# Cayley-Hamilton Theorem for Symplectic Quantum Matrix Algebras 

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#### Abstract

We establish the analogue of the Cayley-Hamilton theorem for the quantum matrix algebras of the symplectic type.


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## 1 Introduction

Let $V$ be a vector space equipped with a bilinear nondegenerate (symmetric or antisymmetric) form. The Brauer algebra $[\mathrm{Br}]$ generalizes the tower of the centralizing algebras which appears in the Brauer-Schur-Weyl duality, related to $V$. The Birman-Murakami-Wenzl algebra [BW, Mur] is the quantum deformation of the Brauer algebra. Important particular cases of local representations of the tower of the Birman-Murakami-Wenzl (BMW) algebra are constructed with the use of orthogonal and symplectic R-matrices. These R-matrices give rise to the quantum matrix algebras of the orthogonal and symplectic types. More precisely, a quantum matrix algebra is defined by a compatible pair $\{R, F\}$ of R -matrices, we recall the definitions below. The general structure properties of quantum matrix algebras with $R$ of the BMW type were investigated in [OP]. In the present work we mainly assume $R$ to be of the symplectic type. Our principal goal is to derive, for the quantum matrices of the symplectic type, an analogue of the Cayley-Hamilton identity and to use it for a description of the spectra of the corresponding quantum matrices.

In Section 2 we recall the necessary facts about the Birman-Murakami-Wenzl (BMW) algebras, their R-matrix realizations, specializations to the symplectic and orthogonal cases and some R-matrix technique.

Section 3 contains the information from [OP] about the quantum matrix algebra, its 'characteristic subalgebra' (the subalgebra to which the coefficients of the Cayley-Hamilton identity belong) and the $\star$-multiplication.

The main results are in Section 4. Here we establish the Cayley-Hamilton theorem for the symplectic quantum matrix algebras. Classically, the symplectic group is defined by the condition $M^{t} \Omega M=\Omega$ where $\Omega$ is the symplectic form. However we have to a work with a bigger group defined by the condition $M^{t} \Omega M=g \Omega$ where $g$ is a constant, that is with the group of transformations which preserve the form up to a multiplicative factor. We call ' 2 -contraction' the quantum analogue of this factor. It is an element $g$ of the quantum matrix algebra. For a general compatible pair $\{R, F\}$, the element $g$ is not necessarily central so we cannot harmlessly set it to 1 . We establish a strengthened form of the Cayley-Hamilton theorem which does not assume the invertibility of the element $g$ (and which is equivalent to the Cayley-Hamilton theorem under the assumption of the invertibility of $g$ ).

Next, we define in Section 4 a homomorphism from the characteristic subalgebra to the algebra of symmetric polynomials in some set of commuting ("spectral") variables. The nature of this homomorphism reflects the reciprocity properties of the characteristic polynomials for the symplectic matrices. The Cayley-Hamilton identities under the action of this homomorphism are completely factorized and hence the spectral variables can be treated as eigenvalues of the quantum matrix. We then give the spectral parameterization of the three series of elements of the characteristic subalgebra: the power sums $p_{i}$, the elementary symmetric functions $a_{i}$ and the complete symmetric functions $s_{i}$.

Section 4 contains also the low-dimensional examples illustrating the Cayley-Hamilton theorem for two most known quantum matrix algebras: the algebra of functions on the quantum group (corresponding to the compatible pair $\{R, P\}$ where $P$ is the flip) and the reflection equation algebra [C, KS] (the reflection equation algebra corresponds to the compatible pair $\{R, R\}$ ). Also we discuss the classical limit of the Cayley-Hamilton theorem.

The Cayley-Hamilton identity for the quantum matrix algebras of the orthogonal type will be considered in a separate publication.

## 2 BMW algebra and their R-matrix representations

In this section we present definitions and describe necessary facts about the Birman-MurakamiWenzl algebras and the BMW type R-matrices. We follow notation of ref. [OP] where the reader can find detailed derivations and the references. Later in the section we investigate two families of the BMW type R-matrices, the $S p(2 k)$ type and the $O(k)$ type R-matrices. They are related, respectively, to the symplectic and orthogonal series of the quantum groups. We identify particular conditions on the eigenvalues which are specific for these families of R-matrices. In the following section we will use the symplectic R-matrices for the definition of $S p(2 k)$ type quantum matrix algebras. Specific properties of the $S p(2 k)$ type R-matrices will then dictate a form of the Cayley-Hamilton identites in these algebras.

### 2.1 BMW algebra

The Birman-Murakami-Wenzl (BMW) algebra $\mathcal{W}_{n}(q, \mu)$ [BW, Mur] depending an two complex parameters $q \in \mathbb{C} \backslash\{0, \pm 1\}$ and $\mu \in \mathbb{C} \backslash\left\{0, q,-q^{-1}\right\}$ is defined in terms of generators $\left\{\sigma_{i}, \kappa_{i}\right\}_{i=1}^{n-1}$ and relations

$$
\begin{array}{rll}
\sigma_{i} \sigma_{i+1} \sigma_{i}=\sigma_{i+1} \sigma_{i} \sigma_{i+1}, & \sigma_{i} \sigma_{j}=\sigma_{j} \sigma_{i} & \forall i, j:|i-j|>1, \\
\sigma_{i} \kappa_{i}=\kappa_{i} \sigma_{i}=\mu \kappa_{i}, & \kappa_{i}=\frac{\left(q 1-\sigma_{i}\right)\left(q^{-1} 1+\sigma_{i}\right)}{\mu\left(q-q^{-1}\right)}, &  \tag{2.1}\\
\kappa_{i+1} \kappa_{i}=\kappa_{i+1} \sigma_{i}^{ \pm 1} \sigma_{i+1}^{ \pm 1}, & \kappa_{i} \kappa_{i+1} \kappa_{i}=\kappa_{i} & \forall i .
\end{array}
$$

The first line is the Artin's presentation of the braid group $\mathcal{B}_{n}$; the rest of relations define the quotient algebra $\mathcal{W}_{n}(q, \mu) \subset \mathbb{C}\left[\mathcal{B}_{n}\right]$.

Imposing further restrictions on the parameters

$$
\begin{equation*}
j_{q}:=\frac{q^{j}-q^{-j}}{q-q^{-1}} \neq 0, \quad \mu \neq \mp q^{\mp(2 j-3)} \quad \forall j=2,3, \ldots, n, \tag{2.2}
\end{equation*}
$$

one can define recursively two sets of idempotents

$$
\begin{align*}
a^{(1)} & :=1, & s^{(1)}:=1,  \tag{2.3}\\
a^{(i+1)} & :=\frac{q^{i}}{(i+1)_{q}} a^{(i)} \sigma_{i}^{-}\left(q^{-2 i}\right) a^{(i)}, & s^{(i+1)}:=\frac{q^{-i}}{(i+1)_{q}} s^{(i)} \sigma_{i}^{+}\left(q^{2 i}\right) s^{(i)}, \tag{2.4}
\end{align*}
$$

where

$$
\sigma_{i}^{ \pm}(x):=1+\frac{x-1}{q-q^{-1}} \sigma_{i}+\frac{\mu(x-1)}{\mu \mp q^{\mp 1} x} \kappa_{i} .
$$

The idempotents $a^{(n)}$ and $s^{(n)}$ in the algebra $\mathcal{W}_{n}(q, \mu)$ are primitive. They correspond to the $q$-deformations of the 'trivial' $\left(\sigma_{i} \mapsto q\right)$ and the 'alternating' $\left(\sigma_{i} \mapsto-q^{-1}\right)$ one-dimensional representations. Therefore, they are called an $n$-th order antisymmetrizer and an $n$-th order symmetrizer, respectively.

### 2.2 R-matrices and their compatible pairs

Let $V$ denote a finite dimensional $\mathbb{C}$-linear space, $\operatorname{dim} V=\mathrm{N}$. Fixing some basis $\left\{v_{i}\right\}_{i=1}^{\mathrm{N}}$ in $V$ we identify elements $X \in \operatorname{End}\left(V^{\otimes n}\right)$ with matrices $X_{i_{1} i_{2} \ldots i_{n}}^{j_{1} j_{2} \ldots j_{n}}$.

Let $X \in \operatorname{End}\left(V^{\otimes k}\right), k \leq n$. For $1 \leq m \leq n-k+1$, denote by $X_{m} \in \operatorname{End}\left(V^{\otimes n}\right)$ an operator given by the matrix

$$
\left(X_{m}\right)_{i_{1} \ldots i_{n}}^{j_{1} \ldots j_{n}}:=I_{i_{1} \ldots i_{m-1}}^{j_{1} \ldots j_{m-1}} X_{i_{m} \ldots i_{m+k-1}}^{j_{m} \ldots j_{m+k-1}} I_{i_{m+k} \ldots i_{n}}^{j_{m+k}} .
$$

Here $I$ denotes the identity operator.
An element $R \in \operatorname{Aut}\left(V^{\otimes 2}\right)$ that fulfills an equation

$$
R_{1} R_{2} R_{1}=R_{2} R_{1} R_{2}
$$

is called an $R$-matrix. The permutation operator $P$, defined by $P(u \otimes v)=v \otimes u \quad \forall u, v \in V$ is the R-matrix. The operator $R^{-1}$ is the R -matrix iff $R$ is.

Any R-matrix $R$ generates representations $\rho_{R}$ of the series of braid groups $\mathcal{B}_{n}, n=2,3, \ldots$

$$
\rho_{R}: \mathcal{B}_{n} \rightarrow \operatorname{Aut}\left(V^{\otimes n}\right), \quad \sigma_{i} \mapsto R_{i}, \quad 1 \leq i \leq n-1 .
$$

An R-matrix is called skew invertible if there exists an operator $\Psi_{R} \in \operatorname{End}\left(V^{\otimes 2}\right)$ such that

$$
\begin{equation*}
\operatorname{Tr}_{(2)} R_{12} \Psi_{R 23}=\operatorname{Tr}_{(2)} \Psi_{R 12} R_{23}=P_{13} \tag{2.5}
\end{equation*}
$$

Here we use notation $X_{i j}$ which shows explicitly indices $i$ and $j$ of the spaces where operator $X$ acts, e.g., $P_{13}=P_{i_{1} i_{3}}^{j_{1} j_{3}} I_{i_{2}}^{j_{2}}$. Symbol $\operatorname{Tr}_{(i)}$ means taking the trace in the vector space with index $i$.

For a skew invertible R-matrix $R$ define the operator $D_{R} \in \operatorname{End}(V)$

$$
\begin{equation*}
\left(D_{R}\right)_{1}:=\operatorname{Tr}_{(2)} \Psi_{R 12} \tag{2.6}
\end{equation*}
$$

The operator $R$ is called strict skew invertible if $D_{R}$ is invertible. The R-matrix $R^{-1}$ is skew invertible iff $R$ is strict skew invertible [I, O], the corresponding operator $D_{R^{-1}}$ reads

$$
\left(D_{R^{-1}}\right)_{2}=\left(\operatorname{Tr}_{(1)} \Psi_{R 12}\right)^{-1}
$$

With a skew invertible R-matrix $R$ we associate a linear map on the space of $\mathrm{N} \times \mathrm{N}$ matrices whose entries belong to some $\mathbb{C}$-linear space $W$

$$
\operatorname{Tr}_{R}: \operatorname{End}(V) \otimes W \rightarrow W, \operatorname{Tr}_{R}(M)=\sum_{i, j=1}^{\mathbb{N}}\left(D_{R}\right)_{i}^{j} M_{j}^{i}, M \in \operatorname{End}(V) \otimes W
$$

This map is called an $R$-trace.
It is easy to check that the R-matrix $P$ is strict skew invertible and $\operatorname{Tr}_{P}$ coincides with the usual trace. A characteristic property of the R-trace map is

$$
\begin{equation*}
\operatorname{Tr}_{R^{(2)}} R_{1}=I_{1} \tag{2.7}
\end{equation*}
$$

An ordered pair $\{R, F\}$ of two R-matrices $R$ and $F$ is called a compatible $R$-matrix pair if the following conditions

$$
\begin{equation*}
R_{1} F_{2} F_{1}=F_{2} F_{1} R_{2}, \quad R_{2} F_{1} F_{2}=F_{1} F_{2} R_{1}, \tag{2.8}
\end{equation*}
$$

are satisfied. The equalities (2.8) are called twist relations. Clearly, $\{R, P\}$ and $\{R, R\}$ are compatible pairs of R-matrices.

A compatible pair of R-matrices $\{R, F\}$ gives rise to a new R-matrix

$$
\begin{equation*}
R_{f}:=F^{-1} R F \tag{2.9}
\end{equation*}
$$

called the twisted R-matrix. The R-matrix pair $\left\{R_{f}, F\right\}$ is compatible. If $R$ is skew invertible and $F$ is strict skew invertible, then $R_{f}$ is skew invertible; if additionally $R$ is strict skew invertible, then $R_{f}$ is strict skew invertible as well [OP].

