Chaos, Solitons and Fractals 143 (2021) 110550

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

Solitons of Whitham equation with resonance dispersion

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ARTICLE INFO

Article history:
Received 29 September 2020
Revised 18 November 2020
Accepted 4 December 2020

Keywords:
Whitham equation
Solitons
Velocity spectra

ABSTRACT

Localized stationary solutions of nonlinear nonlocal Whitham equation with resonance dispersion relation are considered. The existence of exponentially localized smooth and singular solitons, bound states of the solitons and localized solutions with oscillating asymptotics is recognized. The velocity spectra of solitons, in contrast to all other known Whitham equations, appear to be discrete. Asymptotic "quantization rules" for calculation of discrete spectra are obtained.

1. Introduction

Nonlinear nonlocal Whitham equation

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} = \frac{\partial}{\partial x}\int_{-\infty}^{\infty} dx' R(x-x')V(x',t) \quad (1)$$

represents a wide class of equations which are of great interest for nonlinear wave theory. It combines the typical hydrodynamic nonlinearity and an integral term descriptive of dispersion of the linear theory. The dispersion relation for linearized Eq. (1) is defined by the Fourier transform of the kernel $R(x)$

$$\omega = k\widetilde{R}(k), \quad \widetilde{R}(k) = \int_{-\infty}^{\infty} dx R(x) e^{ikx}. \quad (2)$$

A number of special cases of Eq. (1) were examined in detail. Among them are the Benjamin-Ono [1] and Joseph [2] equations describing internal waves in stratified fluids of infinite and finite depth. These equations appeared to be integrable by inverse scattering technique and the behavior of their solutions has been studied well. The Benjamin-Ono and Joseph equations are however the only representatives of the Whitham equations possessing this property [3]. Another widely known equations of that class are studied not so exhaustively although the literature on the subject is quite extensive [4]. A list of well-studied equations involves, in particular, the Leibovitz one for the waves in rotating fluid [5], the Klimontovich equation for magnetohydrodynamic waves in non-isothermal collision-less plasma [6] and equations for shallow water waves [7].

More or less comprehensive review of recent works cannot be presented within the limits of a short communication. It should be however noted that the interest in the Whitham equations is still great since the use of relatively simple equations of the form Eq. (1) turns out to be very fruitful for various physical applications. New representatives of that equation have recently been proposed for capillary [8] and hydroelastic [9] waves, shallow water waves with vorticity [10], acoustic waves in a medium with strong spatial dispersion [11]. A great number of works is devoted to the longstanding problem of formation and stability of traveling waves due the interplay between nonlinear and dispersive effects (see, for example, [12] and references herein)

The characteristic feature of Whitham equations is the existence of solitons. The velocity spectra of solitons can be bounded or not but for all known cases the spectra are continuous. This raises the question of whether or not the continuous spectra are the common property of solitons of Whitham equations? Here we examine the particular case of Whitham equation with resonance dispersion with special emphasis on the spectra of solitons.

We consider solitary wave solutions of Whitham equation with

$$\widetilde{R}(k) = \frac{1}{1-k^2+D^2k^4}. \quad (3)$$

That equation has been proposed for the waves in compressible fluid in thin elastic tube with slip boundary condition on the tube wall. Such model is justified in the case of liophobic surface of the wall for liquids or extremely small accommodation coefficient for

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https://doi.org/10.1016/j.chaos.2020.110550
0960-0779/

gases. Dimensionless parameter in Eq. (3) is equal

$$D^2 = \frac{E_x h^2}{12(1-\nu^2)\rho_S^2 a^2 c_0^4},$$

where E_x, ρ_s, ν are Young modulus, density and Poisson ratio of the tube material, h and a are thickness and radius of the tube, c_0 is sound speed of the fluid [13]. In addition, with small D^2 including $D^2 = 0$ that equation is applicable to the specific case of waves in a medium with internal oscillators, that is with an oscillatory response of internal degrees of freedom to the disturbance of pressure [14].

It should be noted that the form of the integral term in Eq. (1) depends on the poles location of Fourier transform of the kernel. While $D^2 < 1/4$ the poles are located in the real axis so that the causative-type dispersion takes place, $R(x<0)=0$. Otherwise, the poles are located symmetrically with respect to $\mathrm{Im}(k) = 0, R(-x) = R(x)$, and dispersion therefore takes a spatial form. The spatial dispersion occurs when the sound velocity of a fluid is greater than the minimum phase velocity of bending oscillations of the shell. This is caused by the origination of radiation physically similar to Cherenkov one.

