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CENTERS OF GENERALIZED REFLECTION EQUATION ALGEBRAS

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As is known, in the reflection equation (RE) algebra associated with an involutive or Hecke R-matrix, the elements $\mathrm{Tr}_R L^k$ *(called quantum power sums) are central. Here, L is the generating matrix of this algebra, and* Tr_R *is the operation of taking the R-trace associated with a given R-matrix. We consider the problem of whether this is true in certain RE-like algebras depending on a spectral parameter. We mainly study algebras similar to those introduced by Reshetikhin and Semenov-Tian-Shansky (we call them algebras of RS type). These algebras are defined using some current R-matrices (i.e., depending on parameters) arising from involutive and Hecke R-matrices by so-called Baxterization. In algebras of RS type. we define quantum power sums and show that the lowest quantum power sum is central iff the value of the "charge" c in its definition takes a critical value. This critical value depends on the birank* $(m|n)$ *of the initial R-matrix. Moreover, if the birank is equal to* $(m|m)$ *and the charge c has a critical value, then all quantum power sums are central.*

Keywords: reflection equation algebra, algebra of Reshetikhin–Semenov-Tian-Shansky type, charge, quantum powers of the generating matrix, quantum power sum

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1. Introduction

The best known quantum matrix (QM) algebras related to quantum groups (QG) $U_q(sl(N))$ are the so-called *RTT* and reflection equation (RE) algebras. In the general case, any QM algebra is defined by a pair of compatible *R*-matrices (see [1]) that yield permutation relations for the algebra generators. The permutation relations can be written in matrix form by introducing the so-called generating matrix whose elements are generators of the given algebra.

By definition, an *R*-matrix is an operator matrix $\widehat{R}\colon V^{\otimes 2} \to V^{\otimes 2}$ (in some fixed basis) satisfying the so-called *braid relation*

$$R_{12}\,R_{23}\,R_{12} = R_{23}\,R_{12}\,R_{23}, \qquad R_{12} = R \otimes I, \qquad R_{23} = I \otimes R. \tag{1}$$

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Here, V is a finite-dimensional linear space of dimension N over the ground field $\mathbb{C}$, and I denotes the identity operator. Moreover, the subscript on operators indicates the position of the factor(s) in the tensor product of the spaces in which the operators act.

We note that the operator PR, where P is the usual flip operator or its matrix, satisfies the quantum Yang–Baxter equation.

The QM algebras associated with R-matrices are said to be *constant* to distinguish them from *current* algebras that we consider below. The defining relations of the current algebras contain some functions depending on spectral parameters.

We recall that the constant RTT and RE algebras are respectively defined by the following systems of relations on their generators t_i^j and l_i^j:

$$\begin{aligned} &R_{12}T_1T_2 - T_1T_2R_{12} = 0, \qquad && T = \|t_i^j\|_{1\le i,j\le N}, \\ &R_{12}L_1R_{12}L_1 - L_1R_{12}L_1R_{12} = 0, \qquad && L = \|l_i^j\|_{1\le i,j\le N}. \end{aligned} \tag{2}$$

Here and hereafter, $A_1 = A\otimes I$ and $A_2 = I\otimes A$ for an arbitrary matrix A. These RTT and RE algebras are respectively denoted by $\mathcal{L}(R)$ and $\mathcal{T}(R)$.

Here, we assume that all R-matrices are *skew-invertible* involutive or Hecke symmetries (see Sec. 2). An involutive symmetry satisfies the additional relation $R^2 = I$, and a Hecke symmetry satisfies a more general quadratic condition of the form

$$(R - qI)(R + q^{-1}I) = 0, \quad q \ne \pm 1,$$

where the nonzero parameter q is assumed to be generic, i.e., $q^k \ne 1$ for any integer k. Consequently, all q-integers $k_q := (q^k - q^{-k})/(q - q^{-1})$ are nonzero.