### 2.3 BMW type R-matrices

Assume that an R-matrix $R$ satisfies a third order minimal characteristic polynomial

$$
\begin{equation*}
(q I-R)\left(q^{-1} I+R\right)(\mu I-R)=0, \tag{2.10}
\end{equation*}
$$

and an element

$$
\begin{equation*}
K:=\mu^{-1}\left(q-q^{-1}\right)^{-1}(q I-R)\left(q^{-1} I+R\right) \tag{2.11}
\end{equation*}
$$

fulfills conditions

$$
\begin{equation*}
K_{2} K_{1}=K_{2} R_{1}^{ \pm 1} R_{2}^{ \pm 1}, \quad K_{1} K_{2} K_{1}=K_{1} . \tag{2.12}
\end{equation*}
$$

In this case $R$ generates representations $\rho_{R}$ of the tower of the BMW algebras $\mathcal{W}_{n}(q, \mu) \rightarrow$ $\operatorname{End}\left(V^{\otimes n}\right) \forall n>1$

$$
\rho_{R}: \mathcal{W}_{n}(q, \mu) \rightarrow \operatorname{Aut}\left(V^{\otimes n}\right), \quad \sigma_{i} \mapsto R_{i}, \quad \kappa_{i} \mapsto K_{i}, \quad 1 \leq i \leq n-1 .
$$

Such R-matrix is said to be of $B M W$ type.
If the R-matrix $R$ is skew invertible and of the BMW type, then it is strict skew invertible and the rank of the associated operator $K(2.11)$ equals 1 ; the R-trace map in this case fulfills equalities [IOP3]

$$
\begin{equation*}
\operatorname{Tr}_{R^{(2)}} K_{1}=\mu I_{1}, \quad \operatorname{Tr}_{R} I=\frac{(q-\mu)\left(q^{-1}+\mu\right)}{q-q^{-1}} . \tag{2.13}
\end{equation*}
$$

Let $\{R, F\}$ be a compatible pair of R-matrices, where $R$ is skew-invertible of the BMW type and $F$ is strict skew-invertible. In [OP] we associated with such a pair an invertible operator $G \in \operatorname{Aut}(V)$ and two invertible linear maps, $\phi$ and $\xi$, acting on the space $\operatorname{End}(V) \otimes W$ where $W$ is an arbitrary vector space. We extensively use the operator $G$ and maps $\phi$ and $\xi$ in investigations of the BMW type quantum matrix algebras (see, e.g., sections 3,4 below). Here we present formulas for them and for their inverses. The operator $G$ and its inverse read

$$
\begin{equation*}
G_{1}:=\operatorname{Tr}_{(23)} K_{2} F_{1}^{-1} F_{2}^{-1}, \quad G_{1}^{-1}=\operatorname{Tr}_{(23)} F_{2} F_{1} K_{2} \tag{2.14}
\end{equation*}
$$

The maps $\phi$ and $\xi$ are defined by

$$
\begin{align*}
\phi(M)_{1} & :=\operatorname{Tr}_{R^{(2)}}\left(F_{1} M_{1} F_{1}^{-1} R_{1}\right),  \tag{2.15}\\
\xi(M)_{1} & :=\operatorname{Tr}_{R^{(2)}}\left(F_{1} M_{1} F_{1}^{-1} K_{1}\right) . \tag{2.16}
\end{align*}
$$

Here $M$ is an arbitrary operator with values in a vector space $W, M \in \operatorname{End}(V) \otimes W$. The inverse maps read

$$
\begin{align*}
\phi^{-1}(M)_{1} & =\mu^{-2} \operatorname{Tr}_{R_{f}^{(2)}}\left(F_{1}^{-1} M_{1} R_{1}^{-1} F_{1}\right)  \tag{2.17}\\
\xi^{-1}(M)_{1} & =\mu^{-2} \operatorname{Tr}_{R_{f}^{(2)}}\left(F_{1}^{-1} M_{1} K_{1} F_{1}\right) . \tag{2.18}
\end{align*}
$$

Here the matrix $D_{R_{f}}$ which is needed for calculations of the $R_{f}$-traces is

$$
D_{R_{f}}=D_{F^{-1}}\left(D_{R^{-1}}\right)^{-1} D_{F} .
$$

### 2.4 Orthogonal and symplectic type R-matrices

Consider R-matrix realizations $\rho_{R}\left(a^{(i)}\right)$ of the antisymmetrizers (2.4). We impose additional constraints on a skew invertible BMW-type R-matrix $R$ demanding that

$$
\begin{equation*}
\operatorname{rk} \rho_{R}\left(a^{(i)}\right) \neq 0 \quad \forall i=2,3, \ldots, k \quad \text { and } \quad \rho_{R}\left(a^{(k)} \sigma_{k}^{-}\left(q^{-2 k}\right) a^{(k)}\right) \equiv 0 \tag{2.19}
\end{equation*}
$$

for some $k \geq 2$. Here we assume that the parameters $q, \mu$ fulfill conditions (c.f. with the conditions (2.2) )

$$
\begin{equation*}
i_{q} \neq 0 \quad \forall i=2,3, \ldots, k ; \quad \mu \neq-q^{-2 i+1} \quad \forall i=1,2, \ldots, k . \tag{2.20}
\end{equation*}
$$

Note that in case $(k+1)_{q} \neq 0$ the last condition in eq. (2.19) means vanishing of the ( $k+1$ )-st antisymmetrizer: $\rho_{R}\left(a^{(k+1)}\right)=0$. We do not use this short form to avoid unnecessary restrictions on the parameter $q$.

An R-matrix satisfying the conditions (2.19) is called an $R$-matrix of finite height; the number $k$ is called the height of the R -matrix.

Let us discuss some consequences of the relations (2.19). Applying $\operatorname{Tr}_{R^{(i)}}$ to $\rho_{R}\left(a^{(i)}\right)$ and using the relations (2.4), (2.7) and (2.13), we calculate

$$
\begin{equation*}
\operatorname{Tr}_{R^{(i)}} \rho_{R}\left(a^{(i)}\right)=\delta_{i} \rho_{R}\left(a^{(i-1)}\right) \tag{2.21}
\end{equation*}
$$

where $\delta_{i} \equiv \delta_{i}(q, \mu):=-\frac{q^{i-1}\left(\mu+q^{1-2 i}\right)\left(\mu^{2}-q^{4-2 i}\right)}{\left(\mu+q^{3-2 i}\right)\left(q-q^{-1}\right) i_{q}}$. In view of eqs.(2.21), the last condition in (2.19) implies, in particular, that $\delta_{k+1}=0$, wherefrom one specifies three admissible values of $\mu: \mu \in\left\{-q^{-1-2 k}, \pm q^{1-k}\right\}$.

Notice that the choice $\mu=-q^{1-k}$ contradicts the conditions (2.20) in the case when the number $k$ is even. In the case when $k$ is odd, the choices $\mu=-q^{1-k}$ and $\mu=q^{1-k}$ are related by a substitution $R \mapsto-R$. On the algebra level, this corresponds to an algebra isomorphism (see [OP], section 2.2) $\iota^{\prime \prime}: \mathcal{W}_{n}(q, \mu) \rightarrow \mathcal{W}_{n}(-q,-\mu), \iota^{\prime \prime}\left(\sigma_{i}\right)=-\sigma_{i}, i=1, \ldots, n-1$. The antisymmetrizers $a^{(i)}$ are invariant under this map. Therefore we are left with only two essentially different choices of the parameter $\mu$ : either $\mu=-q^{-1-2 k}$ or $\mu=q^{1-k}$. With these choices, the consistency of the conditions on $\mu$ in eq.(2.20) follows from the conditions on $q$.

We are now ready to define families of the orthogonal and symplectic R-matrices.
Definition 2.1 Let $R$ be a skew invertible BMW-type $R$-matrix. Assume additionally that the R -matrix $R$ has finite height $k$ for some $k \geq 2$. This implies, in particular, restrictions on $q$ : $i_{q} \neq 0$ for $i=2, \ldots, k$. Then
a) $R$ is called a $S p(2 k)$-type $R$-matrix in the case when $\mu=-q^{-1-2 k}$;
b) $R$ is called an $O(k)$-type $R$-matrix in the case when $\mu=q^{1-k}$ and $r k \rho_{R}\left(a^{(k)}\right)=1$.

For the standard R-matrices related to the quantum groups of the series $S p_{q}(2 k)$ and $S O_{q}(k)$ [RTF], the conditions a) and b), respectively, and the relations (2.19) are fulfilled. This explains our terminology.

The main subject of this paper is an investigation of the general structure of the quantum matrix algebras associated with the R-matrices of symplectic type (see next sections). For illustration purposes in subsection 4.3 we consider examples of such algebras related to the standard $S p(2 k)$-type R-matrices. For reader's convenience we recall formulas for these particular symplectic R-matrices.

The standard $S p(2 k)$-type R-matrix (see [RTF]) reads

$$
\begin{equation*}
R^{(\mathrm{st})}:=\sum_{i, j=1}^{2 k} q^{\left(\delta_{i j}-\delta_{i j^{\prime}}\right)} E_{i j} \otimes E_{j i}+\left(q-q^{-1}\right) \sum_{1 \leq j<i}^{2 k}\left\{E_{j j} \otimes E_{i i}-q^{\left(\rho_{i}-\rho_{j}\right)} \epsilon_{i} \epsilon_{j} E_{i^{\prime} j} \otimes E_{i j^{\prime}}\right\} . \tag{2.22}
\end{equation*}
$$

Here $E_{i j}$ are $2 k \times 2 k$ matrix units; $\delta_{i j}$ is the Kronecker symbol;

$$
\begin{equation*}
i^{\prime}=2 k+1-i ; \quad \epsilon_{i}=-\epsilon_{i^{\prime}}=1 ; \quad \rho_{i}=-\rho_{i^{\prime}}=(k+1-i) \forall i: 1 \leq i \leq k . \tag{2.23}
\end{equation*}
$$

The corresponding matrices $K^{(\mathrm{st})}$ and $D_{R^{(\mathrm{st})}}$ are

$$
\begin{equation*}
K^{(\mathrm{st})}=\sum_{i, j=1}^{2 k} q^{-\left(\rho_{i}+\rho_{j}\right)} \epsilon_{i} \epsilon_{j^{\prime}} E_{i j} \otimes E_{i^{\prime} j^{\prime}}, \quad D_{R^{(\mathrm{st})}}=\sum_{i=1}^{2 k} q^{-\left(2 k+2 \rho_{i}+1\right)} E_{i i} . \tag{2.24}
\end{equation*}
$$

Remark 2.2 For the family of symplectic R-matrices, the case $k=1$ is particular: the antisymmetrizer $\rho_{R}\left(a^{(2)}\right)$ vanishes and the minimal polynomial of $R$ becomes quadratic. The R-matrix $R^{\text {st) }}$, up to normalization and reparameterization $q \mapsto q^{1 / 2}$, is of the Hecke type $G L(2)$ (see $S p(2)$ examples in the subsection 4.3). This is a manifestation of the accidental isomorphism $S L(2) \sim S p(2)$. Accidental isomorphisms for quantum groups, corresponding to the standard deformation, are discussed in [JO].

Remark 2.3 Functions

$$
\Delta^{(i)}(q, \mu):=\operatorname{Tr}_{R^{(1,2, \ldots, i)}} \rho_{R}\left(a^{(i)}\right)=\prod_{j=1}^{i} \delta_{j}(q, \mu)
$$

are, up to an overall factor, particular elements of a set of rational functions $Q_{\lambda}\left(\mu^{-1}, q\right)$ labelled by partitions $\lambda \vdash i$; we have $\Delta(q, \mu)=\mu^{i} Q_{\left[1^{i}\right]}\left(\mu^{-1}, q\right)$. The functions $Q_{\lambda}\left(\mu^{-1}, q\right)$ were introduced in Theorem 5.5 in [W]. They describe the q-dimensions of the highest weight modules $V_{\lambda}$ for the orthogonal and symplectic quantum groups (see [W], Section 5 and [OrW], Lemma 3.1).

## 3 Quantum matrix algebra

In this section we recall definitions and main facts about the quantum matrix algebras from [OP]. A special attention is paid to the family of BMW type quantum matrix algebras. The notion of the characteristic subalgebra is introduced and two of its generating sets are described. The *-product of the quantum matrices is defined. It substitutes for the usual matrix multiplication in the case of quantum matrices. All these data are necessary for a proper generalization of the

Cayley-Hamilton theorem to the case of quantum matrix algebras. The latter is done in the next section.

