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Rationalizability, Observability, and Common Knowledge*

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We study the strategic impact of players' higher-order uncertainty over the observability of actions in general two-player games. More specifically, we consider the space of all belief hierarchies generated by the uncertainty over whether the game will be played as a static game or with perfect information. Over this space, we characterize the correspondence of a solution concept which captures the behavioural implications of Rationality and Common Belief in Rationality (RCBR), where "rationality" is understood as *sequential* whenever the game is dynamic. We show that such a correspondence is generically single-valued, and that its structure supports a robust refinement of rationalizability, which often has very sharp implications. For instance, (1) in a class of games which includes both zero-sum games with a pure equilibrium and coordination games with a unique efficient equilibrium, RCBR generically ensures efficient equilibrium outcomes (*eductive coordination*); (2) in a class of games which also includes other well-known families of coordination games, RCBR generically selects components of the Stackelberg profiles (*Stackelberg selection*); (3) if it is commonly known that player 2's action is *not* observable (*e.g.* because 1 is commonly known to move earlier, etc.), in a class of games which includes all of the above RCBR generically selects the equilibrium of the static game most favourable to player 1 (*pervasiveness of first-mover advantage*).

Key words: Eductive co-ordination, Extensive-form uncertainty, First-mover advantage, Kreps Hypothesis, Higher-order beliefs, Rationalizability, Robustness, Stackelberg selections.

JEL Codes: C72, D82, D83.

1. INTRODUCTION

A large literature in game theory has studied the effects of perturbing common knowledge assumptions on payoffs, from different perspectives (*e.g.* Rubinstein, 1989; Carlsson and van Damme, 1993; Kajii and Morris, 1997; Morris and Shin, 1998; Lipman, 2003; Weinstein and Yildiz, 2007; Penta, 2012, etc.). In contrast, the assumption of common knowledge of other features of the

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game captured by the extensive form, such as the order of moves, the available actions, and their observability, has hardly been challenged.[1] Yet, this kind of uncertainty is key to many strategic situations. It is clearly paramount in military applications, but economic settings abound in which players are uncertain over the moves that are available to their opponents, or about their opponents' information about their moves, etc., in ways which need not match the typical common knowledge assumptions that are implicit in standard economic models. The reliability of such models therefore depends on whether the predictions they generate are robust to this kind of model mis-specification.

For instance, when we study firms interacting in a market, we often model the situation as a static game (Cournot competition, simultaneous entry, technology adoption, etc.), or as a dynamic one (*e.g.* Stackelberg, sequential entry, sequential technology adoption, etc.). But, in the former case, this not only presumes that firms' decisions are made without observing other firms' choice but also that this is common knowledge among them. Yet, firms in reality may often be concerned that their decisions could be leaked to their competitors. Or perhaps consider that other firms may be worried about that, or that their competitors may think the same about them, and so on. In other words, firms may face higher-order uncertainty over the observability of actions in ways which would be hard (if at all possible) to model with absolute precision. It is then natural to ask which predictions, based on models that impose standard common knowledge assumptions, would retain their validity even if players' beliefs over the observability of actions were misspecified. To address this question, we consider the space of all belief hierarchies generated by players' uncertainty over whether a two-player game will be played as a static game, *i.e.*, with no information about others' moves, or sequentially, with perfect information. Over this space, we characterize the correspondence of a solution concept—formally denoted by R—which represents the behavioural implications of Rationality and Common Belief in Rationality (RCBR), where the term "rationality" is understood as *sequential*, whenever the game is dynamic.[2] For general two-player games, we show that R is generically single-valued, and that it admits a robust and non-empty refinement which characterizes the *regular predictions* of RCBR, *i.e.*, those which do not depend on knife-edge, non-generic restrictions on the belief hierarchies. We then explore the implications of these results in classes of games in which they are especially sharp or significant and show that they provide theoretical foundations to intuitive predictions in disparate classes of games.

For example, we show that in a class of games which includes common interest games Aumann and Sorin (1989), coordination games with a unique efficient equilibrium (*e.g.* Stag-Hunt, pure coordination, etc.), but also zero-sum games with a pure equilibrium, RCBR generically selects the efficient equilibrium actions. Aside from the sharpness of the refinement

1. Some papers have studied commonly known structures to represent players' uncertainty over features of the environment captured by the extensive form (most notably, Robson, 1994; Kalai, 2004; Reny and Robson, 2004), but none of these papers have relaxed common knowledge assumptions in the sense that we do here, or that the papers above did for payoff uncertainty. We discuss the related literature in Section 5.

2. Under a genericity assumption on payoffs, the behavioural implications of RCBR in our setting are conveniently obtained applying iterated strict dominance to the interim normal form of the game with uncertainty over the observability of actions, preceded by one round of weak dominance only for those types who observe the opponent's action—the round of weak dominance serves to capture *sequential* rationality. R is thus a hybrid of Interim Correlated Rationalizability (Dekel *et al.*, 2007) and the $S^{\infty}W$ procedure (Dekel and Fudenberg, 1990). This is the instantiation of *weak* rationalizability (Battigalli and Siniscalchi, 1999) in the present setting and, thus, of Rationality and Common *Initial* Belief in Rationality (RC*i*BR). In this sense, it is the weakest solution concept that makes sense for games with some sequential moves. We nonetheless avoid stressing the *i* in the acronym because the difference between *weak* and *strong* rationalizability (and, hence, between *Initial* and *Strong* Belief in Rationality, ibid.) is moot in our setting, and it is not obvious whether, in games with more than two stages, analogues of our results would extend based on RC*i*BR, RC*s*BR, or else.

it supports for these games, this result shows that higher-order uncertainty over the observability of actions may serve as a mechanism for equilibrium coordination based on purely introspective reasoning. This is especially significant because the possibility that correct conjectures can be achieved on the basis of purely "eductive" mechanisms (Binmore, 1987, 1988), in the absence of focal points and with no information on past interactions, is generally met with scepticism.[3] Our result shows that, in the presence of higher-order uncertainty over the observability of actions, equilibrium coordination emerges endogenously as the generic implication of standard assumptions of RCBR. For zero-sum games with a pure equilibrium, this result also implies that, for a generic set of belief hierarchies, the maxmin solution coincides with the unique implication of RCBR, thereby solving a tension between RCBR and the maxmin logic which has long been discussed in the literature (*e.g.* von Neumann and Morgenstern (1944, Chapter 17); Luce and Raiffa (1957, Chapter 4); Schelling (1960, Chapter 7), etc.). In a class of games which includes all of the above, as well as other well-known families of coordination games (*e.g.* "unanimity" games in Harsanyi, 1981, or Kalai and Samet, 1984), we find that for a generic set of belief hierarchies, RCBR implies that players choose components of the Stackelberg profiles, regardless of the actual observability of actions.

We also characterize the robust predictions in environments with "one-sided" uncertainty, in the sense that we maintain common knowledge that one player's action is *not* observable, but there may be higher-order uncertainty over the observability of the other player's action. Such one-sided uncertainty arises naturally in a number of settings, for instance when moves are chosen at different points in time, with a commonly known order. But, it is also relevant in any situation in which players commonly believe that only the actions of one player are effectively irreversible, or that only one player cannot condition his choices on those of the other. In these settings, the analysis delivers particularly striking results: in a class of games which encompasses as special cases all of those discussed above, we show that RCBR generically selects the equilibrium of the static game which is most favourable to the earlier mover (or, more generally, to the player who is commonly known to *not* observe the opponent's move). Hence, a first-mover advantage is *pervasive* in these games: it arises for a generic set of types, regardless of whether the action is actually observable, including for types who share arbitrarily many (but finite) orders of mutual belief that the action is *not* observable.

This result has important strategic implications, in that it points at the impact of any mechanism which ensures that higher-order uncertainty over the observability of actions is only *one-sided.* As discussed, various mechanisms may establish this kind of uncertainty, but perhaps the simplest and most obvious to consider is the one associated with a commonly known order of moves. Within this context, our result suggests that, by determining the direction of the one-sided uncertainty, *timing* of moves alone (plus irreversibility of choices) may determine the attribution of the strategic advantage, independently of the actual observability of actions. This message is clearly at odds with the received game theoretic wisdom that observability, not timing, is key to ensure the upper hand in a strategic situation. Our results show that this classical insight is somewhat fragile, and in fact overturned, when one considers even arbitrarily small departures from the standard assumptions of common knowledge of the extensive form.

A large experimental literature has explored the impact of timing on individuals' choices in a static game, with findings that are often difficult to reconcile with the received game theoretic

3. The term "eductive" was introduced by Binmore (1987, 1988), to refer to the rationalistic, reasoning-based approach to the foundations of solution concepts. It was contrasted with the "evolutionary approach," in which solution concepts are interpreted as the steady state of an underlying learning or evolutionary process. Questions of eductive stability have been pursued in economics both in partial and general equilibrium settings (see *e.g.* Guesnerie, 2005, and references therein).

wisdom. For instance, it is well-known (see *e.g.* Camerer, 2003) that asynchronous moves in the Battle of the Sexes systematically select the Nash equilibrium most favourable to the first mover, thereby confirming an earlier conjecture by Kreps (1990), who also pointed at the difficulty of making sense of this intuitive idea in a classical game theoretic sense:

> "From the perspective of game theory, the fact that player B moves first chronologically is not supposed to matter. It has no effect on the strategies available to players nor to their payoffs. [...] however, and my own casual experiences playing this game with students at Stanford University suggest that in a surprising proportion of the time (over 70 percent), players seem to understand that the player who 'moves' first obtains his or her preferred equilibrium. [...] And *formal mathematical game theory has said little or nothing about where these expectations come from, how and why they persist, or when and why we might expect them to arise*." (Kreps, 1990, pp. 100–101 (italics in the original)).

Our results achieve this goal, as they show that the behaviour observed in these experiments is the unique regular prediction consistent with RCBR, when one considers higher-order uncertainty over the observability of actions.

The rest of the article is organized as follows: Section 1.1 presents an illustrative example; Section 2 introduces the model; Section 3 contains the general characterization of the R correspondence, and Section 4 explores some of its implications for eductive co-ordination and robust refinements, as well as the variations with one-sided uncertainty. Section 5 discusses the most closely related literature, and in particular the connections with the closely related work by Weinstein and Yildiz (2007). Section 6 concludes.

1.1. *Leading example*

We begin with a simple example to illustrate the basic elements of our model and some of our results. Consider the following "augmented" Battle of the Sexes, in which we denote the row and column players as players 1 and 2, respectively:

	L		C		R	
U	4	2	0	0	0	0
M	0	0	2	4	0	0
D	0	0	0	0	1	1

The (pure) Nash equilibria are on the main diagonal. The equilibrium (D,R) is inefficient, while (U,L) and (M,C) are both efficient, but the two players have conflicting preferences over which equilibrium they would like to coordinate on. Clearly, if it is common knowledge that the game is static, everything is rationalizable (and, thus, consistent with RCBR).

In an influential paper, which will be further discussed below, Weinstein and Yildiz (2007) characterize, for static games like this one, the set of predictions that would retain their validity under small perturbations of common knowledge assumptions on players' payoffs. Their results

imply that no outcome can be robustly ruled-out in this game, under their form of perturbations. Here, we consider different perturbations of the common knowledge assumptions: namely, we maintain that payoffs are common knowledge, but we introduce higher-order uncertainty over the observability of actions. As we will show, this change has profound effects on the insights that emerge from the analysis.

For instance, suppose that players commonly believe that player 1 chooses before 2, but there is uncertainty over whether his action will be observed. Let ω^0 denote the state of nature in which actions are not observable, and ω^1 denote the case in which 1's action is observable. If the true state is ω^1, and this is common knowledge, the only strategy profile consistent with RCBR is the backward induction solution, which induces 1's favourite equilibrium outcome, (U,L). Imagine next a situation in which the game is actually static (*i.e.* the true state is ω^0), and both players know it, but 2 thinks that 1 thinks it is common belief that the state is ω^1. Then, 2 expects 1 to choose U, and hence L is his only best reply. Moreover, if 1 believes that 2's beliefs are just as described, she also picks U as the only action consistent with RCBR. But then, if 2 believes the above, his unique best reply is to indeed play L, and so on. Iterating this argument, one can see that 1 and 2 may share arbitrarily many levels of mutual belief that the game is static, and yet have (U,L) as the only outcome consistent with RCBR. Thus, 1 *de facto* has a first-mover advantage, if she is merely believed to have it at some arbitrarily high order of beliefs. Proposition 3 in Section 4 implies that, if the only uncertainty concerns the observability of 1's action, then this selection actually occurs for a *generic* set of belief hierarchies in this game. In this sense, 1's first-mover advantage is *pervasive*, regardless of the actual observability of her action.

Clearly, if we considered symmetric uncertainty, and also included a state ω^2 in which it is 1 who observes 2's action, a similar argument would uniquely select (M,C). Hence, with two-sided uncertainty, no player would necessarily obtain a first-mover advantage, but it can still be shown that no open set of belief hierarchies would select actions D and R. Proposition 2 in Section 4 shows that, for a class of games which includes this example, the predictions consistent with RCBR generically select components of the Stackelberg profiles.

