TWO DYNAMICAL SYSTEMS IN THE SPACE OF TRIANGLES

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Abstract. Let M be the space of triangles, defined up to shifts, rotations and dilations. We define two maps $f : M \to M$ and $g : M \to M$. The map f corresponds to a triangle of perimeter π the triangle with angles numerically equal to edges of the initial triangle. The map g corresponds to a triangle of perimeter 2π the triangle with *exterior* angles numerically equal to edges of the initial triangle. For $p \in M$ the sequence $\{p, f(p), f(f(p)), \ldots\}$ converges to the equilateral triangle and the sequence $\{p, g(p), g(g(p)), \ldots\}$ converges to the "degenerate triangle" with angles $(0, 0, \pi)$. In Supplement an analogous problem about inscribed-circumscribed quadrangles is discussed.

1. Introduction

Dynamical systems in space of triangles are objects of an interest for many years. For example in [2] and [3] the map is studied that corresponds to a triangle its pedal triangle. And in [1] the map is studied, where a new triangle is constructed from cevians of the given one.

We adopt another approach: we interchange roles of edges and angles. Namely, to a triangle with perimeter π we correspond the triangle whose angles are numerically equal to edges of the initial triangle, and to a triangle with perimeter 2π we correspond the triangle whose *exterior* angles are numerically equal to edges of the initial one.

Let M be the space triangles defined up to shifts, rotations and dilations. Thus, an element of M is a triple of positive numbers with the sum π. Triples (α, β, γ), (β, γ, α) and (γ, α, β) are the same, but mirror symmetric triangles are different elements in M. We denote by α the smallest angle of a triangle and by γ — the biggest, thus, $\alpha \leqslant \beta \leqslant \gamma$.

We will consider two maps $f : M \to M$ and $g : M \to M$. Let p be a triangle of perimeter π, with angles α, β, γ and let α', β' and γ' be lengths of edges opposite to angles α, β and γ, respectively. Then $f(p) = q$, where $q = (\alpha', \beta', \gamma')$. Let now p be the same element in M, but with perimeter 2π. Let a, b, c be exterior angles, adjacent to α, β and γ and a', b', c' be lengths of edges, opposite to α, β and γ, respectively. Then $g(p) = r$, where $r = (\pi - a', \pi - b', \pi - c')$, i.e. (a', b', c') are exterior angles of the new triangle.

Remark 1.1. The triangle inequality is not valid for values of interior angles, but valid for values of exterior angles. Hence, f is not a bijection, but g is.

Theorem 2.1. *Let $p \in M$, then sequence $\{p, f(p), f(f(p)), \ldots\}$ converges to the equilateral triangle.*

Theorem 4.1. *Let $p \in M$, then the sequence $\{p, g(p), g(g(p)), \ldots\}$ converges to the point $(0, 0, \pi)$, which does not belong to M, but belong to its boundary.*

In Supplement we consider the map h that correspond to a inscribed-circumscribed quadrangle of perimeter 2π the inscribed-circumscribed quadrangle which angles are numerically equal to edges of the initial quadrangle.

Theorem 5.1. *Let Q be an inscribed-circumscribed quadrangle then the sequence $\{Q, h(Q), h(h(Q)), \ldots\}$ converges to the "degenerate" quadrangle with angles $0, 0, \pi, \pi$.*

2. Properties of the map f

Let us remind that in a triangle the bigger edge lies opposite the bigger angle. Thus, $\alpha' \leqslant \beta' \leqslant \gamma'$.

