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Complete Complexity Dichotomy for 7-Edge Forbidden Subgraphs in the Edge Coloring Problem

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Abstract—The edge coloring problem for a graph is to minimize the number of colors that are sufficient to color all edges of the graph so that all adjacent edges receive distinct colors. The computational complexity of the problem is known for all graph classes defined by forbidden subgraphs with at most 6 edges. We improve this result and obtain a complete complexity classification of the edge coloring problem for all sets of prohibitions each of which has at most 7 edges.

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INTRODUCTION

We consider only *simple graphs*, i.e., the unlabeled undirected graphs without loops and multiple edges. Recall that a *subgraph of a graph* is the result of removing some vertices and edges from the graph, where *removing a vertex* means removing all incident edges. An *induced subgraph* is the result of removing some vertices from a graph.

A *graph class* is an arbitrary set of graphs which is closed under isomorphism. A graph class closed under vertex removal is called *hereditary*. A *strongly hereditary* (or *monotone*) graph class is a hereditary class closed also under vertex removal. Every hereditary graph class can be defined by means of its *forbidden induced subgraphs*, i.e., the graphs minimal under removal of vertices not belonging to them. If $\mathcal{X}$ is a hereditary class and $\mathcal{Y}$ is the set of the forbidden induced fragments of $\mathcal{X}$ then we write $\mathcal{X} = \mathrm{Free}(\mathcal{Y})$. Every monotone class $\mathcal{X}$ can be defined by the family of its forbidden subgraphs $\mathcal{Y}$; and we write $\mathcal{X} = \mathrm{Free}_s(\mathcal{Y})$.

A *proper coloring of the vertices of a graph G in k colors* (or simply a *k-coloring of the vertices of G*) is an arbitrary mapping from $V(G)$ into $\{1, 2, \dots, k\}$ assigning different colors to adjacent vertices. A *proper coloring of edges of a graph G in k colors* (or, briefly, a *k-coloring of the edges of G*) is a mapping from $E(G)$ into $\{1, 2, \dots, k\}$ assigning different colors to adjacent edges. The minimal numbers of colors in k-colorings of vertices and edges of a graph G are called the *chromatic number* and the *index* of G and are denoted by $\chi(G)$ and $\chi'(G)$ respectively.

The *vertex k-coloring problem* or simply the k-VC *Problem* (respectively, the *edge k-coloring problem* or the k-EC *Problem*) for a given graph G consists in recognizing the presence of a vertex (edge) k-coloring for this graph. The *vertex* and *edge coloring problems* (briefly, *Problems* VC and EC) for some given graph G and number k consist in checking whether $\chi(G) \le k$ and $\chi'(G) \le k$. Problems VC and EC and also Problems k-VC and k-EC for $k \ge 3$ are the classical NP-complete problems (see [1, 2]).

By the Vizing Theorem [3], each graph G satisfies the inequality

$$\Delta(G) \le \chi'(G) \le \Delta(G) + 1,$$

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where $\Delta(G)$ is the maximum vertex degree in G. Thus, Problem EC for every graph G is equivalent to the equality $\chi'(G) = \Delta(G)$.

Let P_n designate a simple path on n vertices. It is known that Problem VC is polynomially solvable for the class $\mathrm{Free}(\{H\})$ if H is an induced subgraph of P_4 or of $P_3 + P_1$ (the disjoint union of P_3 and P_1); otherwise, Problem VC is NP-complete in this class (see [4]). However, it is no longer possible to obtain the complete classifications of the complexity of Problem VC as under the prohibition of a pair of induced fragments as under the prohibition of small induced structures. For example, for all but three hereditary classes defined by prohibitions with at most 4 vertices each, the computartional status of Problem VC is known (see [5]). In [6–11], the algorithmic complexity was considered of Problem VC for pairs of connected forbidden induced fragments, each on 5 vertices, and at present there are exactly 4 open cases here. Some results on the complexity of Problem VC for the hereditary classes defined by prohibitions of small size are presented in [12–17].

For Problem k-VC, the complexity status remains an open question even for some classes defined by one forbidden induced subgraph. In [18, 19], the complete complexity dichotomies were obtained for $k = 3$ and the family $\{\mathrm{Free}(\{H\}) \mid |V(H)| \leq 6\}$ and also for $k = 4$ and the family $\{\mathrm{Free}(\{H\}) \mid |V(H)| \leq 5\}$. Problem 3-VC is polynomially solvable in the class $\mathrm{Free}(\{P_7\})$ [20], whereas Problem 4-VC is polynomially solvable in the class $\mathrm{Free}(\{P_6\})$ [21]. For each k, Problem k-VC is polynomially solvable in the class $\mathrm{Free}(\{P_5\})$ [22]. For each fixed $k \geq 5$, Problem k-VC is NP-complete in the class $\mathrm{Free}(\{P_6\})$ [23], whereas Problem 4-VC is NP-complete in the class $\mathrm{Free}(\{P_7\})$ [23]. At present, the complexity status of Problem k-VC is an open question for the class $\mathrm{Free}(\{P_8\})$ and $k = 3$, and also for the class $\mathrm{Free}(\{P_7\})$ and $k = 4$.

The articles [24–26] deal with Problem 3-VC. In [24], for Problem 3-VC, the complete complexity dichotomy was obtained in the family

$$\{\mathrm{Free}(\{H_1, H_2\}) \mid \max(|V(H_1)|, |V(H_2)|) \leq 5\}.$$

In [25], the analogous result was obtained for the family

$$\big\{\mathrm{Free}(\{H_1, H_2, H_3\}) \mid \max\big(|V(H_1)|,\ |V(H_2)|,\ |V(H_3)|\big) \leq 5\big\}.$$

In [26], the quadruples of forbidden 5-vertex subgraphs were considered; and, for all given hereditary classes but three, the computational status of Problem 3-VC was established.

There are rarer works classifying the computational complexity of Problems k-EC and EC. For example, in [27], for every k, a complexity dichotomy was obtained for Problem k-EC and all classes of the form $\mathrm{Free}(\{H\})$, and in [28], a complete classification of algorithmic complexity of Problem 3-EC was deduced for all sets of forbidden induced structures, each with at most 6 vertices, among which at most two have exactly 6 vertices. In [29], Problem EC and the family

$$\{\mathrm{Free}_s(\mathcal{Y}) \mid \text{for every } G \in \mathcal{Y} \text{ either } |V(G)| \leq 7 \text{ or } |E(G)| \leq 6\}$$

were under consideration and a complete complexity classification was obtained.

In the present article, we improve the result of [29] and establish the computational complexity of Problem EC for all classes of forbidden subgraphs each of which has at most 7 edges.

1. NOTATIONS

Let G be some graph and let x be a vertex in G. The neighborhood of x is denoted by $N_G(x)$ and $\deg_G(x)$ stands for the degree of x. The maximum degree of the vertices of G is denoted by $\Delta(G)$. If $\Delta(G) \leq 3$ then G is called *subcubic*. If the degrees of all vertices in G equal 3 then G is called *cubic*.

As usual, by P_n and O_n we denote a simple path and an empty graph on n vertices respectively. The graphs B_1, B_1^+, B_1^{++}, ${}^+B_1^+$, B_{1+}^+, B_2, $\overset{+}{B}_2$, B_2^+, and B_3 are depicted in Fig. 1.

Let G_1 and G_2 be graphs with disjoint vertex sets. Then $G_1 + G_2$ stands for the disjoint union of G_1 and G_2, i.e., the graph $(V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$. Given a graph G and a number k, the notation kG stands for the graph

$$\underbrace{G + G + \dots + G}_{k \text{ times}}.$$

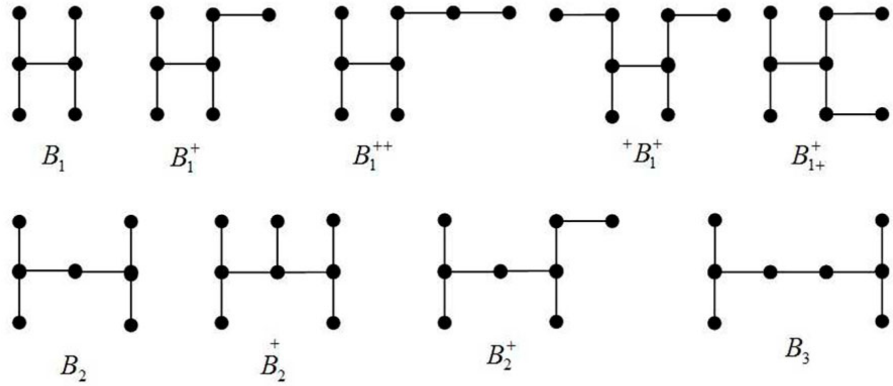

Fig. 1.

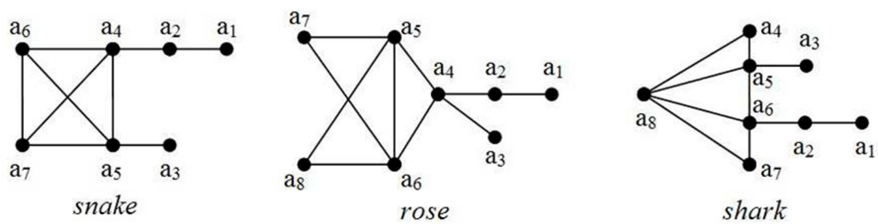

Fig. 2. The graphs snake, rose, and shark.

