomorphic.*

We now investigate the question of determinability of the group structures by the properties of underlying varieties for reductive algebraic groups.

Theorem 10. *Let G_1 and G_2 be connected affine algebraic groups, and let R_i be a maximal reductive algebraic subgroup of G_i , $i = 1, 2$. If the underlying varieties of G_1 and G_2 are isomorphic, then R_1 and R_2 are the connected algebraic groups with isomorphic Lie algebras.*

Proof. From $\text{char}(k) = 0$ it follows that G_i is a semidirect product of R_i and $\mathcal{R}_u(G_i)$ (see [5, 11.22]). Hence, R_i is connected (because G_i

is connected), and the topological manifolds G_i and R_i are homotopy equivalent (see the proof of Theorem 13).

Consider a compact form R_i of the reductive algebraic group R_i . Topological manifolds R_i and R_i are homotopy equivalent.

Suppose that the underlying varieties of G_1 and G_2 are isomorphic, and therefore they are homeomorphic as topological varieties. Then the topological manifolds R_1 and R_2 are homotopy equivalent. In view of [18, Satz], this implies that the real Lie algebras $\text{Lie}(R_1)$ and $\text{Lie}(R_2)$ are isomorphic. Now the claim of the theorem follows from the fact that the real Lie algebra $\text{Lie}(R_i)$ is a real form of the complex Lie algebra $\text{Lie}(R_i)$. \square

Theorem 11. *Let R be a connected reductive algebraic group.*

(i) *If G is an algebraic group such that the underlying varieties of G and R are isomorphic, then*

- (a) *G is connected and reductive, and the Lie algebras $\text{Lie}(R)$ and $\text{Lie}(G)$ are isomorphic;*
- (b) *in the case of a semisimple simply connected group R the algebraic groups R and G are isomorphic.*

(ii) *The number of all algebraic groups, considered up to isomorphism, whose underlying varieties are isomorphic to that of R , is finite.*

Proof. (i)(a) It follows from connectedness of R and the condition on G that G is connected. In view of Theorem 10 and reductivity of R , the Lie algebra of a maximal reductive subgroup in G is isomorphic to $\text{Lie}(R)$. In particular, the dimension of this subgroup is $\dim(R)$. Since $\dim(R) = \dim(G)$, this subgroup coincides with G .

(i)(b) From the condition on G and simply connectedness of the topological manifold R it follows that the topological manifold G is simply connected. In view of (a), the Lie algebras $\text{Lie}(R)$ and $\text{Lie}(G)$ are isomorphic. Consequently, the algebraic groups R and G are isomorphic (see [14, Chap. 1, §3, 3°, Chap. 3, §3, 4°]).

Statement (ii) follows from (i)(a) and finiteness of the numbers of all, considered up to isomorphism, connected reductive algebraic groups of a fixed dimension (this finiteness theorem, which I could not find in the literature, is proved below in Appendix ; see Theorem 14). \square

Remark 3. The proof of Theorem 14 yields an upper bound for the number specified in statement (ii) of Theorem 11 (see also Remark 4).

Theorem 1 proves the existence non-isomorphic reductive non-semisimple algebraic groups, whose underlying varieties are isomorphic (in accordance with statement (a) of Theorem 11, the Lie algebras of these groups are isomorphic). There are also semisimple algebraic groups

with similar properties. Below is given a general construction that allows to construct them. It is applicable for the field k of any characteristic.

6. Construction of non-isomorphic semisimple algebraic groups with isomorphic underlying varieties. Fix a positive integer n and an abstract group H . Consider the group

$$G := H^{\times n} := H \times \cdots \times H \quad (n \text{ factors}).$$

We have $\mathcal{C}(G) = \mathcal{C}(H)^{\times n}$.

Let F_n be a free group of rank n with a free system of generators x_1, \dots, x_n . For any elements $g = (h_1, \dots, h_n) \in G$, where $h_j \in H$, and $w \in F_n$ denote by $w(g)$ the element of H , which is the image of element w under the homomorphism $F_n \rightarrow H$, mapping x_j to h_j for every j .