In both QM algebras, we can define analogues of *symmetric polynomials* (elementary and full polynomials, power sums, and also Schur polynomials). But the properties of these quantum symmetric polynomials differ essentially in the RTT and RE algebras: quantum symmetric polynomials generate a commutative subalgebra in the algebra $\mathcal{T}(R)$ and are central in the algebra $\mathcal{L}(R)$.

We are especially interested in quantum analogues of *power sums*. We note that quantum power sums[1] in the algebras $\mathcal{L}(R)$ look like their classical counterparts, namely, $\mathrm{Tr}_R\, L^k$, $k \ge 1$, while they cannot be represented in such a simple form in the RTT algebras $\mathcal{T}(R)$. Hereafter, Tr_R denotes the so-called R-trace, which is associated with a given skew-invertible R-matrix (see Sec. 2).

The term "RE algebra" appeared in connection with constructing exactly solvable models of particles on the half-line with reflection at the boundary (see [1]–[4]). A method for constructing constant RE algebras using a pair of RTT algebras was proposed in [5]. A generalization of this construction to some current algebras was described in [6] (formula (11) in Sec. 2).

Our main objective is to define analogues of power sums in some current RE-like algebras and study their centrality. Each considered algebra is defined by the system

$$(R + g_1(u,v)I)L_1(u)(R + g_2(u,v)I)L_1(v) = L_1(v)(R + g_3(u,v)I)L_1(u)(R + g_4(u,v)I), \tag{3}$$

where R is a skew-invertible involutive or Hecke symmetry and $g_k(u,v)$, $k = 1,2,3,4$, are some functions. We call these algebras *generalized RE algebras*.

Requirements on the factors $R + g_k(u,v)I$ are usually imposed for them to be current R-matrices (possibly with an additional dependence on a charge). We consider algebras similar to those introduced by

[1]The terminology is motivated by the commutative case: if the elements of a matrix L are commutative and μ_i are its eigenvalues, then $\mathrm{Tr}\, L^k = \sum_k \mu_i^k$.

Reshetikhin and Semenov-Tian-Shansky [6] but associated with current R-matrices arising from involutive and Hecke symmetries via the Baxterization procedure. For brevity, such algebras are called algebras of RS type.

Let $R(u,v)$ be a current R-matrix associated with a Hecke symmetry R. In the current algebra of RS type, we define *quantum matrix powers* of its generating matrix $L(u)$ by the formula

$$L^{[k]}(u) = L_1(c^{k-1}u)\, L_1(c^{k-2}u) \cdots L_1(cu)\, L_1(u), \quad k \le 1. \tag{4}$$

The factor c is called the *charge*. In the case of an involutive symmetry R, the quantum powers $L^{[k]}(u)$ are defined by an analogous formula, but the shifts of the argument u are additive: $u \mapsto u + kc$, $k \ge 1$. After defining quantum powers of the generating matrix $L(u)$, we introduce quantum analogues of power sums standardly by calculating the R-trace $\mathrm{Tr}_R\, L^{[k]}(u)$.

We note that the notion of the quantum power of a matrix depending on a parameter and the corresponding quantum power sum was introduced by Talalaev [7] in studying the rational Gaudin model. Our definition (4) of the quantum powers of matrices differs from Talalaev's definition and is analogous to the definition introduced in [8], [9] for the generating matrices of generalized Yangians.

Our main result here is a proof that in any generalized RE algebra of RS type, the quantum power sum $\mathrm{Tr}_R\, L(u)$ is central iff the charge c takes a special critical value. This result is analogous to the statement in [6] proved for algebras related to the QG $U_q(g)$ (g is a classical simple Lie algebra). But in contrast to [6], the critical charge value in our approach is not related to the dual Coxeter number of g; it is determined by the birank of the initial symmetry R. For example, in the case $g = sl(N)$ (we call this case standard below), our result is a generalization of the result in [6].

We stress that the involutive and Hecke symmetries in the structures of our algebras are not necessarily standard. Moreover, they can even not be deformations of the usual flips or superflips. A way to construct such symmetries was proposed in [10] (see our footnote 4). To describe generalized RE algebras and the obtained first quantum power sum $\mathrm{Tr}_R\, L(u)$, we use the so-called R-matrix technique.[2]

In addition, we prove that if the birank of the initial symmetry R is equal to $(m|m)$, then for the critical charge value,[3] all quantum power sums are central in the corresponding RE algebras of RS type. This question remains open for an arbitrary birank of the symmetry R.

