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Letter

Entanglement of multiphoton two-mode polarization Fock states and of their superpositions

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Abstract

Density matrices of pure multiphoton Fock polarization states and their resultant reduced density matrices of mixed states are expressed in similar ways in terms of matrices of correlators, defined as averaged products of equal numbers of creation and annihilation operators. The degree of entanglement of considered states is evaluated for various combinations of parameters of states and characters of their reduction.

Keywords: entanglement, multiphoton states, density matrices, Schmidt entanglement parameter

1. Introduction

There is a growing interest in the science of quantum information to multiparticle entanglement. It finds applications in quantum computing and quantum error correction [1, 2], as well as in quantum networks [3]. The latter includes, in particular, communication among many parties that is enhanced by shared multiparticle entanglement. The most promising recourse for establishing this type of entanglement is, of course, multiphoton systems. Thus, there is a natural interest in studying entanglement properties of the states of many photons. Recent experiments have also shown that entanglement of up to ten photons can be observed in the laboratory [4].

In this work we consider pure multiphoton two-mode polarization Fock states and their superpositions. We give a general definition of the density matrices of such states, as well as of the density matrices of mixed states arising from pure Fock states after their partial reduction over a series of photon variables. Elements of such density matrices are expressed in terms of correlators, defined as averaged products of equal numbers of creation and annihilation operators with different distributions of operators over two polarization modes. We will calculate parameters characterizing the degree of entanglement in such states and investigate their dependence on features of the original pure states and on the ways of their reduction.

Note that for biphoton states the method of density matrices of the described type was suggested by D N Klyshko in 1997 [5] and somewhat later used in the works [6, 7]. More recently there was a series of works on some aspects of entanglement in multipohoton states [8–12]. However, as far as we know, there were no works where the Klyshko method of density matrices would be generalized for multiphoton states with numbers of photons higher than 3. Such generalization is one of the main goals of this work. The second goal is characterizing entanglement of multiphoton states in terms of Schmidt

decompositions and their parameters, which will also be new. Note also that, although the Schmidt decomposition has been known in mathematics since 1907 [13], in the fields of modern quantum optics and quantum information it was introduced by J H Eberly and coworkers first in 1994 [14] and then in 2004 [15]. A much more general and detailed description of the Schmidt decomposition, as well as its applications, was given in the review paper [16].

2. Density matrices

Let us consider an arbitrary pure state $|\Psi^{(n)}\rangle$ of n photons with identical frequencies and identical given propagation directions but distributed arbitrarily between two polarization modes, horizontal and vertical or H and V. Two-mode polarization basic Fock states are states with given numbers of horizontally and vertically polarized photons n_H and n_V:

$$|\Psi_{n_H, n_V}\rangle = |n_H, n_V\rangle = \frac{a_H^{\dagger n_H} a_V^{\dagger n_V}}{\sqrt{n_H! n_V!}} |0\rangle. \qquad (1)$$

More general n-photon polarization states to be considered are superpositions of basic Fock states (1) with identical total numbers of photons in all terms of superposition, $n \equiv n_H + n_V = \text{const}$.

$$|\Psi^{(n)}\rangle = \sum_{n_H=0}^{n} C_{n_H} |n_H, n_V\rangle|_{n_V=n-n_H} \qquad (2)$$

with $\sum_{n_H=0}^{n} |C_{n_H}|^2 = 1$. The wave functions of all n-photon states $|\Psi^{(n)}\rangle$ depend on n single-photon variables σ_i, $\Psi^{(n)}(\{\sigma_i\}) = \langle\{\sigma_i\}|\Psi^{(n)}\rangle$ and, explicitly, they are given by symmetrized products of n single-photon wave functions [17]. In the case of polarization modes the single-photon wave functions in these products are $\psi_H(\sigma_i) = \delta_{\sigma_i, H}$ and $\psi_V(\sigma_j) = \delta_{\sigma_j, V}$. In the matrix representation $\psi_H(\sigma_i) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_i$ and $\psi_V(\sigma_j) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}_j$ [18].

Note that sometimes it is possible to find in the literature mentions of particle or mode entanglement, and of their differences or similarities. We do not use such concepts here because in our opinion the type of entanglement to be studied can be much more correctly interpreted as related to uncertainty of distributions of particle variables between modes or, in brief, as variable entanglement. For two-mode polarization states this means an uncertainty of attachment of polarization variables σ_i to H- or V-modes.