Denote by $\mathcal{S}$ the set

$$\{B_1 + P_3,\ B_1 + 2P_2,\ B_1^+ + P_2,\ B_1^{++},\ {}^+B_1^+,\ B_{1+}^+,\ B_2 + P_2,\ ,B_2^+,\ B_3\}.$$

The graphs *snake*, *rose*, and *shark* are depicted in Fig. 2.

Denote by $T_{i,j,k}$ the tree called a *triode* that is obtained by simultaneous identification along the vertex v of the ends of three simple paths $(v = x_0, x_1, \dots, x_i)$, $(v = y_0, y_1, \dots, y_j)$, and $(v = z_0, z_1, \dots, z_k)$ (Fig. 3).

Note that the parameters i, j, and k can take zero values. The symbol $\mathcal{T}$ stands for the class of all forests whose each connected component is a triode.

Let G be a graph and let $V' \subseteq V(G)$. Then $G[V']$ is the subgraph of G induced by the set of vertices V' and $G \setminus V'$ is the result of removing all elements of V' from G. Given a graph G and its edge ab, the result of removing ab from G is denoted by $G \setminus \{ab\}$.

Let $G_1, G_2, \dots, G_k$ be graphs, and let $v_1, v_2, \dots, v_s$ be some vertices of the graph G under consideration. Then the notation $[v_1, v_2, \dots, v_s; G_1, G_2, \dots, G_k]$ means that $G[\{v_1, v_2, \dots, v_s\}]$ includes each of the graphs $G_1, G_2, \dots, G_k$ as a subgraph and $G_1 \cong G_2$ means the isomorphism of G_1 and G_2.

An *independent set* of a graph is an arbitrary subset of its pairwise nonadjacent vertices.

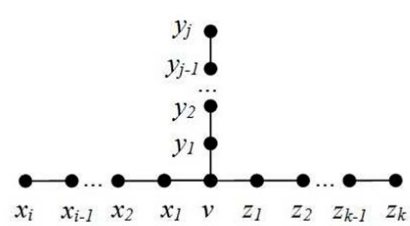

Fig. 3. Graph $T_{i,j,k}$.

2. SOME AUXILIARY RESULTS

2.1. The Structural Properties of Incompressible Graphs without Subgraphs of a Special Kind

It is known (see [30, p. 465]) that a graph G containing a vertex x whose at most one neighbor has degree $\Delta(G)$, has an edge coloring with $\Delta(G)$ colors if and only if the graph $G \setminus \{x\}$ has.

Recall that a *bridge* is an arbitrary edge in a graph whose removal increases the number of its connected components. Obviously, for every graph G and the result G' of removing some bridge from G, the following holds: G has an edge coloring with $\Delta(G)$ colors if and only if G' admits an edge coloring with $\Delta(G')$ colors.

Call an edge ab in a graph G *redundant* if

$$\deg_G(a) + \deg_G(b) \leq \Delta(G) + 1.$$

If ab is a redundant edge of G then

$$\chi'(G \setminus \{ab\}) \leq \Delta(G) \ \Leftrightarrow\ \chi'(G) = \Delta(G).$$

Call a connected graph G without bridges and redundant edges *incompressible* if each vertex in G has at least two neighbors of degree $\Delta(G)$. For the graphs of an arbitrary monotone class, Problem EC is polynomially reduced to the same problem for the incompressible graphs of this monotone class.

A graph will be called *k-close to an empty* graph if it becomes empty after removing some k of its vertices.

Lemma 1. *Let $H \in (\mathcal{S} \setminus \{B_{1+}^{+}\}) \cup \{\overset{+}{B}_2\}$, and let $G \in \mathrm{Free}_s(\{H\})$ be an incompressible graph. Then either $\Delta(G) \leq 10$ or G is 3-close to an empty graph.*

Proof. Suppose that $\Delta(G) \geq 11$. In G, consider a vertex x of degree $\Delta(G)$. Since G is incompressible, among the neighbors of x there exist distinct vertices y and z such that $\deg_G(y) = \deg_G(z) = \Delta(G)$. Consequently,

$$|N_G(x) \setminus \{y,z\}| \geq \Delta(G) - 2, \qquad |N_G(y) \setminus \{x,z\}| \geq \Delta(G) - 2,$$
$$|N_G(z) \setminus \{x,y\}| \geq \Delta(G) - 2.$$

Redenote $(N_G(x) \cup N_G(y) \cup N_G(z)) \setminus \{x,y,z\}$ by N. Suppose that N contains a vertex v adjacent to $u \notin \{x,y,z\}$.

Consider the case of $v \in N_G(y)$. The case of $v \in N_G(z)$ is similar. Since $\Delta(G) \geq 11$, there exist pairwise distinct vertices

$$v_1, v_2 \in (N_G(y) \cap N) \setminus \{v,u\}, \qquad v_3, v_4 \in (N_G(x) \cap N) \setminus \{v,u\},$$
$$v_5, v_6 \in (N_G(z) \cap N) \setminus \{v,u\}.$$

Thus, G contains the subgraph $\overset{+}{B}_2$. It is not hard to check that the vertices x, y, z, v, u, and v_1–v_6 induce in G a graph for which each element of the set $\mathcal{S} \setminus \{B_{1+}^{+}, B_3\}$ is a subgraph. The same also holds for B_3 if $vx \in E(G)$ or $vz \in E(G)$; therefore, we may assume that $vx \notin E(G)$ and $vz \notin E(G)$. Since G is incompressible, v is adjacent to a vertex $w \notin \{x,y,z\}$ of degree $\Delta(G)$. Since $\Delta(G) \geq 11$, there exist vertices $w_1, w_2 \in N_G(w) \setminus \{v\}$ each of which is different from v_3 and v_4. Hence, $[w_1, w_2, w, v, y, x, v_3, v_4; B_3]$.

Consider the case of $v \in N_G(x)$. In view of the arguments of the previous paragraph, we may assume that none of the elements of $N \setminus N_G(x)$ is adjacent to a vertex outside $\{y,z\}$. Since

$$\deg_G(x) = \deg_G(y) = \Delta(G),$$

either $N \cap (N_G(y) \setminus N_G(x)) \neq \varnothing$ or $N \cap N_G(y) = N \cap N_G(x)$ and $yz \in E$. But if

$$N \cap (N_G(y) \setminus N_G(x)) \neq \varnothing$$

then each of its elements is adjacent to at least two vertices of degree $\Delta(G)$. Consequently, each element of this set is adjacent to z.

Since $\Delta(G) \geq 11$, there exist pairwise distinct vertices

$$v_1, v_2, v_3 \in (N_G(y) \cap N) \setminus \{v, u\}, \qquad v_4, v_5 \in (N_G(x) \cap N) \setminus \{v, u\},$$
$$v_6, v_7 \in (N_G(z) \cap N) \setminus \{v, u\}.$$

Thus, G contains the subgraph $\overset{+}{B}_2$. We may assume that either v_1x, v_2x, v_3x, $yz \in E$ or v_1z, v_2z, $v_3z \in E$. It is easy to check that the vertices x, y, z, v, u, and v_1–v_7 induce some subgraph of G for which each element of $\mathcal{S}$ is a subgraph. Therefore, N contains no adjacent vertices and

$$V(G) = N \cup \{x, y, z\}.$$

Thus, G is 3-close to an empty graph.

Lemma 1 is proved. □

Lemma 2. *Suppose that an incompressible graph G is k-close to an empty graph. Then either G has at most $k^3 + 3k^2$ vertices or $\chi'(G) = \Delta(G)$.*

Proof. Assume that $V(G)$ is partitioned into an independent set I and a subset $|V'| \leq k$. We may assume that each element in V' is adjacent to some element in I since otherwise we can remove the former element from V' and adjoin it to I. Since G is connected and the set I is independent, each vertex in I is adjacent to some vertex in V'. Thus, $|V(G)| \leq |V'|(\Delta(G) + 1)$; therefore, if $\Delta(G) \leq k^2 + 3k - 1$ then $|V(G)| \leq k^3 + 3k^2$. Hence, we may assume that $\Delta(G) \geq k^2 + 3k$. Since G contains no redundant edges, we may assume that each vertex in V' has at least $(\Delta(G) - 2|V'| + 3)$ neighbors in I.

Let $V' = \{v_1, v_2, \dots, v_{k'}\}$, where $k' \leq k$. Form subgraphs H_1 and H_2 of G. Recall that each vertex in V' has at least $(k^2 + k + 3)$ neighbors in I and each vertex in I has at most k neighbors in V'. Consequently, there exist pairwise distinct vertices $u_1, u_2, \dots, u_{k'} \in I$ such that $u_i \in N(v_i)$ for each i. Note that for each i the vertex v_i has exactly $(\deg_G(v_i) - \deg_{G[V']}(v_i))$ neighbors in I. Owing to the fact that

$$|I \setminus \{u_1, u_2, \dots, u_{k'}\}| > k^2 \quad \text{and} \quad k' - \deg_{G[V']}(v_i) \leq k,$$

there exist pairwise distinct subsets $V_1^1, \dots, V_{k'}^1$ such that, for each $i = 1, \dots, k'$, the set V_i^1 consists of $(k' - \deg_{G[V']}(v_i))$ neighbors of v_i belonging to $I \setminus \{u_1, u_2, \dots, u_{k'}\}$. For each i the set V_i^2 coincides with $V_i^1 \cup \{u_i\}$.