In conclusion, we note that in this paper, we do not use the usual expansion of the generating matrices in series in the spectral parameter. We thus avoid questions connected with the normal ordering of the quantum powers of matrices. The interested reader can refer to [11], where centers of some quantum affine vertex algebras were constructed using a special ordering of the generating matrices.

2. Symmetries and QM algebras

In this section, we consider a relation between constant RTT and RE algebras associated with symmetries. We therefore first recall some elements of the R-matrix technique (see [12]for details).

Let $R\colon V^{\otimes 2} \to V^{\otimes 2}$ be a Hecke symmetry. We consider the associated R-skew-symmetric algebra

$$\Lambda_R(V) = T(V)/\langle \mathrm{Im}(q^{-1}I + R)\rangle, \tag{5}$$

where $T(V)$ is the free tensor algebra of the linear space V and $\langle J\rangle$ denotes the two-sided ideal generated by a subset $J \subset T(V)$. We let $\Lambda_R^{(k)}(V)$ denote its kth-degree homogenous component and introduce the

[2] By R-matrix technique, we mean a collection of formulas and properties applicable to all skew-invertible R-matrices independently of their concrete form. Moreover, the R=matrix technique does not rely on objects of the QG type.

[3] In this case, the critical value is unity if R is a Hecke symmetry.

Hilbert–Poincaré series by the equality

$$P_{\Lambda_R}(t) = \sum_{k\geq 0} t^k \dim \Lambda_R^{(k)}(V).$$

For an involutive symmetry R in (5), we set $q = 1$.

As is known (see [13]), the series $P_{\Lambda_R}(t)$ is a rational function of the parameter t for any involutive or Hecke symmetry R. We say that R has the birank $(m|n)$ if the rational function $P_{\Lambda_R}(t)$ is a ratio of two coprime polynomials and the respective degrees of the numerator and denominator of that ratio are n and m.

Moreover, we assume that the R-matrix is skew-invertible. This means that there exists an operator $\Psi\colon V^{\otimes 2} \to V^{\otimes 2}$ such that

$$\mathrm{Tr}_{(2)} R_{12}\Psi_{23} = P_{13}.$$

Calculating the partial traces of Ψ, we define the operators

$$B = \mathrm{Tr}_{(1)} \Psi, \qquad C = \mathrm{Tr}_{(2)} \Psi,$$

and introduce the operation of taking the *R-trace* by the rule

$$\mathrm{Tr}_R A = \mathrm{Tr}(C \cdot A), \tag{6}$$

where A is an $N\times N$ matrix (possibly with noncommutative elements) and Tr is the usual trace.

The operator B plays an important role in constructing the representation category of the algebra $\mathcal{L}(R)$, but we do not use B in what follows.

If R is a skew-invertible Hecke symmetry of birank $(m|n)$, then B and C are invertible and are related by the formula

$$B \cdot C = q^{2(n-m)}\, I. \tag{7}$$

We also present normalizing relations for the R-trace:

$$\mathrm{Tr}_{R(2)} R_{12} = I_1, \qquad \mathrm{Tr}_R I = q^{n-m}(m-n)_q. \tag{8}$$

Below, we use the formula (proved in [12])

$$\mathrm{Tr}_{R(2)} R_{12}A_1R_{12}^{-1} = \mathrm{Tr}_{R(2)} R_{12}^{-1}A_1R_{12} = I_1(\mathrm{Tr}_R A), \tag{9}$$

where A is an arbitrary $N\times N$ matrix.

