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Quantum matrix algebras of BMW type: Structure of the characteristic subalgebra

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ABSTRACT

A notion of quantum matrix (QM-) algebra generalizes and unifies two famous families of algebras from the theory of quantum groups: the RTT-algebras and the reflection equation (RE-) algebras. These algebras being generated by the components of a 'quantum' matrix M possess certain properties which resemble structure theorems of the ordinary matrix theory. It turns out that such structure results are naturally derived in a more general framework of the QM-algebras. In this work we consider a family of Birman–Murakami–Wenzl (BMW) type QM-algebras. These algebras are defined with the use of R-matrix representations of the BMW algebras. Particular series of such algebras include orthogonal and symplectic types RTT- and RE-algebras, as well as their super-partners.

For a family of BMW type QM-algebras, we investigate the structure of their 'characteristic subalgebras' — the subalgebras where the coefficients of characteristic polynomials take values. We define three sets of generating elements of the characteristic subalgebra and derive recursive Newton and Wronski relations between them. We also define an associative $\star$-product for the matrix M of generators of the QM-algebra which is a proper generalization of the classical matrix multiplication. We determine the set of all matrix 'descendants' of the quantum matrix M, and prove the $\star$-commutativity of this set in the BMW type.

1. Introduction

A notion of a quantum matrix group, also called the RTT-algebra, is implicit in the quantum inverse scattering method. A formal definition has been given in the works of V. Drinfel'd, L. Faddeev, N. Reshetikhin and L. Takhtajan [6,47]. Since then, various aspects of the quantum matrix group theory have been elaborated, especially in attempts to define differential geometric structures on non-commutative spaces (see, e.g., [48,51]). In particular, a different family of algebras generated by matrix components, the so-called reflection equation (RE-) algebras [5,33], has been brought into consideration. Soon it was realized that, for both the RTT- and the RE-algebras, some of the basic concepts of the classical matrix algebra, like the notion of the spectral invariants and the characteristic identity (the Cayley–Hamilton theorem) can be properly generalized (see [9,40,44,52]). So, it comes out that the matrix notation used for the definition of the RTT- and the RE-algebras is not only technically convenient, but it dictates certain structure properties for the algebras themselves. It is

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then natural to search for a possibly most general algebraic setting for the matrix-type objects. Such family of algebras was introduced in Refs. [18] and [24], and in the latter case the definition was dictated by a condition that the standard matrix theory statements should have their appropriate generalizations. These algebras were called quantum matrix (QM-) algebras although one should have in mind that the QM-algebras are generated by the matrix components rather than by the matrix itself.

The RTT- and the RE-algebras are probably the most important subfamilies in the variety of QM-algebras. They are distinguished both from the algebraic point of view (the presence of additional non-braided bi-algebra and bi-comodule structures) and from the geometric point of view (their interpretation as, respectively, the algebras of quantized functions and of quantized invariant differential operators on a group); also, the RE-algebras naturally appear in the representation theory, in the description of the diagonal reduction algebras [32]. However, for the generalization of the basic matrix algebra statements, it is not only possible but often more clarifying to use a weaker structure settings of the QM-algebras.

So far, the program of generalizing the Cayley–Hamilton theorem was fully accomplished for the 'linear' (or Iwahori–Hecke) type QM-algebras. For the $GL(m)$-type algebras, the results were described in [14,24,26] and for the $GL(m|n)$-type algebras in [15,16]. These works generalize earlier results on characteristic identities by A.J. Bracken, H.S. Green, et al. in the Lie (super)algebra case [2,11,13,28,42] (for a review see [19]) and in the quantized universal enveloping algebra case [12], and by I. Kantor and I. Trishin in the matrix superalgebra case [30,31].

The similar investigation program for the QM-algebras of Birman–Murakami–Wenzl (BMW) type (for their definition see Section 4.1) was initiated in [43]. In the present and forthcoming works we continue and complement this program. The family of BMW type QM-algebras serves as a unifying set-up for the description of the orthogonal and symplectic QM-algebras as well as for their supersymmetric partners. Some partial results about specific examples of such algebras and their limiting cases were already derived. In particular, the characteristic identities for the generators of the orthogonal and symplectic Lie algebras have been considered at the representation theoretical and at the abstract algebraic levels in [2,13] and in [11,27,36,42]. The characteristic identities for the canonical Drinfeld–Jimbo quantizations of the orthogonal and symplectic universal enveloping algebras were obtained in [37] and their images in the series of highest weight representations were discussed in detail in [38]. So, it is pretty clear that proper generalizations of the Cayley–Hamilton theorems do exist for the families of orthogonal and symplectic QM-algebras. However, in a derivation of these results one meets serious technical complications. The reason is that the structure of the Birman–Murakami–Wenzl algebras is substantially more sophisticated then that of the Iwahori–Hecke algebras (Iwahori–Hecke and Birman–Murakami–Wenzl algebras play similar roles in the construction of the QM-algebras of linear and BMW types). In the present work we develop an appropriate techniques to deal with these complications.

In Sections 2 and 3 we collect necessary results concerning the Birman–Murakami–Wenzl (BMW) algebras and their R-matrix representations. In the beginning of Section 2 we define the BMW algebras in terms of generators and relations, describe few helpful morphisms between these algebras, and introduce the baxterized elements. These elements are used in Section 2.2 for the definition of three sets of idempotents called antisymmetrizers, symmetrizers and contractors. Necessary properties of these idempotents are proved in Proposition 2.2. All the material of this section, except the construction and properties of the contractors is fairly well known and we present it to make the presentation self-contained.

In Section 3 we consider the R-matrix representations of the BMW algebras. We define standard notions of the R-trace,[1] skew-invertibility, compatible pair of R-matrices and R-matrix twist (Section 3.1). In Section 3.2 we collect necessary formulas and statements relating the notions introduced before. To investigate the skew-invertibility of the R-matrix after a twist, in Section 3.3 we derive an expression for the twisted R-matrix, which is different from the standard one. Next we describe the BMW type R-matrices (Section 3.4). The major part of a technical preparatory work is done in Sections 3.2—3.4, and 3.5, 3.6. Here we develop the R-matrix technique, which is later used in the main Sections 4, 5.

In the beginning of Section 4 we introduce the QM-algebras of general and BMW types. We then define the characteristic subalgebra of the QM-algebra. In the Iwahori–Hecke case, it is the subalgebra where the coefficients of the Cayley–Hamilton identity take their values. As it was shown in [24], the characteristic subalgebra is abelian. In Section 4.2 we describe three generating sets for the characteristic subalgebra of the BMW type QM-algebra. As compared to the linear QM-algebras, all these generating sets contain a single additional element — the 2-contraction g — which at the classical level gives rise to bilinear invariant 2-forms for the orthogonal and symplectic groups.

Next, in Section 4.3, we construct a proper analogue of the matrix multiplication for the quantum matrices. We call it the quantum matrix product '$\star$'. In general, the $\star$-product is different from the usual matrix product. It is worth noting that for the family of RE-algebras, the $\star$-product coincides with the matrix product. The $\star$-product is proven to be associative and hence the $\star$-powers of the same quantum matrix M commute. We determine then the set of all 'descendants' of the quantum matrix M in the BMW case and prove that this set is $\star$-commutative. It turns out that, unlike the linear QM-algebra case, it is not possible to express all these descendants in terms of the $\star$-powers of M only. The expressions include also a new operation '$\top$', which can be treated as a 'matrix multiplication with a transposition'.

In Section 4.5 we define an extension of the BMW type QM-algebra by the element g^{-1} which is the inverse to the 2-contraction. Then we construct in the extended algebra the inverse $\star$-power of the quantum matrix M.

[1] This operation is also called a quantum trace or, shortly, a q-trace in the literature.

The last Section 5 contains the principal result of the present work, Theorem 5.2, which establishes, for the BMW type QM-algebras, recursive relations between the elements of the three generating sets of their characteristic subalgebras. These formulas generalize the classical Newton and Wronsky relations for the sets of the power sums, elementary and complete symmetric polynomials (see [35]) to the case of quantum matrices and simultaneously, to the situation where additional element of the characteristic subalgebra, the 2-contraction, is present. To prove this result we first derive the matrix relations among the descendants of the BMW type quantum matrix M (see Lemma 5.1). These relations can be viewed as the matrix counterparts of the Newton relations, and they are expected to be important ingredients in a future derivation of the characteristic identities for the QM-algebras of the BMW type.

Some auxiliary results, which are interesting in themselves, although not necessary for considerations in the main text, are collected in the Appendices. In Appendix A we prove the primitivity of the contractors from Section 2.4. In Appendix B their further properties are discussed. Appendix C is devoted to a discussion of universal counterparts of the matrix relations given in Sections 3.2, 3.3.

In forthcoming papers we are going to construct the Cayley–Hamilton identities, and, more generally Cayley–Hamilton–Newton identities in the spirit of [22], for the series of orthogonal and symplectic QM-algebras and, further on, for their super-partners.

2. Some facts about Birman–Murakami–Wenzl algebras

In this preparatory section we collect definitions and derive few results on the Birman–Murakami–Wenzl algebras. We give a minimal information, which is required for the main part of the paper. In particular, in Section 2.2 we describe series of morphisms of the braid groups and their quotient BMW algebras; in Section 2.3 we introduce baxterized elements which are then used in Section 2.4 to define three series of idempotents in the BMW algebras, the so called symmetrizers, antisymmetrizers and contractors.

The reader will find a more detailed presentation of the Birman–Murakami–Wenzl algebras in, e.g., papers [50] and [34].

2.1. Definition

The braid group $\mathcal{B}_n$, $n \geq 2$, in Artin presentation, is defined by generators $\{\sigma_i\}_{i=1}^{n-1}$ and relations

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \quad \forall\, i = 1, 2, \dots, n-1, \tag{2.1}$$

$$\sigma_i\sigma_j = \sigma_j\sigma_i \quad \forall\, i, j: \ |i-j| > 1 \,. \tag{2.2}$$

We put, by definition, $\mathcal{B}_1 := \{1\}$.

The *Birman–Murakami–Wenzl (BMW) algebra* $\mathcal{W}_n(q,\mu)$ [1,39] is a finite dimensional quotient algebra of the group algebra $\mathbb{C}\mathcal{B}_n$. It depends on two complex parameters q and μ. Let

$$\kappa_i := \frac{(q1-\sigma_i)(q^{-1}1+\sigma_i)}{\mu(q-q^{-1})}, \qquad i = 1, 2, \dots, n-1 \,. \tag{2.3}$$

The quotient algebra $\mathcal{W}_n(q,\mu)$ is specified by conditions

$$\sigma_i\kappa_i = \kappa_i\sigma_i = \mu\kappa_i \,, \tag{2.4}$$

$$\kappa_i\sigma_{i+1}^{\epsilon}\kappa_i = \mu^{-\epsilon}\kappa_i \,, \tag{2.5}$$

where ϵ is the sign,[2] $\epsilon = \pm 1$.

Eqs. (2.3) and (2.4) imply that the characteristic polynomial for the generator σ_i has degree three,

$$(\sigma_i - q1)(\sigma_i + q^{-1}1)(\sigma_i - \mu 1) = 0 \,. \tag{2.6}$$

The relations (2.4)–(2.5) imply also

$$\sigma'_i\kappa_{i+1}\sigma'_i = \sigma'_{i+1}\kappa_i\sigma'_{i+1} \,, \quad \text{where} \quad \sigma' = \sigma - (q-q^{-1})1 \,, \tag{2.7}$$

$$\kappa_i\sigma_{i+\pi}^{\epsilon} = \kappa_i\kappa_{i+\pi}\sigma_i^{-\epsilon} \,, \qquad \sigma_{i+\pi}^{\epsilon}\kappa_i = \sigma_i^{-\epsilon}\kappa_{i+\pi}\kappa_i \,, \tag{2.8}$$

$$\kappa_i\kappa_{i+\pi}\kappa_i = \kappa_i \,, \tag{2.9}$$

$$\kappa_i^2 = \eta\,\kappa_i \,, \qquad \text{where} \quad \eta := \frac{(q-\mu)(q^{-1}+\mu)}{\mu(q-q^{-1})} \,. \tag{2.10}$$

Here ϵ and π are the signs: $\epsilon = \pm 1$ and $\pi = \pm 1$.

The parameters q and μ of the BMW algebra are taken in domains[3]

$$q \in \mathbb{C}\backslash\{0, \pm 1\}, \qquad \mu \in \mathbb{C}\backslash\{0, q, -q^{-1}\}, \tag{2.11}$$

so that the elements κ_i are well defined and non-nilpotent. Further restrictions on q and μ will be imposed in Section 2.3.

[2] If $\mu \neq q - q^{-1}$ then it is enough to impose only one of the relations (2.5), the relation with another sign follows (see [25]).

[3] For particular values $\mu = \pm q^i$, $i \in \mathbb{Z}$, the limiting cases $q \to \pm 1$ to the Brauer algebra [3] can be consistently defined.

2.2. Natural morphisms

- The braid groups and their quotient BMW algebras admit a chain of monomorphisms

$$\begin{array}{l} \mathcal{B}_2 \hookrightarrow \cdots \hookrightarrow \mathcal{B}_n \hookrightarrow \mathcal{B}_{n+1} \hookrightarrow \cdots \ , \\ \mathcal{W}_2 \hookrightarrow \cdots \hookrightarrow \mathcal{W}_n \hookrightarrow \mathcal{W}_{n+1} \hookrightarrow \cdots \end{array} \tag{2.12}$$

defined on the generators as

$$\mathcal{B}_n \ (\text{or} \ \ \mathcal{W}_n) \ni \sigma_i \mapsto \sigma_{i+1} \in \mathcal{B}_{n+1} \ (\text{or} \ \ \mathcal{W}_{n+1}) \ \ \forall\, i = 1, \dots, n-1. \tag{2.13}$$

We denote by $\alpha^{(n)\uparrow i} \in \mathcal{B}_{n+i}$ (or $\mathcal{W}_{n+i}$) an image of an element $\alpha^{(n)} \in \mathcal{B}_n$ (or $\mathcal{W}_n$) under a composition of the mappings (2.12)–(2.13). Conversely, if for some $j < (n-1)$, an element $\alpha^{(n)}$ belongs to the image of $\mathcal{B}_{n-j}$ (or $\mathcal{W}_{n-j}$) in $\mathcal{B}_n$ (or $\mathcal{W}_n$) then by $\alpha^{(n)\downarrow j}$ we denote the preimage of $\alpha^{(n)}$ in $\mathcal{B}_{n-j}$ (or $\mathcal{W}_{n-j}$).

This notation will be helpful in Section 2.4 where we discuss three distinguished sequences of idempotents in the BMW algebras.

- Consider series of elements $\tau^{(n)} \in \mathcal{B}_n$ defined inductively

$$\tau^{(1)} := 1, \quad \tau^{(j+1)} := \tau^{(j)}\, \sigma_j \sigma_{j-1} \dots \sigma_1 \, . \tag{2.14}$$

$\tau^{(n)}$ is the lift of the longest element of the symmetric group S_n. The inner $\mathcal{B}_n$ (and, hence, $\mathcal{W}_n$) automorphism

$$\tau \, : \, \sigma_i \mapsto \tau^{(n)}\, \sigma_i \, (\tau^{(n)})^{-1} = \sigma_{n-i} \ , \tag{2.15}$$

will be used below in derivations in Sections 2.4 and 4 .

- One has three algebra isomorphisms:

$$\iota: \ \mathcal{W}_n(q,\mu) \to \mathcal{W}_n(-q^{-1},\mu)\, , \quad \iota' : \ \mathcal{W}_n(q,\mu) \to \mathcal{W}_n(q^{-1},\mu^{-1}) \quad \text{and} \quad \iota'' : \ \mathcal{W}_n(q,\mu) \to \mathcal{W}_n(-q,-\mu)$$

defined on generators by

$$\iota : \sigma_i \mapsto \quad \sigma_i \ , \tag{2.16}$$

$$\iota' : \sigma_i \mapsto \quad \sigma_i^{-1} \ , \tag{2.17}$$

$$\iota'' : \sigma_i \mapsto -\sigma_i \ . \tag{2.18}$$

The map ι interchanges the two sets of baxterized elements $\sigma^{\pm}(x)$ and the series of symmetrizers $a^{(n)}$ and antisymmetrizers $s^{(n)}$: $\iota(a^{(n)}) = s^{(n)}$ (see Sections 2.3 and 2.4). For the maps ι', ι'' one has: $\iota'(\sigma^{\pm}(x)) = x\sigma^{\pm}(x^{-1})$, $\iota''(\sigma^{\pm}(x)) = \sigma^{\pm}(x)$. The series of (anti)symmetrizers are stable under maps ι' and ι''. One also has $\iota(\kappa_i) = \iota'(\kappa_i) = \iota''(\kappa_i) = \kappa_i$.

- . There exists an algebra antiautomorphism $\varsigma : \ \mathcal{W}_n(q,\mu) \to \mathcal{W}_n(q,\mu)$ ($\varsigma(xy) = \varsigma(y)\varsigma(x)$), defined on generators as

$$\varsigma : \ \sigma_i \mapsto \sigma_i \ . \tag{2.19}$$

This morphism will be used later in the proofs of Propositions 2.2 and 4.11.

2.3. Baxterized elements

A set of elements $\sigma_i(x)$, $i = 1, 2, \dots, n-1$, depending on a complex parameter x, in a quotient of the group algebra $\mathbb{C}\mathcal{B}_n$ is called a set of *baxterized elements* if

$$\sigma_i(x)\, \sigma_{i+1}(xy)\, \sigma_i(y) \ = \ \sigma_{i+1}(y)\, \sigma_i(xy)\, \sigma_{i+1}(x) \tag{2.20}$$

for $i = 1, 2, \dots, n-1$ and

$$\sigma_i(x)\, \sigma_j(y) \ = \ \sigma_j(y)\, \sigma_i(x) \tag{2.21}$$

if $|i-j| > 1$.

Lemma 2.1 ([20,29]). *For the algebra $\mathcal{W}_n(q,\mu)$, the baxterized elements exist. There are two sets of the baxterized elements $\{\sigma_i^{\varepsilon}\}$, $\varepsilon = \pm 1$, given by*

$$\sigma_i^{\varepsilon}(x) \ := \ 1 \ + \ \frac{x-1}{q-q^{-1}}\, \sigma_i \ + \ \frac{x-1}{\alpha_{\varepsilon} x + 1}\, \kappa_i \, , \tag{2.22}$$

where $\alpha_{\varepsilon} \ := \ -\varepsilon q^{-\varepsilon} \mu^{-1}$.

The complex argument x, traditionally called *the spectral parameter*, is chosen in a domain $\mathbb{C} \setminus \{-\alpha_{\varepsilon}^{-1}\}$.

2.4. Symmetrizers, antisymmetrizers and contractors

In terms of the baxterized generators we construct two series of elements $a^{(i)}$ and $s^{(i)}$, $i = 1, 2, \dots, n$, in the algebra $\mathcal{W}_n(q, \mu)$. They are defined iteratively in two ways:

$$a^{(1)} := 1 \quad \text{and} \quad s^{(1)} := 1 \,, \tag{2.23}$$

$$a^{(i+1)} := \frac{q^i}{(i+1)_q} a^{(i)} \sigma_i^-(q^{-2i}) a^{(i)} \quad \text{or} \quad a^{(i+1)} := \frac{q^i}{(i+1)_q} a^{(i)\uparrow 1} \sigma_1^-(q^{-2i}) a^{(i)\uparrow 1} \,, \tag{2.24}$$

$$s^{(i+1)} := \frac{q^{-i}}{(i+1)_q} s^{(i)} \sigma_i^+(q^{2i}) s^{(i)} \quad \text{or} \quad s^{(i+1)} := \frac{q^{-i}}{(i+1)_q} s^{(i)\uparrow 1} \sigma_1^+(q^{2i}) s^{(i)\uparrow 1} \,, \tag{2.25}$$

where i_q are usual q-numbers, $i_q := (q^i - q^{-i})/(q - q^{-1})$. Below we show that in each of Eqs. (2.24), (2.25) the two definitions coincide. We note that the factorized formula for the (anti)symmetrizers, in the spirit of the fusion procedure for the BMW algebra [21], follows from Eqs. (2.24), (2.25).

To avoid singularities in the definition of $a^{(i)}$ (respectively, $s^{(i)}$), $i = 1, 2, \dots, n$, we impose further restrictions on the parameters of $\mathcal{W}_n(q, \mu)$:

$$j_q \neq 0 \,, \quad \mu \neq -q^{-2j+3} \ \ (\text{respectively,} \ \ \mu \neq q^{2j-3}) \quad \forall\, j = 2, 3, \dots, n \,. \tag{2.26}$$

The elements $a^{(i)}$ and $s^{(i)}$ are called an *ith order antisymmetrizer* and an *ith order symmetrizer*, respectively.

The second order antisymmetrizer and symmetrizer

$$a^{(2)} = \frac{q}{2_q} \sigma_1^-(q^{-2}) = \frac{(q1 - \sigma_1)(\mu 1 - \sigma_1)}{2_q(\mu + q^{-1})} \,, \quad s^{(2)} = \frac{q^{-1}}{2_q} \sigma_1^+(q^2) = \frac{(q^{-1}1 + \sigma_1)(\mu 1 - \sigma_1)}{2_q(\mu - q)} \tag{2.27}$$

are the idempotents participating in a resolution of unity in the algebra $\mathcal{W}_2(q, \mu)$ (c.f. with the property (2.6)),

$$1 = a^{(2)} + s^{(2)} + \eta^{-1} \kappa_1 \,. \tag{2.28}$$

Likewise for $a^{(2)}$ and $s^{(2)}$, one can introduce higher order analogues for the third idempotent entering the resolution. Namely, define iteratively

$$c^{(2)} := \eta^{-1} \kappa_1 \,, \qquad c^{(2i+2)} := c^{(2i)\uparrow 1} \kappa_1 \kappa_{2i+1} c^{(2i)\uparrow 1} \,. \tag{2.29}$$

The element $c^{(2i)}$ is called an $(2i)$th *order contractor*. Main properties of the (anti)symmetrizers and contractors are summarized below.

Proposition 2.2. *Two expressions given for the antisymmetrizers and symmetrizers in Eqs.* (2.24) *and* (2.25) *are identical. The elements $a^{(n)}$ and $s^{(n)}$ are central primitive idempotents in the algebra $\mathcal{W}_n(q, \mu)$. One has*

$$a^{(n)} \sigma_i = \sigma_i a^{(n)} = -q^{-1} a^{(n)} \,, \ s^{(n)} \sigma_i = \sigma_i s^{(n)} = q s^{(n)} \qquad \forall\, i = 1, 2, \dots, n-1 \tag{2.30}$$

and

$$a^{(n)} a^{(m)\uparrow i} = a^{(m)\uparrow i} a^{(n)} = a^{(n)} \,, \ s^{(n)} s^{(m)\uparrow i} = s^{(m)\uparrow i} s^{(n)} = s^{(n)} \qquad \text{if} \quad m + i \leq n \,. \tag{2.31}$$

The antisymmetrizers $a^{(n)}$, for all $n = 2, 3, \dots$, are orthogonal to the symmetrizers $s^{(m)}$, for all $m = 2, 3, \dots$,

$$a^{(n)} s^{(m)} = 0 \,. \tag{2.32}$$

The element $c^{(2n)}$ is a primitive idempotent in the algebra $\mathcal{W}_{2n}(q, \mu)$ and in the algebra $\mathcal{W}_{2n+1}(q, \mu)$. One has

$$c^{(2n)} c^{(2i)\uparrow n-i} = c^{(2i)\uparrow n-i} c^{(2n)} = c^{(2n)} \ \ \forall\, i = 1, 2, \dots, n \,; \tag{2.33}$$

$$c^{(2n)} \sigma_i = c^{(2n)} \sigma_{2n-i} \,, \qquad \sigma_i c^{(2n)} = \sigma_{2n-i} c^{(2n)} \ \ \forall\, i = 1, 2, \dots, n-1 \,, \tag{2.34}$$

and

$$c^{(2n)} \sigma_n = \sigma_n c^{(2n)} = \mu c^{(2n)} \,. \tag{2.35}$$

The contractors $c^{(2n)}$ are orthogonal to the antisymmetrizers $a^{(m)}$ and to the symmetrizers $s^{(m)}$ for all $m > n$.

Proof. The explicit formula (2.24) for idempotents, which we call antisymmetrizers here, appears in [49], although without referring to the baxterized elements (see the proof of the lemma 7.6 in [49]).[4] Our proof of the formulas (2.30) and (2.31) relies on the relations (2.20) for the baxterized generators.

We first check that the elements $a^{(i)}$ defined iteratively by the first formula in (2.24) satisfy the relations (2.30) and (2.31). The equalities (2.30) for the antisymmetrizers are equivalent to

$$a^{(n)} s^{(2)\uparrow i-1} = s^{(2)\uparrow i-1} a^{(n)} = a^{(n)} c^{(2)\uparrow i-1} = c^{(2)\uparrow i-1} a^{(n)} = 0 \,, \quad \forall\, i = 1, 2, \dots n-1 \,,$$

[4] Different expressions for the antisymmetrizers and symmetrizers, which are less suitable for our applications, were derived in [17].

which, in turn, are equivalent to

$$a^{(n)}\,\sigma_i^-(q^2) = \sigma_i^-(q^2)\,a^{(n)} = 0\,. \tag{2.36}$$

Indeed, the spectral decomposition of $\sigma_i^-(q^2)$ contains (with nonzero coefficients) only two idempotents, $s^{(2)\uparrow i-1}$ and $c^{(2)\uparrow i-1}$:

$$\sigma_i^-(q^2) = \, q\,2_q\,(s^{(2)\uparrow i-1} + \frac{1+q\mu}{q^3+\mu}\,c^{(2)\uparrow i-1})\,.$$

To avoid a singularity in the expression for $\sigma_i^-(q^2)$, we have to assume additionally $\mu \neq -q^3$ for the rest of the proof. However, the expressions entering the relations (2.30) and (2.31) are well defined and continuous at the point $\mu = -q^3$ (unless $-q^3$ coincides with one of the forbidden by Eq. (2.26) values of μ), so the validity of the relations (2.30) and (2.31) at the point $\mu = -q^3$ follows by the continuity.

Notice that the equalities $a^{(n)}\sigma_i = -q^{-1}a^{(n)}$ are equivalent to the equalities $\sigma_i a^{(n)} = -q^{-1}a^{(n)}$ due to the antiautomorphism (2.19) since $\varsigma(a^{(n)}) = a^{(n)}$ by construction.

We now prove the equalities (2.30) and (2.31) by induction on n.

For $n = 2$, $\ a^{(2)}\sigma_1 = -q^{-1}a^{(2)}$, by (2.27) and (2.6).