The direct products of n two-line columns $\begin{pmatrix} 1 \\ 0 \end{pmatrix}_i$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}_j$ form a basis of columns with 2^n elements ('rows') and with different locations of a single unit in one of these 'rows'. Written down in this basis explicitly, the multiphoton wave function can be used for constructing the density matrix $\rho^{(n)}(\{\sigma_i\}, \{\sigma'_i\}) = \Psi^{(n)}(\{\sigma_i\})\Psi^{(n)\dagger}(\{\sigma'_j\})$. However, at high values of the photon number n this procedure is rather cumbersome to reproduce it explicitly. Fortunately, there is a much more compact algorithm for constructing multiphoton density

matrices to be described and discussed below. Of course, at any given n correctness of the matrix representations used below can be checked and confirmed directly by the described derivations based on the use of the multiphoton wave functions $\Psi^{(n)}(\{\sigma_i\})$.

Thus, for any pure two-mode multiphoton state $|\Psi^{(n)}\rangle$ its $2^n \times 2^n$ density matrix can be presented symbolically in the following form:

$$\rho^{(n)} = \frac{1}{n!} \left(\left\{ \langle (a_H^\dagger)^{n-k_2} (a_V^\dagger)^{k_2} a_H^{n-k_1} a_V^{k_1} \rangle \right\} \right) \qquad (3)$$

with averaged products of creation and annihilation operators defined as $\langle ... \rangle = \langle \Psi^{(n)} | ... | \Psi^{(n)} \rangle$ and sometimes referred to below as correlators. The integers k_1 and k_2 (both $\geqslant 0$ and $\leqslant n$) in equation (3) represent, correspondingly, the number of identical groups of columns and rows in the matrix. At any given values of k_1 and k_2 columns and rows repeat themselves $C_n^{k_{1,2}}$ times, where $C_n^k = n!/k!(n-k)!$ are the binomial coefficients. Note also that the total powers of creation operators and total powers of annihilation operators in all elements are the same: $(n-k_2) + k_2 = n$ and $(n-k_1) + k_1 = n$. However, proportions between powers of the creation operators in the H- and V-modes change from one line of the matrix to another and are controlled by the integer k_2. Similarly, proportions between powers of the annihilation operators in the H- and V-modes change from one column of the matrix to another and are controlled by the integer k_1.

The simplest examples are the density matrices of pure one-photon and two-photon polarization states

$$
\begin{array}{ccc}
k_1 = & 0 & 1 & k_2 \\
\rho^{(1)} = & \begin{pmatrix} \langle a_H^\dagger a_H \rangle & \langle a_H^\dagger a_V \rangle \\ \langle a_V^\dagger a_H \rangle & \langle a_V^\dagger a_V \rangle \end{pmatrix} & \begin{matrix} 0 \\ 1 \end{matrix}
\end{array}
\qquad (4)
$$

and

$$
\begin{array}{cccccc}
k_1 = & 0 & 1 & 1 & 2 & k_2 \\
\rho^{(2)} = \frac{1}{2} & \begin{pmatrix} \langle a_H^{\dagger 2} a_H^2 \rangle & \langle a_H^{\dagger 2} a_H a_V \rangle & \langle a_H^{\dagger 2} a_H a_V \rangle & \langle a_H^{\dagger 2} a_V^2 \rangle \\ \langle a_H^\dagger a_V^\dagger a_H^2 \rangle & \langle a_H^\dagger a_V^\dagger a_H a_V \rangle & \langle a_H^\dagger a_V^\dagger a_H a_V \rangle & \langle a_H^\dagger a_V^\dagger a_V^2 \rangle \\ \langle a_H^\dagger a_V^\dagger a_H^2 \rangle & \langle a_H^\dagger a_V^\dagger a_H a_V \rangle & \langle a_H^\dagger a_V^\dagger a_H a_V \rangle & \langle a_H^\dagger a_V^\dagger a_V^2 \rangle \\ \langle a_V^{\dagger 2} a_H^2 \rangle & \langle a_V^{\dagger 2} a_H a_V \rangle & \langle a_V^2 a_H a_V \rangle & \langle a_V^{\dagger 2} a_V^2 \rangle \end{pmatrix} & \begin{matrix} 0 \\ 1 \\ 1 \\ 2 \end{matrix}
\end{array}
\qquad (5)
$$

etc.