Any element $\sigma \in \text{End}(F_n)$ determines the map

$$\sigma_G: G \rightarrow G, \quad g \mapsto (\sigma(x_1)(g), \dots, \sigma(x_n)(g)). \quad (22)$$

It is not hard to see that

$$\begin{aligned} (\sigma \circ \tau)_G &= \tau_G \circ \sigma_G \quad \text{for any } \sigma, \tau \in \text{End}(F_n), \\ e_G &= \text{id}. \end{aligned} \quad (23)$$

It follows from (22) and the definition of $w(g)$ that

- (i) $\sigma_G(S^{\times n}) \subseteq S^{\times n}$ for every subgroup S of H ;
- (ii) $\sigma_G(gz) = \sigma_G(g)\sigma_G(z)$ for all $g \in G, z \in \mathcal{C}(G)$.

In particular, the restriction of the map σ_G to the group $\mathcal{C}(G)$ is its endomorphism.

From (23) it follows that if $\sigma \in \text{Aut}(F_n)$, then σ_G is a bijection (but, generally speaking, not an automorphism of the group G). Moreover, if H is an algebraic group (respectively, a Lie group), then σ_G is an automorphism of the algebraic variety (respectively, a diffeomorphism of the differentiable manifold) G .

Consider now an element $\sigma \in \text{Aut}(F_n)$ and a subgroup of C in $\mathcal{C}(G)$. Then from (ii) it follows C -equivariance of the bijection $\sigma_G: G \rightarrow G$ if we assume that any element $c \in C$ acts on the left copy of G as the translation (multiplication) by c , and on the right one as the translation by $\sigma_G(c)$. The quotient for the first action is the group G/C , and for the second is the group $G/\sigma_G(C)$. Hence, σ_G induces a bijection $G/C \rightarrow G/\sigma_G(C)$. Moreover, if H is an algebraic group (respectively, a Lie group), then this bijection is an isomorphism of algebraic varieties (respectively, a diffeomorphism of differentiable manifolds); see [5, 6.1]. Thus, G/C and $G/\sigma_G(C)$ are isomorphic algebraic varieties (respectively, diffeomorphic differentiable manifolds). But, generally speaking,

G/C and $G/\sigma_G(C)$ are not isomorphic as algebraic groups (respectively, as Lie groups).

Indeed, take for H a simply connected semisimple algebraic group (respectively, a compact Lie group). Then G is also a simply connected algebraic group (respectively, a compact Lie group), so the group $\mathcal{C}(G)$ is finite. Consider the natural epimorphisms $\pi: G \rightarrow G/C$ and $\pi_\sigma: G \rightarrow G/\sigma_G(C)$. Since the group C is finite, the differentials

$$d_e\pi: \text{Lie}(G) \rightarrow \text{Lie}(G/C) \quad \text{and} \quad d_e\pi_\sigma: \text{Lie}(G) \rightarrow \text{Lie}(G/\sigma_G(C))$$

are the Lie algebra isomorphisms. Suppose there is an isomorphism of algebraic groups $\alpha: G/C \rightarrow G/\sigma_G(C)$. Then

$$(d_e\pi_\sigma)^{-1} \circ d_e\alpha \circ d_e\pi: \text{Lie}(G) \rightarrow \text{Lie}(G)$$

is the Lie algebra automorphism. Since G is simply connected, it is the differential of some automorphism $\tilde{\alpha} \in \text{Aut}(G)$ (see [14, Thm. 6, p. 30]). It follows from the construction that the diagram

$$\begin{array}{ccc} G & \xrightarrow{\tilde{\alpha}} & G \\ \pi \downarrow & & \downarrow \pi_\sigma \\ G/C & \xrightarrow{\alpha} & G/\sigma_G(C) \end{array}$$

commutative, which, in turn, implies that $\tilde{\alpha}(C) = \sigma_G(C)$. Thus, G/C and $G/\sigma_G(C)$ are isomorphic as algebraic groups (respectively, Lie groups) if and only if C and $\sigma_G(C)$ lie in the same orbit of the natural action of the group $\text{Aut}(G)$ on the set of all subgroups of the group $\mathcal{C}(G)$. This action is reduced to the action of the group $\text{Out}(G)$ (isomorphic to the group of automorphisms of the Dynkin diagram of the group G ; see [14, Chap. 4, §4, no. 1]), because $\text{Int}(G)$ acts on $\mathcal{C}(G)$ trivially. It is not difficult to find H , σ , and C such that the groups C and $\sigma_G(C)$ do not lie in the same $\text{Out}(G)$ -orbit. Here is a concrete example.