Let us check the equalities for some fixed $n > 2$ assuming that they are valid for all smaller values of n. Notice that as a byproduct of the definition (2.24) (the first equality) and the induction assumption, the relations (2.36) and (2.31) are satisfied, respectively, for all $i = 1, 2, \dots, n-2$ and for all $m, i :\ \ m+i \leq n-1$. It remains to check the relation (2.36) for $i = n-1$ and the relation (2.31) for $m = n-i$. Respectively, we calculate

$$\begin{aligned} a^{(n)}\,\sigma_{n-1}^-(q^2) &\sim a^{(n-1)}\sigma_{n-1}^-(q^{-2n+2})a^{(n-1)}\,\sigma_{n-1}^-(q^2) \\ &\sim (a^{(n-1)}a^{(n-2)})\,\sigma_{n-1}^-(q^{-2n+2})\sigma_{n-2}^-(q^{-2n+4})\sigma_{n-1}^-(q^2)\,a^{(n-2)} \\ &= (a^{(n-1)}\sigma_{n-2}^-(q^2))\,\sigma_{n-1}^-(q^{-2n+4})\sigma_{n-2}^-(q^{-2n+2})\,a^{(n-2)} = 0\,, \end{aligned}$$

('$\sim$' means 'proportional') and

$$\begin{aligned} a^{(n)}\,a^{(n-i)\uparrow i} &= \frac{q^{n-i-1}}{(n-i)_q}\,(a^{(n)}a^{(n-i-1)\uparrow i})\,\sigma_{n-1}^-(q^{-2(n-i-1)})\,a^{(n-i-1)\uparrow i} \\ &= \frac{q^{n-i-1}}{(n-i)_q}\,a^{(n)}\,(1+q^{i-n}(n-i-1)_q)\,a^{(n-i-1)\uparrow i} = a^{(n)}\,. \end{aligned}$$

Here in both cases, the definition of antisymmetrizers (2.24) (the first equality), induction assumption and relation (2.20) were used. The centrality and primitivity of the idempotents $a^{(n)} \in \mathcal{W}_n(q,\mu)$ follow then from the relations (2.30).

To prove equivalence of the two expressions for the antisymmetrizers given in the formulas (2.24), notice that under conjugation by $\tau^{(i+1)}$ (2.14) the first expression in the formulas (2.24) gets transformed into the second one. However, the elements $a^{(i+1)}$ are central in $\mathcal{W}_{i+1}$, so they do not change under the conjugation which proves the consistency of the equalities (2.24).

All the assertions concerning the symmetrizers follow from the relations for the antisymmetrizers by an application of the map ι (2.16)

$$\iota(a^{(n)}) = s^{(n)}, \qquad \iota(s^{(n)}) = a^{(n)}, \qquad \iota(c^{(2n)}) = c^{(2n)}\,.$$

the latter formulas are direct consequences of the definitions.

The orthogonality of the antisymmetrizers and the symmetrizers is a byproduct of the relations (2.30):

$$-q^{-1}a^{(n)}s^{(m)} = (a^{(n)}\sigma_1)s^{(m)} = a^{(n)}(\sigma_1 s^{(m)}) = q a^{(n)}s^{(m)}\,.$$

The equalities (2.33) can be proved by induction on n. They are obvious in the case $n = 1$. Let us check them for some fixed $n \geq 2$, assuming they are valid for all smaller values of n. Notice that the iterative definition (2.29) together with the induction assumption approve the relations (2.33) for all values of index i, except $i = n$. Checking the case $i = n$ splits in two subcases: $n = 2$ and $n > 2$. In the subcase $i = n = 2$, we have $c^{(4)} = \eta^{-2}\kappa_2\kappa_3\kappa_1\kappa_2$ and

$$\big(c^{(4)}\big)^2 = \eta^{-4}\kappa_2\kappa_3\kappa_1\kappa_2^2\kappa_3\kappa_1\kappa_2 = \eta^{-3}\kappa_2\kappa_3(\kappa_1\kappa_2\kappa_1)\kappa_3\kappa_2 = \eta^{-3}\kappa_2\kappa_3\kappa_1\kappa_3\kappa_2 = \eta^{-2}\kappa_2\kappa_3\kappa_1\kappa_2 = c^{(4)}\,,$$

while in the subcase $i = n > 2$, the calculation is carried out as follows

$$\begin{aligned} \big(c^{(2n)}\big)^2 &= c^{(2n-2)\uparrow 1}\kappa_1\kappa_{2n-1}c^{(2n-2)\uparrow 1}\kappa_1\kappa_{2n-1}c^{(2n-2)\uparrow 1} \\ &= \big(c^{(2n-2)\uparrow 1}c^{(2n-4)\uparrow 2}\big)\,(\kappa_1\kappa_2\kappa_1)(\kappa_{2n-1}\kappa_{2n-2}\kappa_{2n-1})\,\big(c^{(2n-4)\uparrow 2}c^{(2n-2)\uparrow 1}\big) \\ &= c^{(2n-2)\uparrow 1}\kappa_1\kappa_{2n-1}c^{(2n-2)\uparrow 1} = c^{(2n)}\,. \end{aligned}$$

Here in both calculations we used the definition (2.29), the induction assumption and the relations (2.9) and (2.10).

Taking into account the relations (2.33), one can derive an alternative expression for the contractors

$$
\begin{aligned}
c^{(2i)} &= c^{(2i-2)\uparrow 1}\kappa_1\kappa_{2i-1}c^{(2i-2)\uparrow 1} = c^{(2i-2)\uparrow 1}\kappa_1\kappa_{2i-1}c^{(2i-4)\uparrow 2}\kappa_2\kappa_{2i-2}c^{(2i-4)\uparrow 2} \\
&= (c^{(2i-2)\uparrow 1}c^{(2i-4)\uparrow 2})\kappa_1\kappa_{2i-1}\kappa_2\kappa_{2i-2}c^{(2i-4)\uparrow 2} = c^{(2i-2)\uparrow 1}\kappa_{2i-1}\kappa_{2i-2}\kappa_1\kappa_2 c^{(2i-4)\uparrow 2} \\
&= \dots = c^{(2i-2)\uparrow 1}\,(\kappa_{2i-1}\kappa_{2i-2}\dots\kappa_{i+1})\,(\kappa_1\kappa_2\dots\kappa_{i-1})\,c^{(2)\uparrow i-1} \\
&= \eta^{-1}c^{(2i-2)\uparrow 1}\,(\kappa_{2i-1}\kappa_{2i-2}\dots\kappa_{i+1})\,(\kappa_1\kappa_2\dots\kappa_i)\,.
\end{aligned}
\tag{2.37}
$$

Now, using this expression and noticing that, by the relations (2.8),

$$\kappa_{i+1}\kappa_{i-1}\kappa_i\sigma_{i-1} = \kappa_{i+1}\kappa_{i-1}\sigma_i^{-1} = \kappa_{i-1}\kappa_{i+1}\sigma_i^{-1} = \kappa_{i+1}\kappa_{i-1}\kappa_i\sigma_{i+1}\ ,$$

we conclude that the equality (2.34) is satisfied for $i = n-1$. In particular, the relations (2.34) hold for $n = 2$ and $i = 1$. It is enough (by induction on n) to prove the relations (2.34) for $i = 1$. Then observe, again by the relation (2.8), that

$$\kappa_i\kappa_{i\pm1}\kappa_{i\pm2}\,\sigma_i = \kappa_i\kappa_{i\pm1}\sigma_i\,\kappa_{i\pm2} = \kappa_i\sigma_{i\pm1}^{-1}\kappa_{i\pm2} = \sigma_{i\pm2}\,\kappa_i\kappa_{i\pm1}\kappa_{i\pm2}\ .$$

Now, for $n > 2$,

$$
\begin{aligned}
c^{(2n)}\sigma_1 &= \eta^{-1}c^{(2n-2)\uparrow 1}\,(\kappa_{2n-1}\kappa_{2n-2}\dots\kappa_{n+1})\,(\kappa_1\kappa_2\dots\kappa_n)\,\sigma_1 \\
&= \eta^{-1}c^{(2n-2)\uparrow 1}\,(\kappa_{2n-1}\kappa_{2n-2}\dots\kappa_{n+1})\,\sigma_3\,(\kappa_1\kappa_2\dots\kappa_n) \\
&= \eta^{-1}c^{(2n-2)\uparrow 1}\sigma_3\,(\kappa_{2n-1}\kappa_{2n-2}\dots\kappa_{n+1})\,(\kappa_1\kappa_2\dots\kappa_n) \\
&= \eta^{-1}c^{(2n-2)\uparrow 1}\sigma_{2n-3}\,(\kappa_{2n-1}\kappa_{2n-2}\dots\kappa_{n+1})\,(\kappa_1\kappa_2\dots\kappa_n) \\
&= \eta^{-1}c^{(2n-2)\uparrow 1}\,(\kappa_{2n-1}\kappa_{2n-2}\dots\kappa_{n+1})\,\sigma_{2n-1}\,(\kappa_1\kappa_2\dots\kappa_n) = c^{(2n)}\sigma_{2n-1}\ .
\end{aligned}
$$

The relation (2.35) follows from the property (2.4) and the expression (2.37) (with $i = n$) for the contractor. Then, orthogonality of the contractors $c^{(2n)}$ with the antisymmetrizers and the symmetrizers $a^{(m)}$, $s^{(m)}$, $m > n$ is a corollary of the relations (2.30) and (2.35).

A statement of the primitivity of the idempotent $c^{(2n)} \in \mathcal{W}_i(q,\mu)$, $i = 2n, 2n+1$, goes beyond the needs of the present paper, we mention it for a sake of completeness and postpone a purely algebraic proof till Appendix A. ∎

Since the family of higher contractors does not appear to have been previously discussed in the literature, we include Appendix B, which contains their additional properties.

3. R-matrices

Let V denote a finite dimensional $\mathbb{C}$-linear space, $\dim V = \mathrm{N}$. Fixing some basis $\{v_i\}_{i=1}^{\mathrm{N}}$ in V we identify elements $X \in \mathrm{End}(V^{\otimes n})$ with matrices $X_{i_1 i_2 \dots i_n}^{j_1 j_2 \dots j_n}$.

In this section we investigate properties of certain elements in $\mathrm{Aut}(V^{\otimes 2})$ generating representations of the braid groups $\mathcal{B}_n$ or, more specifically, of the Birman–Murakami–Wenzl algebras $\mathcal{W}_n(q,\mu)$ on the spaces $V^{\otimes n}$. Traditionally such operators are called R-matrices.

R-matrices and compatible pairs of R-matrices are introduced in Section 3.1. We also discuss there the notions of the skew-invertibility and the R-trace. Some basic technique, useful in the work with the R-matrices, is presented in Section 3.2.

A twist operation which associates a new R-matrix to a compatible pair of R-matrices, is discussed in Section 3.3. We derive there an alternative expression for the twisted R-matrix and study its skew-invertibility.

Starting from Section 3.4, we concentrate on the R-matrices of the BMW type. In Sections 3.5, 3.6 important ingredients appear: a matrix G and the linear maps ϕ and ξ. As it will be explained in Section 4, the matrix G is responsible for the commutation relation of the quantum matrix with a special element, called 2-contraction, of the quantum matrix algebra. The two maps ϕ and ξ, in turn, are necessary for the definition of the $\star$-product of the BMW type quantum matrices, which is a proper generalization of the usual matrix multiplication to the case of matrices with noncommuting entries.

3.1. Definition and notation

Let $X \in \mathrm{End}(V^{\otimes 2})$. For any $n = 2, 3, \dots$ and $1 \le m \le n-1$, denote by X_m an operator whose action on the space $V^{\otimes n}$ is given by the matrix

$$(X_m)_{i_1\dots i_n}^{j_1\dots j_n} := I_{i_1\dots i_{m-1}}^{j_1\dots j_{m-1}}\ X_{i_m i_{m+1}}^{j_m j_{m+1}}\ I_{i_{m+2}\dots i_n}^{j_{m+2}\dots j_n}\ .$$

Here I denotes the identity operator. In some formulas below (see, for instance, Eqs. (3.1)) we will also use a notation $X_{mr} \in \mathrm{End}(V^{\otimes n})$, $1 \le m < r \le n-1$, referring to an operator given by a matrix

$$(X_{mr})_{i_1\dots i_n}^{j_1\dots j_n} := X_{i_m i_r}^{j_m j_r}\ I_{i_1\dots i_{m-1} i_{m+1}\dots i_{r-1} i_{r+1}\dots i_n}^{j_1\dots j_{m-1} j_{m+1}\dots j_{r-1} j_{r+1}\dots j_n}\ .$$

Clearly, $X_m = X_{m\,m+1}$.

We reserve the symbol P for the permutation operator: $P(u\otimes v)=v\otimes u\quad \forall\, u,v\in V$. Below we repeatedly make use of relations

$$P^2=I\,;\quad P_{12}X_{12}=X_{21}P_{12}\quad \forall\;\, X\in \mathrm{End}(V\otimes V)\,;\quad \mathrm{Tr}_{\,(1)}P_{12}=\mathrm{Tr}_{\,(3)}P_{23}=I_2\,,$$

where the symbol $\mathrm{Tr}_{(i)}$ stands for the trace over an ith component space in the tensor power of the space V.

An operator $X\in \mathrm{End}(V^{\otimes 2})$ is called *skew invertible* if there exists an operator $\Psi_X\in \mathrm{End}(V^{\otimes 2})$ such that

$$\mathrm{Tr}_{\,(2)}X_{12}\Psi_{X\,23}=\mathrm{Tr}_{\,(2)}\Psi_{X\,12}X_{23}=P_{13}\,. \tag{3.1}$$

Define two elements of $\mathrm{End}(V)$

$$C_X:=\mathrm{Tr}_{(1)}\Psi_{X\,12}\,,\qquad D_X:=\mathrm{Tr}_{(2)}\Psi_{X\,12}\,. \tag{3.2}$$

By (3.1),

$$\mathrm{Tr}_{\,(1)}C_{X\,1}X_{12}=I_2\,,\qquad \mathrm{Tr}_{\,(2)}D_{X\,2}X_{12}=I_1\,. \tag{3.3}$$

A skew invertible operator X is called *strict skew invertible* if one of the matrices, C_X or D_X, is invertible (by Lemma 3.5 below, if one of the matrices, C_X or D_X, is invertible then they are both invertible).

An equation

$$R_1\,R_2\,R_1\;=\;R_2\,R_1\,R_2\;.$$

for an element $R\in \mathrm{Aut}(V^{\otimes 2})$ is called the *Yang–Baxter equation.*

An element $R\in \mathrm{Aut}(V^{\otimes 2})$ that fulfills the Yang–Baxter equation is called an *R-matrix.*

All R-matrices in this text are assumed to be invertible.

Clearly, the permutation operator P is the R-matrix; R^{-1} is the R-matrix iff R is. Any R-matrix R generates representations ρ_R of the series of braid groups $\mathcal{B}_n$, $n=2,3,\dots$

$$\rho_R:\,\mathcal{B}_n\to \mathrm{Aut}(V^{\otimes n})\;,\quad \sigma_i\mapsto \rho_R(\sigma_i)=R_i,\quad 1\le i\le n-1. \tag{3.4}$$

If additionally the R-matrix R satisfies a third order *minimal* characteristic polynomial (c.f. with the relation (2.6))

$$(qI-R)(q^{-1}I+R)(\mu I-R)=0\;, \tag{3.5}$$

and an element

$$K:=\mu^{-1}(q-q^{-1})^{-1}\,(qI-R)(q^{-1}I+R) \tag{3.6}$$

fulfills conditions

$$K_2\,K_1\;=\;R_1^{\pm 1}\,R_2^{\pm 1}\,K_1 \tag{3.7}$$

and

$$K_1\,K_2\,K_1\;=\;K_1\,, \tag{3.8}$$

then we call R an R-matrix of a *BMW type* (c.f. with Eqs. (2.3)–(2.10); we make a different but equivalent choice of defining relations).

For an R-matrix of the BMWtype, the formulas (3.4) define representations of the algebras $\mathcal{W}_n(q,\mu)\to \mathrm{End}(V^{\otimes n})$, $n=2,3,\dots$. In particular, $\rho_R(\kappa_i)=K_i$.

An ordered pair $\{R,F\}$ of two operators R and F from $\mathrm{End}(V^{\otimes 2})$ is called *a compatible pair* if conditions

$$R_1\,F_2\,F_1\;=\;F_2\,F_1\,R_2\,,\qquad R_2\,F_1\,F_2\;=\;F_1\,F_2\,R_1\,, \tag{3.9}$$

are satisfied. If, in addition, R and F are R-matrices, the pair $\{R,F\}$ is called a compatible pair of R-matrices. The equalities (3.9) are called *twist relations* (on the notion of the twist see [8,23,45]). Clearly, $\{R,P\}$ and $\{R,R\}$ are compatible pairs of R-matrices; pairs $\{R^{-1},F\}$ and $\{R,F^{-1}\}$ are compatible iff the pair $\{R,F\}$ is.

Definition 3.1. Consider a space of $\mathrm{N}\times\mathrm{N}$ matrices $\mathrm{Mat}_{\mathrm{N}}(W)$, whose entries belong to some $\mathbb{C}$-linear space W. Let R be a skew invertible R-matrix. A linear map

$$\mathrm{Tr}_R:\;\mathrm{Mat}_{\mathrm{N}}(W)\;\to\;W,\qquad \mathrm{Tr}_R(M)=\sum_{i,j=1}^{\mathrm{N}}(D_R)_i^jM_j^i\,,\qquad M\in \mathrm{Mat}_{\mathrm{N}}(W),$$

is called an R-trace.

The relation (3.3) in this notation reads

$$\mathrm{Tr}_{R(2)}R_{12}=I_1\,. \tag{3.10}$$

3.2. R-technique

In this and the next subsections we develop a technique for dealing with the R-matrices, their compatible pairs and the R-trace. Most of results reported here, like Lemma 3.5 and, in a particular case of a compatible pair $\{R, R\}$ – Lemmas 3.2 and 3.3 and Corollary 3.4 – are rather well known (see, e.g., [20,41]). However, we often use them in a more general setting and so, when necessary, we present sketches of proofs.

Proposition 3.6 contains new results. Here we derive an expression, different from the standard one, for the twisted R-matrix, which helps to investigate its skew-invertibility.

A universal (i.e., quasi-triangular Hopf algebraic) content of the matrix relations derived in this and the next subsections is discussed in the Appendix C.

Lemma 3.2. *Let $\{X, F\}$ be a compatible pair, where X is skew invertible. Let $\mathrm{Mat}_N(W)$ be as in Definition 3.1. For any $M \in \mathrm{Mat}_N(W)$, one has*

$$\mathrm{Tr}_{(1)}\left(C_{X\,1}F_{12}^{\varepsilon}\, M_2\, F_{12}^{-\varepsilon}\right) = I_2\,\mathrm{Tr}\,(C_X M)\,, \tag{3.11}$$

$$\mathrm{Tr}_{(2)}\left(D_{X\,2}F_{12}^{-\varepsilon}\, M_1\, F_{12}^{\varepsilon}\right) = I_1\,\mathrm{Tr}\,(D_X M) \tag{3.12}$$

for $\varepsilon = \pm 1$.

Proof. We use the twist relations (3.9) in a form

$$F_{23}^{\varepsilon}\, X_{34}\, F_{23}^{-\varepsilon} = F_{34}^{-\varepsilon}\, X_{23}\, F_{34}^{\varepsilon}\,, \quad \varepsilon = \pm 1\,.$$

Multiplying it by $(\Psi_{X\,12}\Psi_{X\,45})$ and taking the traces in the spaces 2 and 4, we get

$$\mathrm{Tr}_{(2)}(\Psi_{X\,12}\, F_{23}^{\varepsilon}\, P_{35}\, F_{23}^{-\varepsilon}) = \mathrm{Tr}_{(4)}(\Psi_{X\,45}\, F_{34}^{-\varepsilon}\, P_{13}\, F_{34}^{\varepsilon})\,. \tag{3.13}$$

Here the relation (3.1), defining the operator Ψ_X, was applied to calculate the traces. Now taking the trace in the space number 1 or number 5, we obtain (after relabeling)

$$\mathrm{Tr}_{(1)}(C_{X\,1}\, F_{12}^{\varepsilon}\, P_{23}\, F_{12}^{-\varepsilon}) = C_{X\,3}\, I_2\,, \tag{3.14}$$

$$\mathrm{Tr}_{(3)}(D_{X\,3}\, F_{23}^{-\varepsilon}\, P_{12}\, F_{23}^{\varepsilon}) = D_{X\,1}\, I_2\,. \tag{3.15}$$

These two relations are equivalent forms of the relations (3.11) and (3.12). For example, the formula (3.11) is obtained by multiplying the relation (3.14) by the operator M_3 and taking the trace in the space 3. ■

Lemma 3.3. *Let $\{X, F\}$ be a compatible pair of skew invertible operators X and F. Then the following relations*

$$C_{X\,1}\,\Psi_{F\,12} = F_{21}^{-1}\, C_{X\,2}\,, \quad \Psi_{F\,12}\, C_{X\,1} = C_{X\,2}\, F_{21}^{-1}\,, \tag{3.16}$$

$$\Psi_{F\,12}\, D_{X\,2} = D_{X\,1}\, F_{21}^{-1}\,, \qquad D_{X\,2}\,\Psi_{F\,12} = F_{21}^{-1}\, D_{X\,1} \tag{3.17}$$

hold.

Proof. For a skew invertible operator F, the relations (3.16) and (3.17) are equivalent to the relations (3.14) and (3.15). Let us demonstrate how the left one of the relations (3.16) is derived from the relation (3.14) with $\varepsilon = 1$.

Multiply the relation (3.14) by a combination $(P_{23}\Psi_{F\,24})$ from the right, take the trace in the space 2 and simplify the result using the relation (3.1) for $X = F$ and the properties of the permutation

$$\mathrm{Tr}_{(1)}(C_{X\,1}\, P_{14}\, F_{13}^{-1}) = C_{X\,3}\,\mathrm{Tr}_{(2)}(P_{23}\,\Psi_{F\,24}) = C_{X\,3}\,\Psi_{F\,34}\,.$$

Then simplify the left hand side of the equality using the cyclic property of the trace

$$\mathrm{Tr}_{(1)}(C_{X\,1}\, P_{14}\, F_{13}^{-1}) = \mathrm{Tr}_{(1)}(P_{14}\, F_{13}^{-1}\, C_{X\,1}) = F_{43}^{-1}\, C_{X\,4}\,\mathrm{Tr}_{(1)} P_{14} = F_{43}^{-1}\, C_{X\,4}\,.$$

This proves the left relation in (3.16). ■

Corollary 3.4. *Let $\{X, F\}$ and $\{Y, F\}$ be compatible pairs of skew invertible operators X, Y and F. Then the following relations*

$$F_{12}\, C_{X\,1} C_{Y\,2} = C_{Y\,1} C_{X\,2} F_{12}\,, \qquad F_{12}\, D_{X\,1} D_{Y\,2} = D_{Y\,1} D_{X\,2} F_{12}\,, \tag{3.18}$$

$$F_{12}\,(C_X D_Y)_2 = (C_X D_Y)_1\, F_{12}\,, \qquad F_{12}\,(D_Y C_X)_1 = (D_Y C_X)_2\, F_{12}\,, \tag{3.19}$$

$$\mathrm{Tr}_{(1)}(C_{X\,1} F_{12}^{-1}) = (C_X D_F)_2 = (D_F C_X)_2\,, \tag{3.20}$$

$$\mathrm{Tr}_{(2)}(D_{X\,2} F_{12}^{-1}) = (C_F D_X)_1 = (D_X C_F)_1 \tag{3.21}$$

hold.

Proof. A calculation $(F_{12}^{-1}C_{Y\,1})C_{X\,2} = C_{Y\,2}(\Psi_{F\,21}C_{X\,2}) = C_{Y\,2}C_{X\,1}F_{12}^{-1} = C_{X\,1}C_{Y\,2}F_{12}^{-1}$ proves the left one of the relations (3.18). Here the relations (3.16) were applied.

A calculation $(F_{12}^{-1}C_{X\,1})D_{Y\,1} = C_{X\,2}(\Psi^{F}_{21}D_{Y\,1}) = C_{X\,2}D_{Y\,2}F_{12}^{-1}$ proves the left one of the relations (3.19). Here one uses subsequently the left equations from (3.16) and (3.17).

The relations (3.20) follow by taking $\mathrm{Tr}_{(2)}$ of the equations (3.16).

The rest of the relations in (3.18)–(3.21) are derived in a similar way. ■

Lemma 3.5. *Let X be a skew invertible R-matrix. Then statements*
(a) the R-matrix X^{-1} is skew invertible;
(b) the R-matrix X is strict skew invertible,
are equivalent.

Provided these statements are satisfied, both C_X and D_X are invertible and one has

$$C_{X^{-1}} = D_X^{-1}\,, \qquad D_{X^{-1}} = C_X^{-1}\,. \tag{3.22}$$

Proof. See [41], section 4.1, statements after eq. (4.1.77), or [20], proposition 2 in section 3.1. ■

Under an assumption of an existence, for an R-matrix X, of the operators X^{-1}, Ψ_X and $\Psi_{X^{-1}}$, the relations (3.22) were proved in [46].

Since, for a compatible pair $\{X, F\}$, the pair $\{X, F^{-1}\}$ is also compatible, the formulas (3.22) together with the relations (3.20)–(3.21) imply that $C_X C_F = C_F C_X$ and $D_X D_F = D_F D_X$.