Lemma 2.1.
$$\alpha' = \frac{\pi\cdot\sin(\alpha)}{\sin(\alpha)+\sin(\beta)+\sin(\gamma)},\quad \beta' = \frac{\pi\cdot\sin(\beta)}{\sin(\alpha)+\sin(\beta)+\sin(\gamma)},\quad \gamma' = \frac{\pi\cdot\sin(\gamma)}{\sin(\alpha)+\sin(\beta)+\sin(\gamma)}.$$

Proof. It is enough to note that in triangle lengths of edges are proportional to sines of opposite angles. As perimeter must be π, it remains to find the proportionality coefficient. □

Lemma 2.2. *$\alpha' \geqslant \alpha$ and equality is satisfied only when $\alpha = \frac{\pi}{3}$,*

Proof.
$$\frac{\pi\cdot\sin(\alpha)}{\sin(\alpha)+\sin(\beta)+\sin(\gamma)} = \frac{\pi\cdot\sin(\alpha)}{\sin(\alpha)+\sin(\beta)+\sin(\alpha+\beta)} =$$
$$= \frac{\pi\cdot\sin(\alpha)}{2\sin\frac{\beta+\alpha}{2}\cos\frac{\beta-\alpha}{2}+2\sin\frac{\beta+\alpha}{2}\cos\frac{\beta+\alpha}{2}} = \frac{\pi\cdot\sin(\alpha)}{4\sin\frac{\beta+\alpha}{2}\cos\frac{\alpha}{2}\cos\frac{\beta}{2}} = \frac{\pi\sin\frac{\alpha}{2}}{2\sin\frac{\beta+\alpha}{2}\cos\frac{\beta}{2}}.$$

If $0 < x < \frac{\pi}{6}$, then $\sin(x) > \frac{3x}{\pi}$. Hence,
$$\frac{\pi\sin\frac{\alpha}{2}}{2\sin\frac{\beta+\alpha}{2}\cos\frac{\beta}{2}} \geqslant \frac{3\alpha}{4\sin\frac{\beta+\alpha}{2}\cos\frac{\beta}{2}},$$
with equality only when $\alpha = \frac{\pi}{3}$. It is enough to prove that
$$\frac{3\alpha}{4\sin\frac{\beta+\alpha}{2}\cos\frac{\beta}{2}} \geqslant \alpha \Leftrightarrow 3 \geqslant 4\sin\frac{\beta+\alpha}{2}\cos\frac{\beta}{2} = 2\sin\frac{\alpha+2\beta}{2} + 2\sin\frac{\alpha}{2}.$$

Now it remains to note that the first summand is not greater, than 2, and the second is not greater, than 1 (because $\alpha \leqslant \frac{\pi}{3}$). □

Lemma 2.3. *$\gamma' \leqslant \gamma$ and equality is satisfied only when $\gamma = \frac{\pi}{3}$.*

Proof. As
$$\gamma' = \frac{\pi\cdot\sin(\gamma)}{\sin(\alpha)+\sin(\beta)+\sin(\gamma)} = \frac{\pi\cdot\sin(\gamma)}{2\sin\frac{\pi-\gamma}{2}\cdot\cos\frac{\beta-\alpha}{2}+\sin(\gamma)},$$
then γ' is maximal when the difference $\beta-\alpha$ is maximal. If $\gamma \geqslant \frac{\pi}{2}$, then this difference is maximal for $\alpha = 0$ and $\beta = \pi-\gamma$. But then
$$\gamma' < \frac{\pi\cdot\sin(\gamma)}{2\sin\frac{\pi-\gamma}{2}\cdot\cos\frac{\pi-\gamma}{2}+\sin(\gamma)} \leqslant \frac{\pi}{2} \leqslant \gamma.$$

If $\frac{\pi}{3} \leqslant \gamma < \frac{\pi}{2}$, then the difference is maximal when $\beta = \gamma$ and $\alpha = \pi - 2\gamma$. But then
$$\gamma' = \frac{2\pi\cdot\sin\frac{\gamma}{2}\cdot\cos\frac{\gamma}{2}}{2\cdot\cos\frac{\gamma}{2}\cdot\cos\frac{3\gamma-\pi}{2}+2\cdot\sin\frac{\gamma}{2}\cdot\cos\frac{\gamma}{2}} = \frac{\pi\cdot\sin\frac{\gamma}{2}}{\sin\frac{3\gamma}{2}+\sin\frac{\gamma}{2}} = \frac{\pi}{\cos(\gamma)+2\cdot\cos^2\frac{\gamma}{2}+1} = \frac{\pi}{2\cos(\gamma)+2} \leqslant \frac{\pi}{3} \leqslant \gamma.$$
□

