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On Realization of Gradient-like Flows on the Four-dimensional Projective-like Manifold[1]

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1. Introduction and Statement of Results

Let M^n be a smooth closed connected manifold of dimension n. Recall that a flow f^t on M^n is called *Morse-Smale* if its non-wandering set Ω_{f^t} belongs to a finite set of hyperbolic equilibrium states and closed trajectories, and invariant manifolds of different equilibrium states and closed trajectories have only transversal intersection. A Morse-Smale flow without closed trajectories is called *gradient-like*. S. Smale in [1] showed that for an arbitrary manifold M^n there exists a Morse function (a smooth function whose critical points are non-generated) defined on M^n, and it is possible to choose a metric on M^n such that the gradient flow of the Morse function will be a gradient-like flow. Hence, gradient-like flows exist on all manifolds.

Recall that the sets

$$W^s_p = \{q \in M^n\colon \lim_{t\to+\infty} f^t(q) \to p\}, W^u_p = \{q \in M^n\colon \lim_{t\to+\infty} f^{-t}(q) \to p\}$$

are called *stable and unstable manifolds of an equilibrium state* p correspondingly.

According to [2, Theorem 2.3], if there is a gradient-like flow f^t on a manifold M^n then M^n is a disjoint union of stable manifolds of all points from Ω_{f^t} and for any point $p \in \Omega_{f^t}$ its stable and unstable manifolds are smoothly embedded open balls. Dimension $dim\, W^u_p$ of the unstable manifold of the point p is called *a Morse index* of p. It follows from hyperbolicity of the point p that $dim\, W^u_p \in \{0, 1, \dots, n\}$ and $dim\, W^s_p + dim\, W^u_p = n$. An equilibrium p such that $dim\, W^u_p = 0$ ($dim\, W^u_p = n$) is called *a sink* (*a source*), and an equilibrium p such that $dim\, W^u_p \in (0, n)$ is called *a saddle* point.

It follows from the observation above that for any gradient-like flow f^t the set Ω_{f^t} contains at least one source and one sink. If the set Ω_{f^t} is exhausted by these two

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points, then the ambient manifold M^n is a sphere, and all such flows are topologically equivalent. According to [9] any gradient-like flow has an *energy function* — a Morse function decreasing along non-singular trajectories of f^t such that the set of critical points of f coincides with the set Ω_{f^t}. Then the question of an existing of gradient-like flows with non-wandering set consisting of exactly three equilibrium states is reduced to the problem of existing of Morse function with exactly three critical points. Manifolds admitting such Morse function were studied in [7]. In particular, there was proven that the dimension of these manifolds takes the values $n \in \{2, 4, 8, 16\}$ and the indices of the critical points equal $0, \frac{n}{2}, n$. For $n = 2$ this manifold is the projective plane.

Gradient-like flows with non-wandering set consisting of exactly three points were studied in [3], [4]. In these papers manifolds admitting such flows were called *projective-like manifolds*. It was also proved that for $n = 4$ all flows on a projective-like manifold which non-wandering set consists of exactly three hyperbolic equilibrium states are topologically equivalent. Hence, all four-dimensional projective-like manifolds are homeomorphic. This fact is not true in case $n > 4$, since, due to [7], in each dimension $8, 16$ there exist projective-like manifolds with different homotopy types.

In this paper, we do the first step to solution of a problem of topological classification of gradient-like flows on projective-like manifolds with arbitrary number of equilibria. Namely, we study a structure of a non-wandering set of gradient-like flows on a projective-like manifold of dimension four and provide an algorithm of a realization of such flows for given number of equilibria of different Morse indices.

For gradient-like flow f^t on a four-dimensional manifold denote by l_{f^t} the number of sink and source equilibrium states, by h_{f^t} — the number of saddle equilibrium states of Morse index two, and by k_{f^t} the number of saddle equilibrium states of Morse indices one and three.

Main results of the paper are following.