3.3. Twists

Let $\{R, F\}$ be a compatible pair of R-matrices. Define a *twisted* operator

$$R_f := F^{-1}RF\,. \tag{3.23}$$

It is well known that R_f is an R-matrix and the pair $\{R_f, F\}$ is compatible. Therefore, one can twist again; in [24] it was shown that if F is skew invertible then

$$D_{F\,1}\,D_{F\,2}\,((R_f)_f)_{12} \;=\; R_{12}\,D_{F\,1}\,D_{F\,2} \quad \text{and} \quad C_{F\,1}\,C_{F\,2}\,((R_f)_f)_{12} \;=R_{12}\,C_{F\,1}\,C_{F\,2}\,. \tag{3.24}$$

A comparison of two equalities in Eq. (3.24) shows that

$$[\,R_{12}\,,\,(C_F^{-1}\,D_F)_1\,(C_F^{-1}\,D_F)_2\,] = 0\,. \tag{3.25}$$

Proposition 3.6. *Let $\{R, F\}$ be a compatible pair of R-matrices. The following statements hold:*

(a) *if F is strict skew invertible then the twisted R-matrix R_f, defined by the formula (3.23), can be expressed in a form*

$$R_{f\,12} \;=\; \mathrm{Tr}_{(34)}\left(F_{32}^{-1}C_{F^{-1}\,3}R_{34}D_{F\,4}F_{14}\right)\,; \tag{3.26}$$

(b) *if R is skew invertible and F is strict skew invertible then R_f is skew invertible; its skew inverse is*

$$\Psi_{R_f\,12} \;=\; C_{F^{-1}\,2}\,\mathrm{Tr}_{(34)}\left(F_{23}^{-1}\Psi_{R\,34}F_{41}\right)D_{F\,1}\,; \tag{3.27}$$

moreover, Ψ_{R_f} can be expressed in a form

$$\Psi_{R_f\,12} \;=\; C_{F^{-1}\,2}F_{21}D_{F^{-1}\,2}\Psi_{R\,12}C_{F\,1}F_{21}^{-1}D_{F\,1}\,; \tag{3.28}$$

(c) *under the conditions in* (b),

$$C_{R_f} = C_{F^{-1}}D_R\,C_F\,, \quad D_{R_f} = D_{F^{-1}}C_R D_F \tag{3.29}$$

(thus, if, in addition to the conditions in (b)*, R is strict skew invertible then R_f is strict skew invertible as well).*

Proof. To verify the assertion (a) we calculate

$$\begin{aligned} R_{f\,12} \;&=\; (F^{-1}RF)_{12} = F_{12}^{-1}\left(\mathrm{Tr}_{(4)}F_{41}^{-1}C_{F^{-1}\,4}\right)(RF)_{12} \\ &=\; \mathrm{Tr}_{(4)}\left((RF)_{41}F_{12}^{-1}F_{41}^{-1}C_{F^{-1}\,4}\right) = \left(\mathrm{Tr}_{(3)}P_{13}\right)\mathrm{Tr}_{(4)}\left((RF)_{41}F_{12}^{-1}C_{F^{-1}\,1}\Psi_{F\,14}\right) \\ &=\; \mathrm{Tr}_{(34)}\left((RF)_{43}F_{32}^{-1}C_{F^{-1}\,3}P_{13}\Psi_{F\,14}\right) = \mathrm{Tr}_{(3)}\left(F_{32}^{-1}C_{F^{-1}\,3}\underline{P_{13}\mathrm{Tr}_{(4)}\Psi_{F\,14}(RF)_{43}}\right)\,, \end{aligned} \tag{3.30}$$

where in the second equality we used the relation (3.3) for $X = F^{-1}$; in the third equality we applied the twist relations for the compatible pairs $\{R, F\}$ and $\{F, F\}$; in the fourth equality we applied the relations (3.16) for $X = F^{-1}$ and inserted the identity operator $\mathrm{Tr}_{(3)}P_{13}$; in the fifth equality we permuted the operator P_{13} rightwards and then, in the sixth equality, used the cyclic property of the trace to move the combination $(RF)_{43}$ to the right.

To complete the transformation, we derive an alternative form for the underlined expression in the last line in Eq. (3.30). Multiplying the twist relation $R_2F_3F_2 = F_3F_2R_3$ by a combination $(\Psi_{F\,12}D_{F\,4})$ and taking the traces in the spaces 2 and 4, we obtain (using the formulas (3.1) and (3.3) for $X = F$)

$$\mathrm{Tr}_{(2)}\,(\Psi_{F\,12}(RF)_{23}) \;=\; \mathrm{Tr}_{(4)}\,(D_{F\,4}F_{34}P_{13}R_{34})\ ,$$

which is equivalent (multiply by P_{13} from the left and use the cyclic property of the trace) to

$$P_{13}\mathrm{Tr}_{(2)}\,(\Psi_{F\,12}(RF)_{23}) \;=\; \mathrm{Tr}_{(4)}\,(R_{34}D_{F\,4}F_{14})\ . \tag{3.31}$$

Now, substituting the equality (3.31) into the last line of the calculation (3.30), we finish the transformation and obtain the formula (3.26).

Given the formula for R_f, the calculation of Ψ_{R_f} becomes straightforward and one finds the formula (3.27).

Thus, the skew invertibility of R_f is established.

Now we derive the expression (3.28) for Ψ_{R_f}. Multiplying the equality (3.13) with $\varepsilon = 1$ by a combination $P_{35}D_{F^{-1}5}$ from the right and taking the trace in the space 5, we obtain

$$\begin{aligned}\mathrm{Tr}_{(2)}(\Psi_{R\,12}F_{23}) &= \mathrm{Tr}_{(45)}(\Psi_{R\,45}F_{34}^{-1}P_{13}F_{34}P_{35}D_{F^{-1}5})\\ &= \mathrm{Tr}_{(4)}(F_{34}^{-1}P_{13}F_{34}D_{F^{-1}3}\Psi_{R\,43})\ .\end{aligned}$$

Substituting this into the expression (3.27), we find

$$\begin{aligned}\Psi_{R_f\,12} &= C_{F^{-1}2}\,\mathrm{Tr}_{(34)}\left(F_{23}^{-1}F_{14}^{-1}P_{13}F_{14}D_{F^{-1}1}\Psi_{R\,41}\right)D_{F\,1}\\ &= C_{F^{-1}2}\,\mathrm{Tr}_{(4)}\left(F_{14}^{-1}\mathrm{Tr}_{(3)}(F_{23}^{-1}P_{13})F_{14}D_{F^{-1}1}\Psi_{R\,41}\right)D_{F\,1}\\ &= C_{F^{-1}2}\,\mathrm{Tr}_{(4)}\left(F_{14}^{-1}F_{21}^{-1}F_{14}D_{F^{-1}1}\Psi_{R\,41}\right)D_{F\,1}\\ &= C_{F^{-1}2}\,F_{21}\,\mathrm{Tr}_{(4)}\left(F_{14}^{-1}F_{21}^{-1}D_{F^{-1}1}\Psi_{R\,41}\right)D_{F\,1}\\ &= C_{F^{-1}2}\,F_{21}\,D_{F^{-1}2}\mathrm{Tr}_{(4)}\left(F_{14}^{-1}\Psi_{F\,12}\Psi_{R\,41}\right)D_{F\,1}\ .\end{aligned} \tag{3.32}$$

We used the Yang–Baxter equation for the operator F in the fourth equality and the relations (3.16) in the fifth equality.

Multiplying Eq. (3.13) with $\varepsilon = -1$ by $\Psi_{F\,01}P_{13}$ from the left and by $P_{35}\Psi_{F\,56}$ from the right and taking the traces in the spaces 1 and 5, we find

$$F_{14}^{-1}\Psi_{F\,12}\Psi_{R\,41} = \Psi_{R\,12}\Psi_{F\,41}F_{21}^{-1}\ .$$

Substituting this into the last line of the calculation (3.32), we obtain the equality (3.28).

Finally, the expressions (3.29) for the operators C_{R_f} and D_{R_f} are obtained by taking the trace in the space 1 or the space 2 of the expression (3.27) for the skew inverse of the twisted R-matrix and the subsequent use of the relations (3.3) for $X = F^{\pm 1}$ and the relations (3.2), (3.20) and (3.21) for $X = R$. ■

Remark 3.7. If one uses the expression (3.26) for the twisted R-matrix then the relation (3.24) becomes straightforward:

$$\begin{aligned}((R_f)_f)_{12} &= \mathrm{Tr}_{(3456)}\left(F_{32}^{-1}(D_F^{-1})_3F_{54}^{-1}(D_F^{-1})_5R_{56}D_{F\,6}F_{36}D_{F\,4}F_{14}\right)\\ &= \mathrm{Tr}_{(3456)}\left(\underline{\underline{F_{54}^{-1}D_{F\,4}F_{14}}}(D_F^{-1})_5R_{56}D_{F\,6}\underline{F_{32}^{-1}(D_F^{-1})_3F_{36}}\right)\\ &= \mathrm{Tr}_{(56)}\left(P_{15}D_{F\,1}(D_F^{-1})_5R_{56}D_{F\,6}(D_F^{-1})_2P_{26}\right)\\ &= (D_F^{-1})_2\mathrm{Tr}_{(56)}\left(P_{15}(D_F^{-1})_5R_{56}D_{F\,6}P_{26}\right)D_{F\,1}\\ &= (D_F^{-1})_1(D_F^{-1})_2R_{12}D_{F\,1}D_{F\,2}\ .\end{aligned}$$

In the first equality we applied the formula (3.26) twice and replaced the operators $C_{F^{-1}}$ by D_F^{-1} by the relation (3.22); in the second equality we collected together the terms involving the space number 3 (they are underlined) and the terms involving the space number 4 (they are underlined twice); in the third equality we evaluated the traces in the spaces 3 and 4 using the relations from Lemma 3.3; in the fourth equality we moved the operator $(D_F^{-1})_2$ leftwards out of the trace and the operator $D_{F\,1}$ rightwards out of the trace; in the fifth equality we transported the operator P_{15} rightwards and the operator P_{26} leftwards under the trace and then evaluated the remaining traces in the spaces 5 and 6.

3.4. BMW type R-matrices

In this subsection we discuss the R-matrices of the BMW type in more detail.

In Lemma 3.8 we collect additional relations specific to the BMW type R-matrices. Based on these formulas, we will introduce later, in Sections 3.5 and 3.6, an invertible operator $G \in \mathrm{Aut}(V)$ and linear maps ϕ and ξ, which will be used in Section 4 for a definition of a product of quantum matrices and for a quantum matrix inversion.

Lemma 3.8. *Let R be a skew invertible R-matrix of the BMW type. Then*

- *the operator R is strict skew invertible;*
- *the rank of the operator K equals one,* $\operatorname{rk} K = 1$*;*
- *the following relations*

$$\mathrm{Tr}_{(2)} K_{12} = \mu^{-1} D_{R1} \ , \ \ \mathrm{Tr}_{(1)} K_{12} = \mu^{-1} C_{R2} \ , \tag{3.33}$$

$$\mathrm{Tr}_{R(2)} K_{12} = \mu\, I_1 \, , \tag{3.34}$$

$$\mathrm{Tr}_R I = \mu\, \eta \equiv \frac{(q-\mu)(q^{-1}+\mu)}{(q-q^{-1})} \, , \tag{3.35}$$

$$C_R D_R = \mu^2 I \, , \tag{3.36}$$

$$K_{12} D_{R1} D_{R2} = D_{R1} D_{R2} K_{12} \ = \ \mu^2 K_{12} \tag{3.37}$$

hold.

Proof. The proof of all the statements in the lemma but the last one is given in [25].

The last relation (3.37) (which, in another form, figures in [25], in proposition 2) can be established in the following way.

The first equality in (3.37) is a consequence of a relation

$$R_{12} D_{R1} D_{R2} \ = \ D_{R1} D_{R2} R_{12} \, , \tag{3.38}$$

which is just the equality (3.18) written for the pair $\{R, R\}$. Then the conditions $K^2 \sim K$ and $\operatorname{rk} K = 1$ together imply $K_{12} D_{R1} D_{R2} \sim K_{12} D_{R1} D_{R2} K_{12} \sim K_{12}$. A coefficient of proportionality in this relation is recovered by taking the trace of it in the space 2 and the subsequent use of the relations (3.33) and (3.34). ■

In [25], a pair of mutually inverse matrices

$$E_2 := \mathrm{Tr}_{(1)}(K_{12} P_{12}) \ \text{ and } \ E_1^{-1} := \mathrm{Tr}_{(2)}(K_{12} P_{12}) \tag{3.39}$$

was introduced (see eqs. (32) and (33) and proposition 2 in [25]).

We shall now collect several useful identities involving the operators K and E.

Lemma 3.9. (a) *The following relations*

$$K_{12} K_{23} = E_3\, K_{12} P_{23} P_{12} \ , \ K_{23} K_{12} = E_1^{-1}\, K_{23} P_{12} P_{23} \ , \tag{3.40}$$

$$K_{13} K_{23} = \mu^{-1} D_{R2}\, K_{13} P_{12} \ , \ K_{12} K_{13} = \mu^{-1} C_{R3}\, K_{12} P_{23} \ , \tag{3.41}$$

$$K_{23} K_{14} P_{12} P_{34} = K_{23} K_{14} \ , \ K_{23} K_{14} P_{13} P_{24} = K_{23} K_{14} P_{23} P_{14} \ ,$$

$$K_{12} E_1^{-1} = \mu^{-1} K_{12} P_{12} D_{R1} \ , \ E_1 K_{12} = \mu^{-1} D_{R1} P_{12} K_{12}$$

hold.

(b) *We have*

$$K_{12} E_1 E_2 = E_1 E_2 K_{12} = K_{12} \ .$$

(c) *The operator K is skew invertible, its skew inverse is*

$$\Psi_{K\,12} = E_1 K_{12} E_2 = \mu^{-2} D_{R1} K_{21} D_{R1} \ .$$

Proof. (a) All these identities follow from the rank one property of the operator K (written explicitly, with indices, they become evident).

(b) To verify, for instance, that $K_{12} E_1^{-1} E_2^{-1} = K_{12}$, use the definition (3.39) of the matrix E_2^{-1}, $E_2^{-1} = \mathrm{Tr}_{(3)}(K_{23} P_{23})$, and then the relation (3.40) to remove the trace.

(c) This follows from the identities in (a) in the lemma. ■

Remark 3.10. The relations (3.40) admit the following generalizations:

$$K_1 K_2 \dots K_j \ = \ E_3 E_4 \dots E_{j+1} \cdot (P_1 P_2 \dots P_j)^2 K_j \ ,$$

$$K_j \dots K_2 K_1 \ = \ E_1^{-1} E_2^{-1} \dots E_{j-1}^{-1} K_j \cdot (P_j \dots P_2 P_1)^2 \ .$$

The relations (3.41) admit the following generalizations:

$$K_{10} K_{20} \dots K_{j0} \ = \ \mu^{1-j} (D_{R2} D_{R3} \dots D_{Rj}) \cdot (P_1 P_2 \dots P_{j-1})\, K_{j0} \ ,$$

$$K_{01} K_{02} \dots K_{0j} \ = \ \mu^{1-j} (C_{R2} C_{R3} \dots C_{Rj}) \cdot (P_1 P_2 \dots P_{j-1})\, K_{0j} \ .$$

In all four formulas above j is an arbitrary positive integer. These relations can be proved by induction on j.

3.5. Operator G

In the following lemma, we define analogues of the matrices E and E^{-1} for a compatible pair $\{R, F\}$ of R-matrices. When the operator F is the permutation operator, $F = P$, the matrix G of the Definition-Lemma 3.11 coincides with the matrix E.

Definition-Lemma 3.11. *Let $\{R, F\}$ be a compatible pair of R-matrices, where R is skew-invertible of the BMW type and F is strict skew-invertible. Define an element $G \in \mathrm{End}(V)$ by*

$$G_1 \;:=\; \mathrm{Tr}_{(23)} K_2 F_1^{-1} F_2^{-1} . \tag{3.42}$$

The operator G is invertible, the inverse operator reads

$$G_1^{-1} \;=\; \mathrm{Tr}_{(23)} F_2 F_1 K_2 . \tag{3.43}$$

The following relations

$$R_{12} G_1 G_2 \;=\; G_1 G_2 R_{12} , \tag{3.44}$$

$$F_{12}^{\varepsilon} G_1 = G_2 F_{12}^{\varepsilon} \quad \textit{for} \;\; \varepsilon = \pm 1 , \tag{3.45}$$

$$[D_R, G] \;=\; 0 , \tag{3.46}$$

$$[C_F, G] \;=\; [D_F, G] \;=\; 0 , \tag{3.47}$$

$$[E, G] \;=\; 0 ,$$

$$[C_F, E] \;=\; [D_F, E] \;=\; 0$$

are satisfied.

Proof. A check of the invertibility of G is a direct calculation

$$\begin{aligned} G_1\, G_1^{-1} \;&=\; (\mathrm{Tr}_{(23)} K_2 F_1^{-1} F_2^{-1})(\mathrm{Tr}_{(23)} F_2 F_1 K_2) \;=\; \mathrm{Tr}_{(23)} K_2 F_1^{-1} F_2^{-1} K_2 F_2 F_1 \\ &=\; \mathrm{Tr}_{(23)} K_2 F_1^{-1} K_{f\,2} F_1 \;=\; \mathrm{Tr}_{(23)} K_2 F_2 K_{f\,1} F_2^{-1} \;=\; \mathrm{Tr}_{(23)} K_{f\,2} K_{f\,1} \;=\; I . \end{aligned} \tag{3.48}$$

Here in the first line we used the formulas (3.42) and (3.43) and the property $\mathrm{rk}\, K = 1$: if $\Pi = |\zeta\rangle\langle\psi|$ is a rank one projector then $\mathrm{Tr}\,(\Pi A) = \langle\psi|A|\zeta\rangle$ for any operator A and

$$\mathrm{Tr}\,(\Pi A)\,\mathrm{Tr}\,(\Pi B) = \langle\psi|A|\zeta\rangle\,\langle\psi|B|\zeta\rangle = \langle\psi|A\Pi B|\zeta\rangle = \mathrm{Tr}\,(\Pi A \Pi B)$$

for any A and B; in the second line of the calculation (3.48) we passed from K to $K_f = F^{-1} K F$ and applied the twist relations (for the operators K_f and F) and the cyclic property of the trace. In the last equality of (3.48) we evaluated the traces using the relations (3.33) and then the relation (3.34) for the operator K_f (we are allowed to use these relations because the operator R_f is skew-invertible by Proposition 3.6).

Notice that, in view of the relation (3.37), we can rewrite the formula for the operator G using the R-traces instead of the ordinary ones

$$G_1 \;=\; \mu^{-2}\, \mathrm{Tr}_{R(23)} K_2 F_1^{-1} F_2^{-1} . \tag{3.49}$$

Applying the formula (3.12) (written for $F^{\varepsilon} = X = R$) twice to this equality, we begin our next calculation

$$\begin{aligned} G_1 I_2 \;&=\; \mu^{-2}\, \mathrm{Tr}_{R(34)} (R_2 R_3) K_2 F_1^{-1} F_2^{-1} (R_3^{-1} R_2^{-1}) \;=\; \mu^{-2}\, \mathrm{Tr}_{R(34)} K_3 K_2 F_1^{-1} F_2^{-1} R_3^{-1} R_2^{-1} \\ &=\; \mu^{-2}\, \mathrm{Tr}_{R(34)} K_2 F_1^{-1} F_2^{-1} K_2 K_3 \;=\; \mu^{-1}\, \mathrm{Tr}_{R(3)} K_2 F_1^{-1} F_2^{-1} K_2 . \end{aligned} \tag{3.50}$$

Here we used the relation (3.7) in the last equality of the first line. In the second line we again applied the relation (3.7) after moving the operator K_3 to the right (for that we need the relation (3.37) and the cyclicity of the trace) and then we evaluated one R-trace with the help of the relation (3.34).

Now we use the formula (3.50) for the product $G_1 G_2$ in a transformation

$$\begin{aligned} G_1 G_2 R_1 \;&=\; \mu^{-2}\, \mathrm{Tr}_{R(34)} (K_3 F_2^{-1} F_3^{-1} K_3)(K_2 F_1^{-1} F_2^{-1} K_2) R_1 \\ &=\; \mu^{-2}\, \mathrm{Tr}_{R(34)} F_2^{-1} F_3^{-1} K_2 K_3 K_2 F_1^{-1} F_2^{-1} K_2 R_1 \;=\; \mu^{-2}\, \mathrm{Tr}_{R(34)} F_2^{-1} F_3^{-1} F_1^{-1} F_2^{-1} K_1 K_2 R_1 \\ &=\; \mu^{-2}\, \mathrm{Tr}_{R(34)} F_2^{-1} F_3^{-1} F_1^{-1} F_2^{-1} K_1 R_2^{-1} \;=\; \mu^{-2}\, \mathrm{Tr}_{R(34)} K_3 F_2^{-1} F_3^{-1} F_1^{-1} F_2^{-1} R_2^{-1} \\ &=\; \mu^{-2}\, \mathrm{Tr}_{R(34)} F_2^{-1} F_1^{-1} F_3^{-1} F_2^{-1} R_3 K_2 K_3 \;=\; \mu^{-2} R_1\, \mathrm{Tr}_{R(34)} F_2^{-1} F_1^{-1} F_3^{-1} F_2^{-1} K_2 K_3 \\ &=\; \mu^{-2} R_1\, \mathrm{Tr}_{R(34)} K_3 F_2^{-1} F_3^{-1} F_1^{-1} F_2^{-1} K_2 \;=\; R_1 G_1 G_2 , \end{aligned}$$

which demonstrates the relation (3.44). While doing the above calculation, we repeatedly used the twist relations for the pairs $\{K, F^{-1}\}$ and $\{R, F^{-1}\}$, applied the formulas (3.7) and (3.7) and exploited the cyclic property of the trace to move the operator K_3 to the right/left in the fourth/fifth line, respectively.

Due to the expression (3.49) for the operator G, we can write

$$G_1 I_2 \;=\; \mu^{-2}\,\mathrm{Tr}_{R(34)}\left((F_2^{-\varepsilon}F_3^{-\varepsilon})K_2F_1^{-1}F_2^{-1}(F_3^{\varepsilon}F_2^{\varepsilon})\right)$$

by the formula (3.12).

The relation (3.45) is now proved as follows

$$\begin{aligned} G_1F_1^{\varepsilon} \;&=\; \mu^{-2}\,\mathrm{Tr}_{R(34)}\left((F_2^{-\varepsilon}F_3^{-\varepsilon})K_2F_1^{-1}F_2^{-1}(F_3^{\varepsilon}F_2^{\varepsilon})\right)F_1^{\varepsilon} \\ &= \mu^{-2}\,\mathrm{Tr}_{R(34)}(F_2^{-\varepsilon}F_3^{-\varepsilon})F_3^{\varepsilon}F_2^{\varepsilon}F_1^{\varepsilon}K_3F_2^{-1}F_3^{-1} \;=\; F_1^{\varepsilon}\mu^{-2}\,\mathrm{Tr}_{R(34)}K_3F_2^{-1}F_3^{-1} \;=\; F_1^{\varepsilon}G_2\;. \end{aligned} \tag{3.51}$$

Here we subsequently used the twist relations for the pair $\{K, F^{\varepsilon}\}$, the Yang–Baxter equations for F and again the expression (3.49) for the operator G.

Vanishing of the commutators $[C_F, G]$ and $[D_F, G]$ in Eq. (3.47) follow from the above proved equality. To find these commutators, transform Eq. (3.51) to

$$G_1\Psi_{F\,12} \;=\; \Psi_{F\,12}G_2\,, \qquad G_2\Psi_{F\,12} \;=\; \Psi_{F\,12}G_1\,,$$

(multiply the relation (3.51) by a combination $\Psi_{F\,41}\Psi_{F\,23}$ and take $\mathrm{Tr}_{\,(12)}$) and then apply the trace in the space 1 or the space 2 to these relations and compare results.

The relation (3.46) is approved by a calculation

$$\begin{aligned} G_1D_{R\,1} \;&=\; \mu^{-2}\,\mathrm{Tr}_{R(23)}K_2F_1^{-1}F_2^{-1}D_{R\,1} \;=\; \mu^{-2}\,\mathrm{Tr}_{\,(23)}K_2F_1^{-1}F_2^{-1}D_{R\,1}D_{R\,2}D_{R\,3} \\ &= \mu^{-2}\,\mathrm{Tr}_{\,(23)}D_{R\,1}D_{R\,2}D_{R\,3}K_2F_1^{-1}F_2^{-1} \;=\; D_{R\,1}G_1\,. \end{aligned}$$

Here the expression (3.49) for the operator G, the relations (3.18) for $X = Y = R$ and the relation (3.37) were used.

To prove the relation $[E, G] = 0$, we rewrite the expression for G:

$$\begin{aligned} G_1 &= \mathrm{Tr}_{\,(23)}(K_2F_1^{-1}F_2^{-1}) = \eta^{-1}\mathrm{Tr}_{\,(23)}(K_2K_2F_1^{-1}F_2^{-1}) = \eta^{-1}\mathrm{Tr}_{\,(23)}(K_2F_1^{-1}F_2^{-1}K_1) \\ &= \eta^{-1}\mathrm{Tr}_{\,(23)}(F_1^{-1}F_2^{-1}K_1K_2) = \eta^{-1}\mathrm{Tr}_{\,(23)}(F_1^{-1}F_2^{-1}K_1P_{23}P_{12})\,E_1\;. \end{aligned} \tag{3.52}$$

In the second equality we used the relation $K^2 = \eta K$; in the third equality we used the twist relation; in the fourth equality we moved the operator K_2 cyclically under the trace; in the fifth equality we used the first of the relations (3.40).

Due to the relation (3.45), the combination $\mathrm{Tr}_{\,(23)}(F_1^{-1}F_2^{-1}K_1P_{23}P_{12})$ commutes with the operator G_1. Therefore the operators G and E commute.

We have already shown that the operators C_F and D_F commute with the operator G. It follows then from the expression (3.52) for the operator G that to prove that the operators C_F and D_F commute with the operator E it is enough to prove that the operators C_F and D_F commute with the combination $\Xi_1 := \mathrm{Tr}_{\,(23)}(F_1^{-1}F_2^{-1}K_1P_{23}P_{12})$. We have

$$\begin{aligned} \Xi_1D_{F^{-1}\,1} &= \mathrm{Tr}_{\,(23)}(F_1^{-1}F_2^{-1}D_{F^{-1}\,3}K_1P_{23}P_{12}) = \mathrm{Tr}_{\,(23)}(F_1^{-1}F_2^{-1}D_{F^{-1}\,3}C_{R\,3}C_{R\,3}^{-1}K_1P_{23}P_{12}) \\ &= \mathrm{Tr}_{\,(23)}(F_1^{-1}D_{F^{-1}\,2}C_{R\,2}F_2^{-1}K_1P_{23}P_{12})C_{R\,1}^{-1} = D_{F^{-1}\,1}C_{R\,1}\Xi_1C_{R\,1}^{-1} = D_{F^{-1}\,1}\Xi_1\;. \end{aligned} \tag{3.53}$$

In the first equality we moved the operator $D_{F^{-1}}$ leftwards through the permutation operators; in the second equality we inserted $C_{R\,3}C_{R\,3}^{-1}$; in the third equality we used the relations (3.19) and moved the operator $C_{R\,1}^{-1}$ rightwards out of the trace; in the fourth equality we used again the relations (3.19). The operator C_R commutes with the operators G and E by the already proved relation (3.46) for the compatible pairs $\{R, F\}$ and $\{R, P\}$; therefore, due to the expression (3.52) for the operator G, the operator C_R commutes with the operator Ξ, which is used in the fifth equality.

The calculation (3.53) establishes the relation $[C_F, E] = 0$; the proof of the relation $[D_F, E] = 0$ is similar, we do not repeat details. ■

Remark 3.12. One can rewrite further the expression (3.49) for G:

$$\begin{aligned} G_1 \;&=\; \mu^{-2}\,\mathrm{Tr}_{R(23)}F_1^{-1}F_2^{-1}K_1 = \mu^{-2}\mathrm{Tr}_{R(2)}F_1^{-1}C_{F\,2}D_{R\,2}K_1 \\ &= \mathrm{Tr}_{R(2)}F_1^{-1}C_{F\,2}D_{R}^{-1}{}_1K_1 = \mu^{-2}\mathrm{Tr}_{R(2)}F_1^{-1}C_{F\,2}C_{R\,1}K_1 \\ &= \mu^{-2}C_{F\,1}\mathrm{Tr}_{R(2)}C_{R\,2}F_1^{-1}K_1 = C_{F\,1}\mathrm{Tr}_{\,(2)}F_1^{-1}K_1\;. \end{aligned}$$

Here we used subsequently: the twist relation, the relations (3.21), (3.37), (3.36), (3.18) and then again (3.36).