The consideration below is restricted to the specific properties of solitary wave solutions of Eqs. (1)–(3). It is shown that the equation possesses exponentially localized smooth and singularity involving solutions, and solutions with oscillating asymptotics. The spectrum of velocities of exponentially localized solutions appears therewith to be a discrete one. Asymptotic “quantization rules” suitable for $D^2 \ll 1$ are obtained.

2. Localized traveling waves

Hereafter we consider traveling wave solutions $V(y)$ depending on the single variable $y = x + \lambda t$. For such solutions the left hand side of Eq. (1) is the derivative in y of the function $S = \lambda V + \frac{1}{2}V^2$. The inversion of integral operator in Eq. (1), with the account of the form Eq. (3) for $\tilde{R}$, then gives

$$D^2\frac{d^4S}{dy^4} + \frac{d^2S}{dy^2} + S = V, \quad S = \lambda V + \frac{1}{2}V^2 \tag{4}$$

Real solutions $V(S)$ exist for $\lambda^2 + 2S \geq 0$, and there function $V(S)$ is two-valued, $V = -\lambda \pm \sqrt{\lambda^2 + 2S}$. Eq. (4) with boundary conditions of vanishing of function $V(y)$ and all its derivatives as $y \to \pm\infty$ presents a nonlinear eigenvalue problem for parameter λ. The first integral of Eq. (4) has the form

$$D^2\left[2\frac{dS}{dy}\cdot\frac{d^3S}{dy^3} - \left(\frac{d^2S}{dy^2}\right)^2\right] + \left(\frac{dS}{dy}\right)^2 + S^2 - V^2\left(\lambda + \frac{2}{3}V\right) = E. \tag{5}$$

Eq. (4) can be transformed to a 2 degrees of freedom reversible Hamiltonian system with equilibria whose homoclinic orbits just correspond to solitons of the equation. To show this we introduce new variables $Q_1 = S, P_1 = -dS/dy - Dd^3S/dy^3, Q_2 = dS/dy, P_2 = Dd^2S/dy^2$. Then Eq. (4) is reduced to the Hamiltonian system

$$\frac{dQ_1}{dy} = Q_2, \ \frac{dP_1}{dy} = Q_1 - V(Q_1),$$
$$D\frac{dQ_2}{dy} = P_2, \ D\frac{dP_2}{dy} = -Q_2 - P_1. \tag{6}$$

with the Hamiltonian $H = Q_2P_1 - Q_1^2/2 + (Q_2^2 + P_2^2)/2 + \int V(Q_1)dQ_1$ and symplectic form $\Omega = dP_1 \wedge dQ_1 + DdP_2 \wedge dQ_2$. For D^2 small the related Hamiltonian system is slow-fast with one slow and one fast degrees of freedom [15]. A detailed study of this system will be performed elsewhere.

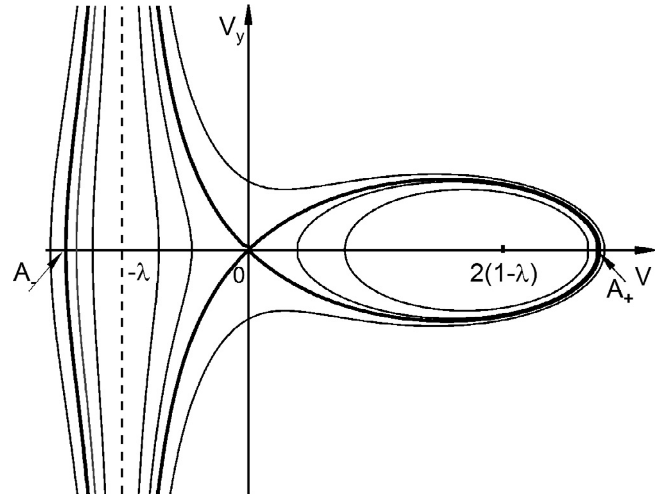

Fig. 1. Slow dynamics: $D^2 = 0$.

Consider singular limit $D^2 = 0$. This corresponds to the slow subsystem of the above Hamiltonian system. The set of integral curves of Eq. (5) for $0 \leq \lambda \leq 1$, where localized solutions exist, has the form shown by Fig. 1 and as is easily seen, there exist two types of localized solutions. The first one is presented by the smooth loop in the region $V > 0$ whereas the second corresponds to the discontinuous loop in the region $V < 0$. For brevity sake we will call them as solitons and cavitons, respectively. The singularity in the caviton profile is described by the relations $V + \lambda \sim \sqrt{|y|}, S + \lambda^2/2 \sim |y|$.