As mentioned above, the biphoton density matrix $\rho^{(2)}$ (5) was written by Klyshko [5] and used in [5–7]. Note, however, that the next step used for working with the density matrix (5) consisted of simply crossing out one of two coinciding rows and one of two coinciding columns. This reduces the fourth-order matrix to the three-dimensional one, but significantly changes features of the arising matrix. In particular, its trace becomes different from one in contrast to the density matrix (5). Additionally, it does not provide a correct transition to the coherence matrix of biphoton qutrits [18]. Indeed, the most general polarization biphoton state is qutrit, the state vector of which is

$$|\Psi^{(2)}\rangle = C_1 |2_H\rangle + C_2 |1_H, 1_V\rangle + C_3 |2_V\rangle \qquad (6)$$

with C_i being arbitrary complex constants obeying the normalization condition $\sum_i |C_i|^2 = 1$. The natural coherence matrix of this state is

$$\rho_{coh}^{(2)} = \begin{pmatrix} |C_1|^2 & C_1^* C_2 & C_1^* C_3 \\ C_2^* C_1 & |C_2|^2 & C_2^* C_3 \\ C_3^* C_1 & C_3^* C_2 & |C_3|^2 \end{pmatrix} =$$

$$\begin{pmatrix} \frac{\langle a_H^{\dagger 2} a_H^2 \rangle}{2} & \frac{\langle a_H^{\dagger 2} a_H a_V \rangle}{\sqrt{2}} & \frac{\langle a_H^{\dagger 2} a_V^2 \rangle}{2} \\ \frac{\langle a_H^\dagger a_V^\dagger a_H^2 \rangle}{\sqrt{2}} & \langle a_H^\dagger a_V^\dagger a_H a_V \rangle & \frac{\langle a_H^\dagger a_V^\dagger a_V^2 \rangle}{\sqrt{2}} \\ \frac{\langle a_V^{\dagger 2} a_H^2 \rangle}{2} & \frac{\langle a_V^{\dagger 2} a_H a_V \rangle}{\sqrt{2}} & \frac{\langle a_V^{\dagger 2} a_V^2 \rangle}{2} \end{pmatrix}. \quad (7)$$

Evidently, the last expression (7) does not coincide with that of equation (5), e.g. deleting the third column and third row. Therefore, the procedure of crossing out repeated columns and rows cannot be considered as mathematically correct. To make it correct, one has to first make the unitary transformation of the matrix (5) [18], after which all elements in one of the rows and one of the columns in the 4×4 matrix become zero. For the matrix (5) the required unitary transformation has the form

$$\rho^{(2)} \to \widetilde{\rho}^{(2)} = U \rho^{(2)} U^\dagger$$

with

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Only after this transformation can the arising single line and single column with zero elements be safely removed without changing general features of the original density matrix and providing the correct expression for the coherence matrix (7) [18]. In principle, similar transformations can also be found for density matrices of higher-order states, $n > 2$. However, in the following discussion we will not use such transformations by keeping all the full 2^n dimensionality of the density matrices unchanged, with repetitions of identical columns and rows of the density matrix completely conserved. This repetition of columns and rows is related directly to the symmetry features of multi-boson wave functions. This symmetry is not seen explicitly in the multiphoton state vectors of the type (1), but in the wave function of polarization variables they are present in the form of terms differing only by transposition of variables [17–19]. Such terms in the wave functions are responsible directly for the appearance of repeated columns and rows in the density matrices.

3. Reduced density matrices

It is known that the degree of entanglement of pure quantum states is related directly to the degree of mixing of reduced states. The concept of reduced states arises when one represents a complicated pure state as if consisting of two parts. Reduction is then the averaging over one of these two parts, giving rise to a possibly mixed state of the other part. In the simplest cases of $n = 2$ and $n = 3$ definitions of two parts

are evident: these parts consist of two single-photon states in the case of biphotons, and they consist of single-photon and two-photon states in the case of a pure three-photon original state. In the cases of states with large numbers of photons, $n \geqslant 4$, there is more than one way of imagining how the original n-photon state can be divided into two parts. E.g. for $n = 4$ there are are two ways of the gedanken splitting this state into two parts: $4 = 2 + 2$ and $4 = 3 + 1$ [12]. Thus, in these cases one can discuss different degrees of entanglement corresponding to different ways of splitting the original state into two parts.

Mathematically, a standard way of reducing density matrices of pure states consists of using their wave-function representation $\rho^{(n)}(\{\sigma_i\}, \{\sigma_i'\}) = \Psi^{(n)}(\{\sigma_i\}) \Psi^{(n)\dagger}(\{\sigma_i'\})$, equalizing one or several variables $\sigma_i = \sigma_i'$ and summing the product $\Psi^{(n)} \Psi^{(n)\dagger}$ over the variable(s) σ_m. However, the procedure is rather cumbersome for states with many photons and with all symmetry requirements of the multi-boson wave functions fully taken into account. Fortunately, the result of such calculations can be presented in a relatively simple form with elements of the reduced density matrices expressed in terms of correlators similar to those arising in the above-mentioned density matrices of pure states. By assuming that for an n-photon state we reduce the density matrix $\rho^{(n)}$ with respect to $n - m$ variables, we can write the following general expression for the resulting reduced 2^m-order density matrix:

$$\rho_r^{(m;n)} = \frac{(n-m)!}{n!} \left(\left\{ \langle (a_H^\dagger)^{m-k_2} (a_V^\dagger)^{k_2} a_H^{m-k_1} a_V^{k_1} \rangle \right\} \right) \quad (8)$$

with the previous definition of averaging in correlators $\langle ... \rangle = \langle \Psi^{(n)} | ... | \Psi^{(n)} \rangle$ and with the previous meaning of the integers k_2 and k_1 ($m \geqslant k_{1,2} \geqslant 0$) representing the number of groups of columns and rows, at given k_1 and k_2 repeated $C_m^{k_{1,2}}$ times. Below are some examples of reduced matrices.

The single-photon reduced density matrices of arbitrary pure n-photon states $|\Psi^{(n)}\rangle$ arising at $m = 1$ have the form

$$\rho_r^{(1;n)} = \frac{1}{n} \begin{pmatrix} \langle a_H^\dagger a_H \rangle & \langle a_H^\dagger a_V \rangle \\ \langle a_V^\dagger a_H \rangle & \langle a_V^\dagger a_V \rangle \end{pmatrix}. \quad (9)$$

For basic Fock states $|\Psi_{n_H, n_V}\rangle$ (1) the matrices are very simple:

$$\rho_r^{(1;n)} = \frac{1}{n_H + n_V} \begin{pmatrix} n_H & 0 \\ 0 & n_V \end{pmatrix}, \quad (10)$$

and they correspond to the Schmidt entanglement parameter

$$K(n_H, n_V) = \frac{1}{Tr[(\rho_r^{(1;n)})^2]} = \frac{(n_H + n_V)^2}{n_H^2 + n_V^2}. \quad (11)$$

In the case of an even total number of photons $n = n_H + n_V$, as a function of n_H, the Schmidt parameter K achieves a maximum at $n_H = n_V = n/2$ and $K_{max} = 2$. For other relations between n_H and n_V the Schmidt parameter K is smaller than K_{max}. In the cases of odd numbers of photons n the maximal values of the Schmidt parameter are achieved at $n_H = [n/2]$ and $n_H = [n/2] + 1$, where the symbol $[x]$ denotes in this case the integer closest to but smaller than x. Maximal values of the Schmidt parameter in these cases are somewhat smaller than

2. The simplest example of the basic Fock state with odd n is that of three-photon states $|1_H, 2_V\rangle$ and $|2_H, 1_V\rangle$. In both cases equation (11) gives $K = 9/5$ in agreement with the results of the work [12]. The main conclusion from this brief analysis concerns the achievable entanglement of n-photon basic Fock states with respect to division into subsystems of a single-photon and an $(n-1)$-photon state: entanglement of such states with respect to such division for subsystems does not exceed that occurring in the case of biphoton states, and the maximal entanglement with $K = 2$ or close to 2 is achieved in the states with maximally close numbers of horizontally and vertically polarized photons, n_H and n_V.

The two-photon reduced density matrices of arbitrary pure n-photon states $\Psi^{(n)}$ arise in the cases of $m = 2$ and their general form is given by

$$\rho_r^{(2;n)} = \frac{1}{n(n-1)} \begin{pmatrix} \langle a_H^{\dagger 2} a_H^2 \rangle & \langle a_H^{\dagger 2} a_H a_V \rangle & \langle a_H^{\dagger 2} a_H a_V \rangle & \langle a_H^{\dagger 2} a_V^2 \rangle \\ \langle a_H^\dagger a_V^\dagger a_H^2 \rangle & \langle a_H^\dagger a_V^\dagger a_H a_V \rangle & \langle a_H^\dagger a_V^\dagger a_H a_V \rangle & \langle a_H^\dagger a_V^\dagger a_V^2 \rangle \\ \langle a_H^\dagger a_V^\dagger a_H^2 \rangle & \langle a_H^\dagger a_V^\dagger a_H a_V \rangle & \langle a_H^\dagger a_V^\dagger a_H a_V \rangle & \langle a_H^\dagger a_V^\dagger a_V^2 \rangle \\ \langle a_V^{\dagger 2} a_H^2 \rangle & \langle a_V^{\dagger 2} a_H a_V \rangle & \langle a_V^{\dagger 2} a_H a_V \rangle & \langle a_V^{\dagger 2} a_V^2 \rangle \end{pmatrix}.$$
(12)