7. Example. Let H be a simply connected simple algebraic group (respectively, a compact Lie group), whose center is non-trivial. Take $n = 2$, so that

$$G = H \times H. \tag{24}$$

Let σ is defined by the equalities

$$\sigma(x_1) = x_1, \quad \sigma(x_2) = x_1x_2^{-1} \tag{25}$$

(clearly, $x_1, x_1x_2^{-1}$ is a free system of generators of the group F_2 , so $\sigma \in \text{Aut}(F_2)$). Let S be a non-trivial subgroup of $\mathcal{C}(H)$. Take

$$C := \{(s, s) \mid s \in S\}. \tag{26}$$

Then from (22), (25), (26) it follows that

$$\sigma_G(C) = \{(s, e) \mid s \in S\}. \quad (27)$$

In view of simplicity of G , the group $\text{Out}(G)$ contains a unique element that does not preserve the factors on the right-hand side of the equality (24),—it is the automorphism $(h_1, h_2) \mapsto (h_2, h_1)$. From this and from (26), (27) it follows that C is not mapped to $\sigma_G(C)$ by an automorphism from $\text{Out}(G)$. Therefore,

$$G/C = (H \times H)/C \quad \text{and} \quad G/\sigma_G(C) = (H/S) \times H$$

are non-isomorphic algebraic groups (respectively, compact Lie groups), which are isomorphic as algebraic varieties (respectively, diffeomorphic as differentiable manifolds).

For example, let $H = \text{SL}_d$, $d \geq 2$, and $S = \langle z \rangle$, where $z = \text{diag}(\varepsilon, \dots, \varepsilon) \in H$, $\varepsilon \in k$ is a primitive d -th root of 1. In this case, we obtain non-isomorphic algebraic groups

$$G/C = (\text{SL}_d \times \text{SL}_d)/\langle (z, z) \rangle, \quad G/\sigma_G(C) = \text{PSL}_d \times \text{SL}_d,$$

whose underlying varieties are isomorphic. Note that if $d = 2$, then $G = \text{Spin}_4$, $G/C = \text{SO}_4$.

For $H = \text{SU}_d$ and the same group S we obtain that

$$G/C = K_1 := (\text{SU}_d \times \text{SU}_d)/C, \quad G/\sigma_G(C) = K_2 := \text{PU}_d \times \text{SU}_d$$

are diffeomorphic non-isomorphic compact Lie groups. For $d = p^r$ with prime p this is proved in [3, p. 331], where non-isomorphism of the groups K_1 and K_2 is deduced from non-isomorphism of their Pontryagin rings $H_*(K_1, \mathbb{Z}/p\mathbb{Z})$ and $H_*(K_2, \mathbb{Z}/p\mathbb{Z})$ (describing these rings is a nontrivial problem). Note that if $d = 2$, then

$$K_1 = \text{SO}_4, \quad K_2 = \text{SO}_3 \times \text{SU}_2. \quad (28)$$

In [8, Chap. 3, §3.D], a diffeomorphism of the underlying manifolds of groups (28) is constructed using quaternions.

8. The case of simple algebraic groups. The following theorem shows that the considered phenomenon is not possible for simple groups

Theorem 12. *Let G_1 and G_2 be algebraic groups, one of which is connected and simple. The following properties are equivalent:*

- (a) *the underlying varieties of G_1 and G_2 are isomorphic;*
- (b) *G_1 and G_2 are isomorphic algebraic groups.*

Proof. Let G_1 be connected and simple.

Suppose (a) holds. Let \tilde{G}_1 be a simply connected algebraic group with the Lie algebra isomorphic to $\text{Lie}(G_1)$. Then G_1 is isomorphic to

\tilde{G}_1/Z_1 for some subgroup Z_1 of $\mathcal{C}(\tilde{G}_1)$. From Theorem 11 it follows that the group G_2 is isomorphic to \tilde{G}_1/Z_2 for some subgroup Z_2 of $\mathcal{C}(\tilde{G}_1)$. As explained above, statement (b) is equivalent to the property that Z_1 and Z_2 lie in the same orbit of the natural action of the group $\text{Out}(\tilde{G}_1)$ (isomorphic to the automorphism group of the Dynkin diagram of the group \tilde{G}_1) on the set of all subgroups of the group $\mathcal{C}(\tilde{G}_1)$. We shall show that Z_1 and Z_2 indeed lie in the same $\text{Out}(\tilde{G}_1)$ -orbit.

