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Research Article

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Weyl n-algebras and the Swiss cheese operad

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Abstract: We apply Weyl n-algebras to prove formality theorems for higher Hochschild cohomology. We present two approaches: via propagators and via the factorization complex. It is shown that the second approach is equivalent to the first one taken with a new family of propagators we introduce.

Keywords: Factorization homology, higher Hochschild cohomology, little disks operad, Swiss cheese operad

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Introduction

The present paper continues studies of Weyl n-algebras began in [20–22]. We describe how these ideas can be applied to prove formality theorems, which are isomorphisms between higher Hochschild cohomology of polynomial algebras and Weyl n-algebras. The substantial part of this paper is rephrasing and generalization of the pioneer paper [16], where the formality for usual Hochschild cohomology was firstly proved, in terms of Weyl n-algebras.

The construction from [16] depends on choice of a propagator. There is another approach to formality via the factorization homology of Weyl n-algebras, which was implicitly stated and used in [20]. We show that, for the usual Hochschild cohomology, this formality is equivalent to the one introduced in [16] but with a different propagator. Due to the geometric nature of this approach, all coefficients of this morphism are rational. It leads us to a surprising conjecture that a family of propagators we define gives formalities with rational coefficients.

Two approaches to the formality described in the present paper resemble two approaches to the Kontsevich integral of a knot. The first one using iterated integrals (see e.g. [4, Part 3]) is similar to the approach via propagator. The second partly conjectural approach (see [20] and references therein) corresponds to the one via the factorization complex.

The first three section of the paper do not contain any new material. In the first and second sections, we recall basic definitions for the present series of papers of the Fulton–MacPherson operad, Weyl n-algebras and the factorization complex.

The third section is devoted to the notion of the Swiss cheese operad, which was introduced in [15, 28]; for the recent progress, see [12], and for the higher Hochschild cohomological complex, see [14]. A module over the Swiss cheese operad is a triple of an e_n-algebra, an e_{n-1}-algebra and some additional data, which is referred to as an action of the e_n-algebra on the e_{n-1}-algebra. The main result of [27] states that, given an action of an e_n-algebra on an e_{n-1}-algebra, there is a morphism from this e_n-algebra to the higher Hochschild complex of e_{n-1}-algebra. We demonstrate that, for $n = 2$, if one takes the usual Hochschild cohomological complex as a model for higher Hochschild complex, the corresponding morphism of L_∞-algebras is the one appearing in the proof of the formality theorem in [16]. The Hochschild cohomological complex introduced

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in [6] is a dg-Lie (not L_∞!) algebra. It seems to be an important feature that it is equipped with a pre-Lie algebra structure, which is not compatible with the differential. It would be highly interesting to find some explanation of the existence of such a model and discover some higher-dimensional generalization of it. This generalization must be a dg-Lie algebra model of the L_∞-algebra of the higher Hochschild complex.

In the fourth section, we construct a quasi-isomorphism between the Weyl n-algebra and the higher Hochschild cohomological complex of the polynomial algebra. We build an action of the Weyl n-algebra on the polynomial algebra using a propagator, which, especially in the light of the direct description of [10], seems to be parallel to the action of [13] in the closed case and in the open/closed case, of which the Swiss cheese is a part in [14]. This is a higher-dimensional generalization of the main construction of [16] given in terms of Weyl n-algebras. As was mentioned in [16] for $n = 2$, this construction works with any propagator. But [16] and later papers explore merely the same propagator and its slight variations like in [24]. Conjecture 1 we formulate implies that there are other interesting propagators to work with.

In the fifth section, following [20], we use the factorization complex to build formality morphisms. These formalities turn out to be equal to the ones from the previous section for some particular propagators. The terms of these formality morphisms are given by integrals similar to the ones from [1].

The key point in the factorization complex approach for an e_3-algebra and 1-sphere (see also [20]) is an isomorphism between the Hochschild cohomological complex of the polynomial algebra and the Hochschild homological complex of the e_3-algebra. The latter is equipped with the cyclic structure. But the quasi-isomorphism given by Proposition 13 does not respect it. Consequently, there is some interesting interaction between this structure and the formality given by the factorization complex approach. The paper [20] may be considered as the first step in studying this interaction. Besides, as was mentioned in [16] and developed in later papers (see [24] and references therein), the set of formalities for the usual Hochschild cohomological complex of a polynomial algebra is equipped with a rich additional structure such as the Grothendieck–Teichmüller Lie algebra action. The interaction of this structure and the structure mentioned above is a subject for future research.

1 Weyl n-algebras

1.1 Fulton–MacPherson operad

Let $\mathbb{R}^n$ be an affine space. For a finite set S, denote by $(\mathbb{R}^n)^S$ the set of ordered S-tuples in $\mathbb{R}^n$. Further, let $\mathscr{C}^0(\mathbb{R}^n)(S) \subset (\mathbb{R}^n)^S$ be the configuration space of distinct ordered points in $\mathbb{R}^n$ labeled by S. In [7, 23] (see also [1, 25]), the Fulton–MacPherson compactification $\mathscr{C}(\mathbb{R}^n)(S)$ of $\mathscr{C}^0(\mathbb{R}^n)(S)$ is introduced. This is a manifold with corners and a boundary with interior $\imath\colon \mathscr{C}^0(\mathbb{R}^n)(S) \hookrightarrow \mathscr{C}(\mathbb{R}^n)(S)$. There is a projection $\pi\colon \mathscr{C}(\mathbb{R}^n)(S) \to (\mathbb{R}^n)^S$ such that $\pi \circ \imath\colon \mathscr{C}^0(\mathbb{R}^n)(S) \to (\mathbb{R}^n)^S$ is the natural embedding. For any $S' \subset S$, there is the projection map

$$\mathscr{C}(\mathbb{R}^n)(S) \to \mathscr{C}(\mathbb{R}^n)(S'),$$

which forgets points.

The natural action of the group of affine transformations on $\mathscr{C}^0(\mathbb{R}^n)(S)$ is lifted to $\mathscr{C}(\mathbb{R}^n)(S)$. Denote by $\mathrm{Dil}(n)$ its subgroup consisting of dilatations with positive coefficients and shifts. The group $\mathrm{Dil}(n)$ acts freely on $\mathscr{C}(\mathbb{R}^n)(S)$, and the quotient is isomorphic to the fiber $\pi^{-1}(\vec{0})$, where $\vec{0} \in (\mathbb{R}^n)^S$ is the S-tuple sitting at the origin (see e.g. [21, Subsection 2.2]). Denote any of these isomorphic manifolds by $\mathbf{FM}_n^S$. The sequence $\mathbf{FM}_n^S$ may be equipped with a structure of an unital operad in the category of topological spaces; for details, see [17], [21, Subsection 2.2] and other references above.

Definition 1. The sequence of topological spaces $\mathbf{FM}_n^S$ with the unital operad structure as above is called the Fulton–MacPherson operad.

Given a topological operad, one may produce a dg-operad by taking complexes of chains of its components.

Definition 2. Denote by $\mathfrak{fm}_n$ the operad of $\mathbb{R}$-chains of $\mathbf{FM}_n$.

Real numbers appear here are to simplify things; all objects and morphisms we shall use may be defined over rationals. By chains, we mean the complex of de Rham currents; that is why we need real chains. Mostly, below, we will consider the cooperad of de Rham cochains of $\mathbf{FM}_n$.

Proposition 1. *Operad $\mathfrak{fm}_n$ is weakly homotopy equivalent to e_n, the operad of chains of the little disks operad.*

Proof. See [25, Proposition 3.9] and [21, Subsection 3.3]. □

Spaces $\mathbf{FM}_n^S$ are acted on by the general linear group, and, in particular, by its maximal compact subgroup $\mathrm{SO}(n)$, we suppose that a scalar product on the space is chosen. One may consider the semi-direct product of $\mathrm{SO}(n)$ and the Fulton–MacPherson operad and algebras over it; see [26]. But we will need only the following special case of such algebras.