For each j the graph H_j is obtained by adding to $G[V']$ all vertices of $\bigcup_{i=1}^{k'} V_i^j$ and all edges incident to v_i and the vertices of V_i^j, $i = 1, \dots, k'$. It is not hard to see that

$$\Delta(H_j) = \deg_{H_j}(v_i) = k' + j - 1$$

for all i and j.

If $\chi'(H_1) = \Delta(H_1) = k'$ then $\chi'(H_2) = k' + 1$ since $\chi'(H_2) \geq k' + 1$ and a $(k' + 1)$-coloring of the edges of H_2 can be obtained from a k'-coloring of the edges of H_1 by coloring the edges $v_1u_1, v_2u_2, \dots, v_{k'}u_{k'}$ with color $k' + 1$. If $\chi'(H_1) = \Delta(H_1) = k' + 1$ then $\chi'(H_2) = k' + 1$. Indeed, $\chi'(H_2) \geq k' + 1$, and, in a $(k' + 1)$-coloring of the edges of H_1, for each i, there exists a color c_i such that an edge of color c_i does not occur among the edges incident to v_i. Color the edge v_iu_i with color c_i for each i and obtain a $(k' + 1)$-coloring of the edges of H_2.

By the König Theorem (see, for example, [31]), the chromatic index of every bipartite graph is equal to its maximum degree. The graph $G' = (V(G), E(G) \setminus E(H_2))$ is bipartite; moreover,

$$\Delta(G') = \Delta(G) - (k' + 1).$$

Therefore, $\chi'(G') = \Delta(G) - (k' + 1)$.

Color the edges of G' with $(\Delta(G) - k' - 1)$ colors and then color the edges of H_2 with other $(k' + 1)$ colors, and so obtain a coloring of the edges of G with $\Delta(G)$ colors. Consequently, $\chi'(G) = \Delta(G)$.

Lemma 2 is proved. □

Let G be an incompressible graph. Refer as a Δ-*component* to every connected component of the subgraph of G induced by the subset of vertices of degree $\Delta(G)$. If G contains exactly one Δ-component CO then $|V(G)| \le |V(CO)|\,(\Delta(G)+1)$ because each vertex of G is adjacent to at least one vertex of CO.

Suppose that $\Delta(G) \ge 4$. Then the degree of every vertex of each Δ-component is at least 2; therefore, each of the Δ-components contains at least 3 vertices. For the same reasons, if there are at least two Δ-components then there exists an induced path $P = (x, v_1, \dots, v_k, y)$, where $k \in \{1, 2\}$, between some vertices $x \in V(CO_1)$ and $y \in V(CO_2)$ from different Δ-components CO_1 and CO_2 in which all vertices $v_1, \dots, v_k$ belong to no Δ-components. We may assume that if $k = 2$ then there is no path of length 2 between the vertices of these Δ-components. Consequently, if $k = 2$ then v_1 is adjacent to no vertex in CO_2 and v_2 is adjacent to no vertex in CO_1.

The vertex x has two neighbors x' and x'' in CO_1, and the vertex y has two neighbors y' and y'' in CO_2. There exist vertices $x'_1, x'_2, x'_3 \in N_G(x') \setminus \{x\}$, where x'_1 and x'_2 are different from x''; and also vertices $x''_1, x''_2, x''_3 \in N_G(x'') \setminus \{x\}$, where x''_1 and x''_2 are different from x'. Since CO_1 and CO_2 are different Δ-components, there is no edge incident to a vertex in CO_1 and a vertex in CO_2. In particular,

$$x'y' \notin E(G), \qquad x'y'' \notin E(G), \qquad x''y' \notin E(G), \qquad x''y'' \notin E(G).$$

These observations and notations will be used in the following two lemmas:

Lemma 3. *Let $H \in \mathcal{S} \setminus \{B^+_{1^+}, {}^+B^+_1\}$, let $G \in \mathrm{Free}_s(\{H\})$ be an incompressible graph, and let $\Delta(G) \ge 4$. Then G has at most*

$$\max\left(7 + 7\,\frac{\Delta^4(G)-1}{\Delta(G)-1},\ \frac{\Delta^4(G)-1}{\Delta(G)-1}\,(\Delta(G)+1)\right)$$

vertices.

Proof. Suppose that

$$H \in \{B_1 + P_3,\ B_1 + 2P_2,\ B^+_1 + P_2,\ B_2 + P_2\}.$$

Then H is representable as $H = H_1 + H_2$, where $|E(H_1)| > |E(H_2)|$. If G contains a subgraph H_1 then contract H_1 to a vertex x and obtain the graph G'. Then, obviously, $\deg_{G'}(x) \le 7\,\Delta(G)$ and the degree of each other vertex in G' is at most $\Delta(G)$. Since $G \in \mathrm{Free}_s(\{H\})$, the graph G' either contains no subgraph P_3 or contains no subgraph $2P_2$; therefore, the distance from x to every other vertex in G' is at most 3. Hence,

$$|V(G')| \le 1 + 7\left(\Delta(G) + \Delta^2(G) + \Delta^3(G)\right) = 1 + 7\,\frac{\Delta^4(G)-1}{\Delta(G)-1}.$$

At the same time, $|V(G)| \le |V(G')| + 6$. If $G \in \mathrm{Free}_s(\{H_1\})$ then

$$G \in \mathrm{Free}_s(\{B^{++}_1\}) \cup \mathrm{Free}_s(\{B^+_2\});$$

these cases will be examined below. Thus, we may assume that $H \in \{B^{++}_1, B^+_2, B_3\}$.

Suppose that $P = (x, v_1, y)$. Then $[x'_1, x'_2, x'_3, x', x, v_1, y, y', y''; B_3]$ since at most two vertices among x'_1, x'_2, and x'_3 differ from v_1; and $[x'_i, x', x'', x, v_1, y, y', y''; B^+_2]$ for a vertex $x'_i \notin \{v_1, x''\}$ that exists because $x'' \notin \{x'_1, x'_2\}$ and one of the vertices x'_1 and x'_2 differs from v_1. If $x'' \notin \{x'_1, x'_2, x'_3\}$ or $v_1 \notin \{x'_1, x'_2, x'_3\}$ then $[x'_1, x'_2, x'_3, x, x'', v_1, y, y'; B^{++}_1]$ since at least two of the vertices x'_1, x'_2, and x'_3 simultaneously differ from v_1 and x''. If $x'' \in \{x'_1, x'_2, x'_3\}$ and $v_1 \in \{x'_1, x'_2, x'_3\}$ then $[y', y'', y, v_1, x', x, x'', x''_i; B^{++}_1]$ since there exists a vertex $x''_i \notin \{v_1, x'\}$.

Suppose that $P = (x, v_1, v_2, y)$. Obviously, $[x', x'', x, v_1, v_2, y, y', y''; B_3]$. If $x'' \notin \{x'_1, x'_2, x'_3\}$ or $v_1 \notin \{x'_1, x'_2, x'_3\}$ then $[x'_1, x'_2, x'_3, x, x'', v_1, v_2, y; B^{++}_1]$ since at least two of the vertices x'_1, x'_2, and x'_3 simultaneously differ from v_1 and x''. If $x'' \in \{x'_1, x'_2, x'_3\}$ and $v_1 \in \{x'_1, x'_2, x'_3\}$ then

$$[x'_i, x', x'', v_1, x, v_2, y, y''; B^{++}_1]$$

because there exists $x_i' \notin \{v_1, x''\}$. Since G is incompressible, either $\deg_G(v_1) \geq 3$ or $\deg_G(v_2) \geq 3$. The vertices x and y have equal rights; therefore, we may assume that there exists a vertex v^* adjacent to v_2 and not belonging to $V(CO_1) \cup \{v_1, y\}$. Then $[v^*, v_2, y, v_1, x, x', x_i', x''; B_2^+]$ since there exists a vertex $x_i' \notin \{v_1, x''\}$.

Thus, G has only one Δ-component CO. Suppose that in CO, between some vertices u_1 and u_5, there exists an induced path $(u_1, u_2, u_3, u_4, u_5)$. Then $G[N_G(u_1) \cup N_G(u_2) \cup \{u_4, u_5\}]$ contains the subgraph B_1^{++}, $G[N_G(u_1) \cup N_G(u_3) \cup \{u_5\}]$ contains B_2^+, and $G[N_G(u_1) \cup N_G(u_4)]$ contains B_3. Thus, the distance in CO between every two vertices is at most 3, and so

$$|V(CO)| \leq 1 + \Delta(G) + \Delta^2(G) + \Delta^3(G) = \frac{\Delta^4(G) - 1}{\Delta(G) - 1}.$$

Hence,

$$|V(G)| \leq \frac{\Delta^4(G) - 1}{\Delta(G) - 1}\,(\Delta(G) + 1).$$

Lemma 3 is proved. □

Lemma 4. *Suppose that $G \in \mathrm{Free}_s(\{\overset{+}{B}_2\})$ is an incompressible graph and $\Delta(G) \geq 4$. Then either G has at most*

$$\frac{\Delta^5(G) - 1}{\Delta(G) - 1}\,(\Delta(G) + 1)$$

vertices or $\Delta(G) = 4$ and G contains one of the subgraphs snake, rose, *and* shark *for which* $\deg_G(a_2) = 2$.