Since the fundamental groups of topological manifolds G_1 and G_2 are isomorphic to, respectively, Z_1 and Z_2 , it follows from (a) that Z_1 and Z_2 are isomorphic groups. Let d be their order.

The structure of the group $\mathcal{C}(\tilde{G}_1)$ is known (see [14, Table 3, pp. 297–298]). Namely, if the type of the simple group \tilde{G}_1 is different from

$$D_\ell \text{ with even } \ell \geq 4, \quad (29)$$

then $\mathcal{C}(\tilde{G}_1)$ is a cyclic group. In the case of the group \tilde{G}_1 of type (29), the group $\mathcal{C}(\tilde{G}_1)$ is isomorphic to the Klein four-group $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Since there is at most one subgroup of a given finite order in any cyclic group, we get that if the type \tilde{G}_1 is different from (29), then $Z_1 = Z_2$, so in this case Z_1 and Z_2 lie in the same $\text{Out}(\tilde{G}_1)$ -orbit.

Now, let \tilde{G}_1 be of type (29). This means that $\tilde{G}_1 = \text{Spin}_{4m}$ for some integer $m \geq 2$. Since $|\mathcal{C}(\tilde{G}_1)| = 4$, only the cases $d = 1, 2, 4$ are possible. It is clear that $Z_1 = Z_2$ for $d = 1$ and 4 , so in these cases, as above, Z_1 and Z_2 lie in the same $\text{Out}(\tilde{G}_1)$ -orbit. Therefore, it remains to consider only the case $d = 2$.

There are exactly three subgroups of order 2 in $\mathcal{C}(\tilde{G}_1)$. The natural action on $\mathcal{C}(\tilde{G}_1)$ of the group $\text{Out}(\text{Spin}_{4m})$ (isomorphic to the automorphism group of the Dynkin diagram of the group \tilde{G}_1) can be easily described explicitly using the information specified in [14, Table 3, p. 297–298]¹. This description shows that the number of $\text{Out}(\text{Spin}_{4m})$ -orbits on the set of these subgroups equals 1 for $m = 2$ and equals 2 for $m > 2$. Thus, for $m = 2$, the groups G_1 and G_2 are isomorphic and it remains for us to consider the case $m > 2$.

The quotient group of the group Spin_{4m} by a subgroup of order 2 in $\mathcal{C}(\tilde{G}_1)$, which is not fixed (respectively, fixed) with respect to the group $\text{Out}(\text{Spin}_{4m})$, is the half-spin group SSpin_{4m} (respectively, the

¹In the notation of [14, Table 3, p. 297–298], for $m = 2$, each permutation of the vectors h_1, h_3, h_4 with fixed h_2 is realized by some automorphism of the Dynkin diagram (identified with the corresponding outer automorphism), and for $m > 2$, the only nontrivial automorphism of the Dynkin diagram swaps h_{2m} and h_{2m-1} and leaves the rest h_i fixed.

orthogonal group SO_{4m}). Let SSpin_{4m} and SO_{4m} be the compact real forms of the groups SSpin_{4m} and SO_{4m} , respectively. If the underlying varieties of the groups SSpin_{4m} and SO_{4m} were isomorphic, then the underlying manifolds of the groups SSpin_{4m} and SO_{4m} would be homotopy equivalent. But according to [3, Thm. 9.1], for $m > 2$, they are not homotopy equivalent, since $H^*(\mathrm{SSpin}_{4m}, \mathbb{Z}/2\mathbb{Z})$ and $H^*(\mathrm{SO}_{4m}, \mathbb{Z}/2\mathbb{Z})$ for $m > 2$ are not isomorphic as algebras over the Steenrod algebra². Hence, the underlying varieties of the groups SSpin_{4m} and SO_{4m} for $m > 2$ are not isomorphic. This completes the proof of implication (a) \Rightarrow (b). The implication (b) \Rightarrow (a) is obvious. \square

Considerations used in the proof of Theorem 12, yield a proof of the following Theorem 13, which was published in [3] without proof.