Let $\{R, F\}$ be a compatible pair of R -matrices. In the sequel we assume that $R$ and $F$ are strict skew invertible although some definitions can be given without this condition. A quantum matrix algebra $\mathcal{M}(R, F)$ is a quotient algebra of the free associative unital algebra $W=\mathbb{C}\left\langle M_{a}^{b}\right\rangle$ by a two-sided ideal generated by entries of the matrix relation

$$
\begin{equation*}
R_{1} M_{\overline{1}} M_{\overline{2}}=M_{\overline{1}} M_{\overline{2}} R_{1} \tag{3.1}
\end{equation*}
$$

Here $M=\left\|M_{a}^{b}\right\|_{a, b=1}^{\mathrm{N}}$ is the matrix of generators; the matrix copies $M_{\bar{i}}$ are constructed with the help of the R-matrix $F$ in the following way

$$
\begin{equation*}
M_{\overline{\mathrm{I}}}:=M_{1}, \quad M_{\bar{i}}:=F_{i-1} M_{\overline{i-1}} F_{i-1}^{-1} . \tag{3.2}
\end{equation*}
$$

The set of relations

$$
\begin{equation*}
R_{i} M_{\bar{i}} M_{\overline{i+1}}=M_{\bar{i}} M_{\overline{i+1}} R_{i} \tag{3.3}
\end{equation*}
$$

for any given value of the index $i \geq 1$ is equivalent to (3.1) and can be as well used for the definition of the quantum matrix algebra.

Denote by $\mathcal{C}(R, F)$ a vector subspace of the quantum matrix algebra $\mathcal{M}(R, F)$ spanned linearly by the unity and elements

$$
\begin{equation*}
\operatorname{ch}\left(\alpha^{(n)}\right):=\operatorname{Tr}_{R^{(1, \ldots, n)}}\left(M_{\overline{1}} \ldots M_{\bar{n}} \rho_{R}\left(\alpha^{(n)}\right)\right), \quad n=1,2, \ldots, \tag{3.4}
\end{equation*}
$$

where $\alpha^{(n)}$ is an arbitrary element of the braid group $\mathcal{B}_{n}$. The space $\mathcal{C}(R, F)$ is a commutative subalgebra in $\mathcal{M}(R, F)$ (this is proved in the article [IOP1] which deals with the Hecke type quantum matrix algebras but the proof is valid for an arbitrary compatible pair $\{R, F\}$ ). The algebra $\mathcal{C}(R, F)$ is called the characteristic subalgebra of $\mathcal{M}(R, F)$.

Denote by $\mathcal{P}(R, F)$ a linear subspace of $\operatorname{End}(V) \otimes \mathcal{M}(R, F)$ spanned by $\mathcal{C}(R, F)$-multiples of the identity matrix, I ch $\forall c h \in \mathcal{C}(R, F)$, and by elements

$$
\begin{equation*}
M^{1}:=M,\left(M^{\alpha^{(n)}}\right)_{1}:=\operatorname{Tr}_{R^{(2, \ldots, n)}}\left(M_{\overline{1}} \ldots M_{\bar{n}} \rho_{R}\left(\alpha^{(n)}\right)\right), n=2,3, \ldots, \tag{3.5}
\end{equation*}
$$

where $\alpha^{(n)}$ belongs to the braid group $\mathcal{B}_{n}$. The space $\mathcal{P}(R, F)$ carries a structure of a right $\mathcal{C}(R, F)$-module

$$
\begin{equation*}
M^{\alpha^{(n)}} \operatorname{ch}\left(\beta^{(i)}\right)=M^{\left(\alpha^{(n)} \beta^{(i) \uparrow n}\right)} \forall \alpha^{(n)} \in \mathcal{B}_{n}, \beta^{(i)} \in \mathcal{B}_{i}, n, i=1,2, \ldots, \tag{3.6}
\end{equation*}
$$

Here, in the right hand side, we denoted by the same symbol $\alpha^{(n)}$ the image of the element $\alpha^{(n)}$ under the natural monomorphism $\mathcal{B}_{n} \hookrightarrow \mathcal{B}_{n+i}: \sigma_{j} \mapsto \sigma_{j}$. The symbol $\beta^{(i) \uparrow n}$ in the right hand side denotes the image of the element $\beta^{(i)}$ under the natural monomorphism $\mathcal{B}_{i} \hookrightarrow \mathcal{B}_{n+i}$ : $\sigma_{j} \mapsto \sigma_{j+n-1}$. Formula (3.6) is just a component-wise multiplication of the matrix $M^{\alpha^{(n)}}$ by the element $\operatorname{ch}\left(\beta^{(i)}\right)$.

We call $\star$-product the binary operation $\mathcal{P}(R, F) \otimes \mathcal{P}(R, F) \xrightarrow{\star} \mathcal{P}(R, F)$ defined by

$$
\begin{array}{rll}
\left(\operatorname{ch}\left(\beta^{(i)}\right) I\right) \star M^{\alpha^{(n)}}:=M^{\alpha^{(n)}} \operatorname{ch}\left(\beta^{(i)}\right) & =: & M^{\alpha^{(n)} \star\left(\operatorname{ch}\left(\beta^{(i)}\right) I\right),} \\
\left(\operatorname{ch}\left(\alpha^{(n)}\right) I\right) \star\left(\operatorname{ch}\left(\beta^{(i)}\right) I\right) & :=\left(\operatorname{ch}\left(\alpha^{(n)}\right) \operatorname{ch}\left(\beta^{(i)}\right)\right) I, \\
M^{\alpha^{(n)} \star} M^{\beta^{(i)}} & :=M^{\left(\alpha^{(n)} \star \beta^{(i)}\right)}, \tag{3.7}
\end{array}
$$

where we use the notation $\quad \alpha^{(n)} \star \beta^{(i)}:=\alpha^{(n)} \beta^{(i) \uparrow n}\left(\sigma_{n} \ldots \sigma_{2} \sigma_{1} \sigma_{2}^{-1} \ldots \sigma_{n}^{-1}\right)$.
The $\star$-product on $\mathcal{P}(R, F)$ is associative [OP].
In what follows we often use the $\star$-multiplication by the matrix of generators of the quantum matrix algebra $\mathcal{M}(R, F)$. Explicitly it reads, see [OP],

$$
\begin{equation*}
M \star N=M \cdot \phi(N) \quad \forall N \in \mathcal{P}(R, F), \tag{3.8}
\end{equation*}
$$

where • denotes the usual matrix multiplication and the map $\phi$ is defined in (2.15). In particular, one can introduce the noncommutative analogue of the matrix power:

$$
\begin{equation*}
M^{\overline{0}}:=I, \quad M^{\bar{n}}:=\underbrace{M \star M \star \cdots \star M}_{n \text { times }}=M^{\left(\sigma_{1} \sigma_{2} \ldots \sigma_{n-1}\right)} . \tag{3.9}
\end{equation*}
$$

Here we use symbol $M^{\bar{n}}$ for the $n$-th power of the matrix $M$.

### 3.1 BMW type

If $R$ is an R-matrix of the BMW, $S p(2 k)$ or $O(k)$ type then $\mathcal{M}(R, F)$ is called, respectively, a $B M W, S p(2 k)$ or $O(k)$ type quantum matrix algebra.

For the BMW type quantum matrix algebra the following relations are satisfied as a consequence of (3.1)

$$
\begin{equation*}
K_{i} M_{\bar{i}} M_{\overline{i+1}}=\mu^{-2} K_{i} g=M_{\bar{i}} M_{\overline{i+1}} K_{i} \quad \forall i \geq 1, \tag{3.10}
\end{equation*}
$$

where

$$
\begin{equation*}
g:=\frac{\mu\left(q-q^{-1}\right)}{(q-\mu)\left(q^{-1}+\mu\right)} \operatorname{Tr}_{R^{(1,2)}}\left(M_{\overline{1}} M_{\overline{2}} K_{1}\right) . \tag{3.11}
\end{equation*}
$$

The element $g$ is called a 2-contraction of $M$.
For the quantum matrix algebra of the BMW type the 2 -contraction $g$ is an element of the characteristic subalgebra. The characteristic subalgebra of the BMW type quantum matrix algebra is generated by either one of the sets $\left\{g, p_{i}\right\}_{i \geq 0}$, where

$$
\begin{equation*}
p_{0}=\operatorname{Tr}_{R} I=\frac{(q-\mu)\left(q^{-1}+\mu\right)}{q-q^{-1}}, p_{1}=\operatorname{Tr}_{R} M, p_{i}=\operatorname{ch}\left(\sigma_{i-1} \ldots \sigma_{2} \sigma_{1}\right) i=2,3, \ldots, \tag{3.12}
\end{equation*}
$$

or $\left\{g, a_{i}\right\}_{i \geq 0}$, where

$$
\begin{equation*}
a_{0}=1, \quad a_{i}=\operatorname{ch}\left(a^{(i)}\right) \quad i=1,2, \ldots \tag{3.13}
\end{equation*}
$$

Elements $p_{i}$ and $a_{i}$ are called power sums and elementary symmetric functions, respectively.

For the Hecke type quantum matrix algebra, the corresponding algebra $\mathcal{P}(R, F)$, as the $\mathcal{C}(R, F)$-module, is spanned by the matrix powers $M^{\bar{n}}, n \geq 0$, of the generating matrix $M$. For the BMW type quantum matrix algebra this is not the case. Namely, as the $\mathcal{C}(R, F)$-module, the BMW type algebra $\mathcal{P}(R, F)$ is spanned by matrices (see [OP], proposition 4.11)

$$
M^{\bar{n}} \quad \text { and } \quad M \top\left(M^{\overline{n+2}}\right), \quad n=0,1, \ldots
$$

Here we introduced a $\mathcal{C}(R, F)$-module map $M \mathrm{~T}: \mathcal{P}(R, F) \rightarrow \mathcal{P}(R, F)$

$$
\begin{equation*}
M \top(N):=M \cdot \xi(N), \quad \forall N \in \mathcal{P}(R, F), \tag{3.14}
\end{equation*}
$$

where the map $\xi$ is given in (2.16).
The BMW type algebra $\mathcal{P}(R, F)$ is commutative [OP].
To define inverse powers of the quantum matrix $M$ one considers the extension of the BMW type algebra $\mathcal{M}(R, F)$ by the inverse $g^{-1}$ of the 2-contraction

$$
\begin{equation*}
g^{-1} g=g g^{-1}=1, \quad g^{-1} M=\left(G^{-1} M G\right) g^{-1} \tag{3.15}
\end{equation*}
$$

where the numeric matrices $G^{ \pm 1} \in \operatorname{Aut}(V)$ are defined in eqs. (2.14). The latter relation in (3.15) is justified by the permutation rules for the 2 -contraction. For an arbitrary matrix $N \in \mathcal{P}(R, F)$ it reads

$$
\begin{equation*}
N g=g\left(G^{-1} N G\right) \tag{3.16}
\end{equation*}
$$

Proof. In a particular case $N=M$ - the matrix of generators of $\mathcal{M}(R, F)$ this formula is proved in [OP], lemma 4.13. Consequently, by lemma 3.11, eq. (3.45), [OP], we have $M_{\bar{j}} g=g\left(G_{j}^{-1} M_{\bar{j}} G_{j}\right), j=1,2, \ldots$ Thus for $\left.u=M_{\overline{1}} \ldots M_{\bar{n}} \rho_{R}\left(\alpha^{(n)}\right)\right), \alpha^{(n)} \in \mathcal{B}_{n}$, we have $u g=g G_{1} G_{2} \ldots G_{n} u G_{n}^{-1} \ldots G_{2}^{-1} G_{1}^{-1}$. By the cyclic property of the trace and lemma 3.11, eq. (3.44), $G_{2} \ldots G_{n}$ cancels with $G_{n}^{-1} \ldots G_{2}^{-1}$ which proves eq.(3.16) for $N=\operatorname{Tr}_{R^{(2, \ldots, n)}}(u)$.

The extended algebra, which we shall further denote by $\mathcal{M}^{\bullet}(R, F)$, contains the inverse matrix to the matrix $M$

$$
\begin{equation*}
M^{-1}=\mu \xi(M) g^{-1}, \quad M \cdot M^{-1}=I=M^{-1} \cdot M \tag{3.17}
\end{equation*}
$$

The matrix $M^{-1}$ is the inversion of $M$ with respect to the usual matrix product. Inversion with respect to the $\star$-product looks differently

$$
\begin{equation*}
M^{-1}=\phi^{-1}\left(M^{-1}\right), \quad M^{-1} \star M=I=M \star M^{-1} \tag{3.18}
\end{equation*}
$$

In general, $M^{-1} \neq M^{-1}$.
One can define the unique extension $\mathcal{P}^{\bullet}(R, F)$ of the algebra $\mathcal{P}(R, F)$ by a repeated $\star$ multiplication with $M^{-1}$

$$
\begin{equation*}
M^{-1} \star N:=\phi^{-1}\left(M^{-1} \cdot N\right)=: N \star M^{-1} \quad \forall N \in \mathcal{P}^{\bullet}(R, F) \tag{3.19}
\end{equation*}
$$

The algebra $\mathcal{P}^{\bullet}(R, F)$ is associative and commutative with respect to the $\star$-product. It is also the right $\mathcal{C}^{\bullet}(R, F)$-module algebra with respect to the extension $\mathcal{C}^{\bullet}(R, F) \supset \mathcal{C}(R, F)$ of the characteristic subalgebra by the element $g^{-1}$.