Proof. Suppose that G contains at least two Δ-components. Let v' denote the third vertex of the path P counting from the vertex x. Prove that $\deg_G(v_1) = 2$. Suppose the contrary.

Owing to the equity of x and y, we may assume that either v_1 is adjacent to no vertex in $N_G(x)$ or v_1 has a neighbor in $N_G(x)$ and v_2 has a neighbor in $N_G(y)$ (when $k = 2$) or v_1 has neighbors both in $N_G(x)$ and in $N_G(y)$ (when $k = 1$). If the second case is realized then

$$G[N_G(x) \cup N_G(v_1) \cup N_G(v_2)]$$

contains the subgraph $\overset{+}{B}_2$; therefore, further we assume only the first and third cases.

Suppose that there exists a vertex $v^* \in N_G(v_1) \setminus (N_G(x) \cup \{x, v'\})$, which embraces the third case. If $k = 1$ then $G[N_G(x) \cup N_G(v_1) \cup N_G(y)]$ contains $\overset{+}{B}_2$. If $\Delta(G) \geq 6$ or $\Delta(G) = 5$ and $x''v_1 \notin E(G)$; then $N_G(x'') \setminus \{x, x', v_1, v^*\}$ contains at least two elements and, together with v', v_1, v^*, x, x', and x'' they induce a subgraph containing $\overset{+}{B}_2$. The case of $\Delta(G) = 5$ and $x'v_1 \notin E(G)$ is analogous. Note that the case when $\Delta(G) = 5$, $x'v_1 \in E(G)$, and $x''v_1 \in E(G)$ is impossible since then $\deg_G(v_1) = 5$, and v_1 must belong to Δ-components.

Assume that $k = 2$ and $\Delta(G) = 4$. Owing to equity considerations for the vertices x and y, we may assume that v_2 has a neighbor $v^{**} \notin \{v_1, y\}$. If $v^* \neq v^{**}$ then $[y, v_2, v^{**}, v_1, v^*, x, x', x''; \overset{+}{B}_2]$. Hence, $v^* = v^{**}$. If $v^*x' \notin E(G)$ or $v^*x'' \notin E(G)$ then

$$[v_2, v_1, v^*, x, x', x'', x_1', x_2'; \overset{+}{B}_2] \quad \text{or} \quad [v_2, v_1, v^*, x, x', x'', x_1'', x_2''; \overset{+}{B}_2].$$

Thus, $v^*x' \in E(G)$ and $v^*x'' \in E(G)$. Equity considerations for x and y also yield: $v^*y' \in E(G)$ and $v^*y'' \in E(G)$. Consequently, $\deg_G(v^*) \geq 6$. We obtain a contradiction to the fact that $\Delta(G) = 4$. Thus, $\deg_G(v_1) = 2$.

Let $\Delta(G) = 5$. If it is false that

$$N_G(x') \setminus \{x, x''\} = N_G(x'') \setminus \{x, x'\}, \qquad x'x'' \in E(G)$$

then there exist vertices $z_1, z_2 \in N_G(x') \setminus \{x, x''\}$ and $z_3, z_4 \in N_G(x'') \setminus \{x, x'\}$; moreover,

$$\{z_1, z_2\} \cap \{z_3, z_4\} = \varnothing.$$

Then $[z_1, z_2, z_3, z_4, x', x'', x, v_1; \overset{+}{B}_2]$. Suppose that

$$V_1 = N_G(x') \setminus \{x, x''\} = N_G(x'') \setminus \{x, x'\} = \{z_1, z_2, z_3\}, \qquad x'x'' \in E(G).$$

The graph G contains $\overset{+}{B}_2$ if x has a neighbor $\widehat{x} \notin \{x', x'', v_1\} \cup V_1$. Indeed, the edges v_1x, $\widehat{x}x$, xx', $x'z_1$, $x'x''$, $x''z_1$, and $x''z_2$ constitute the subgraph $\overset{+}{B}_2$. Thus, we may assume that x is adjacent to z_1 and z_2. Since G contains no bridges, at least one vertex in V_1 has a neighbor $z' \notin \{x, x', x''\} \cup V_1$. It suffices to consider the cases when $z'z_1 \in E(G)$ and when $z'z_3 \in E(G)$, $xz_3 \notin E(G)$. If $z'z_1 \in E(G)$ then the edges v_1x, xx', xz_1, z_1z', z_1x'', $x''z_2$, and $x''z_3$ constitute the subgraph $\overset{+}{B}_2$. If $z'z_3 \in E(G)$ and $xz_3 \notin E(G)$ then the edges $z'z_3$, z_3x'', z_3x', $x'z_1$, $x'x$, xv_1, and xz_2 constitute $\overset{+}{B}_2$.

Assume that $\Delta(G) = 4$ and $x'x'' \notin E(G)$. Then $V_2 = \{x'_1, x'_2, x'_3\} = \{x''_1, x''_2, x''_3\}$ since otherwise G contains the subgraph $\overset{+}{B}_2$. If x has a neighbor in V_2 then some element in V_2 must have a neighbor outside $\{x, x', x''\} \cup V_2$ since xv_1 is not a bridge in G. Then the graph G contains $\overset{+}{B}_2$. If x is adjacent to no vertex in V_2 then $N_G(x) = \{x^*, v_1, x', x''\}$, where $x^* \notin V_2$. The set V_2 must be independent since otherwise $G \notin \mathrm{Free}_s(\{\overset{+}{B}_2\})$. Owing to the incompressibility of G, we have $\deg_G(x'_i) = 4$ for some i and also $N_G(x_i) \cap (\{x', x''\} \cup V_2) = \varnothing$. Then $G[\{v_1, x\} \cup V_2 \cup N_G(x'_i)]$ contains the subgraph $\overset{+}{B}_2$.

Suppose that $x'x'' \in E(G)$. Then, since $G \in \mathrm{Free}_s(\{\overset{+}{B}_2\})$, either $\{x'_1, x'_2\} = \{x''_1, x''_2\}$ or we may assume that $x'_2 = x''_2$. In the first case, G contains snake or rose as a subgraph, and the degree of the vertex a_2 of this subgraph in G equals 2. Indeed, if x is adjacent to one of the vertices x'_1 and x'_2 (say, to x'_1) then it suffices to put

$$a_1 = v', \qquad a_2 = v_1, \qquad a_3 = x'_2, \qquad a_4 = x, \qquad a_5 = x'', \qquad a_6 = x', \qquad a_7 = x'_1$$

in the definition of snake. If x is adjacent to the vertex $\widehat{x} \notin \{x', x'', v_1\}$ then

$$a_1 = v', \quad a_2 = v_1, \quad a_3 = \widehat{x}, \quad a_4 = x, \quad a_5 = x'', \quad a_6 = x', \quad a_7 = x'_1, \quad a_8 = x'_2.$$

In the second case, since $\deg_G(x) = 4$ and $G \in \mathrm{Free}_s(\{\overset{+}{B}_2\})$, the vertex x is adjacent to one of the vertices x'_1, x'_2, and x''_1. Indeed, if there exists a neighbor $\widetilde{x} \notin \{v_1, x', x'', x'_1, x'_2, x''_1\}$ of x then the edges $\widetilde{x}x$, v_1x, xx', $x'x'_1$, $x'x''$, $x''x'_2$, and $x''x''_1$ generate $\overset{+}{B}_2$. Thus, G contains snake or shark as a subgraph; moreover, the degree of the vertex a_2 of this subgraph equals 2. This is obvious if $xx'_2 \in E(G)$. If x is adjacent to one of the vertices x'_1 and x''_1 (say, to x'_1) then, in the notations of the definition of shark, it suffices to put

$$a_1 = v', \quad a_2 = v_1, \quad a_3 = x''_1, \quad a_4 = x''_2, \quad a_5 = x'', \quad a_6 = x, \quad a_7 = x'_1, \quad a_8 = x'.$$

Suppose that G has only one Δ-component CO in which, between the vertices u_1 and u_6, there exists an induced path $(u_1, u_2, u_3, u_4, u_5, u_6)$. Since $\Delta(G) \geq 4$, there exist distinct vertices $v' \in N_G(u_2) \setminus \{u_1, u_3\}$ and $v'' \in N_G(u_3) \setminus \{u_2, u_4\}$. If $N_G(u_4) \neq \{u_3, u_5, v', v''\}$ then G contains the subgraph $\overset{+}{B}_2$. The same is true if $N_G(u_4) = \{u_3, u_5, v', v''\}$ and $N_G(u_5) \neq \{u_4, u_6, v', v''\}$. If $N_G(u_4) = \{u_3, u_5, v', v''\}$ and $N_G(u_5) = \{u_4, u_6, v', v''\}$ then

$$[u_1, u_2, u_3, v', u_4, u_5, v'', u_6; \overset{+}{B}_2].$$

Thus, the distance in CO between every two vertices is at most 4; therefore,

$$|V(CO)| \leq 1 + \Delta(G) + \Delta^2(G) + \Delta^3(G) + \Delta^4(G) = \frac{\Delta^5(G) - 1}{\Delta(G) - 1}.$$