By the same logic, if payoffs were such that the Stackelberg outcomes coincided (which would be the case, for instance, in Stag-Hunt games, in pure coordination games, but also in zero-sum games with pure equilibria), then the Stackelberg profile would be the only outcome consistent with RCBR for a generic set of belief hierarchies, thereby implying equilibrium coordination on the basis of RCBR alone. This is the logic of Proposition 1 in Section 4.

As a comparison, Weinstein and Yildiz (2007, WY) maintain common knowledge that the game is static, and consider perturbations of belief hierarchies over a "rich" space of payoff uncertainty, which contains strict dominance states for every player's action. Hence, any action profile could be used to start the "infection" in the argument above. Thus, because of their richness assumption, in WY's space there would be belief hierarchies similar to the ones above, in which higher-order beliefs also trace back to profile (D,L), which would thus be uniquely selected for all such hierarchies. This implies that no refinement of rationalizability is robust in their setting, and hence—under their perturbations of common knowledge of payoffs—no outcome can be robustly ruled out in the game above. The qualitative message that emerges from our article is thus very different from WY's *unrefinability* result.

As shown by this example, one key difference between our analysis and WY's is due to the fact that, given the nature of the uncertainty that we consider, only the two backward induction outcomes (or just (U,L), in the first case) could be used to ignite the "infection argument" in our setting. But, this is only *one* of the points of departure from WY, and it doesn't suffice to explain the difference in the general results. Because of the particular configuration of payoffs, the infection argument in this example only involved a standard chain of (static) best responses. In general games, however, the robust predictions also depend on the behaviour of types who

are uncertain over whether the game is static or dynamic, whose optimization problem therefore is a hybrid of the standard static and dynamic ones. These hybrid best responses carry over to the higher-order beliefs, and hence the way the infection spreads from one type to another will differ from WY's, and so will the robust predictions. These further differences will be explained in Sections 3 and 5.

2. MODEL

2.1. *Environment*

Let $G:=(A_i,u_i)_{i=1,2}$ denote a static two-player game, where for any $i=1,2$ A_i denotes i's set of actions, and $u_i:A_1\times A_2\to\mathbb{R}$ his payoff function, all assumed common knowledge. We also let $A:=A_1\times A_2$. Similar to the example in Section 1.1, we introduce uncertainty over the observability of actions by letting $\Omega:=\{\omega^0,\omega^1,\omega^2\}$ denote the set of states of nature: ω^0 represents the state in which the game is actually static; ω^i represents the state in which the game has perfect information, with player i moving first. (Some extensions are discussed in Section 7.) We maintain throughout the following assumption on G:

Assumption 1. For each $i\in\{1,2\}$, $j\neq i$, and $a_i\in A_i$, there exists one and only one $a_j^*(a_i)$ such that $\arg\max_{a_j\in A_j}u_j(a_j,a_i)=\{a_j^*(a_i)\}$ and for each $a_i,a_i'\in A_i$ such that $a_i\neq a_i'$, $u_i(a_i,a_j^*(a_i))\neq u_i(a_i',a_j^*(a_i'))$.

In words, the first part says that for each of player i's actions, j has a unique best response; the second part says that no two distinct actions of player i, when combined with the corresponding best replies of player j, yield the same payoff to player i. This assumption, which is weaker than requiring that payoffs in G are in "generic position" (Battigalli, 1996), ensures that backward induction is well-defined, and identifies a unique outcome, in both dynamic games associated with states ω^1 and ω^2, and for any subset of actions of the first mover. In the following, we will denote by $a^i=(a_1^i,a_2^i)$ the backward induction outcome in the game in which ω^i is common knowledge. We will also refer to a_i^i as i's *Stackelberg action*.

Information: There are two possible pieces of "hard information" for a player: either he plays knowing the other's action (he is "second," θ_i''), or not (denoted by θ_i'). We let $\Theta_i:=\{\theta_i',\theta_i''\}$ denote the set of *information types*, generated by the information partition over Ω with cells $\theta_i':=\{\omega^0,\omega^i\}$ and $\theta_i'':=\{\omega^j\}$. Hence, whereas the true state of nature is never *common knowledge* (although it may be common belief), it is always pinned down by agents' pooled information: letting $\theta_i(\omega)$ denote the cell of i's information partition which contains ω, we have $\theta_i(\omega)\cap\theta_j(\omega)=\{\omega\}$ for all $\omega\in\Omega$ (so called *distributed knowledge*).

Beliefs: An *information-based type space* is a tuple $\mathcal{T}:=(T_i,\hat{\theta}_i,\tau_i)_{i=1,2}$ where each T_i is a compact and metrizable set of types, each $\hat{\theta}_i:T_i\to\Theta_i$ is a Borel-measurable map that assigns to each type his information about the extensive form, and beliefs $\tau_i:T_i\to\Delta(T_j\times\Omega)$ are continuous with respect to the weak$*$ topology and concentrated on opponent's types whose information is consistent with t_i's (*i.e.* $\tau_i(t_i)[\{(t_j,\omega):\omega\in\hat{\theta}_i(t_i)\cap\hat{\theta}_j(t_j)\}]=1$).

Each type t_i encodes a *belief hierarchy* about the states of nature, that consists of a *first-order* belief about Ω, a *second-order* belief about Ω and the opponents' first-order beliefs, and so on. Type t_i's first-order belief is obtained by taking the marginal of $\tau_i(t_i)$ over Ω, so as to obtain an element in $Z_i^1:=\Delta(\Omega)$; higher order beliefs are obtained following a customary recursive

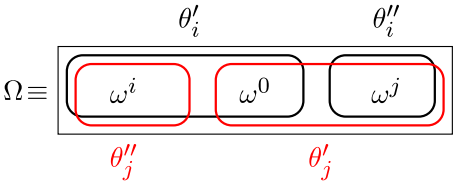

FIGURE 1
Information Partitions on Ω

marginalization procedure, where type t_i's kth order belief is an element of $Z_i^k = Z_i^{k-1} \times \Delta(Z_j^{k-1})$, and hence it consists of t_i's $(k-1)$th order beliefs, plus his belief about player j's $(k-1)$th order beliefs (see Appendix A for details). We let $\hat{\pi}_i^k(t_i)$ denote t_i's kth order belief and $\hat{\pi}_i(t_i) := (\hat{\pi}_i^k(t_i))_{k \in \mathbb{N}}$, its complete belief hierarchy.

As shown by Mertens and Zamir (1985), it is possible to construct a *universal type space* $\mathcal{T}^* = (T_i^*, \hat{\theta}_i^*, \tau_i^*)_{i=1,2}$, in which the set of types T_i^* coincides with the set of all (mutually consistent) information-belief hierarchy pairs, (θ_i, π_i), endowed with the product topology. Hence, by construction, the universal type space is such that each type $t_i = (\theta_i, \pi_i)$ is such that $\theta_i = \hat{\theta}_i^*(t_i)$ and $\pi_i = \hat{\pi}_i^*(t_i)$ (see Appendix A for the detailed construction). A Cartesian subset $T_1' \times T_2' \subseteq T_1^* \times T_2^*$ is said to be *belief-closed* if, for each player i and each type $t_i \in T_i'$ belief $\tau_i(t_i)$ assigns probability one to $T_j' \times \Omega$; a type t_i is said to be *finite* if it is contained in some finite belief-closed subset. Finally, for each $\omega \in \Omega$, we let $t_i^{CB}(\omega)$ denote the type corresponding to i's certainty that ω is common belief: namely, the type whose beliefs assign probability one to ω, they assign probability one to the opponents assigning probability one to ω, and so on. As frequently done in the literature, in the rest of the paper we will refer to such types simply as "*common belief* types."

Strategic form: Players' strategy sets depend on the state of nature:

$$S_i(\omega) := \begin{cases} A_i^{A_j} & \text{if } \omega = \omega^j \text{ and } j \neq i, \\ A_i & \text{otherwise.} \end{cases}$$

Note that i knows his own strategy set at every state of nature (that is, $S_i : \Omega \to \{A_i\} \cup \{A_i^{A_j}\}$ as a function is measurable with respect to the information partition Θ_i). With a slight abuse of notation, we can thus write $S_i(t_i)$ to refer to $S_i(\omega)$ such that $\omega \in \hat{\theta}_i(t_i)$, and we let $S_i := \bigcup_{\omega \in \Omega} S_i(\omega)$. For any $\omega \in \Omega$ and $(s_i, s_j) \in S(\omega)$, we let the (state dependent) strategic-form payoffs be defined as:

$$U_i(s_i, s_j, \omega) := \begin{cases} u_i(s_i, s_j) & \text{if } \omega = \omega^0, \\ u_i(s_i, s_j(s_i)) & \text{if } \omega = \omega^i, \\ u_i(s_i(s_j), s_j) & \text{if } \omega = \omega^j. \end{cases}$$

2.2. *Solution concept*

We are interested in the behavioural implications of players' Rationality and Common Belief in Rationality (RCBR) in this setting, where "rationality" is understood in the sense of *sequential rationality* for types t_i with information $\hat{\theta}_i(t_i) = \theta_i''$. Noting that, under Assumption 1, $a_i^*(\cdot)$ is the only sequentially rational strategy for all types who move second, RCBR can be captured by a tractable iterated deletion procedure in the interim strategic form. Specifically, for types such

that $\hat{\theta}_i(t_i)=\theta_i''$, the procedure uniquely selects $a_i^*(\cdot)$ from the first round on; for all other types, it consists of a standard iterated deletion of never best replies.

Formally, fix a type space $\mathcal{T}=(T_i,\hat{\theta}_i,\tau_i)_{i=1,2}$; for any i and t_i, let $R_i^0(t_i):=S_i(t_i)$. Then, recursively for $k=1,2,...$, and letting $R_j^{k-1}:=\{(s_j,t_j):s_j\in R_j^{k-1}(t_j)\}$, we define $R_i^k(t_i)$ to be such that, for any $t_i\in T_i$: if $\hat{\theta}_i(t_i)=\theta_i''$, then $R_i^k(t_i):=\{a_i^*(\cdot)\}$; otherwise,

$$R_i^k(t_i):=\left\{ s_i\in R_i^{k-1}(t_i):\ \begin{array}{l} \exists \mu_i\in\Delta(R_j^{k-1}\times\Omega) \text{ such that:} \\ (i)\ \text{marg}_{T_j\times\Omega}\mu_i=\tau_i(t_i), \\ (ii)\ s_i\in\arg\max_{s_i'\in S_i(t_i)}\sum_{\omega\in\Omega}\sum_{s_j\in S_j}\mu_i[\{(s_j,\omega)\}\times T_j]\cdot U_i(s_i',s_j,\omega) \end{array} \right\}.$$

Finally, we let $R_i(t_i):=\bigcap_{k\geq 0}R_i^k(t_i)$, and $R(t):=R_i(t_i)\times R_j(t_j)$.

In words, for types who move second (such that $\hat{\theta}_i(t_i)=\theta_i''$), $a_i^*(\cdot)$ is the only strategy consistent with $R_i^k(t_i)$, for any $k\geq 1$; for all other types t_i, a strategy s_i survives the kth round if and only if there exists a conjecture μ_i concentrated on the opponent's strategy-type pairs which are consistent with his $(k-1)$th round, and such that μ_i is both consistent with t_i's beliefs about j's types (condition (i)), and makes s_i a best response (condition (ii)).

It can be shown that, under Assumption 1, this solution concept is equivalent to applying iterated strict dominance to the interim normal form of the game with uncertainty over the observability of actions, preceded by one round of weak dominance only for types such that $\hat{\theta}_i(t_i)=\theta_i''$ (based on a standard duality argument, the round of weak dominance serves to capture *sequential* rationality). Hence, R is effectively a hybrid of Interim Correlated Rationalizability (ICR, Dekel *et al.*, 2007) and the $S^\infty W$ procedure (Dekel and Fudenberg, 1990). Arguments similar to those in Battigalli *et al.* (2011) can be used to show that $R_i(t_i)$ characterizes the behavioural implications of RCBR, given t_i's beliefs.

Example 1. Consider a set of types $T_i:=\{t_i^1,t_i^0,t_i^2\}$ for each $i=1,2$, where types t_i^1 and t_i^2 correspond to common belief that the game is dynamic, respectively with player 1 and player 2 as first mover. Type t_i^0 instead knows that he is not second, and attaches probability p to (t_j^0,ω^0) and $(1-p)$ to (t_j^i,ω^i). Hence, if $p=1$, t_i^0 represents common belief in the static game; but for $p\in(0,1)$, t_i^0 is uncertain whether he is part of a static game or the first-mover in a dynamic game. Formally, the type space is such that $\omega^x\in\hat{\theta}_i(t_i^x)$ for each $x=0,1,2$; $\tau_i(t_i^x)[(t_j^x,\omega^x)]=1$ if $x=1,2$, whereas $\tau_i(t_i^0)[(t_j^0,\omega^0)]=p$ and $\tau_i(t_i^0)[(t_j^i,\omega^i)]=1-p$.

