The computational complexity of weighted vertex coloring for $$\{P_5,K_{2,3},K^+_{2,3}\}$$ { P 5 , K 2 , 3 , K 2 , 3 + } -free graphs

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ORIGINAL PAPER

The computational complexity of weighted vertex coloring for $\{P_5, K_{2,3}, K^+_{2,3}\}$-free graphs

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Abstract

In this paper, we show that the weighted vertex coloring problem can be solved in polynomial on the sum of vertex weights time for $\{P_5, K_{2,3}, K^+_{2,3}\}$-free graphs. As a corollary, this fact implies polynomial-time solvability of the unweighted vertex coloring problem for $\{P_5, K_{2,3}, K^+_{2,3}\}$-free graphs. As usual, P_5 and $K_{2,3}$ stands, respectively, for the simple path on 5 vertices and for the biclique with the parts of 2 and 3 vertices, $K^+_{2,3}$ denotes the graph, obtained from a $K_{2,3}$ by joining its degree 3 vertices with an edge.

Keywords Coloring problem · Hereditary class · Computational complexity

1 Introduction

In this paper, we consider only *simple graphs*, i.e. unlabelled, non-oriented graphs without loops and multiple edges. An *induced subgraph* is formed by a subset of vertices of a graph together with all edges, whose endvertices are both in this subset.

Any subset of pairwise non-adjacent vertices of a graph is called *independent*. A *clique* of a graph is any subset of its pairwise adjacent vertices. A *dominating set* of a graph is any subset of its vertices, such that any vertex outside the set has a neighbour in the set.

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A *coloring of a graph* $G = (V, E)$ is a mapping $c : V \longrightarrow \mathbb{N}$, such that $c(u) \neq c(v)$, for any adjacent vertices u and v of G. All elements of $\{c(v) | v \in V\}$ are said to be *colors*. In other words, a graph coloring is a partition of its vertex set into independent sets of vertices of the same color. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum number k, such that G can be colored in k colors. For a given graph G and a number k, the *coloring problem* (the COL *problem*, for short) is to decide whether $\chi(G) \leq k$ or not.

The weighted coloring problem is a generalization of the coloring problem. For given a graph $G = (V, E)$ and a function $w : V \longrightarrow \mathbb{N} \cup \{0\}$, a pair (G, w) is called a *weighted graph*. A *coloring of a weighted graph* (G, w) is any function $c : V \longrightarrow 2^{\mathbb{N}}$, where $|c(v)| = w(v)$, for any $v \in V$, and $c(v_1) \cap c(v_2) = \emptyset$, for any edge $v_1 v_2$ of G. All elements of $\bigcup_{v \in V} c(v)$ are also called *colors*. The *weighted coloring problem* (the WCOL *problem*, for short), for a given weighted graph (G, w), is to find the minimum number k, denoted by $\chi_w(G)$, such that (G, w) admits a coloring in k colors. The WCOL problem becomes the COL problem for the all-ones vector of vertex weights. Notice that none of the colors should be arranged to any zero-weight vertex, and all these vertices can be removed from any weighted graph with their incident edges.

A class of graphs is called *hereditary* if it is closed under deletion of vertices. Any hereditary (and only hereditary) graph class $\mathcal{X}$ can be defined by a set of its *forbidden induced subgraphs* $\mathcal{S}$, i.e. the minimal under deletion of vertices graphs that do not belong to $\mathcal{X}$. We write $\mathcal{X} = Free(\mathcal{S})$, and the graphs in $\mathcal{X}$ are said to be $\mathcal{S}$*-free*. If $\mathcal{S} = \{G\}$, then we write "G-free" instead of "$\{G\}$-free".

The computational complexity of the COL problem was intensively studied for families of hereditary classes, defined by small graphs only or by a small number of forbidden induced structures. We should mention the papers [1–16,18–23] in this field. We should also mention the interesting papers [17,24–26], concerning graph coloring. The computational complexity of the COL problem was completely determined for all classes of the form $Free(\{G\})$ [16]. A study of forbidden pairs was also initiated in [16]. For all but 3 cases, either NP-completeness or polynomial-time solvability was shown for the COL problem in the family of all hereditary classes, defined by 4-vertex forbidden induced structures [18].

As usual, O_n stands for the empty graph on n vertices, P_n stands for the simple path on n vertices, $K_{p,q}$ stands for the complete bipartite graph with p vertices in the first part and q vertices in the second one. By $K^+_{2,3}$ we denote the graph, obtained from a $K_{2,3}$ by joining its degree 3 vertices with an edge. The graphs $K^+_{2,3}$, $bull$, $butterfly$, W_4 are depicted in Fig. 1.

The computational complexity of the COL problem for pairs of connected 5-vertex forbidden induced fragments was considered in [14,15,19,20,22,23]. At the present time, the complexity of this problem is still open only for the following pairs of the mentioned type:

- $\{K_{1,3}, G\}$, where $G \in \{bull, butterfly\}$,
- $\{P_5, H\}$, where $H \in \{K_{2,3}, K^+_{2,3}, W_4\}$.

Unfortunately, for none of these 5 open cases, we clarify the complexity status of the COL problem. We consider the intersection of two of them and present a polynomial-

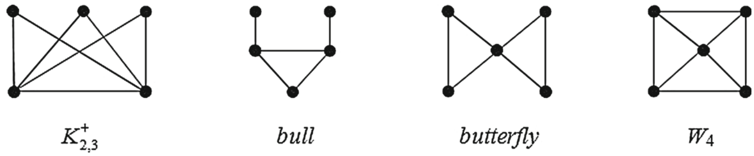

Fig. 1 The graphs $K_{2,3}^{+}$, $bull$, $butterfly$, and W_4

time algorithm for its graphs. Perhaps, this result will help to design polynomial-time algorithms for graphs from the initial classes.

In this paper, we show that the WCOL problem can be solved in polynomial on the sum of vertex weights time for $\{P_5, K_{2,3}, K_{2,3}^{+}\}$-free graphs. Hence, the COL problem can be solved in polynomial time on the length of input data for these graphs.

2 Some definitions and notations

For a vertex x of a graph, $N(x)$ is its neighbourhood. Let A and B be non-intersecting subsets of vertices of a given graph. If all possible edges are present between the sets A and B, then A is said to be *complete* to B. If no edges between A and B are present, then A is said to be *anti-complete* to B. We assume that A is simultaneously complete and anti-complete to B, whenever $B = \emptyset$.

For a graph $G = (V, E)$ and a subset $V' \subset V$, $G[V']$ means its subgraph, induced by V', and $G \setminus V'$ means the result of deletion of all vertices in V' with their incident edges.

3 Irreducible graphs and their properties

Let $G = (V, E)$ be a graph. A set $M \subseteq V$ is a *module* in G if, for any $x \in V \setminus M$, x is adjacent either to all elements of M or to none of them. A module in a graph is *trivial* if it contains only one vertex or all vertices of the graph, otherwise, it is *non-trivial*. A *separating clique* in a graph is a clique, whose removal increases the number of connected components. A graph is called *atomic*, if it does not contain non-trivial modules and separating cliques. The following result is well-known, see, for example, the paper [23].

Lemma 1 *For any hereditary class, the* WCOL *problem can be reduced in polynomial on the length of input data time to its atomic graphs.*

The *anti-neighbourhood* of a vertex $v \in V$ is the set $V \setminus N(v)$, denoted by $\overline{N(v)}$.

Lemma 2 *Let (G, w) be a weighted graph, containing a vertex v, such that $\overline{N(v)} = \{v, v_1, \ldots, v_k\}$ is an independent set. Then, $\chi_w(G) = \chi_{w'}(G \setminus \{v\}) + w(v)$, where $w'(u) = w(u)$, for any u, not belonging to $\overline{N(v)}$, and $w'(u) = \max(w(u) - w(v), 0)$, for any $u \neq v$, belonging to $\overline{N(v)}$.*

Proof As $\overline{N(v)}$ is independent, then any color, used for v, can also be used for all other vertices from $\overline{N(v)} \setminus \{v\}$ without changing the feasibility and the total number of used colors. Therefore, it is sufficient to consider colorings of (G, w), where, for any $u \in \overline{N(v)}$, some of $\min(w(v), w(u))$ colors of u coincide with some of $\min(w(v), w(u))$ colors of v. Removing v from G and decreasing $w(u)$, for any $u \in \overline{N(v)} \setminus \{v\}$, by $\min(w(v), w(u))$ gives the weighted graph $(G \setminus \{v\}, w')$, which can be colored in $\chi_w(G) - w(v)$ colors. Hence, $\chi_w(G) \geq \chi_{w'}(G \setminus \{v\}) + w(v)$. On the other hand, any coloring of $(G \setminus \{v\}, w')$ can be extended to a coloring of (G, w) by using new $w(v)$ colors to color v and to add any of $w(u) - w'(u)$ new colors to u, for any $u \in \overline{N(v)} \setminus \{v\}$. Hence, $\chi_w(G) \leq \chi_{w'}(G \setminus \{v\}) + w(v)$. Therefore, the statement of this lemma is true. □

A graph is *irreducible* if it is connected, atomic, and the anti-neighbourhood of any its vertex is not independent. By Lemmas 1 and 2, the following result is true.