Lemma 2.4. *The point $(\frac{\pi}{3},\frac{\pi}{3},\frac{\pi}{3})$ is a stationary attracting point of the map f.*

Proof. Let
$$\alpha = \frac{\pi}{3}+x\varepsilon,\ \beta = \frac{\pi}{3}+y\varepsilon,\ \gamma = \frac{\pi}{3}+z\varepsilon,\quad x^2+y^2+z^2=1,\quad x+y+z=0.$$
then in the first approximation
$$\alpha' = \frac{\pi}{3}+\frac{x\,\pi\varepsilon}{3\sqrt{3}},\ \beta' = \frac{\pi}{3}+\frac{y\,\pi\varepsilon}{3\sqrt{3}},\ \gamma' = \frac{\pi}{3}+\frac{z\,\pi\varepsilon}{3\sqrt{3}}.$$

It remains to note that $\frac{\pi}{3\sqrt{3}} < 1$. □

The above statements prove the theorem.

Theorem 2.1. *Let $p \in M$, then the sequence $\{p, f(p), f(f(p)), \dots\}$ converges to the point $(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3})$.*

3. Properties of the map g

The reasoning here will be in terms of exterior angles. Let $p = (\alpha, \beta, \gamma) \in M$ and $a = \pi - \alpha$, $b = \pi - \beta$ and $c = \pi - \gamma$ — values of exterior angles. In what follows we will use notations a', b', c' instead of α', β', γ' and we will assume that $a \leqslant b \leqslant c$.

Lemma 3.1.

$$a' = \frac{2\pi \cdot \sin(a)}{\sin(a) + \sin(b) + \sin(c)},\ b' = \frac{2\pi \cdot \sin(b)}{\sin(a) + \sin(b) + \sin(c)},\ c' = \frac{2\pi \cdot \sin(c)}{\sin(a) + \sin(b) + \sin(c)}.$$

Proof. It is enough to mention that $\sin(\alpha) = \sin(a)$, $\sin(\beta) = \sin(b)$ and $\sin(\gamma) = \sin(c)$. □

Lemma 3.2. *The point $(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3})$ is a stationary repelling point of the map g.*

Proof. Let

$$a = \frac{2\pi}{3} + x\varepsilon, \quad b = \frac{2\pi}{3} + y\varepsilon, \quad c = \frac{2\pi}{3} + z\varepsilon, \quad x + y + z = 0, \quad x^2 + y^2 + z^2 = 1.$$

Then in the first approximation

$$a' = \frac{2\pi}{3} - \frac{2\pi}{3\sqrt{3}} \cdot x\varepsilon, \quad b' = \frac{2\pi}{3} - \frac{2\pi}{3\sqrt{3}} \cdot y\varepsilon, \quad c' = \frac{2\pi}{3} - \frac{2\pi}{3\sqrt{3}} \cdot z\varepsilon.$$

It remains to note that $\frac{2\pi}{3\sqrt{3}} > 1$. □

Lemma 3.3. $a' \geqslant b' \geqslant c'$.

Proof. $a < b < c \Leftrightarrow \alpha > \beta > \gamma \Leftrightarrow a' > b' > c'$. □

4. Barycentric coordinates

Let us consider an equilateral triangle $\triangle ABC$ and define a barycentric coordinates a, b, c, $a + b + c = 2\pi$. As the value of an exterior angle is $< \pi$, then we will work with triangle $\triangle A_1B_1C_1$, where A_1B_1, C_1 are midpoints of BC, AC and AB, respectively.