Now consider the example in Section 1.1. Clearly, we have $a^1=(U,L)$, $a^2=(M,C)$, and in the following we let $a'=(D,R)$. First note that $S_i(t_i^i)=S_i(t_i^0)=A_i$ and $S_j(t_j^i)=A_j^{A_i}$. Since no action is dominated for t_i^i, $R_i^1(t_i^i)=A_i$, whereas the only sequentially rational strategy for t_j^i is its best-response function: $R_j^1(t_j^i)=\{a_j^*(\cdot)\}$. Given this, the only undominated action at the next round for t_i^i is $R_i^2(t_i^i)=\{a_i^i\}$, and hence the only outcome consistent with $R(t^i):=R_i(t_i^i)\times R_j(t_j^i)$ is $a^i=(a_i^i,a_j^*(a^i))$. If $p=1$, it is also easy to check that $R_i(t_i^0)=A_i$, as in standard (static) rationalizability (Bernheim, 1984; Pearce, 1984).

If $p\in(0,1)$, t_i^0 attaches probability p to playing a static game against type t_j^0, and probability $(1-p)$ to playing the dynamic game against type t_j^i, which would observe i's action. Then, it

is easy to check that, for $i=1,2$, $R_i^1(t_i^j)=\{a_i^*(\cdot)\}$ and $R_i^1(t_i^0)=R_i^1(t_i^i)=A_i$. At the second round, types t_i^i assign probability one to t_j^i, who plays $a_j^*(\cdot)$, and hence play their Stackelberg action a_i^i: $R_i^2(t_i^i)=R_i(t_i^i)=\{a_i^i\}$, $R_i^1(t_i^j)=R_i(t_i^j)=\{a_i^*(\cdot)\}$. Type t_i^0 thinks that, with probability $(1-p)$, he faces t_i^j (who plays $a_j^*(\cdot)$), otherwise he faces t_j^0, for whom $R_j^1(t_j^0)=A_j$, and so he will have to form conjectures $\eta_i\in\Delta(A_j)$ over that type's behaviour. The resulting optimization problem for type t_i^0, with conjectures η_i over t_j^0's action, is therefore to choose $a_i'\in A_i$ that maximizes the following expected payoff:

$$EU_i(a_i';p,\eta_i):=p\cdot\sum_{a_j\in A_j}\eta_i[a_j]\cdot u_i(a_i',a_j)+(1-p)\cdot u_i(a_i',a_j^*(a_i')). \tag{2.1}$$

Hence, $R_i^2(t_i^0)=\{a_i\in A_i:\exists\eta_i\in\Delta(A_j)\ s.t.\ a_i\in\arg\max_{a_i'\in A_i}EU_i(a_i';p,\eta_i)\}$, *that is:*

$$R_i^2(t_i^0)=R_i(t_i^0)=\begin{cases}A_i & \text{if } p\geq 3/4,\\ \{a_i^i,a_j^i\} & \text{if } p\in[1/2,3/4),\\ \{a_i^i\} & \text{if } p<1/2.\end{cases}$$

□

The combination of static and dynamic best-res ponses illustrated in this example will play a central role in our analysis, since the behaviour of the R_i correspondence around the natural benchmarks (*i.e.* the types which commonly believe ω^0 and ω^i) will in general depend on its solutions for other belief hierarchies, including those in which players are uncertain over whether the game is static or not. Next, we present two important properties of R_i:

Lemma 1 (Type space invariance) *For any two type spaces $\mathcal{T}$ and $\tilde{\mathcal{T}}$, if $t_i\in T_i$ and $\tilde{t}_i\in\tilde{T}_i$ are such that $(\hat{\theta}_i(t_i),\hat{\pi}_i(t_i))=(\hat{\theta}_i(\tilde{t}_i),\hat{\pi}_i(\tilde{t}_i))$, then $R_i(t_i)=R_i(\tilde{t}_i)$.*

Lemma 1 ensures that the predictions of R_i only depend on a type's information and belief hierarchy, not on the particular type space used to represent it. It thus enables us to study R_i as a correspondence on the universal type space, $R_i:T_i^*\rightrightarrows S_i$.[4]

Lemma 2 (Upper hemicontinuity) *$R_i:T_i^*\rightrightarrows S_i$ is an* upper hemicontinuous *(u.h.c.) correspondence. That is: for each $t_i\in T_i^*$, each $s_i\in S_i(t_i)$ and each sequence $(t_i^n)_{n\in\mathbb{N}}$ in T_i^*, if $t_i^n\to t_i$ and $s_i\in R_i(t_i^n)$ for every $n\in\mathbb{N}$, then $s_i\in R_i(t_i)$.*

This result shows that, similar to ICR and ISR on the universal type space generated by a space of payoff uncertainty, R_i is u.h.c. on our universal type space. This is a robustness property in that it ensures that anything that is ruled out by R_i for some type $t_i\in T_i^*$, is also ruled out for all types in a neighbourhood of t_i. This is an important property in the above mentioned literature, in which it is customary to identify *robustness* with upper hemicontinuity. For instance, WY's unrefinability results (respectively, Penta's, 2012) can be summarized by saying that ICR (respectively, ISR) is the strongest robust solution concept among its refinements. As will be shown, however, whereas R_i is robust in this sense, with the kind of uncertainty we consider here

4. This is a standard property for solution concepts with correlated conjectures, such as ICR (Dekel *et al.*, 2007) and interim sequential rationalizability (ISR, Penta, 2012).

it will not be the strongest robust solution concept on T_i^*: a proper refinement of R_i is also robust in our space.

3. ROBUST PREDICTIONS: CHARACTERIZATION

In this section, we characterize the strongest predictions consistent with RCBR that are robust to higher-order uncertainty over the observability of actions. We begin by constructing a set of actions, $\mathcal{B}_i \subseteq A_i$, which consists of all actions that can be uniquely rationalized for some type in the universal type space. The intuitive idea behind this construction is best understood thinking about the example in Section 1.1. There, an 'infection argument' showed that the uniqueness of the backward induction solution for types that commonly believe in ω^i propagates to types sharing n levels of mutual belief in ω^0 through a chain of unique best-responses. This type of argument is standard in the literature, and it generally involves two main ingredients: (1) the *seeds* of the infection, and (2) a chain of *strict best-responses*, which spreads the infection to other types. In WY, for instance, best-responses are the standard ones that define rationality in static games, whereas a "richness condition" ensures that any action is dominant at some state (hence, the infection can start from many "seeds," one for every action of every player). Due to the nature of the uncertainty we consider, both elements will differ from WY's in our analysis: first, only the backward induction outcomes can serve as seeds (see Example in Section 1.1); second, best-responses must account for the "hybrid" problems illustrated in Example 1.[5] The set $\mathcal{B}_i$ is defined recursively, based precisely on these two elements. Formally, for each i, let $\mathcal{B}_i := \bigcup_{k\geq 1} \mathcal{B}_i^k$, where $\mathcal{B}_i^1 := \{a_i^i\}$ and for $k \geq 1$,

$$\mathcal{B}_i^{k+1} := \mathcal{B}_i^k \cup \left\{ a_i \in A_i : \begin{array}{c} \exists p \in [0,1], \exists \eta_i \in \Delta(\mathcal{B}_j^k) \text{ such that:} \\ \underset{a_i' \in A_i}{\arg\max} \left(p \sum_{a_j \in A_j} \eta_i[a_j] u_i(a_i', a_j) + (1-p) u_i(a_i', a_j^*(a_i')) \right) = \{a_i\} \end{array} \right\},$$

where we recall that $a_j^*(\cdot)$ denotes j's sequential best response, as a function of a_i. The requirement that the argmax in the definition be *equal* to the singleton $\{a_i\}$ formalizes the idea that actions added to the $\mathcal{B}$-sets must be a *unique* best-response to some conjecture, thereby mimicking the role they play in the infection argument we discussed earlier.

Since A is finite, there exists $m < \infty$ such that $\mathcal{B}_i^m = \mathcal{B}_i$ for all i. If $p = 1$ in the definition of $\mathcal{B}_i^{k+1}$, then $\mathcal{B}_i^{k+1}$ contains the strict best replies in the static game to conjectures concentrated on $\mathcal{B}_j^k$. The case $p < 1$ instead corresponds to a situation in which i attaches probability $(1-p)$ to player j observing his choice a_i, and hence respond by choosing $a_j^*(a_i)$. Hence, as p varies between 0 and 1, $\mathcal{B}_i^{k+1}$ may also contain actions that are *not* a static best-response to conjectures concentrated in $\mathcal{B}_j^k$.[6] The following example illustrates the point:

5. A full comparison with the papers in the related literature is provided in Section 5.

6. The definition of $\mathcal{B}_i^k$ may seem to incorporate an implicit assumption of independence between player i's beliefs about the observability of his action (p), and his beliefs about j's choice (η_i) in the event in which a_i is not observable. But since player j is already assumed to play $a_j^*(a_i)$ whenever a_i is observable (which happens with probability $(1-p)$), such independence assumption entails no loss of generality in this case.

Example 2. Consider the following game, where $x \in [0,1]$:

	L		C		R	
U	4	2	0	0	0	0
M	6	0	2	4	0	0
D	0	0	0	0	3	3

Then, $a^1 = (U,L)$ and $a^2 = (M,C)$, and hence $\mathcal{B}_1^1 = \{U\}$, $\mathcal{B}_2^1 = \{C\}$. Since M (respectively, L) is a unique best-response to C (resp., U), it follows that $M \in \mathcal{B}_1^2$ (resp., $L \in \mathcal{B}_2^2$). Moreover, it can be checked that no other actions are a best-response for any $p \in [0,1]$, hence $\mathcal{B}_1^2 = \{U,M\}$, $\mathcal{B}_2^2 = \{C,L\}$. At the third iteration, suppose that η_1 attaches probability one to $C \in \mathcal{B}_2^2$, and let $p \in [0,1]$. Then, the expected payoffs from player 1's actions are:

$$\begin{aligned} EU_1(U;p,\eta_1) &= p \cdot 0 + (1-p)4 = 4 - 4p \\ EU_1(M;p,\eta_1) &= p \cdot 2 + (1-p)2 = 2 \\ EU_1(D;p,\eta_1) &= p \cdot x + (1-p)3 = 3 - (3-x)p. \end{aligned}$$

If $x = 1$, D is the only maximizer when $p \in (1/6, 1/2)$, and hence $D \in \mathcal{B}_1^3$ and $\mathcal{B}_i = A_i$ for both i. If instead $x = 0$, then it is easy to check that $\mathcal{B}_1 = \{U,M\}$ and $\mathcal{B}_2 = \{L,C\}$. □

We introduce next a solution concept, $RP_i : T_i^* \rightrightarrows A_i$, obtained by applying the same iterated deletion procedure as R_i, but starting from the set $\mathcal{B}$ instead of A. We exploit again Assumption 1, which ensures that $a_i^*(\cdot)$ is the only sequentially rational strategy for types who move second, and for those types we initialize the procedure directly from this point. Formally, for each i and t_i, let

$$RP_i^0(t_i) := \begin{cases} \mathcal{B}_i & \text{if } \hat{\theta}_i(t_i) = \theta_i', \\ \{a_i^*(\cdot)\} & \text{if } \hat{\theta}_i(t_i) = \theta_i''. \end{cases}$$

Then, for all $k \geq 1$, having set $RP_j^{k-1} := \{(s_j,t_j) : s_j \in RP_j^{k-1}(t_j)\}$, we have,

$$RP_i^k(t_i) := \left\{ s_i \in RP_i^{k-1}(t_i) : \begin{array}{c} \exists \mu_i \in \Delta(RP_j^{k-1} \times \Omega) \text{ such that:} \\ (i)\ \operatorname{marg}_{T_j \times \Omega} \mu_i = \tau_i(t_i), \\ (ii)\ s_i \in \arg\max_{s_i' \in S_i(t_i)} \sum_{\omega \in \Omega} \sum_{s_j \in S_j} \mu_i[\{(s_j,\omega)\} \times T_j] \cdot U_i(s_i', s_j, \omega) \end{array} \right\}.$$

Finally, set $RP_i(t_i) := \bigcap_{k \geq 0} RP_i^k(t_i)$, and $RP(t) := RP_i(t_i) \times RP_j(t_j)$.

As for the definition of R_i, here too, for types t_i who do not observe the opponent's action, s_i survives the kth round if and only if there exists a conjecture μ_i concentrated on strategy-type

pairs of the opponent which are consistent with RP_j^{k-1}, and such that it is consistent with t_i's beliefs about t_j (condition (i)), and makes s_i a best response (condition (ii)).

It is immediate to check that the solution concept RP_i coincides with R_i whenever $\mathcal{B}=A$, and hence the "robust predictions" would be no finer than R_i itself in that case. In general, however, $RP_i(t_i)\subseteq R_i(t_i)\cap\mathcal{B}_i$ for all t_i such that $\hat{\theta}_i(t_i)=\theta_i'$, whereas $RP_i(t_i)=\{a_i^*(\cdot)\}=R_i(t_i)$ for all t_i such that $\hat{\theta}_i(t_i)=\theta_i''$. Hence, RP_i is formally a refinement of R_i, and a proper one if, for instance, $\mathcal{B}_i\subsetneq R_i(t_i)$ for some t_i.

