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On existence of Morse energy function for topological flows ☆

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It is a well known fact that any smooth manifold admits a Morse function, whereas the problem of existence of a Morse function for a topological manifold stated by Marston Morse in 1959 ([6]) is still open. In the present paper we prove that a topological manifold admits a continuous Morse function if it admits a topological flow with a finite hyperbolic chain recurrent set. We construct this function as a Lyapunov function whose set of the critical points coincides with the chain recurrent set of the flow.

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1. Introduction and main results

It is well known that for dimensions 4 and greater there are topological manifolds admitting no smooth structure. Therefore, dynamical systems as well as functions on such manifolds may only be considered as topological and continuous, respectively. Nevertheless, some properties of these systems and functions are the same for both topological and smooth cases, especially if these properties are related to the topology of the ambient manifold. For example, if a topological manifold admits a continuous Morse function then the classical Morse inequities for the number of critical points of index ν and the ν-th Betti number hold for both cases. However, the problem of existence of such a function is open.

The main result of this paper is the following.

Theorem 1. *If a topological manifold M admits a topological flow with a finite hyperbolic chain recurrent set then it admits a Morse function whose set of the critical points coincides with the chain recurrent set of this flow.*

In order to prove this theorem in Section 2 we study the dynamics of topological flows and in Section 3 we actually construct a continuous Morse energy function for these flows.

Recall that a *Lyapunov function* for a smooth flow is a continuous function which decreases along the orbits outside the chain recurrent set and which is constant on each chain component. It follows from the results of C. Conley [1] that this function exists for every flow defined by a continuous vector field (this fact is known as "The Fundamental Theorem of Dynamical Systems"). From the results of W. Wilson [8] it follows that for every such flow there is a smooth Lyapunov function whose set of critical points coincides with the chain recurrent set. This Lyapunov function is called the *energy function*.

The suggested construction unfortunately doesn't allow one to control the type of critical points of the energy function. On the other hand for structurally stable flows with finite chain recurrent set (gradient-like flows) it is natural to expect existence of an energy function with no degenerate critical points (Morse function). By methods different from that of C. Conley this function was constructed by S. Smale [7] and K. Meyer [5].

In this paper we study a class of topological flows that generalizes gradient-like flows. We prove that for these flows even on non-smoothable topological manifolds there exists a continuous Morse energy function.

Our result gives rise to a new approach to studying the topology of ambient manifolds of topological flows. For instance, in [2] there is a complete classification of topological manifolds allowing continuous Morse functions with exactly 3 critical points. On the other hand the problem of topology of ambient manifolds of topological Morse-Smale flows with exactly 3 fixed points was stated in [9]. The solution of this problem immediately follows from our results as ambient manifolds of these flows coincide with the manifolds allowing a Morse function with exactly 3 critical points (non-smoothable among them).

2. Dynamics of topological flows with finite hyperbolic chain recurrent sets

Let M be a closed n-manifold with a metric. *A topological flow* on M is a one-parameter group of homeomorphisms $f^t : M \to M$, $t \in \mathbb{R}$. The *trajectory* or the *orbit* of a point $x \in M$ with respect to the flow f^t is the set $\mathcal{O}_x = \{f^t(x), t \in \mathbb{R}\}$. The trajectories are oriented as increasing t. Any two trajectories either coincide or they do not intersect one another, therefore, the phase space is the union of pairwise disjoint trajectories. There are three types of trajectories:

1. a *fixed point* $\mathcal{O}_x = \{x\}$.
2. a *periodic trajectory* $\mathcal{O}_x$ for which there exists such $per(x) > 0$ that $f^{per(x)}(x) = x$ but $f^t(x) \neq x$ for every $0 < t < per(x)$. The number $per(x)$ is called the *period* of $\mathcal{O}_x$ and it is independent of the choice of $x \in \mathcal{O}_x$.
3. a *regular* trajectory $\mathcal{O}_x$ is a trajectory that is neither a fixed point nor a periodic trajectory.

An *ε-chain of length T* connecting a point x to a point y with respect to the flow f^t is a sequence of points $x = x_0, \dots, x_n = y$ for which there is a sequence $t_1, \dots, t_n$ such that $d(f^{t_i}(x_{i-1}), x_i) < \varepsilon$, $t_i \geq 1$ for $1 \leq i \leq n$ and $t_1 + \dots + t_n = T$.

