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ORIGINAL PAPER

The vertex colourability problem for {*claw*, *butterfly*}-free graphs is polynomial-time solvable

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Abstract
The vertex colourability problem is to determine, for a given graph and a given natural k, whether it is possible to split the graph's vertex set into at most k subsets, each of pairwise non-adjacent vertices, or not. A hereditary class is a set of simple graphs, closed under deletion of vertices. Any such a class can be defined by the set of its forbidden induced subgraphs. For all but four hereditary classes, defined by a pair of connected five-vertex induced prohibitions, the complexity status of the vertex colourability problem is known. In this paper, we reduce the number of the open cases to three, by showing polynomial-time solvability of the problem for $\{claw, butterfly\}$-free graphs. A $claw$ is the star graph with three leaves, a $butterfly$ is obtained by coinciding a vertex in a triangle with a vertex in another triangle.

Keywords Vertex colourability problem · Computational complexity · Hereditary graph class

1 Introduction

All graphs, considered in this paper, are *simple*, i.e. unweighted, non-oriented graphs without loops and multiple edges. The *vertex colouring* of a graph $G = (V, E)$ is a mapping $c : V \longrightarrow \mathbb{N}$, such that $c(v_1) \neq c(v_2)$, for any adjacent vertices v_1 and v_2 of G. If $c : V \longrightarrow \{1, 2, \ldots, k\}$, then c is called a *k-colouring*. All the elements of $\{c(v) : v \in V\}$ are said to be *colours*. The minimum number of colours in vertex colourings of a graph G is called the *chromatic number* of G and is denoted by $\chi(G)$.

The *vertex k-colourability problem* (the k- VERTCOL *problem*, for short) is to recognize, for a given graph G, whether $\chi(G) \leq k$ or not. The *vertex colourability problem* (briefly, the VERTCOL *problem*) is to recognize, for a given graph G and a

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given natural k, whether $\chi(G) \leq k$ or not. The VERTCOL problem and k- VERTCOL problem, for any $k \geq 3$, are classical NP-complete problems [13]. Notice that the k- VERTCOL problem is polynomial-time solvable for $k \in \{1, 2\}$.

A class of graphs is called *hereditary* if it is closed under deletion of vertices. Any hereditary (and only hereditary) graph class $\mathcal{X}$ can be defined by a set of its *forbidden induced subgraphs* $\mathcal{S}$. We write $\mathcal{X} = Free(\mathcal{S})$, and the graphs in $\mathcal{X}$ are said to be $\mathcal{S}$*-free*.

There are many "white spots" on "maps" of the computational complexity of the considering colourability problems in the family of hereditary classes. There are two ways to reduce the amount of these "white spots". The first is to increase the number of prohibited induced subgraphs, and the second one is to increase sizes of such subgraphs. A restriction on the size or number of induced prohibitions forms a certain subfamily of the hereditary classes family. A possible reduction of "white spots" is to obtain complete complexity dichotomies for larger values of this border.

As usual, P_n, C_n mean the simple path and the simple cycle with n vertices, correspondingly, O_n, K_n stand for the empty and the complete graph on n vertices, respectively, $K_{p,q}$ is the complete bipartite graph with p vertices in the first part and q vertices in the second one. The graph $K_{1,3}$ is also called a $claw$. By $K_{2,3}^+$ we denote the graph, obtained from a $K_{2,3}$ by joining its degree 3 vertices with an edge. The graphs $K_{2,3}^+$, $bull$, $butterfly$, W_4 are depicted in Fig. 1.

For graphs G_1 and G_2 with non-intersected sets of vertices, we denote by $G_1 + G_2$ their disjunctive union. For a graph G and a natural k, kG means the sum $\underbrace{G + \ldots + G}_{k\ times}$.

The computational complexity of the VERTCOL problem was intensively studied for families of hereditary classes, defined by small graphs or by a small number of forbidden induced structures. Namely, the complexity of this problem has been completely determined for all the classes of the form $Free(\{H\})$ [23]. The VERTCOL problem is polynomial-time solvable for $Free(\{H\})$, if H is an induced subgraph of a P_4 or a $P_3 + P_1$, otherwise, the problem is NP-complete for $Free(\{H\})$. A study of forbidden pairs was also initiated in [23]. For all but four cases, either NP-completeness or polynomial-time solvability was shown for the VERTCOL problem and all the hereditary classes, defined by four-vertex forbidden induced structures [25]. The exceptional classes are

$$Free(\{claw, O_4\}), Free(\{claw, K_2 + O_2, O_4\}),$$

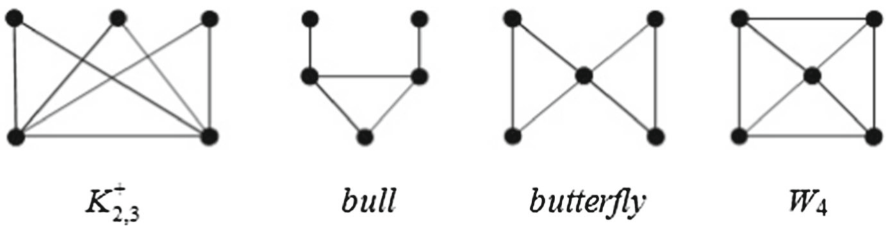

Fig. 1 The graphs $K_{2,3}^+$, $bull$, $butterfly$, and W_4

$$Free(\{claw, K_2 + O_2\}), Free(\{C_4, O_4\}).$$

Moreover, the VertCol problem for $\{claw, K_2 + O_2\}$-free graphs is polynomially equivalent to the same problem for $\{claw, K_2 + O_2, O_4\}$-free graphs [25].

The situation for the k- VertCol problem is unclear, even when only one induced subgraph is forbidden. The complexity of the 3- VertCol problem is known for all the classes of the form $Free(\{H\})$, where $|V(H)| \leq 6$ [4]. A similar result for $\{H\}$-free graphs with $|V(H)| \leq 5$ was recently obtained for the 4- VertCol problem [16]. For each fixed k, the k- VertCol problem can be solved in polynomial time for $\{P_5\}$-free graphs [18]. The 3- VertCol problem can be solved in polynomial time for $\{P_7\}$-free graphs [3]. The 4- VertCol problem can be solved in polynomial time for $\{P_6\}$-free graphs [34]. For every $k \geq 5$, the k- VertCol problem is NP-complete in the class of $\{P_6\}$-free graphs [20]. Additionally, the 4- VertCol problem is NP-complete for $\{P_7\}$-free graphs [20]. On the other hand, at the present time the complexity status of the k- VertCol problem is open for $\{P_8\}$-free graphs and $k = 3$, for $\{P_7\}$-free graphs and $k = 4$.

In the papers [27,29,32,33], the computational complexity of the 3- VertCol problem has been classified for all the hereditary classes, defined by 5-vertex forbidden induced subgraphs.

The computational complexity of the VertCol problem for a pair of connected 5-vertex forbidden induced fragments was considered in [19,22,26,28,30,31]. Presently, the complexity of this problem is still open only for the following four pairs of this type:

– $\{claw, butterfly\}$,
– $\{P_5, H\}$, where $H \in \{K_{2,3}, K^+_{2,3}, W_4\}$.

Additionally, we would mention the papers [2,6–12,14,15,17,24] in the complexity of the VertCol problem for hereditary classes, defined by small-size prohibitions.

In this paper, we reduce the number of the open cases from four to three by showing that the VertCol problem is polynomial-time solvable for $\{claw, butterfly\}$-free graphs. It was known that the 3- VertCol problem can be solved in polynomial time for $\{claw, butterfly\}$-free graphs [21].