The next theorem provides the main results of the paper, and formalizes the sense in which RP_i characterizes the strongest robust predictions consistent with RCBR in our space of uncertainty, and that both RP_i and R_i generically coincide and are single-valued:

Theorem 1 (Robust predictions) *For each player i, $RP_i:T_i^*\rightrightarrows A_i$ is nonempty-valued and upper hemicontinuous. Moreover, for each finite type $t_i\in T_i^*$, and for each strategy $s_i\in RP_i(t_i)$, there exists a sequence of finite types $(t_i^n)_{n\in\mathbb{N}}$ in T_i^*, with limit t_i, and such that $R_i(t_i^n)=RP_i(t_i^n)=\{s_i\}$ for every $n\in\mathbb{N}$.*

The first part of Theorem 1 ensures that the predictions of RP_i are non-empty and robust to higher-order uncertainty over the observability of actions: anything that is ruled out by RP_i for a particular type t_i would still be ruled out for all types in a neighbourhood of t_i. The second part states that, for any finite type t_i, any strategy $s_i\in RP_i(t_i)$ is uniquely selected by both R_i and RP_i for some finite type arbitrarily close to t_i. This has a few important implications: (1) RP_i is the strongest robust refinement of R_i, since no refinement of RP_i is u.h.c.; (2) R_i and RP_i generically coincide on the universal type space, and deliver the same unique prediction—hence, not only is RP_i a strongest u.h.c. refinement of R_i, but it also characterizes the predictions of R_i which do not depend on the fine details of the infinite belief hierarchies (what we call the "robust predictions" of RCBR); (3) since RP_i is u.h.c., the "nearby uniqueness" result only holds for the strategies in $RP_i(t_i)$, not for those in $R_i(t_i)\backslash RP_i(t_i)$. We summarize this discussion in the following corollary:

Corollary 1. *The following hold:*

- (i) *No proper refinement of RP_i is upper hemicontinuous on T_i^*.*
- (ii) R_i coincides with RP_i *and is* single-valued *over an open and dense subset of T_i^*.*
- (iii) *For each t_i and each $s_i\in S_i(t_i)$, if there exists a sequence $(t_i^n)_{n\in\mathbb{N}}$ in T_i^* with limit t_i such that $R_i(t_i^n)=\{s_i\}$ for every $n\in\mathbb{N}$, then $s_i\in RP_i(t_i)$.*

Hence, while there is a clear formal similarity between Theorem 1 and the result of WY, the implications are very different: higher-order uncertainty over the observability of actions supports a *robust refinement* of R. Clearly, in games in which $\mathcal{B}=A$ (*e.g.* in a standard Battle of the Sexes), $R_i(t_i^{CB}(\omega^0))=RP_i(t_i^{CB}(\omega^0))$, and hence the results have the same implications. But, in some cases, the difference can be especially sharp.

Example 3. Consider the following game:

	L		C		R	
U	4	2	0	0	0	0
M	6	0	2	4	0	0
D	0	0	0	0	3	3

If players commonly believe in ω^0, the rationalizable set for this game is $R(t^{CB}(\omega^0))=\{M,D\}\times\{C,R\}$. The Stackelberg profiles are $a^1=(U,L)$ and $a^2=(M,C)$, and it is easy to check that $\mathcal{B}=\{U,M\}\times\{L,C\}$, and hence $RP(t^{CB}(\omega^0))=R(t^{CB}(\omega^0))\cap\mathcal{B}=\{(M,C)\}$.

We also note an interesting non-monotonicity of the set of robust predictions: for instance, if U were dropped from this game, then the rationalizable set under common belief would not be affected, but the Stackelberg profiles would be $a^1=(D,R)$ and $a^2=(M,C)$. It follows that $\mathcal{B}=\{M,D\}\times\{C,R\}$ and it is easy to show that $RP(t^{CB}(\omega^0))=\{M,D\}\times\{C,R\}$. Hence, eliminating actions from a game may enlarge the RP set. □

The result that R_i and RP_i generically coincide (part (*ii*) of Corollary 1) is particularly relevant from a conceptual viewpoint: Suppose that, for purely epistemic considerations (or other *a priori* reasons), we had decided to only care about the predictions generated by RCBR, except that we do not want to rely on the fine details of the infinite belief hierarchies, and hence discard the actions which are only rationalizable for nowhere dense sets of types. Then, part (*ii*) of Corollary 1 implies that whereas RCBR may deliver less sharp predictions than RP for non-generic types (such as $t^{CB}(\omega^0)$ in the example, where RCBR only rules out U and L), it would still be unique and coincide with RP_i generically on the universal type space. In this sense, RP_i characterizes the "regular predictions" of RCBR. Formally:[7]

Definition 1. For any type $t_i\in T_i^*$ and any strategy $s_i\in R_i(t_i)$, s_i is a *regular prediction of RCBR* for t_i if, for each neighbourhood N of t_i, there exists an open set $U\subseteq N$ such that $s_i\in R_i(t_i')$ for every $t_i'\in U$.

Corollary 2. *For any type $t_i\in T_i^*$ and any strategy $s_i\in R_i(t_i)$, s_i is a* regular prediction of RCBR *for t_i if and only if $s_i\in RP_i(t_i)$.*

Hence, the RP solution concept is the answer to our opening question: it characterizes the predictions that an analyst could make, for instance in a "standard" model (*i.e.* one which maintains standard common knowledge assumptions on the extensive form), to capture the strategic implications of a situation in which players entertain higher order uncertainty over the observability of their actions. As we will show in the next section, such robust predictions can prove especially insightful in important classes of games.

Theorem 1, however, is obviously predicated under the assumption that the underlying payoffs are common knowledge. This is useful to distill the specific implications of higher-order uncertainty about the observability of actions. However, one should be cautious in just taking RP as a robust solution concept, *tout court*: presumably, players in reality may face higher order uncertainty about both the observability of actions, and their payoffs—which, in the language of the earlier discussion, would lead to a substantially richer set of seeds.

One may thus wonder how the results in Theorem 1 would be affected if uncertainty over the observability of actions interacted with payoff uncertainty. It can be shown that, as long as the added payoff states satisfy a slight strengthening of Assumption 1, the result of Theorem 1 would still go through, with the only difference that the sets $\mathcal{B}$ may grow larger (though not necessarily), and hence entail weaker robust predictions. For example, if one added a richness condition à la WY, then trivially $\mathcal{B}_i=A_i$, and hence the strongest robust predictions around the common-belief types $t_i^{CB}(\omega^0)$ would be the same as in WY. Richness, however, often entails

7. We note that the open sets U in Definition 1 are not required to include t_i. If they did, regularity would be equivalent to lower hemicontinuity, which neither R_i nor RP_i satisfy.

an unnecessarily demanding robustness requirement, and the plausibility of considering payoff states which induce new "seeds" (and, hence, might affect the robust predictions) depends on the specific application. For instance, suppose that the matrix of the game in Section 1.1 does not represent players' payoffs, but monetary payments, according to some commonly known "rules of the game" $g : A \to \mathbb{R} \times \mathbb{R}$. The actual payoffs would thus depend on players' Bernoulli utility functions $v_i : \mathbb{R} \to \mathbb{R}$, with $u_i(a) = v_i(g_i(a))$. In such a setting, it certainly makes sense to consider uncertainty over utility functions v_i (*e.g.* Börgers, 1993). In most economic applications, however, it would still be sensible to maintain common knowledge that such v_i are increasing. But note that, even letting the space of payoff uncertainty include all possible profiles of such functions, the sets $\mathcal{B}$ (and, hence, the robust predictions) would still not be affected. That is because the Stackelberg profiles in that game are pinned down by the ordinal preferences, and no other actions can be made sequential best responses without violating monotonicity of the v_i functions, or also relaxing common knowledge of the outcome function g. We note that this observation applies to any game which satisfies the conditions of any of the propositions in Section 4.

The discussion above also applies to extensions of the model with richer possibilities of uncertainty. For instance, besides having states in which players observe others' actions perfectly or not at all, one may consider states in which the second mover has partial information about the earlier mover's action. This situation too would boil down to a larger set of states Ω. But once again, as long as the added states satisfy a strengthening of Assumption 1, it can be shown that the main result goes through unchanged, with the only difference that the sets $\mathcal{B}$ may grow larger (though not necessarily, as we discussed).

4. APPLICATIONS

In Example 3, not only are the robust predictions particularly sharp, but they also imply that, for a generic set of types, equilibrium coordination arises as the *only* behaviour consistent with RCBR, *i.e.*, without imposing correctness of beliefs. In Section 4.1, we consider classes of games in which the robust predictions take this especially strong form, and hence equilibrium coordination arises purely from individual reasoning. Section 4.2 explores other classes of games, in which Theorem 1 also has strong implications, which may or may not lead to eductive coordination. Section 4.3 contains our results on environments with one-sided uncertainty, and conditions under which a first-mover advantage is "pervasive."

4.1. *Eductive coordination*

Understanding the mechanisms by which individuals achieve coordination of behaviour and expectations is one of the long-lasting questions in game theory. When individuals interact repeatedly over time, learning theories or evolutionary arguments may be provided to sustain coordination (*e.g.* Fudeberg and Levine, 1998; Samuelson, 1998; Hart and Mas-Colell, 2013, and references therein). But when interactions are one-shot or isolated, or when players have no information about past interactions, their choices can only be guided by their own reasoning, and whether equilibrium coordination can be achieved is far from understood.

That a purely *eductive* approach, based only on internal inferences, may result in equilibrium co-ordination is generally met with scepticism. As a result, two main reactions can be found in the literature. At one extreme, non-equilibrium approaches such as rationalizability (*e.g.* Bernheim, 1984; Pearce, 1984) or level-k theories (*e.g.* Nagel, 1995) have been developed to analyse initial responses in games. At the opposite extreme, other approaches have developed the idea of *focal points* in Schelling (1960) (*e.g.* Sudgen, 1995), which maintains the equilibrium

assumption and shifts the discussion on the mechanisms that bring about co-ordination to external properties of the game, which are not included in the extensive form or related to players' payoffs in the game.

The next result shows that there is an interesting class of games for which higher-order uncertainty over the extensive form provides a purely eductive mechanism for equilibrium coordination, based on classical game theoretic assumptions (namely, RCBR), without appealing to any external factors or theory of focal points:

Proposition 1 (Generic coordination) *For any G which satisfies Assumption 1 and in which the two Stackelberg profiles coincide ($a^1 = a^2 \equiv \bar{a}$), there exists an open and dense subset $T' \subseteq T^*$ such that, for all $t \in T'$, $\bar{a}$ is the only outcome induced by $R(t)$.*

Note that, since by definition a_j^i is a best response to a_i^i, the condition $a^1 = a^2 \equiv \bar{a}$ implies that $\bar{a}$ is a Nash equilibrium. Hence, Proposition 1 implies that RCBR generically yields an equilibrium *outcome*. In this sense, higher-order uncertainty over the observability of actions provides a channel through which equilibrium coordination is justified from a purely eductive viewpoint. While the result follows immediately from Theorem 1 and from the observation that $\mathcal{B} = \{\bar{a}\}$ if $a^1 = a^2 \equiv \bar{a}$, this proposition is interesting because important and seemingly disparate classes of games (which include, for instance, archetypal models of both common interest and pure conflict situations) satisfy the condition $a^1 = a^2$:

Remark 1. If G satisfies Assumption 1, then the condition $a^1 = a^2 \equiv \bar{a}$ holds if G belongs to any of the following classes of games:

1. Coordination games with a unique Pareto efficient equilibrium, $\bar{a}$.
2. Common interest games (cf. Aumann and Sorin, 1989).[8]
3. Zero-sum games with a pure Nash equilibrium, $\bar{a}$.

Proposition 1 is also interesting from the viewpoint of equilibrium refinements. For instance, in *common interest games*, efficient coordination is a particularly intuitive prediction. Yet, supporting it without involving refinements directly based on efficiency has required in the past surprisingly complex arguments, and in any case always relying on the observability of the opponent's actions or on modifications of the available actions or payoffs of the game.[9] In contrast, our efficient coordination result holds for a generic subset of the universal type space, with no changes to the underlying game, regardless of whether players' actions are actually observable, and as the only outcome consistent with RCBR for those types.

8. Formally, a *coordination game* is a game in which every profile in which players choose the same or corresponding (pure) strategies is a strict Nash equilibrium (*i.e.* there exists an ordering of players' actions, $\{a_i(1), \ldots, a_i(n)\} = A_i$, such that all profiles of the form $(a_i(n), a_j(n))$ are Nash equilibria). A *common interest* game is a co-ordination game which also satisfies $u_1^*(a) = u_2^*(a)$ for all $a \in A$.