Similarly,

$$G_1^{-1} = \mathrm{Tr}_{\,(2)}(K_1F_1)D_F^{-1}{}_1\;.$$

3.6. Two linear maps

The next lemma introduces two linear maps which will be important in the study of the matrix $\star$-product.

Definition-Lemma 3.13. *Let $\{R, F\}$ be a compatible pair of skew invertible R-matrices, where the operator R is of the BMW type and the operator F is strict skew invertible. Define two endomorphisms ϕ and ξ of the space $\mathrm{Mat}_N(W)$:*

$$\phi(M)_1 := \mathrm{Tr}_{R(2)}\left(F_{12}M_1F_{12}^{-1}R_{12}\right), \qquad M \in \mathrm{Mat}_N(W), \tag{3.54}$$

and

$$\xi(M)_1 := \mathrm{Tr}_{R(2)}\left(F_{12}M_1F_{12}^{-1}K_{12}\right), \qquad M \in \mathrm{Mat}_N(W). \tag{3.55}$$

The mappings ϕ and ξ are invertible; their inverse mappings read

$$\phi^{-1}(M)_1 = \mu^{-2}\mathrm{Tr}_{R_f^{(2)}}\left(F_{12}^{-1}M_1R_{12}^{-1}F_{12}\right) \tag{3.56}$$

and

$$\xi^{-1}(M)_1 = \mu^{-2}\mathrm{Tr}_{R_f^{(2)}}\left(F_{12}^{-1}M_1K_{12}F_{12}\right). \tag{3.57}$$

The following relations for the R-traces

$$\mathrm{Tr}_{R_f}\phi(M) = \mathrm{Tr}_R M, \qquad \mathrm{Tr}_{R_f}\xi(M) = \mu\,\mathrm{Tr}_R M. \tag{3.58}$$

are satisfied.

Proof. The expressions on the right hand sides of the formulas (3.56) and (3.57) are well defined, since, by Proposition 3.6(b), the R-matrix R_f is skew invertible.

Let us check the relation $\phi^{-1}(\phi(M)) = M$ directly.

Using the formulas (3.54) and (3.56) and applying the relation (3.12) for the pair $\{R, F\}$ we begin a calculation

$$\begin{aligned}\phi^{-1}(\phi(M))_1 &= \mu^{-2}\,\mathrm{Tr}_{R_f^{(2)}}\left(F_{12}^{-1}(\mathrm{Tr}_{R(2')}F_{12'}M_1F_{12'}^{-1}R_{12'})R_{12}^{-1}F_{12}\right)\\ &= \mu^{-2}\,\mathrm{Tr}_{R_f^{(2)}}\mathrm{Tr}_{R(3)}\left(F_1^{-1}F_2^{-1}\underline{F_1}M_1F_1^{-1}R_1F_2R_1^{-1}F_1\right).\end{aligned}$$

In the next step we move the element F_1, underlined in the expression above, to the left and it becomes F_2 due to the Yang–Baxter equation; then we transport the operator to the right using the cyclic property of the trace (when F_2 moves cyclically, $\mathrm{Tr}_{R_f^{(2)}}\mathrm{Tr}_{R(3)}$ becomes $\mathrm{Tr}_{R(2)}\mathrm{Tr}_{R_f^{(3)}}$ due to the relations (3.18)). Applying the Yang–Baxter equation for the operator F and the relations (3.18) in the case $X = R$ and $Y = R_f$, we continue the calculation

$$\begin{aligned}\phi^{-1}(\phi(M))_1 &= \mu^{-2}\,\mathrm{Tr}_{R(2)}\mathrm{Tr}_{R_f^{(3)}}\left(F_1^{-1}F_2^{-1}M_1\underline{F_1^{-1}R_1F_2}\,\underline{R_1^{-1}F_1F_2}\right)\\ &= \mu^{-2}\,\mathrm{Tr}_{R(2)}\mathrm{Tr}_{R_f^{(3)}}\left(F_1^{-1}M_1F_2^{-1}R_{f\,1}\,\underline{F_1^{-1}F_2F_1R_{f\,1}^{-1}}\,F_2\right)\\ &= \mu^{-2}\,\mathrm{Tr}_{R(2)}\mathrm{Tr}_{R_f^{(3)}}\left(F_1^{-1}M_1\underline{F_2^{-1}R_{f\,1}F_2}\,F_1\,\underline{F_2^{-1}R_{f\,1}^{-1}\,F_2}\right)\\ &= \mu^{-2}\,\mathrm{Tr}_{R(2)}\left(F_1^{-1}M_1F_1(\underline{\mathrm{Tr}_{R_f^{(3)}}R_{f\,2}F_1R_{f\,2}^{-1}})F_1^{-1}\right).\end{aligned} \tag{3.59}$$

Here we consequently transformed the underlined expressions using the definition of the twisted R-matrix R_f, the Yang–Baxter equation for the operator F and the twist relations for the compatible pair $\{R_f, F\}$. To calculate the trace underlined in the last line of Eq. (3.59), we apply the relation (3.12) for the compatible pair $\{R_f, R_f\}$ and then use the relation (3.21) written for the compatible pair $\{R_f, F^{-1}\}$. The result reads

$$\phi^{-1}(\phi(M))_1 = \mu^{-2}\,\mathrm{Tr}_{R(2)}\left(F_1^{-1}M_1F_1\underline{(D_{R_f}C_{F^{-1}})_1F_1^{-1}}\right).$$

Now, using the relations (3.19), written for the compatible pairs $\{R_f, F\}$ and $\{F^{-1}, F\}$, the relations (3.29) and (3.22) for $X = F$, the relations (3.36) and the (3.3) for $X = F^{-1}$, we complete the calculation

$$\phi^{-1}(\phi(M))_1 = \mu^{-2}\,\mathrm{Tr}_{(2)}\left((D_{R_f}C_{F^{-1}}D_R)_2F_1^{-1}\right)M_1$$

$$= \mu^{-2}\,\mathrm{Tr}_{(2)}\left((D_{F^{-1}}C_RD_FC_{F^{-1}}D_R)_2F_1^{-1}\right)M_1 = \mathrm{Tr}_{(2)}(D_{F^{-1}\,2}F_{12}^{-1})M_1 = M_1.$$

A proof of the equality $\xi^{-1}(\xi(M)) = M$ proceeds quite similarly until the line (3.59), where one has to use a relation

$$\mathrm{Tr}_{R(2)}(K_1M_1K_1) = (\mathrm{Tr}_{R}M)I_1 \quad \forall\, M \in \mathrm{Mat}_N(W)$$

instead of the relation (3.12). This in turn follows from the relations (3.33) and (3.34) and the property $\mathrm{rk}\,K = 1$.

The relations (3.58) can be directly checked starting from the definitions (3.54) and (3.55), applying the relation (3.18) in the case $X = R$ and $Y = R_f$ and then using the formulas (3.10) and (3.34). ■

Remark 3.14. For the mapping ϕ, the statement of Lemma 3.13 remains valid if one weakens the conditions, imposed on the R-matrix R, replacing the BMW type condition by the strict skew invertibility. In this case, one should substitute the term $\mu^{-2}D_{R_f}$ by $D_{R_f^{-1}}$ in the expression (3.56) for the inverse mapping ϕ^{-1}. The proof repeats the proof of the formula (3.56).

4. Quantum matrix algebra

In this section we deal with the main objects of our study, the quantum matrix algebras, and construct the $\star$-product for them. We mainly discuss the quantum matrix algebras of the type BMW.

In Section 4.2 we introduce a *characteristic* subalgebra of the quantum matrix algebra. In the theory of the polynomial identities, a ring, generated by the traces of products of generic matrices, is known as the ring of matrix invariants (see, *e.g.*, [10]). The characteristic subalgebra can be understood as a generalization of the ring of matrix invariants (in the simplest case of a single matrix) to the setting of the quantum matrix algebras and, simultaneously, to a situation when the invariants can be formed not only by taking a trace (on the quantum level, the invariants can be conveniently formed by taking the R-trace of a product of a 'string' $M_{\bar 1}M_{\bar 2}\dots M_{\bar n}$ by a matrix image of a word in the braid group $\mathcal{B}_n$).

In Propositions 4.7, 4.8 we exhibit three generating sets of the characteristic subalgebra in the BMW case. Explicit relations between the generators of these sets will be constructed in Section 5. Some preparatory work for this constructions is performed in the rest of Section 4.

In Section 4.3 we introduce an algebra $\mathcal{P}(R,F)$ for the quantum matrix algebras of the general type. The algebra $\mathcal{P}(R,F)$ has the same relationship to the characteristic subalgebra as the trace ring (see, *e.g.*, [10]) to the ring of matrix invariants.

In Section 4.4 we prove the commutativity of the algebra $\mathcal{P}(R,F)$ in the case of the quantum matrix algebras of the BMW type.

In Section 4.5 we define an extended quantum matrix algebra of the BMW type by adding an inverse of the quantum matrix.

4.1. Definition

Consider a linear space $\mathrm{Mat}_{\mathrm{N}}(W)$, introduced in Definition 3.1. For a fixed element $F \in \mathrm{Aut}(V\otimes V)$, we consider series of 'copies' $M_{\bar i}$, $i = 1, 2, \dots, n$, of a matrix $M \in \mathrm{Mat}_{\mathrm{N}}(W)$. They are defined recursively by

$$M_{\bar 1} := M_1, \quad M_{\bar i} := F_{i-1}M_{\overline{i-1}}F_{i-1}^{-1}\,. \tag{4.1}$$

For $F = P$, these are usual copies, $M_{\bar i} = M_i$, but, in general, $M_{\bar i}$ can be nontrivial in all the spaces $1, \dots, i$.

We shall, slightly abusing notation, denote by the same symbol $M_{\bar i}$ an element in $\mathrm{Mat}_{\mathrm{N}}(W)^{\otimes k}$ for any $k \ge i$, which is defined by an inclusion of the spaces

$$\mathrm{Mat}_{\mathrm{N}}(W)^{\otimes j} \hookrightarrow \mathrm{Mat}_{\mathrm{N}}(W)^{\otimes (j+1)} : \quad \mathrm{Mat}_{\mathrm{N}}(W)^{\otimes j} \ni X \mapsto X\otimes I \in \mathrm{Mat}_{\mathrm{N}}(W)^{\otimes (j+1)}\,.$$

From now on we specify W to be the associative $\mathbb{C}$-algebra freely generated by the unity and by N^2 elements M_a^b, $W := \mathbb{C}\langle 1, M_a^b\rangle$, $1 \le a, b \le \mathrm{N}$.

Definition 4.1. Let $\{R, F\}$ be a compatible pair of strict skew invertible R-matrices (see Section 3.1). A quantum matrix algebra $\mathcal{M}(R,F)$ is a quotient algebra of the algebra $W = \mathbb{C}\langle 1, M_a^b\rangle$ by a two-sided ideal generated by entries of the matrix relation

$$R_1 M_{\bar 1}M_{\bar 2} = M_{\bar 1}M_{\bar 2}R_1\,, \tag{4.2}$$

where $M = \|M_a^b\|_{a,b=1}^{\mathrm{N}}$ is a matrix of the generators of $\mathcal{M}(R,F)$ and the matrix copies $M_{\bar i}$ are constructed with the help of the R-matrix F as in Eq. (4.1).

If R is an R-matrix of the BMW type (see Eqs. (3.5)–(3.8)) then $\mathcal{M}(R,F)$ is called a BMW type quantum matrix algebra.

Remark 4.2. The quantum matrix algebras were introduced in Ref. [18] under the name 'quantized braided groups'. In the context of the present paper they have been first investigated in [24]. The matrix M' of the generators of the algebra $\mathcal{M}(R,F)$ used in [24] is different from the matrix M that we use here. A relation between these two matrices is explained in section 3 of [23]: $M' = D_R M (D_F)^{-1}$.

Lemma 4.3 ([24]). *The matrix copies of the matrix $M = \|M_a^b\|_{a,b=1}^{\mathrm{N}}$ of the generators of the algebra $\mathcal{M}(R,F)$ satisfy relations*

$$F_i\,M_{\bar j} \;=\; M_{\bar j}\,F_i \qquad \text{for } \; j \neq i, i+1, \tag{4.3}$$

$$R_i\,M_{\bar j} \;=\; M_{\bar j}\,R_i \qquad \text{for } \; j \neq i, i+1, \tag{4.4}$$

$$R_j\, M_{\bar{j}}\, M_{\overline{j+1}} \;=\; M_{\bar{j}}\, M_{\overline{j+1}}\, R_j \quad \textit{for}\;\; j = 1, 2, \dots\;, \tag{4.5}$$

$$F_i\, F_{i+1} \dots F_k \cdot M_{\bar{i}}\, M_{\overline{i+1}} \dots M_{\bar{k}} \;=\; M_{\overline{i+1}}\, M_{\overline{i+2}} \dots M_{\overline{k+1}} \cdot F_i\, F_{i+1} \dots F_k \quad \textit{for}\;\; i \le k\,. \tag{4.6}$$

4.2. Characteristic subalgebra

From now on we assume that M is the matrix of generators of the quantum matrix algebra $\mathcal{M}(R, F)$ and its copies $M_{\bar{n}}$ are calculated by the rule (4.1).

Denote by $\mathcal{C}(R, F)$ a vector subspace of the quantum matrix algebra $\mathcal{M}(R, F)$ linearly spanned by the unity and elements

$$ch(\alpha^{(n)}) := \mathrm{Tr}_{R(1,\dots,n)}(M_{\bar{1}} \dots M_{\bar{n}}\, \rho_R(\alpha^{(n)}))\;, \quad n = 1, 2, \dots\;, \tag{4.7}$$

where $\alpha^{(n)}$ is an arbitrary element of the braid group $\mathcal{B}_n$.

Notice that elements of the space $\mathcal{C}(R, F)$ satisfy a *cyclic property*

$$ch(\alpha^{(n)}\beta^{(n)}) = ch(\beta^{(n)}\alpha^{(n)}) \quad \forall\, \alpha^{(n)}, \beta^{(n)} \in \mathcal{B}_n\,, \quad n = 1, 2, \dots\;, \tag{4.8}$$

which is a direct consequence of the relations (4.4), (4.5) and (3.38) and the cyclic property of the trace.

Definition-Proposition 4.4 (*[24]*)**.** *The space $\mathcal{C}(R, F)$ is a commutative subalgebra of the quantum matrix algebra $\mathcal{M}(R, F)$:*

$$ch(\alpha^{(n)})\, ch(\beta^{(i)}) = ch(\alpha^{(n)}\, \beta^{(i)\uparrow n}) = ch(\alpha^{(n)\uparrow i}\, \beta^{(i)})\,. \tag{4.9}$$

Recall that $\alpha^{(n)\uparrow i}$ denotes the image of an element $\alpha^{(n)}$ under the embedding $\mathcal{B}_n \hookrightarrow \mathcal{B}_{n+i}$ defined in (2.13). We shall call $\mathcal{C}(R, F)$ the characteristic subalgebra of $\mathcal{M}(R, F)$.

A proof of the proposition given in [24] is based in particular on the following lemma:

Lemma 4.5 (*[24]*)**.** *Consider an arbitrary element $\alpha^{(n)}$ of the braid group $\mathcal{B}_n$. Let $\{R, F\}$ be a compatible pair of R-matrices, where R is skew invertible. Then relations*

$$\mathrm{Tr}_{R(i+1,\dots,i+n)}(M_{\overline{i+1}} \dots M_{\overline{i+n}}\, \rho_R(\alpha^{(n)\uparrow i})) \;=\; I_{1,2,\dots,i}\; ch(\alpha^{(n)}) \tag{4.10}$$

hold for any matrix $M \in \mathrm{Mat}_N(W)$.[5]

We will make use of Lemma 4.5 several times below.

Let us introduce a shorthand notation for certain elements of $\mathcal{C}(R, F)$

$$p_0 \;:=\; \mathrm{Tr}_R\, I \;(= \mu\eta \text{ in the BMW case})\,, \qquad p_1 := \mathrm{Tr}_R\, M\,, \tag{4.11}$$

$$p_i \;:=\; ch(\sigma_{i-1} \dots \sigma_2\sigma_1) = ch(\sigma_1\sigma_2 \dots \sigma_{i-1})\,, \quad i = 2, 3, \dots . \tag{4.12}$$

The last equality in Eq. (4.12) is due to the inner automorphism (2.15) and the cyclic property (4.8).

The elements p_i are called *traces of powers of M* or, shortly, *power sums*.

From now on in this subsection we assume the R-matrix R and, hence, the algebra $\mathcal{M}(R, F)$ to be of the BMW type. Denote

$$g \;:=\; ch(c^{(2)}) \;\equiv\; \eta^{-1} ch(\kappa_1) \;\equiv\; \eta^{-1}\, \mathrm{Tr}_{R(1,2)}\left(M_{\bar{1}} M_{\bar{2}}\, K_1\right)\,. \tag{4.13}$$

The notation used here was introduced in the formulas (2.3), (2.10), (2.29) and (3.6). We call the element g a *contraction of two matrices M* or, simply, a *2-contraction*.

Lemma 4.6. *Let M be the matrix of generators of the BMW type quantum matrix algebra $\mathcal{M}(R, F)$. Then its copies, defined in Eq. (4.1), fulfill relations*

$$K_n\, M_{\bar{n}} M_{\overline{n+1}} \;=\; M_{\bar{n}} M_{\overline{n+1}}\, K_n \;=\; \mu^{-2} K_n\, g \quad \forall\, n \ge 1\,. \tag{4.14}$$

Proof. We employ induction on n. Due to the property $\mathrm{rk}\, K = 1$, one has

$$K_1\, M_{\bar{1}} M_{\bar{2}} \;=\; M_{\bar{1}} M_{\bar{2}}\, K_1 \;=\; K_1\, t\,,$$

where $t \in \mathcal{M}(R, F)$ is a scalar. Evaluating the R-trace of this equality in the spaces 1 and 2 and using the relations (3.34) and (3.35), one finds $t = \mu^{-2} g$, which proves the relation (4.14) in the case $i = 1$. It remains to check the induction step $n \to (n+1)$:

$$\begin{aligned} K_{n+1} M_{\overline{n+1}} M_{\overline{n+2}} \;&=\; K_{n+1}(F_n M_{\bar{n}} F_n^{-1}) M_{\overline{n+2}} \;=\; K_{n+1} F_n M_{\bar{n}} (F_{n+1} M_{\overline{n+1}} F_{n+1}^{-1}) F_n^{-1} \\ &=\; (K_{n+1} F_n F_{n+1}) M_{\bar{n}} M_{\overline{n+1}} F_{n+1}^{-1} F_n^{-1} \;=\; F_n F_{n+1} (K_n M_{\bar{n}} M_{\overline{n+1}}) F_{n+1}^{-1} F_n^{-1} \\ &=\; \mu^{-2} F_n F_{n+1} K_n F_{n+1}^{-1} F_n^{-1} g \;=\; \mu^{-2} K_{n+1}\, g\,. \end{aligned}$$

[5] Here there is no need to specify M to be the matrix of the generators of the algebra $\mathcal{M}(R, F)$.

Here Eqs. (4.1) and (4.3), the twist relation (3.9) for the pair $\{K, F\}$ and the induction assumption were used for the transformation. ∎

Proposition 4.7. *Let $\mathcal{M}(R, F)$ be the quantum matrix algebra of the BMW type. Its characteristic subalgebra $\mathcal{C}(R, F)$ is generated by the set $\{g, p_i\}_{i\geq 0}$.*

Proof. Consider the chain of the BMW algebras monomorphisms (2.12)–(2.13). We adapt, for $n \geq 3$, the following presentation for an element $\alpha^{(n)} \in \mathcal{W}_n$

$$\alpha^{(n)} = \beta\sigma_1\beta' + \gamma\kappa_1\gamma' + \delta \,, \tag{4.15}$$

where $\beta, \beta', \gamma, \gamma', \delta \in \mathrm{Im}(\mathcal{W}_{n-1}) \subset \mathcal{W}_n$. For $n = 3$, the formula (4.15) follows from the relations (2.1)–(2.7). For $n > 3$, it can be proved by induction on n (one has to prove that the expressions of the form (4.15) form an algebra, for which it is enough to show that the products $\sigma_1\beta\sigma_1$, $\sigma_1\beta\kappa_1$, $\kappa_1\beta\sigma_1$ and $\kappa_1\beta\kappa_1$ with $\beta \in \mathrm{Im}(\mathcal{W}_{n-1}) \subset \mathcal{W}_n$ can be rewritten in the form (4.15); this is done by further decomposing β, using the induction assumption, $\beta = \tilde{\beta}\sigma_2\tilde{\beta}' + \tilde{\gamma}\kappa_2\tilde{\gamma}' + \tilde{\delta}$, where $\tilde{\beta}, \tilde{\beta}', \tilde{\gamma}, \tilde{\gamma}', \tilde{\delta} \in \mathrm{Im}(\mathcal{W}_{n-2}) \subset \mathcal{W}_n$).

Using the expression (4.15) for $\alpha^{(n)}$ and the cyclic property (4.8), we conclude that, in the BMW case, any element (4.7) of the characteristic subalgebra can be expressed as a linear combination of terms

$$ch(\alpha_1\alpha_2\dots\alpha_{n-1})\,, \quad \text{where} \quad \alpha_i \in \{1, \sigma_i, \kappa_i\}\,. \tag{4.16}$$

Let us analyze the expressions (4.16) for different choices of α_i.

(i) If $\alpha_i = 1$ for some value of i, then, applying the relation (4.10), we get

$$ch(\alpha_1\dots\alpha_{i-1}\alpha_{i+1}\dots\alpha_{n-1}) = ch(\alpha_1\dots\alpha_{i-1})\, ch((\alpha_{i+1}\dots\alpha_{n-1})^{\downarrow i})\,, \tag{4.17}$$

where $(\alpha_{i+1}\dots\alpha_{n-1})^{\downarrow i} \in \mathcal{W}_{n-i}$ is the preimage of $(\alpha_{i+1}\dots\alpha_{n-1}) \in \mathcal{W}_n$.

(ii) In the case when $\alpha_{n-1} = \kappa_{n-1}$, we apply the relation (4.14) and then the relations (3.10), (3.34) or (3.35) to reduce the expression (4.16) to

$$ch(\alpha_1\dots\alpha_{n-2}\kappa_{n-1}) = f(\alpha_{n-2})\, ch(\alpha_1\dots\alpha_{n-3})\, g\,, \tag{4.18}$$

where $f(\sigma_{n-2}) = \mu^{-1}$, $f(\kappa_{n-2}) = 1$ and $f(1) = \eta$.

(iii) In the case when $\alpha_i = \kappa_i$ for some i, and $\alpha_j = \sigma_j$ for all $j = i+1, \dots, n-1$, we perform the following transformations

$$\begin{aligned} &ch(\alpha_1\dots\alpha_{i-1}\underline{\kappa_i\sigma_{i+1}}\sigma_{i+2}\dots\sigma_{n-1}) = ch(\alpha_1\dots\alpha_{i-2}\,\sigma_i^{-1}\alpha_{i-1}\kappa_i\underline{\kappa_{i+1}\sigma_{i+2}}\dots\sigma_{n-1}) \\ &\quad = \dots = \underline{ch(\alpha_1\dots\alpha_{i-2}(\sigma_{n-2}^{-1}\dots\sigma_i^{-1})\alpha_{i-1}\kappa_i\kappa_{i+1}\dots\kappa_{n-1})}. \end{aligned} \tag{4.19}$$

Here the relations (2.8) and the cyclic property (4.8) are repeatedly used; expressions suffering a transformation are underlined.

Now, depending on a value of α_{i-1}, we proceed in different ways.

If $\alpha_{i-1} = \kappa_{i-1}$ then by Eqs. (2.8) and (4.18) we have

$$\begin{aligned} (4.19) \quad &= \quad ch(\alpha_1\dots\alpha_{i-2}\,\sigma_{i-1}\sigma_i\dots\sigma_{n-3}\kappa_{n-2}\kappa_{n-1}) \\ &= \quad ch(\alpha_1\dots\alpha_{i-2}\,\sigma_{i-1}\sigma_i\dots\sigma_{n-3})\, g\,. \end{aligned}$$

If $\alpha_{i-1} = \sigma_{i-1} = \sigma_{i-1}^{-1} + (q - q^{-1})(1 - \kappa_{i-1})$ then, using the relations $\sigma_i^{-1}\sigma_{i-1}^{-1}\kappa_i = \kappa_{i-1}\kappa_i$ and applying the previous results (4.18) and (4.17), we obtain

$$\begin{aligned} (4.19) \quad = \quad & ch(\alpha_1\dots\alpha_{i-2}\,\kappa_{i-1}\sigma_i\dots\sigma_{n-3})\, g \\ & + (q - q^{-1})\,\mu^{-1}\, ch(\alpha_1\dots\alpha_{i-2})\, p_{n-i-1}\, g \\ & - (q - q^{-1})\, ch(\alpha_1\dots\alpha_{i-2}\,\sigma_{i-1}\sigma_i\dots\sigma_{n-3})\, g\,. \end{aligned}$$

Iterating transformations (i)–(iii) finitely many times, we eventually prove the assertion of the proposition. ∎

We keep considering the BMW type quantum matrix algebra $\mathcal{M}(R, F)$ with the R-matrix R generating representations of the algebras $\mathcal{W}_n(q, \mu)$, $n = 1, 2, \dots$. Assume that the antisymmetrizers $a^{(i)}$ and symmetrizers $s^{(i)}$ in these latter algebras are consistently defined (see Eqs. (2.24), (2.25) and (2.26)). In this case, we can introduce two following sets of elements in the characteristic subalgebra $\mathcal{C}(R, F)$

$$a_0 := 1 \quad \text{and} \quad s_0 := 1\,; \tag{4.20}$$

$$a_i := ch(a^{(i)}) \quad \text{and} \quad s_i := ch(s^{(i)})\,, \quad i = 1, 2, \dots. \tag{4.21}$$

Proposition 4.8. *Let $\mathcal{M}(R, F)$ be the quantum matrix algebra of the BMW type. Assume that $j_q \neq 0$, $\mu \neq -q^{-2j+3}$ (respectively, $j_q \neq 0$, $\mu \neq q^{2j-3}$) for all $j = 2, 3, \dots$. Then the characteristic subalgebra $\mathcal{C}(R, F)$ is generated by the set $\{g, a_i\}_{i\geq 0}$ (respectively, $\{g, s_i\}_{i\geq 0}$).*

Proof. These statements are byproducts of the previous proposition and the Newton relations, which are proved in Section 5, Theorem 5.2. ∎

4.3. Matrix $\star$-product, general case

Consider the quantum matrix algebra $\mathcal{M}(R,F)$ of the general type (no additional conditions on an R-matrix R).