Formally, this density matrix looks identical to that of equation (5), although the normalization factors in these two matrices are different. However, an even more important difference concerns the meaning of averaging in correlators in these matrices. In the case of the density matrix of a pure two-photon states $\rho^{(2)}$ (5) averaging is defined as $\langle \Psi^{(2)} | ... | \Psi^{(2)} \rangle$. In contrast, in the case of the second-order reduced density matrix (12) correlators in this matrix are defined as $\langle \Psi^{(n)} | ... | \Psi^{(n)} \rangle$, where $n > 2$. Note also that all described matrices, both of pure states (3)–(5) and of mixed states (8)–(12), obey the same important feature: their traces are equal to 1.

To evaluate the degree of entanglement of multiphoton states $|\Psi^{(n)}\rangle$ their reduced density matrices have to be diagonalized numerically, after which the found eigenvalues $\lambda_i^{(m;n)}$ can be used to find the Schmidt entanglement parameter or the entropy of the reduced density matrices:

$$K = \frac{1}{\sum_i \lambda_i^2} \quad \text{and} \quad S_r = -\sum_i \lambda_i \log_2 \lambda_i.$$
(13)

Before presenting specific results of calculations, it is worth making a note concerning features of the above-mentioned density matrices and differences between their features in the cases of basic Fock states (1) and their superpositions (2). In the case of single basic Fock states, their pure-state and reduced density matrices have many zeros. In fact, averaging over basic Fock zeroes all correlators containing products of creation and annihilation operators in one of two modes in different powers, e.g. $(a_H^\dagger)^p a_H^q$ with $p \neq q$ and the same for the vertical-polarization mode. Owing to this, the density matrices of single Fock states have a diagonal-block structure. The following equation represents an example of such a diagonal-block second-order reduced density matrix $\rho_r^{(2;4)}$ (12) for the state $|2_H, 2_V\rangle$ reduced with respect to two variables ($m = 2$):
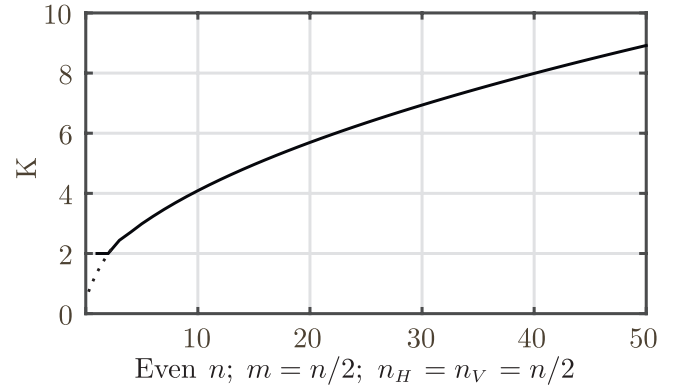
Figure 1. Calculated Schmidt entanglement parameter $K(n)$ for states $|\Psi^{(n)}\rangle = |n_H, n_V\rangle$ with even n, equal numbers of horizontally and vertically polarized photons, $n_H = n_V = n/2$, and with the gedanken splitting of the states for two m-photon states with equal numbers of photons, $m = (n - m) = n/2$; the dotted line corresponds to K_{appr} of equation (17).

$$\rho_r^{(2;4)} = \begin{pmatrix} 1/6 & 0 & 0 & 0 \\ 0 & 1/3 & 1/3 & 0 \\ 0 & 1/3 & 1/3 & 0 \\ 0 & 0 & 0 & 1/6 \end{pmatrix}.$$
(14)

In this matrix three diagonal blocks are located (a) at the crossing of the first line and first column, (b) at the crossing of the second and third lines with the second and third columns, and (c) at the crossing of the fourth line and fourth column. Each block gives only one non-zero eigenvalue, and they are equal to, correspondingly, 1/6, 2/3, and 1/6, which gives $K = 2$ in accordance with the result shown in figure 1.