Hence,

$$|V(G)| \leq \frac{\Delta^5(G) - 1}{\Delta(G) - 1}\,(\Delta(G) + 1).$$

Lemma 4 is proved. □

2.2. The Click-Width of Graphs and New Cases of Polynomial Solvability for Problem EC *for the Graphs of Minimal Degree* 4

The click-width has an important meaning for constructing efficient algorithms on graphs since, for every constant C, many problems on graphs are polynomially solvable in the class of graphs for which the click-width is at most C (see [32]). The following assertion gives a sufficient condition for the uniform boundedness of click-width (see [33]):

Lemma 5. *For every monotone class $\mathcal{X}$ not including $\mathcal{T}$, there exists a constant $C(\mathcal{X})$ such that the click-width of each graph in $\mathcal{X}$ is at most $C(\mathcal{X})$.*

In [29, Lemma 4], we proved

Lemma 6. *If $\mathcal{X}$ is a monotone class and $\mathcal{T} \not\subseteq \mathcal{X}$ then Problem* EC *is polynomially solvable for the graphs of $\mathcal{X}$.*

Lemma 7. *Let G be an incompressible graph, $G \in \mathrm{Free}_s(\{B_{1+}^{+}\}) \cup \mathrm{Free}_s(\{^{+}B_1^{+}\})$ and $\Delta(G) \geq 4$. Then $G \in \mathrm{Free}_s(\{T_{4,4,4}\})$.*

Proof. Suppose that G has a subgraph isomorphic to $T_{4,4,4}$. If there exists a neighbor of x_1 not belonging to the triode $T_{4,4,4}$ then G contains the subgraphs B_{1+}^{+} and $^{+}B_1^{+}$ simultaneously. The same holds if x_1 has a neighbor in $V(T_{4,4,4}) \setminus \{y_1, z_1\}$ or if $N_G(x_1) = \{v, x_2, y_1, z_1\}$. The same arguments can be carried out for the vertices y_1 and z_1. Therefore, we may assume that these cases are not realized. Thus, for every vertex $t \in \{x_1, y_1, z_1\}$, either $\deg_G(t) = 2$ or $\deg_G(t) = 3$, and $\{x_1, y_1, z_1\} \setminus \{t\}$ contains a neighbor of t.

Since G is incompressible, $N_G(v)$ contains at least two vertices of degree $\Delta(G) \geq 4$. Let u be one of these vertices. Clearly, $u \notin \{x_1, y_1, z_1\}$ and u is adjacent to none of the vertices x_1, y_1, and z_1. Let $u_1 \neq v$ and $u_2 \neq v$ be neighbors of u; moreover, $\deg_G(u_1) = \Delta(G)$. Among $\{x_2, x_3, x_4\}$, $\{y_2, y_3, y_4\}$, and $\{z_2, z_3, z_4\}$, there is a set not containing u_1 and u_2. We may assume that this is $\{x_2, x_3, x_4\}$. If $\{u_1, u_2\} \neq \{y_2, z_2\}$ then $[u_1, u_2, u, v, x_1, x_2, y_1, y_2, z_1, z_2; B_{1+}^{+}]$. If $\{u_1, u_2\} = \{y_2, z_2\}$ then we have $[y_3, y_2, y_1, u, v, x_1, z_2, z_3; B_{1+}^{+}]$.

Since $\deg_G(u_1) = \Delta(G) \geq 4$, the vertex u_1 has a neighbor $u' \notin \{v, u, u_2\}$. Clearly, $u' \notin \{x_1, y_1, z_1\}$. Since $u_i = y_2$ for $i = 1, 2$, we have $[x_2, x_1, v, y_1, y_2, u, u_j, z_1; {}^{+}B_1^{+}]$, where $\{i, j\} = \{1, 2\}$. The situation is similar in the case of $u_i = z_2$ for $i = 1, 2$. If $\{u_1, u_2\} \cap \{y_2, z_2\} = \varnothing$ then $[u', u_1, u, u_2, v, x_1, y_1, y_2; {}^{+}B_1^{+}]$ (if $u' \neq y_2$) or $[u', u_1, u, u_2, v, x_1, z_1, z_2; {}^{+}B_1^{+}]$ (if $u' \neq z_2$).

Lemma 7 is proved. □

Lemma 8. *For every $H \in \mathcal{S} \cup \{\overset{+}{B}_2\}$, Problem* EC *is polynomially solvable in the class*

$$\mathcal{K}_H = \{G \mid G \in \mathrm{Free}_s(\{H\}),\ \Delta(G) \geq 4\}.$$

Proof. As follows from Lemmas 1–4, 6, and 7, it suffices to consider the case of $H = \overset{+}{B}_2$ and incompressible graphs $G \in \mathcal{K}_H$ with $\Delta(G) = 4$ containing one of the subgraphs snake, rose, shark in which $\deg_G(a_2) = 2$. Indeed, if $H \in \mathcal{S} \setminus \{B_{1+}^{+}\}$ then Lemmas 1–3 imply that there exists a constant C_H such that, for every incompressible graph $G \in \mathcal{K}_H$, either $\chi'(G) = \Delta(G)$ or $|V(G)| \leq C_H$. The chromatic index of a graph with at most C_H vertices can be found in time $O(1)$. If $H = B_{1+}^{+}$ then Lemma 8 follows from Lemmas 6 and 7. The possibility of considering the corresponding constraints in the case of $H = \overset{+}{B}_2$ stems from Lemma 4.

Suppose that G contains the subgraph snake. Let

$$V' = \{a_4, a_5, a_6, a_7\}, \qquad N = \{x \mid xa_i \in E(G),\ a_i \in V',\ x \notin V'\}.$$

Since $G \in \mathrm{Free}_s(\{\overset{+}{B}_2\})$, $\Delta(G) = 4$, we have $|N| \le 3$. It is not hard to see that a 4-coloring of the edges of the subgraph $(V(G), E(G) \setminus \{a_i a_j \mid a_i, a_j \in V'\})$ extends to a 4-coloring of the edges of G if and only if between N and V' there are edges of at most two colors. Thus, we may assume that no vertex in N is adjacent to three vertices in V'.

Assume that $b \in N \setminus \{a_2\}$ and $N_G(b) \cap V' = \{a_i\}$. Then necessarily $\deg_G(b) = 2$ owing to the incompressibility of G and the fact that if there exist vertices $c_1, c_2 \in N_G(b) \setminus \{a_i\}$ then

$$[c_1, c_2, b, a_2, a_4, a_5, a_6, a_7; \overset{+}{B}_2].$$

Suppose now that $b \in N$ and $N_G(b) \cap V' = \{a_i, a_j\}$. Then necessarily $\deg_G(b) \le 3$ because if $N_G(b) = \{c_1, c_2, a_i, a_j\}$ then $[c_1, c_2, b, a_2, a_4, a_5, a_6, a_7; \overset{+}{B}_2]$.

We may assume without loss of generality that $N = \{a_2, a_3, a_8\}$ and

$$N_G(a_2) \cap V' = \{a_4\}, \qquad N_G(a_3) \cap V' = \{a_5\}, \qquad N_G(a_8) \cap V' = \{a_6\}$$

or

$$N_G(a_2) \cap V' = \{a_4\}, \qquad N_G(a_3) \cap V' = \{a_5\}, \qquad N_G(a_8) \cap V' = \{a_6, a_7\}.$$

In the first case, $\deg_G(a_2) = \deg_G(a_3) = \deg_G(a_8) = 2$; therefore, every 4-coloring of the edges of the subgraph $G \setminus V'$ can be extended to a 4-coloring of the edges of G by coloring a_2a_4, a_3a_5, and a_6a_8 with one color. In the second case, $\deg_G(a_2) = \deg_G(a_3) = 2$ and $\deg_G(a_8) \le 3$. Thus, every 4-coloring of the edges of $G \setminus V'$ can be extended to a 4-coloring of the edges of G by coloring a_2a_4 and a_6a_8 with one color and a_3a_5 and a_6a_7, with another color.