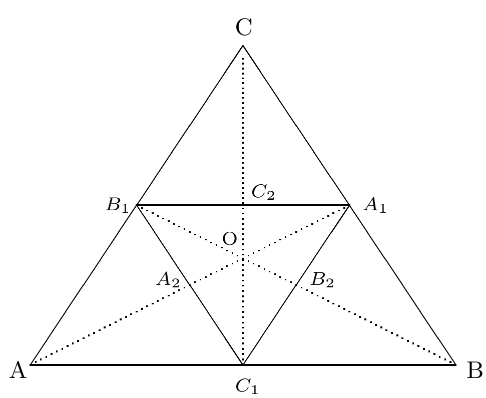


Figure 4.1

Here points A_1, B_1, C_1 have coordinates $(0, \pi, \pi)$, $(\pi.0, \pi)$ and $(\pi, \pi, 0)$, respectively. And points A_2, B_2, C_2 have coordinates $(\pi, \frac{\pi}{2}, \frac{\pi}{2})$, $(\frac{\pi}{2}, \pi, \frac{\pi}{2})$ and $(\frac{\pi}{2}, \frac{\pi}{2}, \pi)$, respectively. The point O — the center of ABC has coordinates $(\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3})$.

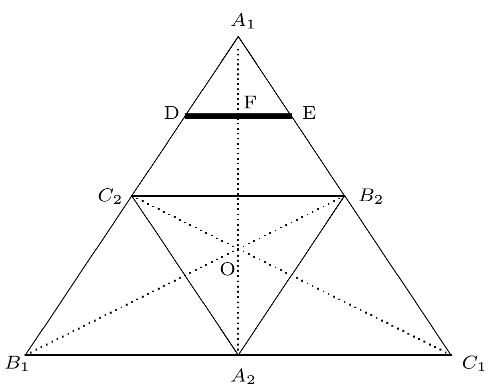
g maps

- the triangle A_1OC_2 onto the triangle A_2OC_1: $A_1 \to A_2$, $C_2 \to C_1$, $O \to O$ (and back);
- the triangle A_2OB_1 onto the triangle A_1OB_2: $A_2 \to A_1$, $B_1 \to B_2$, $O \to O$ (and back);
- the triangle B_1OC_2 onto the triangle B_2OC_1: $B_1 \to B_2$, $C_2 \to C_1$, $O \to O$ (and back).

In what follows we will always assume that $a \leqslant b$ and $a \leqslant c$. Let $g(g(a, b, c)) = (a'', b'', c'')$. Our aim is to prove that $a'' < a$.

Let

$$I_t = \{(a, b, c) \in \triangle A_1B_1C_1 \,:\, a = t, t \leqslant \frac{\pi}{2}\} \quad \text{and} \quad J_t = \{(a, b, c) \in \triangle A_1B_1C_1 \,:\, a = t, \frac{2\pi}{3} > t > \frac{\pi}{2}\}.$$

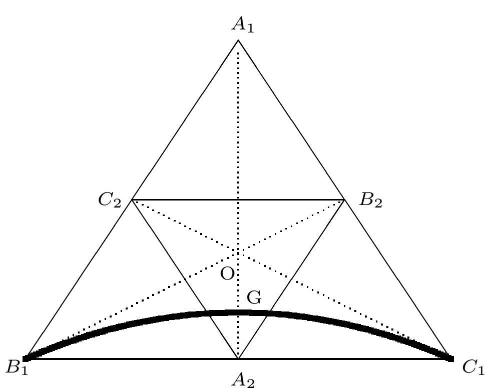


set I_t set J_t

Figure 4.2

Coordinates of points D, E and F are $(t, \pi - t, \pi)$, $(t, \pi, \pi - t)$ and $(t, \pi - \frac{t}{2}, \pi - \frac{t}{2})$, respectively. Coordinates of points K, L and M are $(t, t, 2\pi - 2t)$, $(t, 2\pi - 2t, t)$ and $(t, \pi - \frac{t}{2}, \pi - \frac{t}{2})$, respectively.