Denote by $\mathcal{P}(R,F)$ a linear subspace of $\mathrm{Mat}_N(\mathcal{M}(R,F))$ spanned by $\mathcal{C}(R,F)$-multiples of the identity matrix, $I\,ch\;\forall\, ch\in\mathcal{C}(R,F)$, and by elements

$$M^1:=M,\qquad (M^{\alpha^{(n)}})_1:=\mathrm{Tr}_{R(2,\dots,n)}(M_{\overline{1}}\dots M_{\overline{n}}\,\rho_R(\alpha^{(n)}))\,,\quad n=2,3,\dots\,, \tag{4.22}$$

where $\alpha^{(n)}$ belongs to the braid group $\mathcal{B}_n$. The space $\mathcal{P}(R,F)$ inherits a structure of a right $\mathcal{C}(R,F)$–module

$$M^{\alpha^{(n)}}ch(\beta^{(i)})=M^{(\alpha^{(n)}\beta^{(i)\uparrow n})}\quad\forall\,\alpha^{(n)}\in\mathcal{B}_n,\ \beta^{(i)}\in\mathcal{B}_i\,,\quad n,i=1,2,\dots\,, \tag{4.23}$$

which is just a component-wise multiplication of the matrix $M^{\alpha^{(n)}}$ by the element $ch(\beta^{(i)})$ (use the relation (4.10) to check this). The $\mathcal{C}(R,F)$–module structure agrees with an R-trace map Tr_R (which means that $\mathrm{Tr}_R(Xa)=\mathrm{Tr}_R(X)a\;\;\forall\,X\in\mathcal{P}(R,F)$ and $\forall\,a\in\mathcal{C}(R,F)$)

$$\mathcal{P}(R,F)\xrightarrow{\mathrm{Tr}_R}\mathcal{C}(R,F)\;:\quad\begin{cases}\;\;M^{\alpha^{(n)}}&\mapsto\quad ch(\alpha^{(n)})\,,\\ I\,ch(\alpha^{(n)})&\mapsto\quad(\mathrm{Tr}_R I)\,ch(\alpha^{(n)})\,,\end{cases} \tag{4.24}$$

where $\alpha^{(n)}\in\mathcal{B}_n\,,\quad n=1,2,\dots$

Besides, elements of the space $\mathcal{P}(R,F)$ satisfy a *reduced cyclic property*

$$M^{(\alpha^{(n)}\beta^{(n-1)\uparrow 1})}=M^{(\beta^{(n-1)\uparrow 1}\alpha^{(n)})}\quad\forall\,\alpha^{(n)}\in\mathcal{B}_n,\ \beta^{(n-1)}\in\mathcal{B}_{n-1},\quad n=2,3,\dots\,. \tag{4.25}$$

Definition-Proposition 4.9. *Formulas*

$$M^{\alpha^{(n)}}\star M^{\beta^{(i)}}\;:=\;M^{(\alpha^{(n)}\star\beta^{(i)})}\,, \tag{4.26}$$

where

$$\alpha^{(n)}\star\beta^{(i)}\;:=\;\alpha^{(n)}\beta^{(i)\uparrow n}(\sigma_n\dots\sigma_2\sigma_1\sigma_2^{-1}\dots\sigma_n^{-1})\,, \tag{4.27}$$

$$(I\,ch(\beta^{(i)}))\star M^{\alpha^{(n)}}\;:=\;M^{\alpha^{(n)}}\star(I\,ch(\beta^{(i)}))\;:=\;M^{\alpha^{(n)}}ch(\beta^{(i)})\,, \tag{4.28}$$

$$(I\,ch(\alpha^{(i)}))\star(I\,ch(\beta^{(n)}))\;:=\;I\,(ch(\alpha^{(i)})\,ch(\beta^{(n)}))\,, \tag{4.29}$$

define an associative multiplication on the space $\mathcal{P}(R,F)$, which agrees with the $\mathcal{C}(R,F)$–module structure (4.23).[6]

Proof. To prove the associativity of the multiplication (4.26), it is enough to check

$$(\alpha^{(n)}\star\beta^{(i)})\star\gamma^{(m)}\;=\alpha^{(n)}\star(\beta^{(i)}\star\gamma^{(m)})\,,$$

which is a straightforward exercise in an application of the relations (2.1) and (2.2).

It is less trivial to prove a compatibility condition for the formulas (4.26) and (4.28)

$$\left\{M^{\alpha^{(n)}}\star(I\,ch(\beta^{(i)}))\right\}\star M^{\gamma^{(m)}}\;=\;M^{\alpha^{(n)}}\star\left\{(I\,ch(\beta^{(i)}))\star M^{\gamma^{(m)}}\right\}\,,$$

which, in terms of the matrix 'exponents', amounts to

$$\begin{aligned}&\alpha^{(n)}\beta^{(i)\uparrow n}\gamma^{(m)\uparrow(i+n)}(\sigma_{i+n}\dots\sigma_2\sigma_1\sigma_2^{-1}\dots\sigma_{i+n}^{-1})\\ &\qquad\overset{\mathrm{mod}\,(4.25)}{=}\;\;\alpha^{(n)}\gamma^{(m)\uparrow n}\beta^{(i)\uparrow(m+n)}(\sigma_n\dots\sigma_2\sigma_1\sigma_2^{-1}\dots\sigma_n^{-1})\,.\end{aligned} \tag{4.30}$$

Here the symbol $\overset{\mathrm{mod}\,(4.25)}{=}$ means the equality modulo the reduced cyclic property (4.25).

To check Eq. (4.30), we apply a technique, which was used in [24] to prove the commutativity of the characteristic subalgebra. Consider an element

$$\begin{aligned}u^{(i+m)}_{i,m}\;&:=\;(\sigma_i\dots\sigma_2\sigma_1)(\sigma_{i+1}\dots\sigma_3\sigma_2)\dots(\sigma_{i+m-1}\dots\sigma_{m+1}\sigma_m)\\ &=\;(\sigma_i\sigma_{i+1}\dots\sigma_{i+m-1})(\sigma_{i-1}\sigma_i\dots\sigma_{i+m-2})\dots(\sigma_1\sigma_2\dots\sigma_m)\,,\end{aligned} \tag{4.31}$$

which intertwines certain elements of the braid group $\mathcal{B}_{(i+m)}$:

$$\beta^{(i)}\,u^{(i+m)}_{i,m}\;=\;u^{(i+m)}_{i,m}\,\beta^{(i)\uparrow m}\,,\qquad\gamma^{(m)\uparrow i}\,u^{(i+m)}_{i,m}\;=\;u^{(i+m)}_{i,m}\,\gamma^{(m)}\,. \tag{4.32}$$

Substitute an expression $(u^{(i+m)\uparrow n}_{i,m}\gamma^{(m)\uparrow n}\beta^{(i)\uparrow(n+m)}(u^{(i+m)\uparrow n}_{i,m})^{-1})$ for the factor $(\beta^{(i)\uparrow n}\gamma^{(m)\uparrow(i+n)})$ on the left hand side of Eq. (4.30), move the element $u^{(i+m)\uparrow n}_{i,m}$ cyclically to the right and then use an equality

$$(\sigma_1^{-1}\sigma_2^{-1}\dots\sigma_i^{-1})u^{(i+m)}_{i,m}=u^{(i+m-1)\uparrow 1}_{i,m-1} \tag{4.33}$$

to cancel it on the right hand side. Such transformation results in the right hand side of Eq. (4.30).

[6] In other words, a map $ch(\alpha^{(n)})\mapsto I\,ch(\alpha^{(n)})$ is an algebra monomorphism $\mathcal{C}(R,F)\hookrightarrow\mathcal{P}(R,F)$.

Consistency of the multiplication and the $\mathcal{C}(R, F)$–module structures on $\mathcal{P}(R, F)$ follows obviously from the last equality in (4.28). ■

To illustrate the relation between the $\star$-product and the usual matrix multiplication, we present formulas (4.26) and (4.27) in the case $n = 1$ ($\alpha^{(1)} \equiv 1$) in a form

$$M \star N = M \cdot \phi(N) \quad \forall N \in \mathcal{P}(R, F)\,, \tag{4.34}$$

where $\cdot$ denotes the usual matrix multiplication and the map ϕ is defined by the formula (3.54) in Section 3.6.

The noncommutative analogue of the matrix power is given by a repeated $\star$ -multiplication by the matrix M

$$M^{\overline{0}} := I\,, \qquad M^{\overline{n}} := \underbrace{M \star M \star \ldots \star M}_{n \text{ times}} = M^{(\sigma_1\sigma_2\ldots\sigma_{n-1})} = M^{(\sigma_{n-1}\ldots\sigma_2\sigma_1)}\,. \tag{4.35}$$

Here we introduce symbol $M^{\overline{n}}$ for the nth *power of the matrix M*. The standard matrix powers multiplication formula follows immediately from the definition

$$M^{\overline{n}} \star M^{\overline{i}} = M^{\overline{n+i}}\,. \tag{4.36}$$

Proposition 4.10. *A $\mathcal{C}(R, F)$–module, generated by the matrix powers $M^{\overline{n}}$, $n = 0, 1, \ldots$, belongs to the center of the algebra $\mathcal{P}(R, F)$.*

Proof. It is sufficient to check a relation $M \star M^{\alpha^{(i)}} = M^{\alpha^{(i)}} \star M$, which, in turn, follows from a calculation

$$\alpha^{(i)}\sigma_i \ldots \sigma_2\sigma_1\sigma_2^{-1} \ldots \sigma_i^{-1} = \sigma_i \ldots \sigma_2\sigma_1\alpha^{(i)\uparrow 1}\sigma_2^{-1} \ldots \sigma_i^{-1} \overset{\text{mod}\,(4.25)}{=} \alpha^{(i)\uparrow 1}\sigma_1\,. \quad ■$$

4.4. Matrix $\star$-product, BMW case

It is natural to expect that the algebra $\mathcal{P}(R, F)$ is commutative as all of its elements are generated by the matrix M alone. We can prove the commutativity in the BMW case. Notice that (in contrast to the Iwahori–Hecke case), in the BMW case, the algebra $\mathcal{P}(R, F)$ cannot be generated by the $\star$-powers of M only.

By an analogy with formula (4.34), we define a $\mathcal{C}(R, F)$–module map $M^{\intercal} : \mathcal{P}(R, F) \to \mathcal{P}(R, F)$

$$M^{\intercal}(N) := M \cdot \xi(N), \qquad N \in \mathcal{P}(R, F)\,, \tag{4.37}$$

where the endomorphism ξ is defined by formula (3.55) in Section 3.6. Equivalently, we can write

$$M^{\intercal}(M^{\alpha^{(n)}}) = M^{(\alpha^{(n)\uparrow 1}\kappa_1)} \qquad \forall\, \alpha^{(n)} \in \mathcal{W}_n, \quad n = 1, 2, \ldots\,. \tag{4.38}$$

Proposition 4.11. *Let the quantum matrix algebra $\mathcal{M}(R, F)$ be of the BMW type. Then the algebra $\mathcal{P}(R, F)$ is commutative. As a $\mathcal{C}(R, F)$–module, it is spanned by matrices*

$$M^{\overline{n}} \quad \textit{and} \quad M^{\intercal}(M^{\overline{n+2}})\,, \quad n = 0, 1, \ldots\,. \tag{4.39}$$

Proof. A proof of the last statement of the proposition goes essentially along the same lines as the proof of Proposition 4.7 and we will not repeat it. The only modification is a reduction of the cyclic property (c.f., Eqs. (4.8) and (4.25)), which finally leads to an appearance of the additional elements $\{M^{\intercal}(M^{\overline{n}})\}_{n\geq 2}$ in the generating set.

To prove the commutativity of $\mathcal{P}(R, F)$, we derive an alternative expression for the exponent in the matrix product formula (4.26)

$$\alpha^{(n)} \star \beta^{(i)} = (\sigma_i^{-1} \ldots \sigma_2^{-1}\sigma_1\sigma_2 \ldots \sigma_i)\alpha^{(n)\uparrow i}\beta^{(i)}\,. \tag{4.40}$$

The calculation proceeds as follows

$$\begin{aligned}
\alpha^{(n)} \star \beta^{(i)} &= \alpha^{(n)}\beta^{(i)\uparrow n}(\sigma_n \ldots \sigma_1\sigma_2^{-1} \ldots \sigma_n^{-1}) = u^{(n+i)}_{n,i}\alpha^{(n)\uparrow i}\beta^{(i)}(u^{(n+i)}_{n,i})^{-1}(\sigma_n \ldots \sigma_1\sigma_2^{-1} \ldots \sigma_n^{-1}) \\
&\overset{\text{mod}\,(4.25)}{=} (u^{(n+i-1)\uparrow 1}_{n,i-1})^{-1}(\sigma_2^{-1} \ldots \sigma_n^{-1})u^{(n+i)}_{n,i}\alpha^{(n)\uparrow i}\beta^{(i)} = (u^{(n+i-1)\uparrow 1}_{n,i-1})^{-1}\sigma_1 u^{(n+i-1)\uparrow 1}_{n,i-1}\alpha^{(n)\uparrow i}\beta^{(i)} \\
&= (\sigma_i^{-1} \ldots \sigma_2^{-1})(u^{(n+i-2)\uparrow 2}_{n-1,i-1})^{-1}\sigma_1 u^{(n+i-2)\uparrow 2}_{n-1,i-1}(\sigma_2 \ldots \sigma_i)\alpha^{(n)\uparrow i}\beta^{(i)} \text{right hand side of Eq. (4.40).}
\end{aligned}$$

Here we applied again the intertwining operators (4.31) and used their properties (4.32) and (4.33) and the reduced cyclicity. One more property

$$u^{(n+i)}_{n,i} = u^{(n+i-1)\uparrow 1}_{n-1,i}(\sigma_1\sigma_2 \ldots \sigma_i)$$

is used in the last line of the calculation.

Due to Proposition 4.10, to prove the commutativity of the algebra $\mathcal{P}(R, F)$, it remains to check the commutativity of the set $\{M^{\intercal}(M^{\overline{n}})\}_{n\geq 2}$.

Notice that the factors of the exponents of the matrices $M^{\mathsf{T}}(M^{\overline{n}})$ can be taken in an opposite order, $M^{\mathsf{T}}(M^{\overline{n}}) = M^{(\kappa_1\sigma_2\sigma_3\dots\sigma_n)} = M^{(\sigma_n\dots\sigma_3\sigma_2\kappa_1)}$. This observation, together with formula (4.40), allow us to choose the exponents of two matrices $M^{\mathsf{T}}(M^{\overline{n}}) \star M^{\mathsf{T}}(M^{\overline{i}})$ and $M^{\mathsf{T}}(M^{\overline{i}}) \star M^{\mathsf{T}}(M^{\overline{n}})$ to be mirror (left–right) images of each other. Finally, $M^{\alpha^{(n)}} = M^{\varsigma(\alpha^{(n)})}$, $\forall\, \alpha^{(n)} \in \mathcal{W}_n(q,\mu)$, where ς is the antiautomorpism (2.19), since both sides of this equality can be expanded into linear combinations of the generators (4.39), which are invariant with respect to the mirror reflection of their exponents, and since the expansion rules (i.e. the defining relations for the BMW algebras) are mirror symmetric. ■

Lemma 4.12. *For the BMW type quantum matrix algebra $\mathcal{M}(R,F)$, one has*

$$M^{\mathsf{T}}(I) = \mu M\,, \qquad M^{\mathsf{T}}(M) = \mu^{-1} I\, g\,, \tag{4.41}$$

$$M^{\mathsf{T}}(M^{\mathsf{T}}(N)) = N\, g \quad \forall\, N \in \mathcal{P}(R,F)\,. \tag{4.42}$$

Proof. The relations (4.41) follow immediately from the relations (4.14) and (3.34) and the definitions (4.37) and (3.55).

As for the equality (4.42), it is enough to check it in the case when the matrix N is a power of the matrix M.

To evaluate the expression $M^{\mathsf{T}}(M^{\mathsf{T}}(M^{\overline{n}})) = M^{(\kappa_1\kappa_2\sigma_3\dots\sigma_{n+1})}$, we transform its exponent, using the relations (2.8) in the BMW algebra and the reduced cyclic property, to

$$\begin{aligned} \kappa_1\kappa_2\sigma_3\dots\sigma_{n+1} &= \kappa_1(\kappa_2\kappa_3\sigma_2^{-1})\sigma_4\dots\sigma_{n+1} \overset{\mathrm{mod}\,(4.25)}{=} (\sigma_2^{-1}\kappa_1\kappa_2)\kappa_3\sigma_4\dots\sigma_{n+1} \\ &= \sigma_1\kappa_2\kappa_3\sigma_4\dots\sigma_{n+1} = \dots \overset{\mathrm{mod}\,(4.25)}{=} \sigma_1\sigma_2\dots\sigma_{n-1}\kappa_n\kappa_{n+1}\,. \end{aligned} \tag{4.43}$$

For the exponent (4.43), the matrix power is easily calculated, again with the help of the relations (4.14) and (3.34), and gives the expression $M^{\overline{n}}g$. ■

The last relation in (4.41) shows that to introduce the inverse matrix to the matrix M it is sufficient to add the inverse g^{-1} of the 2-contraction g to the algebra $\mathcal{M}(R,F)$. This is realized in the next subsection.

4.5. Matrix inversion

In this subsection we define an *extended* quantum matrix algebra, to which the inverse of the quantum matrix belongs.

Lemma 4.13. *Let $\mathcal{M}(R,F)$ be the BMW type quantum matrix algebra. Its 2-contraction g fulfills a relation*

$$M\, g = g\,(G^{-1}MG)\,, \tag{4.44}$$

where G is defined by formula (3.42).

Proof. The proof consists of a calculation

$$\begin{aligned} M_1\,(gK_2) &= \mu^2 M_{\overline{1}} M_{\overline{2}} M_{\overline{3}} K_2 = \mu^2 M_{\overline{1}} M_{\overline{2}} M_{\overline{3}} K_2 K_1 K_2 = \mu^2 K_2 \left(M_{\overline{1}} M_{\overline{2}} K_1\right) M_{\overline{3}} K_2 \\ &= gK_2 K_1 M_{\overline{3}} K_2 = (gK_2)\,\mathrm{Tr}_{(2,3)}\left(K_2 K_1 M_{\overline{3}}\right) \\ &= (gK_2)\,\mathrm{Tr}_{(2,3)}\left(K_2 F_2 F_1 K_2 M_1 F_1^{-1} F_2^{-1}\right) \\ &= (gK_2)\,\mathrm{Tr}_{(2,3)}(F_2 F_1 K_2) M_1 \mathrm{Tr}_{(2,3)}(K_2 F_1^{-1} F_2^{-1}) = (gK_2)\,(G^{-1}MG)_1 . \end{aligned} \tag{4.45}$$

Here the relations (4.14) and (2.9) were used in the first two lines; the property $\mathrm{rk}\, K = 1$ was used in the last/first equality of the second/fourth line; the definition of $M_{\overline{3}}$ was substituted and the twist relation for the pair $\{K, F\}$ was used in the third line; the formulas (3.42) and (3.43) for G and G^{-1} were substituted in the last equality. ■

Definition-Proposition 4.14. *Let $\mathcal{M}(R,F)$ be the BMW type quantum matrix algebra. Consider an extension of the algebra $\mathcal{M}(R,F)$ by a generator g^{-1} subject to relations*

$$g^{-1} g = g\, g^{-1} = 1\,, \qquad g^{-1} M = (G^{-1}MG)\, g^{-1}\,. \tag{4.46}$$

The extended algebra, which we shall further denote by $\mathcal{M}^{\bullet}(R,F)$, contains an inverse matrix to the matrix M

$$M^{-1} := \mu\, \xi(M)\, g^{-1} : \qquad M \cdot M^{-1} = M^{-1} \cdot M = I\,. \tag{4.47}$$

Proof. Lemma 4.13 ensures the consistency of the relations (4.46). The equality $M \cdot M^{-1} = I$ for the inverse matrix (4.47) follows immediately from the formulas (4.41) and (4.37).

To prove the equality $M^{-1} \cdot M = I$, consider a mirror partner of the map ξ:

$$\theta(M) := \mu^{-2}\, \mathrm{Tr}_{R(2)} K_1 M_{\overline{2}}\,. \tag{4.48}$$

By the (left–right) symmetry arguments in the assumptions of Lemma 3.13, the map θ is invertible and the inverse map reads

$$\theta^{-1}(M) = \mathrm{Tr}_{R_f^{(2)}} \left(F_1^{-1} K_1 M_1 F_1\right)\,. \tag{4.49}$$

Applying in a standard way the transformation formula (3.12), we calculate a composition of the maps ξ and θ,

$$\xi(\theta(M))_1 = \theta(\xi(M))_1 = \mu^{-2}\,\mathrm{Tr}_{R(2,3)} K_2 K_1 M_{\overline{3}} = \mathrm{Tr}_{\,(2,3)} K_2 K_1 M_{\overline{3}} = (G^{-1} M G)_1\,. \tag{4.50}$$

Here the relation (3.37) was used to substitute the R-traces by the usual traces; the last equality follows from a comparison of the second and the last lines in the calculation (4.45).

Now we observe that, in view of the relations (4.14) and (3.34), a matrix $({}^{-1}M) := \mu\, g^{-1}\,\theta^{-1}(M)$ fulfills the relation $({}^{-1}M)\cdot M = I$. The identity $({}^{-1}M) = M^{-1}$ follows then from the relations (4.50) and (4.44). ■

Remark 4.15. One can generalize the definitions of the characteristic subalgebra and of the matrix powers to the case of the extended quantum matrix algebra $\mathcal{M}^{\bullet}(R,F)$. Not going into details, we just mention that the *extended* characteristic subalgebra $\mathcal{C}^{\bullet}(R,F)$ is generated by the set $\{g, g^{-1}, p_i\}_{i\geq 0}$ and the *extended* algebra $\mathcal{P}^{\bullet}(R,F)$, as a $\mathcal{C}^{\bullet}(R,F)$–module, is spanned by matrices

$$M^{\overline{n}} \quad \text{and} \quad M^{\mathsf{T}}(M^{\overline{n}}) \quad \forall\, n\in\mathbb{Z}\,.$$

Here inverse powers of M are defined through the repeated $\star$-multiplication by $M^{\overline{-1}}$, which is given by

$$M^{\overline{-1}} \star N \;:=\; N\star M^{\overline{-1}} \;:=\; \phi^{-1}(M^{-1}\cdot N) \quad \forall\, N\in\mathcal{P}^{\bullet}(R,F)\,. \tag{4.51}$$

Explicitly, one has

$$M^{\overline{-n}} \;:=\; \underbrace{M^{\overline{-1}}\star\ldots\star M^{\overline{-1}}}_{n\ times}\star I \;=\; \mathrm{Tr}_{R_f^{-1}\,(2,\ldots,n+1)}\Big(M_{\underline{2}}^{-1} M_{\underline{3}}^{-1}\ldots M_{\underline{n+1}}^{-1}\,\rho_{R_f^{-1}}(\sigma_n\ldots\sigma_2\sigma_1)\Big)\,;$$

where the copies $M_{\underline{i}}^{-1}$ of the matrix M^{-1} are defined as (c.f. with Eq. (4.1))

$$M_{\underline{1}} \;:=\; M_1, \qquad M_{\underline{i+1}} \;:=\; F_i^{-1} M_{\underline{i}} F_i, \qquad i=2,3,\ldots\,. \tag{4.52}$$

Notice that in general $M^{\overline{-1}} = \phi^{-1}(M^{-1}) \neq M^{-1}$. Here are some particular examples of the multiplication by $M^{\overline{-1}}$

$$M^{\overline{-n}}\star M^{\overline{i}} \;=\; M^{\overline{i-n}}\,, \qquad M^{\overline{-1}}\star M^{\alpha^{(n)}\uparrow 1} \;=\; ch(\alpha^{(n)})\,I\,.$$

5. Relations for generating sets of the characteristic subalgebra: BMW case

In this last section we use the basic identities from Section 5.1 to establish relations between the three sets of elements in the characteristic subalgebra — $\{g, a_i\}_{i\geq 0}$, $\{g, s_i\}_{i\geq 0}$ and the power sums $\{g, p_i\}_{i\geq 0}$. As a byproduct, we prove Proposition 4.8.

Before we proceed, let us recall the initial data of the construction.

- Given a compatible pair of R-matrices $\{R, F\}$, in which the operator F is strict skew invertible and the operator R is skew invertible of the BMW type (and, hence, strict skew invertible), we introduce the BMW type quantum matrix algebra $\mathcal{M}(R,F)$ (see Definition 4.1);
- Assuming additionally that the eigenvalues q and μ of the R-matrix R (i.e., the parameters of the BMW algebras, whose representations are generated by the matrix R) satisfy conditions $i_q\neq 0,\ \mu\neq -q^{3-2i}\ \ \forall\, i=2,3,\ldots,n$ (see (2.26)) we can consistently define the antisymmetrizers $a^{(i)}$ and introduce skew powers of the quantum matrix M: $M^{a^{(i)}}$, $0\leq i\leq n$.

5.1. Basic identities

In this subsection we establish relations between 'descendants' of the matrices $M^{a^{(i)}}$ in the algebra $\mathcal{P}(R,F)$. These relations are used later in a derivation of the Newton relations.

For $1\leq i\leq n$ and $m\geq 0$, we consider two series of descendants of $M^{a^{(i)}}$:

$$A^{(m,i)} \;:=\; i_q\, M^{\overline{m}}\star M^{a^{(i)}}\quad,\qquad B^{(m+1,i)} \;:=\; i_q\, M^{\overline{m}}\star M^{\mathsf{T}}(M^{a^{(i)}})\,. \tag{5.1}$$

It is suitable to define $A^{(m,i)}$ and $B^{(m,i)}$ for boundary values of their indices

$$A^{(-1,i)} \;:=\; i_q\,\phi^{-1}\left(\mathrm{Tr}_{R(2,3,\ldots i)} M_{\overline{2}} M_{\overline{3}}\ldots M_{\overline{i}}\,\rho_R(a^{(i)})\right)\quad,\qquad B^{(0,i)} \;:=\; i_q\,\phi^{-1}\big(\xi\big(M^{a^{(i)}}\big)\big) \tag{5.2}$$

and

$$A^{(m,0)} \;:=0 \quad\text{and}\quad B^{(m,0)} \;:=\; 0 \qquad \forall\, m\geq 0\,. \tag{5.3}$$

Notice that although the elements $A^{(-1,i)}$ and $B^{(0,i)}$ do not, in general, belong to the algebra $\mathcal{P}(R,F)$, their descendants $A^{(-1,i)}g$ and $B^{(0,i)}g$ do (see Eqs. (5.4) and (5.5) in the case $m=0$).