Lemma 3 *For any hereditary class, the* WCOL *problem can be reduced in polynomial on the length of input data time to its irreducible graphs.*

4 Some complexity results for the weighted coloring problem

The next two Lemmas have been proven in [23].

Lemma 4 *The* WCOL *problem for any O_3-free weighted graph $(G = (V, E), w)$ can be solved in $O((\sum_{v \in V} w(v))^3)$ time.*

Lemma 5 *For each fixed C, the* WCOL *problem can be solved in polynomial time on the sum of vertex weights in the class of all graphs, having at most C vertices.*

Let $\mathcal{X}$ be a graph class. By $\mathcal{X}^*$ we denote the set of all graphs, obtained from graphs in $\mathcal{X}$ as follows. We take a graph $G = (V, E) \in \mathcal{X}$, add vertices v_1, v_2, u_1, u_2, add all edges of the form vv_i, where $v \in V, i \in \{1, 2\}$, and the edges v_1u_1, u_1u_2, u_2v_2. It is easy to check that the new graph contains exactly one induced P_4 with degree 2 internal vertices, assuming that G has at least 2 vertices. Hence, for graphs in $\mathcal{X}^*$, it is possible to uniquely restore their prefiguration graphs from $\mathcal{X}$ in polynomial on the number of vertices time.

Lemma 6 *If $\mathcal{X}$ is a hereditary class, then the* WCOL *problem for graphs in $\mathcal{X}^*$ can be reduced in polynomial on the sum of vertex weights time to the same problem in $\mathcal{X}$.*

Proof Let (H, w) be a weighted graph, where $H \in \mathcal{X}^*$. The prefiguration graph G_H for H can be found in polynomial on the number of its vertices time. By symmetry, one can assume that $w(v_1) \geq w(v_2)$. Let x mean the number of common colors of v_1 and v_2 in a considering coloring of (H, w). Hence, there are exactly $w(v_1) + w(v_2) - x$ distinct colors for $\{v_1, v_2\}$, each of which cannot be used to color any vertex in G_H. To minimize the number of colors, used for $\{u_1, u_2\}$, all of the remaining $w(v_1) - x$ colors for v_1 can be used to color u_2. Similarly, all of the remaining $w(v_2) - x$ colors for v_2 can be used to color u_1. Hence, to color u_1 and u_2, we need exactly

$$\chi'_x = \max(w(u_1) - w(v_2) + x, 0) + \max(w(u_2) - w(v_1) + x, 0)$$

colors. Therefore, $\chi_w(H) = \min_{x \le w(v_2)}(w(v_1) + w(v_2) - x + \max(\chi_w(G), \chi'_x))$. So, this lemma holds. □

5 Some properties of irreducible $\{P_5, K_{2,3}, K^+_{2,3}\}$-free graphs

Let $G = (V, E)$ be an irreducible $\{P_5, K_{2,3}, K^+_{2,3}\}$-free graph and $C = (v_1, v_2, v_3, v_4)$ be an arbitrary its induced cycle with 4 vertices. We associate with G and C the following notations, assuming throughout the paper for indices to be taken modulo 4:

1. for any $1 \le i \le 4$, V_i is the set of all vertices v, such that $N(v) \cap V(C) = \{v_i\}$.
2. for any $1 \le i \le 4$, V'_i is the set of all vertices v, such that $N(v) \cap V(C) = \{v_i, v_{i+1}\}$.
3. for any $1 \le i \le 4$, V''_i is the set of all vertices v, such that $N(v) \cap V(C) = \{v_i, v_{i+1}, v_{i+2}\}$.
4. W_C is the set of all vertices, adjacent to all vertices of C, and S_C is the set of all vertices, not having a neighbour on C.

Further, we will prove several relations between the sets, defined above.

Lemma 7 *Any vertex of G, having a neighbour on C, belongs to*

$$V(C) \cup \bigcup_{i=1}^{4}(V_i \cup V'_i \cup V''_i) \cup W_C.$$

Any element in $\bigcup_{i=1}^{4}(V_i \cup V'_i)$ has no neighbours in S_C.

Proof Assume that there is a vertex $v \notin V(C) \cup \bigcup_{i=1}^{4}(V_i \cup V'_i \cup V''_i) \cup W_C$ with $N(v) \cap V(C) \neq \emptyset$. Clearly, v must be adjacent to exactly two non-adjacent vertices of C. Then, v, v_1, v_2, v_3, v_4 induce a $K_{2,3}$.

Assume that some element of $v \in V_i \cup V'_i$ has a neighbour $u \in S_C$. Then either $u, v, v_i, v_{i+1}, v_{i+2}$ or $u, v, v_{i+1}, v_{i+2}, v_{i+3}$ induce a P_5. □

Lemma 8 *For any i, V_i is anti-complete to*

$$V_{i-1} \cup V_{i+1} \cup V'_i \cup V'_{i+3} \cup V''_{i+1} \cup V''_{i+3} \cup W_C$$

and complete to $V_{i+2} \cup V'_{i+1} \cup V'_{i+2}$. For any i, V'_i is complete to $V'_{i+1} \cup V'_{i+3}$. For any i, V''_i is a clique.

Proof Let $v \in V_i$. If v is adjacent to a vertex $u \in V_{i-1} \cup V_{i+1}$, then either $u, v, v_i, v_{i+1}, v_{i+2}$ or $u, v, v_i, v_{i+3}, v_{i+2}$ induce a P_5. If v is adjacent to a vertex $u \in V'_i \cup V'_{i+3}$, then either $v, u, v_{i+1}, v_{i+2}, v_{i+3}$ or $v, u, v_{i+3}, v_{i+2}, v_{i+1}$ induce a P_5. If v is adjacent to a vertex $u \in V''_{i+3} \cup W_C$, then $v, u, v_{i+3}, v_i, v_{i+1}$ induce a $K^+_{2,3}$. If v is adjacent to a vertex $u \in V''_{i+1}$, then $v, u, v_i, v_{i+1}, v_{i+3}$ induce a $K_{2,3}$.

Let $v \in V_i$ and $u \in V_{i+2}$. If v and u are not adjacent, then $v, v_i, v_{i+1}, v_{i+2}, u$ induce a P_5. Now, let $v \in V_i$ and $u \in V'_{i+1} \cup V'_{i+2}$. If v and u are not adjacent, then either $u, v_{i+2}, v_{i+3}, v_i, v$ or $v, v_i, v_{i+1}, v_{i+2}, u$ induce a P_5.

Let $v \in V_i'$ and $u \in V_{i+1}' \cup V_{i+3}'$. If $vu \notin E$, then either $v, v_i, v_{i+3}, v_{i+2}, u$ or $v, v_{i+1}, v_{i+2}, v_{i+3}, u$ induce a P_5.

Assume that vertices $v \in V_i''$ and $u \in V_i''$ are not adjacent. Then, $v, u, v_i, v_{i+2}, v_{i+3}$ induce a $K_{2,3}$. □

Lemma 9 *If $V_i \neq \emptyset$, then each of the pairs (V_i', V_{i+2}') and (V_{i+1}', V_{i+3}') contains the empty set. If $V_i \neq \emptyset$ and $V_{i+1} \neq \emptyset$, then $V_{i+2} = V_{i+3} = \emptyset$ and $V_{i+2}' \neq \emptyset$, $V_i' = \emptyset$.*

Proof Let us prove the first statement. By symmetry, it is enough to consider the case, when $a \in V_i'$ and $b \in V_{i+2}'$. By Lemma 8, we have $bc \in E$ and $ac \notin E$, where $c \in V_i$. Then, $c, b, v_{i+2}, v_{i+1}, a$ induce a P_5, if $ba \notin E$, or c, b, a, v_i, v_{i+3} induce a $K_{2,3}$, if $ba \in E$.

Assume that $V_i \neq \emptyset$ and $V_{i+1} \neq \emptyset$. By Lemma 8, V_i is anti-complete to V_{i+1}. Additionally, suppose that $V_{i+2} \cup V_{i+3} \neq \emptyset$. If $V_{i+2} \neq \emptyset$, then V_{i+2} is complete to V_i and anti-complete to V_{i+1}, by Lemma 8. Then, any vertex of V_{i+2}, any vertex of V_i, v_i, v_{i+1}, any vertex of V_{i+1} induce a P_5. Hence, $V_{i+2} = \emptyset$. Similarly, $V_{i+3} = \emptyset$.