Particular examples of the $\star$-multiplication by $M^{\overline{-1}}$ are the inverse $\star$-powers of $M$

$$
M^{-n}:=\underbrace{M^{\overline{-1}} \star \cdots \star M^{-1}}_{n \text { times }} \star .
$$

The $\star$-powers obey the usual rules of the $\star$-product of matrix powers: $M^{\bar{i}} \star M^{\bar{n}}=M^{\overline{i+n}} \forall i, n \in$ $\mathbb{Z}$.

## 4 Cayley-Hamilton theorem

The Cayley-Hamilton theorem for the orthogonal and symplectic quantum groups was stated in the unpublished text [OP2]. Here we establish and discuss in details a strengthened version of the Cayley-Hamilton theorem in the symplectic case.

Throughout this section we assume that $\{R, F\}$ is a compatible pair of R -matrices, in which the operator $F$ is strict skew invertible and the operator $R$ is skew invertible of the BMW-type and, hence, strict skew invertible.

In the subsection 4.1 we investigate matrix relations in the algebra $\mathcal{P}(R, F)$ involving 'wedge' powers of the quantum matrix $M: M^{a^{(i)}}, 0 \leq i \leq n$. We confine the eigenvalues $q$ and $\mu$ of the matrix $R$ by conditions

$$
i_{q} \neq 0, \mu \neq-q^{3-2 i} \forall i=2,3, \ldots, n
$$

in which case all the antisymmetrizers $a^{(i)} \in \mathcal{W}_{n}(q, \mu), i=2,3, \ldots, n$, and, hence, the elements $a_{i} \in \mathcal{C}(R, F)$ and the matrices $M^{a^{(i)}} \in \mathcal{P}(R, F)$ are well defined.

Conditions on $R$ specific for the R-matrices of the type $S p(2 k)$, are imposed in Subsection 4.2.

### 4.1 Basic identities

Consider a set of 'wedge' powers of the quantum matrix $M$ : $M^{a^{(i)}} \in \mathcal{P}(R, F)$. Following [OP], we introduce series of matrices in $\mathcal{P}(R, F)$, which we further refer to as 'descendants' of the matrices $M^{a^{(i)}}$.

$$
\begin{array}{ll}
A^{(m, i)}:=i_{q} M^{\bar{m}} \star M^{a^{(i)}} & \forall i, m: 1 \leq i \leq n, m \geq 0 . \\
B^{(m+1, i)}:=i_{q} M^{\bar{m}} \star M \top\left(M^{a^{(i)}}\right) & \tag{4.1}
\end{array}
$$

It is suitable to set, by definition,

$$
\begin{equation*}
A^{(m, 0)}:=0 \quad \text { and } \quad B^{(m, 0)}:=0 \quad \forall m \geq 0 \tag{4.2}
\end{equation*}
$$

and to complement the series by the elements ${ }^{1}$

$$
\begin{align*}
A^{(-1, i)} & :=i_{q} \phi^{-1}\left(\operatorname{Tr}_{R^{(2,3, \ldots i)}} M_{\overline{2}} M_{\overline{3}} \ldots M_{\bar{i}} \rho_{R}\left(a^{(i)}\right)\right)  \tag{4.3}\\
B^{(0, i)} & :=i_{q} \phi^{-1}\left(\xi\left(M^{a^{(i)}}\right)\right) .
\end{align*}
$$

The following recursive relations among the descendants are derived in [OP]:

[^0]Lemma 4.1 For $0 \leq i \leq n-1$ and $m \geq 0$, the matrices $A^{(m-1, i+1)}$ and $B^{(m+1, i+1)}$ satisfy equalities

$$
\begin{align*}
& A^{(m-1, i+1)}=q^{i} M^{\bar{m}} a_{i}-A^{(m, i)}-\frac{\mu q^{2 i-1}\left(q-q^{-1}\right)}{1+\mu q^{2 i-1}} B^{(m, i)}  \tag{4.4}\\
& B^{(m+1, i+1)}=\left(\mu^{-1} q^{-i} M^{\bar{m}} a_{i}+\frac{q-q^{-1}}{1+\mu q^{2 i-1}} A^{(m, i)}-B^{(m, i)}\right) g \tag{4.5}
\end{align*}
$$

By a repeated use of these recurrent relations one can derive for a certain subset of the descendants their expansions in terms of non-negative matrix powers $M^{\bar{j}}, j \geq 0$, only ${ }^{2}$. For the Hecke type QM-algebras analogues of these expansions are known as the Cayley-HamiltonNewton identities [IOP, IOP1, IOPS].

Proposition 4.2 For $1 \leq i \leq n$ and $m \geq i-2$, one has

$$
\begin{equation*}
A^{(m, i)}=(-1)^{i-1} \sum_{j=0}^{i-1}(-q)^{j}\left\{M^{\overline{m+i-j}}+\frac{1-q^{-2}}{1+\mu q^{2 i-3}} \sum_{r=1}^{i-j-1} M^{\overline{m+i-j-2 r}}\left(q^{2} g\right)^{r}\right\} a_{j} \tag{4.6}
\end{equation*}
$$

For $1 \leq i \leq n$ and $m \geq i$, one has

$$
\begin{align*}
& B^{(m, i)}=(-1)^{i-1} \sum_{j=0}^{i-1}(-q)^{j}\left\{\mu^{-1} q^{-2 j} M^{\overline{m-i+j}} g^{i-j}\right. \\
&\left.-\frac{q^{-1}\left(1-q^{-2}\right)}{1+\mu q^{2 i-3}} \sum_{r=1}^{i-j-1} M^{\overline{m+i-j-2 r}}\left(q^{2} g\right)^{r}\right\} a_{j} \tag{4.7}
\end{align*}
$$

Proof. We employ induction on $i$. In the case $i=1$, the relations (4.6) and (4.7) reproduce the definitions (4.1):

$$
A^{(m, 1)}=M^{\overline{m+1}}, \quad B^{(m, 1)}=\mu^{-1} M^{\overline{m-1}} g
$$

It is then straightforward to verify the induction step $i \rightarrow i+1$ with the help of the relations (4.4) and (4.5).

Remark 4.3 When $m \geq i-2$ (respectively, $m \geq i$ ), all the $\star$-powers of $M$ in the right hand side of the relation (4.6) (respectively, the relation (4.7)) are non-negative. This is why we specify these restrictions on $m$. For an invertible matrix $M$, the restrictions on $m$ can be removed.

Remark 4.4 The Hecke type version of these relations can be reproduced by setting $g=0$ in formulas of Proposition 4.2. Relation for $B^{(m, j)}$ becomes trivial. Relation (4.6) for $A^{(m, j)}$

[^1]simplifies drastically, the terms with the element $g$ disappear and the condition $m \geq i-2$ weakens to $m \geq-1$. For $m=0$, the relation (4.6) reproduces the Cayley-Hamilton-Newton identities found in [IOP, IOP1]. The R-trace maps of these identities are the Newton relations. In the $G L(k)$-case, that is, if the operator $R$ fulfills the condition $\rho_{R}\left(a^{(k+1)}\right)=0$, the left hand side of the relation (4.6) vanishes in the case $i=k+1$. Then, with the choice $m=-1$ the relation (4.6) reproduces the Cayley-Hamilton identity.

### 4.2 Cayley-Hamilton theorem: type $S p(2 k)$

Specifying to the case of the $S p(2 k)$-type quantum matrix algebra, we notice that the condition $\mu=-q^{-1-2 k}$ leads to the following linear dependency between $A^{(m-1, k+1)}$, see (4.4), and $B^{(m+1, k+1)}$, see (4.5):

$$
\begin{equation*}
\left.\left(B^{(m+1, k+1)}+q A^{(m-1, k+1)} g\right)\right|_{\mu=-q^{-1-2 k}}=0 \quad \forall m \geq 0 \tag{4.8}
\end{equation*}
$$

The height $k$ condition (2.19) on the $S p(2 k)$-type R-matrix $R$ cuts the series of 'descendants' $A^{(m, i)}$ and $B^{(m, i)}$ at the level $i=k+1: A^{(m-1, k+1)}=B^{(m+1, k+1)}=0 \quad \forall m \geq 0$. The CayleyHamilton theorem follows exactly from these cutting conditions. The relations (4.8) show that all the conditions for $B^{(m+1, k+1)}$ follow from the conditions for $A^{(m-1, k+1)}$. In turn, by eqs. (4.1) and (4.3) we have

$$
\begin{equation*}
A^{(m-1, k+1)}=M^{\bar{m}} \star A^{(-1, k+1)} \tag{4.9}
\end{equation*}
$$

Thus, all the cutting conditions arise from the single one

$$
\begin{equation*}
A^{(-1, k+1)}=0 \tag{4.10}
\end{equation*}
$$

Unfortunately, the latter condition cannot be expressed in terms of nonnegative powers of the matrix $M$ only. By Proposition 4.2, for the condition

$$
\begin{equation*}
A^{(k-1, k+1)}=0 \tag{4.11}
\end{equation*}
$$

such an expression does exist.
The relations (4.10) and (4.11) are equivalent if the 2-contraction $g$ and, hence, the matrix $M$ are invertible. We shall first investigate the condition (4.11). Substituting $\mu=-q^{-1-2 k}$ and (4.6) into (4.11) and rearranging terms of the sum we obtain the Cayley-Hamilton theorem for the quantum matrices of the type $S p(2 k)$ :

Theorem 4.5 Let $\mathcal{M}(R, F)$ be the $S p(2 k)$-type quantum matrix algebra. Then the quantum matrix $M$ of the algebra generators satisfies the Cayley-Hamilton identity

$$
\begin{equation*}
\sum_{i=0}^{2 k}(-q)^{i} M^{\overline{2 k-i}} \epsilon_{i}=0 \tag{4.12}
\end{equation*}
$$

where

$$
\begin{equation*}
\epsilon_{i}:=\sum_{j=0}^{[i / 2]} a_{i-2 j} g^{j}, \quad \epsilon_{k+i}:=\epsilon_{k-i} g^{i} \quad \forall i=1,2, \ldots, k \tag{4.13}
\end{equation*}
$$

Let us now consider the matrix identity (4.10). In case of non-invertible $g$, this identity is more informative than the Cayley-Hamilton identity (4.12). Matrix components of its left hand side are $k$-th order homogeneous polynomials in the components of the quantum matrix $M$, containing, apart of $M$, *-powers of yet another quantum matrix obtained from $M$ by a linear map $\pi:=\mu \phi^{-1} \circ \xi$ (see eqs. (2.16), (2.17)).

Lemma 4.6 For the compatible pair $\{R, F\}$ of strict skew invertible $R$-matrices, where $R$ is of the BMW-type, the map $\pi:=\mu \phi^{-1} \circ \xi$ does not depend on $F$. The explicit formulas for $\pi$ and $\pi^{-1}$ read:

$$
\begin{align*}
\pi(M)_{1} & =\operatorname{Tr}_{R^{(2)}} R_{12} M_{1} K_{12}=\operatorname{Tr}_{R^{(2)}} K_{12} M_{1} R_{12}  \tag{4.14}\\
\pi^{-1}(M)_{1} & =\mu^{-2} \operatorname{Tr}_{R^{(2)}} R_{12}^{-1} M_{1} K_{12}=\mu^{-2} \operatorname{Tr}_{R^{(2)}} K_{12} M_{1} R_{12}^{-1} \tag{4.15}
\end{align*}
$$

Proof. Instead of proving the first equality in (4.14) directly it is easier to verify the relation $\phi\left(\operatorname{Tr}_{R^{(2)}} R_{1} M_{1} K_{1}\right)=\mu \xi(M)_{1}:$

$$
\begin{align*}
\phi\left(\operatorname{Tr}_{R^{(2)}} R_{1} M_{1} K_{1}\right) & =\operatorname{Tr}_{R^{(2)}} F_{12}\left\{\operatorname{Tr}_{R^{\left(2^{\prime}\right)}} R_{12^{\prime}} M_{1} K_{12^{\prime}}\right\} F_{12}^{-1} R_{12} \\
& =\operatorname{Tr}_{R^{(23)}} \underline{F_{1}\left\{F_{2} R_{1} M_{1} K_{1} F_{2}^{-1}\right\} F_{1}^{-1} R_{1}} \\
& =\operatorname{Tr}_{R^{(23)}} \underline{R_{2}} F_{1} \underline{F_{2} M_{1} F_{2}^{-1} F_{1}^{-1} K_{2} R_{1}} \\
& =\operatorname{Tr}_{R^{(23)}} F_{1} M_{1} F_{1}^{-1} \underline{K_{2} R_{1} R_{2}}  \tag{4.16}\\
& =\operatorname{Tr}_{\left.R^{(23}\right)} F_{1} M_{1} F_{1}^{-1} \underline{K_{2} K_{1}} \\
& =\mu \operatorname{Tr}_{R^{(2)}} F_{1} M_{1} F_{1}^{-1} K_{1}=\mu \xi(M) .
\end{align*}
$$

Here in calculations we underline terms which undergo a transformation in the next step. For the transformations we used the compatibility relations for the pair $\{R, F\}$ (2.8), BMW algebra relations for the matrices $R$ and $K(2.12)$, first formula in (2.13), and the following properties of the R-trace (see [OP], lemma 3.2 and corollary 3.4)

$$
\operatorname{Tr}_{R^{(2)}} F_{1}^{ \pm 1} X_{1} F_{1}^{\mp 1}=I_{1} \operatorname{Tr}_{R} X \quad \forall X \in \operatorname{End}(V) \otimes W
$$

where $W$ is a $\mathbb{C}$-linear space, and

$$
\left[R_{12}, D_{R 1} D_{R 2}\right]=0
$$

which is equivalent to

$$
\operatorname{Tr}_{R^{(12)}} Y_{1} R_{1}=\operatorname{Tr}_{R^{(12)}} R_{1} Y_{1} \quad \forall Y \in \operatorname{End}\left(V^{\otimes 2}\right) \otimes W
$$

The second equality in relation (4.14) holds for an arbitrary BMW type R-matrix $R$.
To prove formula (4.15) one notices that the map $\pi$ is proportional to the map $\xi$ for the pair $\{R, R\}$. Thus the first equality in (4.15) follows from the formula for $\xi^{-1}$ for the pair $\{R, R\}$, see (2.18). The second equality in (4.15) is obtained from the first one by the same remark as for the map $\pi$.