Definition 3 ([21, Definition 3]). We say that a dg-algebra A over $\mathfrak{fm}_n$ is invariant if all structure maps of complexes $\mathfrak{fm}_n \otimes A \otimes \cdots \otimes A \to A$ are invariant under the action of group $\mathrm{SO}(n)$ on complexes of operations of $\mathfrak{fm}_n$.

Note that we mean invariance on the level of complexes, not up to homotopy.

1.2 Weyl n-algebras

The algebras over operad $\mathfrak{fm}_n$ we need below are Weyl n-algebras. Recall its definition, which slightly differs from the one given in [21]. The difference is in the quantization parameter h: in the mentioned paper, we considered algebras over formal series of h since below we suppose that $h = 1$.

Let $n > 1$ be a natural number. Let V be a $\mathbb{Z}$-graded finite-dimensional vector space over the base field $\Bbbk$ of characteristic zero containing $\mathbb{R}$ equipped with a non-degenerate skew-symmetric pairing $\omega\colon V \otimes V \to \Bbbk$ of degree $1 - n$. Let $\Bbbk[V]$ be the polynomial algebra generated by V. Denote by

$$\partial_\omega\colon \Bbbk[V] \otimes \Bbbk[V] \to \Bbbk[V] \otimes \Bbbk[V] \tag{1.1}$$

the differential operator that is a derivation in each factor and acts on generators as ω.

Consider $\mathbf{FM}_n(\underline{2})$, the space of 2-ary operations of the Fulton–MacPherson operad. This is homeomorphic to the $(n-1)$-dimensional sphere. Denote by $\mathfrak{v}$ the standard $\mathrm{SO}(n)$-invariant $(n-1)$-differential form on it. For any two-element subset $\{i, j\} \subset S$, denote by $p_{ij}\colon \mathbf{FM}_n(S) \to \mathbf{FM}_n(\underline{2})$ the map that forgets all points except ones marked by i and by j (compare with [3]). Denote by $\mathfrak{v}_{ij}$ the pullback of $\mathfrak{v}$ under projection p_{ij}. Let α be an element of endomorphisms of

$$\Bbbk[V]^{\otimes S} \underset{\mathrm{Aut}(S)}{\otimes} C^*(\mathbf{FM}_n(S))$$

(where $C^*(-)$ is the de Rham complex) given by

$$\alpha = \sum_{i,j \in S} \partial_\omega^{ij} \wedge \mathfrak{v}_{ij},$$

where ∂_ω^{ij} is the operator ∂_ω applied to the i-th and j-th factors.

Proposition 2. *The composition*

$$\Bbbk[V]^{\otimes S} \xrightarrow{\exp(\alpha)} \Bbbk[V]^{\otimes S} \otimes C^*(\mathbf{FM}_n(S)) \xrightarrow{\mu} \Bbbk[V] \otimes C^*(\mathbf{FM}_n(S)),$$

where μ is the product in the polynomial algebra, defines an algebra over the operad $\mathfrak{fm}_n$ with the underlying space $\Bbbk[V]$.

Proof. This is a simple check. □

The algebra defined in this way is obviously invariant under the action of $\mathrm{SO}(n)$; thus it is invariant (see Definition 3).

Definition 4 ([21]). For a pair (V, ω) and $n > 1$ as above, the invariant $\mathfrak{fm}_n$-algebra given by Proposition 2 is called the Weyl $\mathfrak{fm}_n$-algebra or the Weyl n-algebra. Denote it by $\mathcal{W}^n(V)$.

The Weyl 1-algebra is the usual Weyl algebra generated by a $\mathbb{Z}$-graded finite-dimensional vector space V with relations $[x, y] = (x, y)$, where $x, y \in V$ and $(\cdot, \cdot)$ is a perfect pairing of degree 0 on V. Below, we will use this definition only in Proposition 13. Note that Proposition 1 provides us with a notion of Weyl e_n-algebras.

The natural map of operads $\mathfrak{fm}_m \to \mathfrak{fm}_n$ for $m < n$ induces the functor from $\mathfrak{fm}_n$-algebras to $\mathfrak{fm}_m$-algebras. As in [20], denote it by obl_n^m.

Proposition 3. *For $m < n$, the $\mathfrak{fm}_m$-algebra $\mathrm{obl}_n^m \mathcal{W}^n(V)$ is isomorphic to the commutative polynomial algebra $\Bbbk[V]$.*

Proof. It follows from the very definition of the Weyl $\mathfrak{fm}_n$-algebra. □

1.3 Lie algebra

Recall the construction of a morphism from the shifted L_∞ operad to $\mathfrak{fm}_n$; see e.g. [21, Subsection 2.3].

Spaces of operations of the Fulton–MacPherson operad are equipped with a stratification labeled by trees as follows. As an operad of sets, $\mathbf{FM}_n$ is freely generated by $\mathscr{C}^0(\mathbb{R}^n)(S)/\mathrm{Dil}(n)$. Denote by μ the map from this free operad to the free operad with one generator in each arity, which sends generators to generators. Elements of the latter operad are enumerated by rooted trees. The map above sends $\mathscr{C}^0_{\underline{\mathbf{k}}}(\mathbb{R}^n)/\mathrm{Dil}(n)$ to the star tree with k leaves. For a tree $t \in T(S)$, denote by $[\mu^{-1}(t)] \in C_*(F_n(S))$ the chain presented by its preimage under μ. The operad L_∞ is a semi-free operad with generators labeled by trees; see e.g. [11]. One may see that $[\mu^{-1}(\cdot)]$ commutes with differentials. It gives us the following statement.

Proposition 4. *The map $[\mu^{-1}(\cdot)]$ as above gives a morphism*

$$L_\infty[1-n] \to \mathfrak{fm}_n \tag{1.2}$$

from shifted L_∞ operad to the dg-operad $\mathfrak{fm}_n$ of chains of the Fulton–MacPherson operad.

Proof. See e.g. [21, Proposition 2]. □

Definition 5. For an $\mathfrak{fm}_n$-algebra A, call its pullback under (1.2) the associated L_∞-algebra, and denote it by $L(A)$.

Consider the L_∞-algebra $L(\mathcal{W}^n(V))$ associated with the Weyl n-algebra. By the very definition, all operations on it are given by integration of closed forms by chains of the Fulton–MacPherson operad. But one may see that chains representing higher operations (that is, operations which are not compositions of Lie brackets) in L_∞ are all homologous to zero because L_∞ is a resolution of the Lie operad. Thus $L(\mathcal{W}^n(V))$ is a $\mathbb{Z}$-graded Lie algebra, that is, all higher operations vanish. This Lie algebra $L(\mathcal{W}^n(V))$ is a deformation of the Abelian one. The first-order deformation gives the Poisson Lie algebra: the underlying space is the $\mathbb{Z}$-graded commutative algebra $\Bbbk[V]$; the bracket is defined by $\omega\colon V \otimes V \to k$ on generators and satisfies the Leibniz rule.

Proposition 5. *For $n > 1$, the Lie algebra $L(\mathcal{W}^n(V))$ is isomorphic to the Poisson Lie algebra of $(\Bbbk[V], \omega)$.*

Proof. Clear because, for $n > 1$, the square of the de Rham cochain v is zero. □

2 Factorization complex

2.1 Factorization complex

The factorization complex of an algebra over the framed n-disks operad on a manifold is the tensor product of the right module over the framed n-disks operad corresponding to the manifold and the left one defined by the algebra; see e.g. [8]. For an invariant $\mathfrak{fm}_n$-algebra, we will use a simplified version of this definition following [21].