The amplitudes of solitons and cavitons are defined by the relation

$$A_\pm = \frac{2}{3}(2-3\lambda) \pm \frac{2}{3}\sqrt{4-3\lambda}. \tag{7}$$

As would be expected, in the limit $\lambda \to 1$ the solitons are turned to the well known KdV-ones

$$V = \frac{3(1-\lambda)}{\cosh^2\left(\sqrt{1-\lambda}\, y/2\right)}. \tag{8}$$

In the opposite limit $\lambda \to 0$ soliton acquires simple finite form

$$V = \frac{8}{3}\cos^2\frac{y}{4}, \quad |y| \leq 2\pi. \tag{9}$$

We now turn to the general case of Eq. (4) with $D^2 > 0$. Qualitative inspection shows that the localized solutions can exist in the following cases:

1. $D^2 > 0, \lambda \in (0, 1)$
2. $0 < D^2 < \frac{1}{4}, \lambda \in \left(-\frac{4D^2}{1-4D^2}, 0\right)$
3. $D^2 > \frac{1}{4}, \lambda \in (-\infty, 0) \cup \left(\frac{4D^2}{4D^2-1}, \infty\right)$

In the former case, for the related Hamiltonian system the origin is the saddle-center equilibrium (its eigenvalues are a pair of pure imaginary and a pair real nonzero ones) and the localized solutions, if any, share an exponential asymptotics as $|y| \to \infty$; in the other cases asymptotics possesses an oscillating character, since the related equilibrium is a saddle-focus (with a quadruple of complex eigenvalues). The integral curves of Eq. (4) pertain to $3D$ phase space determined by the first integral Eq. (5).

Hereafter we concentrate upon the exponentially localized solutions. With $D^2 = 0, \lambda \in (0, 1)$ the homoclinic orbits exist for any value of λ; the velocity spectra of solitons and cavitons are bounded but continuous. With $D^2 > 0$ the situation changes: the velocity spectra should generally be discrete due to the splitting

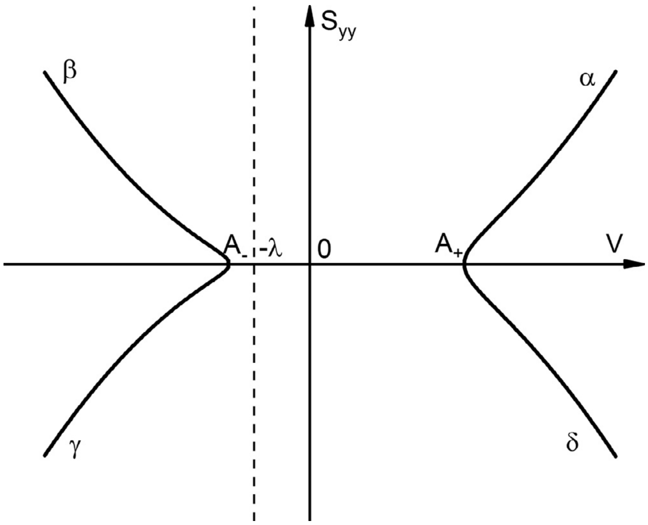

Fig. 2. Nodal lines in the plane of symmetry $dV/dy = 0$, $E = 0$.

of the separatrices. Indeed, in three-dimensional space defined by the equality $E = 0$ the existence of a separtrix loop for the saddle-center equilibrium means a coincidence of its one unstable and one stable one-dimensional separatrices corresponding to real eigenvalues. As λ varies, this closed separatrix loop is usually destroyed. In other words, for $D^2 > 0$, the continuous spectra should generally decay.

Qualitative inspection enables one to make some conclusions on the behavior of integral curves to Eq. (5) at $E = 0$. As is seen from Eq. (5), the integral curves cross the symmetry plane $E = 0$, $dV/dy = 0$ only along the lines of nodal points. At other points, the component of the tangent vector of integral curves normal to that plane vanishes since the derivative d^3S/dy^3 tends to infinity at $dV/dy = 0$ (see Eq. (5)). It should be emphasized that all amplitude-bounded solutions, both symmetric and asymmetric ones must intersect the symmetry plane since the derivative dV/dy changes the sign on this plane.