Until the end of this subsection we let $M$ to be the matrix of generators of the BMW type quantum matrix algebra $\mathcal{M}(R, F)$.

In general, the matrix $\pi(M)$ does not belong to the algebra $\mathcal{P}(R, F)$. On the other hand, $\pi(M)$ is related to the $\star$-inverse of the matrix $M$ (see (3.18))

$$
\begin{equation*}
\pi(M)=M^{\overline{-1}} g=M^{-1} \star I g \tag{4.17}
\end{equation*}
$$

and thus belongs to the extended algebra $\mathcal{P}^{\boldsymbol{0}}(R, F)$. The formula for the $\star$-product for the matrix $\pi(M)$ is clearly induced from that for $M^{-1}$ (see (3.19)) and the permutation rules for $g$ (see (3.16)):

$$
\begin{equation*}
\pi(M) \star N:=N \star \pi(M):=\mu \phi^{-1}\left(\xi(M) \cdot G^{-1} N G\right), \quad \forall N \in \mathcal{P}(R, F) . \tag{4.18}
\end{equation*}
$$

Complementing the algebra $\mathcal{P}(R, F)$ with the $\star$-multiples of $\pi(M)$

$$
\pi(M)^{\bar{n}}:=\underbrace{\pi(M) \star \cdots \star \pi(M)}_{n \text { times }}
$$

one obtains an intermediate extension $\mathcal{P}^{\circ}(R, F) \supset \mathcal{P}(R, F), \mathcal{P}^{\circ}(R, F) \subset \mathcal{P}^{\bullet}(R, F)$. It is this algebra where the matrix $A^{(-1, k+1)}$ belongs to.

Now we are ready to write down the identity (4.10) in terms of $\star$-powers of the matrices $M$ and $\pi(M)$.

Proposition 4.7 Let $\mathcal{M}(R, F)$ be the $S p(2 k)$-type quantum matrix algebra. The matrix $M$ of generators of this algebra and its image $\pi(M)$ under the map (4.14) satisfy the following $k$-th order matrix polynomial identity

$$
\begin{equation*}
\sum_{i=0}^{k}(-q)^{i} M^{\overline{k-i}} \epsilon_{i}+q^{2 k} \sum_{i=0}^{k-1}(-q)^{-i} \pi(M)^{\overline{k-i}} \epsilon_{i}=0 \tag{4.19}
\end{equation*}
$$

Here the coefficients $\epsilon_{i}, i=1, \ldots, k$, are given by eq. (4.13).

### 4.3 Simple examples and classical limit.

In this section we present the Cayley-Hamilton and 'pre-Cayley-Hamilton' identities (4.12) and (4.19) for the standard RTT- and RE-algebras corresponding to the $S p(2 k)$-type R-matrix (2.22) in cases $k=1,2$.

Standard Sp(2)-type RTT-algebra is the quantum matrix algebra $\mathcal{M}\left(R^{\text {st })}, P\right)$, where the Rmatrix $R^{(\mathrm{st})}(2.22)$ and permutation $P$ act on a tensor square of the 2-dimensional vector space. We use the symbol $T$ for the $2 \times 2$ matrix of generators of this algebra. Permutation relations for its components $T_{j}^{i}, i, j \in\{1,2\}$ are identical to the permutaton relations of the standard $G L_{q^{2}}(2)$-type RTT-algebra:

$$
\begin{equation*}
q^{2} T_{2}^{i} T_{1}^{i}=T_{1}^{i} T_{2}^{i}, \quad q^{2} T_{i}^{2} T_{i}^{1}=T_{i}^{1} T_{i}^{2}, \quad\left[T_{1}^{2}, T_{2}^{1}\right]=0, \quad\left[T_{2}^{2}, T_{1}^{1}\right]=\left(q^{-2}-q^{2}\right) T_{2}^{1} T_{1}^{2} \tag{4.20}
\end{equation*}
$$

The R-matrix image $\rho_{R^{(\mathrm{st)}}}\left(a^{(2)}\right)$ of the second order antisymmetrizer vanishes in this particular case and, therefore, there is no any additional $g$-covariance conditions.

The two generators of the characteristic subalgebra $g$ and $a_{1}$ read

$$
\begin{align*}
g & =\frac{q^{-6}}{q^{2}+q^{-2}}\left(q^{-2} T_{1}^{1} T_{2}^{2}+q^{2} T_{2}^{2} T_{1}^{1}-T_{2}^{1} T_{1}^{2}-T_{1}^{2} T_{2}^{1}\right) \\
& =q^{-6}\left(T_{1}^{1} T_{2}^{2}-q^{2} T_{2}^{1} T_{1}^{2}\right),  \tag{4.21}\\
a_{1} & =\operatorname{Tr}_{R} T=q^{-5} T_{1}^{1}+q^{-1} T_{2}^{2},
\end{align*}
$$

where the second simplified expression for $g$ is obtained with the help of the permutation relations (4.20). The 2 -contraction $g$ is central in this case (the matrix $G$ for the R-matrix pair $\left\{R^{\text {(st) }}, P\right\}$ equals the unity), while the element $a_{1}$ is not.

To write down the characteristic identities for this algebra we need explicit expressions for the maps $\phi$ (2.15) and $\pi$ (4.14)

$$
\phi(T)=\left(\begin{array}{cc}
q^{-4} T_{1}^{1}+\left(1-q^{-4}\right) T_{2}^{2} & q^{-6} T_{2}^{1}  \tag{4.22}\\
q^{-2} T_{1}^{2} & T_{2}^{2}
\end{array}\right), \quad \pi(T)=\left(\begin{array}{cc}
\left(q^{-6}+q^{-2}\right) T_{1}^{1}+q^{-2} T_{2}^{2} & -q^{-2} T_{2}^{1} \\
-q^{-2} T_{1}^{2} & q^{-6} T_{2}^{2}
\end{array}\right) .
$$

The Cayley-Hamilton identity (4.12) and its parent identity (4.19), respectively, read

$$
\begin{align*}
& T^{\overline{2}}-q T a_{1}+q^{2} I g=0,  \tag{4.23}\\
& T-q I a_{1}+q^{2} \pi(T)=0, \tag{4.24}
\end{align*}
$$

where $T^{\overline{2}}=T \cdot \phi(T)$.
We note that the identity (4.23) coincides with the Cayley-Hamilton identity for the standard $G L_{q^{2}}(2)$-type RTT-algebra (see [EOW, IOP, IOP2]), where the 2 -contraction $g$ plays the role of the quantum determinant of the matrix $T$. In this particular case the Cayley-Hamilton identity (4.23) encodes the half of the permutation relations (4.20); in general, a half-quantum matrix of $G L$ type satisfies the Cayley-Hamilton identity [CFR, IO].

Another specific feature of the $S p(2)$ case is that the 'parent' Cayley-Hamilton identity (4.24) being linear in generators is satisfied without any reference to the quadratic permutation relations.

The standard $S p(2)$-type Reflection Equation (RE) algebra is the quantum matrix algebra $\mathcal{M}\left(R^{(\mathrm{st})}, R^{(\mathrm{st})}\right)$, where the R-matrix $R^{(\mathrm{st})}(2.22)$ acts on the tensor square of the 2 -dimensional vector space. We use the symbol $L$ for the $2 \times 2$ matrix of generators of this algebra. The permutation relations for its components $L_{j}^{i}, i, j \in\{1,2\}$, are identical to the permutation relations for the standard $G L_{q^{2}}(2)$-type RE-algebra:

$$
\begin{gather*}
L_{j}^{i} L_{1}^{1}=q^{4(j-i)} L_{1}^{1} L_{j}^{i}, \quad\left[L_{2}^{2}, L_{2}^{1}\right]=\left(1-q^{-4}\right) L_{1}^{1} L_{2}^{1}, \\
{\left[L_{2}^{2}, L_{1}^{2}\right]=-q^{-4}\left(1-q^{-4}\right) L_{1}^{1} L_{1}^{2},}  \tag{4.25}\\
{\left[L_{1}^{2}, L_{2}^{1}\right]=\left(1-q^{-4}\right) L_{1}^{1}\left(L_{1}^{1}-L_{2}^{2}\right) .}
\end{gather*}
$$

The two generators of the characteristic subalgebra $g$ and $a_{1}$ are

$$
\begin{align*}
g & =\frac{q^{-4}}{q^{2}+q^{-2}}\left(L_{1}^{1} L_{2}^{2}+L_{2}^{2} L_{1}^{1}-\left(1-q^{-4}\right)\left(L_{1}^{1}\right)^{2}-L_{2}^{1} L_{1}^{2}-q^{4} L_{1}^{2} L_{2}^{1}\right) \\
& =q^{-2}\left(L_{1}^{1} L_{2}^{2}-\left(1-q^{-4}\right)\left(L_{1}^{1}\right)^{2}-L_{2}^{1} L_{1}^{2}\right),  \tag{4.26}\\
a_{1} & =\operatorname{Tr}_{R} L=q^{-5} L_{1}^{1}+q^{-1} L_{2}^{2},
\end{align*}
$$

where the second expression for $g$ is obtained with the use of permutation relations (4.25). As for any RE-algebra, the generators $g$ and $a_{1}$ are central.

Another distinguishing property of the RE-algebras - the identity of the map $\phi$ - makes their characteristic identity (4.23) particularly simple and similar to the classical case. In our situation it reads

$$
\begin{equation*}
L^{2}-q L a_{1}+q^{2} I g=0, \tag{4.27}
\end{equation*}
$$

where $L^{2}$ means the usual matrix square of $L$ and the coefficients $g$ and $a_{1}$ are given by (4.26). Again, this matrix equality encodes half of the permutation relations (4.25).

As stated in lemma 4.6 the map $\pi$ depends on the first R-matrix from the compatible pair $\{R, F\}$ only. Hence, for the RTT- and RE-algebra generating matrices $T$ and $L$ the map $\pi$ is literally the same (see (4.22)), and the parent Cayley-Hamilton identities for the $\mathrm{Sp}(2)$-type RTT- and RE-algebras coincide (see 4.24).