Now, additionally suppose that $V_{i+2}' = \emptyset$. By Lemma 7, the set S_C is anti-complete to $V_i \cup V_{i+1}$. By Lemma 8, V_i is anti-complete to $V_i' \cup V_{i+3}' \cup V_{i+1}'' \cup V_{i+3}'' \cup W_C$. By the same reasons, V_{i+1} is anti-complete to $V_i' \cup V_{i+1}' \cup V_i'' \cup V_{i+2}'' \cup W_C$. The set V_{i+2}'' is anti-complete to V_i, otherwise, any element of V_i, any element of V_{i+2}'', v_{i+2}, v_{i+1}, and any element of V_{i+1} induce a P_5. Similarly, V_{i+1}'' is anti-complete to V_{i+1}.

As $\{v_i, v_{i+1}\}$ is a clique, but not separating, since G is irreducible, then some element of V_i and some element of V_{i+1} must be adjacent to elements in $\bigcup_{i=1}^{4}(V_i' \cup V_i'')$. Let x be an arbitrary element of V_i and y be an arbitrary element of V_{i+1}, both having neighbours in $\bigcup_{i=1}^{4}(V_i' \cup V_i'')$. Suppose that $z' \in V_{i+3}'$. Then, $yz' \in E$, by Lemma 8. Then, by the first part of this lemma, $V_{i+1}' = \emptyset$. Hence, there is a vertex $z'' \in V_i''$, such that $xz'' \in E$. If $z'z'' \notin E$, then $y, z', v_{i+3}, v_{i+2}, z''$ induce a P_5, otherwise, x, v_i, v_{i+1}, z', z'' induce a $K_{2,3}^+$. Therefore, we will consider that $V_{i+3}' = V_{i+1}' = \emptyset$. If $yz_1 \in E$ and $xz_2 \in E$, where $z_1 \in V_{i+3}''$, $z_2 \in V_i''$, then $z_1z_2 \in E$, otherwise, $y, z_1, v_{i+3}, v_{i+2}, z_2$ induce a P_5. By this fact and as V_i'', V_{i+3}'' are both cliques, by Lemma 8, $\{v_i, v_{i+1}\} \cup V_i^1 \cup V_i^2$ is a separating clique, where

$$V_i^1 = \{v \in V_{i+3}'' |\ \exists u \in V_{i+1}, vu \in E\} \text{ and } V_i^2 = \{v \in V_i'' |\ \exists u \in V_i, vu \in E\}.$$

Hence, V_{i+2}' must be non-empty. By Lemma 8, V_{i+2}' is complete to $V_i \cup V_{i+1}$. Suppose that $V_i' \neq \emptyset$. Then, by Lemma 8, V_i' is anti-complete to $V_i \cup V_{i+1}$. Hence, V_i' is anti-complete to V_{i+2}', as G is $K_{2,3}$-free. Then, G contains an induced P_5. Thus, $V_i' = \emptyset$. □

Lemma 10 *For any i, V_i is either empty or independent.*

Proof Assume the opposite, i.e. that $V_i \neq \emptyset$ and it is not independent, for some i. Let $\tilde{V}$ be the vertex set of an arbitrary connected component with at least 2 vertices of $G[V_i]$. Notice that $\tilde{V}$ exists, as V_i is not independent. Let us show that $\tilde{V}$ is a non-trivial module in G.

By Lemmas 7, 8, and the choice of $\tilde{V}$, $\tilde{V}$ is anti-complete to

$$(V_i \setminus \tilde{V}) \cup S_C \cup V_{i+3} \cup V_{i+1} \cup V_i' \cup V_{i+3}' \cup V_{i+1}'' \cup V_{i+3}'' \cup W_C$$

and complete to $V_{i+2} \cup V'_{i+1} \cup V'_{i+2}$. Let $j \in \{i, i+2\}$. If $\tilde{V}$ is not complete to V''_j, then either $\tilde{V}$ is anti-complete to V''_j or there are vertices $x, y \in \tilde{V}, z \in V''_j$, such that $xy \in E, yz \in E, xz \notin E$. Hence, either $x, y, z, v_{i+2}, v_{i+3}$ or $x, y, z, v_{i+2}, v_{i+1}$ induce a P_5. We have a contradiction. Hence, the assumption was false. □

Lemma 11 *If C dominates the maximum number of vertices among all induced cycles with 4 vertices, then S_C must be empty.*

Proof Assume the opposite. By $\tilde{V}$ we denote the set of all vertices, each of which lies outside S_C and has a neighbour in S_C. This set is not empty. By Lemma 7, we have that $\tilde{V} \subseteq \bigcup_{i=1}^{4} V''_i \cup W_C$. If an element $v \in \bigcup_{i=1}^{4} V''_i$ has a neighbour $s \in S_C$ and $ss' \in E$, where $s' \in S_C$ and $s'v \notin E$, then s', s, v, and some two vertices of C induce a P_5. If non-adjacent elements $v_1, v_2 \in \bigcup_{i=1}^{4} V''_i \cup W_C$ have neighbours $u_1 \in S_C \cap (N(v_1) \setminus N(v_2))$ and $u_2 \in S_C \cap (N(v_2) \setminus N(v_1))$, then $u_1u_2 \in E$, otherwise, v_1, v_2, u_1, u_2, and some vertex of C induce a P_5. Hence, if v_1 or v_2 belongs to $\bigcup_{i=1}^{4} V''_i$, then G is not P_5-free, by the previous statement.

As G does not contain separating cliques, then $\tilde{V}$ is not a clique. Therefore, there are non-adjacent vertices in $\tilde{V}$. Suppose that $a \in \tilde{V}$ and $b \in \tilde{V}$ are arbitrary non-adjacent vertices.

Suppose that there is a vertex $c \in S_C$, simultaneously adjacent to a and b. If $\{a, b\} \cap W_C \neq \emptyset$, then both a and b are simultaneously adjacent to two non-adjacent vertices of C. Hence, G contains an induced copy of a $K_{2,3}$. Therefore, $a \in V''_i$ and $b \in V''_j$. By Lemma 8, $j \neq i$. If $j = i+2$, then $v_{i+1}, a, c, b, v_{i+3}$ induce a P_5. Thus, we may consider that $j = i+1$. As C dominates the maximum number of vertices and $(v_i, a, v_{i+2}, v_{i+3})$, $(v_i, v_{i+1}, b, v_{i+3})$ are induced cycles, then $V_{i+1} \neq \emptyset$ and $V_{i+2} \neq \emptyset$. Hence, by Lemma 9, there exists a vertex $d \in V'_{i+3}$. By Lemma 7, $dc \notin E$. To avoid the induced paths (d, v_i, v_{i+1}, b, c) and $(d, v_{i+3}, v_{i+2}, a, c)$, da and db are edges of G. Then, a, d, c, d, v_{i+1} induce a $K_{2,3}$. Therefore, any two non-adjacent elements of $\tilde{V}$ have no a common neighbour in S. Hence, $a, b \in W_C$ and $S_C \cap (N(a) \setminus N(b))$ is complete to $S_C \cap (N(b) \setminus N(a))$. Let $a' \in S_C \cap (N(a) \setminus N(b))$ and $b' \in S_C \cap (N(b) \setminus N(a))$.

Suppose that some vertex $x \in \tilde{V} \setminus \{a, b\}$ has a neighbour $x' \in S_C$. If $x' \in N(a) \cup N(b)$, then x', two non-adjacent vertices of C, x, and a vertex in $\{a, b\}$ induce either a $K_{2,3}$ or a $K^+_{2,3}$. Hence, x has no neighbours in $(N(a) \cup N(b)) \cap S_C$. The vertex x is simultaneously adjacent to a and b, otherwise, x, v_j, a, a', b' or x, v_j, b, b', a' induce a P_5, for some j. The vertex x' is adjacent to at least one vertex in $\{a', b'\}$ (say, a'), otherwise, x', x, a, a', b' induce a P_5. Hence, b, v_1, a, a', x' induce a P_5. Thus, $\tilde{V} = \{a, b\}$.