Let M be an n-dimensional oriented manifold. In the same way, as for $\mathbb{R}^n$, there is the Fulton–MacPherson compactification $\mathscr{C}(M)(S)$ of the space $\mathscr{C}^0(M)(S)$ of ordered pairwise distinct points in M labeled by S. Locally, it is the same thing. The inclusion $\mathscr{C}^0(M)(S) \hookrightarrow \mathscr{C}(M)(S)$ is a homotopy equivalence; there is a projection $\mathscr{C}(M)(S) \xrightarrow{\pi} M^S$.

Recall that a point in the Fulton–MacPherson compactification $\mathscr{C}(\mathbb{R}^n)(S)$ of the configuration space of $\mathbb{R}^n$ looks like a configuration from the configuration space $\mathscr{C}^0(\mathbb{R}^n)(S')$ with elements of $\mathbf{FM}_n$ sitting at each point of the configuration. It follows that spaces $\mathscr{C}(\mathbb{R}^n)(\bullet)$ form a right $\mathbf{FM}_n$-module, which is freely generated by $\mathscr{C}^0(\mathbb{R}^n)(\bullet)$ as a set. The same is nearly true for the Fulton–MacPherson compactification of any oriented manifold M. But to define such an action, one needs to choose coordinates at the tangent space of any point of a configuration of $\mathscr{C}(M)(S)$. To fix it, one has to consider either only framed manifolds or introduce a framed configuration space. For invariant algebras, these problems vanish.

Definition 6 ([21, Proposition 3]). For an invariant unital $\mathfrak{fm}_n$-algebra A and an oriented manifold M, the factorization complex $\int_M A$ is the complex given by the colimit of the diagram

$$\begin{array}{c} \bigoplus\limits_{S'} C_*(\mathscr{C}(M)(S')) \underset{\mathrm{Aut}(S')}{\otimes} A^{\otimes S'} \\ \big\uparrow \\ \bigoplus\limits_{i:\, S'\to S} C_*(\mathscr{C}^0(M)(S)) \underset{\mathrm{Aut}(S)}{\otimes} \bigotimes\limits_{s\in S} (\mathfrak{fm}_n(i^{-1}s) \underset{\mathrm{Aut}(i^{-1}s)}{\otimes} A^{\otimes(i^{-1}s)}) \\ \big\downarrow \\ \bigoplus\limits_{S} C_*(\mathscr{C}^0(M)(S)) \underset{\mathrm{Aut}(S)}{\otimes} A^{\otimes S}, \end{array} \tag{2.1}$$

where the summation in the middle runs over maps between finite sets, the downwards arrow is given by the left action of $\mathfrak{fm}_n$ on A for $\operatorname{Im} i$, and the unit for $S \setminus \operatorname{Im} i$ and the upwards arrow is given by the right action of $\mathfrak{fm}_n$ on $C_*(\mathscr{C}(M)(\bullet))$.

Note that relations (2.1) include in particular colimits

$$\begin{array}{c} \bigoplus\limits_{S'} C_*(\mathscr{C}(M)(S')) \underset{\mathrm{Aut}(S')}{\otimes} A^{\otimes S'} \\ \big\uparrow \\ \bigoplus\limits_{i:\, S'\hookrightarrow S} C_*(\mathscr{C}(M)(S)) \underset{\mathrm{Aut}(S')}{\otimes} A^{\otimes S'} \\ \big\downarrow{\scriptstyle \otimes 1^{(S\setminus S')}} \\ \bigoplus\limits_{S} C_*(\mathscr{C}(M)(S)) \underset{\mathrm{Aut}(S)}{\otimes} A^{\otimes S}, \end{array} \tag{2.2}$$

where the upwards arrow is the projection, which forgets points labeled by $S \setminus S'$.

Proposition 6. *For a $\mathbb{Z}$-graded vector space V, the cohomology of the factorization complex $\int_M \mathbb{k}[V]$ of the polynomial algebra $\mathbb{k}[V]$ is isomorphic to $\mathbb{k}[V \otimes H_*(M)]$.*

Proof. The factorization complex of a polynomial algebra $\mathbb{k}[V]$ on a manifold M is isomorphic to

$$\bigoplus_i C_*(M^{\times \underline{i}}) \otimes_{\Sigma_i} V^{\otimes \underline{i}}$$

by the very definition. It follows the statement. □

Proposition 7. *The following statements hold.*

(1) *The factorization complex $\int_{M^k} A$ of an invariant $\mathfrak{fm}_n$-algebra on a closed compact oriented k-manifold M^k is naturally equipped with a structure of $\mathfrak{fm}_{n-k}$-algebra.*

(2) *For a fiber bundle $E^n \xrightarrow{F^k} B^{n-k}$ with closed compact oriented base and fiber and an invariant $\mathfrak{fm}_n$-algebra A,*

$$\int_{B^{n-k}} \Big(\int_{F^k} A \Big) = \int_{E^n} A,$$

where $\int_{F^k} A$ is an $\mathfrak{fm}_{n-k}$-algebra by the previous item.

Proof. See [10, Section 5] and references therein. □

This theorem may be formulated for maps more general than projections of fiber bundles. To define push-forward in a more general situation, one needs to introduce factorization sheaves; see [2, 8] for details. The construction from Subsection 5.2 below is an example of such a push-forward.

2.2 Factorization complex of a disk

The factorization complex is homotopy invariant. In particular, it means that the factorization complex of a disk is trivial. It is stated in two subsequent propositions.

Denote by $\mathbb{D}^n$ the open disk $\{x \in \mathbb{R}^n \mid |x| < 1\}$ and by $\overline{\mathbb{D}}^n$ the closed disk $\{x \in \mathbb{R}^n \mid |x| \leq 1\}$.

Proposition 8. *For an $\mathfrak{fm}_n$-algebra A, the factorization complex $\int_{\mathbb{D}^n} A$ is homotopy equivalent to A, and an embedding of any point $p \to \mathbb{D}^n$ induces a quasi-isomorphism $A = \int_p A \xrightarrow{\sim} \int_{\mathbb{D}^n} A$.*

Proof. See e.g. [21, Proposition 5]. □

Proposition 9. *For an invariant $\mathfrak{fm}_n$-algebra A, the factorization complex $\int_{\overline{\mathbb{D}}^n} A$ is homotopy equivalent to A, and the embedding $\mathbb{D}^n \to \overline{\mathbb{D}}^n$ induces a quasi-isomorphism $\int_{\mathbb{D}^n} A \xrightarrow{\sim} \int_{\overline{\mathbb{D}}^n} A$.*

Proof. The following proof is taken from [9, Subsection 5.2]. Consider the projection $p\colon \overline{\mathbb{D}}^n \to [0, 1]$, which sends point x to $|x|$. The fiber over a non-zero point is the sphere S^{n-1}. The factorization complex of $\int_{S^{n-1}} A$ is an e_1-algebra, and A as a complex is a module over it (see [5, 19] and also [9, Proposition 5.8]). As it follows from the gluing property of the factorization complex, the factorization complex $\int_{\overline{\mathbb{D}}^n} A$ is quasi-isomorphic to

$$A \overset{L}{\underset{\int_{S^{n-1}} A}{\otimes}} \int_{S^{n-1}} A,$$

which follows the statement of the proposition. □

From the proof of this proposition, it follows that the factorization complex $\int_{\overline{\mathbb{D}}^n} A$ is equipped with a structure of $(\int_{S^{n-1}} A)$-module. As this complex is quasi-isomorphic to A, it follows that the underlying complex of A itself is an $\int_{S^{n-1}} A$-module [9, Lemma 5.12].