2 Some definitions, notations, and facts

For a vertex x of some graph, we denote by $N(x)$ and $\deg(x)$ the neighbourhood and the degree of x, respectively. For a vertex x of some graph and a subset V' of its vertices, $N_{V'}(x)$ means $N(x) \cap V'$. The maximum vertex degree of a graph G is denoted by $\Delta(G)$.

Let $G = (V, E)$ be a graph and $V' \subseteq V$. Then, $G[V']$ is the subgraph of G, induced by V', $G \setminus V'$ is the result of deletion all the vertices in V' from G, together with their incident edges.

Let A and B be non-intersected subsets of vertices of some graph. The predicate $A \bullet B$ means that any vertex from A is adjacent to any vertex from B.

3 Informal review of the algorithm

The polynomial-time algorithm from this paper deals with several standard graph reduction techniques, described in Sect. 4. By its results, we may consider only graphs, which are large enough, have three pairwise-nonadjacent vertices, without special separators and vertices of degree at most three, simultaneously. The proof separately considers two cases: when a given irreducible graph is not $\{C_4\}$-free and when it is $\{C_4\}$-free. In Sect. 5, we show that for the first case the input can be additionally reduced in polynomial time to its induced subgraph by removing special complete or small-size subgraphs, keeping the chromatic number under control. This gives polynomial-time reducibility to $\{claw, butterfly, C_4\}$-free graphs. In Sect. 6, we show that the VERTCOL problem can be solved in polynomial time for $\{claw, butterfly, C_4, K_4\}$-free graphs. In Sect. 7, we present more reductions, devoted to deletion of redundant vertices and elimination of cliques with at least 4-vertices. Together with previously obtained results, it leads to polynomial-time solvability of the VERTCOL problem for $\{claw, butterfly, C_4\}$-free graphs. Combining results of Sects. 5–7, we obtain the main result of the whole paper, see Sect. 8.

4 Some basic complexity results for the vertex colourability problem

Let $G = (V, E)$ be a graph. A *clique* of G is a subset of its pairwise adjacent vertices. A clique with k vertices will be called a k-*clique*. A *separating clique* Q of G is a clique of G, such that $G \setminus Q$ contains more connected components, than G. If a clique Q is separating in G, then $V \setminus Q$ can be arbitrarily partitioned into simultaneously non-empty subsets A and B, such that the vertex set of any connected component of $G \setminus Q$ is completely contained in either A or in B. We have $\chi(G) = \max(\chi(G[A \cup Q]), \chi(G[B \cup Q]))$, see [35]. The decomposition process above can be represented by a binary tree, which can be computed in $O(mn)$ time, for any graph with n vertices and m edges, see [35].

The following assertion follows from [23]:

Lemma 1 *For any $\{O_3\}$-free n-vertex graph, the* VERTCOL *problem can be solved in $O(n^3)$ time.*

The next Lemma is obvious:

Lemma 2 *For each fixed C, the* VERTCOL *problem can be solved in $O(1)$ time for graphs on at most C vertices.*

We call a connected graph $G = (V, E)$ *irreducible* if

$$G \notin Free(\{O_3\}), \chi(G) \geq 4, |V| \geq 14,$$

G does not contain separating cliques and vertices of degree at most 3, simultaneously.

Lemma 3 *The* VERTCOL *problem for any hereditary subclass $\mathcal{X}$ of $\{claw, butterfly\}$-free graphs can be reduced in polynomial time to the same problem for irreducible graphs in $\mathcal{X}$.*

Proof By the reasonings from the first paragraph, Lemmas 1 and 2, one may consider graphs in $\mathcal{X}$ without separating cliques, not belonging to $Free(\{O_3\})$, and having at least 14 vertices, simultaneously. It is known that the 3- VERTCOL problem can be solved in polynomial time for $\{claw, butterfly\}$-free graphs, see [21]. Therefore, we may consider graphs in $\mathcal{X}$ with the chromatic number at least 4. Deletion of vertices of degree at most 3 in these graphs does not change their chromatic number. So, this Lemma holds. □

5 The vertex colouring problem for {*claw*, *butterfly*}-free graphs, containing induced 4-cycles

Let $G = (V, E)$ be a $\{claw, butterfly\}$-free graph, containing an induced 4-cycle $C = (v_1, v_2, v_3, v_4)$ with $\hat{V} = \{v_1, v_2, v_3, v_4\}$. We associate the following notations with C, assuming throughout this Section for indices of its vertices to be modulo 4:

- V_i is the set $\{x : N_{\hat{V}}(x) = \{v_i, v_{i+1}\}\}$,
- V_i' is the set $\{x : N_{\hat{V}}(x) = \{v_i, v_{i+1}, v_{i+2}\}\}$,
- U is the set $\{x : N_{\hat{V}}(x) = \{v_1, v_2, v_3, v_4\}\}$,
- W is the set $\{x : N_{\hat{V}}(x) = \emptyset\}$.

As G is $\{claw\}$-free, any vertex, having a neighbour on C, belongs to $\hat{V} \cup \bigcup_{i=1}^{4}(V_i \cup V_i') \cup U$. Additionally to the notations above, denote by $\tilde{V}_i$ the set $\{x \in V_i : N_W(x) \neq \emptyset\}$.

Lemma 4 *The following properties are true:*

1. *For any i, V_i and V_i' are both cliques.*
2. *For any i, $V_i \bullet (V_{i\pm1} \cup \bigcup_{j=1}^{4} V_j')$ and $V_i' \bullet V_{i+2}'$.*
3. *If there are non-adjacent vertices, one from V_i and the other one from V_{i+2}, then $\bigcup_{j=1}^{4} V_j' = \emptyset$.*
4. *If $U \neq \emptyset$, then, for any i, $|V_i| \leq 2$, and every element of U is adjacent to exactly one element of V_i, whenever $|V_i| = 2$. If a vertex in V_i and a vertex in V_{i+2} have the same neighbour in U, then these vertices are adjacent.*

Proof Let us prove the first item. If $ab \notin E$, where $a \in V_i$ and $b \in V_i$, then a, b, v_i, v_{i-1} induce a $claw$. If $ab \notin E$, where $a \in V_i'$ and $b \in V_i'$, then a, b, v_i, v_{i-1} induce a $claw$. Hence, this item holds.

Let us prove the second item. If $ab \notin E$, where $a \in V_i$ and $b \in V_{i\pm1}$, then either $a, b, v_i, v_{i+1}, v_{i+2}$ or $a, b, v_{i-1}, v_i, v_{i+1}$ induce a $butterfly$. If $ab \notin E$, where $a \in V_i$ and $b \in V_{i-1}' \cup V_i'$, then either a, b, v_i, v_{i-1} or a, b, v_{i+1}, v_{i+2} induce a $claw$. If $ab \notin E$, where $a \in V_i$ and $b \in V_{i+1}' \cup V_{i+2}'$, then either $a, b, v_i, v_{i+1}, v_{i+2}$ or $a, b, v_{i-1}, v_i, v_{i+1}$ induce a $butterfly$. If $ab \notin E$, where $a \in V_i'$ and $b \in V_{i+2}'$, then $a, b, v_{i-1}, v_i, v_{i+1}$ induce a $butterfly$. Hence, this item holds.

Let us prove the third item. Suppose that $a \in V_i$ and $b \in V_{i+2}$ are nonadjacent. Then, by the second item of this Lemma, any vertex $c \in \bigcup_{j=1}^{4} V_j'$ must be adjacent to a and to b, simultaneously. Therefore, $c, a, v_{i+1}, b, v_{i+3}$ or c, a, v_i, b, v_{i+2} induce a $butterfly$. Hence, this item holds.