If there is a vertex $v' \in \bigcup_{i=1}^{4} V_i$, then $av' \notin E, bv' \notin E, v'a' \notin E, v'b' \notin E$, by Lemmas 7 and 8. Then, v', a vertex on C, a, a', b' induce a P_5. Hence, $\bigcup_{i=1}^{4} V_i = \emptyset$. If there is a vertex $v'' \in V'_i$, non-adjacent to a, then $v''a' \notin E, v''b' \notin E$, and v'', v_i, a, a', b' induce a P_5. Hence, $\{a, b\}$ is complete to V'_i. Similarly, $\{a, b\}$ is complete to $\bigcup_{i=1}^{4} V''_i \cup (W_C \setminus \{a, b\})$. Hence, $\bigcup_{i=1}^{4} (\{v_i\} \cup V'_i \cup V''_i) \cup (W_C \setminus \{a, b\})$ is a non-trivial module in G.

So, our initial assumption was false. □

We will assume up to the end of this section that C is an induced cycle with 4 vertices, dominating the maximum number of vertices of G. By Lemma 11, $S_C = \emptyset$.

Lemma 12 *If $V_i \neq \emptyset$ and $V_{i+1} \neq \emptyset$, then $|V| \leq 11$.*

Proof By Lemmas 8 and 9, $V_{i+2} = V_{i+3} = \emptyset$, $V'_{i+2} \neq \emptyset$, $V'_i = \emptyset$, and $V_i \cup V_{i+1}$ is complete to V'_{i+2}. Hence, by Lemma 10, we have $V_i = \{a\}$, otherwise, V_i is independent and any two its elements, v_i, v_{i+3}, any element of V'_{i+2} induce a $K_{2,3}$. Similarly, $V_{i+1} = \{b\}$. By Lemma 8, $ab \notin E$ and V''_{i+3} is anti-complete to $\{a\}$. By Lemma 8, V'_{i+2} is complete to $V'_{i+1} \cup V'_{i+3}$, W_C is anti-complete to $\{a, b\}$, V'_{i+1} is complete to $\{a\}$ and anti-complete to $\{b\}$, V'_{i+3} is complete to $\{b\}$ and anti-complete to $\{a\}$.

The set V''_{i+3} is complete to V'_{i+2}, otherwise, v_{i+2}, some vertex of V'_{i+2}, a, v_i, and some vertex of V''_{i+3} induce a P_5. Hence, V''_{i+3} is complete to $\{b\}$, otherwise, any element of V''_{i+3}, b, v_{i+1}, v_{i+2}, any element of V'_{i+2} induce a $K_{2,3}$. Therefore, V''_{i+3} is complete to $V'_{i+2} \cup \{b\}$ and anti-complete to $\{a\}$. Similarly, V''_i is complete to $V'_{i+2} \cup \{a\}$ and anti-complete to $\{b\}$. The set V'_{i+2} is complete to W_C, otherwise, b, an element of V'_{i+2}, a, v_i, an element of W_C induce a P_5. Hence, V'_{i+1} is complete to W_C, otherwise, v_{i+3}, an element of W_C, v_{i+1}, an element of V'_{i+1}, a induce a P_5. Thus, W_C is complete to $V'_{i+1} \cup V'_{i+3}$. The set V''_{i+3} is complete to W_C, otherwise, any element of V'_{i+2}, an element of W_C, v_i, a, an element of V''_{i+3} induce a $K_{2,3}$. Hence, $V''_{i+3} \cup V''_i$ is complete to W_C.

Let us show that $V''_{i+2} = \emptyset$. Suppose the opposite, and let $v \in V''_{i+2}$. By Lemma 8, $bv \notin E$. Then, $va \notin E$, otherwise, $a, v, v_{i+2}, v_{i+1}, b$ induce a P_5. Thus, $\{a, b\}$ is anti-complete to V''_{i+2}. The vertex v is adjacent to all vertices of V'_{i+2}, otherwise, v, v_i, a, an element of V'_{i+2}, b induce a P_5. The vertex v is adjacent to all vertices of W_C, otherwise, v, v_{i+3}, an element of W_C, v_{i+1}, b induce a P_5. Therefore, V''_{i+2} is complete to $V'_{i+2} \cup W_C$. The set V''_{i+2} is complete to V''_{i+3}, otherwise, a, v_i, any element of V'_{i+2}, an element of V''_{i+2}, an element of V''_{i+3} induce a $K_{2,3}$. The set V''_{i+2} is anti-complete to V''_i, otherwise, v_{i+3}, some element of V''_{i+2}, some element of V''_i, v_{i+1}, b induce a P_5.

Suppose that $u \in V'_{i+1} \cup V'_{i+3} \cup V''_{i+1}$. If $u \in V'_{i+3}$, then $bu \in E$. The vertices u and v must be adjacent, otherwise, v, u, a, v_i, an element of V'_{i+2} induce a $K_{2,3}$. If $u \in V''_{i+1}$, then $au \notin E$. The vertices u and v must be adjacent, otherwise, a, v_i, v, v_{i+2}, u induce a P_5. Suppose that $u \in V'_{i+1}$. Then, v and u must be non-adjacent, otherwise, $v_{i+3}, v, u, v_{i+1}, b$ induce a P_5. Hence, V''_{i+2} is complete to $V'_{i+3} \cup V''_{i+1}$ and anti-complete to V'_{i+1}. Recall that V''_{i+2} is a clique, by Lemma 8. Then, $V''_{i+2} \cup \{v_{i+3}\}$ is a non-trivial module in G. Thus, $V''_{i+2} = \emptyset$. Similarly, $V''_{i+1} = \emptyset$.

Suppose that $V''_{i+3} \neq \emptyset$. Suppose that there is a vertex $x \in V'_{i+1} \cup V''_i$. Then, $\{x\}$ is complete to $V'_{i+2} \cup \{a\}$ and $xb \notin E$. To avoid an induced P_5, formed by a, b, x, any vertex of V'_{i+2}, and any vertex of V''_{i+3}, $\{x\}$ must be complete to V''_{i+3}. Then, a, any vertex of V''_{i+3}, x, v_{i+2}, and any vertex of V'_{i+2} induce a $K^+_{2,3}$. Hence, $V'_{i+1} \cup V''_i = \emptyset$. The set V'_{i+3} is complete to $V'_{i+2} \cup \{b\}$ and anti-complete to $\{a\}$. Hence, V'_{i+3} must be complete to V''_{i+3}, otherwise, some element of V''_{i+3}, some element of V'_{i+3}, some element of V'_{i+2}, a, v_{i+3} induce a $K_{2,3}$. Thus, each of the sets W_C, V'_{i+1}, V'_{i+2}, V'_{i+3}, V''_i, V''_{i+3} has at most one element, as it is a module in G. By Lemma 9, at least one of the sets V'_{i+1} and V'_{i+3} is empty. Hence, G has at most 11 vertices. The same is true, if $V''_i \neq \emptyset$.

Suppose that $V''_i = V''_{i+3} = \emptyset$. Each of the sets W_C, V'_{i+1}, V'_{i+2}, V'_{i+3} has at most one element, as it is a module in G. Therefore, $|V'_{i+2}| = 1$. By Lemma 9, at least one of the sets V'_{i+1} and V'_{i+3} is empty. So, $|V| \leq 9$. □

Lemma 13 *If $V_i \neq \emptyset$ and $V'_{i+1} \neq \emptyset$, $V'_{i+2} \neq \emptyset$, then $|V| \leq 12$.*

Proof By Lemma 8, V_i is complete to $V'_{i+1} \cup V'_{i+2}$ and V'_{i+1} is complete to V'_{i+2}. By Lemma 9, $V'_i = V'_{i+3} = \emptyset$. By Lemma 12, we may assume that $V_{i+1} = V_{i+3} = \emptyset$. The set V_i contains only one element (say, a), otherwise, by Lemmas 8 and 10, V_i is independent and any two its elements, v_i, v_{i+3}, any element of V'_{i+2} induce a $K_{2,3}$. If $|V_{i+2}| \geq 2$, then V_{i+2} is independent, by Lemma 10, V_{i+2} is complete to V_i and anti-complete to $V'_{i+1} \cup V'_{i+2}$, by Lemma 8. Hence, a, v_{i+2}, any two elements of V_{i+2}, any element of V'_{i+2} induce a $K_{2,3}$. Therefore, $|V_{i+2}| \leq 1$.

Let us show that $V''_{i+3} = \emptyset$ and each of the sets $V'_{i+1}, V'_{i+2}, V''_i, V''_{i+1}, V''_{i+2}, W_C$ is a module in G. By Lemma 8, W_C is anti-complete to $V_i \cup V_{i+2}$. The set W_C is complete to $V'_{i+1} \cup V'_{i+2}$, otherwise, v_{i+3}, an element of W_C, v_{i+1}, an element of $V'_{i+1} \cup V'_{i+2}$, a induce a P_5.