Theorem 1. *Let f^t be a gradient-like flow on the four-dimensional projective-like manifold M^4. Then $l_{f^t} - k_{f^t} + h_{f^t} = 3$. If for any two different saddle equilibria $p, q \in \Omega_{f^t}$ the intersection $W^s_p \cap W^u_q$ is empty then $h_{f^t} = 1$.*

Theorem 2. *Let $l \geq 2$, $k \neq 0$, $h \geq 1$ be integers such that $l - k + h = 3$. Then there is a gradient-like flow f^t on the four-dimensional projective-like manifold such that $l_{f^t} = l, k_{f^t} = k, h_{f^t} = h$.*

2. The Structure of non-wandering set of gradient-like flows on four-dimensional projective-like manifolds

This section is devoted to the proof of Theorem 1.

2.1. Auxiliary results

Let us recall that *a sphere* S^k is the manifold homeomorphic to the standard sphere $\mathbb{S}^k = \{(x_1, \dots, x_{k+1}) \subset \mathbb{R}^{k+1} |\ x_1^2 + \dots + x_{k+1}^2 = 1\}$, *a ball* (*an open ball*) B^n is the manifold homeomorphic to the standard ball (the interior of the standard ball) $\mathbb{B}^n = \{(x_1, \ldots, x_n) \subset \mathbb{R}^n | \; x_1^2 + \cdots + x_n^2 \leq 1\}$.

The sphere Σ^k topologically embedded in a topological manifold M^n $(1 \leq k \leq n-1)$ is called *locally flat* if for any point $z \in \Sigma^k$ there exists a neighborhood $U_z \subset M^n$ and a homeomorphism $\varphi_z \colon U_z \to \mathbb{R}^n$ such that $\varphi_z(\Sigma^k \cap U_z) = \mathbb{R}^k \subset \mathbb{R}^n$. If the sphere Σ^k is not flat at a point z, then the point z is called *the point of wildness* and the sphere Σ^k is called *wild*.

The statement below follows from [2, Theorem 2.3].

Statement 1. *Let f^t be a gradient-like flow on a closed manifold M^n. Then*

1. $M^n = \bigcup\limits_{p \in \Omega_{f^t}} W^s_p = \bigcup\limits_{p \in \Omega_{f^t}} W^u_p$;

2. *for any point $p \in \Omega_{f^t}$ the manifold W^u_p is a smooth submanifold of M^n;*

3. *for any point $p \in \Omega_{f^t}$ and any connected component l^u_p of set $W^u_p \setminus p$ the closure $cl\ l^u_p$ of l^u satisfy the equality $cl\ l^u_p \setminus (l^u_p \cup p) = \bigcup\limits_{q \in \Omega_f : W^s_q \cap l^u_p \neq \emptyset} W^u_q$.*

Item 1 of the Statement 1 and the fact that an unstable manifold of a hyperbolic equilibrium state p is a ball of dimension $ind_p \in \{0, \ldots, 4\}$ lead to the fact that the set Ω_{f^t} of any gradient-like flow f^t contains at least one source and one sink. Indeed, in the absence of sinks (or sources), a manifold M^n of dimension n would be represented as a finite union of smoothly embedded balls of smaller dimension that is impossible.

Everywhere below we suppose that f^t is a gradient-like flow on projective-like manifold M^4.

Denote by $\Omega^i_{f^t}$ the set of all equilibrium states of the flow f^t which have the dimension of the unstable manifold equal to $i \in \{0, 1, 2, 3, 4\}$ and by $|\Omega^i_{f^t}|$ the capacity of the set $|\Omega_{f^t}|$. Put $l_{f^t} = |\Omega^0_{f^t}| + |\Omega^4_{f^t}|$, $k_{f^t} = |\Omega^1_{f^t}| + |\Omega^3_{f^t}|$, and $h_{f^t} = |\Omega^2_{f^t}|$. It follows from [7] that Euler characteristic $\chi(M^4)$ of M^4 is 3. Then due to Poincare-Hopf Theorem we have

$$l_{f^t} - h_{f^t} + k_{f^t} = 3. \qquad (2.1)$$

It immediately follows from Equation (2.1) that if the set $\Omega^1_{f^t} \cup \Omega^3_{f^t}$ is empty then the set Ω_{f^t} consists of exactly three equilibrium states: a source, a sink, and a saddle with a Morse index two.