Let us prove the fourth item. Let a be an arbitrary element of U. Recall that V_i is a clique, by the first item. For any i, it must be adjacent to at least $|V_i| - 1$ elements of V_i, otherwise, any two non-neighbours of a in V_i, the vertices a, v_i, v_{i-1} induce a $butterfly$. On the other hand, $|N_{V_i}(a)| \leq 1$, otherwise, a, any two elements of $N_{V_i}(a)$, and v_{i+2}, v_{i+3} induce a $butterfly$. If $a \in V_i, b \in V_{i+2}, c \in U$ and $ac \in E, bc \in E, ab \notin E$, then a, b, c, v_i, v_{i+2} induce a $butterfly$. Hence, this item holds.

So, this Lemma holds. □

Lemma 5 *The following properties are true:*

1. *For any i, each vertex in $\tilde{V}_i$ has the only one neighbour in W and it does not have a neighbour in V_{i+2}.*
2. *If there is a vertex in U, adjacent to an element of $\tilde{V}_i$, then it does not have a neighbour in V_{i+2}.*
3. *If x is an arbitrary element of $\tilde{V}_i$, x' is its unique neighbour, and $V_{i+1} \cup V_{i+3} \neq \emptyset$, then any element of V_{i+2} is adjacent to x' and at least one of the sets $V_i \setminus \{x\}$ and V_{i+2} is empty.*

Proof Let us prove the first item. Let x be an arbitrary element of $\tilde{V}_i$ and x' be its neighbour in W. If x has two neighbours $x', x'' \in W$, then either x, x', x'', v_i induce a $claw$, if $x'x'' \notin E$, or x, x', x'', v_i, v_{i+1} induce a $butterfly$, if $x'x'' \in E$. If x has a neighbour $x^* \in V_{i+2}$, then either x, x', x^*, v_i induce a $claw$, if $x'x^* \notin E$, or $x, x', x^*, v_{i+2}, v_{i+3}$ induce a $butterfly$, if $x'x^* \in E$.

Let us prove the second item. If there is a vertex $z \in U$, adjacent to $x \in \tilde{V}_i$, then z does not have a neighbour $x^* \in V_{i+2}$, otherwise $xx^* \in E$, by Lemma 4 (item 4), which is impossible by the first item of this Lemma.

Let us prove the third item. Without loss of generality, suppose that $y \in V_{i+1}$. There is not a vertex $x^* \in V_{i+2}$, such that $x^*x' \notin E$. Suppose the opposite. By Lemma 4 (item 2), $yx \in E$ and $yx^* \in E$. To avoid a $claw$, induced by x, x', y, v_i, we have $x'y \in E$. Then, the vertices y, x, x', v_{i+2}, x^* induce a $butterfly$. If there are vertices $x^{**} \in V_i \setminus \{x\}$ and $x^{***} \in V_{i+2}$, then $xx^{**} \in E, xy \in E, x^{**}y \in E, x^{***}y \in E$, by Lemma 4 (items 1 and 2). To avoid a $claw$, induced by x, x', y, v_i, we have $x'y \in E$. To avoid a $claw$, induced by y, x', x^{**}, v_{i+2}, we have $x'x^{**} \in E$. To avoid a $butterfly$, induced by $y, x, x^{**}, v_{i+2}, x^{***}$, either $xx^{***} \in E$ or $x^{**}x^{***} \in E$. We have a contradiction with the second item of this Lemma.

So, this Lemma holds. □

We say that C *dominates* all the vertices of G, if $W = \emptyset$. A set of pairwise non-adjacent vertices is called *independent*.

Lemma 6 *The* VERTCOL *problem for $\{claw, butterfly\}$-free graphs, containing dominating induced 4-cycles, can be solved in polynomial time.*

Proof Let $G = (V, E)$ be a $\{claw, butterfly\}$-free graph, containing an induced 4-cycle C, which dominates all its vertices. For a given graph, containing an induced 4-cycle, dominating all its vertices, it is possible to find such a cycle in polynomial time. Checking, whether a given graph is O_3-free or has at most 13 vertices can be done in polynomial time. Hence, by this observation, Lemmas 1 and 2, one may assume that G contains an independent set $\{a, b, c\}$ and $|V| \geq 14$. We will check that $a \in V_i, b \in V_{i+2}, c \in U$, for some i. In the following reasonings, we will consider all the possible cases for the location of a, b, c, up to symmetry.

Clearly, at most one vertex among a, b, c lies on C. If $a = v_i$, then $b, c \in V_{i+1} \cup V_{i+2} \cup V'_{i+1}$ and, by Lemma 4 (items 1 and 2), b and c are adjacent. If $\{a, b, c\} \cap \bigcup_{j=1}^{4} V_j = \emptyset$, then $a \in V'_i$, for some i, otherwise, $a, b, c \in U$ and a, b, c, v_1 induce a $claw$. Hence, by Lemma 4 (items 1 and 2), we have $b, c \in V'_{i\pm1} \cup U$. If $b \in U$ or $c \in U$, then a, b, c and v_{i+1} induce a $claw$. Therefore, $b \in V'_{i-1}$ and $c \in V'_{i+1}$, by Lemma 4 (item 1), and v_{i+1}, a, b, c induce a $claw$.

Without loss of generality, one may assume that $a \in V_i, b \in V_{i+2}, c \in U$. By Lemma 4 (item 3), we have $\bigcup_{j=1}^{4} V'_j = \emptyset$. If $U = \{c\}$, then, by Lemma 4 (item 4), we have $|V| \leq 13$. Hence, there is a vertex c', such that $c' \in U \setminus \{c\}$. By Lemma 4 (item 4), c' cannot be simultaneously adjacent to a and b. Then, $cc' \in E$. Indeed, if $cc' \notin E$, then $c'a \in E, c'b \in E$, to avoid a $claw$, induced by either a, c, c', v_i or b, c, c', v_{i+3}.

Let us show that $V_{i+1} \cup V_{i+3} = \emptyset$. Without loss of generality, suppose that $d \in V_{i+1}$. By Lemma 4 (item 2), $ad \in E, bd \in E$. Clearly, $cd \notin E$, otherwise, a, b, c, d induce a $claw$. We have $c'd \in E$, otherwise either a, v_{i+1}, d, c, c' or b, v_{i+2}, d, c, c' induce a $butterfly$. Then, $c'a \in E$ or $c'b \in E$, otherwise, d, a, b, c' induce a $claw$. Hence, either v_{i-1}, c, c', a, d or v_i, c, c', b, d induce a $butterfly$.

Let us show that $\chi(G)$ can be computed in polynomial time. Recall that c is adjacent to all the elements of $U \setminus \{c\}$. Hence, by Lemma 4 (item 4), we have $N(c) = V \setminus \{a, b, c\}$. Therefore, $\chi(G) = \chi(G^*) + 1$, where $G^* = G \setminus \{a, b, c\}$. By Lemma 4 (item 4), we have $|V_i| \leq 2$ and $|V_{i+2}| \leq 2$. By the result, claimed at the end of the first paragraph, the graph G^* is not $\{O_3\}$-free iff there are pairwise non-adjacent vertices $a^* \in V_i \setminus \{a\}, b^* \in V_{i+2} \setminus \{b\}, c^* \in U \setminus \{c\}$. If G^* is $\{O_3\}$-free, then $\chi(G^*)$ and $\chi(G)$ can be computed in polynomial time, by Lemma 1. If G^* is not $\{O_3\}$-free, then

$$V_i = \{a, a^*\}, V_{i+2} = \{b, b^*\}, \chi(G^*) = \chi(G^{**}) + 1,$$

where $G^{**} = G^* \setminus \{a^*, b^*, c^*\}$ and G^{**} is $\{O_3\}$-free by the result, claimed at the end of the first paragraph. Hence, $\chi(G^{**}), \chi(G^*), \chi(G)$ can be computed in polynomial time, by Lemma 1. So, this Lemma holds. □