9. Aumann and Sorin (1989) for instance support the efficient equilibrium in this very special class of games as the only equilibrium outcome of a repeated game in which one player is uncertain about his opponent's type, and types may have bounded memory. For the same class of games, Lagunoff and Matsui (1997) support the efficient outcome considering a repeated game setting with perfect monitoring in which players choose simultaneously in the first period, and they alternate after that. Similar conclusions can be supported by forward induction arguments if the game is appended with a preliminary stage in which one of the players can "burn" payoffs (Ben-Porath and Dekel, 1992). Evolutionary models have also explored related questions, but they are very distant from our approach. Sandholm (2010) provides a masterful account of that approach.

For *zero-sum games*, this result bridges a gap between RCBR and the maxmin solution which has long been discussed in the literature. To illustrate the point, we adapt arguments from Luce and Raiffa (1957) to the following example:[10]

Example 4. Consider the following game, in which $\varepsilon > 0$:

	L	C	R
U	$100 \quad -100$	$-\varepsilon \quad \varepsilon$	$-2\varepsilon \quad 2\varepsilon$
M	$\varepsilon \quad -\varepsilon$	$0 \quad 0$	$\varepsilon \quad -\varepsilon$
D	$-2\varepsilon \quad 2\varepsilon$	$-\varepsilon \quad \varepsilon$	$2\varepsilon \quad -2\varepsilon$

First note that: (*i*) everything is rationalizable in this game; (*ii*) (M, C) is the maxmin solution; and (*iii*) $\mathcal{B} = \{(M, C)\}$. In Luce and Raiffa's words, choice M has two properties for player 1: "(*i*) It maximizes player 1's security level; (*ii*) it is the best counterchoice against [C]. Certainly (*ii*) is not a very convincing argument if player 1 has any reason to think that player 2 will not choose [C]. Also, (*i*) implies a very pessimistic point of view; to be sure, M yields at least [0], but it also yields at most [ε]." (ibid., p. 62). If 1 had any uncertainty that 2 might be playing L in this game, it would be unreasonable to assume he would not play U for sufficiently small ε. But then it might be unreasonable to rule out R, and hence D, and ultimately L, reinforcing the rationale for U. "[...] So it goes, for nothing prevents us from continuing this sort of 'I-think-that-he-thinks-that-I-think-that-he-thinks...' reasoning to the point where all strategy choices appear to be equally reasonable" (ibid., p. 62). □

Hence, the strategic uncertainty associated with RCBR, reflected in the fact that all actions are rationalizable in the example, clashes with the sharpness of the maxmin criterion. On the other hand, the latter is grounded on a simple, if extreme, decision theoretic principle. A classical argument to reconcile the two views is to note that the maxmin action ensures expected utility maximization in the eventuality that one's action is leaked to the opponent (see *e.g.* von Neumann and Morgenstern, 1944). The logic behind our result is reminiscent of that argument. We point out, however, that whereas the standard "fear of leaks" argument can be thought of as a first-order beliefs effect, Proposition 1 implies that the maxmin action is the only *regular prediction* of RCBR everywhere on T^*, including for types that share arbitrarily many (but finite) orders of mutual belief that leaks have *zero* probability.

The role of the $a^1 = a^2 \equiv \bar{a}$ condition in Proposition 1 is to ensure that $\mathcal{B} = \{\bar{a}\}$, which in turn implies that RP_i is single-valued also at the static common-belief type $t_i^{CB}(\omega^0)$, yielding the eductive coordination result. As shown by Example 3, however, eductive co-ordination is possible even if $a^1 \neq a^2$: all we need is that RP uniquely selects a Nash equilibrium, which can be ensured for instance if the game is such that, as in Example 3, $\mathcal{B} \cap R(t^{CB}(\omega^0)) = \{\bar{a}\}$ for some

10. Apart from using a different labelling of actions, the original argument by Luce and Raiffa (1957) refers to a game that violates Assumption 1, but it applies unchanged to our example, which satisfies Assumption 1. This explains the use of square brackets for the actions and payoffs in the quoted text.

Nash equilibrium $\bar{a}$. Various restrictions on payoffs could yield this property. We focused on the $a^1 = a^2$ condition because of its special significance, as discussed.

4.2. *Stackelberg selections*

The next result follows from Theorem 1, for a class of games which includes the example in Section 1.1, as well as the unanimity games in Harsanyi (1981) or Kalai and Samet (1984):[11]

Proposition 2. *If G satisfies Assumption 1, both players are indifferent over non-equilibrium profiles, and they strictly prefer any Nash equilibrium to any non-equilibrium profile, then, for each $i \in \{1,2\}$ there is an open and dense set $T_i' \subseteq T_i^*$ such that, for every $t_i \in T_i'$, $R_i(t_i) = RP_i(t_i) \in \{\{a_i^i\},\{a_i^j\}\}$ if $\hat{\theta}_i(t_i) = \theta_i'$ and $R_i(t_i) = \{a_i^*(\cdot)\}$ if $\hat{\theta}_i(t_i) = \theta_i''$.*

Besides including as special cases important classes of games, such as the unanimity games in Harsanyi (1981) or Kalai and Samet (1984), the conditions in Proposition 2 describe a broader, interesting class of strategic situations, in which players agree that any Nash equilibrium outcome is better than receiving the "disagreement payoff" associated with any non-equilibrium outcome. In such a class of "agreement games," the robust predictions only contemplate that players choose one of the actions associated with the Stackelberg profiles, a_i^i or a_i^j. Note that, beyond finiteness, there is no restriction on the number of actions in the baseline game, or on the rationalizable set, which could be arbitrarily large. That the robust predictions involve at most two actions is thus a remarkably sharp refinement for these games.

Proposition 2 follows from the observation that, in games which satisfy the conditions in the proposition, a^i and a^j are Nash equilibria and $\mathcal{B}_i = \{a_i^i, a_i^j\}$. This, together with the fact that $RP_i = R_i$ generically on T_i^* (Corollary 1), implies the result. Note that the statement of Proposition 2 does not only refer to the neighbourhood of the benchmark static types $t_i^{CB}(\omega^0)$, but to the generic predictions of RCBR. Thus, for instance, although inefficient equilibrium actions are consistent with RCBR when ω^0 is common belief, generically, they are not:

Corollary 3. *In any game which satisfies the conditions in Proposition 2, actions associated with inefficient Nash equilibria are generically ruled out by RCBR.*

4.3. *One-sided uncertainty and pervasiveness of first-mover advantage*

In this section, we consider the implications of maintaining common knowledge that one of the two player's actions is *not* observable, so that the higher-order uncertainty only refers to the observability of one of the players' actions. Such one-sided uncertainty is relevant, for instance, if players' choices are irreversible and made with a commonly known order, so that the earlier mover cannot observe the later mover's action; or if players commonly believe that only one of them has successfully committed to ignoring the other player's choice, or that only the actions of one player are effectively irreversible; etc.

Formally, let 1 denote the player who is commonly known to *not* observe the opponent's action, and consider the smaller space of uncertainty $\Omega^\dagger := \{\omega^0, \omega^1\}$ (only player 2 knows the state), and let $T_i^\dagger$ denote the universal type space generated by $\Omega^\dagger$. For each i, define the subset

11. Formally, a *unanimity* game (cf. Harsanyi, 1981 or Kalai and Samet, 1984) is a co-ordination game (cf. Footnote 8) such that, for every player i, $u_i(a') = u_i(a'')$ for all non-equilibrium profiles a', a''.

of actions $\mathcal{B}_i^\dagger := \bigcup_{k\geq 1} \mathcal{B}_i^{\dagger,k}$, where $\mathcal{B}_1^{\dagger,1} := \{a_1^1\}$, $\mathcal{B}_2^{\dagger,1} := \emptyset$ and for each $k \geq 1$:

$$\mathcal{B}_1^{\dagger,k+1} := \mathcal{B}_1^{\dagger,k} \cup \left\{ a_1 \in A_1 : \begin{array}{c} \exists p \in [0,1], \exists \eta_1 \in \Delta(\mathcal{B}_2^{\dagger,k}) \text{ such that:} \\ \arg\max_{a_1' \in A_1} \left(p \sum_{a_2 \in A_2} \eta_1[a_2] u_1(a_1', a_2) + (1-p) u_1(a_1', a_2^*(a_1')) \right) = \{a_1\} \end{array} \right\},$$

$$\mathcal{B}_2^{\dagger,k+1} := \mathcal{B}_2^{\dagger,k} \cup \left\{ a_2 \in A_2 : \exists \eta_2 \in \Delta(\mathcal{B}_1^{\dagger,k}) \text{ s.t. } \arg\max_{a_2' \in A_2} \sum_{a_1 \in A_1} \eta_2[a_1] \cdot u_2(a_2', a_1) = \{a_2\} \right\}$$

Note that $\mathcal{B}_i^\dagger$ is basically the same as the set $\mathcal{B}_i$ defined in Section 3, except that only a_1^1 is taken as a "seed," not a_2^2. For each i, we define the correspondence $RP_i^\dagger$, which is obtained by replacing the sets $\mathcal{B}_i$ with $\mathcal{B}_i^\dagger$ in the definition of $RP_i(t_i)$, for each $t_i \in T_i^\dagger$. The next result, analogous to Theorem 1, implies that, whenever it is non-empty valued, on this space of uncertainty $RP_i^\dagger$ is both the strongest u.h.c. refinement of R_i and it characterizes its *regular* predictions:[12]

Theorem 2 (Asymmetric perturbations) *For each player i, $RP_i^\dagger : T_i^\dagger \rightrightarrows S_i$ is upper hemicontinuous. Moreover, for each finite type $t_i \in T_i^\dagger$ and each strategy $s_i \in RP_i^\dagger(t_i)$, there exists a sequence of finite types $(t_i^n)_{n\in\mathbb{N}}$ in $T_i^\dagger$ with limit t_i and such that $R_i(t_i^n) = RP_i^\dagger(t_i^n) = \{s_i\}$ for every $n \in \mathbb{N}$.*

The following corollary states properties of $RP_i^\dagger$ analogous to those of Corollaries 1–2:

Corollary 4. *If $RP_i^\dagger$ is non-empty valued, then the following hold:*

(*i*) *No proper refinement of $RP_i^\dagger$ is upper hemicontinuous on $T_i^\dagger$.*
(*ii*) *R_i coincides with $RP_i^\dagger$ and is single-valued over an open and dense set of types $T_i' \subseteq T_i^\dagger$.*
(*iii*) *For each $t_i \in T_i^\dagger$ and each $s_i \in S_i(t_i)$, if there exists a sequence $(t_i^n)_{n\in\mathbb{N}}$ in $T_i^\dagger$ with limit t_i such that $R_i(t_i^n) = \{s_i\}$ for every $n \in \mathbb{N}$, then $s_i \in RP_i^\dagger(t_i)$*
(*iv*) *For each $t_i \in T_i^\dagger$ and each $s_i \in S_i(t_i)$, s_i is a* regular prediction of RCBR *for type t_i if and only if $s_i \in RP_i^\dagger(t_i)$.*

This result has especially strong implications in games in which a^1 is also a Nash equilibrium, which is a larger class of games than those considered in Propositions 1 and 2:

Proposition 3 (Pervasiveness of first-mover advantage) *If G satisfies Assumption 1 and a^1 is one of its Nash equilibria, then there is an open and dense subset of types $T_i' \subseteq T_i^\dagger$ such that, for all $t_i \in T_i'$, $R_i(t_i) = \{a_i^1\}$ if $\hat{\theta}_i(t_i) = \theta_i'$, and $R_i(t_i) = \{a^*(\cdot)\}$ if $\hat{\theta}_i(t_i) = \theta_i''$.*

12. $RP_i^\dagger$ is ensured to be non-empty valued, for instance, under the maintained assumptions of Proposition 3 below, or under the generic condition that $u_2(a_1^1, a_2) \neq u_2(a_1^1, a_2')$ for all $a_2 \neq a_2'$.

Hence, in this class of games, the presence of a state in which 1 has a first-mover advantage, implies that 1 has a *de facto* first-mover advantage generically on $T_i^{\dagger}$. In this sense, we say that a first-mover advantage is *pervasive*, and it arises (generically) independently of the actual observability of 1's actions, also for types who share arbitrarily many (but finite) orders of mutual belief that 1's action is *not* observable. The message of Proposition 3 may appear to be in sharp contrast with Bagwell (1995), who argued that the first-mover advantage is rather fragile.[13] Aside from the use of a common prior model, the most important difference is that the information at states $(\omega^i)_{i=1,2}$ violates Bagwell's identical support assumptions on the distributions of signals under different actions. Also, Bagwell (1995) considers games which do not fall within the scope of Proposition 3. For such games, the first-mover advantage may not be "pervasive," but it would still be uniquely selected in an open neighbourhood of $t^{CB}(\omega^1)$, and hence locally robust in our model.