In a general case of the reduced density matrices $\rho_r^{(m;n)}$ (8) corresponding to the original states Ψ_{n_H, n_V} (1), the non-zero square blocks arise at crossings of the lines and columns with equal numbers of integers k_1 and k_2, $k_1 = k_2 \equiv k$ with $0 \leqslant k \leqslant m$, and the dimensionality of each block is C_m^k. The number of blocks equals $m + 1$. All elements inside each block are equal to each other. Owing to the equality of elements inside a block, each block has only one non-zero eigenvalue, and eigenvalues of the reduced density matrix can be expressed via these non-zero eigenvalues of blocks. Explicitly they are given by

$$\lambda_k = \frac{(n-m)!}{n!} C_m^k \langle \Psi_{n_H, n_V} | (a_H^\dagger a_H)^{m-k} (a_V^\dagger a_V)^k | \Psi_{n_H, n_V} \rangle$$
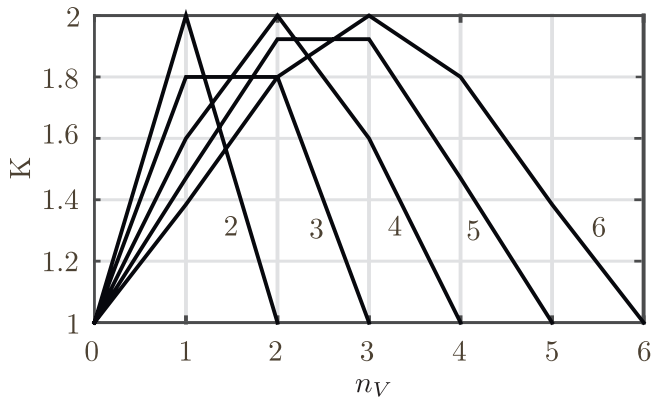$$= \frac{(n-m)!}{n!} \frac{m!}{k!(m-k)!} \frac{n_H!}{(n_H - m + k)!} \frac{n_V!}{(n_V - k)!},$$
(15)

with additional limitations

$$k \leqslant \min\{n_V, m\} \quad \text{and}$$
$$k \geqslant \max\{m - n_H, 0\} \equiv \max\{n_V - (n - m), 0\}.$$
(16)

Notice that at $m = n = n_H + n_V$ the reduced matrix $\rho_r^{(m;n)}$ becomes the density matrix of a pure state $\rho^{(n)}$. In this case the limitations (16) take the form $k \leqslant n_V$ and $k \geqslant n_V$, and they
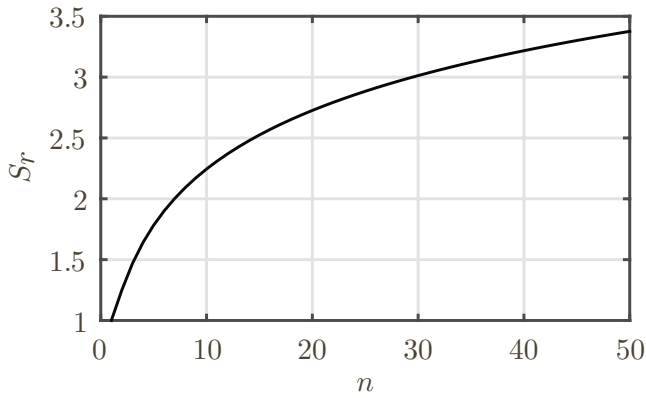
Figure 2. As in figure 1 but for the entropy of the reduced state rather than for the Schmidt entanglement parameter.
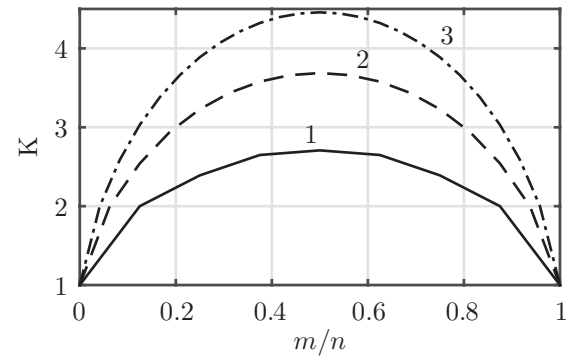

Figure 3. Schmidt entanglement parameter K as a function of the number n_V of vertically polarized photons in the states $|\Psi^{(n)}\rangle$; total numbers of photons n are shown near the curves. The assumed division into subsystems is $n \to 1 + (n-1)$.

are compatible with each other only at $k = n_V$. This means that at a given value of n_V the density matrix $\rho^{(n)}$ has only one non-zero block characterized by $k = n_V$. Simple algebra shows that in this case equation (15) yields $\lambda_k = 1$ as it has to be for a pure state.

The described features of the reduced density matrices corresponding to the basic two-mode Fock states Ψ_{n_H,n_V} (1) significantly simplify diagonalization of these matrices and their Schmidt-mode analysis. The situation appears to be completely different in the case of superpositions of basic states $\Psi^{(n)}$ (2). In this case the diagonal-block structure of matrices does not exist anymore and the reduced density matrices have to be diagonalized without any useful simplifications.