A point $x \in M$ is *chain recurrent* for the flow f^t if for every $\varepsilon > 0$ there are T and ε-chain of length T connecting x to itself. The set of all chain recurrent points of f^t is the *chain recurrent set* denoted by $\mathcal{R}_{f^t}$, its connected components being the *chain components*. The set $\mathcal{R}_{f^t}$ is f^t-invariant, i.e. it is composed of the f^t trajectories which are called *chain recurrent*. Fixed points and periodic orbits are obviously chain recurrent.

Two flows $f^t : M \to M$, $g^t : M \to M$ are said to be *topologically equivalent* if there is a homeomorphism $h : M \to M$ such that $g^t h = h f^t$ for every $t \in \mathbb{R}$ while h is called the *conjugating* homeomorphism.

Let the model flow in a neighborhood of a fixed point be the linear flow $a^t_\lambda : \mathbb{R}^n \to \mathbb{R}^n, \lambda \in \{0, 1, ..., n\}$ defined by

$$a^t_\lambda(x_1, ..., x_\lambda, x_{\lambda+1}, ..., x_n) = (2^t x_1, ..., 2^t x_\lambda, 2^{-t} x_{\lambda+1}, ..., 2^{-t} x_n).$$

Let

$$E^s_\lambda = \{(x_1, ..., x_n) \in \mathbb{R}^n : x_1 = \dots = x_\lambda = 0\},$$
$$E^u_\lambda = \{(x_1, ..., x_n) \in \mathbb{R}^n : x_{\lambda+1} = \dots = x_n = 0\}.$$

A fixed point p is *topologically hyperbolic* if for some neighborhood $U_p \subset M$ of p, some number $\lambda_p \in \{0, 1, \dots, n\}$ and some homeomorphism $h_p : U_p \to \mathbb{R}^n$ the equation

$h_p f^t|_{U_p} = a^t_{\lambda_p} h_p|_{U_p}$ holds whenever both the right and the left sides are defined. The *locally invariant manifolds* of a fixed point p are the sets $h_p^{-1}(E^s_{\lambda_p}), h_p^{-1}(E^u_{\lambda_p})$. The sets

$$W^s_p = \bigcup_{t\in\mathbb{R}} f^t(h_p^{-1}(E^s_{\lambda_p})),\ W^u_p = \bigcup_{t\in\mathbb{R}} f^t(h_p^{-1}(E^u_{\lambda_p}))$$

are called the *stable* and the *unstable* invariant manifolds of p, respectively.

The number λ_p is said to be the *index* of the fixed hyperbolic point p. Points of indexes n and 0 are *sources* and *sinks*, respectively. A point p with $\lambda_p \in \{1, \cdots, n-1\}$ is said to be a *saddle*.

Proposition 2.1. *The unstable W^u_p and the stable W^s_p manifolds of a hyperbolic fixed point p are independent of the choice of the local homeomorphism h_p and they are defined in topological terms as follows: $W^u_p = \{y \in M : \lim\limits_{t\to+\infty} f^{-t}(y) = p\}$ and $W^s_p = \{y \in M : \lim\limits_{t\to+\infty} f^t(y) = p\}$.*

Proof. Let $\tilde{h}_p : \tilde{U}_p \to \mathbb{R}^n$ be a homeomorphism such that $\tilde{h}_p f^t|_{\tilde{U}_p} = a^t_{\tilde{\lambda}_p} \tilde{h}_p|_{\tilde{U}_p}$ whenever the left and right sides are defined. Then in a neighborhood U_O of the original point O in $\mathbb{R}^n$ the homeomorphism $h = h_p \tilde{h}_p^{-1}$ is well-defined and h conjugates $a^t_{\lambda_p}$ with $a^t_{\tilde{\lambda}_p}$. Since a conjugating homeomorphism preserves the invariant manifolds we have $\tilde{\lambda}_p = \lambda_p$ and $h(E^s_{\lambda_p}) = E^s_{\lambda_p}$, $h(E^u_{\lambda_p}) = E^u_{\lambda_p}$. Thus, $\tilde{h}_p^{-1}(E^s_{\lambda_p}) = h_p^{-1}(h(E^s_{\lambda_p})) = h_p^{-1}(E^s_{\lambda_p})$. The same is true for $E^u_{\lambda_p}$. □

It follows from Proposition 2.1 that $W^u_p \cap W^u_q = \emptyset$ and $W^s_p \cap W^s_q = \emptyset$ for any two distinct hyperbolic points p, q.