From a broader perspective, this result has important implications in relation with the idea, which has received strong support by the experimental literature, that timing and commitment may have strategic importance beyond actual observability of actions. Cooper *et al.* (1993), for instance, have shown that asynchronous play in the Battle of the Sexes drastically affects subjects' behaviour, in that it induces coordination on the earlier mover's Stackelberg profile, even when his action is *not* observable (see also Camerer, 2003, and references therein).[14] As we discussed in the introduction, this is in line with the *Kreps Hypothesis* (Kreps, 1990), but clearly at odds with the received game theoretic wisdom. To the best of our knowledge, Proposition 3 is the first result to make sense of this solid experimental evidence, without appealing to behavioural theories or notions of bounded rationality, while maintaining non-observability of actions and without extending the game under consideration.[15] This is not to say that the logic of our results necessarily provides a behaviourally accurate model of individuals' strategic reasoning (see *e.g.* Crawford *et al.*, 2013, and references therein), but only that, once combined with this kind of uncertainty, standard assumptions such as RCBR may provide an effective *as if* model of how timing impacts individuals' strategic behaviour.

Finally, note that the result in Proposition 3 implies that, with one-sided uncertainty, higher-order uncertainty over the observability of actions yields eductive coordination even in games which do not satisfy the condition of Proposition 1.

5. RELATED LITERATURE

On perturbations of common knowledge: Several papers have studied perturbations of common knowledge assumptions on payoffs, following the seminal papers by Lipman (2003) and Weinstein and Yildiz (2007, WY). WY, in particular, characterize the correspondence interim correlated rationalizability (ICR, Dekel *et al.*, 2007) on the universal type space generated by a space of payoff uncertainty, which satisfies a richness condition for static games (namely, for each player's action, it contains a payoff state at which that action is strictly dominant). They show

13. This interpretation of the result in Bagwell (1995) has been criticized, among others, by van Damme and Hurkens (1997), who showed that the perturbed model in Bagwell (1995) admits a mixed equilibrium which converges to the backward induction solution as the perturbations vanish. Hence, the apparent fragility of the first-mover advantage in Bagwell (1995) stems from a particular equilibrium selection in the perturbed model.

14. In the experimental results in Cooper *et al.* (1993), 62% of the row players and 65% of the column players choose the actions associated with their favourite equilibrium in the simultaneous moves version of the Battle of the Sexes, whereas in the sequential version in which row players choose first, followed by column (who still do not observe row's choice), the figures change to 88% and 30%, respectively.

15. Amershi *et al.* (1992) developed solution concepts that assign a specific role to timing as a co-ordinating device, and hence they appeal to "external" considerations.

that ICR is generically single-valued in this space, and whenever it admits multiple rationalizable outcomes for some belief hierarchy, any of those outcomes is uniquely rationalizable for an arbitrarily close sequence of types.

The key insights of WY have been applied to mechanism design by Oury and Tercieux (2012) and the analysis has been extended to dynamic games by Weinstein and Yildiz (2011, 2016), Chen (2012), and Penta (2012). The latter paper also allows for information types and characterizes the strongest robust predictions in general information partitions with a product structure, under an extensive form richness condition. Penta (2013) relaxes the richness condition in static games, and studies sufficient conditions for Weinstein and Yildiz's selection *without* richness; Chen *et al.* (2014) provide a full characterization. Aside from the shift from payoff to extensive form uncertainty, the present paper is the first to study the impact of higher order uncertainty with information types without richness.

All these papers exploit infection arguments which, as discussed, typically consist of two main ingredients: the "seeds" of the infection, and a "chain of strict best replies."[16] Our argument differs from the earlier work both in that it relies on fewer "seeds" (only the Stackelberg actions in our case; all players' actions for the papers based on a richness assumption), and in the chain of best replies (a hybrid of static and sequential best replies in our case; either one or the other in the earlier papers). These differences lead to a structure theorem which, while displaying important similarities with WY's, at the same time describes a very different correspondence: In both cases, multiplicity is only possible within nowhere dense sets of belief hierarchies. But while in WY, when multiplicity occurs it cannot be robustly refined away, because any of the rationalizable outcomes is uniquely selected in an open set of arbitrarily close types, in our space of uncertainty there may be actions (specifically, those in R but not in RP) which are rationalizable *only* within nowhere dense sets of belief hierarchies. No analogs of this phenomenon can be found in WY's space.

It can be shown that our exercise can be mapped to one of payoff uncertainty for a properly designed artificial *auxiliary game*. The auxiliary game, however, does not satisfy the richness conditions in WY or Penta (2012), and it must account for players' information partition over the space of uncertainty (Penta and Zuazo-Garin, 2021). Thus, none of the existing results can be directly applied to the auxiliary game. Yet, it may still be tempting to think that the existence of an u.h.c. refinement of R should perhaps be expected (lack of richness after all entails a smaller set of perturbations than in WY, thereby making it easier to preserve continuity).[17] But the fact that both R and RP generically coincide and are single-valued is *not* a direct implication of the lack of richness: without richness, payoff perturbations alone would often induce open sets of types with multiple rationalizable actions.

On extensive-form uncertainty:[18] A few papers have studied models with uncertainty over the observability of actions. Robson (1994), in particular, introduced a refinement for two-player non zero-sum games, using the same set of states and information partition as in our model. In a similar vein, Reny and Robson (2004) model a situation in which players' types may be uncertain of whether their action will be observed by the opponent, and study the behaviour of equilibria in these settings as the distribution approaches the static benchmark. Both these

16. These arguments are similar to the email game in Rubinstein (1989), and are also common in the contagion literature (*e.g.* Morris, 2000, or Steiner and Stewart, 2008) and in global games (*e.g.* Carlsson and van Damme, 1993; Morris and Shin, 1998; Frankel *et al.*, 2003; Mathevet and Steiner, 2013; etc.).

17. Penta (2013), however, cautioned against this perhaps natural conjecture, by showing that weak conditions on a space of payoff uncertainty without richness may entail exactly the same structure theorem as WY.

18. Here and in the following, we use the expression *extensive form uncertainty* as short for "uncertainty about features of the strategic situation, other than players' payoffs, which are captured by the extensive form."

papers, however, adopt an equilibrium approach in a standard common prior setting. Kalai (2004) introduced a notion of "extensive robust equilibrium" to denote a profile of choices which remains an equilibrium in a large set of extensive forms, and then shows that, as the game becomes large, all equilibria become approximately extensively robust. Like the previous papers, Kalai assumes that there is no higher-order uncertainty over observability among players; only the analyst faces such uncertainty. Solan and Yariv (2004) studied a game in which the monitoring structure is endogenous, and commonly known in equilibrium. Zuazo-Garin (2017) introduced incomplete information about the information sets over a game-tree and studied sufficient conditions for the backward induction outcome. None of these papers, however, relax common knowledge assumptions in the sense that we do here, or in the literature on payoff uncertainty we discussed in the previous paragraph.

An alternative approach to extensive-form robustness is that of Doval and Ely (2020) and Makris and Renou (2018), who seek to bound or characterize the set of equilibrium distributions over a large class of extensive forms which are consistent with some minimal information about the game. These papers differ in the set of extensive forms considered in the analysis and in the equilibrium concepts they adopt. Doval and Ely (2020) and Makris and Renou (2018) also allow for payoff uncertainty. In all these papers, however, it is maintained that the actual extensive form is common knowledge among the players.

6. CONCLUSIONS

In this article, we studied the implications of perturbing common knowledge assumptions on the observability of actions in two-player games. Our main results show that higher-order uncertainty over the observability of actions supports a robust refinement of rationalizability, with several implications in important classes of games, such as: (1) eductive coordination in games in which inverting the order of moves does not affect the Stackelberg profiles; (2) maxmin selection in zero-sum games with pure equilibria; (3) efficient coordination in common interest games; (4) Stackelberg selections in a class of coordination games.

In environments in which only player 1's actions may be observable (but not 2's), we showed that, in a class of games which generalizes all of those in the previous paragraph, RCBR generically selects the equilibrium of the static game which is most favourable to player 1. Such one-sided uncertainty may arise, for instance, because 1 is commonly known to move earlier, or to be the only one whose choices are irreversible, etc. In the former case, this result also provides a rational basis for the *Kreps Hypothesis* (Kreps, 1990), which maintains that timing and commitment may have strategic importance beyond actual observability of actions—an idea which has found extensive experimental support, but which has been difficult to reconcile with standard game theoretic analysis. Here it emerges as the *only* behaviour consistent with RCBR for a generic set of types.

The problem of *extensive-form uncertainty* (cf., Section 5) is very broad, and little understood. In this article, we have focused on one particular form it can take, but more work is needed to address the broader question. In Section 7 we discussed how our results extend to environments with payoff uncertainty, as well as to richer extensive-form uncertainties. An important and more challenging extension would be to games with more than two players, which would require dealing with the richness of extensive forms associated with a larger set of players. From a more applied perspective, it would be interesting to further explore the implications of Theorems 1 and 2 to classes of games not covered by Propositions 1, 2, and 3 above.

More broadly, different notions of extensive-form robustness can be developed, mimicking the several notions of robustness which have been developed by the literature on payoff uncertainty. For instance, while in this article, we pursued a "local" notion of robustness

(similar to WY for payoff uncertainty, and Oury and Tercieux, 2012, in mechanism design), the recent papers by Doval and Ely (2020) and Makris and Renou (2018), which we discussed in Section 5, pursue a more "global" approach to extensive-form robustness, and are in this sense closer to the paper by Bergemann and Morris (2013, 2016) for games with payoff uncertainty, and Bergemann and Morris (2005, 2009) and Penta (2015) in mechanism design. Similarly, intermediate notions of robustness with payoff uncertainty, which have been put forward in the mechanism design literature (*e.g.* Ollár and Penta, 2017, 2019), may suggest further directions of research on extensive-form robustness.

In conclusion, the problem of extensive-form robustness is broad and complex. We provided one of the first attempts at a systematic understanding of this question, and we have shown that basic and plausible forms of extensive-form uncertainty may deliver novel qualitative insights on important classes of games, which include many archetypal models of both conflict and coordination, as well as on many classical questions, which we summarized at the beginning of this section. But the modeling possibilities are very rich, and our results as well as the richness of such possibilities suggest that further exploring the problem of extensive-form uncertainty may prove to be a fertile direction for future research.

APPENDIX

A. THE UNIVERSAL TYPE SPACE

In this Appendix, we recall the construction of the universal type space informally introduced in Section 2, obtained minimally adapting the standard one by Brandenburger and Dekel (1993). The construction for the space in Section 4.3 proceeds in an analogous way, applying the obvious changes in the set of states Ω and the information types in each Θ_i. Conceptually, the elements of the universal type space formalize players' belief-hierarchies in a specific way. That is, for every i, his beliefs about Ω (*first order beliefs*), his beliefs about Ω and the opponent's first order beliefs (*second order beliefs*), and so on. Remember that each $\theta_i \in \Theta_i$ is a subset of Ω. Then, the set of possible *first order beliefs consistent with* θ_i is defined as $Z_i^1(\theta_i) := \Delta(\theta_i)$. Also define player j's first-order beliefs that are consistent with i's information θ_i as:

$$Z_j^1(\theta_i) := \left\{ \pi_j^1 \in Z_j^1(\theta_j) : \theta_j \cap \theta_i \neq \emptyset \right\}.$$

These are the first order beliefs of player j that are not inconsistent with player i's information θ_i; and thus, the only ones that might eventually receive positive probability by a belief consistent with θ_i. Recursively, also define for any $k \in \mathbb{N}$,

$$Z_i^{k+1}(\theta_i) := \left\{ \left(\pi_i^\ell\right)_{\ell=1}^{k+1} \in Z_i^k(\theta_i) \times \Delta\left(\theta_i \times Z_j^k(\theta_i)\right) : \operatorname{marg}_{\theta_i \times Z_j^{k-1}(\theta_i)} \pi_i^{k+1} = \pi_i^k \right\},$$

$$Z_j^{k+1}(\theta_i) := \left\{ \left(\pi_j^\ell\right)_{\ell=1}^{k+1} \in Z_j^{k+1}(\theta_j) : \theta_j \cap \theta_i \neq \emptyset \right\}.$$

The *first-order beliefs* of a type with information θ_i are elements of $\Delta(\theta_i)$. An element of $\Delta(\theta_i \times Z_j^{k-1}(\theta_i))$ is the *kth order belief* of a type with information θ_i. The set of (*collectively coherent*) *belief hierarchies* for type θ_i is then defined as:

$$H_i(\theta_i) := \left\{ \pi_i \in Z_i^1(\theta_i) \times \prod_{k \in \mathbb{N}} \Delta\left(\theta_i \times Z_j^k(\theta_i)\right) : \forall k \in \mathbb{N}, \left(\pi_i^\ell\right)_{\ell=1}^{k} \in Z_i^k(\theta_i) \right\},$$

and the set of all (*consistent*) *information-hierarchy* pairs, as,

$$T_i^* := \bigcup_{\theta_i \in \Theta_i} \{\theta_i\} \times H_i(\theta_i).$$

It follows from Mertens and Zamir (1985) that when T_i^* is endowed with the product topology there exists a homeomorphism $\tau_i^* : T_i^* \longrightarrow \Delta(T_j^* \times \Omega)$ that preserves beliefs of all orders; *i.e.*, such that for every information-hierarchy pair (θ_i, π_i) we have both that:

(1) $\pi_i^1[\omega] = \tau_i^*(t_i)[\text{Proj}_\Omega^{-1}(\omega)]$ for any state ω.