If $v \in V''_{i+3}$, then $va \notin E$, by Lemma 8. The set $\{v\}$ is complete to V'_{i+2}, otherwise, v, v_i, a, an element of V'_{i+2}, v_{i+2} induce a P_5. Similarly, $\{v\}$ is complete to V'_{i+1}. Hence, v, a, v_{i+2}, any element of V'_{i+1}, any element of V'_{i+2} induce a $K^+_{2,3}$. Hence, $V''_{i+3} = \emptyset$.

By Lemma 8, V''_{i+1} is anti-complete to $V_i \cup V_{i+2}$. The set V''_{i+1} is complete to W_C, otherwise, an element of V''_{i+1}, v_{i+2}, an element of W_C, v_i, a induce a P_5. The set V''_{i+1} is complete to $V'_{i+1} \cup V'_{i+2}$, otherwise, v_{i+1}, v_{i+3}, a, an element of V''_{i+1}, an element of $V'_{i+1} \cup V'_{i+2}$ induce a P_5.

Suppose that u is an arbitrary vertex of V''_i. If u is adjacent to a vertex $u' \in V'_{i+2}$, then it must be adjacent to a, otherwise, u, a, v_i, v_{i+3}, u' induce a $K_{2,3}$. Then, a, v_{i+1}, u', u, and any vertex of V'_{i+1} induce either a $K_{2,3}$ or a $K^+_{2,3}$. Therefore, $\{u\}$ is anti-complete to V'_{i+2}. Then, $au \notin E$, otherwise, v_{i+1}, u, a, an element of V'_{i+2}, v_{i+3} induce a P_5. The set $\{u\}$ is complete to V'_{i+1}, otherwise, v_{i+3}, an element of V'_{i+2}, any element of V'_{i+1}, v_{i+1}, u induce a P_5. The vertex u is not adjacent to the vertex in V_{i+2}, otherwise, u, the vertex in V_{i+2}, a, any vertex in V'_{i+2}, v_{i+3} induce a P_5. The set $\{u\}$ is complete to W_C, otherwise, u, v_{i+1}, some vertex of W_C, v_{i+3}, and any element of V'_{i+2} induce a P_5. The set $\{u\}$ is complete to V''_{i+1}, otherwise, a, v_i, u, v_{i+2}, an element of V''_{i+1} induce a P_5.

Hence, V''_i is complete to $V'_{i+1} \cup V''_{i+1} \cup W_C$ and anti-complete to $V_i \cup V_{i+2} \cup V'_{i+2}$. Similarly, V''_{i+2} is complete to $V'_{i+2} \cup V''_{i+1} \cup W_C$ and anti-complete to $V_i \cup V_{i+2} \cup V'_{i+1}$. If u is adjacent to $u'' \in V''_{i+2}$, then v_{i+1}, u, u'', any vertex in V'_{i+2}, a induce a P_5. Hence, V''_i is anti-complete to V''_{i+2}.

So, each of the sets $V'_{i+1}, V'_{i+2}, V''_i, V''_{i+1}, V''_{i+2}, W_C$ is a module in G. Therefore, each of them has at most one element and $|V| \leq 12$. □

Lemma 14 *If $V_i \neq \emptyset$, then either $V_{i+1} \cup V_{i+2} \cup V_{i+3} \neq \emptyset$ or $V'_{i+1} \neq \emptyset$, $V'_{i+2} \neq \emptyset$.*

Proof Assume the opposite. Then, $V_{i+1} = V_{i+2} = V_{i+3} = \emptyset$ and ($V'_{i+1} = \emptyset$ or $V'_{i+2} = \emptyset$). By Lemma 10, V_i is an independent set. By Lemma 8, V_i is anti-complete to $V'_i \cup V'_{i+3} \cup V''_{i+1} \cup V''_{i+3} \cup W_C$. Let

$$V_i^1 = \{v \in V''_i |\ \exists u \in V_i, vu \in E\} \text{ and } V_i^2 = \{v \in V''_{i+2} |\ \exists u \in V_i, vu \in E\}.$$

Let us show that $V_i^1 \cup V_i^2$ is the empty set or a clique. Suppose the opposite. Then, there are non-adjacent vertices $u_1 \in V_i^1, u_2 \in V_i^2$, such that there are vertices $v_1, v_2 \in V_i$, for which we have $v_1u_1 \in E$ and $v_2u_2 \in E$. By Lemma 8, one may consider that $u_1 \in V_i''$, $u_2 \in V_{i+2}''$, as, by Lemma 8, V_i^1 and V_i^2 are cliques. As G is $K_{2,3}^+$-free, $v_2u_1 \notin E$, $v_1u_2 \notin E$. Therefore, $v_1, u_1, v_{i+2}, u_2, v_2$ induce a P_5. Hence, $V_i^1 \cup V_i^2$ is the empty set or a clique.

As G is irreducible, $\{v_i\} \cup V_i^1 \cup V_i^2$ is a clique, but not separating. Therefore, at least one of the sets V_{i+1}' and V_{i+2}' is not empty. By our assumption, at least one (say, V_{i+2}') of these sets is empty. Hence, $V_{i+1}' \neq \emptyset$. Then, $V_{i+3}' = \emptyset$, by Lemma 9. By Lemma 8, V_i is complete to V_{i+1}'. By this fact and as G is $K_{2,3}$-free, we have that $V_i = \{a\}$ and V_{i+1}' is a clique.

Let $x \in W_C$. Then, $xa \notin E$. By Lemma 8, V_{i+1}' is complete to V_i'. The set $\{x\}$ is complete to V_{i+1}', to avoid a P_5, induced by v_{i+3}, x, v_{i+1}, an element of V_{i+1}', and a. The set $\{x\}$ is complete to V_i', to avoid a $K_{2,3}$, induced by a, x, v_i, any element of V_{i+1}', and an element of V_i'. Therefore, W_C is complete to $V_i' \cup V_i''$

As G is irreducible, the set $\overline{N(v_{i+1})}$ is not independent. Therefore, $V_{i+2}'' \neq \emptyset$. Let $v \in V_{i+2}''$. Then, $\{v\}$ is anti-complete to $\{a\} \cup V_{i+1}'$ or complete to $\{a\} \cup V_{i+1}'$. Indeed, if $va \in E$, then v_{i+3}, v, a, any element of V_{i+1}', non-adjacent to v, v_{i+1} induce a P_5. If $va \notin E$ and v is adjacent to an element in V_{i+1}', then this element, a, v_i, v_{i+1}, x induce a $K_{2,3}$.

Let us show that $\{v\}$ is complete to $V_{i+1}'' \cup V_{i+3}''$. By Lemma 8, $\{a\}$ is anti-complete to $V_{i+1}'' \cup V_{i+3}''$. Let $u \in V_{i+1}'' \cup V_{i+3}''$, such that $vu \notin E$. Notice that $\{u\}$ is complete to V_{i+1}', otherwise, v_{i+3}, u, v_{i+1}, an element of V_{i+1}', a induce a P_5. Then, $\{v\}$ is complete to $\{a\} \cup V_{i+1}'$, otherwise, u, v_{i+2}, v, v_i, a or v, v_{i+3}, u, v_{i+1}, and an element of V_{i+1}' induce a P_5. Then, $v_{i+1}, u, v_{i+3}, v, a$ induce a P_5. The set $\{v\}$ is complete to W_C, otherwise, v, v_{i+3}, an element of W_C, v_{i+1}, a induce a P_5. Therefore, V_{i+2}'' is complete to $V_{i+1}'' \cup V_{i+3}'' \cup W_C$.

Let us show that $\{v\}$ is complete to $\{a\} \cup V_{i+1}'$. Suppose the opposite. If there is a vertex $u \in V_i'$, adjacent to v, then $ua \notin E$ and v_{i+3}, v, u, v_{i+1}, any element of V_{i+1}' induce a P_5. Hence, $\{v\}$ is anti-complete to V_i'. If there is a vertex $u \in V_i''$, adjacent to v, then $\{u\}$ is complete to V_{i+1}', otherwise, v_{i+3}, v, u, v_{i+1}, an element of V_{i+1}' induce a P_5. Similarly, $au \in E$, otherwise, v_{i+3}, v, u, any element of V_{i+1}', and a induce a P_5. Then, a, v, u, v_i, v_{i+1} induce a $K_{2,3}^+$. Hence, $\{v\}$ is anti-complete to V_i''. Therefore, V_{i+2}'' is complete to $V_{i+1}' \cup W_C$ and anti-complete to V_i'. By Lemma 8, V_{i+2}'' is a clique. Therefore, $\{v, v_{i+3}\}$ is a module in G. Thus, $\{v\}$ is really complete to $\{a\} \cup V_{i+1}'$.