The form of these nodal lines at $E = 0$ is defined by relation

$$D^2\left(\frac{d^2S}{dy^2}\right)^2 = -\frac{V^2}{4}(A_+ - V)(V - A_-) \tag{10}$$

where the constants are given by Eq. (7). The geometry of these lines is presented by Fig. 2.

In the vicinity of arbitrary singular point V_* the set of integral curves is presented by the expansion

$$V - V_* = \pm\frac{V_*}{4D(V_* + \lambda)}\sqrt{(V_* - A_+)(V_* - A_-)}y^2 + cy^3 + \cdots .$$

Among the integral curves entering the point V_* only one ($c = 0$) enters along the normal to the plane and pertains to symmetry solution. As Fig. 2 suggests, all local extremums of function $V(y)$ are greater than A_+ or lesser than A_-. In particular, the amplitudes of solitons and cavitons are greater than A_+ and $|A_-|$, respectively.

The integral curve starting from the origin to half-space $V > 0$ ($V < 0$) cross the symmetry plane for the first time only in the point of branch $\delta(\beta)$ shown by Fig. 2. Exponentially localized solutions are conveniently classified by the sequence of intersections of the symmetry plane. We denote each intersection by the symbol of branch; the symbol O is attributed to passing near the origin. Then the sequences "$O\delta O$" and "$O\beta O$" correspond respectively to soliton and caviton solutions. Another exponentially localized solutions can be interpreted as the bound states of solitons and cavitons. For example, the "$O\delta\alpha\delta O$" and "$O\delta\beta\delta O$" sequences correspond to configurations "soliton-soliton" and "soliton-caviton-soliton", etc. Some examples of localized solutions are presented by Fig. 3.

Remark 1. It is worth noting that asymmetric localized solutions like $O\delta\beta O$ can exist only for specific values of the pair of parameters (λ, D^2). Indeed, a related solution of the Hamiltonian system means an existence of a nonsymmetric orbit in the 3-dimensional level of the Hamiltonian which belongs to the 1-dimensional unstable manifold of the saddle-center, makes one passage near the solitonic homoclinic orbit of the slow manifold, then passes one time near the caviton homoclinic orbit of the slow manifold, and finally gets lie on the stable 1-dimensional manifold of the saddle-center. This means that two 1-dimensional stable and unstable curves in 3-dimensional level have to coincide, this an event that is possible only at specific values of the pair (λ, D^2).

Computational investigation shows the existence of asymmetric solutions (see picture $0\delta\beta 0$ in Fig. 3).

3. Asymptotic "quantization rules"

As noted above, the spectrum of eigenvalues at $D^2 = 0$ is continuous and coincides with $0 \le \lambda \le 1$. It is of interest to trace how the continuous spectra of λ decay at small finite values of D^2. To do that it is necessary to calculate the splitting of separatrices shown by Fig. 1 in relation to the value of λ.

To that purpose we introduce the new variable $U = d^2S/dy^2$ and write the Lagrangian of Eq. (4) as

$$L = \frac{1}{2}(\lambda + V)^2\dot{V}^2 + \frac{1}{8}V^2(A_+ - V)(V - A_-) + D^2(\lambda + V)\dot{V}\dot{U} + \frac{1}{2}D^2U^2 \tag{11}$$

where differentiation is traditionally indicated by a dot above the function. We introduce the Routhian which is Hamiltonian with respect to variables $P = \partial L/\partial\dot{V}$, V and Lagrangian with respect to U. Then, we turn from the variables P, V to the variables of the type of action-angle and choose the variable E from Eq. (5) with $D^2 = 0$ instead of the action. After that canonical transformation we obtain

$$D^2\ddot{U} + U = (1 - \lambda)V + \tfrac{1}{2}V^2, \qquad \dot{E} = D^2\left[(1 - \lambda)V + \tfrac{1}{2}V^2\right]\dot{U} \tag{12}$$

where $V(y, E)$ is defined by the integration of Eq. (5) with $D^2 = 0$ and $E = const$. The first integral of Eq. (12) takes the form

$$E = \frac{1}{2}D^2\left(D^2\dot{U}^2 + U^2\right) + const \tag{13}$$

where for the localized solutions the constant is zero, because E, U and $\dot{U}$ vanish at the origin. For homoclinic orbits $E(y = -\infty) = E(y = +\infty) = 0$. At $D^2 = 0$ the condition $E(y = +\infty) = 0$ is satisfied identically when $E(y = -\infty) = 0$. However, at $D^2 > 0$, an integral curve leaving the origin does not have to return there with an arbitrary value of λ It is easy to see that the increment of E in one bypass of separatrix loop of Eq. (4) with $D^2 = 0$ is the measure of the separatrix splitting under the action of small singular perturbation. Recall that now E is not conserved, since it is here the action-like variable (compare with [16]).