We have



$g(I_t)$ Figure 4.3 $g(J_t)$

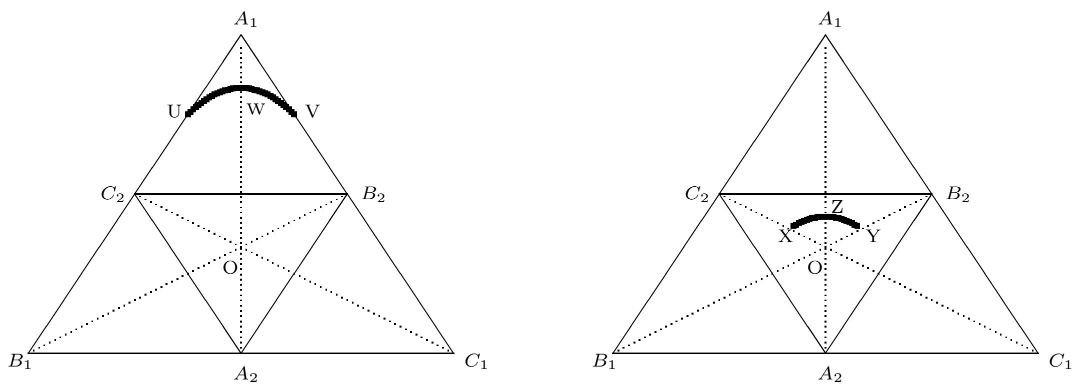
Here $g(F) = G$ and

$$G = \left(\frac{2\pi \cdot \cos\frac{t}{2}}{\cos\frac{t}{2} + 1}, \frac{\pi}{\cos\frac{t}{2} + 1}, \frac{\pi}{\cos\frac{t}{2} + 1} \right) = (2s, \pi - s, \pi - s), \text{ where } s = \frac{\pi \cdot \cos\frac{t}{2}}{\cos\frac{t}{2} + 1}.$$

Then $g(K) = P$, $g(L) = Q$, $g(M) = N$ and

$$P = \left(\frac{\pi}{1 - \cos(t)}, \frac{\pi}{1 - \cos(t)}, -\frac{2\pi \cdot \cos(t)}{1 - \cos(t)} \right), \quad Q = \left(\frac{\pi}{1 - \cos(t)}, -\frac{2\pi \cdot \cos(t)}{1 - \cos(t)}, \frac{\pi}{1 - \cos(t)} \right).$$

Remark 4.1. When $t = \frac{\pi}{2}$ (i.e. when DE coincides with B_2C_2), then altitudes B_1B_2 and C_1C_2 are tangent to the arc B_1GC_1 at points B_1 and C_1.



$g(g(I_t))$ $g(g(J_t))$

Figure 4.4

We will denote by $GG(t)$ — the first coordinate of the point $W = g(g(F))$:

$$GG(t) = \frac{2\pi \cdot \cos(s)}{\cos(s) + 1} = \frac{2\pi \cdot \cos \dfrac{\pi \cdot \cos\frac{t}{2}}{\cos\frac{t}{2} + 1}}{\cos \dfrac{\pi \cdot \cos\frac{t}{2}}{\cos\frac{t}{2} + 1} + 1}.$$

Formulas for coordinates of points $N = g(M)$ and $Z = g(g(M))$ are the same, only here $\frac{\pi}{2} \leqslant t \leqslant \frac{2\pi}{3}$. Thus the function GG is defined in the segment $[0, \frac{2\pi}{3}]$. In the figure below the plot of $GG(t)$ is presented.

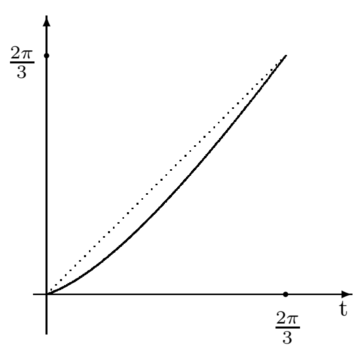


Figure 4.5

Lemma 4.1. $GG(t) < t$.