The set V_{i+3}'' is empty, otherwise, if $y \in V_{i+3}''$, then $\{y\}$ is complete to V_{i+1}', to avoid a P_5, induced by y, v_{i+3}, v_{i+2}, any element of V_{i+1}', and a. Hence, a, y, v, v_{i+2}, and any element of V_{i+1}' induce a $K_{2,3}^+$. The set V_i' is empty, otherwise, if $y \in V_i'$, then $ya \notin E$ and either $a, v, v_{i+2}, v_{i+1}, y$ induce a P_5 or a, v, y, v_i, v_{i+3} induce a $K_{2,3}^+$. If there is a vertex $y \in V_i''$, such that $vy \in E$, then $ay \in E$, to avoid the $K_{2,3}^+$, induced by a, v, y, v_i, v_{i+3}. Hence, any vertex in V_{i+2}'', having a neighbour in V_i'', is adjacent to a. If $z \in V_{i+2}''$, $z \neq v$, then $zy \in E$, to avoid a P_5, induced by $v_{i+1}, y, a, z, v_{i+3}$. Thus, all vertices of V_{i+2}'' have the same neighbourhoods in V_i''.

Let $\hat{V}_i''$ be the set of all vertices of V_i'', each of which does not have a neighbour in V_{i+2}''. Let us show that $\hat{V}_i'' = \emptyset$. Suppose the opposite. Recall that $V_i' = \emptyset$, $V_{i+3}'' = \emptyset$, $V_{i+1}' \neq \emptyset$, $V_{i+2}'' \neq \emptyset$, and V_{i+2}'' is complete to $V_{i+1}'' \cup W_C$. The set $\hat{V}_i''$ is complete to V_{i+1}', otherwise, an element of $\hat{V}_i''$, v_{i+1}, any element of V_{i+1}', any element of V_{i+2}'', v_{i+3} induce a P_5. The set $\hat{V}_i''$ is anti-complete to $\{a\}$, otherwise, v_{i+1}, an element of $\hat{V}_i''$, a, any element of V_{i+2}'', v_{i+3} induce a P_5. The $\hat{V}_i''$ is complete to W_C, otherwise, an element of $\hat{V}_i''$, v_{i+1}, an element of W_C, any element of V_{i+2}'', a induce a P_5. If $x \in V_{i+1}''$, then $ax \notin E$ and $\{x\}$ is complete to V_{i+2}''. Then, $\{x\}$ is complete to $\hat{V}_i''$, otherwise, a, v_i, an element of $\hat{V}_i''$, v_{i+2}, x induce a P_5. By Lemma 8, V_i'' is a clique. Therefore, $\hat{V}_i''$ is complete to $(V_i'' \setminus \hat{V}_i'') \cup V_{i+1}' \cup V_{i+1}'' \cup W_C$ and anti-complete to $\{a\} \cup V_{i+2}''$. Thus, $\{v_{i+1}\} \cup \hat{V}_i''$ is a non-trivial module in G. Hence, $\hat{V}_i'' = \emptyset$, as G is irreducible.

Thus, $V_i' = \emptyset$, $V_{i+3}'' = \emptyset$, V_{i+2}'' is complete to $\{a\} \cup V_i' \cup V_i'' \cup V_{i+1}'' \cup V_{i+3}'' \cup W_C$, and V_{i+2}'' is a clique. Hence, for any $v \in V_{i+2}''$, its anti-neighbourhood consists of v and v_{i+1}. So, G is not irreducible. Our initial assumption was false. $\square$

Lemma 15 *If V_i and V_{i+2} are simultaneously not empty, then either $|V| \leq 12$ or $G \in (Free(\{O_3\}))^*$.*

Proof By Lemma 9, we have $V_{i+1} = V_{i+3} = \emptyset$. By Lemma 8, V_i is complete to $V_{i+2} \cup V_{i+1}' \cup V_{i+2}'$ and anti-complete to $V_i' \cup V_{i+3}'$. Similarly, V_{i+2} is complete to $V_i' \cup V_{i+3}'$ and anti-complete to $V_{i+1}' \cup V_{i+2}'$. By Lemma 8, $V_{i+1}'' \cup V_{i+3}''$ is anti-complete to $V_i \cup V_{i+2}$. Hence, $V_{i+1}'' \cup V_{i+3}'' = \emptyset$, otherwise, any element of V_i, any element of V_{i+2}, any element of $V_{i+1}'' \cup V_{i+3}''$, and v_i, v_{i+2} induce a P_5. By Lemma 10, V_i and V_{i+2} are both independent. Hence, to avoid an induced $K_{2,3}$, $|V_i| + |V_{i+2}| \leq 3$. By Lemma 8, W_C is anti-complete to $V_i \cup V_{i+2}$. By Lemma 8, V_i'' and V_{i+2}'' are cliques.

If $V_i' \neq \emptyset$, $V_{i+3}' \neq \emptyset$ or $V_{i+1}' \neq \emptyset$, $V_{i+2}' \neq \emptyset$, then $|V| \leq 12$, by Lemma 13. Suppose that $V_i' \neq \emptyset$ and $V_{i+1}' \neq \emptyset$. Then, by Lemma 8, V_i' is complete to V_{i+2} and anti-complete to V_i, V_{i+1}' is complete to V_i and anti-complete to V_{i+2}. Then, any vertex of V_i, any vertex of V_{i+2}, any vertex of V_i', any vertex of V_{i+1}', and v_{i+2} induce a $K_{2,3}$. Therefore, by Lemma 9, at most one of the sets V_i', V_{i+1}', V_{i+2}', V_{i+3}' is not empty. Hence, if $\bigcup_{i=1}^{4} V_i' \neq \emptyset$, then, by symmetry, $V_i' \neq \emptyset$. Recall that

$$V_{i+1} = V_{i+3} = V_{i+1}' = V_{i+2}' = V_{i+3}' = V_{i+1}'' = V_{i+3}'' = \emptyset.$$

Suppose that $V_i' \neq \emptyset$. Then, $|V_i| = |V_{i+2}| = 1$, otherwise, by Lemmas 8 and 10, G contains an induced $K_{2,3}$. Similarly, V_i' is a clique. The set V_i' is complete to W_C, otherwise, v_{i+3}, an element of W_C, v_{i+1}, an element of V_i', and the element of V_{i+2} induce a P_5. Therefore, $W_C = \emptyset$, otherwise, the element of V_i, the element of V_{i+2}, any element of V_i', any element of W_C, and v_{i+3} induce a P_5. If there is a vertex $c \in V_i''$, adjacent to $a \in V_i'$ and non-adjacent to $b \in V_i'$, then $b, a, c, v_{i+2}, v_{i+3}$ induce a P_5. Suppose that there is a vertex $c \in V_{i+2}''$, adjacent to $a \in V_i'$ and non-adjacent to $b \in V_i'$. Then, c and the element of V_i are not adjacent, otherwise, a, c, v_i, v_{i+3}, and the element of V_i induce a $K^+_{2,3}$. Hence, c and the element of V_{i+2} are adjacent, otherwise, v_{i+3}, c, a, the element of V_{i+2}, and the element of V_i induce a P_5. Then, v_i, b, c, the element of V_i, the element of V_{i+2} induce a $K_{2,3}$. Hence, V'_i contains exactly one element.

Let us show that if $V_i \cup V_{i+2}$ is not anti-complete to $V''_i \cup V''_{i+2}$, then $|V| \leq 9$. Suppose that some vertex $v \in V_i \cup V_{i+2}$ has a neighbour in $V''_i \cup V''_{i+2}$. Without loss of generality, let $v \in V_i$ and $\tilde{V}_i = \{u \in V''_i |\, uv \in E\} \neq \emptyset$. Denote by $\tilde{V}_{i+2}$ the set of all vertices in V''_{i+2}, adjacent to v.

Let us show that $\tilde{V}_{i+2} = V''_{i+2}$. Suppose the opposite. The set $\tilde{V}_i$ is complete to $\tilde{V}_{i+2}$, otherwise, v, v_{i+1}, v_{i+3}, an element of $\tilde{V}_i$, and an element of $\tilde{V}_{i+2}$ induce a P_5. The set $\tilde{V}_i$ is anti-complete to $V''_{i+2} \setminus \tilde{V}_{i+2}$, to avoid a $K^+_{2,3}$, induced by v, v_i, v_{i+1}, and adjacent elements of $\tilde{V}_i$ and $V''_{i+2} \setminus \tilde{V}_{i+2}$. Similarly, $\tilde{V}_{i+2}$ is anti-complete to $V''_i \setminus \tilde{V}_i$. The set $V''_i \setminus \tilde{V}_i$ is anti-complete to $V''_{i+2} \setminus \tilde{V}_{i+2}$, otherwise, v, any element of $\tilde{V}_i$, an element of $V''_i \setminus \tilde{V}_i$, an element of $V''_{i+2} \setminus \tilde{V}_{i+2}$, and v_{i+3} induce a P_5.