Denote by G the set of topological flows on M with finite hyperbolic chain recurrent set. The points of the chain recurrent set are the fixed points. The dynamics of these flows are similar to that of the gradient-like flows in the following sense. Let Smale's relation on the set of fixed points of $f^t \in G$ be defined by

$$p \prec q \iff W^s_p \cap W^u_q \neq \emptyset.$$

A m-cycle $(m \geq 1)$ is a collection $p_1, p_2, \cdots, p_m$ of pairwise distinct fixed points satisfying $p_1 \prec p_2 \prec \cdots \prec p_m \prec p_1$.

Proposition 2.2. *Every flow $f^t \in G$ has no cycles.*

Proof. Suppose the contrary: there exists a sequence of fixed points $p_1 \prec \cdots \prec p_m \prec p_1$. By construction, each point of $\bigcup\limits_{i=1}^{m} (W^s_{p_i} \cap W^u_{p_{i+1}})$ (here $p_{m+1} = p_1$) is chain recurrent and this contradicts the finiteness of the chain recurrent set of f^t. □

Due to Proposition 2.2 the introduced Smale's relation can be extended (not uniquely) to a total order relation, that is for each pair of fixed points p_i, p_j either $p_i \prec p_j$, or $p_j \prec p_i$ is true. Consider the fixed points of a flow $f^t \in G$ numbered in accordance with the introduced order:

$$p_1 \prec \cdots \prec p_k.$$

Without loss of generality assume that any sink orbit is located in this order below any saddle orbit and that any source orbit is higher than any saddle one.

The main result of this section is the following fact.

Theorem 2. *Let $f^t \in G$. Then*

1. $M = \bigcup\limits_{i=1}^{k} W^u_{p_i} = \bigcup\limits_{i=1}^{k} W^s_{p_i}$;
2. $W^u_{p_i}$ *and* $W^s_{p_i}$ *are topological submanifolds of* M *homeomorphic to* $\mathbb{R}^{\lambda_{p_i}}$ *and* $\mathbb{R}^{n-\lambda_{p_i}}$, *respectively;*
3. $cl(W^u_{p_i}) \setminus W^u_{p_i} \subset \bigcup\limits_{j=1}^{i-1} W^u_{p_j}$ *and* $cl(W^s_{p_i}) \setminus W^s_{p_i} \subset \bigcup\limits_{j=i+1}^{k} W^s_{p_j}$.

Each item of the theorem is proved in a separate subsection of this Section.

All statements formulated for unstable manifolds hold for stable manifolds as well. One gets them if one formally changes "u" to "s" because $\mathcal{R}_{f^t} = \mathcal{R}_{f^{-t}}$ and the stable manifolds of the chain recurrent points for f^t are the unstable manifolds of the chain recurrent points for f^{-t}.

Notice that the similar result for Morse-Smale diffeomorphisms was proved in [4] and for Morse-Smale homeomorphism in [3].

2.1. Representation of the ambient manifold as the union of the invariant manifolds of the periodic points

Proof of the item (1) of Theorem 2.

Proof. Now we prove that $M = \bigcup\limits_{i=1}^{k} W^u_{p_i}$ for every flow $f^t \in G$.

Let $x \in M$. Recall that a point $y \in M$ is called an *α-limit point* for the point x if there is a sequence $t_n \to -\infty,\ t_n \in \mathbb{Z}$ for which

$$\lim_{t_n \to -\infty} d(f^{t_n}(x), y) = 0.$$

The set $\alpha(x)$ of all α-limit points for the point x is called the *α-limit set* of x. Since M is compact the set $\alpha(x)$ is not empty and $\alpha(x) \subset \mathcal{R}_{f^t}$. Indeed, since f is uniformly continuous and $\lim\limits_{t_n \to -\infty} d(f^{t_n}(x), y) = 0$ for every $\varepsilon > 0$ there is $n_\varepsilon \in \mathbb{N}$ such that for every $n \geq n_\varepsilon$ the inequalities $d(f^{t_n}(x), y) < \varepsilon$ and $d(f^{t_n+1}(x), f(y)) < \varepsilon$ hold. Thus, $y, f(y), f^{t_n+1}(x), f^{t_n+2}(x), \dots, f^{t_{n+1}}(x), y$ is the ε-chain connecting y to itself.