(2) $\pi_i^{k+1}[E] = \tau_i^*(t_i)[\text{Proj}^{-1}_{\Omega \times \theta_i \times Z_i^k(\theta_i)}(E)]$ for any measurable $E \subseteq \Omega \times \theta_i \times Z_i^k(\theta_i)$ and any $k \geq 1$.

Hence, the tuple $\mathcal{T}^*:=(T_i^*,\hat{\theta}_i,\tau_i^*)_{i\in I}$, where $\hat{\theta}_i(\theta_i,\pi_i):=\theta_i$ for every information-hierarchy pair (θ_i,π_i), is an information-based type space. It will be referred to as the (*information-based*) *universal type space*.

Now, every type t_i from a type space $\mathcal{T}=(T_i,\hat{\theta}_i,\tau_i)_{i\in I}$, induces a consistent information-hierarchy pair determined by information $\hat{\theta}_i(t_i)$ and:

- First order beliefs specified by map $\hat{\pi}_i^1:T_i\to\Delta(\Omega)$, where for any $E\subseteq\Omega$,

$$\hat{\pi}_i^1(t_i)[E]:=\tau_i(t_i)\left[\left\{t_j\in T_j:\hat{\theta}_i(t_i)\cap\hat{\theta}_j\left(t_j\right)\subseteq E\right\}\right],$$

- Higher order beliefs specified by, for each $k\geq 1$, map $\hat{\pi}_i^k:T_i\to\Delta(\Omega\times Z_j^k)$, where for any measurable $E\subseteq\Omega\times Z_j^k$,

$$\hat{\pi}_i^{k+1}(t_i)[E]:=\tau_i(t_i)\left[\left\{t_j\in T_j:\hat{\theta}_i(t_i)\cap\hat{\theta}_j\left(t_j\right)\times\left\{\hat{\pi}_j^k(t_j)\right\}\subseteq E\right\}\right].$$

Then, continuous map $\phi_i:T_i\to T_i^*$ given by $t_i\mapsto(\hat{\theta}_i(t_i),\hat{\pi}_i(t_i))$, where $\hat{\pi}_i(t_i):=(\hat{\pi}_i^k(t_i))_{k\in\mathbb{N}}$, assigns to each type in an information-based type space the induced information-hierarchy pair that corresponds. Mertens and Zamir (1985) showed that for arbitrary non-redundant information-based type space $\mathcal{T}=(T_i,\hat{\theta}_i,\tau_i)_{i\in I}$,[19] set $\phi_i(T_i)$ is a *belief-closed* subset of T_i^*, in the sense that for every type $t_i\in\phi_i(T_i)$ belief $\tau_i^*(t_i)$ assigns full probability to $\phi_j(T_j)$. A type $t_i\in T_i^*$ is *finite* if it belongs to a finite belief-closed subset of T_i^*.

B. PROOF OF THEOREMS 1 AND 2

B.1. *Robustness*

Proof. We complete the proof for Theorem 1 (for Theorem 2 simply substitute RP for $RP^\dagger$ and Ω for $\Omega^\dagger$). Upper hemicontinuity and non-emptiness for types with information θ_i'' come directly from Assumption 1, which implies a unique best-reply. For types with information θ_i', we proceed by induction on k. The initial case ($k=0$) is trivially true; for the inductive step, suppose that $k\geq 0$ is such that the claims hold, and show that this implies that they hold for $k+1$. For upper hemicontinuity, fix player i and take convergent sequence of types $(t_i^n)_{n\in\mathbb{N}}$ with limit t_i and strategy $s_i\in\bigcap_{n\in\mathbb{N}}RP_i^{k+1}(t_i^n)$. For each, $n\in\mathbb{N}$ take conjecture μ_i^n that justifies the inclusion of s_i in $RP_i^{k+1}(t_i^n)$. We know from compactness of $\Delta(S_j\times T_j\times\Omega)$ that there exists some convergent subsequence of $(\mu_i^n)_{n\in\mathbb{N}}$, $(\mu_i^{n_m})_{m\in\mathbb{N}}$, whose limit we denote by μ_i. Continuity of marginalization guarantees that μ_i is consistent with t_i, and by u.h.c. of best-responses a_i is a best-response to μ_i for type t_i. Under the induction hypothesis, RP_j^k is u.h.c., and hence RP_j^k is closed. It follows that $\mu_i[RP_j^k\times\Omega]\geq\limsup_{m\to\infty}\mu_i^{n_m}[RP_j^k\times\Omega]=1$. This way, we conclude that $s_i\in RP_i^{k+1}(t_i)$, and thus, that RP_i^{k+1} is u.h.c. For non-emptiness of $RP_i^{k+1}(t_i)$ notice that we know that RP_j^k is nonempty-valued and hence there exist conjectures μ_i for t_i concentrated on RP_j^k. Set then $p:=\mu_i[S_j\times T_j\times\{\omega^0\}]$ and $\eta_i[a_j]=\mu_i[T_j\times\{(a_j,\omega^0)\}]$ for all $a_j\in A_j$. Obviously, $\eta_i\in\Delta(\mathcal{B}_j)$. Hence, if the "hybrid" best response to p and μ_i is unique, then it is in $\mathcal{B}_i$ and hence also in $RP_i^{k+1}(t_i)$. Otherwise, consider sequence of types $(t_i^n)_{n\in\mathbb{N}}$ such that $\tau_i(t_i^n)=(1-1/n)\cdot\tau_i(t_i)+(1/n)\cdot t_i^i$, where t_i^i is the type consistent with common belief in ω^i. Obviously, $(\tau_i^n)_{n\in\mathbb{N}}$ approaches t_i. Moreover, p^n and η_i^n are defined from t_i^n analogously as p and η_i are for type t_i, and hence (using Assumption 1) for n large enough the "hybrid" best-response is unique. Hence there exist $\bar{n}$ and a_i such that $s_i\in\bigcap_{n\geq\bar{n}}RP_i^{k+1}(t_i^n)$ and thus $s_i\in RP_i^{k+1}(t_i)$ from upper hemicontinuity of RP_i^{k+1}. ∥

B.2. *Unique selections*

The proof exploits the following auxiliary solution concept: for each type t_i let $W_i^{\mathcal{B}}(t_i):=\bigcap_{k\geq 0}W_i^{\mathcal{B},k}(t_i)$, where $W_i^{\mathcal{B},0}(t_i):=\mathcal{B}_i$ if t_i has information θ_i' and $W_i^{\mathcal{B},0}(t_i)=\{a_i^*(\cdot)\}$ otherwise, and then, for every $k\geq 0$, having defined $W_j^{\mathcal{B},k}=\{(s_j,t_j):s_j\in W_j^{\mathcal{B},k}(t_j)\}$, we have:

$$W_i^{\mathcal{B},k+1}(t_i):=\left\{s_i\in W_i^{\mathcal{B},k}(t_i):\begin{array}{ll}\exists\mu_i\in\Delta(W_j^{\mathcal{B},k}\times\Omega)\text{ such that:}\\ (i)\quad \operatorname{marg}_{T_j\times\Omega}\mu_i=\tau_i(t_i)\\ (ii)\quad \arg\max_{s_i'\in S_i(t_i)}\sum_{\omega\in\Omega}\sum_{s_j\in S_j}\mu_i[\{(s_j,\omega)\}\times T_j]\cdot U_i(s_i',s_j,\omega)=\{s_i\}\end{array}\right\}.$$

19. Type space $\mathcal{T}$ is *non-redundant* if for any player i map ϕ_i is injective.

Lemma 3. *For every $k \geq 0$, every player i, every state ω and every strategy $s_i \in \mathcal{B}_i^k$, if $\omega \in \theta_i'$, and $s_i \in \{a_i^*(\cdot)\}$ otherwise, there exists some finite type $t_i^{s_i,\omega} \in T_i^*$ with information $\theta_i(\omega)$ such that $R_i^{k+1}(t_i^{s_i,\omega}) = \{s_i\}$.*

Proof. We proceed by induction on k. The initial step ($k=0$) holds trivially: for states $\omega \in \theta_i'$ let $t_i^{a_i^*,\omega}$ be the type that represents common belief in ω^i, and for state ω^j let $t_i^{a_i^*(\cdot),\omega^j}$ be the type that represents common belief in ω^j. For the inductive step, let $k \geq 0$ be such that the claim holds; we verify that it also holds for $k+1$. Fix strategy s_i in the appropriate set and conjecture η_i which justifies the inclusion of s_i in said set. We know from the inductive hypothesis that, for all $(s_j,\omega) \in \text{supp } \eta_i$, there exists some finite type $t_j^{s_j,\omega}$ with information $\theta_j(\omega)$ and for which $R_j^{k+1}(t_j^{s_j,\omega}) = \{s_j\}$. Define $t_i^{s_i,\omega}$ as the type with information $\theta_i(\omega)$ and beliefs $\tau_i[E] := \eta_i[\{(s_j,\omega) \in S_j \times \Omega : (t_j^{s_j,\omega},\omega) \in E\}]$ for every measurable $E \subseteq T_j \times \Omega$. Obviously, $t_i^{s_i,\omega}$ is well-defined and finite. Pick now an arbitrary conjecture μ_i that puts probability one on the graph of R_j^{k+1} and is consistent with $t_i^{s_i,\omega}$. Notice that for every $(s_j,\omega') \in \text{supp } \eta_i$ we have that:

$$\begin{aligned}\mu_i\left[T_j \times \{(s_j,\omega')\}\right] &= \mu_i\left[\left\{t_j^{s_j,\omega''} : \omega' \in \theta_j(\omega'') \cap \theta_i(\omega)\right\} \times \{(s_j,\omega')\}\right] \\ &= \mu_i\left[\left\{t_j^{s_j,\omega''} : \omega' \in \theta_j(\omega'') \cap \theta_i(\omega)\right\} \times S_j \times \{\omega'\}\right] \\ &= \tau_i\left[\left\{t_j^{s_j,\omega''} : \omega' \in \theta_j(\omega'') \cap \theta_i(\omega)\right\} \times \{\omega'\}\right] = \eta_i[(s_j,\omega')].\end{aligned}$$

Clearly, it follows that $R_i^{k+2}(t_i^{s_i,\omega}) = \{s_i\}$. ∥

Lemma 4. *For every i, every finite type $t_i \in T_i^*$ and every $s_i \in RP_i(t_i)$ there exists a sequence of finite types $(t_i^n)_{n\in\mathbb{N}}$ in T_i^* converging to t_i such that $s_i \in W_i^{\mathcal{B}}(t_i^n)$ for every $n \in \mathbb{N}$.*

Proof. Fix arbitrary finite type space $(T_i, \hat{\theta}_i, \tau_i)_{i=1,2}$. Then, for every $n \in \mathbb{N}$ define type space $(T_i^n, \hat{\theta}_i^n, \tau_i^n)_{i=1,2}$ by setting for each player i:

- Set of types $T_i^n := \{n\} \times \left\{(s_i,t_i),(s_i,t_i^{s_i}) : t_i \in T_i \text{ and } s_i \in RP_i(t_i)\right\}$, where $t_i^{s_i}$ is constructed as in Lemma 3. Obviously, T_i^n is a finite set.
- Information-map $\hat{\theta}_i^n : T_i^n \to \Theta_i$ given by $(n,s_i,t_i) \mapsto \hat{\theta}_i(t_i)$.
- Finally, in order to define belief-maps, for state $\omega \in \theta_i'$ and strategy $s_i \in \mathcal{B}_i$ let $\eta_i^{s_i,\omega}$ denote a conjecture over $S_j \times \Omega$ that justifies the inclusion of s_i in $\mathcal{B}_i$. For state $\omega \in \theta_i''$ and strategy $s_i = a_i(\cdot)$ let $\mu_i^{s_i,\omega}$ be an arbitrary conjecture over $S_j \times \Omega$ consistent with θ_i''. Then, define player i's belief-map $\tau_i^n : T_i^n \to \Delta(T_j^n \times \Omega)$ as follows:

$$(n,s_i,t_i) \mapsto \tau_i^n(n,s_i,t_i)[(n,s_j,t_j,\omega')] := \begin{cases} \left(1-\frac{1}{n}\right)\tau_i(t_i)\left[t_j\right] & \text{if } t_j \in T_j, \\ \left(\frac{1}{n}\right) 1_{\{t_j^{s_j,\omega'}\}}(t_j) \cdot \eta_i^{s_i,\omega}[(s_j,\omega')] & \text{otherwise,} \end{cases}$$

  for every $(n,s_j,t_j,\omega') \in T_j^n \times \Omega$ such that (t_j,ω') is in the support of $\eta_i^{s_i,\omega}$, and $t_j^{s_j,\omega'}$ is constructed as in Lemma 3. The finiteness of the set of types guarantees that these belief-maps are well-defined and continuous, and that every type in T_i^n and T_j^n is finite.