Next, we consider a less trivial example in order to demonstrate the results of this section in a greater generality. It is the standard $S p(4)$-type $R T T$-algebra - the quantum matrix algebra $\mathcal{M}\left(R^{(s t)}, P\right)$, where the R-matrix $R^{(s t)}(2.22)$ and the permutation $P$ now act on the tensor square of the 4 -dimensional vector space. We keep notation $M$ for the $4 \times 4$ matrix of generators of this algebra. Quadratic relations in this algebra consist of 120 permutation relations for 16 matrix components, and of 10 additional conditions. The latter ones together with expression for the 2 -contraction $g$ can be extracted from the matrix equalities (3.10), where $i=1$ and $\mu=-q^{-5}$ in our case. All the quadratic relations and the expressions for $g$ are collected in the Appendix. There and in the formulas below it is suitable to break the $4 \times 4$ matrix $M$ into four $2 \times 2$ blocks $A, B, C$ and $D:$

$$
M=\left(\begin{array}{ll}
A & B  \tag{4.28}\\
C & D
\end{array}\right) .
$$

The coefficients $\epsilon_{1}$ and $\epsilon_{2}$ of the Cayley-Hamilton identity, together with the 2-contraction $g$ generate the characteristic subalgebra. The 2-contraction $g$ is central, while $\epsilon_{i}, i=1,2$, are not. Expression for $g$ is given in the Appendix (see eq.(A.2)); formulas for $\epsilon_{i}$ read

$$
\begin{aligned}
\epsilon_{1}= & a_{1}=q^{-9} A_{1}^{1}+q^{-7} A_{2}^{2}+q^{-3} D_{1}^{1}+q^{-1} D_{2}^{2} \\
\epsilon_{2}= & a_{2}+g= \\
& q^{-16}\left(A_{1}^{1} A_{2}^{2}-q A_{2}^{1} A_{1}^{2}\right) \\
& +q^{-4}\left(D_{1}^{1} D_{2}^{2}-q D_{2}^{1} D_{1}^{2}\right)+q^{-12}\left(D_{1}^{1}+q^{2} D_{2}^{2}\right)\left(A_{1}^{1}+q^{2} A_{2}^{2}\right) \\
& \quad-q^{-12}\left(q^{-1} C_{1}^{1} B_{1}^{1}-\left(q-q^{-1}\right) C_{1}^{1} B_{2}^{2}+C_{2}^{1} B_{1}^{2}+C_{1}^{2} B_{2}^{1}+q^{3} C_{2}^{2} B_{2}^{2}\right) .
\end{aligned}
$$

To write down the characteristic identities we also need expressions for the maps $\xi^{ \pm 1}, \phi^{ \pm 1}$. They are

$$
\begin{align*}
& \xi(M)=\left(\begin{array}{cc}
-q^{-5} \sigma_{q}(D) & q^{-8} \sigma_{q}(B) \\
q^{-2} \sigma_{q}(C) & -q^{-5} \sigma_{q}(A)
\end{array}\right), \\
& \phi(M)=\left(\begin{array}{cc}
q^{-6} \alpha_{q}^{+}(A)+\left(1-q^{-2}\right) \beta_{q}(D) & q^{-7} \alpha_{q}^{-}(B) \\
q^{-1} \alpha_{q}^{-}(C) & \alpha_{q}^{+}(D)
\end{array}\right), \\
& \xi^{-1}(M)=\left.\xi(M)\right|_{q \leftrightarrow q^{-1}}, \quad \phi^{-1}(M)=\left.\phi(M)\right|_{q \leftrightarrow q^{-1}} . \tag{4.29}
\end{align*}
$$

Here $\sigma_{q}, \alpha_{q}^{ \pm}, \beta_{q}$ are linear maps of the $2 \times 2$ matrices

$$
\begin{gather*}
\sigma_{q}(X)=\left(\begin{array}{cc}
X_{2}^{2} & q^{-1} X_{2}^{1} \\
q X_{1}^{2} & X_{1}^{1}
\end{array}\right), \alpha_{q}^{ \pm}(X)=\left(\begin{array}{cc}
q^{-2} X_{1}^{1} \pm\left(1-q^{-2}\right) X_{2}^{2} & q^{-3} X_{2}^{1} \\
q^{-1} X_{1}^{2} & X_{2}^{2}
\end{array}\right), \\
\beta_{q}(X)=\left(q^{-2} X_{1}^{1}+X_{2}^{2}\right) I+q^{-4} \sigma_{q}(X), \tag{4.30}
\end{gather*}
$$

The following properties of these maps make the check of the relations (4.29) staightforward:

$$
\left(\sigma_{q}\right)^{-1}=\sigma_{1 / q}, \quad\left(\alpha_{q}^{ \pm}\right)^{-1}=\alpha_{1 / q}^{ \pm}, \quad \beta_{q} \circ \alpha_{1 / q}^{+}=q^{-4} \alpha_{q}^{+} \circ \beta_{1 / q}
$$

The composite map $\pi=-q^{-5}\left(\phi^{-1} \circ \xi\right)(M)$ reads explicitly

$$
\pi(M)=\left(\begin{array}{cc}
q^{-4}\left(\alpha_{1 / q}^{+} \circ \sigma_{q}\right)(D)-q^{-8}\left(1-q^{-2}\right)\left(\beta_{1 / q} \circ \sigma_{q}\right)(A) & -q^{-6}\left(\alpha_{1 / q}^{-} \circ \sigma_{q}\right)(B) \\
-q^{-6}\left(\alpha_{1 / q}^{-} \circ \sigma_{q}\right)(C) & q^{-10}\left(\alpha_{1 / q}^{+} \circ \sigma_{q}\right)(A)
\end{array}\right) .
$$

Now we are ready to write down the characteristic identities (4.12) and (4.19) for the case of the standard $S p(4)$-type RTT-algebra:

$$
\begin{array}{r}
M^{\overline{4}}-q M^{\overline{3}} \epsilon_{1}+q^{2} M^{\overline{2}} \epsilon_{2}-q^{3} M \epsilon_{1} g+q^{4} I g^{2}=0, \\
M^{\overline{2}}-q M \epsilon_{1}+q^{2} I \epsilon_{2}-q^{3} \pi(M) \epsilon_{1}+q^{4} \pi^{\overline{2}}(M)=0 . \tag{4.32}
\end{array}
$$

For reader's convenience we recall formulas for the powers of quantum matrices:

$$
M^{\overline{i+1}}=M \cdot \phi\left(M^{\bar{i}}\right) \forall i \geq 1, \quad \pi^{2}(M)=-q^{-5} \phi^{-1}(\xi(M) \cdot \pi(M)),
$$

where in the last formula we took into account that $\mu=-q^{-5}$ and $G=I$ in our particular case.
Using the definitions of the maps $\xi, \phi^{ \pm 1}, \pi$ and of the elements $\epsilon_{1}, \epsilon_{2}$ given above, and applying the quadratic relations from the Appendix one can check the parent characteristic identity (4.32) directly. The Cayley-Hamilton identity (4.31) follows from it by the $\star$-multiplication by $M^{\overline{2}}$.

Finally, we consider the classical limit of the parent Cayley-Hamilton identities. In the limit $q \rightarrow 1$ the standard $S p(2 k)$-type R-matrix (2.22) becomes the usual permutation and the quadratic relations (3.1) in the corresponding algebra $\mathcal{M}(P, P)$ imply the commutativity of the components of matrix $M$. The rank $=1$ projector $K^{(\mathrm{st})}(2.24)$ decouples from the R -matrix and the $g$-invariance conditions (3.10) become independent of (3.1) and should be treated separately. We rewrite them in the familiar form

$$
\begin{equation*}
M^{t} \Omega M=g \Omega=M \Omega M^{t} \tag{4.33}
\end{equation*}
$$

Here $M^{t}$ is the transposed matrix and $\Omega$ is the $2 k \times 2 k$ matrix of the symplectic quadratic form. With our choice of the rank $=1$ matrix $K^{(s t)}$ it reads

$$
\Omega=\left(\begin{array}{cc}
0 & w \\
-w & 0
\end{array}\right)
$$

where $w$ is the $k \times k$ antidiagonal matrix: $w_{j}^{i}=\delta_{j^{\prime}}^{i}, \quad j^{\prime}=k+1-j$.
Notice that in case $g \neq 0$ (more formally, if $g$ is invertible) the left and right equalities in (4.33) result in equivalent sets of conditions. On the contrary, in case $g=0$ these equalities are not equivalent and only together they give the complete set of the $g$-invariance conditions.

Again, it is suitable to use the block notation (4.28) for the matrix $M$, where now $A, B, C$ and $D$ are $k \times k$ matrices. The matrix $\pi(M)$ in this notation is

$$
\pi(M)=-\Omega M^{t} \Omega=\left(\begin{array}{cc}
D^{\prime} & -B^{\prime} \\
-C^{\prime} & A^{\prime}
\end{array}\right),
$$

where $X^{\prime}=w X^{t} w$. This operation is a classical counterpart of the map $\sigma_{q}$ from our previous example.

The classical parent Cayley-Hamilton identity reads

$$
\begin{equation*}
\sum_{i=0}^{k}(-1)^{i} M^{k-i} \epsilon_{i}+\sum_{i=0}^{k-1}(-1)^{i} \pi(M)^{k-i} \epsilon_{i}=0 \tag{4.34}
\end{equation*}
$$

where now all matrix powers are calculated according to the usual rules (the map $\phi$ in the classical limit is identical) and the coefficients $\epsilon_{i}$ become usual traces of the $i$-th wedge powers of the matrix $M: \epsilon_{i}=\operatorname{Tr}\left(\wedge^{i} M\right)$ (the antisymmetrizers computed with the permutation matrix $P$ automatically include contributions from the 2-contraction $g$ ).

Assuming the invertibility of the matrix $A$ one can solve the $g$-invariance relations explicitly

$$
M=\left(\begin{array}{cc}
A & A Y  \tag{4.35}\\
X A & X A Y+g A^{\prime-1}
\end{array}\right)=\left(\begin{array}{cc}
I & 0 \\
X & I
\end{array}\right)\left(\begin{array}{cc}
A & 0 \\
0 & g A^{\prime-1}
\end{array}\right)\left(\begin{array}{cc}
I & Y \\
0 & I
\end{array}\right) .
$$

where matrices $X, Y$ are such that $X^{\prime}=X, \quad Y^{\prime}=Y$.
Substituting this parameterization for $M$ into the identity (4.34) one can reduce it, at least in cases $k=1,2$, to the Cayley-Hamilton identities for $k \times k$ matrices $A$ and $X Y$.

### 4.4 Spectral parameterization

In this section we describe the parameterization the coefficients of the characteristic polynomial (4.12) by means of a $\mathbb{C}$-algebra $\mathcal{E}_{2 k}$ of polynomials in $2 k+1$ pairwise commuting variables $\nu_{i}$, $i=0,1, \ldots, 2 k$, satisfying conditions

$$
\begin{equation*}
\nu_{k+i} \nu_{k+1-i}=\nu_{0}^{2} \quad \forall i=1,2, \ldots, k \tag{4.36}
\end{equation*}
$$

We call $\nu_{i}, i=0,1, \ldots, 2 k$, spectral variables. These variables play a role of the eigenvalues of the symplectic type quantum matrix $M$. This parameterization was initially aimed at comparing our results with expressions given for the power sums for the RE-algebras in [Mudr] (see the subsection 8.3 there). Although the derivation methods are very different the results agree up to some obvious changes in a notation. Notice that compared to [Mudr] we are working in a more general setting. The generalization goes in several directions. First, we do not assume a
"standard" Drinfel'd-Jimbo's form for the R-matrices defining the algebra and, moreover, we do not use any deformation assumptions in our constructions. Next, we are working with a wider family of QM-algebras. And, finally, we are working directly in the algebra without passing to representations ${ }^{3}$.

We are going to factorize the polynomial in the left hand side of the equation (4.12). To this end, we realize elements of the characteristic subalgebra $\mathcal{C}(R, F)$ as polynomials in the spectral variables and construct a corresponding extention of the algebra $\mathcal{P}(R, F)$.

Proposition 4.8 In the setting of the theorem 4.5, assume that the elements $a_{i}, i=1,2, \ldots, k$, are algebraically independent. Consider an algebra homomorphism of the characteristic subalgebra $\mathcal{C}(R, F)$ to the algebra of the spectral variables $\mathcal{E}_{2 k}, \pi_{S p(2 k)}: \mathcal{C}(R, F) \rightarrow \mathcal{E}_{2 k}$, defined on the generators by

$$
\begin{equation*}
\pi_{S p(2 k)}: \quad g \mapsto \nu_{0}^{2}, \quad a_{i} \mapsto e_{i}\left(\nu_{0},-\nu_{0}, \nu_{1}, \nu_{2}, \ldots, \nu_{2 k}\right) \quad \forall i=1, \ldots, k, \tag{4.37}
\end{equation*}
$$

where $e_{i}$ are the elementary symmetric polynomials of their arguments (for the symmetric polynomials we adopt a notation of [Mac]). The map $\pi_{S p(2 k)}$ defines naturally a left $\mathcal{C}(R, F)$-module structure on the algebra $\mathcal{E}_{2 k}$. Consider a corresponding completion of the algebra $\mathcal{P}(R, F)$,

$$
\mathcal{P}_{S p(2 k)}(R, F):=\mathcal{P}(R, F) \underset{\mathcal{C}(R, F)}{\otimes} \mathcal{E}_{2 k},
$$

where the $\star$-product on the completed space is given by the formula

$$
\begin{equation*}
\left(N_{\mathcal{C}(R, F)}^{\otimes} \nu\right) \star\left(N_{\mathcal{C}(R, F)}^{\otimes} \nu^{\prime}\right):=\left(N \star N^{\prime}\right) \underset{\mathcal{C}(R, F)}{\otimes}\left(\nu \nu^{\prime}\right) \quad \forall N, N^{\prime} \in \mathcal{P}(R, F) \text { and } \forall \nu, \nu^{\prime} \in \mathcal{E}_{2 k} . \tag{4.38}
\end{equation*}
$$

Then, in the completed algebra $\mathcal{P}_{S p(2 k)}(R, F)$, the Cayley-Hamilton identity (4.12) acquires a factorized form

$$
\begin{equation*}
{\underset{i=1}{2 k}}_{\star}^{\star_{1}}\left(M-q \nu_{i} I\right)=0 \tag{4.39}
\end{equation*}
$$

where the symbol $\mp$ denotes the product with respect to the $\star$-multiplication (4.38).
Remark 4.9 For the classical symplectic groups, the functions $a_{i}, i=1, \ldots, k$, on the manifold $S p(2 k)$ are functionally independent. This justifies, at least perturbatively, the corresponding assumptions about the independence of the elements $a_{i}$ in the proposition above.