Let $p, q \in \Omega_{f^t}$ are saddle points such that $W^s_p \cap W^u_q \neq \emptyset$. Then the intersection $W^s_p \cap W^u_q$ is called *heteroclinic intersection.*

Lemma 1. *Let a flow f^t has no heteroclinic intersections, and $p \in \Omega^1_{f^t}$ $(p \in \Omega^3_{f^t})$. Then the closure $cl\, W^s_p$ $(cl\, W^u_p)$ of stable (unstable) manifolds W^s_p (W^u_p) of the point p is a locally flat sphere of dimension 3 that divides the manifold M^4 into two connected components.*

Proof. Assume that the set $\Omega^1_{f^t}$ is non-empty and prove the lemma for an arbitrary point $p \in \Omega^1_{f^t}$ (the proof for the point $p \in \Omega^{n-1}_{f^t}$ is carried out similarly). It follows from item 3 of Statement 1 that for any point $p \in \Omega^1_{f^t}$ the closure $cl\, W^s_p$ of its stable manifold W^s_p is the union of the manifold W^s_p itself and a source equilibrium state α_p. Therefore $cl\, W^s_p$ is a sphere of dimension $(n-1)$. Due to item 2 of Statement 1 the sphere $cl\, W^s_p$ is smooth (and, therefore it is locally flat) at all points of W^s_p. According to [8, St 3A.6] a sphere S^{n-1} embedded in a manifold M^n of dimension $n \geq 4$ is either locally flat at each point or has more than a countable number of wildness points[2]. Hence, $cl\, W^s_p$ is a locally flat sphere.

Let us show that the sphere $cl\, W^s_p$ divides the manifold M^4 into two connected components. Since, by virtue of [7], the fundamental group $\pi_1(M^4)$ is trivial then M^4 is orientable. By [5, Theorem 3] a locally flat sphere S^{n-1} in an orientable manifold M^n ($n \geq 3$) is cylindrically embedded, which means that there is a closed neighborhood $V \subset M^n$ of a sphere S^{n-1} and a homeomorphism $h\colon S^{n-1} \times [-1,1] \to V$ such that $h(S^{n-1} \times \{0\}) = S^{n-1}$. Therefore there is a neighborhood V_p of the sphere $cl\, W^s_p$, which is divided by the sphere $cl\, W^s_p$ into two connected components. Choose points x, y that belong to different connected components $V_p \setminus cl\, W^s_p$ and connect them with a smooth arc $l_p \subset V_p$ that intersects the sphere $cl\, W^s_p$ at the only one point. If $cl\, W^s_p$ does not divide M^4, then there is an arc $b_p \subset M^4 \setminus cl\, W^s_p$ connecting the points x, y. By construction, the intersection index of the arc $\lambda_p = l_p \cup b_p$ and the sphere $cl\, W^s_p$ is 1 or -1 (depending on the choice of orientations). On the other hand, since $\pi_{n-1}(M^4)$ is trivial, it is not difficult to choose a sphere $S^{n-1} \subset M^4 \setminus \lambda_p$, homotopic to the sphere $cl\, W^s_p$. Since the intersection index is a homotopy invariant, the intersection index of the sphere S^{n-1} and the arc λ_p must be equal ± 1, but since $S^{n-1} \cap \lambda_p = \emptyset$, it equals to zero. This contradiction proves that the sphere $cl\, W^s_p$ divides the manifold M^4 into two connected components. □

Remind that the set A is called an *attractor* of a flow f^t if there is a closed neighborhood (*a trapping neighborhood*) $V \subset M^n$ such that all trajectories of the flow f^t intersect its boundary ∂V transversally, and $A = \bigcap\limits_{t>0} f^t(V)$. The set R is called a *repeller* of the flow f^t if it is an attractor for the flow f^{-t}.

Set

$$A_{f^t} = \bigcup_{p \in \Omega^0_{f^t} \cup \Omega^1_{f^t}} W^u_p, R_{f^t} = \bigcup_{p \in \Omega^3_{f^t} \cup \Omega^2_{f^t} \cup \Omega^4_{f^t}} W^s_p$$

.

Lemma 2. *If f^t has no heteroclinic intersections then the set A_{f^t} is a connected attractor with a trapping neighborhood diffeomorphic to the ball.*

[2]In the paper [8] it is noted that this statement is a consequence of results of A. V. Chernavsky and R. Kirby obtained independently in 1968. Earlier, in 1963, J. Cantrell proved the following: if the sphere $S^{n-1} \subset S^n$, $n \geq 4$, is wild and B is a set of points such that S^{n-1} is locally flat in each point of the set $S^{n-1} \setminus B$, then the set B consists of more than one point (see [6]).