3 Swiss cheese operad

3.1 Swiss cheese operad

Let $\mathbb{R}^n$ be an affine space. Denote by $\mathbb{R}^n_{\geq 0}$ and $\mathbb{R}^n_{>0}$ subsets $\{\vec{x} \in \mathbb{R}^n \mid x_0 \geq 0\}$ and $\{\vec{x} \in \mathbb{R}^n \mid x_0 > 0\}$ correspondingly, where x_0 is the coordinate function. Denote by $\mathscr{C}^0(\mathbb{R}^n_{\geq 0})(S)$ the configuration space of distinct ordered points in $\mathbb{R}^n_{\geq 0}$ labeled by S. Points inside $\mathbb{R}^n_{>0} \subset \mathbb{R}^n_{\geq 0}$ are called closed, and points on the boundary $\mathbb{R}^{n-1}_{=0} \subset \mathbb{R}^n_{\geq 0}$ are called open. Denote by $\mathscr{SC}(\mathbb{R}^n_{\geq 0})(S)$ the closure of $\mathscr{C}^0(\mathbb{R}^n_{\geq 0})(S)$ in $\mathscr{C}(\mathbb{R}^n)(S)$. This is a manifold with corners and a boundary. There is a projection $\pi\colon \mathscr{SC}(\mathbb{R}^n_{\geq 0})(S) \to (\mathbb{R}^n_{\geq 0})^S$, which restricts on $\mathscr{C}^0(\mathbb{R}^n_{\geq 0})(S)$ to the natural embedding.

Let us define a stratification of $\mathscr{C}^0(\mathbb{R}^n_{\geq 0})(S)$ that is a continuous map to a poset. The poset is $\{O < C\}^S$, where $\{O < C\}$ is the poset consisting of two elements. For $s \in S$, the s-component of this map is C if the point of the configuration labeled by s is closed, and is O if it is open. One may see that taking closures of strata in $\mathscr{SC}(\mathbb{R}^n_{\geq 0})(S)$ defines a Whitney stratification of the latter space with the same indexing poset. Denote the indexing map by

$$\varpi\colon \mathscr{SC}(\mathbb{R}^n_{\geq 0})(S) \to \{O < C\}^S. \tag{3.1}$$

Denote by $\mathrm{Dil}(n-1)$ the subgroup of affine transformations of $\mathbb{R}^n$ consisting of dilatations with positive coefficients and shifts along the hyperplane $\{x_0 = 0\}$. The group $\mathrm{Dil}(n-1)$ acts freely on $\mathscr{SC}(\mathbb{R}^n_{\geq 0})(S)$. The quotient is isomorphic to the fiber $\pi^{-1}(\vec{0})$, where $\vec{0} \in (\mathbb{R}^n_{\geq 0})^S$ is the S-tuple sitting at the origin. It follows that $\pi^{-1}(\vec{0})$ is a retract of $\mathscr{SC}(\mathbb{R}^n_{\geq 0})(S)$. Denote the quotient by $\mathbf{SC}^S_n$. Note that $\pi^{-1}(\vec{x})$ for any S-tuple $\vec{x} \in (\mathbb{R}^n_{>0})^S$ in the interior is isomorphic to $\mathbf{FM}^S_n$.

The sequence of manifolds with corners $\mathbf{SC}^S_n$ form a colored operad called the Swiss cheese operad introduced in [15, 28]. Describe it as an operad of sets. This colored operad has two colors: points may be open and closed. Note that the set of colors is a poset, that is a category, rather than a set, and there are only operations compatible with this structure. This operad of sets is free and is generated by the following operations. The set of S-ary generating operations from S closed points to a close point is equal to the quotient of $\mathscr{C}^0(\mathbb{R}^n)(S) \hookrightarrow \mathscr{C}(\mathbb{R}^n)(S)$ by $\mathrm{Dil}(n)$, which is embedded in $\mathbf{FM}^S_n$. The set of operations from C closed and O open points to an open point is equal to the quotient of the configuration space of C distinct points in $\mathbb{R}^n_{>0}$ and O distinct points in $\mathbb{R}^{n-1}_{=0}$ factored out by the $\mathrm{Dil}(n-1)$ group action. The action of the symmetric group is straightforward, and the composition is analogous to the one of the Fulton–MacPherson operad.

Below, we do not need exactly the notion of this colored operad, but the action of an $\mathfrak{fm}_n$-algebra on an $\mathfrak{fm}_m$-algebra we define below is essentially the action of this operad.

3.2 Action

For a space X with a stratification given by $\varpi\colon X \to P$, where P is a posetal category, we say that a constructible sheaf with values in a category C is ϖ-combinatorial if its restriction to each stratum is constant. A combinatorial sheaf is defined by a functor $P \to \mathsf{C}$; see e.g. [11, Subsection 1.5].

Consider a triple (A, M, ε) consisting of a unital $\mathfrak{fm}_n$-algebra A, an $\mathfrak{fm}_{n-1}$-algebra M and a map of unital $\mathfrak{fm}_{n-1}$-algebras $\varepsilon\colon \mathrm{obl}^{n-1}_n A \to M$. Denote by $A^\vee$ and $M^\vee$ the linear dual complexes. The triple defines a functor from category $\{O < C\}$ to complexes, which sends C to $A^\vee$, O to $M^\vee$ and $\varepsilon^\vee$ to the only non-trivial morphism of this category. The tensor power of this functor gives a functor from $\{O < C\}^S$ to complexes. Denote by $\mathcal{F}_{\varepsilon^\vee}$ the combinatorial sheaf of complexes over $\mathscr{SC}^\bullet$ associated with this functor.

Definition 7 ([15, 27, 28]). For a triple (A, M, ε) as above, an action of A on M is defined by maps of complexes $M^\vee \to C^*(\mathbf{SC}^S_n, \mathcal{F}_{\varepsilon^\vee})$ such that
(1) (compatibility) their restriction on $\mathbb{R}^{n-1}_{=0} \subset \mathbb{R}^n_{\geq 0}$ are given by the $\mathfrak{fm}_{n-1}$-algebra structure on M,
(2) (factorization) they factor through the limit of the diagram

$$\begin{array}{c}
\bigoplus_{S'} C^*(\mathscr{C}(\mathbb{R}^n_{\geq 0})(S'), \mathcal{F}_{\varepsilon^\vee}) \\
\downarrow \\
\bigoplus_{i\colon S' \to (C\cup O)} C^*(\mathscr{C}^0(\mathbb{R}^n_{\geq 0})(C \cup O)) \underset{\mathrm{Aut}(C)\times\mathrm{Aut}(O)}{\otimes} \begin{array}{l} \bigotimes_{s\in C} (\mathfrak{fm}_n(i^{-1}s) \underset{\mathrm{Aut}(i^{-1}s)}{\otimes} A^{\otimes(i^{-1}s)})^\vee \\ \bigotimes_{s\in O} C^*(\mathbf{SC}^{i^{-1}s}_n, \mathcal{F}_{\varepsilon^\vee}) \end{array} \\
\uparrow \\
\bigoplus_{C\cup O} C^*(\mathscr{C}^0(\mathbb{R}^n_{\geq 0})(C \cup O)) \underset{\mathrm{Aut}(C)\times\mathrm{Aut}(O)}{\otimes} (A^{\otimes C} \otimes M^{\otimes O})^\vee,
\end{array}$$

where $\mathscr{C}^0(\mathbb{R}^n_{\geq 0})(C \cup O)$ means the configuration space of C closed and O open distinct points, summation in the middle runs over maps between finite sets, which are surjective on O, the upwards arrow is given by the left coaction of $\mathfrak{fm}_n$ on A for $\mathrm{Im}\, i \cap C$, action of A on M for $\mathrm{Im}\, i \cap O$ and the unit for $S \setminus \mathrm{Im}\, i$, and the downwards arrow is given by the right coaction of $\mathfrak{fm}_n$ on $C^*(\mathscr{C}(\mathbb{R}^n_{\geq 0})(\bullet))$.

This definition resembles Definition 6 of the factorization complex. It is not a coincidence, the action may be defined as a factorization sheaf of a special form; see for details [2, 8].

3.3 Higher Hochschild cohomology

Let M be an invariant $\mathfrak{fm}_{n-1}$-algebra. The factorization complex $\int_{S^{n-2}} M$ is an e_1-algebra, and the underlying complex of M is a module over it; see e.g. [9] and the remark after Proposition 9.