Lemma 7 *Let H be a graph and Q be its k-clique, such that any $v_i \in Q$ has at most one neighbour $u_i \notin Q$. Then, $\chi(H) = \max(\chi(H \setminus Q), |Q|)$, assuming that there is an optimal colouring of $H \setminus Q$, in which not all u_i receive the same colour.*

Proof The inequality $\chi(H) \geq \max(\chi(H \setminus Q), |Q|)$ is obvious. Let us consider any optimal colouring of $H \setminus Q$, in which not all u_i receive the same colour. Without loss of generality, suppose that u_i exists, for any i, and $u_1, \ldots, u_{i_1}$ receive the colour 1, $u_{i_1+1}, \ldots, u_{i_2}$ receive the colour 2, ..., $u_{i_{s-1}+1}, \ldots, u_{i_s}$ receive the colour s. Clearly, $s \geq 2$. For any $1 \leq j \leq s$, we arrange the colour $(i+1) \mod s$ for the vertex v_{i_j}. For the remaining vertices of Q, we arbitrarily arrange all the colours in $\{s+1, \ldots, k\}$. The obtained assignment of colours is a colouring of H. Hence, $\chi(H) = \max(\chi(H \setminus Q), |Q|)$. So, this Lemma holds. □

Lemma 8 *The* VERTCOL *problem for* $\{claw, butterfly\}$*-free graphs can be reduced to the same problem for* $\{claw, butterfly, C_4\}$*-free graphs.*

Proof For a given graph, containing an induced 4-cycle, it is possible to find such a cycle in polynomial time. Let $G = (V, E)$ be a $\{claw, butterfly\}$-free graph and C be its induced 4-cycle. By Lemmas 3 and 6, one may assume that G is irreducible and $W \neq \emptyset$. For some i, both sets $\tilde{V}_i$ and $\tilde{V}_{i+2}$ are non-empty, otherwise, by Lemma 4 (items 1 and 2), there is an index i^*, such that $\tilde{V}_{i^*} \cup \tilde{V}_{i^*\pm 1}$ is a separating clique of G. Hence, by Lemma 4 (item 3), we have $\bigcup_{j=1}^{4} V'_j = \emptyset$.

Let $a \in \tilde{V}_i$ and $a' \in W$ be a neighbour of a, which is unique by Lemma 5 (item 1). Let $b \in \tilde{V}_{i+2}$ and $b' \in W$ be a neighbour of b, which is unique by Lemma 5 (item 1). Perhaps, $a' = b'$.

Suppose that $V_{i+1} \cup V_{i+3} \neq \emptyset$. Let us show that $\chi(G) = \chi(G \setminus \{v_i, v_{i-1}\})$. This relation gives a polynomial-time reduction of the VERTCOL problem for G to the same problem for $G \setminus \{v_i, v_{i-1}\}$ and the proof of this Lemma is applied to $G \setminus \{v_i, v_{i-1}\}$.

Without loss of generality, suppose that $|V_{i+1}| \geq |V_{i+3}|$. Then, we have:

$$V_i = \{a\}, V_{i+2} = \{b\}, ab \notin E, a' = b', \{a, b, a'\} \bullet (V_{i+1} \cup V_{i+3}).$$

Indeed, $V_i = \{a\}, V_{i+2} = \{b\}, ab \notin E$, by Lemma 5 (items 1 and 3). By Lemma 4 (item 2), we have $\{a, b\} \bullet (V_{i+1} \cup V_{i+3})$. To avoid claws, induced by a', a, v_i and an element of $V_{i+1} \cup V_{i+3}$, by b', b, v_{i+3} and an element of $V_{i+1} \cup V_{i+3}$, we have $\{a', b'\} \bullet (V_{i+1} \cup V_{i+3})$. Then $a'b \in E$ and $b'a \in E$, as a, a', b, v_{i+2} and an element of $V_{i+1} \cup V_{i+3}$ or b, b', a, v_{i+1} and an element of $V_{i+1} \cup V_{i+3}$ induce a $butterfly$, otherwise. Hence, by Lemma 5 (item 1), we have $a' = b'$.

There is no a vertex $a'' \in W$, adjacent to a', otherwise, $aa'' \notin E$, by Lemma 5 (item 1), and $a''b \in E$, to avoid a $claw$, induced by a', a'', a, b, and $v_{i+2}, v_{i+3}, b, a', a''$ induce a $butterfly$. There is not a vertex $v' \in W$, adjacent to a vertex $v \in V_{i+1} \cup V_{i+3}$, otherwise, by Lemma 5 (item 1), v, v', b, and v_i or v_{i+1} induce a $claw$.

Let u be an arbitrary vertex in U. If there is a vertex $v \in V_{i+1} \cup V_{i+3}$, adjacent to u, then $ua \in E$ or $ub \in E$, but not both by Lemma 5 (item 2), to avoid a $claw$, induced by v, u, a, b. We have $ua' \notin E$, otherwise, either $u, a, a', v_{i+2}, v_{i+3}$ or u, b, a', v_i, v_{i+1} induce a $butterfly$. Then, G contains an induced $butterfly$. Indeed, without loss of generality, suppose that $v \in V_{i+1}$. Then, either v, a', b, u, v_{i+1}, if $ua \in E, ub \notin E$, or v, a, a', u, v_{i+2}, if $ub \in E, ua \notin E$, induce a $butterfly$. Therefore, there is not an edge xy, where $x \in V_{i+1} \cup V_{i+3}, y \in U$.

Let v be an arbitrary vertex from $V_{i+1} \cup V_{i+3}$. Without loss of generality, let $v \in V_{i+1}$. If u is adjacent to a or to b, but not both by Lemma 5 (item 2), then $ua' \notin E$, otherwise, either $u, a, a', v_{i+2}, v_{i+3}$ (if $ua \in E$) or $u, b, a', v_{i+2}, v_{i+3}$ (if $ub \in E$) induce a $butterfly$. Then, either u, a, v_i, a', v (if $ua \in E$) or u, b, v_{i-1}, a', v (if $ub \in E$) induce a $butterfly$. Therefore, there is not an edge xy, where $x \in \{a, b\}, y \in U$. Hence, U has at most one element, otherwise, any two its elements together with a, v_{i+1}, v induce a subgraph, for which a $claw$ and a $butterfly$ are induced subgraphs, simultaneously.

As $|V| \geq 14$ and $|V_{i+1}| \geq |V_{i+3}|$, we have $|V_{i+1}| \geq 3$. By Lemma 4 (item 1), V_{i+1} and V_{i+3} are both cliques and $\chi(G) \geq 5$. Since the degree of the element from U, if one exists, equals 4, its deletion does not change the chromatic number of G. Hence, one may suppose that $U = \emptyset$. If not $V_{i+1} \bullet V_{i+3}$, then an element of V_{i+1} and its non-neighbour in V_{i+3} together with a and b induce a 4-cycle, dominating all the vertices of G. This case has been considered in Lemma 6. Suppose that $V_{i+1} \bullet V_{i+3}$. Any two colours, meeting among the colours of V_{i+1} in any colouring of $G \setminus \{v_i, v_{i-1}\}$, can be used for v_{i-1} and v_i to extend the colouring to a colouring of G. Hence, $\chi(G) = \chi(G \setminus \{v_i, v_{i-1}\})$.

Assume that $V_{i+1} \cup V_{i+3} = \emptyset$. By the first paragraph and Lemma 4 (item 1), V_i and V_{i+2} are both non-empty cliques. In any colouring of $G^* = G \setminus (\{v_{i+2}, v_{i+3}\} \cup V_{i+2})$, the vertices v_i and v_{i+1} must receive distinct colours. We may suppose that $U \neq \emptyset$, otherwise, by Lemma 5 (item 1) and Lemma 7, $\chi(G) = \max(\chi(G^*), |V_{i+2}| + 2)$, and the proof of this Lemma is applied to G^*.