Proof. It follows from [1, 9] that there is a Morse function $\varphi\colon M^4 \to [0,4]$ such that the set of critical points of φ coincides with the set Ω_{f^t}, $\varphi(p) = ind(p)$ for any $p \in \Omega_{f^t}$, and $\varphi(f^t(x)) < \varphi(x)$ for any point $x \notin \Omega(f^t)$ and $t > 0$. Let us show that the set $V = \varphi^{-1}([0;1,5])$ is a trapping neighborhood for A_{f^t}.

It follows from the definition that $A_{f^t} \subset V$. Since A_{f^t} is invariant then $A_{f^t} \subset \bigcap\limits_{t>0} f^t(V)$. Let us prove that $A_{f^t} = \bigcap\limits_{t>0} f^t(V)$. Assume the opposite. Then there is a point $x \in \bigcap\limits_{t>0} f^t(V) \setminus A_{f^t}$. Statement 1 implies that there is an equilibrium state $p \in \Omega_{f^t}$ such that $x \in W^u_p$. Since the set $\bigcap\limits_{t>0} f^t(V)$ is closed and invariant then $p \in \bigcap\limits_{t>0} f^t(V) \setminus A_{f^t}$, which is impossible, since the set V does not contain equilibrium states other than those which belong to A_{f^t}. Therefore, $A_{f^t} = \bigcap\limits_{t>0} f^t(V)$ and A_{f^t} is an attractor.

Let us prove that the trapping neighborhood V is connected. Then A_f will be connected as the intersection of connected compact nested sets. Assume that V is disconnected, that is it can be represented as a union of two disjoint non-empty invariant subsets E_1, E_2. Then the union $\bigcup\limits_{p\in A_{f^t}} W^s_p = \bigcup\limits_{t\in\mathbb{R}} f^t(E_1 \cup E_2)$ is disconnected. Due to Statement 1, $M^4 = \bigcup\limits_{p\in A_{f^t}} W^s_p \cup R_{f^t}$, then $M^4 \setminus R_{f^t} = \bigcup\limits_{p\in A_{f^t}} W^s_p$, so $M^n \setminus R_{f^t}$ is disconnected. On the other hand, since the dimension of the set R_{f^t} does not greater than one, then R_{f^t} does not divide M^4, therefore the set $M^4 \setminus R_{f^t}$ is connected. This contradiction proves that V and A_{f^t} are connected.

To prove that V is a ball let us prove that A_{f^t} does not contain subsets homeomorphic to a circle. Assume the opposite: let $c \subset A_{f^t}$ be a simple closed curve. It follows from Items 1,3 of Statement 1 that the set $A_{f^t} \setminus \Omega^1_p$ is a finite set of arcs lying in the disjoint union of stable manifolds of sink equilibria. Therefore there is an equilibrium state $p \in \Omega^1_{f^t}$ such that $p \in c$. Due to Lemma 2, the set $cl\, W^s_p$ divides the ambient manifold M^4 in two connected components. Therefore $cl\, W^s_p$ also divides the curve c, so there is a point $x \in c \cap cl\, W^s_p$ different from p. The point x cannot be a source, since A_{f^t} does not contain sources by construction. The point x cannot be a sink or since $x \in W^s_p \setminus p$ and only non-wandering point in W^s_p is p. Hence, x belongs to a one-dimensional unstable manifold of some point $q \in \Omega^1_{f^t}$, but we supposed that f^t has no heteroclinic intersections, so we get a contradiction.

Thus the set A_{f^t} can be represented as a connected graph without cycles, whose vertices are sink points and edges are one-dimensional unstable manifolds of saddle points. Then $|\Omega^0_{f^t}| = |\Omega^1_{f^t}| + 1$. It follows from Morse theory that the set V is a smooth subset of M^4 obtained from the disjoint union of $|\Omega^0_{f^t}|$ balls by gluing $|\Omega^1_{f^t}|$ handles of index 1. Using induction one can easy prove that V is a ball. $\square$

Set $\tilde{R}_{f^t} = \bigcup\limits_{p\in\Omega^3_{f^t}\cup\Omega^4_{f^t}} W^s_p$. Considering f^{-t} and applying the Lemma 2 one can get that $\tilde{R}_{f^t}$ is a connected attractor for f^{-t} (hence, it is a repeller for f^t) with a trapping neighborhood W diffeomorphic to the ball. This observation and Lemma 2 immediately lead to the following statement.