Now we show that $\alpha(x)$ consists of exactly one fixed point which depends on x. Assume the contrary, i.e. that there are two distinct fixed points $p_v,\ p_w \in \alpha(x)$. Since $\mathcal{R}_{f^t}$ is finite the inequality $d(p_i, p_j) > \rho$ holds for some $\rho > 0$ whenever $i \neq j$. Denote $V_i = \{y \in M\ :\ d(y, p_i) < \frac{\rho}{3}\}$. Since all the points $p_i,\ i = \overline{1, k}$ are fixed there is a neighborhood U_i for which $cl(U_i) \subset V_i$ and $f^{-1}(cl(U_i)) \cap V_j = \emptyset$ for every $j \neq i$. By the assumption there is an increasing sequence q_ℓ of the iterations of f^{-1} such that $f^{-q_{2m}}(x) \in U_v$, $f^{-q_{2m+1}}(x) \in U_w$ and $q_{2m+1} - q_{2m} \geq 2$. Pick the sequence n_m so that n_m is the maximal natural number belonging to the interval (q_{2m}, q_{2m+1}) for each $f^{-(n_m-1)}(x) \in cl(U_v)$. Then $f^{-n_m}(x) \notin cl(U_v)$. On the other hand $f^{-n_m}(x) = f^{-1}(f^{-(n_m-1)}(x)) \notin V_j$ for $j \neq v$, and hence $f^{-n_m}(x) \in (M \setminus \bigcup\limits_{i=1}^{k} U_i)$. But then $\alpha(x)$ is not a subset of $\mathcal{R}_{f^t}$ and we have a contradiction.

Thus, for each point $x \in M$ there is the unique point $p_v(x) \in \mathcal{R}_{f^t}$ such that $\alpha(x) = p_v(x)$, i.e. there is a sequence $k_n \to +\infty$ for which $\lim\limits_{k_n \to +\infty} d(f^{-k_n}(x), p_v(x)) = 0$. It follows from the definition of a hyperbolic fixed point that $f^{-k_n}(x) \in W^u_{p_v(x)}$ for all n greater than some n_0. Then $x \in W^u_{p_v(x)}$ because the unstable manifold is invariant. □

2.2. Embedding of the invariant manifolds of periodic points into the ambient manifold

To prove item (2) of Theorem 2 we need the following proposition.

Proposition 2.3. *Let σ be a hyperbolic saddle fixed point of a flow $f^t \in G$, let $T_\sigma \subset W^s_\sigma$ be a compact neighborhood of σ and let $\xi \in T_\sigma$. Then for every sequence of points $\{\xi_m\} \subset (M \setminus T_\sigma)$ converging to ξ there are a subsequence $\{\xi_{m_j}\}$, a sequence of natural numbers $k_{m_j} \to +\infty$ and a point $\eta \in (W^u_\sigma \setminus \sigma)$ for which the sequence of points $\{f^{k_{m_j}}(\xi_{m_j})\}$ converges to the point η.*

Proof. Without loss of generality assume $(U_\sigma \cap W^s_\sigma) \subset T_\sigma$, $\xi \in (U_\sigma \cap f(U_\sigma))$ and $\{\xi_m\} \subset (U_\sigma \cap f(U_\sigma))$. Pick a number $r > 0$ so that the ball $B_r(O) = \{(x_1, \dots, x_n) \in \mathbb{R}^n\ :\ (x_1^2 + \dots + x_n^2) \leq r\}$ is a subset of $h_\sigma(U_\sigma)$.