Since G is irreducible, we have $\min(\deg(a), \deg(b)) \geq 4$. By this fact, Lemma 4 (item 4), and Lemma 5 (items 1 and 2), either $V_i = \{a\}$, $V_{i+2} = \{b\}$ and both vertices a and b have neighbours in U or

$$V_i = \{a, a^*\}, V_{i+2} = \{b, b^*\}, \tilde{V}_i = \{a\}, \tilde{V}_{i+2} = \{b\}, U \bullet \{a^*, b^*\}.$$

Let $V' = \hat{V} \cup V_i \cup V_{i+2} \cup U$ and $G' = G[V']$. By Lemma 6, $\chi(G')$ can be computed in polynomial time. Clearly, $\chi(G') \geq 4$ and $\chi(G) \geq \max(\chi(G \setminus V'), \chi(G'))$. Consider an optimal colouring c of $G \setminus V'$ and an optimal colouring c' of G'. Independently on $c(a')$ and $c(b')$, the colouring c can be extended to a colouring of G in $\max(\chi(G \setminus V'), \chi(G'))$ colours by using c' and interchanging some colours in it. Hence, $\chi(G) = \max(\chi(G \setminus V'), \chi(G'))$. As $\chi(G')$ can be computed in polynomial time, computing $\chi(G)$ can be reduced in polynomial time to computing $\chi(G \setminus V')$.

Applied several times the described techniques and Lemma 6, we will obtain in polynomial time an induced $\{C_4\}$-free subgraph H of G with a polynomial-time computable connection between $\chi(H)$ and $\chi(G)$. So, this Lemma holds. □

6 The vertex colouring problem for {*claw*, *butterfly*, C_4, K_4}-free graphs

Lemma 9 *The* VERTCOL *problem can be solved in polynomial time for* $\{claw, butterfly, C_4, K_4\}$*-free graphs.*

Proof By Lemma 3, one may consider irreducible $\{claw, butterfly, C_4, K_4\}$-free graphs only. Let $G = (V, E)$ be such a graph and x be its vertex of the maximum degree.

Suppose that $\Delta(G) \geq 5$, which can be tested in polynomial time. That is, whether $\Delta(G) \geq 5$ or not can be verified in polynomial time. Since G is $\{butterfly, C_4, K_4\}$-free, we have that $H = G[N(x)]$ is $\{C_3, 2K_2, C_4\}$-free, implying that H is $\{C_3, C_4, C_6, C_7, \ldots\}$-free. Additionally suppose that H is $\{C_5\}$-free. Then, H is a forest on at least 5 vertices. The graph H has a vertex of the degree at most one in this graph. Deletion of this vertex and its neighbourhood from H produces a forest on at least 3 vertices. Hence, the new forest contains non-adjacent vertices and H contains an independent set with 3 vertices. Therefore, G is not $\{claw\}$-free. We have a contradiction with the assumption.

Now, additionally suppose that H contains an induced 5-cycle $C = (x_1, x_2, x_3, x_4, x_5)$ with $\hat{V} = \{x_1, x_2, x_3, x_4, x_5\}$. All the indices for vertices of C are taken modulo 5. Let $\check{V} = V \setminus (\hat{V} \cup \{x\})$. Since $G \in Free(\{claw, C_4, K_4\})$, it is easy to check that $N(x) = \hat{V}$ and, for any $x' \in \check{V}$, we have either $N_{\hat{V}}(x') = \emptyset$ or $N_{\hat{V}}(x') = \{x_i, x_{i+1}\}$, for some i. Additionally, by the same reasons, for any $x'', x''' \in \check{V}$, $x'' \neq x'''$, both having neighbours on C, we have $N_{\hat{V}}(x'') \neq N_{\hat{V}}(x''')$.

Since degrees of all the vertices of G are at least 4, some vertex of C has two neighbours in $\check{V}$. Let us consider the maximum $2 \leq k \leq 5$, such that there are vertices $x'_1, \ldots, x'_k \in \check{V}$ with $N_{\hat{V}}(x'_j) = \{x_{i+j-1}, x_{i+j}\}$, for any $1 \leq j \leq k$ and some i. Without loss of generality, put $i = 1$. The case $k = 3$ is impossible, as $\deg(x_5) = 3$, otherwise.

Let us consider the case $k = 2$. Then, $\deg(x_1) = \deg(x_3) = 4$ and there is a vertex $x' \in \check{V}$ with $N_{\hat{V}}(x') = \{x_4, x_5\}$. We have $x'x'_1 \notin E$, $x'x'_2 \notin E$, since G is $\{C_4\}$-free. Since $\deg(x') \geq 4$, there are vertices $y_1, y_2 \in N(x') \setminus (\hat{V} \cup \{x, x'_1, x'_2\})$. Then, $G[\{x', x_4, x_5, y_1, y_2\}]$ simultaneously contains a $claw$ and a $butterfly$ as induced subgraphs.

Let us consider the cases $k \in \{4, 5\}$. Notice that $\deg(x_1) = \deg(x_5) = 4$, whenever $k = 4$. Put $V^* = \{x'_1, \ldots, x'_k\}$. As G is $\{butterfly, C_4\}$-free, $x'_i x'_j \in E$ if and only if $|i - j| \in \{1, 4\}$. Since G is irreducible, it is connected and $|V| \geq 14$. Hence, there is a vertex $x^* \in \check{V} \setminus V^*$, adjacent to an element in V^*. Since G is $\{claw\}$-free, any such an element is adjacent to all the vertices in V^*. None of the vertices in $\check{V} \setminus V^*$, having a neighbour in V^*, has a neighbour outside V^*, since G is $\{claw, buttefly\}$-free. Similarly, $|\check{V} \setminus V^*| \leq 1$. Hence, $|V| \leq 12$. We have a contradiction with the assumption.

Finally, we need to consider the case, when $\Delta(G) \leq 4$. As G is irreducible, degrees of all its vertices equal 4. Then, $\chi(G) = 4$, by the Brooks' theorem, see [5], and as $\chi(G) \geq 4$, $G \in Free(\{K_4\})$. So, this Lemma holds. □

7 The vertex colouring problem for {*claw*, *butterfly*, C_4}-free graphs

Let $G = (V, E)$ be a $\{claw, butterfly, C_4\}$-free graph. It is known that the set of all maximal cliques of any $\{C_4\}$-free graph can be found in polynomial time [1]. Hence, for each fixed p, the parameter

$$\chi_{G,p} = \max_{Q \text{ is a maximal clique of } G \text{ and } N' \subseteq N(Q), |N'| \leq p} \chi(G[Q \cup N'])$$

can be computed in polynomial time. Indeed, there are at least $|Q| - p$ elements of Q, whose colours do not appear among the colours of N', and therefore

$$\chi(G[Q \cup N']) = \min_{Q' \subseteq Q, |Q'| \geq |Q| - p} (|Q'| + \chi(G[(Q \setminus Q') \cup N'])).$$

There are polynomially many subsets Q' and $\chi(G[(Q \setminus Q') \cup N'])$ can be computed in constant on p time by Lemma 2. Clearly, $\chi(G) \geq \chi_{G,p}$, for any p.

We call *redundant* any vertex of G of degree at most $\chi_{G,3} - 1$. The significance of the redundant vertex notion is that deletion of any such a vertex from G does not change the chromatic number. Hence, by this observation and Lemma 3, the VERTCOL problem for inputs in any hereditary class $\mathcal{X} \subseteq Free(\{claw, butterfly, C_4\})$ can be reduced in polynomial time for irreducible graphs in $\mathcal{X}$ without redundant vertices.