Corollary 1. *In assumption of Lemma 2 there are smoothly embedded balls $V, W \subset M^4$ such that:*

1. $A_{f^t} \subset V$, $\tilde{R}_{f^t} \subset W$;
2. *trajectories of the flow f^t are transversal to the boundaries of the balls V, W and are oriented out of the interior of W to the interior of V;*
3. *the non-wandering set of the flow f^t restricted on the set $M^4 \setminus (V \cup W)$ consists of the equilibrium states of index 2.*

2.2. Proof of Theorem 1

Remind that *a connected sum* of smooth orientable connected manifolds M_1^n, M_2^n is the manifold $M_1^n \sharp M_2^n$ obtained as follows. Let $B_1^n \subset M_1^n, B_2^n \subset M_2^n$ be two balls. Then the manifold $M_1^n \sharp M_2^n$ is the result of gluing manifolds $M_1^n \setminus B_1^n), M_2^n \setminus B_2^n$ by a reversing the natural orientation diffeomorphism $h\colon \partial B_1^n \to \partial B_2^n$. According to [10, Lemm 2.1], the connected sum operation is defined the unique (up to diffeomorphism) manifold and does not depend on the choice of balls and a gluing homeomorphism.

Proof of Theorem 1. Suppose that a gradient-like flow f^t on the projective-like manifold M^4 has no heteroclinic intersections. Let us show that the set of saddle equilibrium states of the flow f^t contains exactly one equilibrium state whose Morse index equals two. Then the equality $l_{f^t} - k_{f^t} = 2$ will immediately follow from Hopf-Poincare formula 2.1.

Let $W, V \subset M^4$ be balls described in Corollary 1. Then $M^4 \setminus int\,(W \cup V)$ is the manifold with boundary consisting of two $(n-1)$-spheres. Let $\mathbb{B}^n_+, \mathbb{B}^n_-$ be two standart balls enriched by vector fields $\dot{x} = x$, $\dot{x} = -x$ correspondingly. Glue balls $\mathbb{B}^n_+, \mathbb{B}^n_-$ to $M^4 \setminus int\,(W \cup V)$ with reversing the natural orientation diffeomorphism $\varphi\colon \partial\mathbb{B}^n_+ \cup \partial\mathbb{B}^n_- \to \partial W \cup \partial V$, denote by $\tilde{M}^4$ the resulting manifold and by $\pi_\varphi\colon \mathbb{B}^n_+ \cup \mathbb{B}^n_- \cup M^4 \setminus int\,(W \cup V) \to \tilde{M}^4$ the natural projection. It is possible to choose the diffeomorphism φ in such a way that it induce on $\tilde{M}^4$ a gradient-like flow $\tilde{f}^t$ such that $\tilde{f}^t|_{M^4 \setminus int\,(W \cup V)} = \pi f^t|_{M^4 \setminus int\,(W \cup V)}$ and the restrictions $\tilde{f}^t|_{\mathbb{B}^n_+}, \tilde{f}^t|_{\mathbb{B}^n_-}$ are topologically equivalent to dilatation and contraction correspondingly. So, non-wandering set of the flow $\tilde{f}^t$ consists exectly of one source, one sink, and $|\Omega^2_{f^t}|$ saddles of index 2. The operation of gluing balls is equivalent to taking a connected sum with two spheres, so the manifold $\widetilde{M}^4$ is diffeomorphic to the original manifold M^4. Then, due to Poincare-Hopf formula 2.1, $|\Omega^2_{f^t}| = 1$. The Theorem 1 is proven.

3. Realization of gradient-like flows on four-dimensional projective-like manifolds

This section is devoted to the proof of the Theorem 2. Let $l \geq 2$, $k \geq 0$ and $h \geq 1$ be integers such that $l - k + h = 3$.