Let $h_\sigma(\xi_m) = \bar{\xi}_m = (\bar{\xi}_{1,m}, \dots, \bar{\xi}_{\lambda_\sigma,m}, \bar{\xi}_{\lambda_\sigma+1,m}, \dots, \bar{\xi}_{n,m})$ (see Fig. 1). The set $K^u = \{(x_1, \dots, x_{\lambda_\sigma}) \in Ox_1 \dots x_{\lambda_\sigma}\ :\ \frac{r^2}{4} \leq x_1^2 + \dots + x_{\lambda_\sigma}^2 \leq r^2\}$ is the fundamental domain of the restriction of the diffeomorphism a_{λ_σ} to $Ox_1 \dots, x_{\lambda_\sigma} \setminus O$. Then for each $m \in \mathbb{N}$ there is the unique integer k_m for which $\frac{r^2}{4} \leq 4^{k_m}\left((\bar{\xi}_{1,m})^2 + \dots + (\bar{\xi}_{\lambda_\sigma,m})^2\right) < r^2$. Let $\bar{\eta}_m = a^{k_m}_{\lambda_\sigma,+1}(\bar{\xi}_m)$. Since $\lim\limits_{m\to\infty} \bar{\xi}_m = h_\sigma(\xi) \in (Ox_{\lambda_\sigma+1} \dots x_n \backslash O)$ the limit $\lim\limits_{m\to\infty} \bar{\xi}_{i,m}$ equals to 0 for each $i \in \{1, \dots, \lambda_\sigma\}$, and hence $\lim\limits_{m\to\infty} k_m = +\infty$. Furthermore, the sequence $\{\bar{\xi}_{i,m}\}$ is bounded for each $i \in \{\lambda_\sigma + 1, \dots, n\}$ and therefore $\bar{\eta}_{i,m} = \left(\frac{1}{2}\right)^{k_m} \bar{\xi}_{i,m} \to 0$ for $m \to +\infty$ and $i \in \{\lambda_\sigma + 1, \dots, n\}$.

Thus, the coordinates of the points $\bar{\eta}_m = (\bar{\eta}_{1,m}, \dots, \bar{\eta}_{n,m})$ satisfy $\frac{r^2}{4} \leq (\bar{\eta}_{1,m})^2 + \dots + (\bar{\eta}_{\lambda_\sigma,m})^2 < r^2$ for $i \in \{1, \dots, \lambda_\sigma\}$ and $\bar{\eta}_{i,m} \to 0$ as $m \to \infty$ for $i \in \{\lambda_\sigma + 1, \dots, n\}$, i.e.

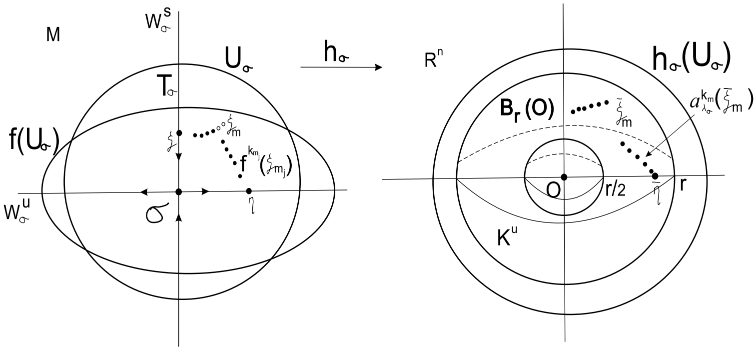

Fig. 1. Illustration to Proposition 2.3.

the points η_m are inside some compact subset of $\mathbb{R}^n$. Since any sequence of points of a compact set has a converging subsequence there are a subsequence $\{k_{m_j}\}$ of the sequence $\{k_m\}$ and a point $\bar\eta \in (W^u_O \setminus O)$ such that $\lim\limits_{j\to\infty} \bar\eta_{m_j} = \bar\eta$. Then $\xi_{m_j} = h^{-1}_\sigma(a^{-k_{m_j}}_{\lambda_\sigma}(\bar\eta_{m_j}))$ is the desired subsequence. □

Proof of item (2) of Theorem 2

Proof. Here we prove that $W^u_{p_i}$ is a submanifold of the manifold M homeomorphic to $\mathbb{R}^{\lambda_{p_i}}$.

Let $T_{p_i} = h_{p_i}(E^u_{\lambda_{p_i}})$. Then for every point $x \in W^u_{p_i}$ there is a natural number n_x for which $x \in f^{n_x}(T_{p_i})$. Let $T_{p_i}(x) = f^{n_x}(T_{p_i})$ then there is a chart $\psi_x : U_x \to \mathbb{R}^n$ of the manifold M such that $\psi_x(U_x \cap T_{p_i}(x)) = \mathbb{R}^{\lambda_{p_i}}$. If $\lambda_{p_i} = n$ or $\lambda_{p_i} = 0$ then $\psi_x(U_x \cap T_{p_i}(x)) = \psi_x(U_x \cap W^u_{p_i})$. Therefore, the unstable manifold of every node point is an n-submanifold.