4. Results

The results of calculations are presented in figures 1–6. The first of these (figure 1) corresponds to multiphoton states $|\Psi^{(n)}\rangle$ with total number of photons n, where n is taken as even, and with equal numbers of photons with horizontal and vertical polarizations, $n_H = n_V = n/2$. The state is assumed to be imagined as consisting of two parts with the same numbers of photons in each, $n/2$. The reduced density matrix of such a subsystem is $\rho^{(\frac{n}{2},n)}$ ($m = n - m = n/2$ in notations of

Figure 4. Schmidt entanglement parameter K of the states $|\Psi_{n_H,n_V}\rangle$ (1) versus the ratio 'number m of variables in the reduced density matrix divided by the total number of polarization variables or the total number of photons n'; the curves correspond to $n = 6\,(1),\ 8\,(2)$ and $24\,(3)$.

equation (8)). Its eigenvalues are λ_i and the Schmidt entanglement parameter is determined by the first expression in equation (13). In figure 1 the Schmidt parameter is shown in relation to its dependence on the total number of photons in the state $|\Psi^{(n)}\rangle$. As seen from figure 1, in the considered case the Schmidt entanglement parameter and, hence, the degree of entanglement are monotonically growing functions of the number of photons. In other words, multiphoton Fock states can have much higher resources of entanglement than usually considered biphoton states.

Note that the curve in figure 1 can be perfectly approximated by the analytical expression

$$K_{\text{appr}} \approx 0.62 + n^{0.54}. \tag{17}$$

The coincidence of this model curve with the numerically calculated one is so perfect that in figure 1 they look indistinguishable, except for a small region $n < 4$. The main qualitative conclusion from this comparison is that as a function of the total number of photons n, the Schmidt entanglement parameter $K(n)$ grows roughly as the root square of n.

Similar conclusions can be deduced from calculations of the entropy of reduced state S_r defined by the second expression in equation (13). For the same state as in the previous calculations the function $S_r(n)$ plotted in figure 2 is seen to be monotonically growing and very similar to the curve of figure 1. This confirms the conclusion regarding the growing degree of entanglement with a growing number of photons, and confirms compatibility of the entropy and Schmidt parameter for characterization of the degree of entanglement.

Figure 3 depicts the dependency of the Schmidt entanglement parameter K on the relation between horizontally and vertically polarized photons in the Fock states with given total number of photons n: if the number of vertically polarized photons is $n_V = k \leqslant n$, the number of horizontally polarized photons is $n_H = n - k$, and the number k varies along the horizontal axis in figure 3. In this series of calculations the degree of reduction is taken to be as high as possible, $m = 1$, i.e. the reduced state is a single-photon one and its reduced density matrix is $\rho_r^{(1;n)}$ of equation (9). As seen clearly from the figure, at all values of n the Schmidt number K and the degree
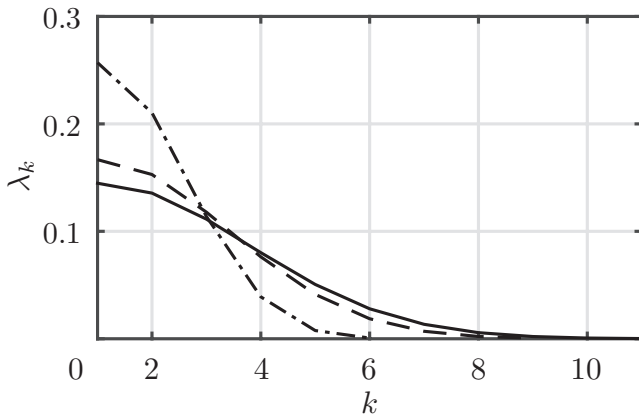
Figure 5. Arranged in descending order, eigenvalues λ_k of the reduced density matrices $\rho_r^{(m,n)}$ (8) of the state $|\Psi_{n_H,n_V}\rangle$ (1) with $n_H = n_V = 60$ and $n = n_H + n_V = 120$ and different degrees of reduction: $m = 50$ (solid line), 30 (dashed line) and 10 (dash-dotted line).
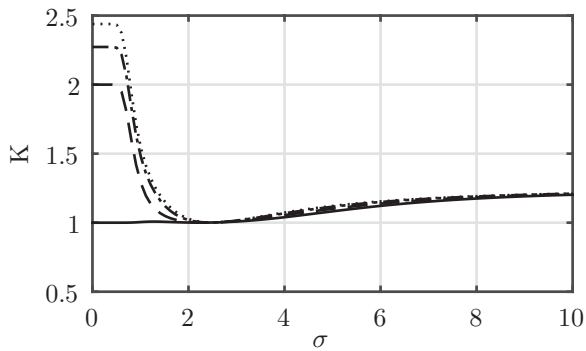
Figure 6. Schmidt entanglement parameter K for the state (18) with $n = 6$ and $m_0 = 3, 2, 1, 0$ (from top to bottom at small values of σ).

of entanglement are maximal when the numbers of vertically and horizontally polarized photons in the state $|\Psi^{(n)}\rangle$ are equal ($k = n/2$) or maximally close to each other (in the case of odd n).