Suppose that $G = (V, E)$ contains a maximal clique $Q = \{v_1, \ldots, v_k\}$ with at least 4 vertices. Let $N(Q) = \{x \notin Q : N_Q(x) \neq \emptyset\}$. We call a vertex $x \in N(Q)$ as a *s-vertex* if $|N_Q(x)| = s$. Since Q is maximal, there is not a k-vertex.

In the next four Lemmas, we will assume that G is an irreducible $\{claw, butterfly, C_4\}$-free graph without redundant vertices.

Lemma 10 *Any two 1-vertices cannot have the same neighbour on Q. If $x_1 \in N(Q)$ and $x_2 \in N(Q)$ are adjacent vertices, then either $N_Q(x_1) \supseteq N_Q(x_2)$ or $N_Q(x_2) \supseteq N_Q(x_1)$ and x_1 is a $(k-1)$-vertex in the first case, x_2 is a $(k-1)$-vertex in the second case. For any $(k-1)$-vertices x_1 and x_2, we have $N_Q(x_1) \neq N_Q(x_2)$.*

Proof If 1-vertices x' and x'' have the same neighbour v_i, then $x'x'' \in E$, otherwise, v_i, x', x'', v_j induce a $claw$, where $v_j \in Q \setminus \{v_i\}$. Then, $v_i, x', x'', v_{j'}, v_{j''}$ induce a $butterfly$, where $v_{j'}, v_{j''} \in Q \setminus \{v_i\}$.

Since G is $\{C_4\}$-free, we have $N_Q(x_1) \supseteq N_Q(x_2)$ or $N_Q(x_2) \supseteq N_Q(x_1)$. Without loss of generality, let us consider the first case. If x_1 is not a $(k-1)$-vertex, then there exist vertices v_i and v_j, each outside $N_Q(x_1)$. Then, v_s, v_i, v_j, x_1, x_2 induce a $butterfly$, where v_s is an arbitrary element of $N_Q(x_2)$.

If x_1 and x_2 are both $(k-1)$-vertices and $N_Q(x_1) = N_Q(x_2)$, then $x_1x_2 \in E$, as any element of $N_Q(x_1)$, any element of $Q \setminus N_Q(x_1)$, x_1, and x_2 induce a $claw$. Then, $N_Q(x_1) \cup \{x_1, x_2\}$ is a clique with $k+1$ vertices. Hence, $\chi_{G,3} \geq k+1$ and $\deg(v_k) \geq k+1$, as v_k is redundant, otherwise. The vertex v_k cannot be simultaneously adjacent to two 1-vertices, by the first paragraph of this proof. Therefore, v_k is adjacent to a vertex $x_3 \in N(Q)$, having at least two neighbours in Q, one of which is simultaneously adjacent to x_1 and to x_2. The vertex x_3 is adjacent to at least one of the vertices x_1 and x_2, to avoid an induced $butterfly$. Hence, G contains an induced C_4. So, this Lemma holds. □

In the next three Lemmas and the remark after them, we will show that Q can be eliminated in polynomial time in the sense that an induced subgraph H of $G \setminus Q$ can be computed in polynomial-time with a polynomial-time computable connection between $\chi(H)$ and $\chi(G)$.

Lemma 11 *If there are $(k-1)$-vertices x' and x'', then $\chi(G) = \max(\chi(G \setminus (Q \cup \{x', x''\})), k)$.*

Proof Without loss of generality and by Lemma 10, $N_Q(x') = \{v_1, \ldots, v_{k-1}\}$ and $N_Q(x'') = \{v_2, \ldots, v_k\}$. As G is $\{C_4\}$-free, we have $x'x'' \notin E$. By this fact, the fact that $|Q| \geq 4$, and the $claw$-freeness of G, the set of $(k-1)$-vertices coincides with $\{x', x''\}$.

If there is a 2-vertex x''', adjacent to v_1 and v_k, then x', x''', v_1, v_2, v_3 induce a subgraph, for which a $butterfly$ and a $claw$ are induced subgraphs, simultaneously.

Let $x \in N(Q)\setminus\{x', x''\}$ be an arbitrary vertex, adjacent to at least one of the vertices $v_2, \ldots, v_{k-1}$, say v_2. To avoid a $claw$, induced by v_2, x, x', x'', either $xx' \in E$ or $xx'' \in E$. If $xx' \in E, xx'' \notin E$, then $xv_k \in E$, to avoid a $butterfly$, induced by v_2, x, x', x'', v_k. Hence, x must be adjacent to all the vertices $v_3, \ldots, v_{k-1}$, as G is $\{C_4\}$-free, which is impossible. Similarly, the case $xx'' \in E, xx' \notin E$ is impossible. If $xx' \in E, xx'' \in E$, then x is adjacent to all the vertices $v_2, \ldots, v_{k-1}$, as G is $\{C_4\}$-free, but $xv_1 \notin E, xv_k \notin E$. Hence, any vertex, having a neighbour in $\{v_2, \ldots, v_{k-1}\}$, must be simultaneously adjacent to $v_2, \ldots, v_{k-1}$ and simultaneously not adjacent to v_1 and to v_k. Therefore, any vertex in $N(Q) \setminus \{x, x', x''\}$ is not adjacent to each of the vertices $v_2, \ldots, v_{k-1}$, otherwise any two elements of $N(Q) \setminus \{x', x''\}$ together with v_1, v_2, v_k induce a subgraph, for which a $butterfly$ and a $claw$ are induced subgraphs, simultaneously.

Let us show that there is not a 1-vertex y, adjacent to v_1 or to v_k, independently on the existence the vertex x. Without loss of generality, let $yv_1 \in E$. Since $\deg(y) \geq 4$ and G is $\{claw, C_4\}$-free, we have $yx' \in E, yx \notin E, yx'' \notin E$ and there are two neighbours y_1 and y_2 of y, both lying outside $Q \cup N(Q)$. Since G is $\{claw\}$-free, we have $y_1y_2 \in E$ and $y_1x' \in E$ or $y_2x' \in E$. Hence, v_2, v_3, x', y, and y_1 or y_2 induce a $butterfly$.

Notice that $\chi(G[Q \cup \{x, x', x''\}]) = k + 1$ and $\chi_{G,3} \geq k + 1$. As v_1 is not redundant, we have $\deg(v_1) \geq k + 1$. Hence, v_1 must have a neighbour 1-vertex. We have a contradiction with the existence of x.

So, $N(Q) = \{x', x''\}$. Since $G \in Free(\{claw, butterfly\})$ and it does not contain separating cliques, there are the unique vertex $y' \in N(x') \setminus Q$ and the unique vertex $y'' \in N(x'') \setminus Q$. Let us consider an optimal colouring c of $G' = G \setminus (Q \cup \{x', x''\})$. Independently on $c(x')$ and $c(x'')$, the colouring c can be extended to a colouring of G in $\max(\chi(G'), k)$ colours. Hence, $\chi(G) = \max(\chi(G'), k)$. So, this Lemma holds. □

Lemma 12 *If there are a s-vertex x with $2 \leq s \leq k-2$ and a $(k-1)$-vertex x', but there are not two $(k-1)$-vertices, then $\chi(G) = \max(\chi(G \setminus (Q \cup \{x, x'\})), k)$.*

Proof Without loss of generality, let $N_Q(x) = \{v_1, \ldots, v_s\}$, where $2 \leq s \leq k-2$. As G is $\{claw, butterfly\}$-free, the vertex x has at most one neighbour outside $Q \cup N(Q)$. As x is not redundant, we have $deg(x) \geq k$. Any neighbour $\tilde{x} \in N(Q)$ of x must be a $(k-1)$-vertex, such that $N_Q(x) \subset N_Q(\tilde{x})$, by Lemma 10. Therefore, we may consider that x has the degree k, i.e. $s = k-2$, x' is its unique neighbour in $N(Q)$, and x'' is its unique neighbour outside $Q \cup N(Q)$. Without loss of generality, let $N_Q(x') = \{v_1, \ldots, v_{k-1}\}$. There is not a vertex $\hat{x} \in N(x') \setminus (\{x''\} \cup Q \cup N(Q))$, as $x', \hat{x}, x, v_{k-1}$ induce a $claw$, otherwise.