Now we show that $W^u_{p_i}$ is a submanifold of M homeomorphic to $\mathbb{R}^{\lambda_{p_i}}$ for every saddle point p_i as well. Suppose the contrary: $W^u_{p_i}$ is not a submanifold of M. Then there is a point $x \in W^u_{p_i}$ such that for every chart $\psi_x : U_x \to \mathbb{R}^n$ of the manifold M for which $\psi_x(U_x \cap T_{p_i}(x)) = \mathbb{R}^{\lambda_{p_i}}$ the intersection $(U_x \setminus T_{p_i}(x)) \cap W^u_{p_i}$ is not empty. Hence, there is a sequence $\{x_m\} \subset (W^u_{p_i} \setminus T_{p_i}(x))$ for which $d(x_m, x) \to 0$ as $m \to +\infty$.

From Proposition 2.3 it follows that for some subsequence x_{m_j} and some sequence k_j the sequence $y_j = f^{-k_j}(x_{m_j}) \subset W^u_{p_i}$ converges to a point $y \in (W^s_{p_i} \setminus p_i)$.

According to the item (1) of Theorem 2 there is a point $p_v \in \mathcal{R}_{f^t}$ such that $y \in W^u_{p_v}$. Consider three possibilities: (a) $\dim W^u_{p_v} = 0$; (b) $0 < \dim W^u_{p_v} < n$; (c) $\dim W^u_{p_v} = n$.

(a) If $\dim W^u_{p_v} = 0$ then $y_j \in W^u_{p_v}$ for all j great enough. Hence, $i = v$ and y is a homoclinic point which contradicts Proposition 2.2. Thus, case (a) is impossible.

(c) If $\dim W^u_{p_v} = n$ then $W^u_{p_v} = p_v$ and $y = p_v$ which contradicts $y \in W^s_{p_i}$. Thus, case (c) is impossible.

(b) If $0 < \dim W^u_{p_v} < n$ then $v > i$ as f has no homoclinic points. According to Proposition 2.3 the sequence $\{f^{m_r}(y_{j_r})\}$ converges to a point $z \in W^s_{p_v}$ for some subsequence

$\{y_{j_r}\}$ and some sequence $m_r \to +\infty$. Since $M = \bigcup_{i=1}^{k} W^u_{p_i}$ we have $z \in W^u_{p_w}$. Similarly to the previous arguments we get $v \neq j, v \neq i$ and $0 < \dim W^u_{p_w} < n$. Due to finiteness of the set $\mathcal{R}_{f^t}$ the case (b) is also impossible.

Thus, $W^u_{p_i}$ is a topological submanifold of the manifold M homeomorphic to $\mathbb{R}^{\lambda_{p_i}}$. □

2.3. Asymptotic behavior of the invariant manifolds of chain recurrent points

Proof of the item (3) of Theorem 2

Proof. Now we prove that $(cl(W^u_{p_i}) \setminus W^u_{p_i}) \subset \bigcup_{j=1}^{i-1} W^u_{p_j}$.

If p_i is a sink then the set $cl(W^u_{p_i}) \setminus W^u_{p_i}$ is empty and the claim is true. If it is not empty consider a point $x \in (cl(W^u_{p_i}) \setminus W^u_{p_i})$. Then $x \in W^u_{p_v}$ for some $v < i$.

Indeed, since $x \in (cl(W^u_{p_i}) \setminus (W^u_{p_i} \cup p_i))$ there is a sequence $\{x_m\} \subset W^u_{p_i}$ for which $d(x_m, x) \to 0$ for $m \to +\infty$. By item (1) of Theorem 2 $x \in W^u_{p_v}$ for some $v \in \{1, \dots, k\}$. There are three possibilities: (a) p_v is a sink, (b) p_v is a saddle, (c) p_v is a source.

In the case (c) $x_m \in W^u_{p_v}$ for all m great enough. But then $p_v = p_i$ and $x \in W^u_{p_i}$ which contradicts the assumption.