A sketch of the proof. As $GG(0)=0$ and $GG(\frac{2\pi}{3})=\frac{2\pi}{3}$ it is enough to prove, that GG is a strictly increasing function and its plot is downward convex. We have

$$GG'=\left(\frac{2\pi\cdot\cos(s)}{\cos(s)+1}\right)'=-2\pi\cdot\frac{\sin(s)\cdot s'}{(\cos(s)+1)^2}=-2\pi\cdot\frac{\sin(s)}{(\cos(s)+1)^2}\cdot\frac{-\frac{\pi}{2}\cdot\sin\frac{t}{2}}{(\cos\frac{t}{2}+1)^2}>0$$

Then

$$GG''=-2\pi\cdot\frac{[\cos(s)\cdot(s')^2+\sin(s)\cdot s'']\cdot(\cos(s)+1)+2\cdot\sin^2(s)\cdot(s')^2}{(\cos(s)+1)^3}\,.$$

The numerator of this expression is

$$2\pi\cdot(1+\cos(s))\cdot[-(s')^2\cdot(2-\cos(s)-\sin(s)\cdot s'']\,.$$

And the expression in square brackets is

$$-\pi\cdot(2-\cos(s))\cdot\left(1-\cos\frac{t}{2}\right)+\sin(s)\cdot\left(2-\cos\frac{t}{2}\right)\cdot\left(1+\cos\frac{t}{2}\right)\,.$$

The proof of the positivity of the above expression is a technical task. □

The first coordinate of the point U is $\frac{\pi}{2}\cdot(1-\cos(t))$. In the figure below are presented plots of the first coordinates of points U (the upper curve) and W (the lower curve) for $0<a\leqslant\frac{\pi}{2}$.

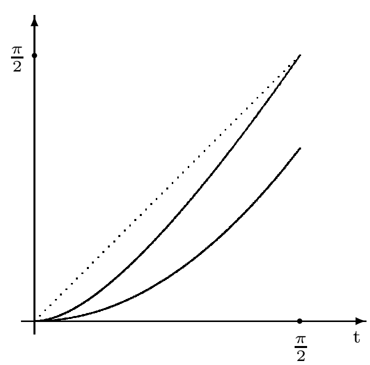


Figure 4.6

The first coordinate of the point X is

$$\frac{\pi}{1-\cos\dfrac{\pi}{1-\cos(t)}}\,,\quad \frac{\pi}{2}<t\leqslant\frac{2\pi}{3}\,.$$

In the figure below are presented plots of the first coordinates of points X (the upper curve) and Z (the lower curve).

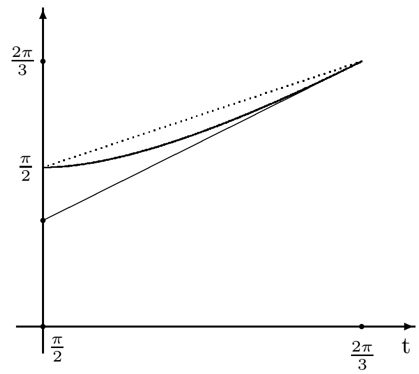


Figure 4.7

Theorem 4.1. *Let $p \in M$, then the sequence $\{p, g(p), g(g(p)), \ldots\}$ converges to the point $(0, 0, \pi)$. This point does not belong to the set M, but belong to its boundary.*

Proof. We see that the map $h = g(g(p))$ decreases the least exterior angle. Thus the sequence $\{T, h(T), h(h(T)), \ldots\}$, where T is a triangle, converges to a "degenerate triangle" with angles $(0, 0, \pi)$. On the other hand, the sequence $\{g(T), h(g(T)), h(h(g(T))), \ldots\}$ also converges to the same "degenerate triangle" (with another zero angles). □

Example 4.1. Let T be a triangle with exterior angles $(1, 2.3, 2\pi{-}3.3)$. Then triangles $g(T), g(g(T)), g(g(g(T))), \ldots$ have the following exterior angles.

$$(3.0300, 2.6851, 0.5680)$$
$$(0.6418, 2.5404, 3.1008)$$
$$(3.1217, 2.9489, 0.2124)$$
$$(0.2953, 2.8492, 3.1385)$$
$$(3.1408, 3.1097, 0.0324)$$
$$(0.0673, 3.0742, 3.1415)$$