Figure 4 shows the dependence of the Schmidt entanglement parameter of the Fock state $|\Psi^{(n)}\rangle$ on m/n, i.e. on the ratio of number m of variables remaining in the state after its reduction to the total number of photons (or their variables) n in the original pure state. The figure shows clearly that entanglement of the state $|\Psi^{(n)}\rangle$ is maximal when it is considered as split for two parts with an equal number of photons in each part ($m/n = 0.5$).

Figure 5 shows the dependence of eigenvalues λ_k on their numbers k for the reduced density matrices $\rho_r^{(m;n)}$ of the state with the total number of photons $n = 120$, $n_H = n_V$ and different degrees of reduction $n - m$.

The results shown in figure 5 show that in spite of a growing degree of entanglement in strongly multiphoton states, eigenvalues of all reduced density matrices remain concentrated in a restricted region of not especially high values. This means that the effective dimensionality of the corresponding Hilbert spaces remains not too high. This conclusion is important for approximate numerical calculations because it opens

the possibility of performing these calculations in smaller-dimensionality matrices, forming the main cores for finding relatively large eigenvalues λ_k.

Let us now consider an example of states more complicated than a single basic Fock state. Let the state under consideration be given by

$$|\Psi\rangle = \sum_{m=1}^{n} C_m |(n-m)_H, m_V\rangle. \tag{18}$$

Let us take the coefficients C_m in the Gaussian form

$$C_m = N \exp\left(-\frac{(m - m_0)^2}{2\sigma^2}\right) \tag{19}$$

with the normalization factor N given by

$$N = \left[\sum_{m=0}^{n} \exp\left(-\frac{(m - m_0)^2}{\sigma^2}\right)\right]^{-1/2} \tag{20}$$

and m_0 denoting the value of m at which the squared coefficients $|C_m|^2$ are maximal. As mentioned above, in this case diagonalization of the reduced density matrix is more complicated because this matrix no longer has a diagonal-block structure, and it has to be diagonalized as a whole, without any simplifications. Nevertheless, the results of such calculations are presented in figure 6 for three different values of the parameter m_0 in the Gausssian distribution of equation (19).

One of the most interesting features of the curves in figure 6 concerns the disappearance of entanglement ($K = 1$) at some definite point σ_0. In principle, this is not contradictory, e.g. to the known features of the simplest superposition of Fock states—a biphoton polarization qutrit (6) characterized by three constants C_1, C_2, C_3. As is known [19], its degree of entanglement can be characterized either by the Schmidt entanglement parameter K or by the so-called concurrence $C = |2C_1C_3 - C_2^2|$ [20], the two of which are related to each other by a simple formula $C = \sqrt{2(1 - K^{-1})}$. It is also known that entanglement of a qutrit disappears when $C = 0$ or $2C_1C_3 = C_2^2$. This effect of disappearing entanglement at some specific relation between the qutrit's parameters seems to be analogous to the effect of missing entanglement of the state (18) at $\sigma = \sigma_0$.

5. Conclusion

In conclusion, in this paper the density matrix approach used previously for biphoton states is generalized for the case of multiphoton two-mode polarization states. Both basic two-mode Fock states and their superpositions with given total numbers of photons are considered. In this method elements of density matrices are expressed in terms of mean values of products of photon creation and annihilation operators. The structures of the arising reduced density matrices with parts of the polarization variables are discussed. Eigenvalues λ_k of the reduced density matrices are found analytically for two-mode Fock states and numerically for their superpositions. These results are used to find the degree of entanglement of multiphoton states with respect to their division into pairs of states

with smaller numbers of photons. The degree of entanglement is estimated either by the Schmidt entanglement parameter $K = 1/\sum_k \lambda_k^2$ or by the entropy of the reduced states $S_r = -\sum_k \lambda_k \log_2 \lambda_k$. The main qualitative conclusion is that the degree of entanglement is maximal if the numbers of photons in two modes, n_H and n_V, are maximally close to each other and if multiphoton states are considered as consisting of two parts with approximately (or exactly) equal numbers of photons in each part. The maximal degree of entanglement is found to be a growing function of the number of photons as shown in figures 1 and 2.

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