In the case (a) $W^u_{p_v} = p_v$, $x = p_v$ and $x_m \in W^s_{p_v}$ for all m great enough. Then $W^u_{p_i} \cap W^s_{p_v} \neq \emptyset$ and $v < i$ is true.

In the case (b) by Proposition 2.3 there are a subsequence x_{m_j} and a sequence k_j for which the sequence $y_j = f^{-k_j}(x_{m_j})$ converges to a point $y \in (W^s_{p_v} \setminus p_v)$. By the item (1) of Theorem 2 there is a point $p_w \in \mathcal{R}_{f^t}$ such that $y \in W^u_{p_w}$, that is $p_v \prec p_w$. If $w = i$ then the statement is true. If not then arguing as before we get that the point p_w cannot be a source. The point p_w is evidently not a sink because p_v is a saddle point. Thus, the point p_w is a saddle distinct from p_v. Since $\mathcal{R}_{f^t}$ is finite and since there are no cycles we repeat the process and prove the statement after a finite number of steps. □

3. Construction of an energy function for flows in *G*

Following M. Morse [6] we now introduce a continuous Morse function on M.

Let $\varphi : M \to \mathbb{R}$ be a continuous function with real values. A point $p \in M$ is said to be *regular* if there is a neighborhood $V_p \subset M$ of p and there is a homeomorphism onto its image $\phi_p : y \in V_p \mapsto \phi_p(y) = (x_1(y), \cdots, x_n(y)) \in \mathbb{R}^n$ for which

$$x_i(p) = 0, i \in \{1, \cdots, n\},\ \varphi(y) = \varphi(p) + x_n(y),\ y \in V_p.$$

Otherwise the point p is called *critical*. Denote by Cr_φ the set of critical points of φ. A critical point p is said to be *non-degenerate critical point of index* ν_p ($\nu_p \in \{0, \cdots, n\}$) if there are coordinates $x_i, i \in \{1, \cdots, n\}$ of the critical point p such that

$$\varphi(y) = \varphi(p) - \sum_{i=1}^{\nu_p} x_i^2(y) + \sum_{i=\nu_p+1}^{n} x_i^2(y),\ y \in V_p.$$

A Morse function whose every critical point is non-degenerate is called a *continuous Morse function*.

From [6] it follows that for a continuous Morse function the classical Morse inequities for the number of critical points of index ν and the ν-th Betti number hold.

Proposition 3.1 states the existence of a local Morse energy function for a hyperbolic fixed point p.

Proposition 3.1. *Let p be a fixed point of index λ_p of a flow $f^t \in G$ and let $h_p : y \in U_p \mapsto h_p(y) = (x_1(y), \cdots, x_n(y)) \in \mathbb{R}^n$ be a homeomorphism conjugating f^t in a neighborhood U_p of p to the linear flow $a^t_{\lambda_p}$. Then for every number $c \in \mathbb{R}$ the function*

$$\varphi_{p,c}(y) = c - \sum_{i=1}^{\lambda_p} x_i^2(y) + \sum_{i=\lambda_p+1}^{n} x_i^2(y),\ y \in U_p$$

is the local Morse energy function for f^t in the neighborhood of p.

Now we prove Theorem 1, that is for every topological flow f^t in G we construct a continuous Morse function $\varphi : M \to \mathbb{R}$ with the following properties

1. $\varphi(f^t(x)) < \varphi(x)$ for every $x \in (M \setminus \mathcal{R}_{f^t})$ and every $t > 0$;
2. $Cr_\varphi = \mathcal{R}_{f^t}$ and $\nu_{p_i} = \lambda_{p_i}$ for any fixed point p_i.

For each $i \in \{1, \dots, k-1\}$ let

$$A_i = \bigcup_{j=1}^{i} W^u_{p_j},\ R_i = \bigcup_{j=i+1}^{i} W^s_{p_j}.$$

By induction on $i \in \{1, \dots, k-1\}$ we are going to construct a neighborhood U_i of the set A_i and a Morse function $\varphi_i : U_i \to [1, i+1/3]$ with the following properties:

- $U_i = \varphi_i^{-1}([1, i+1/3])$ is a closed compact topological n-submanifold, its boundary being $\partial U_i = \varphi_i^{-1}(i+1/3)$, such that $A_i \subset \text{int } U_i$, $U_i \cap R_i = \emptyset$.
- φ_i is an energy function for the flow $f^t|_{U_i}$ which coincides with $\varphi_{p_i,i}$ in some neighborhood of p_i.