Suppose that G contains the subgraph rose. We may assume that $a_4a_7 \notin E(G)$ and $a_4a_8 \notin E(G)$ since this case implies the existence of the corresponding subgraph snake. Assume that the vertex a_7 is adjacent to $b \notin \{a_3, a_5, a_6, a_8\}$. Then $[a_2, a_3, a_4, a_5, a_8, a_7, b, a_6; \overset{+}{B}_2]$. Thus $N_G(a_i) \subseteq \{a_3, a_5, a_6, a_j\}$ for every $i \in \{7, 8\}$, where $\{i, j\} = \{7, 8\}$. The vertex a_3 cannot simultaneously have a neighbor in the set $\{a_7, a_8\}$ (say, a_7) and a neighbor $c \notin \{a_4, a_7, a_8\}$ since then the edges ca_3, a_3a_7, a_3a_4, a_4a_2, a_4a_6, a_6a_5, and a_6a_8 constitute $\overset{+}{B}_2$. Consequently, either $N_G(a_3) \subseteq \{a_4, a_7, a_8\}$ or a_7 and a_8 have neighbors only in the set $\{a_5, a_6, a_7, a_8\}$. Remove the vertices a_5–a_8 from G and obtain a graph G'.

Verify that $\chi'(G') \le 4$ if and only if $\chi'(G) \le 4$. For this it suffices to show that the 4-colorability of the edges of G' implies the 4-colorability of the edges of G. We may assume that, in a 4-coloring of the edges of G', the edges a_2a_4 and a_3a_4 have colors 1 and 2. Assign color 1 to the edge a_7a_8 (if it exists) and the edge a_5a_6; color 2, to the edges a_5a_8 and a_6a_7; color 3, to the edge a_3a_7 (if it exists) and the edges a_4a_5 and a_6a_8; and color 4, to a_3a_8 (if it exists) and a_4a_6 and a_5a_7.

Suppose that G contains a subgraph shark. Obviously, there is no vertex $b \notin V(\text{shark}) \setminus \{a_1\}$ adjacent to at least one of the vertices a_4 and a_7 since otherwise $[b, a_2, a_3, a_4, a_5, a_6, a_7, a_8; \overset{+}{B}_2]$. If $a_3a_4 \in E(G)$ then there is no vertex $b \in N_G(a_3) \setminus \{a_4, a_5, a_7\}$ since otherwise

$$[b, a_3, a_4, a_5, a_7, a_6, a_2, a_8; \overset{+}{B}_2].$$

If $a_3a_7 \in E(G)$ then there is no vertex $b \in N_G(a_3) \setminus \{a_4, a_5, a_7\}$ since otherwise

$$[b, a_3, a_7, a_6, a_8, a_5, a_4, a_2; \overset{+}{B}_2].$$

If $b_1, b_2 \in N_G(a_3) \setminus \{a_4, a_5, a_7\}$ then $[b_1, b_2, a_3, a_5, a_7, a_8, a_4, a_6; \overset{+}{B}_2]$. It follows that $N_G(a_3) = \{a_5, c\}$, where $c \notin \{a_4, a_7\}$ since a_3a_5 is not a bridge in G. Remove all edges of $G[\{a_4, a_5, a_6, a_7, a_8\}]$ from G and obtain the graph G''.

Verify that $\chi'(G'') \le 4$ if and only if $\chi'(G) \le 4$. For this it suffices to show that the 4-colorability of the edges of G'' implies the 4-colorability of the edges of G. Since $\deg_G(a_2) = \deg_G(c) = 2$, we may assume that, in a 4-coloring of the edges of G'', the edges a_3a_5 and a_2a_6 have colors 1 and 2. Assign color 1 to the edges a_6a_8 and a_4a_7 (if the latter exists); color 2, to the edges a_4a_5 and a_7a_8; color 3, to the edges a_4a_8 and a_5a_6; and color 4, to the edges a_5a_8 and a_6a_7.

Lemma 8 is proved. □

2.3. The Complexity of Problem EC *for Classes of Subcubic Graphs*

Let us define some transformation that is called the *replacement of a vertex whith a triangle*. It is applied to a vertex x in a graph whose neighborhood consists exactly of vertices x_1, x_2, and x_3 and is defined as follows: The vertex x is removed, while some vertices x'_1, x'_2, and x'_3 are added together with the new edges x'_1x_1, x'_2x_2, x'_3x_3, $x'_1x'_2$, $x'_2x'_3$, and $x'_1x'_3$. It is not hard to see that a 3-coloring of the initial graph exists if and only if it exists for the so-obtained graph.

Let $\mathcal{X}_k$ denote the set of cubic graphs not containing induced cycles of length at most k. Clearly, the classes $\mathcal{X}_1$ and $\mathcal{X}_2$ coincide with the set of all cubic graphs. Let $\mathcal{X}^*_k$ denote the set of graphs obtained from graphs in $\mathcal{X}_k$ by consecutively replacing a vertex with a triangle for each of its vertices. Obviously, $\mathcal{X}^*_k \subseteq \mathrm{Free}_s(\{\overset{+}{B}_2\})$ for each k since, among every three vertices constituting a path in an arbitrary graph in $\mathcal{X}^*_k$, there are two vertices belonging to a common triangle.

Lemma 9. *For each k, Problem* EC *is* NP*-complete for the graphs of $\mathcal{X}^*_k$.*

Proof. It is known that, for each k, Problem EC is NP-complete for the graphs of $\mathcal{X}_k$ (see [36]). In the class $\mathcal{X}_k$, Problem EC is polynomially reduced to the same problem in $\mathcal{X}^*_k$. Therefore, Problem EC is NP-complete for the graphs of $\mathcal{X}^*_k$. Lemma 9 is proved. □

The *monotone closure of a graph class $\mathcal{Z}$* is the set of graphs that are the subgraphs of graphs in $\mathcal{Z}$. The closure of $\mathcal{Z}$ is denoted by $[\mathcal{Z}]_m$.

Lemma 10. *Let H^* be a graph without isolated vertices having exactly* 7 *edges and belonging to $[\mathcal{X}^*_1]_m$, $H \in \mathcal{S}$. Then Problem* EC *is polynomially solvable for subcubic graphs in the class* $\mathrm{Free}_s(\{H^*, \overset{+}{B}_2\})$ *and for subcubic graphs in the class* $\mathrm{Free}_s(\{H\})$.

Proof. By Lemma 6, we may consider incompressible subcubic graphs of the classes $\mathrm{Free}_s(\{H^*, \overset{+}{B}_2\})$ and $\mathrm{Free}_s(\{H\})$ containing the subgraph $T_{5,5,5}$. Let G be such a graph where $\Delta(G) = 3$. Since G is incompressible, among the vertices x_1, y_1, and z_1, there are at least two vertices of degree 3. Show that, for subcubic graphs in the classes under consideration, Problem EC is reduced to the same problem for graphs in $\mathrm{Free}_s(\{T_{5,5,5}\})$. The former class is the case of the polynomial solvability of Problem EC by Lemma 6. Thus, Lemma 10 will be proved.

Consider the two cases: The set $\{x_1, y_1, z_1\}$ is not independent and it is independent.

I. Consider first the case when, among x_1y_1, x_1z_1, and y_1z_1, there is an edge of G.

Assume that $x_1y_1 \in E(G)$. Contract the triangle induced by v, x_1, and y_1 to a vertex u_1 and obtain a graph G^*_1 with $\Delta(G^*_1) \le 3$. It is not hard to see that $\chi'(G) = 3$ if and only if $\chi'(G^*_1) \le 3$.

We may assume that, among x_2, y_2, and z_1, at least two vertices have degree 3. Indeed, otherwise, a 3-coloring of the edges of G^*_1 (and hence a 3-coloring of the edges of G) exists if and only if it exists for $G^*_1 \setminus \{u_1\} \cong G \setminus \{v, x_1, y_1\}$. By symmetry consideration, we may assume that $\deg_G(x_2) = \deg_G(y_2) = 3$. Let $N_G(x_2) = \{x', x_1, x_3\}$ and $N_G(y_2) = \{y', y_1, y_3\}$.

If $x' = x_4$ or $x'x_3 \in E(G)$ then G contains as a subgraph every graph in $[\mathcal{X}^*_1]_m$ without isolated vertices having exactly 7 edges. The cases of $x' = y_2$ and $x' = z_1$ will be treated below, and it will turn out that, in these cases, G includes $\overset{+}{B}_2$. If $\deg_G(x_3) = 3$ then the subgraph $G[N_G(x_3) \cup \{x_3, x', x_1, y_1, v\}]$ contains $\overset{+}{B}_2$. If $\deg_G(x_3) = 2$ then $N_G(x') = \{a, b, x_2\}$ because of the incompressibility of G. Clearly, $\{a, b\} \cap \{x_3, y_1\} = \varnothing$. Then $[a, b, x', x_2, x_3, x_1, y_1, v; \overset{+}{B}_2]$.

If $x' \notin V(T_{5,5,5})$ then G simultaneously includes

$$B_1 + P_3,\ B_1 + 2P_2,\ B_1^+ + P_2,\ B_1^{++},\ {}^+B_1^+,\ B_{1+}^+,\ B_2 + P_2,\ B_2^+$$

as the subgraphs. The same is true if $x' \in V(T_{5,5,5})$; for this it is easy to examine the following two cases separately:

$$x' \in \{x_4, x_5\} \qquad \text{and} \qquad x' \in \bigcup_{i=2}^{5} \{y_i, z_i\}.$$

Let us now consider all possible subcases of the disposition of the vertex x'.