We are going to construct a gradient-like flow f^t such that the number l_{f^t} of sink and source equilibrium states of f^t equals l, the number h_{f^t} of saddle equilibrium states of Morse index two equals h, and the number k_{f^t} of saddle equilibrium states of Morse index different from two equals k.

To construct the desired flow we define below auxiliary flows g_1^t, g_2^t on the projective-like manifold M^4 and the sphere S^4, respectively, with the following properties:

1. the non-wandering set of the flows g_1^t consists exactly of one source, $(k-h+1)$ saddles of Morse index one, one saddle of Morse index two and $(k-h+2)$ sinks;

2. the non-wandering set of the flows g_1^t consists exactly of one sink, one source, $(h-1)$ saddles of Morse index one and $(h-1)$ saddles of Morse index two.

Choose the balls $B_1^n \subset M^4$, $B_2^n \subset S^4$ that intersect with the sets $\Omega_{g_1^t}, \Omega_{g_2^t}$ exactly at one point: the sink and source respectively, lying in the interior of the balls B_1^4, B_2^4. We form a connected sum of manifolds M^4, S^4 by cutting out the interiors of the balls B_1^4, B_2^4 and gluing the resulting manifolds by a diffeomorphism to induce on the manifold $M^4\sharp S^4$ a gradient-like flow f^t such that the non-wandering set of the flows f^t consists exactly of $l = 3+k-h$ sinks and sources, k saddles of the index 1, and h saddle of the index 2 (see, for example [14]). The connected sum operation with a sphere does not change the topological type of the manifold, so the manifold $M^4\sharp S^4$ is the projective-like manifold, so f^t is the desired flow.

3.1. Construction of the flow g_1^t

Let us describe the buiding of the flow g_1^t step by step.

Step 1. *Realization of a gradient-like flow g_0^t whose non-wandering set consists of exactly three equilibria: a source, a sink and a saddle of Morse index two.*

Let us define the flow f_k^t on the handle $H_k^4 = \mathbb{B}^k \times \mathbb{B}^{4-k}$ of the index $k \in \{0, \dots, 4\}$ by the following system of differential equations

$$\begin{cases} \dot{x} = x, x \in \mathbb{B}^k \\ \dot{y} = -y, y \in \mathbb{B}^{4-k}. \end{cases}$$

A non-wandering set of the flow f_k^t consists of a single equilibrium state O which Morse index is k. For $k > 0$ trajectories of the flow f_k^t having non-empty intersection with the foot $F_k^4 = \partial\mathbb{B}^k \times \mathbb{B}^{4-k}$ of H_k^4 intersect the foot transversally and directed outside of H_k^4.

First, we are going to obtain a projective-like manifold M^4 by sequentially gluing to the handle H_0^4 the handles H_2^4 and H_4^4. After the gluing handles, the flows f_0^t, f_2^t, f_4^t will induce on M^4 the desired flow g_0^t.

The foot $F_2^4 = \mathbb{S}^1 \times \mathbb{B}^2$ is a solid torus whose core $\mathbb{S}^1 \times \{O\}$ (here O — the center of the ball $\mathbb{B}^2$) belongs to the unstable separatrix of the saddle equilibrium state of the flow f_2^t. Remark that $\partial H_0^4 = \mathbb{S}^3$. Let $c \in \mathbb{S}^3$ be a node (a simple closed curve), N_c is its closed neighborhood, and $P_c = \mathbb{S}^3 \setminus int\, N$.

Let us denote by X_φ a manifold with a boundary obtained by gluing the handle H_0^4 to the handle H_2^4 by means a diffeomorphism $\varphi\colon F_2^4 \to N_c$. We are going to glue the handle H_4^4 to X and obtain a closed manifold, then the boundary of X must be diffeomorphic to the sphere S^3. For this purpose we should choose the gluing diffeomorphism $\varphi\colon F_2^4 \to \partial H_0^4$ and the node c.

As the gluing diffeomorphism $\varphi\colon F_2^4 \to N_c$ is a solid torus diffeomorphism, it maps the meridian of F_2^4 to the meridian of N_c. But the meridian of F_2^4 is the longitude of solid torus $\partial H_2^4 \setminus int\, F_2^4$. So, the gluing operation is a nontrivial surgery. By virtue of [11, Theorem 1], no nontrivial surgery along a nontrivial node will give a sphere. It follows that the knot c must be the boundary of a 2-disk in S^3. Hence, P_c is the solid torus. Let φ send the longitude of N_c to the curve of homotopy type $(1,1)$ in ∂N_c. Then, due to [12], ∂X will be the sphere.