Remark 4.10 For a general quantum matrix algebra $\mathcal{M}(R, F)$ the characteristic subalgebra does not belong to its center. So, there is no general rule to define an extension by the spectral variables $\left\{\nu_{i}\right\}$ of the algebra $\mathcal{M}(R, F)$. Nevertheless the commutative algebra $\mathcal{P}_{S p(2 k)}(R, F)$ admits the central extension by the spectral variables. Therefore we formulate the factorized Cayley-Hamilton identity for this extension.

However, for the reflection equation algebra $\mathcal{M}(R, R)$ the characteristic subalgebra lies in the center, the $\star$-product coincides with the usual matrix product and therefore one can assume that eq.(4.39) is satisfied in the central extension of $\mathcal{M}(R, R)$ by the spectral variables $\left\{\nu_{i}\right\}$.

[^2]Proof. Using the equalities

$$
e_{i}\left(\nu_{0},-\nu_{0}, \nu_{1}, \nu_{2}, \ldots, \nu_{2 k}\right)=e_{i}\left(\nu_{1}, \nu_{2}, \ldots, \nu_{2 k}\right)-\nu_{0}^{2} e_{i-2}\left(\nu_{1}, \nu_{2}, \ldots, \nu_{2 k}\right) \quad \forall i \geq 0
$$

and

$$
e_{k+i}\left(\nu_{1}, \nu_{2}, \ldots, \nu_{2 k}\right)=\nu_{0}^{2 i} e_{k-i}\left(\nu_{1}, \nu_{2}, \ldots, \nu_{2 k}\right) \quad \forall i=1, \ldots, k,
$$

if $\left\{\nu_{i}\right\}$ verifies eqs.(4.36), it is straightforward to check that the map $\pi_{S p(2 k)}$ sends the coefficients (4.13) of the Cayley-Hamilton identity to the elementary symmetric functions in the spectral variables: $\epsilon_{i} \mapsto e_{i}\left(\nu_{1}, \nu_{2}, \ldots, \nu_{2 k}\right) \quad \forall i=1, \ldots, 2 k$.

In [OP] we have derived the quantum analogs of the Newton and Wronsky relations among three series of elements of the characteristic subalgebra: the power sums $p_{i}$, the elementary symmetric functions $a_{i}$ and the complete symmetric functions $s_{i}$. Using these relations we now obtain the parameterization of the series $p_{i}$ and $s_{i}$ in terms of the spectral variables.

Proposition 4.11 Let $\mathcal{M}(R, F)$ be the $S p(2 k)$-type quantum matrix algebra. Assume that the algebra parameter $q$ fulfills the conditions $i_{q} \neq 0, i=2, \ldots, n$, for some $n .{ }^{4}$ Then the elements $a_{n}$ and $s_{n}$ can be defined recursively by the use of the Newton relations (see [OP], theorem 5.2)

$$
\begin{align*}
\sum_{i=0}^{n-1}(-q)^{i} a_{i} p_{n-i} & =(-1)^{n-1} n_{q} a_{n}+(-1)^{n} \sum_{i=1}^{\lfloor n / 2\rfloor}\left(\mu q^{n-2 i}-q^{1-n+2 i}\right) a_{n-2 i} g^{i},  \tag{4.40}\\
\sum_{i=0}^{n-1} q^{-i} s_{i} p_{n-i} & =n_{q} s_{n}+\sum_{i=1}^{\lfloor n / 2\rfloor}\left(\mu q^{2 i-n}+q^{n-2 i-1}\right) s_{n-2 i} g^{i} . \tag{4.41}
\end{align*}
$$

In this situation the elements $s_{n}$ and $p_{n}$ have the following images under the homomorphism $\pi_{S p(2 k)}$ (4.37):

$$
\begin{equation*}
\pi_{S p(2 k)}: \quad s_{n} \mapsto h_{n}\left(\nu_{1}, \nu_{2}, \ldots, \nu_{2 k}\right), \quad p_{n} \mapsto q^{n-1} \sum_{i=1}^{2 k} d_{i} \nu_{i}^{n}, \tag{4.42}
\end{equation*}
$$

where $h_{n}$ denotes the complete symmetric polynomial in its arguments and

$$
\begin{equation*}
d_{i}:=\frac{\nu_{i}-q^{-4} \nu_{2 k+1-i}}{\nu_{i}-\nu_{2 k+1-i}} \prod_{\substack{j=1 \\ j \neq i, 2 k+1-i}}^{2 k} \frac{\nu_{i}-q^{-2} \nu_{j}}{\nu_{i}-\nu_{j}} . \tag{4.43}
\end{equation*}
$$

The power sums contain the rational functions $d_{i}$ in the spectral variables and are themselves rational functions in $\left\{\nu_{i}\right\}$. However, as it follows from the Newton recursion (4.40), the power sums simplify, modulo the relations (4.36), to polynomials in the spectral variables.

Proof. For the proof, we use the following auxiliary statement:

[^3]Lemma 4.12 In the assumptions of proposition 4.11, consider the iterations

$$
\begin{array}{rll}
s_{0}^{\prime}=s_{0}, & s_{1}^{\prime}=s_{1}, & s_{i}^{\prime}=s_{i}+s_{i-2}^{\prime} g ; \\
p_{0}^{\prime}=\left(1-\mu^{2} q^{2}\right) /\left(q-q^{-1}\right), & p_{1}^{\prime}=p_{1}, & p_{i}^{\prime}=p_{i}+\left(q^{-2} p_{i-2}^{\prime}-p_{i-2}\right) g \quad \forall i \geq 2 . \tag{4.45}
\end{array}
$$

The modified sequences $\left\{s_{i}^{\prime}\right\}_{i=0}^{n},\left\{p_{i}^{\prime}\right\}_{i=0}^{n}$ satisfy the following versions of the Newton and Wronski relations

$$
\begin{gather*}
\sum_{i=0}^{n-1} q^{-i} s_{i} p_{n-i}^{\prime}=n_{q} s_{n} \quad \forall n \geq 1  \tag{4.46}\\
\sum_{i=0}^{n}(-1)^{i} a_{i} s_{n-i}^{\prime}=\delta_{n, 0} \quad \forall n \geq 0 \tag{4.47}
\end{gather*}
$$

Proof. For $n<2$, the equalities (4.46)-(4.47) are clearly satisfied. For $n \geq 2$, one can check them inductively, applying the iterative formulas (4.44), (4.45).

We now notice that the images of the elements $a_{i}, i=1, \ldots, n$, are given by the elementary symmetric functions (see eq.(4.37)). Hence, by the Wronski relations (4.47), the images of the modified elements $s_{n}^{\prime}, i=1, \ldots, n$, are the complete symmetric functions in the same arguments. Using then eq.(4.44) and taking into account the relation $h_{n}\left(\nu_{0}, \nu_{1}, \ldots\right)=\sum_{i=0}^{n} \nu_{0}^{i} h_{n-i}\left(\nu_{1}, \ldots\right)$, it is easy to check the formulas for the images of the elements $s_{n}$, which are given in eq.(4.42).

To check the formulas for the power sums, we use the following statement, which was proved in [GS]: if the elements $s_{i}$ for $i=0,1, \ldots, n \geq 1$ are realized as the complete symmetric polynomials $h_{i}$ in some set of variables $\left\{\nu_{i}\right\}_{i=1}^{2 k}$, then the elements $p_{n}^{\prime}$, defined by eqs.(4.46), have the following expressions in terms of the variables $\nu_{i}$

$$
\begin{equation*}
p_{n}^{\prime}=q^{n-1} \sum_{i=1}^{p} \widehat{d}_{i} \nu_{i}^{n}, \quad \text { where } \quad \widehat{d}_{i}:=\prod_{\substack{j=1 \\ j \neq i}}^{p} \frac{\nu_{i}-q^{-2} \nu_{j}}{\nu_{i}-\nu_{j}} . \tag{4.48}
\end{equation*}
$$

The proof of (4.42), (4.43) for the power sums $p_{n}$ proceeds as follows.
Assuming that the relation (4.48) stays valid for $p_{0}^{\prime}$ (note, $p_{0}^{\prime}$ is not fixed by the recursion (4.46)) and making the Ansatz (4.42) for the power sums $p_{i}$ for $i=0,1, \ldots, n$, we make use of the recursion (4.45). Upon substitutions, we find that the relations (4.45) hold valid provided that

$$
\begin{equation*}
d_{i}=\frac{\nu_{i}^{2}-q^{-4} \nu_{0}^{2}}{\nu_{i}^{2}-q^{-2} \nu_{0}^{2}} \widehat{d}_{i} . \tag{4.49}
\end{equation*}
$$

Taking into account the relations (4.36) for the spectral variables $\nu_{i} \in \mathcal{E}_{2 k}$, we observe that the conditions (4.49) dictate the choice (4.43) for $d_{i}$.

It remains to verify the initial settings for the recursion (4.45). They are:

$$
\begin{align*}
& p_{0}^{\prime}=q^{-1} \sum_{i=1}^{2 k} \widehat{d}_{i}=\left.\frac{1-\mu^{2} q^{2}}{q-q^{-1}}\right|_{\mu=-q^{-1-2 k}}=q^{-2 k}(2 k)_{q},  \tag{4.50}\\
& p_{1}=p_{1}^{\prime} \Leftrightarrow \sum_{i=1}^{2 k} \nu_{i}\left(d_{i}-\widehat{d}_{i}\right)=0, \tag{4.51}
\end{align*}
$$

as well as the expression (3.12) for $p_{0}$ :

$$
\begin{equation*}
p_{0}=q^{-1} \sum_{i=1}^{2 k} d_{i}=\left.\operatorname{Tr}_{R} I\right|_{\mu=-q^{-1-2 k}}=q^{-1-2 k}\left((2 k+1)_{q}-1\right) . \tag{4.52}
\end{equation*}
$$

To verify them, we use expansions of the following rational functions

$$
w_{1}(z):=\prod_{i=1}^{2 k} \frac{z-q^{-2} \nu_{i}}{z-\nu_{i}}, \quad w_{2}(z):=\frac{\nu_{0}^{2} w_{1}(z)}{z^{2}-q^{-2} \nu_{0}^{2}}, \quad w_{3}(z):=z w_{2}(z)
$$

in simple ratios.
Expanding $w_{1}(z)$ and evaluating the result at $z=0$, we prove immediately the condition (4.50).

A less trivial check of the condition (4.52) we comment in more details. Expanding $w_{2}(z)$, we obtain

$$
w_{2}(z)=\sum_{i=1}^{2 k} q^{2}\left(d_{i}-\widehat{d}_{i}\right) \frac{\nu_{i}}{z-\nu_{i}}+\frac{q \nu_{0}}{2}\left(\frac{w_{1}\left(q^{-1} \nu_{0}\right)}{z-q^{-1} \nu_{0}}-\frac{w_{1}\left(-q^{-1} \nu_{0}\right)}{z+q^{-1} \nu_{0}}\right) .
$$

Here, for the transformation of the first term in the right hand side, we used the formulas (4.48) and (4.49) and applied the relations (4.36), which confine the variables $\nu_{i} \in \mathcal{E}_{2 k}$. The relations (4.36) also allow us to calculate $w_{1}\left( \pm q^{-1} \nu_{0}\right)=q^{-2 k}$. Thus, evaluating $w_{2}(z)$ at $z=0$, we obtain

$$
w_{2}(0)=-q^{2-4 k}=-q^{3}\left(p_{0}-p_{0}^{\prime}\right)-q^{2-2 k},
$$

wherefrom the condition (4.52) follows.
A check of the condition (4.51), by the expansion and evaluation of $w_{3}(z)$ at $z=0$, is a similar calculation.

## A Standard $S p(4)$-type RTT-algebra.

Here we present quadratic relations for the $S p(4)$-type RTT-algebra $\mathcal{M}(R, P)$ corresponding to the standard symplectic R-matrix (2.22), where we take $k=2$. The 2-contraction $g$ turns out to be central in this algebra. With the additional condition $g=\mu^{2} 1=q^{-10} 1$ this algebra can be interpreted as the quantized algebra of functions on the Lie group $S p(4)$.

For the $4 \times 4$ matrix of generators of this algebra we use the following notation

$$
M=\left(\begin{array}{ll}
A & B  \tag{A.1}\\
C & D
\end{array}\right)
$$

where $A, B, C$, and $D$ are $2 \times 2$ matrices. For any matrix $X \in\{A, B, C, D\}$ we denote its matrix components as $X_{j}^{i}, i, j \in\{1,2\}$.

Quadratic relations in the standard $S p(4)$-type RTT-algebra contain 120 permutation relations for 16 components of the quantum matrix $M$ and 10 additional conditions (3.10) which are responsible for invariance of the symplectic form encoded in the rank= 1 projector $K$.