Let us show that there is not a vertex $x^* \in N(Q) \setminus \{x, x'\}$, which is not a 1-vertex. Suppose the opposite. The vertex x^* must be a $(k-2)$-vertex, adjacent to a $(k-1)$-vertex, by the previous paragraph. By Lemma 10, we have $N_Q(x') \supset N_Q(x^*)$ and $x^*x' \in E$. If $N_Q(x) = N_Q(x^*)$, then $v_1, x, x^*, v_{k-1}, v_k$ induce a subgraph, for which a $claw$ and a $butterfly$ are induced subgraphs, simultaneously. If $N_Q(x) \neq N_Q(x^*)$, then $N_Q(x^*) = \{v_2, \ldots, v_{k-1}\}$ and either v_2, x, x^*, v_k induce a $claw$, if $xx^* \notin E$, or v_1, x, x^*, v_{k-1} induce a C_4, if $xx^* \in E$.

Clearly, there is not a 1-vertex y, adjacent to v_i, where $1 \le i \le k-2$. Indeed, either v_i, y, x, v_k induce a $claw$, if $yx \notin E$, or v_i, y, x, v_{k-1}, v_k induce a $butterfly$, if $yx \in E$. Suppose that there is a 1-vertex y, adjacent to v_{k-1}. To avoid a $claw$, induced by v_{k-1}, x', y, v_k, we have $yx' \in E$. Since G is $\{C_4\}$-free and by Lemma 10, we have $yx \notin E$ and y cannot be adjacent to the other 1-vertex. As $\deg(y) \ge 4$, because of the irreducibility of G, there are vertices $y_1, y_2 \in N(y) \setminus (Q \cup N(Q))$. Since G is $\{claw\}$-free, $y_1 y_2 \in E$. Then, y, y_1, y_2, x', v_{k-1} induce a $butterfly$.

As v_k is not redundant, v_k has a neighbour 1-vertex x''', which is unique, by Lemma 10. Similar to the reasonings from the end of the proof of Lemma 11, it is easy to show that $x''x \notin E$, $x''x' \notin E$ and $\chi(G) = \max(\chi(G \setminus (Q \cup \{x, x'\})), k)$. So, this Lemma holds. □

Lemma 13 *If there is not a* s*-vertex with* $2 \le s \le k-2$ *and there are not two* $(k-1)$*-vertices, but there is a* $(k-1)$*-vertex* x*, then* $\chi(G) = \max(\chi(G \setminus (Q \cup \{x\})), k-1)$.

Proof Without loss of generality, let $N_Q(x) = \{v_1, v_2, \ldots, v_{k-1}\}$. Similar to the reasonings from the proofs of Lemmas 11 and 12, it is easy to show that none of the 1-vertices is adjacent to a vertex v_i, where $1 \le i \le k-1$. As v_k is not redundant, the vertex v_k has a neighbour 1-vertex y, which is unique by Lemma 10. As G is $\{C_4\}$-free, $xy \notin E$. As x is not redundant, x has a neighbour $x' \notin Q$. Since $G \in Free(\{claw, butterfly\})$, we have $N(x) = N_Q(x) \cup \{x'\}$. Similar to the reasonings from the end of the proof of Lemma 11, it is easy to show that $\chi(G) = \max(\chi(G \setminus (Q \cup \{x\})), k-1)$. So, this Lemma holds. □

The statements of Lemmas 11–13 give three polynomial-time reductions, which will be called *Q-reductions*.

Lemma 14 *The* VERTCOL *problem can be solved in polynomial time for* $\{claw, butterfly, C_4\}$*-free graphs.*

Proof Given a graph, it is possible to check in polynomial time whether it contains a maximal clique with at least 4 vertices or not and determine it, if one exists. If there is not such a clique, then we apply a polynomial-time algorithm, existing by Lemma 9. Let G be a $\{claw, butterfly, C_4\}$-free graph. By the remark from the beginning of this Section, one may assume that G is an irreducible graph without redundant vertices. Recall that the set of its maximal cliques can be computed in polynomial time [1]. By this fact and Lemmas 11–13, we may assume that for any maximal clique Q' of G with $|Q'| \ge 4$ the set $N(Q')$ consists of 1-vertices only, otherwise, some Q'-reduction can be applied. By Lemma 10 and the fact that G does not contain separating cliques, $N(Q')$ consists of $|Q'|$ elements, each adjacent to its own vertex of Q'. Hence, $N(Q')$ is an independent set, as G is $\{C_4\}$-free.

Let $Q = \{v_1, v_2, v_3, \ldots, v_k\}$ be a maximal clique of G, where $k \ge 4$. We have $N(Q) = \{u_1, \ldots, u_k\}$, where $v_i u_i \in E$, for any i. As G is $\{claw\}$-free and both vertices v_i and u_i are not redundant, the set $(N(u_i) \setminus \{v_i\}) \cup \{u_i\}$ is a clique Q_i of the size k, for any i. The set $N(Q_i)$ consists of 1-vertices, each adjacent to its own vertex of Q_i, for any i. Applied the reasonings above, we conclude that G consists of k-cliques with at most one edge between any two of them.

Let us show that $\chi(G) = k$, by using the induction on the number $q(G)$ of k-cliques. Its base is $q(G) = k + 1$. Hence, G consists of the cliques $Q, Q_1, \ldots, Q_k$ only and there is only one edge between any two of them. To prove that $\chi(G) = k$, we also use the mathematical induction method, applied to k. For the base $k = 2$, we have that G is isomorphic to a C_6 and $\chi(G) = 2$. Suppose that $k > 2$. Let us show that there is an independent set $I = \{a, a_1, \ldots, a_k\}$, such that $a \in Q$ and $a_i \in Q_i$, for any i. Let a be an arbitrary vertex from Q and a_1 be an arbitrary vertex from $Q_1 \setminus \{a'\}$, where a' is a neighbour of a in Q_1. Determined $a, a_1, \ldots, a_{i-1}$, where $i \geq 2$, exactly $k - i + 1$ vertices of Q_i have not a neighbour in $\{a, a_1, \ldots, a_{i-1}\}$, as there is only one edge between any two of the cliques $Q, Q_1, \ldots, Q_k$. Hence, a_i exists, for any i. The graph $G \setminus I$ consists of the $(k-1)$-cliques $Q \setminus \{a\}, Q_1 \setminus \{a_1\}, \ldots, Q_k \setminus \{a_k\}$, and there is only one edge between any two of them. Clearly, $\chi(G \setminus I) + 1 \geq \chi(G) \geq k$. By the induction assumption, we have $\chi(G \setminus I) = k - 1$. Thus, $\chi(G) = k$.