Using (9), we can show that the quantum power sums $\mathrm{Tr}_R L^k$ belong to the center of the algebra $\mathcal{L}(R)$. Indeed, from the defining relations of $\mathcal{L}(R)$, we easily obtain

$$R_{12}L_1R_{12}(L_1)^k = (L_1)^k R_{12}L_1R_{12} \quad \forall\, k \geq 1.$$

Multiplying this relation by R_{12}^{-1} from both sides and applying Tr_R in the second space, we obtain the expression

$$\mathrm{Tr}_{R(2)} L_1R_{12}(L_1)^k R_{12}^{-1} = \mathrm{Tr}_{R(2)} R_{12}^{-1}(L_1)^k R_{12}L_1.$$

Finally, taking (9) into account, we obtain the required equality:

$$L\,(\mathrm{Tr}_R L^k) = (\mathrm{Tr}_R L^k)L \quad \forall\, k \geq 1,$$

which means that the elements $\mathrm{Tr}_R\, L^k$ belong to the center of $\mathcal{L}(R)$.

We emphasize that this method does not work in the RTT algebras $\mathcal{T}(R)$. Although quantum analogues of power sums can also be defined in these algebras, they are not reducible to the form $\mathrm{Tr}_R\, T^k$ (explicit formulas for the power sums in these algebras can be found, e.g., in [1]). Moreover, the quantum power sums are not central elements of RTT algebras and generate a commutative subalgebra called a Bethe subalgebra. The algebraic properties of the QM algebras $\mathcal{L}(R)$ and $\mathcal{T}(R)$ thus differ essentially. Their coalgebraic structures and representation theories also differ drastically.

We now recall the way to construct an RE algebra using a pair of RTT algebras [5], which we reproduce in a more general context for an arbitrary skew-invertible R-matrix.

We consider an algebra constructed from two copies of an RTT algebra. The defining relations on the generating matrices T^+ and T^- of these copies have the forms

$$R_{12}T_1^{\pm}T_2^{\pm} = T_1^{\pm}T_2^{\pm}R_{12}, \qquad R_{12}T_1^{+}T_2^{-} = T_1^{-}T_2^{+}R_{12}. \tag{10}$$

The last equality defines the so-called permutation relations between the generators of the two copies of the RTT algebra.

We now assume that the symmetry R has the birank $(m|0)$.[4] We can then define the so-called R-determinant $\det_R T$ in the RTT algebra $\mathcal{T}(R)$ (if $R = P$, then $\det_R T$ becomes the usual determinant). We also assume that $\det_R T$ is a central element in the algebra. The quotient algebra $\mathcal{T}(R)/\langle \det_R T - 1\rangle$ is then well defined; moreover, we can construct an antipode S in the quotient algebra. Using the relation $TS(T) = S(T)T = I$, we can rewrite the last equality in system (10) as

$$S(T_1^-)R_{12}T_1^+ = T_2^+R_{12}S(T_2^-).$$

It is now easy to prove the following statement.

Proposition 1. *The matrix $L = T^+S(T^-)$ satisfies matrix equality* (2).

This statement was established in [5] in the standard case. We note that in this case, some supplementary conditions are usually imposed on the generating matrices T^+ and T^- by assuming that they are respectively upper and lower triangular matrices and by imposing additional relations on the diagonal elements of these matrices. All these conditions allow reducing algebra (10) to the “classical sizes.”

We mention two properties of the algebra $\mathcal{L}(R)$ for a standard R-matrix. First, this algebra is a deformation of the commutative algebra $\mathrm{Sym}(gl(N))$, i.e., the algebra $\mathcal{L}(R)$ becomes commutative in the limit $q \to 1$, and for q in general position, the corresponding homogeneous components of the two algebras coincide. The RTT algebra $\mathcal{T}(R)$ also has this same property. Second, the algebra $\mathcal{L}(R)$ is covariant under the adjoint action of the QG $U_q(gl(N))$. The corresponding RTT algebra $\mathcal{T}(R)$ does not have this covariance.

3. Centrality of $\mathrm{Tr}_R\, L(u)$

We consider the algebra generated by elements of a matrix $L(u)$ satisfying matrix relation (3), where R is a skew-invertible involutive or Hecke symmetry. We note that if all functions g_k are identically zero, then we obtain a constant RE algebra.