Theorem 13 ([3, Thm. 9.3]). *Two connected compact simple real Lie groups are isomorphic if and only if their underlying manifolds are homotopy equivalent.*

Proof. It repeats the proof of Theorem 12 if in it assume that G_1 and G_2 are connected simple compact real Lie groups, whose underlying manifolds are homotopy equivalent, and replace Spin_{4m} , SSpin_{4m} , and SO_{4m} respectively with \mathbf{Spin}_{4m} , \mathbf{SSpin}_{4m} , and \mathbf{SO}_{4m} . \square

9. Questions.

1. Previous considerations naturally lead to the question of finding a classification of pairs of non-isomorphic connected reductive algebraic groups, whose underlying varieties are isomorphic. Is it possible to obtain it?

2. The same for connected compact Lie groups, whose underlying manifolds are homotopy equivalent.

3. It seems plausible that, using, in the spirit of [4], étale cohomology in place of singular homology and cohomology, it is possible to prove Theorem 5 and implication (c) \Rightarrow (a) of Theorem 8 in the case of positive characteristic of the field k . Are Theorems 10, 11, 12 true for such k ?

10. Appendix: finiteness theorems for connected reductive algebraic groups and compact Lie groups. In this section, the characteristic of k can be arbitrary.

Theorem 14. *The number of all, considered up to isomorphism, connected reductive algebraic groups of a fixed rank is finite.*

²Note that $H^*(\mathrm{SSpin}_n, \mathbb{Z}/2\mathbb{Z})$ and $H^*(\mathrm{SO}_n, \mathbb{Z}/2\mathbb{Z})$ are isomorphic as algebras over $\mathbb{Z}/2\mathbb{Z}$ if (and only if) n is a power of 2, see [3, p. 330].

Proof. For every connected reductive group G , there is a torus Z and a simply connected semisimple algebraic group S such that G is obtained by factorizing the group $Z \times S$ by a finite central subgroup. Indeed, let S be the universal covering group of the connected semisimple group $\mathcal{D}(G)$, let $\pi: S \rightarrow \mathcal{D}(G)$ be the natural projection, and let $Z = \mathcal{C}(G)^\circ$. Then $Z \times S \rightarrow G$, $(z, s) \mapsto z \cdot \pi(s)$ is an epimorphism with a finite kernel, i.e., a factorization with respect to a finite central subgroup.

Being simply connected, the group S is, up to isomorphism, uniquely determined by the type of its root system. Insofar as the set of types of root systems of any fixed rank is finite, tori of the same dimension are isomorphic, and $\mathcal{C}(S)$ is a finite group, the problem comes down to proving that, although for $\dim(Z) > 0$ there are infinitely many finite subgroups F in $\mathcal{C}(Z \times S)$, the set of all, up to isomorphism, groups of the form $(Z \times S)/F$ is finite. Note that for any $\sigma \in \text{Aut}(Z \times S)$, the groups $(Z \times S)/F$ and $(Z \times S)/\sigma(F)$ are isomorphic.

Proving this, we put

$$n := \dim(Z) > 0,$$

and let $\varepsilon_1, \dots, \varepsilon_n$ be a basis of $\text{Hom}(Z, \mathbb{G}_m) \simeq \mathbb{Z}^n$.

Let $\mathcal{D}_{r \times n}$ be the set of all matrices $(m_{ij}) \in \text{Mat}_{r \times n}(\mathbb{Z})$ such that

- (a) $m_{ij} = 0$ for $i \neq j$;
- (b) m_{ii} divides $m_{i+1, i+1}$;
- (c) $m_{ii} = m'_{ii}$ (see the notation in Section 1).

Consider a matrix $M = (m_{ij}) \in \text{Mat}_{r \times n}(\mathbb{Z})$. Then

$$Z_M := \bigcap_{i=1}^r \ker(\varepsilon_1^{m_{i1}} \cdots \varepsilon_n^{m_{in}}) \quad (30)$$

is an algebraic $(n - \text{rk}(M))$ -dimensional subgroup of Z . Every algebraic subgroup of Z is obtained in this way. If $M = (m_{ij})$ has properties (a) and (b), then $Z_M = Z_{M'}$, where $M' := (m'_{ij})$, because $\ker(\varepsilon_i^d) = \ker(\varepsilon_i^{d'})$. If $r = n$, and M is nondegenerate and has properties (a), (b), (c), then Z_M is a finite abelian group with invariant factors $|m_{11}|, \dots, |m_{nn}|$.