Definition 8 ([5, 9]). Define the higher Hochschild cohomological complex of an invariant $\mathfrak{fm}_{n-1}$-algebra M by

$$\mathrm{CH}^{\bullet}_{e_{n-1}}(M, M) = \mathrm{RHom}^{\bullet}_{\int_{S^{n-2}} M}(M, M). \tag{3.2}$$

The higher Hochschild complex of an e_{n-1}-algebra is equipped with an e_{n-1}-algebra structure. It may be defined rather explicitly (see [9]): it is given by the composition of the target of RHom. By [9, 19], the higher Hochschild cohomology is the derived centralizer of the identity map of an e_{n-1}-algebra to itself. By [19], it is equipped with a canonical e_n-algebra structure. It is shown there that the mentioned e_{n-1} structure on $\mathrm{CH}^{\bullet}_{e_{n-1}}(M, M)$ may be lifted to an e_n-algebra structure.

Action in the sense of Definition 7 of an e_n-algebra A on an e_{n-1}-algebra M induces a morphism of e_n-algebras $A \to \mathrm{CH}^{\bullet}_{e_n}(A, A)$; see [27]. In terms of the triple (A, M, ε), it may be defined as follows. Given a chain in the complex $\int_{\overline{\mathbb{D}}^{n-1}} M$, consider the following chain in the complex dual to $C^*(\mathbf{SC}^S_n, \mathcal{F}_{\varepsilon^\vee})$: its open points are given by this chain, where $\overline{\mathbb{D}}^{n-1}$ is the unit disk in $\mathbb{R}^{n-1}_{=0}$ and the only closed point is $(t, 0, \dots, 0)$ labeled by an element of A, where $t \in \mathbb{R}_{>0}$. Consider the limit of this configuration as t approaches 0. Convolution with the action gives a map

$$A \to \mathrm{Hom}\Big(\int\limits_{\overline{\mathbb{D}}^{n-1}} M, M\Big) \simeq \mathrm{Hom}^{\bullet}(M, M).$$

One may see that the resulting element of $\mathrm{Hom}^{\bullet}(M, M)$ is a homomorphism of $(\int_{S^{n-2}} M)$-modules. It gives us a map

$$A \to \mathrm{CH}^{\bullet}_{e_{n-1}}(M, M). \tag{3.3}$$

Proposition 10. *Map* (3.3) *is a morphism of e_n-algebras. The e_n-algebra $\mathrm{CH}^{\bullet}_{e_{n-1}}(M, M)$ is the final object in the category of e_n-algebras acting on M.*

Proof. This is the main result of [27]. □

Being defined as in (3.2), the higher Hochschild cohomological complex is not equipped with an explicit e_n-algebra structure (whereas the e_{n-1}-algebra structure can be made explicit; see [9]). In particular, the L_∞-structure on $L(\mathrm{CH}^{\bullet}_{e_{n-1}}(M, M))$ is rather implicit.

But for $n = 2$, the higher Hochschild cohomological complex is the usual Hochschild cohomological complex, and it is equipped with a Lie bracket due to Gerstenhaber [6]. If an $\mathfrak{fm}_2$-algebra A acts on an algebra M, one may build an explicit L_∞-morphism from $L(A)$ to the Hochschild cohomological complex of M equipped with the Gerstenhaber bracket. Note that the latter is a dg-Lie algebra; it has no higher L_∞-operations. It would be interesting to generalize this construction for higher dimensions.

Let A be an $\mathfrak{fm}_2$-algebra acting on an $\mathfrak{fm}_1$-algebra M. Define a chain in the complex dual to $C^*(\mathbf{SC}^S_n, \mathcal{F}_{\varepsilon^\vee})$ depending on k elements of A and l elements of M. Let $B_2 = \{x \in \mathbb{R}^2_{>0} \mid |x| < 1\}$ and $B_1 = \{x \in \mathbb{R}^1_{=0} \mid |x| < 1\}$. Define chain $\tilde{c}$ by

$$\tilde{c}(a_1, \dots, a_k; m_1, \dots, m_l) = [\mathscr{C}^0(B^2)(\underline{\mathbf{k}})] \otimes_{\Sigma_k} (a_1 \otimes \cdots \otimes a_k) \cup [\mathscr{C}^0(B^1)(\underline{\mathbf{l}})] \otimes_{\Sigma_l} (m_1 \otimes \cdots \otimes m_l),$$

where $[\mathscr{C}^0(B^2)(\underline{\mathbf{k}})]$] and $[\mathscr{C}^0(B^1)(\underline{\mathbf{l}})]$ are cycles in $C_*(\mathscr{C}(\mathbb{R}^2_{\geq 0})(S))$ presented by the configuration space of distinct points lying in B_2 and B_1. Consider the fiberwise closure of this chain with respect to the projection of configuration spaces, which forgets closed points. Take its intersection with the subset consisting of configurations with all closed points lying over the origin. Denote the resulting chain by $c(a_1, \dots, a_k; m_1, \dots, m_l)$.

The convolution of this chain with action defines a map from $A^{\otimes k} \otimes M^{\otimes l}$ to M, which is symmetric in A's. Thus it defines a map

$$c\colon A^{\otimes k}[l+2k-2] \to \bigoplus_l \mathrm{Hom}(M^{\otimes l}, M). \tag{3.4}$$

Proposition 11. *For an $\mathfrak{fm}_2$-algebra A acting on an $\mathfrak{fm}_1$-algebra M, map* (3.4) *defines a L_∞-morphism from $L(A)$ to the Hochschild cohomological complex of M.*

Proof. This is [16, Theorem 6.4] slightly rephrased. □

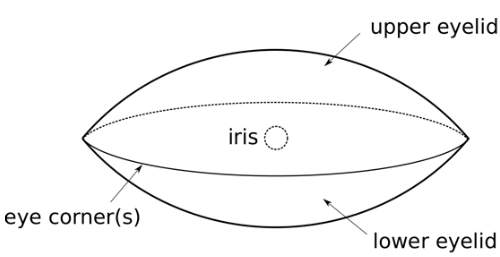
Note that the logic of this construction is similar to [21, Proposition 8].

4 Propagator approach

4.1 Propagator

Consider the stratified space $\mathbf{SC}^2_n$ defined in Subsection 3.1. One may see that this is the higher-dimensional generalization of the "eye" from [16]. The stratum with two closed points is the interior of the "eye", two strata with one open and one closed point are "eyelids", which are hemispheres, the stratum with two open points is the "eye corner(s)", which is an $(n-2)$-dimensional sphere.

The manifold with corners $\mathbf{SC}^2_n$ has two connected components of the boundary. The first one consists of two "eyelids" glued by the "eye corner(s)". The second one is the "iris", the $(n-1)$-dimensional sphere, which magnifies collisions of two closed points.



The following definition is a straightforward high-dimensional generalization of the differential of the angle map from [16, Subsection 6.2].

Definition 9. An n-propagator is a smooth closed differential $(n-1)$-form on $\mathbf{SC}^2_n$ such that
(1) its restriction on the "lower eyelid" is zero,
(2) its restriction on the "iris" is the standard volume form on the sphere.

Consider some examples of propagators. We define a differential form on the interior of the "eye" and leave to the reader to check that it continues to the boundary. The interior consists of pairs of distinct points in $\mathbb{R}^n_{>0}$ modulo the $\mathrm{Dil}(n-1)$ action, that is $\mathscr{C}^0(\mathbb{R}^n_{>0})(\underline{\mathbf{2}})/\mathrm{Dil}(n-1)$. Denote such a pair by (s,t). When t tends to the boundary $\mathbb{R}^{n-1}_{=0} \subset \mathbb{R}^n_{\geq 0}$, the pair (s,t) tends to the "lower eyelid".

Example 0. The first example is a high-dimensional generalization of the propagator used in [16].