In the case when the contraction g (and, hence, the matrix M) is invertible, the formulas (5.1), with m now an arbitrary integer, can be used to define descendants of $M^{a^{(i)}}$ in the extended algebra $\mathcal{P}^{\bullet}(R,F)$ (see Remark 4.15). In this case, the matrices $A^{(-1,i)}$ and $B^{(0,i)}$ are expressed uniformly: $A^{(-1,i)} = i_q M^{\overline{-1}}\star M^{a^{(i)}}$, $\ B^{(0,i)} = i_q M^{\overline{-1}}\star M^{\mathsf{T}}(M^{a^{(i)}})$.

Lemma 5.1. *For* $0 \le i \le n-1$ *and* $m \ge 0$*, the matrices* $A^{(m-1,i+1)}$ *and* $B^{(m+1,i+1)}$ *satisfy recurrent relations*

$$A^{(m-1,i+1)} = q^i M^{\overline{m}}\, a_i \,-\, A^{(m,i)} \,-\, \frac{\mu q^{2i-1}(q-q^{-1})}{1+\mu q^{2i-1}}\, B^{(m,i)}\,, \tag{5.4}$$

$$B^{(m+1,i+1)} = \Big(\mu^{-1} q^{-i} M^{\overline{m}}\, a_i \,+\, \frac{q-q^{-1}}{1+\mu q^{2i-1}}\, A^{(m,i)} \,-\, B^{(m,i)}\Big) g\,. \tag{5.5}$$

Proof. For $i = 0$ relations (5.4) and (5.5) by (5.3) simplify to

$$A^{(m-1,1)} = M^{\overline{m}}\,, \quad B^{(m+1,1)} = \mu^{-1} M^{\overline{m}} g\,.$$

They follow from Eqs. (4.36), (4.41).

Let us check (5.4) for $i > 0$. For $m \ge 0$, we calculate

$$\begin{aligned} A^{(m,i+1)} &= (i+1)_q M^{(a^{(i+1)\uparrow m}\, \sigma_m \ldots \sigma_2 \sigma_1)} \,=\, q^i M^{(a^{(i)\uparrow(m+1)}\, \sigma^-_{m+1}(q^{-2i})\, \sigma_m \ldots \sigma_2 \sigma_1)} \\ &\quad = q^i M^{\overline{m+1}}\, a_i \,-\, A^{(m+1,i)} \,-\, \frac{\mu q^{2i-1}(q-q^{-1})}{1+\mu q^{2i-1}}\, B^{(m+1,i)}\,. \end{aligned}$$

Here in the first line we used the second formula from (2.24) for $a^{(i+1)\uparrow m}$ and applied the reduced cyclic property (4.25) and the relations (2.31) to cancel one of two terms $a^{(i)\uparrow(m+1)}$. In the second line we substituted the formula (2.22) for the baxterized elements $\sigma^-_{m+1}(q^{-2k})$ and applied the relation (4.10) to simplify the first term in the sum.

For $A^{(-1,i+1)}$, the relations (5.4) are verified similarly

$$\begin{aligned} A^{(-1,i+1)} &= q^i\, \phi^{-1}\left(\mathrm{Tr}_{R(2,3,\ldots,i+1)} M_{\overline{2}} M_{\overline{3}} \ldots M_{\overline{i+1}} \rho_R(a^{(i)\uparrow 1} \sigma^-_1(q^{-2i}))\right) \\ &= q^i\, \phi^{-1}(I)\, a_i \,-\, i_q\, \phi^{-1}(\phi(M^{a^{(i)}})) \,-\, \frac{\mu q^{2i-1}(q-q^{-1})}{1+\mu q^{2i-1}}\, i_q\, \phi^{-1}(\xi(M^{a^{(i)}})) \\ &= q^i\, I\, a_i \,-\, A^{(0,i)} \,-\, \frac{\mu q^{2i-1}(q-q^{-1})}{1+\mu q^{2i-1}}\, B^{(0,i)}\,. \end{aligned}$$

Here the definitions (3.54) and (3.55) of the endomorphisms ϕ and ξ were additionally taken into account.

To prove (5.5) for $i > 0$ we proceed in the same way

$$\begin{aligned} &B^{(m+1,i+1)} = (i+1)_q M^{(a^{(i+1)\uparrow m+1}\, \kappa_{m+1} \sigma_m \ldots \sigma_2 \sigma_1)} \,=\, q^i M^{(a^{(i)\uparrow m+2}\, \sigma^-_{m+2}(q^{-2i})\, \kappa_{m+1} \sigma_m \ldots \sigma_2 \sigma_1)} \\ &= q^{-i} M^{\overline{m}} \star M^{\mathsf{T}}(M) a_i - i_q M^{(a^{(i)\uparrow m+2}\, \sigma^{-1}_{m+2} \kappa_{m+1} \sigma_m \ldots \sigma_1)} + \frac{q^i - q^{-i}}{1+\mu q^{2i-1}}\, M^{\overline{m}} \star M^{\mathsf{T}}(M^{\mathsf{T}}(M^{a^{(i)}})). \end{aligned} \tag{5.6}$$

Here in the second line we used another expression for the baxterized generators

$$\sigma^\varepsilon_i(x) \,=\, x1 \,+\, \frac{x-1}{q-q^{-1}}\, \sigma^{-1}_i \,-\, \frac{\alpha_\varepsilon x(x-1)}{\alpha_\varepsilon x + 1}\, \kappa_i\,,$$

which follows by a substitution $\sigma_i = \sigma^{-1}_i + (q-q^{-1})(1-\kappa_i)$ into the original expression (2.22).

Now, notice that

$$\sigma^{-1}_3 \kappa_2 \sigma_1 \,=\, \sigma^{-1}_3 \kappa_2 \kappa_1 \sigma^{-1}_2 \,\overset{\mathrm{mod}\,(4.25)}{=}\, \sigma^{-1}_2 \sigma^{-1}_3 \kappa_2 \kappa_1 \,=\, \kappa_3 \kappa_2 \kappa_1\,, \tag{5.7}$$

and, hence, in the case $m \ge 1$, the second term in the last line of the equality (5.6) can be expressed as

$$-\, i_q M^{(a^{(i)\uparrow m+2}\, \sigma^{-1}_{m+2} \kappa_{m+1} \sigma_m \ldots \sigma_1)} \,=\, -i_q M^{\overline{m-1}} \star M^{\mathsf{T}}(M^{\mathsf{T}}(M^{\mathsf{T}}(M^{a^{(i)}})))\,. \tag{5.8}$$

Applying then the formulas (4.41) and (4.42) to the expressions (5.6) and (5.8), we complete verification of (5.5) for $m \ge 1$.

For the case $m = 0$, the transformation of the second term in (5.6) should be slightly modified. Notice that by Eq. (5.7),

$$\phi(M^{a^{(i)\uparrow 2} \sigma^{-1}_2 \kappa_1}) \,=\, \xi(M^{\mathsf{T}}(M^{\mathsf{T}}(M^{a^{(i)}})))\,.$$

Inverting the endomorphism ϕ in this formula and using the relation (4.42) and the definition of $B^{(0,i)}$ (5.2), we complete the transformation of the second term in (5.6) and, again, get the equality (5.5). ■

5.2. Newton and Wronski relations

Theorem 5.2. *Let* $\mathcal{M}(R,F)$ *be a BMW type quantum matrix algebra. Assume that its two parameters* q *and* μ *satisfy the conditions (2.26), which allow to introduce either the set* $\{a_i\}_{i=0}^n$ *or, respectively, the set* $\{s_i\}_{i=0}^n$ *in the characteristic subalgebra* $\mathcal{C}(R,F)$ *(see the definitions (4.20) and (4.21)). Then the following Newton recurrent formulas relating, respectively, the sets* $\{a_i, g\}_{i=0}^n$*, or* $\{s_i, g\}_{i=0}^n$ *to the set of the power sums (see the definitions (4.11) and (4.12))*

$$\sum_{i=0}^{n-1} (-q)^i a_i\, p_{n-i} = (-1)^{n-1} n_q\, a_n \,+\, (-1)^n \sum_{i=1}^{\lfloor n/2 \rfloor} \Big(\mu q^{n-2i} - q^{1-n+2i}\Big)\, a_{n-2i}\, g^i \tag{5.9}$$

and

$$\sum_{i=0}^{n-1} q^{-i} s_i\, p_{n-i} \;=\; n_q\, s_n \;+\; \sum_{i=1}^{\lfloor n/2\rfloor} \Big(\mu q^{2i-n} + q^{n-2i-1}\Big)\, s_{n-2i}\, g^i \tag{5.10}$$

are fulfilled.

In the case, when both sets $\{a_i, g\}_{i=0}^n$ *and* $\{s_i, g\}_{i=0}^n$ *are consistently defined, they satisfy the Wronski relations*

$$\sum_{i=0}^{n} (-1)^i a_i\, s_{n-i} \;=\; \delta_{n,0} - \delta_{n,2}\, g\,, \tag{5.11}$$

where $\delta_{i,j}$ *is a Kronecker symbol.*

Remark 5.3. One can use the formulas (5.9) and (5.10) for an iterative definition of the elements a_i and s_i for $i \ge 1$, with initial conditions $a_0 = s_0 = 1$. In this case, the elements a_n and s_n are well defined, assuming that $i_q \neq 0 \;\; \forall\, i = 2, 3, \dots n$. The additional restrictions on the parameter μ, which appeared in their initial definition (4.21), are artifacts of the use of the antisymmetrizers and symmetrizers $a^{(n)}, s^{(n)} \in \mathcal{W}_n(q)$.

Proof. We prove the relation (5.9). Denote

$$J^{(0)} := 0, \quad J^{(i)} := \sum_{j=0}^{i-1} (-q)^j M^{\overline{i-j}} a_j, \quad i = 1, 2, \dots, n\,.$$

We are going to find an expression for the matrix $J^{(n)}$ in terms of the matrices $A^{(0,i)}$ and $B^{(0,i)}$, $1 \le i \le n$.

As we shall see, there exist matrices $H^{(i)}$, which fulfill equations

$$(1-q^2) H^{(i)}\, g \;=\; \Big(J^{(i)} + (-1)^i A^{(0,i)}\Big)\,, \quad i = 0, 1, \dots, n. \tag{5.12}$$

To calculate the matrices $H^{(i)}$, we substitute repeatedly the relations (5.4) for the elements $A^{(0,i)}, A^{(1,i-1)}, \dots, A^{(i-1,1)}$ on the right hand side of Eq. (5.12). It then transforms to

$$H^{(i)}\, g \;=\; -\mu q^{-1} \sum_{j=1}^{i-1} (-1)^j \frac{q^{2j-1}}{1+\mu q^{2j-1}} B^{(i-j,j)}\,, \quad i = 0, 1, \dots, n. \tag{5.13}$$

Now, using the expressions (5.5) for the elements $B^{(i-j,j)}$, one can check that matrices

$$\begin{aligned} &H^{(0)} \;:=\; H^{(1)} \;:=\; 0, \\ &H^{(i)} \;:=\; \sum_{j=0}^{i-2} \frac{(-q)^j}{1+\mu q^{2j+1}} \Big(M^{\overline{i-j-2}} a_j \;+\; \frac{\mu q^j (q-q^{-1})}{1+\mu q^{2j-1}} A^{(i-j-2,j)} \;-\; \mu q^j B^{(i-j-2,j)}\Big), \quad i = 2, \dots, n. \end{aligned} \tag{5.14}$$

satisfy Eq. (5.13).

Next, consider a combination $(H^{(i+2)} - H^{(i)} g)$. Using Eq. (5.14) for the first term and Eq. (5.13) for the second term, we calculate

$$\begin{aligned} H^{(i+2)} - H^{(i)} g \;=\; &\sum_{j=0}^{i-1} \frac{(-q)^j}{1+\mu q^{2j+1}} \Big(M^{\overline{i-j}} a_j \;+\; \frac{\mu q^j (q-q^{-1})}{1+\mu q^{2j-1}} (A^{(i-j,j)} - q^{-1} B^{(i-j,j)})\Big) \\ &+ \frac{(-q)^i}{1+\mu q^{2i+1}} \Big(I a_i + \frac{\mu q^i (q-q^{-1})}{1+\mu q^{2i-1}} A^{(0,i)} \;-\; \mu q^i B^{(0,i)}\Big)\,, \quad \forall\, i = 0, \dots, n. \end{aligned}$$

To continue, we need the following auxiliary result:

Lemma 5.4. *For* $1 \le i \le n$*, one has*

$$\frac{(-1)^{i-1} A^{(0,i)}}{1+\mu q^{2i-1}} \;=\; \sum_{j=0}^{i-1} \frac{(-q)^j}{1+\mu q^{2j+1}} \Big(M^{\overline{i-j}} a_j \;+\; \frac{\mu q^j (q-q^{-1})}{1+\mu q^{2j-1}} (A^{(i-j,j)} - q^{-1} B^{(i-j,j)})\Big). \tag{5.15}$$

Proof. Use the recursion (5.4) for $A^{(i-j-1,j+1)}$ to calculate

$$\frac{A^{(i-j-1,j+1)}}{1+\mu q^{2j+1}} + \frac{A^{(i-j,j)}}{1+\mu q^{2j-1}} \;=\; \frac{q^j}{1+\mu q^{2j+1}} \Big(M^{\overline{i-j}} a_j + \frac{\mu q^j (q-q^{-1})}{1+\mu q^{2j-1}} (A^{(i-j,j)} - q^{-1} B^{(i-j,j)})\Big)\,.$$

Compose an alternating sum of the above relations for $0 \le j \le i-1$ and take into account the condition $A^{(i,0)} = 0$. ■

Using the relation (5.15), we finish the calculation

$$H^{(i+2)} - H^{(i)}g \;=\; (1+\mu q^{2i+1})^{-1}\Big((-q)^i I a_i \;+\; (-1)^{i+1}(A^{(0,i)} + \mu q^{2i} B^{(0,i)})\Big) \quad \forall\, i = 0,\dots,n-2. \tag{5.16}$$

Now it is straightforward to get

$$H^{(i)} \;=\; \sum_{j=1}^{[i/2]} \frac{(-1)^{i-1}}{1+\mu q^{2(i-2j)+1}}\Big(A^{(0,i-2j)} + \mu q^{2(i-2j)} B^{(0,i-2j)} - q^{i-2j} I a_{i-2j}\Big)\, g^{j-1}\,, \quad \forall\, i = 0,\dots,n, \tag{5.17}$$

where $[k]$ denotes the integer part of the number k. Finally, substituting the expression (5.17) back into Eq. (5.12), we obtain a formula

$$J^{(i)} \;=\; (-1)^{i-1}A^{(0,i)} \;+\; \sum_{j=1}^{[i/2]} \frac{(-1)^{i-1}(1-q^2)}{1+\mu q^{2(i-2j)+1}}\Big(A^{(0,i-2j)} + \mu q^{2(i-2j)} B^{(0,i-2j)} - q^{i-2j} I a_{i-2j}\Big) g^{j}, \tag{5.18}$$

which is valid for $0 \le i \le n$.

Taking the R-trace of Eq. (5.18), we obtain the Newton relations (5.9). Here, in the calculation of the R-trace of $B^{(0,i-2j)}$, we took into account the formulas (3.58).

The formulas (5.10) can be deduced from the relations (5.9) by a substitution $q \to -q^{-1},\ a_j \to s_j$. This is justified by the existence of the BMW algebras homomorphism (2.16)$\iota:\ \mathcal{W}_n(q,\mu) \to \mathcal{W}_n(-q^{-1},\mu)$ and a fact that one and the same R-matrix R generates representations of both algebras $\mathcal{W}_n(q,\mu)$ and $\mathcal{W}_n(-q^{-1},\mu)$.

The relation (5.11) is proved by induction on n. The cases $n = 0, 1, 2$ are easily checked with the use of Eqs. (5.9) and (5.10). Then, making an induction assumption, we derive the Wronski relations for arbitrary $n > 2$. To this end, we take a difference of Eqs. (5.10) and (5.9)

$$\sum_{i=0}^{n-1}\Big(q^{-i} s_i\, p_{n-i} \;-\; (-q)^i a_i\, p_{n-i}\Big) \;=\; n_q(s_n \;+\; (-1)^n a_n) \;+\; \text{terms proportional to } g$$

and substitute for p_{n-i} in the first/second term of the left hand side its expression from the Newton relation (5.9)/(5.10) (with n replaced by $n-i$). As a result, all terms, containing the power sums, cancel and, after rearranging the summations, we get

$$n_q \sum_{i=0}^{n}(-1)^i a_i s_{n-i} \;=\; -\sum_{i=1}^{[n/2]}(q^{1-n+2i} + q^{n-1-2i})g^i \sum_{j=0}^{n-2i}(-1)^j a_j s_{n-2i-j}\,.$$

By the induction assumption, the double sum on the right hand side of this relation vanishes identically: when n is odd, the second sum vanishes for all values of the index i; when n is even, the second sum is different from zero only for two values of the index i, $i = n/2$ and $i = n/2 - 1$, and these two summands cancel. ■

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Appendix A. Primitivity of contractors

In this appendix we return to the consideration of the contractors in the BMW algebra. We shall establish useful properties of the contractors in Lemmas A.1, A.2 and then use it to demonstrate their primitivity (announced in Proposition 2.2 in Section 2.4) in Proposition A.3.

In this appendix we shall denote by $\mathcal{W}(\sigma_i,\sigma_{i+1},\dots,\sigma_j)$, where $i \le j$, the BMW algebra with the generators $\sigma_i,\sigma_{i+1},\dots,\sigma_j$ (the values of the parameters q and μ are fixed).

Lemma A.1. *Let $\alpha \in \mathcal{W}(\sigma_1,\sigma_2,\dots,\sigma_j)$, where $j \ge n$. Then there exists an element $\tilde\alpha \in \mathcal{W}(\sigma_{n+1},\sigma_{n+2},\dots,\sigma_j)$ such that*

$$c^{(2n)}\alpha = c^{(2n)}\tilde\alpha\ .$$

Proof. Assume that $\alpha \in \mathcal{W}(\sigma_i,\sigma_{i+1},\dots,\sigma_j)$ and $\alpha \notin \mathcal{W}(\sigma_{i+1},\dots,\sigma_j)$. If $i > n$ then there is nothing to prove.

For $i \le n$, we shall prove that there exists an element $\alpha' \in \mathcal{W}(\sigma_{i+1}, \dots, \sigma_j)$ such that

$$c^{(2n)}\alpha = c^{(2n)}\alpha' \ .$$

Given this statement, the proof follows by induction on i.

Due to the formula (4.15), we can express the element α as a linear combination of elements of the form $xu_i\bar{x}$, where $x, \bar{x} \in \mathcal{W}(\sigma_{i+1}, \dots, \sigma_j)$ and u_i is equal to 1, σ_i or κ_i. The terms with $u_i = 1$ belong already to $\mathcal{W}(\sigma_{i+1}, \dots, \sigma_j)$ so we may assume that the element u_i is non-trivial (that is, equals σ_i or κ_i).

We express now the element x as a linear combination of the elements of the form $yu_{i+1}\bar{y}$, where $y, \bar{y} \in \mathcal{W}(\sigma_{i+2}, \dots, \sigma_j)$ and u_{i+1} is equal to 1, σ_{i+1} or κ_{i+1}. Each element $\bar{y}$ commutes with the element u_i thus the element α becomes a linear combination of elements of the form $yu_{i+1}u_i\bar{\bar{x}}$ with $y \in \mathcal{W}(\sigma_{i+2}, \dots, \sigma_j)$ and $\bar{\bar{x}} \in \mathcal{W}(\sigma_{i+1}, \dots, \sigma_j)$. In the terms with $u_{i+1} = 1$ we move the element y rightwards through the element u_i and continue the process for the terms with u_{i+1} equal to σ_{i+1} or κ_{i+1}. After a finite number of steps the process terminates and we will have an expression for the element α as a linear combination of terms

$$u_{i+k} \dots u_{i+1}u_i z \ , \tag{A.1}$$

where the element z belongs to $\mathcal{W}(\sigma_{i+1}, \dots, \sigma_j)$ and each of the elements u_{i+s}, $s = 0, 1, \dots k$, is equal to σ_{i+s} or κ_{i+s}.

Let us first analyze expressions (A.1) with $i + k > n$. The contractor $c^{(2n)}$ is divisible by the element κ_n from the right due to the relation (2.33). The element κ_n can move rightwards in the product $c^{(2n)}u_{i+k} \dots u_{i+1}u_i z$ until it reaches the element u_{n+1} and we arrive at the expression $\dots\kappa_n u_{n+1}u_n \dots$. For all four possibilities ($\sigma_{n+1}\sigma_n$, $\sigma_{n+1}\kappa_n$, $\kappa_{n+1}\sigma_n$ or $\kappa_{n+1}\kappa_n$) for the product $u_{n+1}u_n$, the expression $\kappa_n u_{n+1}u_n$ can be rewritten, with the help of the relations (2.5)–(2.9), in a form $\kappa_n v_{n+1}$, where v_{n+1} is a polynomial in σ_{n+1}. Moving the element κ_n back to the contractor $c^{(2n)}$, we obtain

$$c^{(2n)}u_{i+k} \dots u_{i+1}u_i z = c^{(2n)}u_{i+k} \dots u_{n+2}v_{n+1} \cdot u_{n-1} \dots u_i z = c^{(2n)}u_{n-1} \dots u_i \bar{z}$$

with some other $\bar{z} \in \mathcal{W}(\sigma_{i+1}, \dots, \sigma_j)$.

Thus we can rewrite the product of the contractor $c^{(2n)}$ by an expression (A.1) with $i+k > n$ as a product of $c^{(2n)}$ with an expression of the same form (A.1) but with $i + k < n$.

Now using the relations (2.34) we remove the elements u_{i+k} one by one to the right:

$$c^{(2n)}u_{i+k} \dots u_{i+1}u_i = c^{(2n)}u_{n-i-k}u_{i+k-1} \dots u_{i+1}u_i = c^{(2n)}u_{i+k-1} \dots u_{i+1}u_i u_{n-i-k} \ .$$

At the end we will obtain for the product $c^{(2n)}\alpha$ an expression of the form $c^{(2n)}\alpha'$, where the element α' belongs to $\mathcal{W}(\sigma_{i+1}, \dots, \sigma_j)$, as stated. ■

Lemma A.2. *Relations (2.5) and (2.9) involving the elements κ_i have the following analogues for the higher contractors:*

$$c^{(2i)}\,\sigma_{2i}\,c^{(2i)} = \eta^{-1}\mu^{-1}c^{(2i)} \ , \tag{A.2}$$
$$c^{(2i)}\,\kappa_{2i}\,c^{(2i)} = \eta^{-1}c^{(2i)} \ . \tag{A.3}$$

Proof. We prove the identity (A.3) by induction on i (the base of induction, $i = 1$, is the relation (2.9) itself):

$$\begin{aligned} c^{(2i+2)}\kappa_{2i+2}c^{(2i+2)} &= c^{(2i)\uparrow 1}\kappa_{2i+1}\kappa_1 c^{(2i)\uparrow 1}\kappa_{2i+2}c^{(2i+2)} = c^{(2i)\uparrow 1}\kappa_{2i+1}\kappa_1\kappa_{2i+2}c^{(2i+2)} \\ &= c^{(2i)\uparrow 1}\kappa_{2i+1}\kappa_{2i+2}\kappa_{2i+1}c^{(2i+2)} = c^{(2i)\uparrow 1}\kappa_{2i+1}c^{(2i+2)} = \eta^{-1}c^{(2i+2)} \ . \end{aligned}$$

In the first equality we used the definition (2.29); in the second equality we used the property (2.33); in the third equality we moved the element κ_1 rightwards to the contractor $c^{(2i+2)}$ and used the property (2.34); in the fourth equality we used the relation (2.9); the fifth equality is the induction assumption.

The identity (A.2) is proved again by induction on i (the base of induction, $i = 1$, is now the relation (2.5)):

$$\begin{aligned} c^{(2i+2)}\sigma_{2j+2}c^{(2i+2)} &= c^{(2i)\uparrow 1}\kappa_{2i+1}\kappa_1 c^{(2i)\uparrow 1}\sigma_{2i+2}c^{(2i+2)} = c^{(2i)\uparrow 1}\kappa_{2i+1}\kappa_1\sigma_{2i+2}c^{(2i+2)} \\ &= c^{(2i)\uparrow 1}\kappa_{2i+1}\sigma_{2i+2}\kappa_{2i+1}c^{(2i+2)} = \mu^{-1}c^{(2i)\uparrow 1}\kappa_{2i+1}c^{(2i+2)} = \mu^{-1}\eta^{-1}c^{(2i+2)} \ . \end{aligned}$$

In the first equality we used the definition (2.29); in the second equality we used the property (2.33); in the third equality we moved the element κ_1 rightwards to the contractor $c^{(2i+2)}$ and used the property (2.34); in the fourth equality we used the relation (2.5); the fifth equality is the identity (A.3).

The proof is finished. ■

Proposition A.3. *The contractor $c^{(2n)}$ is a primitive idempotent in the algebra $\mathcal{W}_{2n}(q, \mu)$ and in the algebra $\mathcal{W}_{2n+1}(q, \mu)$.*

Proof. To prove both statements about the primitivity, one has to check that a combination $c^{(2n)}\alpha^{(2n+1)}c^{(2n)}$ is proportional to the contractor $c^{(2n)}$ for an arbitrary element $\alpha^{(2n+1)}$ from the algebra $\mathcal{W}_{2n+1}(q, \mu)$.

Let α be an arbitrary element from the algebra $\mathcal{W}(\sigma_1, \dots, \sigma_j)$, where $j \ge 2n + 1$. Due to Lemma A.1, we have $c^{(2n)}\alpha = c^{(2n)}\beta$ with $\beta \in \mathcal{W}(\sigma_{n+1}, \dots, \sigma_j)$.

Let i ($i > 0$) be such that $\beta \in \mathcal{W}(\sigma_{n+i}, \sigma_{n+i+1}, \dots, \sigma_j)$ and $\beta \notin \mathcal{W}(\sigma_{n+i+1}, \dots, \sigma_j)$. We shall demonstrate that there exists an element $\bar{\beta} \in \mathcal{W}(\sigma_{n+i+1}, \dots, \sigma_j)$ for which

$$c^{(2n)}\beta c^{(2n)} = c^{(2n)}\bar{\beta} c^{(2n)} .$$

Given this statement, the proof follows by induction on i.