Now we are able to glue the handle H_4^4 to X by an arbitrary orientation reversing diffeomorphism $\psi\colon \partial H_4^4 \to \partial X$. As a result, we get a closed manifold M^4 carrying a gradient-like flow g_1^t whose non-wandering set consist of exactly three equilibrium states. Hence, M^4 is the projective-like manifold.

Step 2. *A realization of a gradient-like flow h^t on the sphere S^4 whose non-wandering set consists of exactly one source, k saddles of index 1, and $k+1$ sink.*

Define a gradient-like flow ψ^t on the sphere S^4, which has a non-wandering set consisting of exactly one source, k saddles of index 1, and $k+1$ sinks.

We construct k copies of the sphere $S_1^4, \dots, S_k^4$, each of which carries the flow ψ_i^t, $i \in \{1, \dots, k\}$ whose non-wandering set consists of exactly one source α_i, one saddle σ_i of index 1, and two sinks ω_i^+, ω_i^-. To do this, we glue one handle of index 1 to two handles of index 0 to get the ball carrying a gradient-like flow whose trajectories are transversal to the boundary of the ball and the non-wandering set consists of two sinks and one saddle. Then we glue the handle H_4^4 to the obtained manifold. As a result, we get the desired flow ψ_i^t.

Select a ball $B_1^4 \subset S_1^4$ ($B_2^4 \subset S_2^4$) that intersect the set $\Omega_{\psi_1^t}(\Omega_{\psi_2^t})$ exactly at one point which is the sink ω_1^+ (the source α_2) lying in the interior of the ball $B_1^n(B_2^n)$. We define a connected sum of spheres S_1^4, S_2^4 by cutting out the interiors of balls B_1^4, B_2^4 and gluing the resulting manifolds with the boundary by an orientation-inverting diffeomorphism $h_{1,2}\colon \partial B_1^n \to \partial B_2^n$ such that $h_{1,2}(W_{\sigma_1}^u) \cap W_{\sigma_2}^s = \emptyset$. The gluing operation induce a gradient-like flow $\psi_{1,2}^t$ without heteroclinic intersection on the connected sum $S_1^4 \sharp S_2^4$. Set $S_{1,2}^4 = S_1^4 \sharp S_2^4$. The non-wandering set of the flow $\psi_{1,2}^t$ consists of one source, two saddles of index 1, and three sinks. Similarly, we form a connected sum of the spheres $S_{1,2}^4$ and S_3^4, and so on. After k steps, we get the desired flow ψ^t.

Step 3. *Construction of the desired flow g_1^t*

Let us consider the projective-like manifold M^4 carrying the flows g_0^t defined on the Step 1 and the sphere S^4 carrying the flow h^t defined on the Step 2. As it described above, it is possible to construct the connected sum $M^4 \sharp S^4$ and induce the desired flow g_1^t on the $M^4 \sharp S^4$.

3.2. Construction of the flow g_2^t

Let us construct an auxiliary gradient-like flows η^t on the sphere S^4 whose non-wandering set consists exactly on one source, one sink, and two saddles of Morse index one and two respectively. In [13] it is proved that the intersection of invariant manifolds of these two saddles is non-empty and consists of finite number of non-compact curves (trajectories) that are called heteroclinic curves. Then we take $(h-1)\geq 1$ copies of spheres with carrying such flows and construct the connected sum of the spheres as it described above. As a result we obtain the desired flow g_2^t.

To construct a flow η^t let us construct a mainfold M_1 by gluing the hande H_1 to the hand H_0 by meanse of an arbitrary smooth embedding $g\colon \mathbb{S}^0\times\mathbb{B}^4\to\mathbb{S}^3$. Then $\partial\, M_1$ is homeomorphic to $\mathbb{S}^2\times\mathbb{S}^1$ and flows f_0^t, f_1^t induce on M_1 a gradient-like flow η_1^t whose non-wandering set consists of exactly two equilibria: a source ω and a saddle σ_1 of Morse index one.