[4] As an example of such a symmetry, we can mention the R-matrix related to the QG $U_q(sl(N))$. In this case, $m = N$. But this equality is not satisfied in the general case. In particular, involutive and Hecke symmetries of birank $(2|0)$ acting in the space $V^{\otimes 2}$, where $\dim V = N$, were constructed for any $N \geq 2$ in [10]. For such symmetries, the corresponding skew-symmetric algebra Λ_R has a Hilbert–Poincaré series of the form $P_{\Lambda_R} = 1 + Nt + t^2$. Moreover, a method for “gluing” Hecke symmetries was proposed in [10]. As a result of gluing Hecke symmetries of biranks $(m|0)$ and $(0|n)$, a Hecke symmetry of birank $(m|n)$ can be obtained.

We now assume that all $g_k = g_k(u,v)$, $k = 1,2,3,4$, are nontrivial, but we do not fix them in any concrete way. We consider the case where R is a Hecke symmetry. We use the method in Sec. 2, which was applied in studying the center of the algebra $\mathcal{L}(R)$. Namely, we apply the R-trace $\mathrm{Tr}_{R(2)}$ to both sides of equality (3). In the left-hand side, we obtain

$$\begin{aligned}\mathrm{Tr}_{R(2)}(R_{12}L_1(u)((q-q^{-1})I + R_{12}^{-1})L_1(v)) + g_1 L_1(u)(\mathrm{Tr}_{R(2)}\, R_{12})L_1(v) + \\ + g_2(\mathrm{Tr}_{R(2)}\, R_{12})L_1(u)L_1(v) + (\mathrm{Tr}_{R(2)}\; I_2)g_1g_2L_1(u)L_1(v) = \\ = \mathrm{Tr}_R(L_1(u))L_1(v) + ((q-q^{-1}) + g_1 + g_2 + \alpha\, g_1g_2)L_1(u)L_1(v), \qquad (11)\end{aligned}$$

where $\alpha = q^{n-m}(m-n)_q$. Here, we use formulas (8) and (9) and the corollary of the Hecke relation

$$R = (q-q^{-1})I + R^{-1}.$$

If R is a skew-invertible involutive symmetry, then the presented formula is still applicable, but the term $q - q^{-1}$ vanishes, and the value of the coefficient α should be changed to $\alpha = m - n$. In what follows, we mainly work with Hecke symmetries.

Applying the same operation to the right-hand side of (11), we obtain the expression

$$L_1(v)\,\mathrm{Tr}_R(L_1(u)) + ((q-q^{-1}) + g_3 + g_4 + \alpha\, g_3g_4)L_1(v)L_1(u).$$

Here, $\mathrm{Tr}_R(L_1(u)) = \mathrm{Tr}_{R(1)}(L_1(u))$.

Therefore, the relation expressing the centrality of the quantum power sum

$$\mathrm{Tr}_R(L_1(u))L_1(v) = L_1(v)\,\mathrm{Tr}_R(L_1(u)) \qquad (12)$$

holds if the equalities for the functions $g_k(u,v)$

$$(q-q^{-1}) + g_1 + g_2 + \alpha\, g_1g_2 = 0, \qquad (q-q^{-1}) + g_3 + g_4 + \alpha\, g_3g_4 = 0 \qquad (13)$$

are satisfied. We note that these conditions are also necessary for satisfaction of (12) because any nontrivial linear combination of the products $L_1(v)L_1(u)$ and $L_1(u)L_1(v)$ is nonzero for general values of the parameters u and v.

It is usually assumed that the factors $R + g_k(u,v)$ in relation (3) are current R-matrices with a possible shift of the parameters. Such current R-matrices can be constructed using the Baxterization procedure,[5] which is described in the following statement.