The element β is a linear combination of elements of the form $xu_{n+i}y$, where the elements x and y belong to $\mathcal{W}(\sigma_{n+i+1}, \dots, \sigma_j)$ and u_i is equal to σ_{n+i} or κ_{n+i}. We have

$$c^{(2n)}xu_{n+i}yc^{(2n)} = c^{(2n)}xc^{(2i)\uparrow n-i}u_{n+i}c^{(2i)\uparrow n-i}yc^{(2n)} \sim c^{(2n)}xc^{(2i)\uparrow n-i}yc^{(2n)} = c^{(2n)}xyc^{(2n)} .$$

In the first equality we used the relations (2.33); the proportionality follows from the relations (A.3) and (A.2). Then we used again the relations (2.33) to absorb the contractor $c^{(2i)\uparrow n-i}$ into $c^{(2n)}$.

The proof is finished. ■

Appendix B. Further properties of contractors

The relations, involving the elements κ_i, for the generators of the BMW algebras have analogues for the higher contractors. Two examples of such relations are proved in Lemma A.2. In Proposition B.1 we prove further analogues.

The identities in the lemma below have several versions obtained by an application of the automorphisms (2.16) and (2.15) and the antiautomorphism (2.19). For an identity of each type we present one version.

Proposition B.1. *Another analogue of the identity (2.9):*

$$\kappa_{2j}c^{(2j)}\kappa_{2j} = \eta^{-1}\kappa_{2j}c^{(2j-2)\uparrow 1} . \tag{B.1}$$

More general than (A.2) analogues of the identity (2.5):

$$c^{(2j)}\sigma_{j+k}\sigma_{j+k+1}\dots\sigma_{2j}c^{(2j)} = (\eta^{-1}\mu^{-1})^{j+1-k}c^{(2j)} \quad \text{for} \quad 0 < k \le j \tag{B.2}$$

and

$$c^{(2j)}\sigma_{j-k}\sigma_{j-k+1}\dots\sigma_{2j}c^{(2j)} = \eta^{-j}(\mu^{-1})^{j-1-k}c^{(2j)} \quad \text{for} \quad 0 \le k < j . \tag{B.3}$$

An analogue of the identities (2.8):

$$c^{(2j)}c^{(2j)\uparrow 1} = \eta^{-j}c^{(2j)}\sigma_{2j}^{-1}\sigma_{2j-1}^{-1}\dots\sigma_1^{-1} . \tag{B.4}$$

An analogue of the identity (2.7):

$$\sigma'_j\sigma'_{j-1}\dots\sigma'_1c^{(2j)\uparrow 1}\sigma'_1\dots\sigma'_{j-1}\sigma'_j = \sigma'_{j+1}\sigma'_{j+2}\dots\sigma'_{2j}c^{(2j)}\sigma'_{2j}\dots\sigma'_{j+2}\sigma'_{j+1} . \tag{B.5}$$

Another analogue of the identity (2.9):

$$c^{(2j)\uparrow 1}c^{(2j)}c^{(2j)\uparrow 1} = \eta^{-2j}c^{(2j)\uparrow 1} . \tag{B.6}$$

An analogue of the identity (2.35):

$$c^{(2j)}\tau^{(2k)\uparrow j-k} = \mu^k c^{(2j)} \quad \text{for} \quad k \le j . \tag{B.7}$$

where the elements $\tau^{(i)}$ are defined in Eq. (2.14).

Proof. The identity (B.1) is proved by induction on j (the base of induction, $j = 1$, is the relation (2.9)):

$$\begin{aligned}\kappa_{2j+2}c^{(2j+2)}\kappa_{2j+2} &= \kappa_{2j+2}c^{(2j)\uparrow 1}\kappa_{2j+1}\kappa_1c^{(2j)\uparrow 1}\kappa_{2j+2} = c^{(2j)\uparrow 1}\kappa_{2j+2}\kappa_{2j+1}\kappa_{2j+2}\kappa_1c^{(2j)\uparrow 1}\\ &= c^{(2j)\uparrow 1}\kappa_{2j+2}\kappa_1c^{(2j)\uparrow 1} = \eta^{-1}\kappa_{2j+2}c^{(2j)\uparrow 1} .\end{aligned}$$

In the first equality we used the definition (2.29); in the second equality we formed the combination $\kappa_{2j+2}\kappa_{2j+1}\kappa_{2j+2}$; in the third equality we used the relation (2.9); the fourth equality is the induction assumption.

The identity (B.2) is proved by induction on k down; the base of induction, when $k = j$, is the identity (A.2).

$$\begin{aligned}c^{(2j)}\sigma_{j+k}\sigma_{j+k+1}\dots\sigma_{2j}c^{(2j)} &= c^{(2j)}c^{(2k)\uparrow j-k}\sigma_{j+k}\sigma_{j+k+1}\dots\sigma_{2j}c^{(2k)\uparrow j-k}c^{(2j)}\\ &= c^{(2j)}c^{(2k)\uparrow j-k}\sigma_{j+k}c^{(2k)\uparrow j-k}\sigma_{j+k+1}\dots\sigma_{2j}c^{(2j)}\\ &= \eta^{-1}\mu^{-1}c^{(2j)}c^{(2k)\uparrow j-k}\sigma_{j+k+1}\dots\sigma_{2j}c^{(2j)}\\ &= \eta^{-1}\mu^{-1}c^{(2j)}\sigma_{j+k+1}\dots\sigma_{2j}c^{(2j)} = (\eta^{-1}\mu^{-1})^{j+1-k}c^{(2j)} .\end{aligned}$$

In the first equality we used the property (2.33); in the second equality we formed the combination $c^{(2k)\uparrow j-k}\sigma_{j+k}c^{(2k)\uparrow j-k}$; in the third equality we used the identity (A.2); in the fourth equality we used again the property (2.33); the fifth equality is the induction assumption.

The identity (B.3) is proved by induction on k. We have $c^{(2j)}\sigma_j = \mu c^{(2j)}$ by the relation (2.35), so the identity (B.3) with $k = 0$ follows from the identity (B.2) with $k = 1$.

Next, we have, for $i < j$:

$$\begin{aligned} c^{(2j)}\underline{\sigma}_i\sigma_{i+1}\dots\sigma_{2j}c^{(2j)} &= c^{(2j)}\underline{\sigma}_{2j-i}(\sigma_{i+1}\dots\sigma_{2j})c^{(2j)} \\ &= c^{(2j)}(\sigma_{i+1}\dots\sigma_{2j})\underline{\sigma}_{2j-i-1}c^{(2j)} = c^{(2j)}(\sigma_{i+1}\dots\sigma_{2j})\underline{\sigma}_{i+1}c^{(2j)} \ . \end{aligned} \tag{B.8}$$

Here we used the property (2.34) and the defining relation (2.1).

The last expression in Eq. (B.8) can be rewritten in a form

$$c^{(2j)}\underline{\sigma}_{i+2}(\sigma_{i+1}\dots\sigma_{2j})c^{(2j)} \ ,$$

again by the braid relation (2.1).

If $i + 2$ is still smaller than j, we continue in the same manner:

$$\begin{aligned} c^{(2j)}\underline{\sigma}_{i+2}(\sigma_{i+1}\dots\sigma_{2j})c^{(2j)} &= c^{(2j)}\underline{\sigma}_{2j-i-2}(\sigma_{i+1}\dots\sigma_{2j})c^{(2j)} \\ &= c^{(2j)}(\sigma_{i+1}\dots\sigma_{2j})\underline{\sigma}_{2j-i-3}c^{(2j)} = c^{(2j)}(\sigma_{i+1}\dots\sigma_{2j})\underline{\sigma}_{i+3}c^{(2j)} \end{aligned} \tag{B.9}$$

and the last expression in Eq. (B.9) can again be rewritten in a form

$$c^{(2j)}\underline{\sigma}_{i+4}(\sigma_{i+1}\dots\sigma_{2j})c^{(2j)} \ .$$

We repeat this process till the moment when the index of the underlined σ becomes equal to j. Then we use the property (2.35) and conclude

$$c^{(2j)}\sigma_i\sigma_{i+1}\dots\sigma_{2j}c^{(2j)} = \mu c^{(2j)}\sigma_{i+1}\dots\sigma_{2j}c^{(2j)} \ ,$$

which, due to the induction assumption, finishes the proof of the identity (B.3).

The proof of the identity (B.4) consists of a calculation

$$c^{(2j)}c^{(2j)\uparrow 1} = c^{(2j)}\sigma_1\sigma_2\dots\sigma_{2j}c^{(2j)}\sigma_{2j}^{-1}\dots\sigma_2^{-1}\sigma_1^{-1} = \eta^{-j}c^{(2j)}c^{(2j)}\sigma_{2j}^{-1}\dots\sigma_2^{-1}\sigma_1^{-1} \ .$$

The first equality here is valid due to the defining relations (2.1); in the second equality we used the identity (B.3) with $k = j - 1$.

Using a combination of the isomorphisms (2.15) and (2.17), we can rewrite the identity (B.4) in forms

$$c^{(2j)\uparrow 1}c^{(2j)} = \eta^{-j}c^{(2j)\uparrow 1}\sigma_1\sigma_2\dots\sigma_{2j} \ , \tag{B.10}$$

$$c^{(2j)}c^{(2j)\uparrow 1} = \eta^{-j}c^{(2j)}\sigma_{2j}\dots\sigma_2\sigma_1 \tag{B.11}$$

and

$$c^{(2j)\uparrow 1}c^{(2j)} = \eta^{-j}c^{(2j)\uparrow 1}\sigma_1^{-1}\sigma_2^{-1}\dots\sigma_{2j}^{-1} \ . \tag{B.12}$$

We now turn to the proof of the identity (B.5). First, we prove by induction on i the following identity:

$$\sigma_1'(\kappa_2\kappa_3\dots\kappa_{j+1})\sigma_1'\sigma_2'\dots\sigma_j' = \sigma_2'\sigma_3'\dots\sigma_{j+1}'(\kappa_1\kappa_2\dots\kappa_j)\sigma_{j+1}' \ . \tag{B.13}$$

The base of induction ($j = 1$) is the identity (2.7). The induction step is

$$\begin{aligned} \sigma_1'(\kappa_2\kappa_3\dots\kappa_{j+2})\sigma_1'\sigma_2'\dots\sigma_{j+1}' &= \sigma_1'(\kappa_2\kappa_3\dots\kappa_{j+1})(\sigma_1'\sigma_2'\dots\sigma_j')\kappa_{j+2}\sigma_{j+1}' \\ &= \sigma_2'\sigma_3'\dots\sigma_{j+1}'(\kappa_1\kappa_2\dots\kappa_j)\sigma_{j+1}'\kappa_{j+2}\sigma_{j+1}' = \sigma_2'\sigma_3'\dots\sigma_{j+1}'(\kappa_1\kappa_2\dots\kappa_j)\sigma_{j+2}'\kappa_{j+1}\sigma_{j+2}' \\ &= \sigma_2'\sigma_3'\dots\sigma_{j+2}'(\kappa_1\kappa_2\dots\kappa_{j+1})\sigma_{j+2}' \ , \end{aligned}$$

where we used the identity (2.7) in the third equality.

The image of the identity (B.13) under the antiautomorphism (2.19) reads

$$\sigma_j'\sigma_{j-1}'\dots\sigma_1'(\kappa_{j+1}\kappa_j\dots\kappa_2)\sigma_1' = \sigma_{j+1}'(\kappa_j\kappa_{j-1}\dots\kappa_1)\sigma_{j+1}'\sigma_j'\dots\sigma_2' \ . \tag{B.14}$$

The proof of the identity (B.5) is again by induction on j (the base of induction is the identity (2.7)):

$$\begin{aligned}
&(\sigma'_{j+1}\sigma'_j\dots\sigma'_1)c^{(2j+2)\uparrow 1}(\sigma'_1\dots\sigma'_j\sigma'_{j+1})\\
&\quad=\eta^{-1}(\sigma'_{j+1}\dots\sigma'_1)c^{(2j)\uparrow 2}(\kappa_{2j+2}\dots\kappa_{j+3})(\kappa_2\dots\kappa_{j+2})(\sigma'_1\dots\sigma'_{j+1})\\
&\quad=\eta^{-1}(\sigma'_{j+1}\dots\sigma'_2)c^{(2j)\uparrow 2}(\kappa_{2j+2}\dots\kappa_{j+3})\underline{\sigma'_1(\kappa_2\dots\kappa_{j+2})(\sigma'_1\dots\sigma'_{j+1})}\\
&\quad=\eta^{-1}(\sigma'_{j+1}\dots\sigma'_2)c^{(2j)\uparrow 2}(\kappa_{2j+2}\dots\kappa_{j+3})(\sigma'_2\dots\sigma'_{j+2})(\kappa_1\dots\kappa_{j+1})\sigma'_{j+2}\\
&\quad=\eta^{-1}\underline{(\sigma'_{j+1}\dots\sigma'_2)c^{(2j)\uparrow 2}(\sigma'_2\dots\sigma'_{j+1})}(\kappa_{2j+2}\dots\kappa_{j+3})\sigma'_{j+2}(\kappa_1\dots\kappa_{j+1})\sigma'_{j+2}\\
&\quad=\eta^{-1}(\sigma'_{j+2}\dots\sigma'_{2j+1})c^{(2j)\uparrow 1}\underline{(\sigma'_{2j+1}\dots\sigma'_{j+2})(\kappa_{2j+2}\dots\kappa_{j+3})\sigma'_{j+2}}(\kappa_1\dots\kappa_{j+1})\sigma'_{j+2}\\
&\quad=\eta^{-1}(\sigma'_{j+2}\dots\sigma'_{2j+1})c^{(2j)\uparrow 1}\sigma'_{2j+2}(\kappa_{2j+1}\dots\kappa_{j+2})(\sigma'_{2j+2}\dots\sigma'_{j+3})(\kappa_1\dots\kappa_{j+1})\sigma'_{j+2}\\
&\quad=\eta^{-1}(\sigma'_{j+2}\dots\sigma'_{2j+2})c^{(2j)\uparrow 1}(\kappa_{2j+1}\dots\kappa_{j+2})(\kappa_1\dots\kappa_{j+1})(\sigma'_{2j+2}\dots\sigma'_{j+2})\\
&\quad=(\sigma'_{j+2}\dots\sigma'_{2j+2})c^{(2j+2)}\sigma'_{2j+2}\dots\sigma'_{j+2}\ .
\end{aligned}$$

Here in the first equality we used the expression (2.37) for the contractor; in the second equality we moved the element σ'_1 rightwards to the string $(\kappa_2\dots\kappa_{j+2})$; in the third equality we transformed the underlined expression using the identity (B.13); in the fourth equality we moved the string $(\sigma'_2\dots\sigma'_{j+1})$ leftwards to the contractor $c^{(2j)\uparrow 2}$; in the fifth equality we used the induction assumption to transform the underlined expression; in the sixth equality we transformed the underlined expression using the shift ${}^{\uparrow j+1}$ of the identity (B.14); in the seventh equality we rearranged terms and then used again the expression (2.37) for the contractor in the eighth equality.

The following calculation establishes the identity (B.6):

$$c^{(2j)\uparrow 1}c^{(2j)}c^{(2j)\uparrow 1}=\eta^{-j}c^{(2j)\uparrow 1}c^{(2j)}\sigma_{2j}\dots\sigma_2\sigma_1=\eta^{-2j}c^{(2j)\uparrow 1}\ .$$

Here in the first equality we used the relation (B.11) while in the second one we used the relation (B.12).

To prove the identity (B.7), it is enough to prove its particular case

$$c^{(2j)}\tau^{(2j)}=\mu^j c^{(2j)} \tag{B.15}$$

since the element $c^{(2j)}$ is divisible by the element $c^{(2j-2k)\uparrow k}$ due to the relations (2.33).

We shall need two identities. The first one is

$$c^{(2j+2)}\sigma_1\sigma_2\dots\sigma_{2j}=c^{(2j+2)}c^{(2j)\uparrow 1}\sigma_1\sigma_2\dots\sigma_{2j}=\eta^j c^{(2j+2)}c^{(2j)}\ . \tag{B.16}$$

In the first equality we used the relations (2.33); in the second equality we used the relations (B.10) and again (2.33).

Here is the second identity:

$$\begin{aligned}c^{(2j+2)}c^{(2j)}\sigma_{2j+1}&=c^{(2j+2)}\sigma_1c^{(2j)}=c^{(2j+2)}\sigma_{2j-1}c^{(2j)}=c^{(2j+2)}\sigma_3c^{(2j)}\\
&=\dots=\mu c^{(2j+2)}c^{(2j)}\ .\end{aligned} \tag{B.17}$$

In the first equality we moved the element σ_{2j+1} leftwards through the contractor $c^{(2j)}$ and then we replaced the combination $c^{(2j+2)}\sigma_{2j+1}$ by $c^{(2j+2)}\sigma_1$ due to the relation (2.34); repeatedly using the relation (2.34), we replaced the combination $\sigma_1c^{(2j)}$ by $\sigma_{2j-1}c^{(2j)}$, then $c^{(2j+2)}\sigma_{2j-1}$ by $c^{(2j+2)}\sigma_3$ *etc.* The index of the element σ jumps by 2; at one moment it becomes equal to either j or $j+1$ and we use then the relation (2.35).

We now prove the relation (B.15) by induction on j (the base of induction, $j=1$, is the relation (2.4)):

$$\begin{aligned}
&c^{(2j+2)}\tau^{(2j+2)}=c^{(2j+2)}(\sigma_1\dots\sigma_{2j+1})\tau^{(2j+1)}=\eta^j c^{(2j+2)}c^{(2j)}\sigma_{2j+1}\tau^{(2j+1)}\\
&\quad=\mu\eta^j c^{(2j+2)}c^{(2j)}\tau^{(2j+1)}=\mu\eta^j c^{(2j+2)}c^{(2j)}\tau^{(2j)}(\sigma_{2j}\dots\sigma_1)\\
&\quad=\mu^{j+1}\eta^j c^{(2j+2)}c^{(2j)}(\sigma_{2j}\dots\sigma_1)=\mu^{j+1}\eta^{2j}c^{(2j+2)}c^{(2j)}c^{(2j)\uparrow 1}\\
&\quad=\mu^{j+1}\eta^{2j}c^{(2j+2)}c^{(2j)\uparrow 1}c^{(2j)}c^{(2j)\uparrow 1}=\mu^{j+1}c^{(2j+2)}\ .
\end{aligned}$$

In the first equality we used the iterative definition of the elements $\tau^{(i)}$ (it is different but equivalent to the one given in Eq. (2.15)); in the second equality we used the relation (B.16); in the third equality we used the relation (B.17); in the fourth equality we used again the iterative definition of the elements $\tau^{(i)}$; the fifth equality is the induction assumption; in the sixth equality we used the relation (B.11); in the seventh equality we used the relations (2.33); finally, in the eighth equality we used the relation (B.6).

The proof is finished. ■

Remark B.2. We have also

$$c^{(2j+2)}\tau^{(2j+1)}=c^{(2j+2)}(\sigma_1\dots\sigma_{2j})\tau^{(2j)}=\eta^j c^{(2j+2)}c^{(2j)}\tau^{(2j)}=(\eta\mu)^j c^{(2j+2)}c^{(2j)}\ .$$

In the first equality we used the iterative definition of the elements $\tau^{(i)}$; in the second equality we used the relation (B.16); in the third equality we used the identity (B.7).

Appendix C. On twists in quasitriangular Hopf algebras

Here we shall discuss universal (i.e., quasi-triangular Hopf algebraic) counterparts of relations from Sections 3.2, 3.3, especially from Proposition 3.6: we shall see, in item **8** of the appendix, that these relations have a quite transparent meaning, they reflect the properties of the twisted universal R-matrix.

We do not give an introduction to the theory of quasitriangular Hopf algebras assuming that the reader has some basic knowledge on the subject (see, e.g., [4], the chapter 4).

C.1. Generalities

1. Let $\mathcal{A}$ be a Hopf algebra; m, Δ, ϵ and S denote the multiplication, comultiplication, counit and antipode, respectively.

Assume that $\mathcal{A}$ is quasitriangular with a universal R-matrix $\mathcal{R} = a\otimes b$ (this is a symbolic notation, instead of $\sum_i a_i\otimes b_i$). One has $(S\otimes S)\mathcal{R} = \mathcal{R}$. The universal R-matrix $\mathcal{R}$ is invertible, its inverse is related to $\mathcal{R}$ by formulas $\mathcal{R}^{-1} = S(a)\otimes b$ or $(\mathrm{id}\otimes S)(\mathcal{R}^{-1}) = \mathcal{R}$.

For elements in $\mathcal{A}\otimes\mathcal{A}$, the 'skew' product $\odot$ is defined as the product in $\mathcal{A}^{\mathrm{op}}\otimes\mathcal{A}$, where $\mathcal{A}^{\mathrm{op}}$ denotes the algebra with the opposite multiplication. In other words, the skew product of two elements, $x\otimes y$ and $\tilde{x}\otimes\tilde{y}$ is $(x\otimes y)\odot(\tilde{x}\otimes\tilde{y}) = \tilde{x}x\otimes y\tilde{y}$. For a skew invertible element $\mathcal{X}\in\mathcal{A}\otimes\mathcal{A}$, we shall denote its skew inverse by $\psi_{\mathcal{X}}$. The universal R-matrix $\mathcal{R}$ has a skew inverse, $\psi_{\mathcal{R}} = a\otimes S(b)$. The element $\psi_{\mathcal{R}}$ is invertible, $(\psi_{\mathcal{R}})^{-1} = a\otimes S^2(b)$. The element $\mathcal{R}^{-1}$ is skew invertible as well, its skew inverse is $\psi_{(\mathcal{R}^{-1})} = S^2(a)\otimes b$. All these formulas are present in [7]. We shall see below that there are similar formulas for the twisting element $\mathcal{F}$. However, the properties of the twisting element $\mathcal{F}$ and of the universal R-matrix $\mathcal{R}$ are different, for instance, the square of the antipode is given by $S^2(x) = u_{\mathcal{R}}\, x\,(u_{\mathcal{R}})^{-1}$, where $u_{\mathcal{R}} = S(b)a$, but there is no analogue of such formula for $\mathcal{F}$. Because of this difference, we felt obliged to give some proofs of the relations for $\mathcal{F}$.

Let ρ be a representation of the algebra $\mathcal{A}$ in a vector space V. For an element $\mathcal{X}\in\mathcal{A}\otimes\mathcal{A}$, denote by $\hat{\rho}(\mathcal{X})\in\mathrm{End}(V^{\otimes 2})$ an operator $\hat{\rho}(\mathcal{X}) = P\cdot(\rho\otimes\rho)(\mathcal{X})$ (recall that P is the permutation operator). The skew product $\odot$ translates into the following product $\hat{\odot}$ for elements of $\mathrm{End}(V^{\otimes 2})$:

$$(X\hat{\odot}Y)_{13} := \mathrm{Tr}_{(2)}(X_{12}Y_{23})\ . \tag{C.1}$$

In other words, if $\mathcal{X}\odot\mathcal{Y} = \mathcal{Z}$ then $\hat{\rho}(\mathcal{X})\,\hat{\odot}\,\hat{\rho}(\mathcal{Y}) = \hat{\rho}(\mathcal{Z})$. For an operator $X\in\mathrm{End}(V^{\otimes 2})$, its skew inverse Ψ_X, in the sense explained in Section 3.1, is precisely the inverse with respect to the product $\hat{\odot}$.

2. The following lemma is well known (see, e.g., [4], the chapter 4, and references therein).

Lemma C.1. *Consider an invertible element $\mathcal{F} = \alpha\otimes\beta\in\mathcal{A}\otimes\mathcal{A}$ (we use the symbolic notation, $\alpha\otimes\beta = \sum_i \alpha_i\otimes\beta_i$, like for the universal R-matrix) and let $\mathcal{F}^{-1} = \gamma\otimes\delta$. Assume that the element $\mathcal{F}$ satisfies*

$$\mathcal{F}_{12}\,(\Delta\otimes\mathrm{id})(\mathcal{F}) = \mathcal{F}_{23}\,(\mathrm{id}\otimes\Delta)(\mathcal{F})\ . \tag{C.2}$$

Assume also that

$$\epsilon(\alpha)\,\beta = \alpha\,\epsilon(\beta) = 1\ . \tag{C.3}$$

Then an element $v_{\mathcal{F}} = \alpha\,S(\beta)$ is invertible, its inverse is

$$(v_{\mathcal{F}})^{-1} = S(\gamma)\,\delta\ . \tag{C.4}$$

One also has

$$S(\alpha)\,(v_{\mathcal{F}})^{-1}\,\beta = 1 \quad \text{and} \quad \gamma\, v_{\mathcal{F}}\,S(\delta) = 1\ . \tag{C.5}$$

Twisting the coproduct by the element $\mathcal{F}$,

$$\Delta_{\mathcal{F}}(a) = \mathcal{F}\,\Delta(a)\,\mathcal{F}^{-1}\ , \tag{C.6}$$

one obtains another quasitriangular structure on $\mathcal{A}$ with

$$\mathcal{R}_{\mathcal{F}} = \mathcal{F}_{21}\,\mathcal{R}\,\mathcal{F}^{-1} \tag{C.7}$$

and

$$S_{\mathcal{F}}(a) = v_{\mathcal{F}}\,S(a)\,(v_{\mathcal{F}})^{-1} \tag{C.8}$$

(the counit does not change).

An element $\mathcal{F}$, satisfying conditions (C.2) and (C.3) is called *twisting* element. We shall denote by $\mathcal{A}_{\mathcal{F}}$ the resulting 'twisted' quasitriangular Hopf algebra.

Remark C.2. On the representation level, the formula (C.7) transforms (compare with Eq. (3.23)) into $\hat{\rho}(\mathcal{R}_{\mathcal{F}}) = \hat{\rho}(\mathcal{F})_{21}\hat{\rho}(\mathcal{R})_{21}\hat{\rho}(\mathcal{F})_{21}^{-1}$. Below, when we talk about matrix counterparts of universal formulas, one should keep in mind this difference in conventions.

3. Assume, in addition to Eq. (C.2), that

$$(\Delta \otimes \mathrm{id})\,(\mathcal{F}) = \mathcal{F}_{13}\,\mathcal{F}_{23} \tag{C.9}$$

and

$$(\mathrm{id} \otimes \Delta)\,(\mathcal{F}) = \mathcal{F}_{13}\,\mathcal{F}_{12}\ . \tag{C.10}$$

Now the conditions (C.3) follow from the relations (C.9) and (C.10) and the invertibility of the twisting element $\mathcal{F}$: applying $\epsilon \otimes \mathrm{id} \otimes \mathrm{id}$ to the relation (C.9), we find $(\epsilon \otimes \mathrm{id})(\mathcal{F}) = 1$; applying $\mathrm{id} \otimes \mathrm{id} \otimes \epsilon$ to the relation (C.10), we find $(\mathrm{id} \otimes \epsilon)(\mathcal{F}) = 1$.