To avoid an induced $K^+_{2,3}$, $\tilde{V}_i$ is anti-complete to $V_i \setminus \{v\}$. If V_i has 2 elements, then $\tilde{V}_i$ is complete to V_{i+2}, otherwise, v, the element of V_{i+2}, the element of $V_i \setminus \{v\}$, any element of $\tilde{V}_i$, and v_{i+1} induce a P_5. If V_{i+2} has 2 elements, then any element of $\tilde{V}_i$ has a neighbour in V_{i+2}, otherwise, $V_i \cup V_{i+2} \cup \{v_{i+1}\}$ and an element of $\tilde{V}_i$ induce a $K_{2,3}$. Hence, the set $V''_{i+2} \setminus \tilde{V}_{i+2}$ is anti-complete to $V_i \cup V_{i+2}$. Indeed, otherwise, either $V''_{i+2} \setminus \tilde{V}_{i+2}$ and $\tilde{V}_i$ have a common neighbour in $V_i \cup V_{i+2}$ or one of the sets V_i and V_{i+2} contains vertices u_1 and u_2, such that u_1 has a neighbour in $V''_{i+2} \setminus \tilde{V}_{i+2}$ and u_2 has a neighbour in $\tilde{V}_i$. Therefore, G will contain an induced P_5.

To avoid a P_5, induced by v, v_{i+1}, v_{i+3}, an element of W_C, and an element of $\tilde{V}_i \cup \tilde{V}_{i+2}$, the set $\tilde{V}_i \cup \tilde{V}_{i+2}$ is complete to W_C. To avoid a P_5, induced by v, any element of $\tilde{V}_i$, an element of W_C, v_{i+3}, an element of $V''_{i+2} \setminus \tilde{V}_{i+2}$, the set $V''_{i+2} \setminus \tilde{V}_{i+2}$ is complete to W_C. To avoid a $K^+_{2,3}$, induced by v, v_i, any element of $\tilde{V}_i$, an element of $V''_i \setminus \tilde{V}_i$, an element of W_C, the set $V''_i \setminus \tilde{V}_i$ is complete to W_C. Therefore, $(V''_{i+2} \setminus \tilde{V}_{i+2}) \cup \{v_{i+3}\}$ is a module in G. As G is irreducible, then $\tilde{V}_{i+2} = V''_{i+2}$.

So, $\tilde{V}_{i+2} = V''_{i+2}$. Thus, if $\bigcup_{i=1}^4 V'_i = \emptyset$, then $\overline{N(w)} \subseteq \{w, v_{i+3}\} \cup V_{i+2}$ is independent, for any $w \in \tilde{V}_i$. Suppose that $V'_i = \{u\}$. Then, $|V_i| = |V_{i+2}| = |V'_i| = 1$ and $W_C = \emptyset$. Then, $\{u\}$ is anti-complete to $\tilde{V}_i$, to avoid a $K_{2,3}$ or a $K^+_{2,3}$, induced by v, u, v_{i+2}, an element of $\tilde{V}_i$, and any element of V_{i+2}. If $V''_{i+2} \neq \emptyset$, then either u, v_{i+1}, v_{i+3}, a vertex in $\tilde{V}_i$, a vertex in $\tilde{V}_{i+2} = V''_{i+2}$ induce a P_5 or v, u, v_{i+1}, v_{i+3}, a vertex in $\tilde{V}_{i+2}$ induce a $K^+_{2,3}$. If there is a vertex $w \in V''_i$, adjacent to the vertex in V_{i+2}, then $\overline{N(w)}$ is independent. If V_{i+2} is anti-complete to V''_i, then $\tilde{V}_i$ and $V''_i \setminus \tilde{V}_i$ are modules in G. Hence, $|V''_i| \leq 2$ and $|V| \leq 9$. So, we will assume that $V_i \cup V_{i+2}$ is anti-complete to $V''_i \cup V''_{i+2}$.

Suppose that $V'_i = \{v\}$. Let us show that $|V| = 7$. Recall that

$$V_{i+1} = V_{i+3} = V'_{i+1} = V'_{i+2} = V'_{i+3} = V''_{i+1} = V''_{i+3} = W_C = \emptyset,$$

and $|V_i| = |V_{i+2}| = 1$. Let $u \in V_i'' \cup V_{i+2}''$. If $u \in V_i''$, then $vu \in E$, otherwise, u, v_{i+1}, v, the vertex in V_{i+2}, and the vertex in V_i induce a P_5. If $u \in V_{i+2}''$, then $vu \notin E$, otherwise, v_{i+3}, u, v, the vertex in V_{i+2}, and the vertex in V_i induce a P_5. Hence, if there are adjacent vertices $u \in V_i''$ and $u' \in V_{i+2}''$, then $uv \in E$ and $u'v \notin E$. Then, v_{i+3}, u', u, v, and the vertex in V_{i+2} induce a P_5. Hence, $V_i'' \cup \{v_{i+1}\}$ and $V_{i+2}'' \cup \{v_{i+3}\}$ are modules in G. Then, $V_i'' = V_{i+2}'' = \emptyset$, as G is irreducible, and $|V| = 7$.

Suppose that $\bigcup_{i=1}^4 V_i' = \emptyset$. Recall that $V_{i+1} = V_{i+3} = V_{i+1}'' = V_{i+3}'' = \emptyset$. The sets V_i and V_{i+2} are modules in G, and, hence, $|V_i| = |V_{i+2}| = 1$. The graph $G \setminus (V_i \cup V_{i+2} \cup \{v_i, v_{i+2}\})$ is the prefiguration graph for G. Let us check that the graph $H = G \setminus (V_i \cup V_{i+2})$ is O_3-free. Indeed,

$$V(H) = V_i'' \cup V_{i+2}'' \cup W_C \cup \{v_1, v_2, v_3, v_4\}.$$

Let x, y, z be pairwise non-adjacent vertices of H. As V_i'' and V_{i+2}'' are cliques, by Lemma 8, then $\{x, y, z\} \cap \{v_1, v_2, v_3, v_4\} = \emptyset$, $|V_i'' \cap \{x, y, z\}| \le 1$, and $|V_{i+2}'' \cap \{x, y, z\}| \le 1$. If each of the sets V_i'', V_{i+2}'', W_C contains exactly one element of $\{x, y, z\}$, then H has an induced P_5. Otherwise, H contains an induced $K_{2,3}^+$. Thus, $G \in (Free(\{O_3\}))^*$. □

Lemma 16 *If* $\bigcup_{i=1}^4 V_i = \emptyset$*, then* G *is* O_3*-free.*

Proof Firstly, we will prove the following observations: 1) for any i, V_i' is complete to $V_i'' \cup V_{i+3}''$, 2) for any i, V_i' is a clique.

Let us prove the first observation. Suppose that vertices $a \in V_i'$ and $b \in V_i''$ are not adjacent. If there is a vertex $x \in V_{i+3}'$, then $ax \in E$, by Lemma 8, and $xb \in E$, to avoid a P_5, induced by $v_{i+3}, x, a, v_{i+1}, b$. Then, a, b, v_i, x, v_{i+3} induce a $K_{2,3}^+$. Thus, $V_{i+3}' = \emptyset$. If there is a vertex $x \in V_{i+2}'$, then either $xa \notin E$ or $xa \in E$. In the first case, $bx \in E$, to avoid a P_5, induced by a, v_i, b, v_{i+2}, x. Then, $a, v_{i+1}, b, x, v_{i+3}$ induce a P_5. In the second case, $bx \in E$, to avoid a P_5, induced by $v_{i+3}, x, a, v_{i+1}, b$. Then, a, b, x, v_{i+3}, v_i induce a $K_{2,3}$. Therefore, $V_{i+2}' = \emptyset$. The set $\overline{N(v_{i+1})}$ is not independent, as G is irreducible. Hence, there is a vertex $c \in V_{i+2}''$. We will show that $\{c, v_{i+3}\}$ is a non-trivial module in G.

The vertex c is simultaneously non-adjacent to a and b. Indeed, if $ac \in E$, $bc \in E$, then a, b, c, v_i, v_{i+3} induce a $K_{2,3}^+$. If $ac \in E$, $bc \notin E$, then $v_{i+3}, c, a, v_{i+1}, b$ induce a P_5. If $bc \in E$, $ac \notin E$, then $v_{i+3}, c, b, v_{i+1}, a$ induce a P_5. By Lemma 8, $\{c, v_{i+3}\}$ is complete to $V_{i+2}'' \setminus \{c\}$. Let v be a vertex in V_{i+1}'', non-adjacent to c. To avoid a P_5, induced by $c, v_{i+3}, v, v_{i+1}, a$ or b, we have $va \in E$ and $vb \in E$. Then, a, b, v, v_i, v_{i+3} induce a $K_{2,3}$. Hence, $\{c, v_{i+3}\}$ is complete to V_{i+1}''. Similarly, $\{c, v_{i+3}\}$ is complete to V_{i+3}''. Let $v \neq b$ be a vertex in V_i'', adjacent to c. Then, $vb \in E$, by Lemma 8. To avoid a P_5, induced by $v_{i+3}, c, v, v_{i+1}, a$, we have $va \in E$. Then, a, b, c, v, v_i induce a $K_{2,3}^+$. Therefore, $\{c, v_{i+3}\}$ is anti-complete to V_i''.