Proposition 2. 1. *Let R be an involutive symmetry. Then the sum*

$$R(u,v) = R + \frac{a}{u-v}I, \quad a \in \mathbb{C}, \qquad (14)$$

satisfies the braid relation with a spectral parameter

$$R_{12}(u,v)\,R_{23}(u,w)\,R_{12}(v,w) = R_{23}(v,w)\,R_{12}(u,w)\,R_{23}(u,v). \qquad (15)$$

2. *Let $R = R_q$ be a Hecke symmetry. Then*

$$R(u,v) = R_q - \frac{(q-q^{-1})u}{u-v}I \qquad (16)$$

also satisfies braid relation (15) *presented above.*

[5]This term was introduced in [14], where the Baxterization of the Hecke algebras was considered.

We call solutions of braid relation (15) *current R-matrices.* Below without loss of generality, we assume that the parameter a in (14) is equal to unity.

Current R-matrices (14) and (16) are respectively said to be rational and trigonometric. Substituting $u \to q^{-2u}$ and $v \to q^{-2v}$ in (16), we obtain the trigonometric current R-matrix in the form

$$R_q + \frac{q - q^{-1}}{q^{2(u-v)} - 1}\, I = R_q + \frac{q^{-(u-v)}}{(u-v)_q}\, I. \tag{17}$$

As $q \to 1$, this expression tends to a rational R-matrix if R_q tends to an involutive R-matrix.

We rewrite the trigonometric R-matrix in the form

$$R(u,v) = \mathcal{R}(u/v), \qquad \mathcal{R}(x) = R + f(x)\, I, \tag{18}$$

where $f(x)$ is defined as

$$f(x) = -\frac{(q - q^{-1})x}{x - 1}. \tag{19}$$

We consider the choice of the function $g_k(u,v)$ in system (3)

$$g_1 = f\left(\frac{u}{v}\right), \qquad g_2 = f\left(\frac{vc}{u}\right), \qquad g_3 = f\left(\frac{uc}{v}\right), \qquad g_4 = f\left(\frac{v}{u}\right), \tag{20}$$

where $f(x)$ is defined by (19). The numerical parameter $c \in \mathbb{C}$ is called the *charge.* Defining relations (3) then become

$$\mathcal{R}_{12}(u/v)L_1(u)\mathcal{R}_{12}(vc/u)L_1(v) = L_1(v)\mathcal{R}_{12}(uc/v)L_1(u)\mathcal{R}_{12}(v/u). \tag{21}$$

In the standard case, this result is equivalent to the result obtained in [6]. In this case, our charge is related to the charge C in [6] as $c = e^{hC}$. Moreover, our operators $\mathcal{R}(u/v)$ differ from the operators used in that work by the factor P (the flip operator).

Proposition 3. *In algebra* (21) *with the R-matrix $R(u,v)$ defined by* (18), *the first quantum power sum* $\mathrm{Tr}_R\, L(u)$ *is central iff the value of the charge c is $q^{2(m-n)}$. This value of c is said to be critical.*

Proof. It suffices to verify that at the critical charge value (and only at that value), centrality conditions (13) are satisfied with the functions g_k defined by (20).

We call an algebra defined by relations (21) with an arbitrary skew-invertible Hecke symmetry R and a function $f(x)$ given by (19) an *algebra of RS type.*

If the initial symmetry R is involutive, we set

$$\mathcal{R}(x) = R + f(x)I, \qquad f(x) = \frac{1}{x}. \tag{22}$$

The rational version of an algebra of RS type is defined by the system relation

$$\mathcal{R}_{12}(u-v)L_1(u)\mathcal{R}_{12}(v-u+c)L_1(v) = L_1(v)\mathcal{R}_{12}(u-v+c)L_1(u)\mathcal{R}_{12}(v-u). \tag{23}$$

The numerical term c is also called the *charge.* In this case, the element $\mathrm{Tr}_R\, L(u)$ belongs to the center of the algebra iff $c = n - m$. This is the *critical value* of the charge in the rational case.

Nevertheless, the family of generalized RE algebras with the central element $\mathrm{Tr}_R\, L(u)$ is larger than described in Proposition 3. For instance, as follows from conditions (13), the centrality of this element is preserved if we interchange either the functions g_1 and g_2 or g_3 and g_4 in defining relations (3) and also if we simultaneously apply both interchanges.