For approximate calculation of the solitons and cavitons spectra we can now use the method of separatrix mapping developed in [17]. To do that one need to put $V(y, E) = V(y, 0)$ and calculate the increment of E in bypass of soliton or caviton loop from Eqs. (12),(13). The eigenvalues of parameter λ therewith correspond to the vanishing of the increment of E. From Eqs. (12),(13) it follows that

$$\triangle E = D^2\left[\int_{-\infty}^{\infty} dyV\left(1 - \lambda + \frac{1}{2}V\right)\exp\left\{\frac{\imath y}{D}\right\}\right]^2 \tag{14}$$

where $V(y, 0)$ is the soliton or caviton Eq. (4) with $D^2 = 0$. In order to obtain the asymptotic "quantization rules"we need to calculate

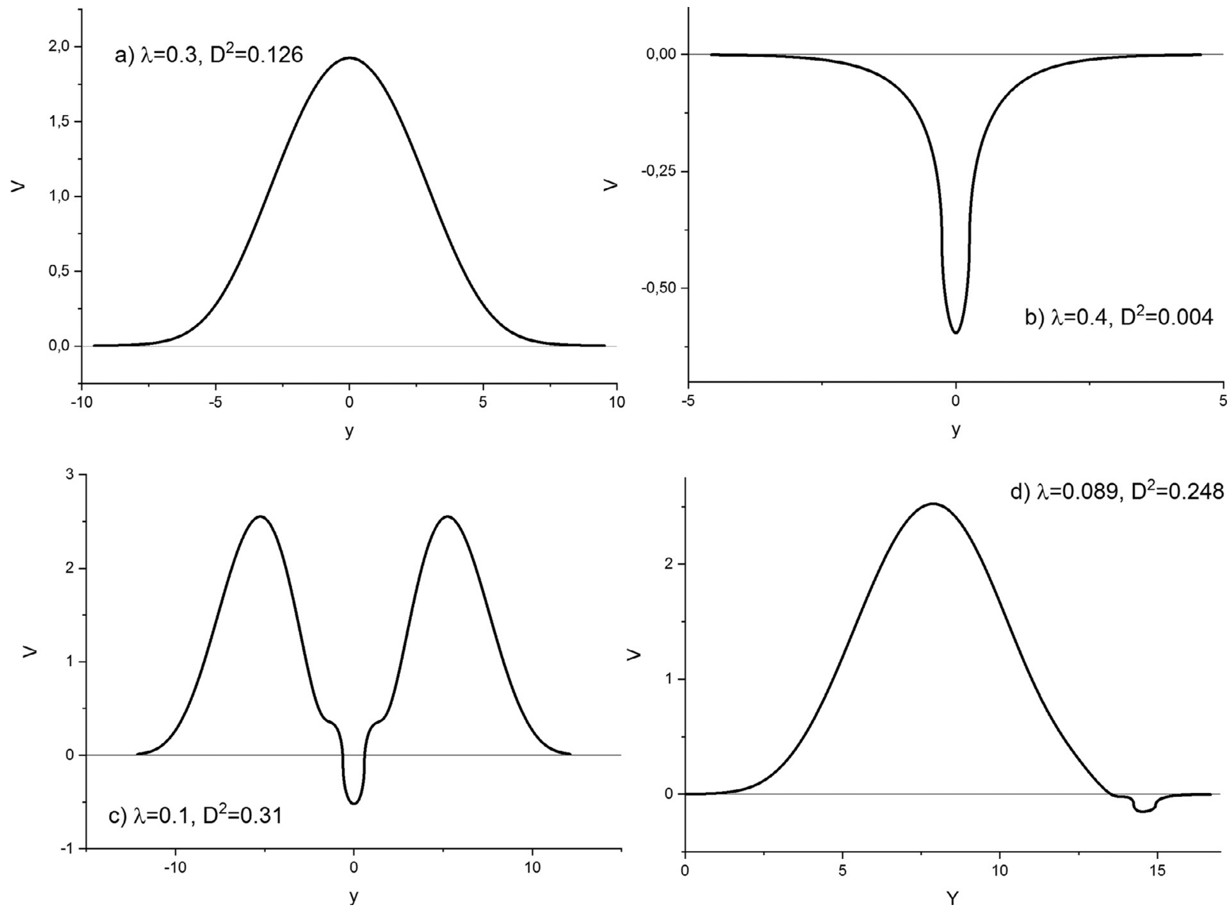

Fig. 3. Shapes of exponentially localized symmetric and asymmetric solutions.