For the presentation of the permutation relations we fix the following linear order on the components of $M$ :

$$
X_{1}^{1}<X_{2}^{1}<X_{1}^{2}<X_{2}^{2} \forall X \in\{A, B, C, D\}, \quad D_{j_{1}}^{i_{1}}<C_{j_{2}}^{i_{2}}<B_{j_{3}}^{i_{3}}<A_{j_{4}}^{i_{4}} \forall i_{k}, j_{k}=\{1,2\}
$$

Permutation relations among the components of the 2 x 2 matrices $A, \ldots, D$ take a universal form:

$$
q X_{2}^{i} X_{1}^{i}=X_{1}^{i} X_{2}^{i}, \quad q X_{i}^{2} X_{i}^{1}=X_{i}^{1} X_{i}^{2}, \quad\left[X_{1}^{2}, X_{2}^{1}\right]=0, \quad\left[X_{2}^{2}, X_{1}^{1}\right]=-\lambda X_{2}^{1} X_{1}^{2}
$$

where in the last formula and below we use the shorthand notation $\lambda:=q-q^{-1}$.
The rest 96 permutation relations between the components of different matrices $A, B, C$ and $D$ are separated into eight groups according to the type of permutation. In the formulas below indices $i, j$ take values 1 or $2 ; i^{\prime}:=3-i$.

## Commutators:

$$
\begin{aligned}
& {\left[A_{i}^{2}, B_{i}^{1}\right]=\left[A_{2}^{i}, C_{1}^{i}\right]=\left[B_{2}^{i}, D_{1}^{i}\right]=\left[C_{i}^{2}, D_{i}^{1}\right]=0} \\
& {\left[B_{j}^{i}, C_{j}^{i}\right]=0, \quad\left[B_{2}^{1}, C_{1}^{2}\right]=0}
\end{aligned}
$$

$q$-commutators:

$$
\begin{aligned}
A_{j}^{i} B_{j}^{i}-q B_{j}^{i} A_{j}^{i} & =A_{j}^{i} C_{j}^{i}-q C_{j}^{i} A_{j}^{i}=B_{j}^{i} D_{j}^{i}-q D_{j}^{i} B_{j}^{i}=C_{j}^{i} D_{j}^{i}-q D_{j}^{i} C_{j}^{i}=0 \\
A_{1}^{2} B_{2}^{1}-q B_{2}^{1} A_{1}^{2} & =A_{2}^{1} C_{1}^{2}-q C_{1}^{2} A_{2}^{1}=B_{2}^{1} D_{1}^{2}-q D_{1}^{2} B_{2}^{1}=C_{1}^{2} D_{2}^{1}-q D_{2}^{1} C_{1}^{2}=0 \\
B_{i}^{1} C_{i}^{2}-q C_{i}^{2} B_{i}^{1} & =0, \quad B_{2}^{i} C_{1}^{i}-q^{-1} C_{1}^{i} B_{2}^{i}=0
\end{aligned}
$$

$q^{2}$-commutators:

$$
A_{1}^{i} B_{2}^{i}-q^{2} B_{2}^{i} A_{1}^{i}=A_{i}^{1} C_{i}^{2}-q^{2} C_{i}^{2} A_{i}^{1}=B_{i}^{1} D_{i}^{2}-q^{2} D_{i}^{2} B_{i}^{1}=C_{1}^{i} D_{2}^{i}-q^{2} D_{2}^{i} C_{1}^{i}=0
$$

commutators with $\pm \lambda$-additional term (this just means that the numeric coefficient of an extra term is equal to $\pm \lambda$ ):

$$
\begin{aligned}
& {\left[A_{i}^{1}, B_{i}^{2}\right]-\lambda B_{i}^{1} A_{i}^{2}=\left[A_{1}^{i}, C_{2}^{i}\right]-\lambda C_{1}^{i} A_{2}^{i}=\left[B_{1}^{i}, D_{2}^{i}\right]-\lambda D_{1}^{i} B_{2}^{i}=\left[C_{i}^{1}, D_{i}^{2}\right]-\lambda D_{i}^{1} C_{i}^{2}=0,} \\
& {\left[A_{j}^{i}, D_{j}^{i}\right]-\lambda C_{j}^{i} B_{j}^{i}=0, \quad\left[B_{2}^{2}, C_{1}^{1}\right]-\lambda C_{1}^{2} B_{2}^{1}=0, \quad\left[B_{1}^{1}, C_{2}^{2}\right]+\lambda C_{1}^{2} B_{2}^{1}=0 ;}
\end{aligned}
$$

$q$-commutators with $\pm q^{ \pm 1} \lambda$-additional term:

$$
\begin{gathered}
A_{i}^{i} B_{i^{\prime}}^{i^{\prime}}-q B_{i^{\prime}}^{i^{\prime}} A_{i}^{i}-\lambda q B_{2}^{1} A_{1}^{2}=A_{i}^{i} C_{i^{\prime}}^{i^{\prime}}-q C_{i^{\prime}}^{i^{\prime}} A_{i}^{i}-\lambda q C_{1}^{2} A_{2}^{1}=0 \\
B_{i}^{i} D_{i^{\prime}}^{i^{\prime}}-q D_{i^{\prime}}^{i^{\prime}} B_{i}^{i}-\lambda q D_{1}^{2} B_{2}^{1}=C_{i}^{i} D_{i^{\prime}}^{i^{\prime}}-q D_{i^{\prime}}^{i^{\prime}} C_{i}^{i}-\lambda q D_{2}^{1} C_{1}^{2}=0 \\
A_{i}^{1} D_{i}^{2}-q D_{i}^{2} A_{i}^{1}-\lambda q C_{i}^{2} B_{i}^{1}=B_{i}^{2} C_{i}^{1}-q C_{i}^{1} B_{i}^{2}-\lambda q C_{i}^{2} B_{i}^{1}=0 \\
B_{1}^{i} C_{2}^{i}-q^{-1} C_{2}^{i} B_{1}^{i}+\lambda q^{-1} C_{1}^{i} B_{2}^{i}=0
\end{gathered}
$$

$q$-commutators with $\pm \lambda$-additional term:

$$
A_{1}^{i} D_{2}^{i}-q D_{2}^{i} A_{1}^{i}-\lambda C_{1}^{i} B_{2}^{i}=0
$$

$q^{2}$-commutators with $\pm q^{2} \lambda$-additional term:

$$
\begin{aligned}
A_{2}^{i} B_{1}^{i}-q^{2} B_{1}^{i} A_{2}^{i}-\lambda q^{2} B_{2}^{i} A_{1}^{i} & =A_{i}^{2} C_{i}^{1}-q^{2} C_{i}^{1} A_{i}^{2}-\lambda q^{2} C_{i}^{2} A_{i}^{1}=0 \\
B_{i}^{2} D_{i}^{1}-q^{2} D_{i}^{1} B_{i}^{2}-\lambda q^{2} D_{i}^{2} B_{i}^{1} & =C_{2}^{i} D_{1}^{i}-q^{2} D_{1}^{i} C_{2}^{i}-\lambda q^{2} D_{2}^{i} C_{1}^{i}=0 .
\end{aligned}
$$

More complicated relations:

$$
\begin{aligned}
& A_{2}^{i} D_{1}^{i}-q^{-1} D_{1}^{i} A_{2}^{i}=\lambda q^{-3} C_{1}^{i} B_{2}^{i}+\lambda 2_{q} q^{-1} C_{2}^{i} B_{1}^{i}, \\
& A_{i}^{2} D_{i}^{1}-q^{-1} D_{i}^{1} A_{i}^{2}=\lambda q^{2} C_{i}^{2} B_{i}^{1}+\lambda 2_{q} C_{i}^{1} B_{i}^{2}, \\
& A_{2}^{1} B_{1}^{2}-q^{-1} B_{1}^{2} A_{2}^{1}=\lambda q^{2} B_{2}^{1} A_{1}^{2}+\lambda 2_{q} B_{1}^{1} A_{2}^{2}, \\
& A_{1}^{2} C_{2}^{1}-q^{-1} C_{2}^{1} A_{1}^{2}=\lambda q^{2} C_{1}^{2} A_{2}^{1}+\lambda 2_{q} C_{1}^{1} A_{2}^{2}, \\
& B_{1}^{2} D_{2}^{1}-q^{-1} D_{2}^{1} B_{1}^{2}=\lambda q^{2} D_{1}^{2} B_{2}^{1}+\lambda 2_{q} D_{1}^{1} B_{2}^{2}, \\
& C_{2}^{1} D_{1}^{2}-q^{-1} D_{1}^{2} C_{2}^{1}=\lambda q^{2} D_{2}^{1} C_{1}^{2}+\lambda 2_{q} D_{1}^{1} C_{2}^{2}, \\
& {\left[B_{1}^{2}, C_{2}^{1}\right]=\lambda C_{2}^{2} B_{1}^{1}-\lambda C_{1}^{1} B_{2}^{2}+\lambda^{2} C_{1}^{2} B_{2}^{1}, } \\
& {\left[A_{1}^{1}, D_{2}^{2}\right]=-\lambda D_{2}^{1} A_{1}^{2}+\lambda q^{-2} C_{1}^{1} B_{2}^{2}+\lambda 2_{q} C_{1}^{2} B_{2}^{1}, } \\
& {\left[A_{2}^{1}, D_{1}^{2}\right]=\lambda q^{-2} C_{1}^{2} B_{2}^{1}+\lambda 2_{q} C_{2}^{2} B_{1}^{1}, } \\
& {\left[A_{2}^{2}, D_{1}^{1}\right]=} {\left[A_{1}^{2}, D_{2}^{1}\right]=\lambda D_{1}^{2} A_{2}^{2}+\lambda q^{-2} C_{1}^{1} B_{2}^{2}+\lambda 2_{q} C_{1}^{1} C_{2}^{1} B_{1}^{2}+\lambda^{2} q^{-2} C_{1}^{2} B_{2}^{1}+\lambda^{2} 2_{q} C_{2}^{2} B_{1}^{1} . }
\end{aligned}
$$

Finally, from the matrix relations

$$
M_{1} M_{2} K_{12}=K_{12} M_{1} M_{2}=\mu^{-2} K_{12} g=q^{10} K_{12} g
$$

we extract two equivalent expressions for $g$

$$
\begin{align*}
g & =q^{-10}\left(D_{1}^{1} A_{2}^{2}+q D_{2}^{1} A_{1}^{2}-q^{-2} C_{2}^{1} B_{1}^{2}-q^{-3} C_{1}^{1} B_{2}^{2}\right) \\
& =q^{-10}\left(D_{2}^{2} A_{1}^{1}+q^{-1} D_{2}^{1} A_{1}^{2}-q^{-2} C_{1}^{2} B_{2}^{1}-q^{-3} C_{1}^{1} B_{2}^{2}\right), \tag{A.2}
\end{align*}
$$

and 10 invariance conditions

$$
\begin{gathered}
B_{1}^{1} A_{2}^{2}+q B_{2}^{1} A_{1}^{2}-q B_{1}^{2} A_{2}^{1}-q^{2} B_{2}^{2} A_{1}^{1}=D_{1}^{1} C_{2}^{2}+q D_{2}^{1} C_{1}^{2}-q D_{1}^{2} C_{2}^{1}-q^{2} D_{2}^{2} C_{1}^{1}=0, \\
C_{1}^{1} A_{2}^{2}+q C_{1}^{2} A_{2}^{1}-q C_{2}^{1} A_{1}^{2}-q^{2} C_{2}^{2} A_{1}^{1}=D_{1}^{1} B_{2}^{2}+q D_{1}^{2} B_{2}^{1}-q D_{2}^{1} B_{1}^{2}-q^{2} D_{2}^{2} B_{1}^{1}=0, \\
C_{1}^{i} B_{2}^{i}+q C_{2}^{i} B_{1}^{i}-q^{3} D_{1}^{i} A_{2}^{i}-q^{4} D_{2}^{i} A_{1}^{i}=C_{i}^{1} B_{i}^{2}+q C_{i}^{2} B_{i}^{1}-q D_{i}^{1} A_{i}^{2}-q^{2} D_{i}^{2} A_{i}^{1}=0, \\
C_{1}^{1} B_{2}^{2}-C_{2}^{2} B_{1}^{1}+\lambda C_{1}^{2} B_{2}^{1}-q^{2} D_{2}^{1} A_{1}^{2}+q^{2} D_{1}^{2} A_{2}^{1}=0, \\
C_{2}^{1} B_{1}^{2}-C_{1}^{2} B_{2}^{1}-q^{2} D_{1}^{1} A_{2}^{2}+q^{2} D_{2}^{2} A_{1}^{1}-\lambda q^{2} D_{2}^{1} A_{1}^{2}=0 .
\end{gathered}
$$

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[^0]:    ${ }^{1}$ Note that $A^{(-1, i)}$ and $B^{(0, i)}$ belong to the extension of the algebra $\mathcal{P}(R, F)$ by the $\star$-inverse matrix $M^{\overline{-1}}$.

[^1]:    ${ }^{2}$ By Proposition 4.11 [OP], one expects also presence of the terms $M^{\top}\left(M^{\bar{j}}\right)$ in the expansions of generic descendants.

[^2]:    ${ }^{3}$ Passing to the representations level is hardly possible except in the RE-algebra case. The reason is that the characteristic subalgebra belongs to the center of the RE-algebra, which is not true for the general QM-algebra.

[^3]:    ${ }^{4}$ For $n \leq k$ these conditions enter the initial settings for the $S p(2 k)$ type quantum matrix algebras.