4. Study of higher quantum power sums

In this section, we work with algebras of RS type given by system (21), where $\mathcal{R}$ is defined by formulas (18) and (19).

We consider the quantum matrix power $L^{[k]}(v)$ (see definition (4)). In the product of quantum matrices

$$L_1^{[k]}(v)\mathcal{R}_{12}\left(\frac{uc}{v}\right)L_1(u)\mathcal{R}_{12}\left(\frac{v}{u}\right),$$

the matrix $L^{[k]}(v)$ can be moved to the rightmost position using the interchange relations in the algebra. This property is one of the motivations for our definition (4) of quantum matrix powers. We illustrate this statement in the case $k = 2$:

$$L_1(cv)L_1(v)\mathcal{R}_{12}\left(\frac{uc}{v}\right)L_1(u)\mathcal{R}_{12}\left(\frac{v}{u}\right) = L_1(cv)\mathcal{R}_{12}\left(\frac{u}{v}\right)L_1(u)\mathcal{R}_{12}\left(\frac{vc}{u}\right)L_1(v) =$$

$$= \mathcal{R}_{12}\left(\frac{u}{cv}\right)L_1(u)\mathcal{R}_{12}\left(\frac{c^2v}{u}\right)L_1(cv)L_1(v).$$

The second equality holds by virtue of (21), where v must be replaced with cv. Repeating this procedure, we obtain a series of relations

$$L_1^{[k]}(v)\,\mathcal{R}_{12}\left(\frac{uc}{v}\right)L_1(u)\mathcal{R}_{12}\left(\frac{v}{u}\right) = \mathcal{R}_{12}\left(\frac{u}{c^{k-1}v}\right)L_1(u)\mathcal{R}_{12}\left(\frac{c^kv}{u}\right)L_1^{[k]}(v), \quad k \ge 2.$$

Multiplying this equality by $\mathcal{R}_{12}^{-1}(u/c^{k-1}v)$ from the left and by $\mathcal{R}_{12}^{-1}(v/u)$ from the right, we obtain

$$\mathcal{R}_{12}^{-1}\left(\frac{u}{c^{k-1}v}\right)L_1^{[k]}(v)\mathcal{R}_{12}\left(\frac{uc}{v}\right)L_1(u) = L_1(u)\mathcal{R}_{12}\left(\frac{c^kv}{u}\right)L_1^{[k]}(v)\mathcal{R}_{12}^{-1}\left(\frac{v}{u}\right).$$

We now take into account that for any function g, we have the identity

$$(R + g\,I)^{-1} = \frac{R - gI - (q - q^{-1})I}{1 - g(g + q - q^{-1})} = \frac{(R^{-1} - gI)}{1 - g(g + q - q^{-1})},$$

which allows obtaining the equality

$$\frac{(R^{-1} - f(u/c^{k-1}v)I)L_1^{[k]}(v)(R + f(uc/v)I)L_1(u)}{1 - f(u/c^{k-1}v)(f(u/c^{k-1}v) + q - q^{-1})} = \frac{L_1(u)(R + f(c^kv/u)I)L_1^{[k]}(v)(R^{-1} - f(v/u)I)}{1 - f(v/u)(f(v/u) + q - q^{-1})}.$$

We now calculate the trace $\mathrm{Tr}_{R(2)}$ of this equality:

$$\frac{(\mathrm{Tr}_R\, L^{[k]}(v))L_1(u)}{1 - f(u/c^{k-1}v)(f(u/c^{k-1}v) + q - q^{-1})} - \frac{L_1(u)(\mathrm{Tr}_R\, L^{[k]}(v))}{1 - f(v/u)(f(v/u) + q - q^{-1})} =$$

$$= \frac{L_1(u)L_1^{[k]}(v)(-f(v/u) + f(c^kv/u)(1 - \alpha(q - q^{-1})) - f(v/u)f(c^kv/u)\alpha)}{1 - f(v/u)(f(v/u) + q - q^{-1})} -$$

$$- \frac{L_1^{[k]}(v)L_1(u)(-f(u/c^{k-1}v) + f(cu/v)(1 - \alpha(q - q^{-1})) - f(u/c^{k-1}v)f(cu/v)\alpha)}{1 - f(u/c^{k-1}v)(f(u/c^{k-1}v) + q - q^{-1})}. \qquad (24)$$