I.a. Suppose that $x' = y_2$. Contract the subgraph $G[\{v, x_1, y_1, x_2, y_2\}]$ to a vertex u_2 and obtain the graph G_2^* for which $\Delta(G_2^*) \leq 3$. It is not hard to see that $\chi'(G) = 3$ if and only if $\chi'(G_2^*) \leq 3$. We may assume that $\deg_G(x_3) = 3$ or $\deg_G(y_3) = 3$, otherwise,

$$\chi'(G) = 3 \Leftrightarrow \chi'(G_2^* \setminus \{u_2\}) \leq 3, \qquad G_2^* \setminus \{u_2\} \cong G \setminus \{v, x_1, y_1, x_2, y_2\}.$$

Then G simultaneously contains the subgraphs B_3 and $\overset{+}{B}_2$; therefore, we may assume that $x' \notin \{y_2, z_1\}$ and $y' \notin \{x_2, z_1\}$.

I.b. Suppose that $x' \neq y'$.

If $x' = y_3$ then $[y_1, y_2, y_3, y_4, x_2, x_1, v,\ z_1; B_3]$; therefore, we may assume that $x' \neq y_3$ and $y' \neq x_3$. Then $[x_3, x_2, x', x_1,\ y_1, y_2, y_3, y'; B_3]$.

I.c. Suppose that $x' = y'$.

If $\deg_G(x_3) = 3$ then the subgraph $G[N_G(x_3) \cup N_G(y_1) \cup \{y_1, x_3\}]$ contains B_3. Thus, $\deg_G(x_3) = 2$ and $\deg_G(y_3) = 2$. Since G is incompressible, there exists a vertex x^* adjacent to x' and different from x_2 and y_2.

If $x^* \neq z_1$ then $[x^*, x_2, y_2, x', y_1, v, x_1, z_1; B_3]$. If $x^* = z_1$ then contract $G[\{v, x_1, y_1, z_1, x_2, y_2, x'\}]$ to a vertex u_3 and obtain the subgraph G_3^*. We have

$$\chi'(G) = 3 \Leftrightarrow \chi'(G_3^* \setminus \{u_3\}) \leq 3, \qquad G_3^* \setminus \{u_3\} \cong G \setminus \{v, x_1, y_1, z_1, x', x_2, y_2\}.$$

Thus, independently of which of the cases $G \in \text{Free}(\{H^*, \overset{+}{B}_2\})$ or $G \in \text{Free}(\{H\})$ is under consideration, we can perform a reduction of G.

II. Let $\{x_1, y_1, z_1\}$ be an independent set and let $N_G(x_1) = \{x'', v, x_2\}$. It is easy to check that G simultaneously contains the subgraphs $B_1 + P_3$, $B_1 + 2P_2$, $B_1^+ + P_2$, B_1^{++}, ${}^+B_1^+$, and B_{1+}^+. We may assume that $N_G(y_1) = \{y'', v, y_2\}$.

Consider the two subcases: $\deg_G(z_1) = 3$ and $\deg_G(z_1) = 2$:

II.a. Suppose that there exists z'' such that $N_G(z_1) = \{z'', v, z_2\}$. If $x'' \neq y''$ then

$$[x_2, x_1, x'', v, y_1, y_2, y'', z_1; \overset{+}{B}_2],$$

excluding the cases of $x'' = y_2$ or $y'' = x_2$. Therefore, G contains $\overset{+}{B}_2$ if, among x'', y'', and z'', there are at least two distinct vertices. If $x'' = y'' = z''$ and the degree of at least one of the vertices x_2, y_2, and z_2 is equal to 3 then G contains $\overset{+}{B}_2$. If

$$x'' = y'' = z'', \qquad \deg_G(x_2) = \deg_G(y_2) = \deg_G(z_2) = 2$$

then contract the subgraph $G[\{v,x_1,y_1,z_1,x''\}]$ to a vertex u_4 and obtain the graph G_4^*. We have

$$\chi'(G)=3\Leftrightarrow\chi'(G_4^*\setminus\{u_4\})\le 3,\qquad G_4^*\setminus\{u_4\}\cong G\setminus\{v,x_1,y_1,z_1,x''\}.$$

Consequently, if $G\in\mathrm{Free}(\{H^*,\overset{+}{B}_2\})$ then it is impossible to perform a reduction of the graph G. Henceforth, in this case, we will assume that $G\in\mathrm{Free}_s(\{H\})$.

II.a.1. Prove that there are no $p\in\{2,3\}$ and $q\in\{2,3,4\}$ such that $x_py_q\in E(G)$. We may assume that $p\le q$. If $p=2$ then

$$[x_3,x_2,y_q,x_1,v,y_1,z_1,z_2,z_3;B_2+P_2,B_2^+],\qquad [y_{q+1},y_q,y_{q-1},x_3,x_2,x_1,v,y_1,z_1,z'',z_2;B_3]$$

(both for $q=2$ and when $q\ne 2$). If $p=3$ then $[x_4,x_3,y_q,x_2,x_1,v,y_1,z_1;B_3]$ and

either $[x_4,x_3,y_q,x_2,x_1,x'',v,y_1,y_2;B_2+P_2,B_2^+]$ (when $x''\notin\{x_4,y_2\}$),
or $[x_4,x_3,y_q,x_2,x_1,x'',v,z_1,z_2;B_2+P_2,B_2^+]$ (when $x''=y_2$),
or $[x_5,x_4,x_1,x_3,y_5,y_4,y_3,y_2,y_1,v;B_2+P_2,B_2^+]$ (when $x''=x_4$).

Thus, for every $2\le p\le 4$ and $2\le q\le 4$, where $(p,q)\ne(4,4)$, we have

$$x_py_q\notin E(G),\qquad x_pz_q\notin E(G),\qquad y_pz_q\notin E(G).$$

II.a.2. Show that $x''\ne y_3$ (and hence $x''\ne z_3$, $y''\notin\{x_3,z_3\}$, and $z''\notin\{x_3,y_3\}$).

Suppose the contrary. Then G simultaneously contains B_2+P_2 and B_2^+. If $z''\notin\{y_2,y_4\}$ then we have $[y_2,y_3,y_4,x_1,v,z_1,z'',z_2;B_3]$. The graph G contains B_3 if $y''\in\{x_3,z_2,z_3\}$.

Consider the case when $z''\in\{y_2,y_4\}$. Then G contains the subgraph B_3 if $y''=x_2$.

Suppose that $z''=y_4$. If $\deg_G(y_2)=3$ then G contains B_3; therefore, $\deg_G(y_2)=2$. Since G is incompressible, $\deg_G(y'')=3$. Then G contains B_3 (independently of the presence of the edge $y''x_2$).

Assume that $z''=y_2$. We have $\deg_G(x_2)=2$; otherwise, G contains B_3. Then $\deg_G(x_3)=3$ since G is incompressible; therefore, G contains B_3.

II.a.3. Prove that $x''\ne y_2$ (and hence $x''\ne z_2$, $y''\notin\{x_2,z_2\}$, and $z''\notin\{x_2,y_2\}$). Suppose that $x''=y_2$. Then $[y_3,y_2,x_1,y_1,v,z_1,z'',z_2;B_3]$. At the same time,

$$[x_2,x_1,y_1,v,z_1,z'',z_2,y_2,y_3,z_3,z_4;B_2+P_2,B_2^+]\quad\text{if }z''\ne x_2,$$
$$[y_3,y_2,y_1,x_1,x_2,x_3,z_1;z_2,z_3;B_2+P_2,B_2^+]\quad\text{if }z''=x_2.$$

II.a.4. Prove that $\deg_G(x_2)=2$ and $\deg_G(x_3)=2$; i.e., G is incompressible.

Let $N_G(x_2)=\{x_3,x_1,\widehat{x}\}$. Then $[x_3,x_2,\widehat{x},x_1,v,y_1,y_2,y_3,z_1;B_2+P_2,B_2^+]$. If there is a vertex among y'' and z'' (say, y'') different from $\widehat{x}$ then $[x_3,x_2,\widehat{x},x_1,v,y_1,y'',y_2;B_3]$. If $\widehat{x}=y''=z''$ then, owing to the incompressibility of G, either $\deg_G(x'')=3$ or there exists a neighbor of x'' of degree 3 different from x_1; but then G contains the subgraph B_3.

Assume that $\deg_G(x_2)=2$ and $N_G(x_3)=\{x_2,x_4,\widetilde{x}\}$. Then $[x_4,x_3,\widetilde{x},x_2,x_1,v,y_1,z_1;B_3]$. At the same time, if $\widetilde{x}\ne x''$ and $x''\ne x_4$ then

$$[x_4,x_3,\widetilde{x},x_2,x_1,x'',v,y_1,y_2;B_2+P_2,B_2^+].$$

If $x''=x_4$ or $\widetilde{x}=x''$ then there is a vertex among y'' and z'' different from x''. Then it is not hard to check that G simultaneously contains subgraphs B_2+P_2 and B_2^+.

II.b. Let $\deg_G(z_1)=2$. Then $N_G(z_2)=\{z''',z_1,z_3\}$ because G is incompressible.

Suppose that $G\in\mathrm{Free}_s(\{H^*,\overset{+}{B}_2\})$ and $x''\ne y''$. Then $[x_2,x_1,x'',v,y_1,y_2,y'',z_1;\overset{+}{B}_2]$ excluding the cases when $x''=y_2$ or $y''=x_2$.