For a pair (s,t) as above, denote by $\bar{t} \in \mathbb{R}^n_{<0}$ the image of the reflection of t with respect to the boundary hyperplane. Consider two maps from $\mathscr{C}^0(\mathbb{R}^n_{>0})(\underline{\mathbf{2}})$ to S^{n-1} which send (s,t) to directions given by vectors $s-t$ and $s-\bar{t}$. The difference between pullbacks of the standard volume form on the sphere under the first and the second maps is a $\mathrm{Dil}(n-1)$-invariant closed $(n-1)$-form on $\mathscr{C}^0(\mathbb{R}^n_{>0})(\underline{\mathbf{2}})$. Its continuous extension to the boundary satisfies conditions of Definition 9. Denote this propagator by Φ^0_n.

Example k. The previous example is the first in a series.

For $k \in \mathbb{N}$, consider the embedding $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+k}$ as the coordinate plane. For any point $t \in \mathbb{R}^n_{>0} \subset \mathbb{R}^n$, denote by S_t the only k-sphere in $\mathbb{R}^{n+k}$, which contains t, has its center on the plane $\mathbb{R}^{n-1}_{=0} \subset \mathbb{R}^n$ and lies in the plane perpendicular to this plane. Consider the space of triples (s, t, p), where $s, t \in \mathbb{R}^n_{>0} \subset \mathbb{R}^n$, $s \neq t$ and $p \in S_t$. Denote by ϖ the projection from this space to the configuration space of two distinct ordered points $\mathscr{C}^0(\mathbb{R}^2_{>0})(\underline{\mathbf{2}})$, which forgets the third term of the triple.

On the configuration space $\mathscr{C}^0(\mathbb{R}^{n+k})(\underline{\mathbf{2}})$ of two distinct points of $\mathbb{R}^{n+k}$, consider the standard differential $(n+k-1)$-form $\mathfrak{v}$, which is the pullback of the standard volume form of the $(n+k-1)$-sphere under the projection given by the direction of the vector connecting two points. Denote by the same letter the $(n+k-1)$-form on the space of triples as above, which is the pullback of $\mathfrak{v}$ under the map which forgets the middle term of the triple. Denote by Φ^k_n the differential $(n-1)$-form on $\mathscr{C}^0(\mathbb{R}^2_{>0})(\underline{\mathbf{2}})$ given by the integration of $\mathfrak{v}$ along the projection ϖ:

$$\Phi^k_n(s,t) = \int_{\varpi} \mathfrak{v}(s,p). \tag{4.1}$$

One may see, that this form is closed, invariant under $\mathrm{Dil}(n-1)$ and allows the smooth extension to $\mathbf{SC}^2_n$, which obeys conditions of Definition 9, that is, it is a propagator.

In this example, one may replace the sphere with any figure in the Euclidean space, which does not contain the origin and once intersects any ray from the origin. For example, one may take an ellipsoid. If one consider a family of ellipsoids in $\mathbb{R}^{n+k}$, or any other figures, which tends to the cylinder over S^l, the corresponding propagator tends to Φ^l_n.

4.2 Action from a propagator

In Subsection 1.2, following [21], we construct the Weyl algebra over the operad $\mathfrak{fm}_n$. Given a propagator, one may, in the same manner, construct an action of this $\mathfrak{fm}_n$-algebra on the polynomial algebra considered as an $\mathfrak{fm}_{n-1}$-algebra.

Let U be a $\mathbb{Z}$-graded finite-dimensional vector space over the base field $\Bbbk$ of characteristic zero containing $\mathbb{R}$, and let $U^\vee$ be the dual space. Then $V = U \oplus U^\vee[1-n]$ is naturally equipped with the perfect skew-symmetric pairing of degree $1-n$. By Definition 4, this data gives us the Weyl $\mathfrak{fm}_n$-algebra $\mathcal{W}^n(U \oplus U^\vee[1-n])$ with the underlying complex $\Bbbk[U \oplus U^\vee[1-n]]$.

Consider the triple $(\Bbbk[U \oplus U^\vee[1-n]], \Bbbk[U], \varepsilon)$, where $\varepsilon\colon \Bbbk[U \oplus U^\vee[1-n]] \to \Bbbk[U]$ is the natural map, which sends all generators from $U^\vee[1-n]$ to zero.

As in Definition 7, denote by $\mathcal{F}_{\varepsilon^\vee}$ the combinatorial sheaf over $\mathbf{SC}^S_n$ associated with $\varepsilon^\vee$. There is a natural map

$$\mathcal{F}_{\varepsilon^\vee} \to (\Bbbk[U \oplus U^\vee[1-n]]^\vee)^S \tag{4.2}$$

from $\mathcal{F}_{\varepsilon^\vee}$ to the constant sheaf given by the augmentation.

Fix an n-propagator $\mathfrak{p}$. For any two-element subset $\{i,j\} \subset S$, denote by $p_{ij}\colon \mathbf{SC}_n(S) \to \mathbf{SC}_n(\underline{\mathbf{2}})$ the map that forgets all points except ones marked by i and by j. Denote by $\mathfrak{p}_{ij}$ the pullback of $\mathfrak{p}$ under projection p_{ij}. Let α be an element of endomorphisms of

$$\Bbbk[U \oplus U^\vee[1-n]]^{\otimes S} \underset{\mathrm{Aut}(S)}{\otimes} C^*(\mathbf{SC}_n(S))$$

(where $C^*(-)$ is the de Rham complex) given by

$$\alpha = \sum_{i,j \in S} \partial^{ij}_\omega \wedge \mathfrak{p}_{ij},$$

where ∂^{ij}_ω is the operator ∂_ω applied to the i-th and j-th factors, where the operator ∂_ω is given by (1.1) for the standard bilinear form of degree $1-n$ on $U \oplus U^\vee[1-n]$.

As in Proposition 2, consider the map

$$\Bbbk[U \oplus U^\vee[1-n]]^{\otimes S} \to C^*(\mathbf{SC}_n^S) \otimes \Bbbk[U \oplus U^\vee[1-n]] \tag{4.3}$$

given by the composition of $\exp(\alpha)$ and μ.

Composing the map dual to (4.3),

$$(\Bbbk[U \oplus U^\vee[1-n]])^\vee \to C^*(\mathbf{SC}_n^S) \otimes (\Bbbk[U \oplus U^\vee[1-n]]^\vee)^{\otimes S} = C^*(\mathbf{SC}_n^S, (\Bbbk[U \oplus U^\vee[1-n]]^\vee)^{\otimes S})$$

with

$$\varepsilon^\vee : \Bbbk[U]^\vee \to (\Bbbk[U \oplus U^\vee[1-n]])^\vee,$$

we get the map

$$\Bbbk[U]^\vee \to C^*(\mathbf{SC}_n^S, (\Bbbk[U \oplus U^\vee[1-n]]^\vee)^{\otimes S}). \tag{4.4}$$

Proposition 12. *For an n-propagator $\mathfrak{p}$, map* (4.4) *uniquely factors through the map induced by the map of sheaves* (4.2)*, and the resulting map*

$$\Bbbk[U]^\vee \to C^*(\mathbf{SC}_n^S, \mathcal{F}_{\varepsilon^\vee})$$

defines an action of $\mathcal{W}^n(U \oplus U^\vee[1-n])$ on the polynomial algebra $\Bbbk[U]$ considered as an $\mathfrak{fm}_{n-1}$-algebra.

Proof. The existence of the unique factorization follows from the first property of the propagator from Definition 9. The second property guarantees the second condition of Definition 7. □

Note that, to get such an action, one could start with a gadget more general than the propagator: one can use a closed form on $\mathbf{SC}_n^2$ taking values in $V^\vee \otimes V^\vee$, which vanishes on the "lower eyelid" and when restricted on the "iris" is equal to the standard volume form multiplied by the standard bilinear form of degree $1-n$ on V.