Here, we use the identity

$$\mathrm{Tr}_{R(2)}\, R^{-1} = \mathrm{Tr}_{R(2)}\, R - (q - q^{-1})\,\mathrm{Tr}_{R(2)}\, I = (1 - (q - q^{-1})\alpha)I,$$

where $\alpha = q^{n-m}(m-n)_q$ and $(m|n)$ is the birank of R.

If $c \neq 1$ and $k \geq 2$, then the denominators of the terms in the left-hand side of relation (24) differ. This does not allow evaluating the difference

$$(\mathrm{Tr}_R\, L^{[k]}(v))L_1(u) - L_1(u)(\mathrm{Tr}_R\, L^{[k]}(v)). \tag{25}$$

Therefore, we cannot draw a definite conclusion about the centrality of the quantum power sums $\mathrm{Tr}_R\, L^{[k]}(v)$, $k \geq 2$.

Nevertheless, if we set $c = 1$ in the algebra defined by relations (21) and (18), then it is easy to obtain a condition on the matrix R ensuring the centrality of the higher quantum power sums $\mathrm{Tr}_R\, L^k(u)$.[6] Indeed, in this case, the denominators in (24) become equal as a consequence of explicit formula (19) for $f(x)$:

$$f\left(\frac{u}{v}\right)\left(f\left(\frac{u}{v}\right) + q - q^{-1}\right) = f\left(\frac{v}{u}\right)\left(f\left(\frac{v}{u}\right) + q - q^{-1}\right) = (q-q^{-1})^2\frac{uv}{(u-v)^2}.$$

As a result, we can evaluate difference (25):

$$(\mathrm{Tr}_R\, L^k(v))L_1(u) - L_1(u)(\mathrm{Tr}_R\, L^k(v)) = (L_1(u)L^k(v) - L^k(v)L_1(u))\frac{\alpha(q-q^{-1})uv}{(u-v)^2}. \tag{26}$$

If $m \neq n$, then $\alpha = q^{(n-m)}(m-n)_q \neq 0$. Consequently, the right-hand side of this equality is nonzero. Therefore, none of the quantum power sums in the corresponding algebra is central.

But if $m = n$, i.e., the birank of the Hecke symmetry R is $(m|m)$, the right-hand side vanishes. In this case, the critical charge value $c = q^{2(m-n)} = 1$. Consequently, we obtain the following conclusion.

Proposition 4. *If the birank of a skew-invertible Hecke symmetry R is $(m|m)$, then in the algebra given by interchange relations* (21)*, where $f(x)$ is defined by* (19)*, the critical charge value is equal to unity. For this charge value, all quantum power sums $\mathrm{Tr}_R\, L^{[k]}(v) = \mathrm{Tr}_R\, L^k(v)$ belong to the center of the algebra.*

We also note that there are generalized RE algebras with a Hecke symmetry R of birank $(m|m)$ that are not of RS type, but all their elements $\mathrm{Tr}_R\, L^{[k]}(v) = \mathrm{Tr}_R\, L^k(v)$ are central.

In conclusion, we mention that the generalized Yangians recently introduced in [8], [9]) are particular cases of generalized RE algebras corresponding to the choice $g_2 = g_3 = 0$.[7] Quantum analogues of matrix powers and power sums and also some other symmetric polynomials are well defined in the generalized Yangians. But these symmetric polynomials are not central in the general case. The quantum determinant $\det_R L(u)$ turns out to be the only central element.

Conflicts of interest. The authors declare no conflicts of interest.

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[6] If the charge $c = 1$, then $L^{[k]}(v) = L^k(v)$, i.e., the quantum power matrices $L(u)$ become the usual power matrices. But it must be remembered that the trace in the definition of quantum power sums remains "deformed" in accordance with definition (6).

[7] We recall that we assume that all g_k are nonzero here.

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