Let $x''=y_2$.

If in addition $y''=x_2$; then, obviously, $\deg_G(x_3)=\deg_G(y_3)=2$ since otherwise G contains $\overset{+}{B}_2$. Then contract $G[\{v,x_1,x_2,y_1,y_2\}]$ to a vertex u_5 and obtain G_5^*. We have

$$\chi'(G)=3\Leftrightarrow\chi'(G_5^*\setminus\{u_5\})\le 3,\qquad G_5^*\setminus\{u_5\}\cong G\setminus\{v,x_1,x_2,y_1,y_2\};$$

therefore, we may assume that $\deg_G(x_2) = 2$ since otherwise G contains $\overset{+}{B}_2$.

If $y'' = y_3$ then contract $G[\{y_3, y_2, y_1, x_1, v\}]$ to the vertex u_6 and obtain G_6^*. We infer

$$\chi'(G) = 3 \Leftrightarrow \chi'(G_6^* \setminus \{u_6\}) \leq 3, \qquad G_6^* \setminus \{u_6\} \cong G \setminus \{y_3, y_2, y_1, x_1, v\}.$$

We may assume that $\deg_G(y_3) = 2$ since otherwise G contains $\overset{+}{B}_2$. By the incompressibility of G, we obtain $\deg_G(y'') \geq 2$.

If $\deg_G(y'') = 3$ then G contains the subgraph $\overset{+}{B}_2$. If $N_G(y'') = \{y_1, a\}$ then a 3-coloring of the edges of G exists if and only if there exists a 3-coloring of the edges of $G \setminus \{v, y_1, y_2, x_1\}$.

Indeed, in every 3-coloring of the edges of $G \setminus \{v, y_1, y_2, x_1\}$, we can choose a color c_1 not coloring x_2x_3 and y_3y_4 simultaneously and color with c_1 the edges x_1x_2 and y_2y_3. We can choose a color c_2 (possibly, $c_2 = c_1$) not coloring $y''a$ and z_1z_2 simultaneously and color with c_2 the edges y_1y'' and vz_1. Then, as is easy to see, the 3-coloring of the edges of G extends to the 3-coloring of the edges of G.

Assume that $x'' = y''$.

If $N_G(x_2) = \{x_1, x_3, \breve{x}\}$ then either $[\breve{x}, x_2, x_3, x_1, x'', v, y_1, z_1; \overset{+}{B}_2]$ if $\breve{x} \neq x''$ or $\breve{x} = x''$.

Suppose that $\breve{x} = x''$. Then $\deg_G(y_2) = 2$. Contract $G[\{x_2, x'', x_1, y_1, v\}]$ to some vertex u_7 and obtain the graph G_7^*. We have

$$\chi'(G) = 3 \Leftrightarrow \chi'(G_7^* \setminus \{u_7\}) \leq 3, \qquad G_7^* \setminus \{u_7\} \cong G \setminus \{x_2, x'', x_1, y_1, v\}.$$

Let $\deg_G(x_2) = \deg_G(y_2) = 2$. Then $N_G(x'') = \{a', x_1, y_1\}$ because of the incompressibility of G; moreover, $\deg_G(a') \geq 2$.

If $\deg_G(a') = 3$ then the subgraph $G[N_G(a') \cup \{a', x_1, y_1, y_2, v\}]$ contains $\overset{+}{B}_2$.

If $\deg_G(a') = 2$ then a 3-coloring of the edges of G exists if and only if there exists a 3-coloring of the edges of $G \setminus \{v, x_1, y_1, x''\}$.

Henceforth, we assume that $G \in \mathrm{Free}_s(\{H\})$.

II.b.1. Let $z''' \notin \{x_1, y_1\}$. Then G simultaneously contains subgraphs $B_2 + P_2$ and B_2^+. The following variants are possible:

If $z''' = x_2$ and $y'' \neq z_3$ then $[y'', y_2, y_1, v, z_1, z_2, z_3, x_2; B_3]$.

If $z''' = x_2$ and $y'' = z_3$ then $[x_1, x_2, x_3, z_2, z_3, y_1, y_2, v; B_3]$.

If $z''' \neq x_2$ and $x'' \neq z_3$ then $[x'', x_1, x_2, v, z_1, z_2, z''', z_3; B_3]$.

If $z''' \neq x_2$ and $x'' = z_3$ then either $[y_2, y_1, y'', v, x_1, z_3, z_2, z_4; B_3]$ (when $y'' \neq z_4$) or $[y_2, y_1, v, z_4, z_3, z_2, z''', z_1; B_3]$ (when $y'' = z_4$, $z''' \neq y_2$) or $[y_3, y_2, y_1, z_2, z_3, x_1, x_2, v]$ (when $y'' = z_4$, $z''' = y_2$).

II.b.2. Consider the case of $z''' = x_1$ that is analogous to the case of $z''' = y_1$. The following variants are possible:

If $y'' \neq x_2$ then $[y_2, y_1, y'', v, x_1, x_2, z_2; B_2]$; therefore, G simultaneously contains $B_2 + P_2$ and B_2^+. If $y'' = x_2$ then G simultaneously contains the subgraphs $B_2 + P_2$ and B_2^+.

If $y'' \neq z_3$ then $[y'', y_2, y_1, v, z_1, z_2, z_3, x_1; B_3]$.

If $y'' = z_3$ then either $\deg_G(x_3) = 3$ or $\deg_G(x_2) = 3$ by the incompressibility of G.

In the first case, $G[N_G(x_3) \cup \{x_3, x_1, v, y_1, z_1\}]$ contains the subgraph B_3. In the second case, either $G[N_G(x_2) \cup N_G(y_1) \cup \{x_2, y_1\}]$ contains B_3 (if $x_2y_2 \notin E(G)$) or $[x_3, x_2, x_1, y_2, y_1, z_3, z_2, z_4; B_3]$ (if $x_2y_2 \in E(G)$).

Lemma 10 is proved. □

3. THE MAIN RESULT

The main result of the article is as follows:

Theorem. *Let $\mathcal{Y}$ be a set of graphs each of which has at most 7 edges. Then Problem* EC *is polynomially solvable for the graphs of $\mathcal{X} = \mathrm{Free}_s(\mathcal{Y})$ if either $\mathcal{Y}$ contains a subcubic forest not belonging to $\{\overset{+}{B}_2 + O_n \mid n \geq 0\}$ or $\mathcal{Y}$ simultaneously contains both a graph of the form $\overset{+}{B}_2 + O_n$ and a graph from $[\mathcal{X}_1^*]_m$. In all other cases, it is* NP*-complete for the graphs of $\mathcal{X}$.*

Proof. Recall that Problem EC is NP-complete in the class $\mathcal{X}_k$ for all k. Therefore, we may assume that $\mathcal{X}_k \nsubseteq \mathcal{X}$ for every k. Note that $\mathcal{Y}$ is finite and, for every graph G^* that is not a subcubic forest, there exists k^* (which can be put equal to the girth of G^*) such that $\mathcal{X}_{k^*+1} \subseteq \mathrm{Free}_s(\{G^*\})$. Consequently, $\mathcal{Y}$ contains a subcubic forest.

Note that each subcubic forest F with 7 edges is representable as

$$F = F' + O_n, \qquad F' \in \mathcal{T} \cup \mathcal{S} \cup \{\overset{+}{B}_2\}.$$

Indeed, if each connected component in F has at most one vertexof degree 3 then $F \in \mathcal{T}$. If F has a connected component with at least two vertices of degree 3 then it belongs to the set

$$\{B_1,\ B_1^+,\ B_1^{++},\ {}^+B_1^+,\ B_{1+}^+,\ B_2,\ B_2^+,\ \overset{+}{B}_2,\ B_3\}.$$

Hence, $F' \in \mathcal{S} \cup \{\overset{+}{B}_2\}$.

Consider an arbitrary subcubic forest $F \in \mathcal{Y}$. If $G \in \mathrm{Free}_s(\{H + P_1\})$ then either $G \in \mathrm{Free}_s(\{H\})$ or $G \cong H$. From this and Lemma 6 we may assume that $F \in \mathcal{S} \cup \{\overset{+}{B}_2\}$. If $F \in \mathcal{S}$ then, by Lemmas 8 and 10, Problem EC is polynomially solvable for graphs in $\mathcal{X}$. If $F = \overset{+}{B}_2$ and $\mathcal{Y} \cap [\mathcal{X}_1^*]_m = \varnothing$ then $\mathcal{X}_1^* \subseteq \mathcal{X}$ since $\overset{+}{B}_2 \notin [\mathcal{X}_1^*]_m$ and no graph in $[\mathcal{X}_1^*]_m$ is forbidden for $\mathcal{X}$. Problem EC is NP-complete for the graphs of $\mathcal{X}$ by Lemma 9. If $F = \overset{+}{B}_2$ and $\mathcal{Y} \cap [\mathcal{X}_1^*]_m \neq \varnothing$ then, by Lemmas 8 and 10, Problem EC is polynomially solvable for the graphs of $\mathcal{X}$.

The theorem is proved. □

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