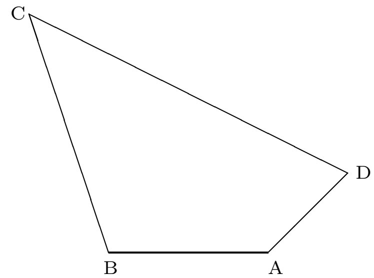
Let now the exterior angles of T be $(1.9, 2.0, 2\pi - 3.9)$. Here the sequence of triples of exterior angles is of the form:

$$(2.3377, 2.2463, 1.6990)$$
$$(1.8152, 1.9674, 2.5004)$$
$$(2.4476, 2.3267, 1.5087)$$
$$(1.6990, 1.9327, 2.6513)$$
$$(2.5988, 2.4505, 1.2338)$$
$$(1.5471, 1.9091, 2.8269)$$
$$(2.7886, 2.6312, 0.8633)$$
$$(1.3625, 1.9252, 2.9954)$$
$$(2.9814, 2.8579, 0.4437)$$
$$(1.1532, 2.0243, 3.1055)$$

5. Supplement

We will consider plane inscribed-circumscribed quadrangles (ic-quadrangles). Sums of opposite angles of an ic-quadrangle are π and sums of lengths of opposite edges are equal. Up to shifts, rotations and dilations such quadrangle is uniquely defined by its angles and their order in going around the quadrangle. Two angles of a ic-quadrangle are acute (and they are adjacent) and two are obtuse (and they are also adjacent).

Let α and β be obtuse angles and $\alpha \geqslant \beta$.



Here α is the value of $\angle A$, β is the value of $\angle B$, $\angle C = \pi - \alpha$, $\angle D = \pi - \beta$. Let r be the radius of the inscribed circle, then

$$|CD| = r \cdot (\tan(\alpha/2) + \tan(\beta/2)) \geqslant |BC| = r \cdot (\tan(\alpha/2) + \cot(\beta/2)) \geqslant$$
$$\geqslant |AB| = r \cdot (\cot(\alpha/2) + \tan(\beta/2)) \geqslant |DA| = r \cdot (\cot(\alpha/2) + \cot(\beta/2)).$$

If the perimeter of $ABCD$ is 2π, then

$$|CD| = \frac{\pi \cdot \sin(\frac{\alpha}{2}) \cdot \sin(\frac{\beta}{2})}{\cos(\frac{\alpha}{2} - \frac{\beta}{2})},\ |BC| = \frac{\pi \cdot \sin(\frac{\alpha}{2}) \cdot \cos(\frac{\beta}{2})}{\sin(\frac{\alpha}{2} + \frac{\beta}{2})}.$$

The map h corresponds to an ic-quadrangle of perimeter 2π the ic-quadrangle of perimeter 2π with angles numerically equal to edges of initial ic-quadrangle.

Theorem 5.1. *Let Q be an ic-quadrangle of perimeter 2π, then the sequence $\{Q, h(Q), h(h(Q)), \ldots\}$ converges to a "degenerate quadrangle" with angles $(0, 0, \pi, \pi)$.*

Sketch of the proof. The sum of obtuse angles in the quadrangle $h(h(Q))$ is strictly greater, than the sum of obtuse angles in the initial quadrangle Q. □

Remark 5.1. The sum of obtuse angles in $h(Q)$ can be less, than the sum of obtuse angles in Q. For example, if $\alpha = 1.85$ and $\beta = 1,75$, then

$$\frac{Pi \cdot \sin(\frac{\alpha}{2}) \cdot \sin(\frac{\beta}{2})}{\cos(\frac{\alpha-\beta}{2}} + \frac{Pi \cdot \sin(\frac{\alpha}{2}) \cdot \cos(\frac{\beta}{2})}{\sin(\frac{\alpha+\beta}{2}} = 3.58 < \alpha + \beta = 3.6$$

References

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