Suppose that $q(G) > k + 1$. Then, for some k-clique Q, there are i and j, such that there is not an edge between Q_i and Q_j. Let G' be the result of deleting Q from G and then adding the edge $u_i u_j$. The graph G' is $\{claw, butterfly, C_4\}$-free, and it also consists of k-cliques with at most one edge between any two of them. Clearly that $q(G') = q(G) - 1 < q(G)$. Applying the induction assumption to G', we have $\chi(G') = k$. Any k-colouring of G' arranges distinct colours for u_i and u_j. Hence, by Lemma 7, $\chi(G) = \max(\chi(G'), k) = k$. So, this Lemma holds. □

8 Main result

The following statement is the main result of the present paper:

Theorem 1 *The* VERTCOL *problem can be solved in polynomial time for* $\{claw, butterfly\}$*-free graphs.*

Proof By Lemma 8, we may consider $\{claw, butterfly, C_4\}$-free graphs only. By Lemma 14, the VERTCOL problem can be solved in polynomial time for $\{claw, butterfly, C_4\}$-free graphs. Therefore, $Free(\{claw, butterfly\})$ is a polynomial-time case for the VERTCOL problem. So, this Theorem is true. □

9 Conclusions and future work

The vertex colourability problem was considered in this paper for hereditary graph classes, defined by a pair of connected 5-vertex forbidden induced subgraphs. Its computational status has been resolved for all such pairs, except for four of them. Here, we have determined it for one class of the remaining four open cases. Namely, we have proved that the problem is polynomial-time solvable for $\{claw, butterfly\}$-free graphs. Clarifying the problem's complexity status for the other three classes is a challenging research problem for future work.

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References

1. Balas, E., Yu, C.S.: On graphs with polynomially solvable maximum weight clique problem. Networks **19**, 247–253 (1989)
2. Blanché, A., Dabrowski, K., Johnson, M., Paulusma, D.: Hereditary graph classes: When the complexities of coloring and clique cover coincide. J. Graph Theory **91**, 267–289 (2019)
3. Bonomo, F., Chudnovsky, M., Maceli, P., Schaudt, O., Stein, M., Zhong, M.: Three-coloring and list three-coloring of graphs without induced paths on seven vertices. Combinatorica **38**, 779–801 (2018)
4. Broersma, H.J., Golovach, P.A., Paulusma, D., Song, J.: Updating the complexity status of coloring graphs without a fixed induced linear forest. Theor. Comput. Sci. **414**, 9–19 (2012)
5. Brooks, R.L.: On colouring the nodes of a network. Proc. Camb. Philos. Soc. Math. Phys. Sci. **37**, 194–197 (1941)
6. Cameron, K., Huang, S., Penev, I., Sivaraman, V.: The class of (P_7, C_4, C_5)-free graphs: decomposition, algorithms, and χ-boundedness. J. Graph Theory **93**, 503–552 (2020)
7. Cameron, K., da Silva, M., Huang, S., Vuskovic, K.: Structure and algorithms for $(cap, even\ hole)$-free graphs. Discrete Math. **341**, 463–473 (2018)
8. Dabrowski, K., Dross, F., Paulusma, D.: Colouring diamond-free graphs. J. Comput. Syst. Sci. **89**, 410–431 (2017)
9. Dabrowski, K., Paulusma, D.: On colouring $(2P_2, H)$-free and (P_5, H)-free graphs. Inf. Process. Lett. **134**, 35–41 (2018)
10. Dai, Y., Foley, A., Hoàng, C.: On coloring a class of claw-free graphs: to the memory of Frédéric Maffray. Electron. Not. Theor. Comput. Sci. **346**, 369–377 (2019)
11. Foley, A., Fraser, D., Hoàng, C., Holmes, K., LaMantia, T.: The intersection of two vertex coloring problems. Graphs Comb. **36**, 125–138 (2020)
12. Fraser, D., Hamela, A., Hoàng, C., Holmes, K., LaMantia, T.: Characterizations of $(4K_1, C_4, C_5)$-free graphs. Discrete Appl. Math. **231**, 166–174 (2017)
13. Garey, M.R., Johnson, D.S.: Computers and Intractability: A Guide to the Theory of NP-Completeness. W. H. Freeman and Co., New York (1979)
14. Gaspers, S., Huang, S., Paulusma, D.: Colouring square-free graphs without long induced paths. J. Comput. Syst. Sci. **106**, 60–79 (2019)
15. Golovach, P., Johnson, M., Paulusma, D., Song, J.: A survey on the computational complexity of coloring graphs with forbidden subgraphs. J. Graph Theory **84**, 331–363 (2017)
16. Golovach, P.A., Paulusma, D., Song, J.: 4-coloring H-free graphs when H is small. Discrete Appl. Math. **161**, 140–150 (2013)
17. Hell, P., Huang, S.: Complexity of coloring graphs without paths and cycles. Discrete Appl. Math. **216**, 211–232 (2017)
18. Hoàng, C., Kamiński, M., Lozin, V.V., Sawada, J., Shu, X.: k-colorability of P_5-free graphs in polynomial time. Algorithmica **57**, 74–81 (2010)
19. Hoàng, C., Lazzarato, D.: Polynomial-time algorithms for minimum weighted colorings of $(P_5, \overline{P_5})$-free graphs and similar graph classes. Discrete Appl. Math. **186**, 106–111 (2015)
20. Huang, S.: Improved complexity results on k-coloring P_t-free graphs. Eur. J. Comb. **51**, 336–346 (2016)
21. Kaminski, M., Lozin, V.: Vertex 3-colorability of claw-free graphs. Algorithm. Oper. Res. **2**, 15–21 (2007)
22. Karthick, T., Maffray, F., Pastor, L.: Polynomial cases for the vertex coloring problem. Algorithmica **81**, 1053–1074 (2017)
23. Král', D., Kratochvíl, J., Tuza, Z., Woeginger, G.: Complexity of coloring graphs without forbidden induced subgraphs. Lect. Not. Comput. Sci. **2204**, 254–262 (2001)
24. Klimosová, T., Malík, J., Masarík, T., Novotná, J., Paulusma, D., Slívová, V.: Colouring $(P_r + P_s)$-free graphs. Algorithmica **82**, 1833–1858 (2020)
25. Lozin, V., Malyshev, D.: Vertex coloring of graphs with few obstructions. Discrete Appl. Math. **216**, 273–280 (2017)

26. Malyshev, D.: The coloring problem for classes with two small obstructions. Opt. Lett. **8**, 2261–2270 (2014)
27. Malyshev, D.: The complexity of the 3-colorability problem in the absence of a pair of small forbidden induced subgraphs. Discrete Math. **338**, 1860–1865 (2015)
28. Malyshev, D.: Two cases of polynomial-time solvability for the coloring problem. J. Comb. Opt. **31**, 833–845 (2016)
29. Malyshev, D.: The complexity of the vertex 3-colorability problem for some hereditary classes defined by 5-vertex forbidden induced subgraphs. Graphs Comb. **33**, 1009–1022 (2017)
30. Malyshev, D.: The weighted coloring problem for two graph classes characterized by small forbidden induced structures. Discrete Appl. Math. **247**, 423–432 (2018)
31. Malyshev, D., Lobanova, O.: Two complexity results for the vertex coloring problem. Discrete Appl. Math. **219**, 158–166 (2017)
32. Malyshev, D.: A complete classification of the complexity of the vertex 3-colourability problem for quadruples of induced 5-vertex prohibitions. Zhurnal Srednevolzhskogo matematicheskogo obshchestva **22**, 38–47 (2020). (in Russian)
33. Sirotkin, D., Malyshev, D.: On the complexity of the vertex 3-coloring problem for the hereditary graph classes with forbidden subgraphs of small size. J. Appl. Ind. Math. **25**, 759–769 (2018)
34. Spirkl, S., Chudnovsky, M., Zhong, M.: Four-coloring of P_6-free graphs. Simp. Discrete Algorithims **1239–1256**, (2019)
35. Tarjan, R.: Decomposition by clique separators. Discrete Math. **55**, 221–232 (1985)