We claim that the following hold: (i) for all $t_i \in T_i$, each $(n,s_i,t_i)_{n\in\mathbb{N}}$ converges to t_i; (ii) for all $t_i \in T_i$ and for all $s_i \in RP_i(t_i)$, $s_i \in W_i^{\mathcal{B}}(n,s_i,t_i)$ for every $n \in \mathbb{N}$. To prove the claim of the lemma, fix player i and pick arbitrary finite type $\bar{t}_i \in T_i^*$ and strategy $\bar{s} \in RP_i(\bar{t}_i)$. Since $\bar{t}_i$ is finite we know that there exists some finite type space $(T_i, \hat{\theta}_i, \tau_i)_{i=1,2}$ where T_i includes some type $\hat{t}_i$ that induces $\bar{t}_i$. Consider the sequence of finite type spaces $((T_i^n, \hat{\theta}_i^n, \tau_i^n)_{i=1,2})_{n\in\mathbb{N}}$ constructed above. By type space invariance, $s_i \in RP_i(\hat{t}_i)$ and by the construction above we know that for all $n \in \mathbb{N}$ there exists some type $t_i^n \in T_i^n$ such that $\bar{s}_i \in W_i^{\mathcal{B}}(t_i^n)$. Let $(\bar{t}_i^n)_{n\in\mathbb{N}}$ be the sequence in the universal type space induced by $(t_i^n)_{n\in\mathbb{N}}$. Again, because of type space invariance we know that $\bar{s}_i \in W_i^{\mathcal{B}}(\bar{t}_i^n)$ for every $n \in \mathbb{N}$. Finally, since we know that $(t_i^n)_{n\in\mathbb{N}}$ converges to $\hat{t}_i$ we also know that $(\bar{t}_i^n)_{n\in\mathbb{N}}$ converges to $\bar{t}_i$ and hence, the proof is complete. ∥

For the following lemma, let $m \in \mathbb{N}$ be such that $\mathcal{B}_i = \mathcal{B}_i^m$ for every player i. Then, we have that:

Lemma 5. *For every $k \geq 1$, every player i, every finite type $t_i \in T_i^*$ and every strategy $s_i \in W_i^{\mathcal{B},k}(t_i)$ there exists some finite type $t_i^k \in T_i^*$ such that: $\hat{\theta}_i(t_i^k) = \hat{\theta}_i(t_i)$, $\pi_i^k(t_i^k) = \pi_i^k(t_i)$, and $R_i^{m+k+2}(t_i^k) = \{s_i\}$.*

Proof. We proceed by induction on k. For the initial step ($k=1$) set $\ell=1$. Fix player i, finite type $\bar{t}_i$, action $\bar{s}_i \in W_i^{\mathcal{B},\ell}(\bar{t}_i)$ and conjecture $\bar{\mu}_i$ that justifies the inclusion of $\bar{s}_i$ in $W_i^{\mathcal{B},\ell}(\bar{t}_i)$. Then, we know by Lemma 3 that for all $(s_j,t_j)\in \text{supp}(\text{marg}_{S_j\times T_j}\bar{\mu}_i)$, there exists a finite type $t_j^{\ell-1}(s_j,t_j)$ with the same information as t_j and s.t. $R_j^{m+\ell}(t_j^{\ell-1}(s_j,t_j))=\{s_j\}$. Then, let type t_i^ℓ have information $\hat{\theta}_i(\bar{t}_i)$ and beliefs $\tau_i^\ell[E]:=\bar{\mu}_i[\{(s_j,t_j,\omega)\in S_j\times T_j\times\Omega:(t_j^{\ell-1}(s_j,t_j),\omega)\in E\}]$, for every measurable $E\subseteq T_j$. Obviously, t_i^ℓ is well-defined and finite, and has the same ℓth-order beliefs as $\bar{t}_i$—and thus, as $\hat{t}_i$. Finally, pick an arbitrary conjecture μ_i inducing t_i^ℓ and putting probability 1 on the graph of $R_j^{m+\ell}$ and notice that for every (s_j,ω) we have that:

$$\begin{aligned}
\mu_i[T_j\times\{(s_j,\omega)\}] &= \mu_i\left[\left\{t_j^{\ell-1}(s_j',t_j')):(s_j',t_j')\in S_j\times T_j, s_j\in R_j^{m+\ell}(t_j^{\ell-1}(s_j',t_j'))\right\}\times\{(s_j,\omega)\}\right]\\
&= \mu_i\left[S_j\times\left\{t_j^{\ell-1}(s_j',t_j'):(s_j',t_j')\in S_j\times T_j, R_j^{m+\ell}(t_j^{\ell-1}(s_j',t_j'))=\{s_j\}\right\}\times\{\omega\}\right]\\
&= \tau_i^\ell\left[\left\{t_j^{\ell-1}(s_j',t_j'):(s_j',t_j')\in S_j\times T_j, R_j^{m+\ell}(t_j^{\ell-1}(s_j',t_j'))=\{s_j\}\right\}\times\{\omega\}\right]\\
&= \bar{\mu}_i\left[\left\{(s_j',t_j')\in S_j\times T_j: R_j^{m+\ell}(t_j^{\ell-1}(s_j',t_j'))=\{s_j\}\right\}\times\{\omega\}\right]\\
&= \bar{\mu}_i[T_j\times\{(s_j,\omega)\}].
\end{aligned}$$

Clearly, it follows that $R_i^{m+\ell+1}(t_i^\ell)=\{\bar{s}_i\}$.

For the inductive step suppose that $k\geq 1$ is such that the claim holds. Then, to verify that it also does for $k+1$ simply repeat, verbatim, the steps of the initial step by replacing index $\ell=1$ by $\ell=k+1$ and noticing that the existence of types $t_j^{\ell-1}(\cdot)$ is not due to Lemma 3, but due to the induction hypothesis, instead. ∥

With all the above, we can easily prove the unique selection result:

Proof. Fix finite type $t_i\in T_i^*$ and strategy $s_i\in RP_i(t_i)$. Then, we know from Lemma 4 that there exists a sequence of finite types $(\hat{t}_i^n)_{n\in\mathbb{N}}$ in T_i^* converging to t_i such that $s_i\in W_i^{\mathcal{B}}(\hat{t}_i^n)$ for every $n\in\mathbb{N}$. It follows from Lemma 5 that for all $n\in\mathbb{N}$ there exists a sequence of finite types $(t_i^{n,k})_{k\in\mathbb{N}}$ in T_i^* converging to $\hat{t}_i^n$ such that $R_i(t_i^{n,k})=\{s_i\}$ for all $k\in\mathbb{N}$. Thus, if for each $n\in\mathbb{N}$ we set $t_i^n=t_i^{n,k}$, $(t_i^n)_{n\in\mathbb{N}}$ is a sequence of finite types in T_i^* converging to t_i such that $R_i(t_i^n)=\{s_i\}$ for every $n\in\mathbb{N}$. For Theorem 2, simply repeat the argument substituting T_i^*, $\mathcal{B}$, and R_i with $T_i^\dagger$, $\mathcal{B}^\dagger$, and $R_i^\dagger$, respectively. ∥

C. ADDITIONAL RESULTS

We start with the proof of Corollary 2:

Proof. For any t_i, let $\mathcal{N}(t_i)$ denote the set of neighbourhoods of t_i, and let $F_i:T_i^*\rightrightarrows S_i$ be the correspondence, where $F_i(t_i)$ is set of strategies $s_i\in R_i(t_i)$ such that for any $N\in\mathcal{N}(t_i)$ of t_i, there exists an open subset $U\subseteq N$ such that $s_i\in R_i(t_i')$ for all $t_i'\in U$. Notice first that F_i is u.h.c. To see this, proceed by contradiction and suppose that $(t_i^n)_{n\in\mathbb{N}}$ converges to t_i, $s_i\in F_i(t_i^n)$ for every $n\in\mathbb{N}$ and $s_i\notin F_i(t_i)$. By upper hemicontinuity of R_i we have $s_i\in R_i(t_i)$. Then there exists an $N'\in\mathcal{N}(t_i)$ s.t. for all $V\subseteq N'$ there is some $t_i'\in V$ s.t. $s_i\notin R_i(t_i')$. But this is a contradiction, since $N'\in\mathcal{N}(t_i^n)$ for large enough n and $s_i\in F_i(t_i^n)$. To see that $F_i(t_i)\subseteq RP_i(t_i)$, pick an arbitrary $s_i\in F_i(t_i)$ and $N\in\mathcal{N}(t_i)$. By Theorem 1, there exists an open and dense $X\subseteq T_i^*$ in which R_i and RP_i coincide. Then, there exists some open $U\subseteq N$ s.t. $s_i\in R_i(t_i')=RP_i(t_i')$ for every $t_i'\in U\cap X\subseteq N$. Hence, for all $N\in\mathcal{N}(t_i)$ there exists t_i^N s.t. $s_i\in RP_i(t_i^N)$. Since RP_i is u.h.c., we have $s_i\in RP_i(t_i)$.

For the other inclusion, pick $s_i\in RP_i(t_i)$. If t_i is finite pick sequence $(t_i^n)_{n\in\mathbb{N}}$ converging to t_i such that $RP_i(t_i^n)=\{s_i\}$ for all $n\in\mathbb{N}$. Obviously, $s_i\in R_i(t_i)$. In addition, upper hemicontinuity of RP_i implies that for all $n\in\mathbb{N}$ there exists some open $U^n\in\mathcal{N}(t_i^n)$ such that $RP_i(t_i')=\{s_i\}$ for all $t_i'\in U^n$. Since for all $N\in\mathcal{N}$ there exists some $n\in\mathbb{N}$ such that $t^n\in N$, there also exists some $U\subseteq N$, $U=U^n\cap N$, such that $s_i\in RP_i(t_i')\subseteq R_i(t_i')$ for all $t_i'\in U$. That is, $s_i\in F_i(t_i)$. Finally, the upper hemicontinuity of F_i implies that the inclusion is also true for nonfinite types. ∥

Next, we prove Proposition 1:

Proof. Fix player i. We know from Theorem 1 that there exists some dense subset $\breve{T}_i\subseteq T_i^*$ such that $|R_i(t_i)|=1$ and $R_i(t_i)=RP_i(t_i)$ for any $t_i\in\breve{T}_i$. Since $a_i^i=a_i^j=a_i^*$, it follows from Assumption 1 that $a_i^j=a_j^i$, and hence, that $\mathcal{B}_i=\{a_i^*\}$, which in turn implies $R_i(t_i)=RP_i(t_i)=\{a_i^*\}$ for any $t_i\in\breve{T}_i$. R_i's upper hemicontinuity then implies that $T_i':=\{t_i\in T_i^*:R_i(t_i)=\{a_i^*\}\}$ is open, and clearly, we have $\breve{T}_i\subseteq T_i'$. Thus, T_i' is an open and dense subset of T_i^* and such that $R_i(t_i)=\{a_i^*\}$ for every $t_i\in T_i'$. ∥

We prove now Proposition 2:

Proof. Under the assumptions of the proposition, w.l.o.g. let $u_i(a)=0$ for any non-equilibrium profile a. Then, note that for any i, $p\in[0,1]$ and $a_i\neq a_i^i, a_i^j$, we have:

$$p\cdot u_i(a_i^i,a_j^j)+(1-p)\cdot u_i(a^i)>p\cdot u_i(a_i,a_j^j)+(1-p)\cdot u_i(a_i,a_j^*(a_i)),$$

because $u_i(a^i)>u_i(a_i,a_j^*(a_i))$ for any $a_i\neq a_i^i$ by definition, and $u_i(a_i^i,a_j^j)\geq u_i(a_i,a_j^j)=0$ for any $a_i\neq a_i^j$. Hence, a_i^i dominates all $a_i\neq a_i^j, a_i^i$ for any p, and it is better than a_j^i for high p, and worse than a_i^j for low p. It follows that $\mathcal{B}_i^2=\{a_i^i,a_i^j\}$. But then, at the next round, for any $p,q\in[0,1]$ and any $a_i\neq a_i^i, a_i^j$ we have:

$$\begin{aligned}pq\cdot u_i(a_i^i,a_j^j)+p(1-q)\cdot u_i(a^i)+(1-p)\cdot u_i(a^i)\\ >pq\cdot u_i(a_i,a_j^j)+p(1-q)\cdot u_i(a^i)+(1-p)\cdot u_i(a_i,a_j^*(a_i)).\end{aligned}$$

Similarly as above, only a_i^i and a_i^j can be a unique best-response for some p and q. It follows that $\mathcal{B}_i=\{a_i^i,a_i^j\}\subseteq R_i$ and $RP_i\subseteq\{a_i^i,a_i^j\}$. The result follows from Theorem 1. ‖

Finally, we prove Proposition 3:

Proof. Fix player i. By Theorem 2, there exists some dense subset $\check{T}_i\subseteq T_i^\dagger$ s.t. $|R_i(t_i)|=1$ and $R_i(t_i)=RP_i^\dagger(t_i)$ for all $t_i\in\check{T}_i$. Since a^1 is a Nash equilibrium, by Assumption 1 $\mathcal{B}_1^\dagger=\{a_1^1\}$ and $\mathcal{B}_2^\dagger=\{a_2^1\}$. The rest of the proof is the same as the proof of Proposition 1, replacing $\mathcal{B}$ with $\mathcal{B}^\dagger$ and RP with $RP^\dagger$. ‖

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