4.3 Formality

Proposition 12 gives an action of $\mathcal{W}^n(U \oplus U^\vee[1-n])$ on the polynomial algebra $\Bbbk[U]$ considered as an $\mathfrak{fm}_{n-1}$-algebra. By Proposition 10, this action gives a morphism of e_n-algebras

$$\mathcal{W}^n(U \oplus U^\vee[1-n]) \to \mathrm{CH}^\bullet_{e_{n-1}}(\Bbbk[U], \Bbbk[U]). \tag{4.5}$$

Theorem 1. *Map* (4.5) *is a quasi-isomorphism.*

Proof. Let us first evaluate $\mathrm{CH}^\bullet_{e_{n-1}}(\Bbbk[U], \Bbbk[U])$. By Proposition 6, $\int_{S^{n-2}} \Bbbk[U] = \Bbbk[U \oplus U[n-2]]$. From (3.2), we get

$$\mathrm{CH}^\bullet_{e_{n-1}}(\Bbbk[U], \Bbbk[U]) = \mathrm{RHom}^\bullet_{\int_{S^{n-2}} \Bbbk[U]}(\Bbbk[U], \Bbbk[U]) = \Bbbk[U \oplus U^\vee[1-n]].$$

The commutative algebra structure on the $\Bbbk[U \oplus U^\vee[1-n]]$ comes from the e_{n-1}-algebra structure on $\mathrm{CH}^\bullet_{e_{n-1}}(\Bbbk[U], \Bbbk[U])$ by the very definition of the latter. But by Proposition 3, $\mathcal{W}^n(U \oplus U^\vee[1-n])$ as an e_{n-1}-algebra is also a free commutative algebra generated by $U \oplus U^\vee[1-n]$. Since (4.5) is a morphism of e_{n-1}-algebras, to prove the statement, we need to show that (4.5) defines an isomorphism on generators. This may be easily checked by the explicit definition of the morphism given before Proposition 10. □

Combining this theorem with Proposition 5, we get the following corollary.

Corollary 1. *Map* (3.4) *defines a quasi-isomorphism between $L(\mathrm{CH}^\bullet_{e_{n-1}}(\Bbbk[U], \Bbbk[U]))$ and the Poisson Lie algebra of* $(\Bbbk[U \oplus U^\vee[1-n]], \omega)$ *as L_∞-algebras, where ω is the standard bilinear form on $U \oplus U^\vee[1-n]$.*

Being combined with Proposition 11, it gives us the main result of [16] about the quasi-isomorphism between polyvector fields and the Hochschild cohomological complex of polynomial algebra. In that paper, only propagator Φ_2^0, which is defined in Subsection 4.1, is used, although it is mentioned there that any other propagator also does the job. It is known that coefficients of the above formality quasi-isomorphism for this propagator are given by integrals similar to multiple zeta values (see e.g. [24]); in particular, they are conjecturally irrational.

Results of the next section lead us to the following conjecture.

Conjecture 1. *The formality morphism given by Corollary 1 with propagator Φ_n^k for $k > 0$ has rational coefficients.*

In the next section, we shall prove this conjecture in the two-dimensional case (Corollary 3).

5 Factorization complex approach

5.1 Factorization complex of a sphere

By Proposition 7, for an invariant $\mathfrak{fm}_n$-algebra A and an oriented closed manifold M^k of dimension $k < n$, the complex $\int_{M^k} A$ is an $\mathfrak{fm}_{n-k}$-algebra. If A is a Weyl n-algebra, the natural candidate for $\int_{M^k} A$ is a Weyl $(n-k)$-algebra again.

Conjecture 2. *For an oriented closed k-manifold M^k and a Weyl n-algebra $\mathcal{W}^n(V)$, where $n > k$, the factorization complex $\int_{M^k} \mathcal{W}^n(V)$ is quasi-isomorphic to $\mathcal{W}^{n-k}(V \otimes H_*(M^k))$ as an $\mathfrak{fm}_{n-k}$-algebra, where $H_*(M^k)$ is the homology of M^k, is negatively graded and equipped with the Poincaré pairing.*

The factorization complex on a sphere is of particular interest; see e.g. [9, Proposition 6.2].

Proposition 13. *For a natural $k < n$, the $\mathfrak{fm}_{n-k}$-algebra $\int_{S^k} \mathcal{W}^n(V)$ is quasi-isomorphic to $\mathcal{W}^{n-k}(V \oplus V[k])$, where $V \oplus V[k]$ is equipped with the natural perfect pairing of degree $1 - n + k$.*

Proof. Fix a point $p \in S^k$. Denote by $[\mathscr{C}^0(S^k \setminus p)(S)]$ the cycle in $C_*(\mathscr{C}(S^k)(S))$ presented by the configuration space of distinct points of S^k labeled by S distinct from p. Denote by $[p]$ the cycle in $C_*(\mathscr{C}(S^k)(S))$ presented by point p. Let $[x \otimes y]$ be a monomial representing an element of $\mathcal{W}^{n-k}(V \oplus V[k])$, where $x \in \Bbbk[V]$ and $y \in \Bbbk[V[k]]$ are monomials. Let $y = l_i \cdots l_i$ be a factorization of y into linear factors.

Define a map from $\mathcal{W}^{n-k}(V \oplus V[k])$ to the factorization complex $\int_{S^k} \mathcal{W}^n(V)$ by

$$[x \otimes y] \mapsto [p] \otimes x \cup [\mathscr{C}^0(S^k \setminus p)(\underline{\mathbf{i}})] \otimes_{\Sigma_i} l_1 \otimes \cdots \otimes l_i$$

The image is closed because a Weyl n-algebra is commutative as Weyl k-algebra by Proposition 3 and the cycle above is the standard cycle representing a class in the factorization complex of a polynomial algebra generalizing the standard class for the Hochschild complex [18, Proposition 1.3.12] (compare with [10, Definition 2]). It is easy to show by the direct calculation that this map respects the $\mathfrak{fm}_{n-k}$-algebra structure. The crucial point here is using relations (2.2). □

5.2 Action

As in Subsection 4.1, for $k \in \mathbb{N}$, define the embedding $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+k}$ as the coordinate plane, and consider the projection

$$\mathbb{R}^{n+k} \to \mathbb{R}^n_{\geq 0}, \tag{5.1}$$

which sends a point to the only intersection with $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+k}$ of the only k-sphere in $\mathbb{R}^{n+k}$, which contains this point, has its center on the plane $\mathbb{R}^{n-1}_{=0} \subset \mathbb{R}^n$ and lies in the plane perpendicular to this plane.

For an invariant $\mathfrak{fm}_{n+k}$-algebra A, the product of the operad gives a $\mathrm{Dil}(n-1)$-invariant map

$$A^\vee \to \bigoplus_S C^*(\mathscr{C}^0(\mathbb{R}^{n+k})(S)) \underset{\mathrm{Aut}(S)}{\otimes} A^{\vee \otimes S}. \tag{5.2}$$

Denote by $A^{\vee\otimes}$ the locally constant sheaf over $\coprod_S \mathscr{C}^0(\mathbb{R}^{n+k})(S)$ with the fiber equal to $A^{\vee\otimes S}$. Then the right side of (5.2) is $H^*(\coprod_S \mathscr{C}^0(\mathbb{R}^{n+k})(S), A^{\vee\otimes})$

Now we need to take the push-forward of the factorization sheaf with respect to map (5.1). The following construction, being a generalization of Proposition 7, could be formulated in terms of the relative factorization complex. But the relative factorization complex is a cosheaf rather than a sheaf. That is why it is more convenient to work with the linear dual thing. Define the dual relative factorization complex $(\int_{\mathbb{R}^{n+k}/\mathbb{R}^n_{\geq 0}} A)^\vee$ of A of map (5.1), which is a complex of sheaves over $\mathbb{R}^n_{\geq 0}$, in analogy with (2.1) as the limit of the diagram

$$\begin{array}{c}
\bigoplus_{S'} C^*_{\mathbb{R}^n_{\geq 0}}(\mathscr{C}(\mathbb{R}^{n+k})(S')) \underset{\mathrm{Aut}(S')}{\otimes} A^{\vee\otimes S'} \\
\downarrow \\
\bigoplus_{i\colon S'\to S} C^*_{\mathbb{R}^n_{\geq 0}}(\mathscr{C}^0(M)(S)) \underset{\mathrm{Aut}(S)}{\otimes} \bigotimes_{s\in S} (\mathfrak{fm}_k(i^{-1}s) \underset{\mathrm{Aut}(i^{-1}s)}{\otimes} A^{\otimes(i^{-1}s)})^\vee \\
\uparrow \\
\bigoplus_{S} C^*_{\mathbb{R}^n_{\geq 0}}(\mathscr{C}^0(M)(S)) \underset{\mathrm{Aut}(S)}{\otimes} A^{\vee\otimes S},
\end{array} \tag{5.3}$$

where the operad $\mathfrak{fm}_k$ coacts along fibers of (5.1); this coaction is trivial over $\mathbb{R}^{n-1}_{=0} \subset \mathbb{R}^n_{\geq 0}$.