the integral in Eq. (14) in the limit $D^2 \to 0,\ \lambda = const$. After the replacing of integration variable by V, the integral is calculated by the saddle point method for solitons and the method of stationary phase for cavitons. The result of calculation takes the form $B\sin\theta$ where

$$B \sim D^{3/2}\exp(-\pi\sqrt{\lambda/(1-\lambda)}/D)$$

for the solitons and $B \sim D^{5/2}$ for the cavitons. The spectra are defined from the evident condition $\theta = \pi n$ which can be presented in the form

$$2\left(\pi - \arcsin\frac{\sqrt{\lambda_s}}{2}\right) - \sqrt{\frac{\lambda_s}{1-\lambda_s}}\operatorname{arcsh}\sqrt{3(1-\lambda_s)} = n\pi D \qquad (15)$$

for the solitons and

$$2\arcsin\frac{\sqrt{\lambda_c}}{2} - \sqrt{\frac{\lambda_c}{1-\lambda_c}}\operatorname{arcsh}\sqrt{3(1-\lambda_c)} = n\pi D \qquad (16)$$

for the cavitons. The number of solitons and cavitons is clearly finite and proportional to $1/D$.

To elucidate the characteristics of the spectra in the vicinity of the boundaries of the interval $0 \le \lambda \le 1$ we calculate the integral in the limits $\lambda \to 0$ and $\lambda \to 1,\ D^2 = const \ll 1$. The results show the lack of eigenvalues with $\lambda_s \to 1$ and the existence of solitons with $\lambda_s = 0$ for a denumerable set of values of parameter D^2 defined by Eq. (15). As for the cavitons, they are absent with $\lambda_c \to 0$ and exist with $\lambda_c = 1$ for a denumerable set of values of D^2 defined by Eq. (16).

The above procedure closely resembles semi-classical quantization; Eqs. (15), (16) therewith correspond to Bohr-Sommerfeld quantization rule. On the other hand, this procedure reproduces semi-classical approach to the problem of transitions of a quantum system under the effect of slow time-varying perturbation [18]. However, the approximate methods as applied to the nonlinear system, are not sufficiently substantiated. It is well known (see, for instance, [19]) that for slow-fast analytic Hamiltonian systems the problem of finding separatrices after splitting is very subtle and hard. The results obtained from the calculation of only dominant term of the asymptotic expansion in the limit $D^2 \to 0$, $y \to \infty$, cannot be interpreted rigorously. Nevertheless, a comparison between computer-calculated and asymptotic spectra shows unexpectedly good agreement both for solitons and cavitons.

4. Conclusion

In summary, the set of stationary localized solutions of Whitham equation contains solitons, cavitons and bound states of the solitons and cavitons including the asymmetric ones at specific values of the pair (λ, D^2). The velocity spectra of exponentially localized solutions at $D^2 \ne 0$ are discrete, and the number of solitons and cavitons is finite. On the other hand, for any λ from the interval (0,1) there exists denumerable set of values of parameter D^2 such that λ is the eigenvalue of velocity of soliton or caviton.

The discreteness of spectra of localized solutions should significantly affect the evolution of disturbances. In particular, the adiabatic drift of soliton parameters under the influence of weak perturbation is impossible in that case.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

CRediT authorship contribution statement

A. Gevorgian: Visualization. **N. Kulagin:** Software, Investigation, Data curation. **L. Lerman:** Investigation, Supervision, Writing - review & editing. **A. Malkin:** Conceptualization, Methodology, Writing - original draft.

Acknowledgement

The work of L.M. Lerman was partially supported by the Laboratory of Dynamical systems and Applications of NRU HSE of the Ministry of Science and Higher Education of RF, grant # 075-15-2019-1931 and by Ministry of Science and Higher Education of RF (Project # 0729-2020-0036).

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