Set $S^2_{\eta_1^t}=W^s_{\sigma_1}\cap\partial M_1$. By construction $S^2_{\psi^t}$ is the 2-sphere which does not bounds any ball in ∂M_1. Then there is a homeomorphism $\theta\colon\mathbb{S}^2\times\mathbb{S}^1\to\partial M_1$ such that $\theta(\mathbb{S}^2\times\{x\})=S^2_{\eta^t}$, $x\in\mathbb{S}^1$. Set $c=\theta(z\times\mathbb{S}^1)$, $z\in\mathbb{S}^2$ and denote by $N_c\subset\partial\, M_1$ a tubular neighborhood of the node c. Let $\mu\colon\mathbb{S}^1\times\mathbb{B}^2\to N_c$ be a diffeomorphism such that $\mu(\mathbb{S}^1\times\{O\})=c$. Denote by M_2 a manifold obtained by gluing the handle H_2 to M_1 by means of μ. The boundary of M_2 is the result of gluing two solid tori $\partial H_2\setminus int\,(\mathbb{S}^1\times\mathbb{B}^2)$ and $\partial M_1\setminus int\, N_c$ by means of the diffeomorphism $\eta|_{\mathbb{S}^1\times\mathbb{S}^1}$ that sends a longitude of $\partial H_2\setminus int\,(\mathbb{S}^1\times\mathbb{B}^2)$ to the meridian of the solid torus ∂N_1. Hence ∂M_2 is $3-$sphere. More over, due to [15, Theorems 3.30., 3.34], the manifold M_2 is diffeomorphic to the ball H_0.

Glue M_2 and the hand H_4^4 to get the sphere S^4 and the desired gradient-like flow η^t.

References

1. *Smale S.* On Gradient Dynamical Systems. Annals of Mathematics. 74, № 1. 199–206 (1961).
2. *Smale S.* Differentiable dynamical systems. Bulletin of the American Mathematical Society. 73, № 6, 747-817 (1967).
3. *Zhuzhoma E. V., Medvedev V.* Morse-Smale systems with few non-wandering points. Topology and its Applications. 160, № 3. 498-507 (2013).
4. *Zhuzhoma E. V., Medvedev V.* Continuous Morse-Smale flows with three equilibrium positions. Sbornik Mathematics. 207, № 5. 702-723 (2016).
5. *Brown M.* Locally flat imbeddings of topological manifolds. Ann. of Math. 75, № 2. 331-341 (1962).
6. *Cantrell J. C.* Almost locally flat sphere S^{n-1} in S^n. Proceeding of the American Mathematical society. 15, № 4. 574-578 (1964).
7. *Eells J., Kuiper N.* Manifolds which are like projective planes. Institut des Haute studes Scientifiques Publications Mathematiques. 14. 5-46 (1962).
8. *Daverman R. J.* Embeddings of (n-1)-spheres in Euclidean n-space. Bulleting of the American Mathematical society. 84, № 3. 377-405 (1978).

9. *Meyer K. R.* Energy Functions for Morse Smale Systems. American Journal of Mathematics. 90, № 4. 1031–1040 (1968).
10. *Kervaire M. A., Milnor J. W.* Groups of homotopy spheres: I. The Annals of Mathematics, 2nd Ser. 77, № 3. 504-537 (1963).
11. *Gordon C., Luecke J.* Knots are determined by their complements. Bull. Amer. Math. Soc. (N.S.). 20, № 1. 83-87 (1989).
12. *Rolfsen D*. Knots and Links. Berkeley, CA: Publish or Perish-Mathematics, 1976.
13. *Gurevich E., Pavlova D.* On embedding of invariant manifolds of the simplest Morse-Smale flows with heteroclinic curves. Zhurnal Srednevolzhskogo Matematicheskogo Obshchestva. 20, № 4. 378-383 (2018).
14. *Grines V., Gurevich E., Zhuzhoma E., Medvedev V.* On Topology of Manifolds Admitting a Gradient-Like Flow with a Prescribed Non-Wandering Set. Siberian Advances in Mathematics. 29. № 2. 116-127 (2019).
15. *Matsumoto Y.* An Introduction to Morse Theory. Oxford University Press. 2001.

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