Since $\Delta^{\mathrm{op}}(x)\mathcal{R} = \mathcal{R}\Delta(x)$ for any element $x \in \mathcal{A}$ (where Δ^{op} is the opposite comultiplication), it follows from the relation (C.9) that

$$\mathcal{R}_{12}\,\mathcal{F}_{13}\,\mathcal{F}_{23} = \mathcal{F}_{23}\,\mathcal{F}_{13}\,\mathcal{R}_{12}\ . \tag{C.11}$$

Similarly, the relation (C.10) implies

$$\mathcal{R}_{23}\,\mathcal{F}_{13}\,\mathcal{F}_{12} = \mathcal{F}_{12}\,\mathcal{F}_{13}\,\mathcal{R}_{23}\ . \tag{C.12}$$

When both relations (C.9) and (C.10) are satisfied, the relation (C.2) is equivalent to the Yang–Baxter equation for the twisting element $\mathcal{F}$:

$$\mathcal{F}_{12}\,\mathcal{F}_{13}\,\mathcal{F}_{23} = \mathcal{F}_{23}\,\mathcal{F}_{13}\,\mathcal{F}_{12}\ . \tag{C.13}$$

Remark C.3. One also has

$$(\Delta_{\mathcal{F}} \otimes \mathrm{id})(\mathcal{F}_{21}) = \mathcal{F}_{31}\,\mathcal{F}_{32} \quad \text{and} \quad (\mathrm{id} \otimes \Delta_{\mathcal{F}})(\mathcal{F}_{21}) = \mathcal{F}_{31}\,\mathcal{F}_{21}\ .$$

Therefore, one can twist $\Delta_{\mathcal{F}}$ again, now by the element $\mathcal{F}_{21}$.

On the matrix level, this corresponds to the second conjugation of $\hat{\rho}(\mathcal{R})$ by $\hat{\rho}(\mathcal{F})$,

$$\hat{\rho}\big((\mathcal{R}_{\mathcal{F}})_{\mathcal{F}_{21}}\big) \;=\; \hat{\rho}(\mathcal{F})^2\;\hat{\rho}(\mathcal{R})\;\hat{\rho}(\mathcal{F})^{-2}\ .$$

Remark C.4. The element $\mathcal{F}_{21}^{-1}$ satisfies the conditions (C.2), (C.9) and (C.10) if the element $\mathcal{F}$ does. Thus, one can twist the coproduct Δ by the element $\mathcal{F}_{21}^{-1}$ as well.

4. The conditions (C.3), (C.9), (C.10) imply the invertibility and skew-invertibility of the element $\mathcal{F}$. The formulas for its inverse and skew inverse are similar to the corresponding formulas for the universal R-matrix $\mathcal{R}$ (in particular, we reproduce the standard formulas for $\mathcal{R}$ since we can take $\mathcal{F} = \mathcal{R}$).

Lemma C.5. *Assume that the conditions (C.3) and (C.9) are satisfied. Then the element $\mathcal{F}$ is invertible, its inverse is*

$$\mathcal{F}^{-1} = S(\alpha) \otimes \beta\ . \tag{C.14}$$

Assume that the conditions (C.3) and (C.10) are satisfied. Then the element $\mathcal{F}$ is skew invertible, with the skew inverse

$$\psi_{\mathcal{F}} = \alpha \otimes S(\beta)\ . \tag{C.15}$$

Assume that the conditions (C.3), (C.9) and (C.10) are satisfied. Then

$$(S \otimes S)(\mathcal{F}) = \mathcal{F}\ . \tag{C.16}$$

Moreover, the element $\psi_{\mathcal{F}}$ is invertible, its inverse is

$$(\psi_{\mathcal{F}})^{-1} = \alpha \otimes S^2(\beta) \tag{C.17}$$

and the element $\mathcal{F}^{-1}$ is skew-invertible, its skew inverse reads

$$\psi_{(\mathcal{F}^{-1})} = S^2(\alpha) \otimes \beta\ . \tag{C.18}$$

Proof. The calculations are similar to those, from textbooks, for the universal R-matrix. We include this proof for a completeness only.

Applications of $m_{12} \circ S_1$ and $m_{12} \circ S_2$ to the relation (C.9) imply the formula (C.14) (here m_{12} is the multiplication of the first and the second tensor arguments; S_1 is an operation of taking the antipode of the first tensor argument, *etc.*).

Applications of $m_{23} \circ S_2$ and $m_{23} \circ S_3$ to the relation (C.10) establish the formula (C.15).

Given the formula (C.15), the statement, that the element $\psi_{\mathcal{F}}$ is a left skew inverse of the element $\mathcal{F}$, reads in components:

$$\alpha\alpha' \otimes S(\beta')\beta = 1\ , \tag{C.19}$$

where primes are used to distinguish different summations terms, the expression $\alpha\alpha'\otimes S(\beta')\beta$ stands for $\sum_{i,j}\alpha_i\alpha_j\otimes S(\beta_j)\beta_i$. Applying S_1 to this equation, we find $(S(\alpha')\otimes S(\beta'))\cdot(S(\alpha)\otimes\beta)=1$ which means that the element $S(\alpha')\otimes S(\beta')$ is the left inverse of the element $S(\alpha)\otimes\beta$. However, the latter element is, by the formula (C.14), the inverse of $\mathcal{F}$. Therefore, the relation (C.16) follows.

Applying S_2 to the equality (C.19), we find that the element $\alpha\otimes S^2(\beta)$ is the right inverse of the element $\psi_{\mathcal{F}}$.

Applying S_1^2 to the equality (C.19) and using the relation (C.16), we find that $S^2(\alpha)\otimes\beta$ is a right skew inverse of the element $\mathcal{F}^{-1}$.

We shall not repeat details for the left inverse of the element $\psi_{\mathcal{F}}$ and the left skew inverse of the element $\mathcal{F}^{-1}$, calculations are analogous. ■

Remark C.6. There is a further generalization of the formulas from Lemma C.5. Start with the element $\mathcal{F}$ and alternate operations 'take an inverse' and 'take a skew inverse'. Then the next operation is always possible, the result is always invertible and skew invertible. One arrives, after n steps, at $S^n(\alpha)\otimes\beta$ if the first operation was 'take an inverse'; if the first operation was 'take a skew inverse' then one arrives at $\alpha\otimes S^n(\beta)$ (see [7], section 8).

From now on, we shall assume that the twisting element $\mathcal{F}$ is invertible and satisfies the conditions (C.2), (C.9) and (C.10).

C.2. Counterparts of matrix relations

5. We turn now to the Hopf algebraic meaning of relations from Sections 3.2, 3.3.

The square of the antipode in an almost cocommutative Hopf algebra, with a universal R-matrix $\mathcal{R}=a\otimes b$, satisfies the property $S^2(x)=u_{\mathcal{R}}x(u_{\mathcal{R}})^{-1}$, where $u_{\mathcal{R}}=S(b)a$, for any element $x\in\mathcal{A}$. In a matrix representation of an algebra $\mathcal{A}$, the element $u_{\mathcal{R}}$ maps to the matrix $D_{\hat{\rho}(\mathcal{R})}$ (and the element $S(u_{\mathcal{R}})$ maps to the matrix $C_{\hat{\rho}(\mathcal{R})}$), so an identity (which follows from the relation (C.16))

$$\begin{array}{rcl}(1\otimes u_{\mathcal{R}})\,\mathcal{F}^{-1}\,(1\otimes(u_{\mathcal{R}})^{-1}) & \equiv & (1\otimes u_{\mathcal{R}})(S(\alpha)\otimes\beta)(1\otimes u_{\mathcal{R}})^{-1}=S(\alpha)\otimes S^2(\beta)\\ & = & \alpha\otimes S(\beta)\equiv\psi_{\mathcal{F}}\end{array}$$

becomes one of the relations from Lemma 3.3. In a similar manner, one can interpret other relations from Lemma 3.3.

Such an interpretation is not, however, unique. For instance, applying $m_{12}\circ S_2$ to the relation (C.13) and using the formula (C.14), one finds

$$v_{\mathcal{F}}\otimes 1=\alpha' v_{\mathcal{F}}S(\alpha)\otimes\beta\beta'\ ,$$

which, after an application of S_2, becomes, due to the formulas (C.15) and (C.16),

$$v_{\mathcal{F}}\otimes 1=\psi_{\mathcal{F}}\,(v_{\mathcal{F}}\otimes 1)\,\mathcal{F}\ . \tag{C.20}$$

Similarly, applying $(\mathrm{id}\otimes S)\circ m_{23}\circ\tau_{23}\circ S_3$ (where τ is the flip, $\tau(x\otimes y)=y\otimes x$) to Eq. (C.13) and using Eqs. (C.16) and (C.17), one finds

$$1\otimes v_{\mathcal{F}}=\alpha\alpha'\otimes S(\beta')v_{\mathcal{F}}S^2(\beta)\ ,$$

which, after an application of S_1, becomes, with the help of Eq. (C.16),

$$1\otimes v_{\mathcal{F}}=\mathcal{F}\,(1\otimes v_{\mathcal{F}})\,\psi_{\mathcal{F}}\ . \tag{C.21}$$

In the matrix picture, the relations (C.20) and (C.21) are also equivalent to particular cases of the relations from Lemma 3.3 – but this time we did not use the fact that the square of the antipode is given by the conjugation by the element $u_{\mathcal{R}}$.

Below we shall make use of another version of the formulas (C.20) and (C.21).

Writing the formulas (C.20) and (C.21) as $(v_{\mathcal{F}}\otimes 1)\mathcal{F}^{-1}=\psi_{\mathcal{F}}(v_{\mathcal{F}}\otimes 1)$ and $\mathcal{F}^{-1}(1\otimes v_{\mathcal{F}})=(1\otimes v_{\mathcal{F}})\psi_{\mathcal{F}}$, respectively, and using the expressions for $\psi_{\mathcal{F}}$, $(\psi_{\mathcal{F}})^{-1}$ and $\mathcal{F}^{-1}$ from Lemma C.5, we find, in components:

$$v_{\mathcal{F}}S(\alpha)\otimes\beta=\alpha v_{\mathcal{F}}\otimes S(\beta) \tag{C.22}$$

and, respectively,

$$S(\alpha)\otimes\beta v_{\mathcal{F}}=\alpha\otimes v_{\mathcal{F}}S(\beta)\ . \tag{C.23}$$

Applying S_1 or S_2 to Eqs. (C.22) and (C.23), we obtain corresponding formulas with $v_{\mathcal{F}}$ replaced by $S(v_{\mathcal{F}})$. These formulas, together with Eqs. (C.22) and (C.23), imply

$$\begin{array}{rcl}\mathcal{F}\cdot(v_{\mathcal{F}}S(v_{\mathcal{F}})\otimes 1) & = & (v_{\mathcal{F}}S(v_{\mathcal{F}})\otimes 1)\cdot\mathcal{F}\ ,\\ \mathcal{F}\cdot(1\otimes v_{\mathcal{F}}S(v_{\mathcal{F}})) & = & (1\otimes v_{\mathcal{F}}S(v_{\mathcal{F}}))\cdot\mathcal{F}\ .\end{array} \tag{C.24}$$

It follows, from a compatibility of the relations (C.20) and (C.21) (express the element $\psi_{\mathcal{F}}$ in terms of $\mathcal{F}$ and $v_{\mathcal{F}}$ in two ways), that

$$\mathcal{F}_{12}\cdot(v_{\mathcal{F}}\otimes v_{\mathcal{F}})=(v_{\mathcal{F}}\otimes v_{\mathcal{F}})\cdot\mathcal{F}_{12}\ . \tag{C.25}$$

The relations (C.24) and (C.25) are universal analogues of the matrix equalities (3.19) and (3.18) (for certain choices of the compatible pairs of the R-matrices) from Corollary 3.4.

6. We need some more information about the element $v_{\mathcal{F}}$. The inverse to the element $v_{\mathcal{F}}$ is given by the formula (C.4); it follows from Lemma C.5 that $(v_{\mathcal{F}})^{-1} = S^2(\alpha)\beta$.

By Eq. (C.16), one has $S(v_{\mathcal{F}}) = S(\beta)\alpha$ and, then, $S^2(v_{\mathcal{F}}) = v_{\mathcal{F}}$. Since $S^2(x) = u_{\mathcal{R}}x(u_{\mathcal{R}})^{-1}$ for any element $x \in \mathcal{A}$, we conclude that the element $u_{\mathcal{R}}$ commutes with the element $v_{\mathcal{F}}$ and, similarly, with the element $S(v_{\mathcal{F}})$.

Making the flip in the relations (C.22) and (C.23), multiplying them out and comparing, we find that the elements $v_{\mathcal{F}}$ and $S(v_{\mathcal{F}})$ commute.

Remark C.7. In fact, more is true. Applying $\mathrm{id}\otimes S^j$ to the relation (C.22), we obtain $v_{\mathcal{F}}\alpha \otimes S^{j-1}(\beta) = \alpha v_{\mathcal{F}} \otimes S^{j+1}(\beta)$ (we used the relation (C.16) to rearrange the powers of the antipode). In a similar way, applying $S^{-j}\otimes \mathrm{id}$ to the relation (C.23), we obtain $\alpha \otimes S^{j-1}(\beta)v_{\mathcal{F}} = \alpha \otimes v_{\mathcal{F}}S^{j+1}(\beta)$. Multiplying out and comparing the right hand sides, we find that the element $v_{\mathcal{F}}$ commutes with the elements $S^k(\alpha)\beta\ \forall\ k \in \mathbb{Z}$.

The same procedure, applied to the flipped versions of the relations (C.22) and (C.23) shows that the element $v_{\mathcal{F}}$ commutes with the elements $S^k(\beta)\alpha\ \forall\ k \in \mathbb{Z}$.

Applying the antipode to these commutativity relations, we find that the element $S(v_{\mathcal{F}})$ commutes with the elements $S^k(\alpha)\beta$ and $S^k(\beta)\alpha\ \forall\ k \in \mathbb{Z}$ as well.

7. We shall now establish a Hopf algebraic counterpart of the relation (3.24).

There is a closed formula for the coproduct of the element $v_{\mathcal{F}}$, again similar to the standard formula for the coproduct of the element $u_{\mathcal{R}}$.

Lemma C.8. *One has*

$$\Delta(v_{\mathcal{F}}) = \mathcal{F}_{12}^{-1}\,\mathcal{F}_{21}^{-1}\cdot(v_{\mathcal{F}}\otimes v_{\mathcal{F}})\ . \tag{C.26}$$

Proof. Together, Eqs. (C.9) and (C.10) imply

$$(\Delta\otimes\Delta)(\mathcal{F}) = \mathcal{F}_{14}\mathcal{F}_{13}\mathcal{F}_{24}\mathcal{F}_{23}\ .$$

Therefore, the coproduct of $v_{\mathcal{F}}$ can be written in a form

$$\Delta(v_{\mathcal{F}}) = \alpha_{(1)}S(\beta_{(2)})\otimes\alpha_{(2)}S(\beta_{(1)}) = \alpha\alpha' S(\beta\beta'')\otimes\alpha'' v_{\mathcal{F}}S(\beta') \tag{C.27}$$

(we use the Sweedler notation for the coproduct, $\Delta(x) = x_{(1)}\otimes x_{(2)}$ for an element $x\in\mathcal{A}$).

Using the relation (C.23), we continue to rewrite the expression (C.27):

$$\Delta(v_{\mathcal{F}}) = \alpha S(\alpha')S(\beta\beta'')\otimes\alpha''\beta' v_{\mathcal{F}}\ . \tag{C.28}$$

The relation (C.13), in a form $\mathcal{F}_{13}\mathcal{F}_{23}\mathcal{F}_{12}^{-1} = \mathcal{F}_{12}^{-1}\mathcal{F}_{23}\mathcal{F}_{13}$, reads, in components,

$$\alpha S(\alpha')\otimes\alpha''\beta'\otimes\beta\beta'' = S(\alpha)\alpha''\otimes\beta\alpha'\otimes\beta'\beta''\ . \tag{C.29}$$

Using Eq. (C.29), we transform the right hand side of Eq. (C.28) to a form

$$\Delta(v_{\mathcal{F}}) = S(\alpha)\alpha'' S(\beta'')S(\beta')\otimes\beta\alpha' v_{\mathcal{F}} = S(\alpha)v_{\mathcal{F}}S(\beta')\otimes\beta\alpha' v_{\mathcal{F}}\ .$$

Using again Eq. (C.23), we obtain

$$\Delta(v_{\mathcal{F}}) = S(\alpha)\beta' v_{\mathcal{F}}\otimes\beta S(\alpha')v_{\mathcal{F}}\ ,$$

which, by the formula (C.14), is a component form of the relation (C.26). ■

Applying the flip to the relation (C.26), we find $\Delta^{\mathrm{op}}(v_{\mathcal{F}}) = \mathcal{F}_{21}^{-1}\,\mathcal{F}_{12}^{-1}\cdot(v_{\mathcal{F}}\otimes v_{\mathcal{F}})$. Since $\Delta^{\mathrm{op}}(v_{\mathcal{F}})\,\mathcal{R} = \mathcal{R}\,\Delta(v_{\mathcal{F}})$, we conclude

$$(\mathcal{R}_{\mathcal{F}})_{\mathcal{F}_{21}}\,(v_{\mathcal{F}}\otimes v_{\mathcal{F}}) = (v_{\mathcal{F}}\otimes v_{\mathcal{F}})\,\mathcal{R}\ . \tag{C.30}$$

The translation of the equality (C.30) into the matrix language is equivalent to the relation (3.24) (see the Remarks C.2 and C.3).

Remark C.9. It follows from the relation (C.26) that

$$\Delta(S(v_{\mathcal{F}})) = (S(v_{\mathcal{F}})\otimes S(v_{\mathcal{F}}))\cdot\mathcal{F}_{12}^{-1}\,\mathcal{F}_{21}^{-1}\ . \tag{C.31}$$

The relation (C.25), together with the relations (C.26) and (C.31), implies that an element

$$\varphi := v_{\mathcal{F}}S(v_{\mathcal{F}})^{-1} \tag{C.32}$$

is group-like, $\Delta(\varphi) = \varphi\otimes\varphi$. Therefore, $S(\varphi) = \varphi^{-1} = S(v_{\mathcal{F}})(v_{\mathcal{F}})^{-1}$ but $S(\varphi) = S(v_{\mathcal{F}}S(v_{\mathcal{F}})^{-1}) = (v_{\mathcal{F}})^{-1}S(v_{\mathcal{F}})$, which shows again that $v_{\mathcal{F}}$ commutes with $S(v_{\mathcal{F}})$.

8. The twisted Hopf algebra $\mathcal{A}_{\mathcal{F}}$ is quasitriangular, so we can write the usual identities for its universal R-matrix $\mathcal{R}_{\mathcal{F}}=\mathcal{F}_{21}\mathcal{R}\mathcal{F}^{-1}$. The relations from Proposition 3.6 are the matrix counterparts of some of these identities.

For the twisted Hopf algebra $\mathcal{A}_{\mathcal{F}}$, one finds, with the help of the first relation in Eq. (C.5), that $u_{(\mathcal{R}_{\mathcal{F}})}=\varphi\, u_{\mathcal{R}}$, where the element φ is defined by the formula (C.32) (on the matrix level, this becomes one of the relations (3.29)). In particular,

$$(S_{\mathcal{F}})^2\,(x)=\varphi\; S^2(x)\;\varphi^{-1}\ . \tag{C.33}$$

(i) The relation (3.26) is a consequence of, for example, the identity

$$(\mathrm{id}\otimes S_{\mathcal{F}})((\mathcal{R}_{\mathcal{F}})^{-1})=\mathcal{R}_{\mathcal{F}}\ . \tag{C.34}$$

We have

$$\begin{aligned}\mathcal{R}_{\mathcal{F}}\ &=\ (\mathrm{id}\otimes S_{\mathcal{F}})((\mathcal{R}_{\mathcal{F}})^{-1})=(\mathrm{id}\otimes S_{\mathcal{F}})(\mathcal{F}\mathcal{R}^{-1}\mathcal{F}_{21}^{-1})=(\mathrm{id}\otimes S_{\mathcal{F}})(\alpha S(a)\beta'\otimes\beta bS(\alpha'))\\ &=\ \alpha S(a)\beta'\otimes v_{\mathcal{F}}S^2(\alpha')S(b)S(\beta)(v_{\mathcal{F}})^{-1}=\alpha a\beta'\otimes v_{\mathcal{F}}S^2(\alpha')bS(\beta)(v_{\mathcal{F}})^{-1}\ .\end{aligned} \tag{C.35}$$

Here we used Eq. (C.8) and the identities from Lemma C.5 for $\mathcal{F}$ and $\mathcal{R}$. Applying $S^2\otimes S$ to Eq. (C.22), we find

$$v_{\mathcal{F}}S^2(\alpha)\otimes\beta=\alpha v_{\mathcal{F}}\otimes\beta\ , \tag{C.36}$$

since $S^2(v_{\mathcal{F}})=v_{\mathcal{F}}$. Using the relation (C.36) and the relation (C.23) in a form $S(\alpha)\otimes(v_{\mathcal{F}})^{-1}\beta=\alpha\otimes S(\beta)(v_{\mathcal{F}})^{-1}$, we rewrite the last expression in Eq. (C.35):

$$\mathcal{R}_{\mathcal{F}}=S(\alpha)a\beta'\otimes\alpha' v_{\mathcal{F}}b(v_{\mathcal{F}})^{-1}\beta$$

or

$$\mathcal{R}_{\mathcal{F}}=\mathcal{F}_{21}\odot\Big((1\otimes v_{\mathcal{F}})\mathcal{R}(1\otimes v_{\mathcal{F}})^{-1}\Big)\odot\mathcal{F}^{-1}\ , \tag{C.37}$$

which, on the matrix level, is equivalent to the relation (3.26).

(ii) Next,

$$\begin{aligned}\psi_{(\mathcal{R}_{\mathcal{F}})}\ &=\ (\mathrm{id}\otimes S_{\mathcal{F}})(\mathcal{R}_{\mathcal{F}})=(\mathrm{id}\otimes S_{\mathcal{F}})(\mathcal{F}_{21}\mathcal{R}\mathcal{F}^{-1})=(\mathrm{id}\otimes S_{\mathcal{F}})(\beta aS(\alpha')\otimes\alpha b\beta')\\ &=\ \beta aS(\alpha')\otimes v_{\mathcal{F}}S(\beta')S(b)S(\alpha)(v_{\mathcal{F}})^{-1}=\beta a\alpha'\otimes v_{\mathcal{F}}\beta'S(b)S(\alpha)(v_{\mathcal{F}})^{-1}\end{aligned}$$

or

$$(1\otimes v_{\mathcal{F}})^{-1}\ \psi_{(\mathcal{R}_{\mathcal{F}})}\,(1\otimes v_{\mathcal{F}})=\mathcal{F}\odot\psi_{\mathcal{R}}\odot\mathcal{F}_{21}^{-1}\ , \tag{C.38}$$

which, on the matrix level, is equivalent to the relation (3.27).

(iii) To obtain another formula for $\psi_{(\mathcal{R}_{\mathcal{F}})}$, we start with the identity $\psi_{(\mathcal{R}_{\mathcal{F}})}=(\mathrm{id}\otimes(S_{\mathcal{F}})^2)((\mathcal{R}_{\mathcal{F}})^{-1})$, which is a direct consequence of the identities from Lemma C.5:

$$\begin{aligned}\psi_{(\mathcal{R}_{\mathcal{F}})}\ &=\ (\mathrm{id}\otimes(S_{\mathcal{F}})^2)(\mathcal{F}\mathcal{R}^{-1}\mathcal{F}_{21}^{-1})=(\mathrm{id}\otimes(S_{\mathcal{F}})^2)\Big(\alpha S(a)\beta'\otimes\beta bS(\alpha')\Big)\\ &=\ \alpha S(a)\beta'\otimes\varphi S^2(\beta)S^2(b)S^3(\alpha')\varphi^{-1}=\alpha a\beta'\otimes\varphi S^2(\beta)S(b)S^3(\alpha')\varphi^{-1}\\ &=\ \alpha a\beta'\otimes S(v_{\mathcal{F}})^{-1}\beta v_{\mathcal{F}}S(b)(v_{\mathcal{F}})^{-1}S(\alpha')S(v_{\mathcal{F}})\ .\end{aligned} \tag{C.39}$$

Here we used the identities from Lemma C.5, relations $\alpha\otimes v_{\mathcal{F}}S^2(\beta)=\alpha\otimes\beta v_{\mathcal{F}}$ and $S^3(\alpha)(v_{\mathcal{F}})^{-1}\otimes\beta=(v_{\mathcal{F}})^{-1}S(\alpha)\otimes\beta$, which follow from Eqs. (C.22) and (C.23), and the formula (C.33) for the square of the twisted antipode.

Eq. (C.39) can be rewritten as

$$\big(1\otimes S(v_{\mathcal{F}})\big)\ \psi_{(\mathcal{R}_{\mathcal{F}})}\ \big(1\otimes S(v_{\mathcal{F}})^{-1}\big)=\mathcal{F}\,(1\otimes v_{\mathcal{F}})\ \psi_{\mathcal{R}}\ (1\otimes(v_{\mathcal{F}})^{-1})\ \mathcal{F}_{21}^{-1}\ , \tag{C.40}$$

which, in the matrix picture, is equivalent to Eq. (3.28).

(iv) The property $(S_{\mathcal{F}}\otimes S_{\mathcal{F}})(\mathcal{R}_{\mathcal{F}})=\mathcal{R}_{\mathcal{F}}$ leads to

$$(v_{\mathcal{F}}\otimes v_{\mathcal{F}})\ \mathcal{F}^{-1}\ \mathcal{R}\ \mathcal{F}_{21}=\mathcal{F}_{21}\ \mathcal{R}\ \mathcal{F}^{-1}\ (v_{\mathcal{F}}\otimes v_{\mathcal{F}})\ . \tag{C.41}$$

Since the twisting element $\mathcal{F}$ commutes with $v_{\mathcal{F}}\otimes v_{\mathcal{F}}$, the formula (C.41) is another manifestation of the relation (3.24).

Remark C.10. We conclude this appendix with several more properties of the group-like element φ defined in Eq. (C.32).

We have

$$\mathcal{R}\cdot(\varphi\otimes\varphi)=(\varphi\otimes\varphi)\cdot\mathcal{R}\ . \tag{C.42}$$

To see this, apply $S\otimes S$ to the relation (C.30) and then compare with the same relation (C.30).

The matrix equivalent of the relation (C.42) is the relation (3.25).

Recall that a quasitriangular Hopf algebra $\mathcal{A}$ is called a ribbon Hopf algebra if it contains a ribbon element r, that is, a central element such that $r^2=u_{\mathcal{R}}S(u_{\mathcal{R}})$ and $\Delta(r)=\mathcal{R}_{12}^{-1}\mathcal{R}_{21}^{-1}\cdot(r\otimes r)$ (see [46], or [4], the chapter 4). The twisted algebra $\mathcal{A}_{\mathcal{F}}$ is a ribbon Hopf algebra if the algebra $\mathcal{A}$ is; for the ribbon element of the algebra $\mathcal{A}_{\mathcal{F}}$, one can choose $r_{\mathcal{F}}=\varphi r$.

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