Let $u \neq a$ be a vertex in V_i', adjacent to c. To avoid a P_5, induced by v_{i+3}, c, u, v_{i+1}, a or b, we have $ua \in E$ and $ub \in E$. Then, a, b, c, u, v_i induce a $K_{2,3}^+$. Therefore, $\{c, v_{i+3}\}$ is anti-complete to V_i'. Let u be a vertex in V_{i+1}', adjacent to c. Then, $ua \in E$, by Lemma 8. To avoid a P_5, induced by $v_{i+3}, c, u, v_{i+1}, b$, we have $ub \in E$. Then,

a, b, c, u, v_i induce a $K_{2,3}$. Therefore, $\{c, v_{i+3}\}$ is anti-complete to V'_{i+1}. If there is a vertex $u \in W_C$, non-adjacent to c, then $ua \in E$, $ub \in E$, to avoid a P_5, induced by $c, v_{i+3}, u, v_{i+1}, a$ or b. Then, v_i, u, v_{i+3}, a, b induce a $K^+_{2,3}$. Therefore, $\{c, v_{i+3}\}$ is complete to W_C. So, $\{c, v_{i+3}\}$ is a non-trivial module in G. Hence, for any i, V'_i is complete to $V''_i \cup V''_{i+3}$.

Let us prove the second statement. Suppose that V'_i is not a clique. Then, $V''_i \cup V''_{i+3}$ is empty. Indeed, if an element belongs to this set, then it must be adjacent to all elements of V'_i, by the first observation, and G contains an induced $K^+_{2,3}$. Similarly, $V'_{i+1} \cup V'_{i+3} = \emptyset$. Let $M \subseteq V'_i$ be a minimal module among modules in $G[V'_i]$, containing non-adjacent vertices. Hence, for any vertex $x \in M$, $\{x\}$ is not complete to $M \setminus \{x\}$.

If there is a vertex $v \in V''_{i+1} \cup V''_{i+2}$, adjacent to a vertex $x \in M$, then there is a vertex $y \in M$, such that $xy \notin E$. To avoid a P_5, induced by $x, y, v, v_{i+1}, v_{i+3}$ or x, y, v, v_i, v_{i+2}, we have $yv \in E$. Then, x, y, v, v_{i+3}, v_i or $x, y, v, v_{i+1}, v_{i+2}$ induce a $K^+_{2,3}$. Therefore, M is anti-complete to $V''_{i+1} \cup V''_{i+2}$. If there is a vertex $v' \in V'_{i+2}$, adjacent to $x' \in M$, then there is a vertex $y' \in M$, such that $x'y' \notin E$. To avoid a P_5, induced by $v_{i+3}, v', x', v_{i+1}, y'$, we have $v'y' \in E$. Then, x', y', v', v_i, v_{i+3} induce a $K_{2,3}$. Therefore, M is anti-complete to V'_{i+2}.

Suppose that $W_C \neq \emptyset$. As G is $K^+_{2,3}$-free, any vertex of W_C has a neighbour in M, as M is not a clique. Let $v'' \in W_C$, x and y be any non-adjacent vertices of M. As G is $K^+_{2,3}$-free, $V'_i \cap N(v'')$ and $V'_i \setminus N(v'')$ are cliques. Hence, $xv'' \in E$, $yv'' \notin E$ or vice versa. The vertex v'' is adjacent to all vertices of $V'_i \setminus M$, otherwise, x, y, v_{i+2}, an element of $V'_i \setminus M$, and an element of W_C induce a P_5. Additionally, $V'_i \setminus M$ is a clique. As G is $K^+_{2,3}$-free, if there are non-adjacent vertices $v'_1, v'_2 \in W_C$, then any vertex of V'_i has a neighbour in $\{v'_1, v'_2\}$. Hence, one may assume that $xv'_1 \in E$, $yv'_1 \notin E$ or $yv'_2 \in E$, $xv'_2 \notin E$. Then, $x, v'_1, v_{i+2}, v'_2, y$ induce a P_5. Therefore, W_C is a clique. Moreover, $W_C \cup \{v_i, v_{i+1}\} \cup (V'_i \setminus M)$ is a clique, which is separating. Thus, $W_C = \emptyset$. So, M is a non-trivial module in G. Hence, V'_i is a clique, for any i.

Now, let us prove that G is O_3-free. Suppose the opposite. Let x, y, z be pairwise non-adjacent vertices of G. Clearly, $\{x, y, z\} \cap \{v_1, v_2, v_3, v_4\}$ has at most one vertex. If this set contains one vertex (say, x), then, by Lemma 8, either $y \in V'_i \cup V'_{i+1}$, $z \in V''_i$ (or vice versa) or $y, z \in V'_i$, for some i. It contradicts to the observations. Suppose that $\{x, y, z\} \cap \{v_1, v_2, v_3, v_4\} = \emptyset$. By the second observation and Lemma 8, the set $\bigcup_{i=1}^4 V'_i \cap \{x, y, z\}$ contains at most two elements.

Suppose that $|\bigcup_{i=1}^4 V'_i \cap \{x, y, z\}| = 2$. We may consider that $x \in V'_i$ and $y \in V'_{i+2}$, by the second observation and Lemma 8. By the first observation, $z \in W_C$. Hence, G contains a P_5, induced by $x, v_{i+1}, z, v_{i+3}, y$.

Suppose that $\bigcup_{i=1}^4 V'_i \cap \{x, y, z\} = \{x\}$, where $x \in V'_i$. Then, by the first observation, we may assume that $y, z \in W_C \cup V''_{i+1} \cup V''_{i+2}$. If $W_C \cap \{y, z\} = \emptyset$, then, by Lemma 8 and the first observation, $y \in V''_{i+1}$, $z \in V''_{i+2}$ or vice versa. Then, $x, v_{i+1}, y, v_{i+3}, z$ induce a P_5. If $y, z \in W_C$, then x, y, z, v_i, v_{i+1} induce a $K^+_{2,3}$. If only one of y, z belongs to W_C, then, by the first observation, we have that $y \in W_C$, $z \in V''_{i+1}$ up to symmetry. Then, z, v_{i+2}, y, v_i, x induce a P_5.

Suppose that $\bigcup_{i=1}^{4} V_i' \cap \{x, y, z\} = \emptyset$. If at least two of the vertices x, y, z belong to W_C, then G contains an induced $K_{2,3}^{+}$. In all other cases, G contains an induced P_5.

So, our assumption about the existence of three pairwise non-adjacent vertices of G was false. □

6 Main result

Theorem 1 *The* WCOL *problem can be solved in polynomial on the sum of vertex weights time for* $\{P_5, K_{2,3}, K_{2,3}^{+}\}$*-free graphs.*

Proof By Lemma 3 and the reasonings from the previous section, the WCOL problem for $\{P_5, K_{2,3}, K_{2,3}^{+}\}$-free graphs can be reduced in polynomial on the number of vertices time to the same problem for graphs in

$$(Free(\{O_3\}))^* \cup Free(\{P_5, K_{2,2}\})$$

and graphs on at most 12 vertices. The WCOL problem can be solved in polynomial on the length of input data time for $\{P_5, K_{2,2}\}$-free graphs [14]. Hence, by the mentioned facts and Lemmas 4–6, this theorem is true. □

As a corollary, Theorem 1 implies that the COL problem can be solved in polynomial on the number of vertices time for $\{P_5, K_{2,3}, K_{2,3}^{+}\}$-free graphs.

7 Conclusions and future work

In the present paper, we considered the weighted coloring problem for hereditary graph classes that are defined by a pair of forbidden induced connected subgraphs, each on 5 vertices. The computational status of the unweighted version of this problem has been resolved for all such pairs, except for 5 of them. We proved here that the weighted coloring problem is polynomial-time solvable for the class of graphs, which is defined by a triple of forbidden such subgraphs. This class is the intersection of some of the unresolved cases, mentioned above. We hope that our result will be helpful in resolving the computational complexity of the (un)weighted coloring problem for the open cases. Clarifying its complexity status for them is a challenging research problem for future work.

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