Elementary transformations of rows of the matrix M do not change the group Z_M . If τ_1, \dots, τ_n is another basis of the group $\text{Hom}(Z, \mathbb{G}_m)$, then $\tau_i = \varepsilon_1^{c_{i1}} \cdots \varepsilon_n^{c_{in}}$, where $C = (c_{ij}) \in \text{GL}_n(\mathbb{Z})$. The automorphism of the group $\text{Hom}(Z, \mathbb{G}_m)$, for which $\varepsilon_i \mapsto \tau_i$, has the form σ_C^* , where σ_C is an automorphism of the group Z . The mapping $\text{GL}_n(\mathbb{Z}) \rightarrow \text{Aut}(Z)$, $C \mapsto \sigma_C$, is a group isomorphism and

$$Z_{MC} = \sigma_C(Z_M). \quad (31)$$

Since elementary transformations of the columns of matrix M are carried out by multiplying M on the right by the corresponding matrices

from $\mathrm{GL}_r(\mathbb{Z})$, and by means of elementary transformations of rows and columns M can be transformed into its diagonal Smith normal form, (31) implies the existence of an automorphism $\nu \in \mathrm{Aut}(Z)$ and a matrix $D \in \mathcal{D}_{r \times n}$ such that $\nu(Z_M) = Z_D$.

Consider now a finite subgroup F in $\mathcal{C}(Z \times S) = Z \times \mathcal{C}(S)$ and the canonical projections

$$Z \xleftarrow{\pi_Z} F \xrightarrow{\pi_S} \mathcal{C}(S).$$

The groups $(Z \times S)/F$ and $((Z \times S)/(F \cap Z))/(F/(F \cap Z))$ are isomorphic. Being an n -dimensional torus, the group $Z/(F \cap Z)$ is isomorphic to the torus Z . Therefore, the groups $(Z \times S)/(F \cap Z)$ and $Z \times S$ are isomorphic. Hence, without changing, up to isomorphism, the group $(Z \times S)/F$, we can (and will) assume that $F \cap Z = \{e\}$. Then $\ker(\pi_S) = \{e\}$, and therefore, π_S is an isomorphism between F and the subgroup $\pi_S(F)$ in $\mathcal{C}(S)$. Let $\alpha: \pi_S(F) \rightarrow \pi_Z(F)$ be an epimorphism that is the composition of the inverse isomorphism with π_Z . Then

$$F = \{\alpha(g) \cdot g \mid g \in \pi_S(F)\}.$$

The subgroup $\pi_Z(F) = \alpha(\pi_S(F))$ in Z is finite and therefore has the form Z_M for some nondegenerate matrix $M \in \mathrm{Mat}_{n \times n}(\mathbb{Z})$. According to the above, there is an element $\nu \in \mathrm{Aut}(Z)$ such that $\nu(\pi_Z(F)) = Z_D$, where D is a nondegenerate matrix from $\mathcal{D}_{n \times n}$; we denote by the same letter the extension of ν to an element of $\mathrm{Aut}(Z \times S)$, which is the identity on S . Replacing F by $\nu(F)$ shows that, without changing, up to isomorphism, the group $(Z \times S)/F$, we can assume that $\pi_Z(F) = Z_D$.

Thus, if \mathcal{F} is the set all subgroups in $Z \times \mathcal{C}(S)$ of the form

$$\{\gamma(h) \cdot h \mid h \in H\},$$

where H runs through all subgroups of $\mathcal{C}(S)$, and γ through all epimorphisms $H \rightarrow Z_D$ with a nondegenerate matrix $D \in \mathcal{D}_{n \times n}$, then $F \in \mathcal{F}$. Since the group $\mathcal{C}(S)$ is finite, and the order of the group Z_D is $|\det(D)|$, the set \mathcal{F} is finite. This completes the proof. \square

Corollary 1. *The number of all, considered up to isomorphism, root data of a fixed rank is finite.*

Proof. This follows from Theorem 14, since connected reductive groups are classified by their root data, see [19, Thms. 9.6.2, 10.1.1]. \square

Corollary 2. *The number of all, considered up to isomorphism, connected compact Lie groups of a fixed rank is finite.*

Proof. This follows from Theorem 14 in view of the correspondence between connected reductive algebraic groups and connected compact

Lie groups, given by passing to a compact real form, see [14, Thm. 5.2.12]. \square

Remark 4. The proof of the Theorem 14 yields an upper bound for the numbers specified in it and its Corollaries 1 and 2.

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