In the diagram

$$\begin{tikzcd}
C^*(\mathscr{C}(\mathbb{R}^{n+k})(S)) \underset{\mathrm{Aut}(S)}{\otimes} A^{\vee\otimes S} & C^*(\mathbb{R}^n_{\geq 0}, (\int_{\mathbb{R}^{n+k}/\mathbb{R}^n_{\geq 0}} A)^\vee) \arrow[l] \\
A^\vee = (\int_{\mathbb{R}^{n+k}} A)^\vee, \arrow[u] \arrow[ur, dashed] &
\end{tikzcd}$$

the isomorphism in the bottom row is given by Proposition 8, the horizontal arrow is the embedding in the top term of diagram (5.3) and the vertical is dual to the action of $\mathfrak{fm}_{n+k}$ operad on A. The dashed arrow, which makes the diagram commutative, exists because relations dual to (5.3) are contained in the relation defining the big factorization complex $\int_{\mathbb{R}^{n+k}} A$. The dashed arrow is $\mathrm{Dil}(n-1)$-invariant, and it gives us a map

$$A^\vee \to C^*\Big(\mathbf{SC}^S_n, \Big(\int_{\mathbb{R}^{n+k}/\mathbb{R}^n_{\geq 0}} A\Big)^\vee\Big). \tag{5.4}$$

Recall that, for an $\mathfrak{fm}_n$-algebra A and $m < n$, we denote by $\mathrm{obl}^m_n A$ the $\mathfrak{fm}_m$-algebra with the same underlying complex as A and the operadic structure induced by the natural map of operads $\mathfrak{fm}_m \to \mathfrak{fm}_n$. A particular case of the following proposition was crucial for the second part of [20].

Proposition 14. *For an invariant $\mathfrak{fm}_{n+k}$-algebra A, consider the triple*

$$\Big(\int_{S^k} A, \mathrm{obl}^{n-1}_{n+k} A, \varepsilon\Big),$$

where $\int_{S^k} A$ is an $\mathfrak{fm}_n$-algebra due to Proposition 7 and $\varepsilon\colon \int_{S^k} A \to A$ is the natural map induced by the map from the sphere to a point. Then the complex $\mathcal{F}^\vee_\varepsilon$ is quasi-isomorphic to the complex $(\int_{\mathbb{R}^{n+k}/\mathbb{R}^n_{\geq 0}} A)^\vee$ defined above, and map (5.4) defines an action of $\int_{S^k} A$ on $\mathrm{obl}^{n-1}_{n+k} A$ in the sense of Definition 7.

Proof. One may see that the complex $(\int_{\mathbb{R}^{n+k}/\mathbb{R}^n_{\geq 0}} A)^\vee$ is constructible with respect to the stratification (3.1) with fibers and the restriction map equal to ones of $\mathcal{F}^\vee_\varepsilon$, what follows that they are quasi-isomorphic. Axioms of the action (Definition 7) follow from the very definition of $\mathfrak{fm}_{n+k}$-algebra and the one of the factorization complex. □

Applying Proposition 10 to the action constructed above, we get, for any $\mathfrak{fm}_{n+k}$-algebra A, a natural map of $\mathfrak{fm}_n$-algebras,

$$\int_{S^k} A \to \mathrm{CH}^\bullet_{e_{n-1}}(\mathrm{obl}^{n-1}_{n+k} A, \mathrm{obl}^{n-1}_{n+k} A), \tag{5.5}$$

where $\mathrm{CH}^\bullet_{e_{n-1}}(\cdot,\cdot)$ is the higher Hochschild cohomology; see Subsection 3.3.

5.3 Formality via the factorization complex

We are going to show that, for Weyl $(n+k)$-algebras, map (5.5) is a quasi-isomorphism.

Recall that, by Proposition 13, the $\mathfrak{fm}_n$-algebra $\int_{S^k} \mathcal{W}^{n+k}(V)$ is isomorphic to $\mathcal{W}^n(V \oplus V[k])$, and by Proposition 3, $\mathrm{obl}_{n+k}^{n-1} \mathcal{W}^{n+k}(V)$ is isomorphic to $\Bbbk[V]$.

Proposition 15. *For a Weyl $(n+k)$-algebra $\mathcal{W}^{n+k}(V)$, the action of*

$$\int_{S^k} \mathcal{W}^{n+k}(V) = \mathcal{W}^n(V \oplus V[k])$$

on $\mathrm{obl}_{n+k}^{n-1} \mathcal{W}^{n+k}(V) = \Bbbk[V]$ defined by Proposition 14 is isomorphic to the action defined by Proposition 12 for propagator Φ_n^k given by (4.1).

Proof. Substituting quasi-isomorphism from Proposition 13 to the definition of the action from Proposition 14, we immediately get the statement. □

Proposition 10 gives a map of $\mathfrak{fm}_n$-algebras

$$\int_{S^k} \mathcal{W}^{n+k}(V) \to \mathrm{CH}^{\bullet}_{e_{n-1}}(\mathrm{obl}_{n+k}^{n-1} \mathcal{W}^{n+k}(V), \mathrm{obl}_{n+k}^{n-1} \mathcal{W}^{n+k}(V)) = \mathrm{CH}^{\bullet}_{e_{n-1}}(\Bbbk[V], \Bbbk[V]). \tag{5.6}$$

Theorem 2. *Map (5.6) is a quasi-isomorphism with rational coefficients.*

Proof. Combining Theorem 1 with Proposition 15, we get the statement. To see that coefficients are rational, note that the action is given by the product in the operad $\mathfrak{fm}_{n+k}$. Thus coefficients of the action (5.4) are given by integration of an integral cocycle by an integer cycle. □

Combining this theorem with Proposition 5 and Proposition 13, we get the following corollary.

Corollary 2. *Map (5.6) gives a quasi-isomorphism between two L_∞-algebras: the higher Hochschild cohomology Lie algebra $L(\mathrm{CH}^{\bullet}_{e_{n-1}}(\Bbbk[V], \Bbbk[V]))$ and the Poisson Lie algebra of $(\Bbbk[V \oplus V^\vee[n]], \omega)$, where ω is the standard bilinear form on $[V \oplus V^\vee[n]$ with rational coefficients.*

Let V be a vector space with a perfect symmetric pairing. Applying this corollary to Weyl 3-algebra $\mathcal{W}^3(V[1])$ and combining it with Corollary 1 and Proposition 15, we get Conjecture 1 in dimension two.

Corollary 3. *For a vector space V, the formality morphism given by Corollary 1 with propagator Φ_2^k for $k > 0$ gives a quasi-isomorphism between $L(\mathrm{CH}^{\bullet}_{e_1}(\Bbbk[V], \Bbbk[V]))$ and the Lie algebra of polyvector fields $\Bbbk[V \oplus V^\vee[1]]$ with rational coefficients.*

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