The last step of the construction is the construction of a neighborhood U_k and a function φ_k when φ_{k-1} is already constructed on the neighborhood U_{k-1}. Then according to Theorem 2 the neighborhood U_k is the entire manifold M and the function φ_k is the desired function φ.

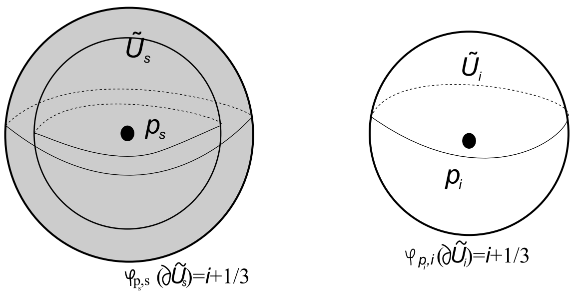

Fig. 2. Induction step if p_i is a sink.

Denote by

$$\mathbb{D}^n = \{(x_1,\dots,x_n) \in \mathbb{R}^n \ : \ \sum_{i=1}^{n} x_i^2 \leq 1\} \text{ the standard } n\text{-disk } (n\text{-ball}), \ \mathbb{D}^0 = \{0\}$$

and by

$$\mathbb{S}^{n-1} = \{(x_1,\dots,x_n) \in \mathbb{R}^n \ : \ \sum_{i=1}^{n} x_i^2 = 1\} \text{ the standard } (n-1)\text{-sphere}, \ \mathbb{S}^0 = \{-1, 1\}.$$

Construction for $i = 1$. By Theorem 2 the point p_1 is a sink. By Proposition 3.1 there is a local energy function $\varphi_{p_1,1}$ in some neighborhood of p_1. Let $U_1 = \varphi^{-1}_{p_1,1}([1, 4/3])$ and $\varphi_1 = \varphi_{p_1,1}|_{U_1}$ and this concludes the proof for this step.

Induction step. Suppose the desired $\varphi_{i-1} : U_{i-1} \to [1, i-2/3]$ is constructed. Now we construct $\varphi_i : U_i \to [1, i+1/3]$. Consider three cases: p_i is (a) a sink; (b) a saddle; (c) a source.

(a) *The point* p_i *is a sink.* Analogous to the step $i = 1$ from Proposition 3.1 it follows that for each $s \in \{1,\dots,i\}$ there is a local energy function $\varphi_{p_s,s}$ in some neighborhood of p_s. Let $\tilde{U}_s = \varphi^{-1}_{p_s,s}([s, i+1/3])$ and $U_i = \bigcup\limits_{s=1}^{i} \tilde{U}_s$. Define the desired function φ_i by

$$\varphi_i(x) = \varphi_{p_s,s}(x), x \in \tilde{U}_s,$$

(see Fig. 2).

(b) *The point* p_i *is a saddle with Morse index* λ_{p_i}. From Proposition 3.1 it follows that there is a local energy function $\varphi_{p_i,i}$ in some neighborhood of p_i. Let $L = \{(x_1,\dots,x_n) \in \mathbb{R}^n : x^2_{\lambda_{p_i}+1} + \dots + x_n^2 \leq 1/4\}$ and let $\Sigma^{\pm} = \{(x_1,\dots,x_n) \in \mathbb{R}^n : -x_1^2 - \dots - x^2_{\lambda_{p_i}} + x^2_{\lambda_{p_i}+1} + \dots + x_n^2 = \pm 1/3\}$. By construction the set $L \subset \mathbb{R}^n$ is homeomorphic to $\mathbb{R}^{\lambda_{p_i}} \times \mathbb{D}^{n-\lambda_{p_i}}$ and the set Σ^- (Σ^+) is homeomorphic to $\mathbb{S}^{\lambda_{p_i}-1} \times \mathbb{R}^{n-\lambda_{p_i}}$ ($\mathbb{S}^{n-\lambda_{p_i}-1} \times \mathbb{R}^{\lambda_{p_i}}$). Let

$$\Sigma_i^{\pm} = \varphi^{-1}_{p_i,i}(i \pm 1/3), \ Q_i = \Sigma_i^- \cap W^u_{p_i}, \ d_i = \Sigma_i^- \cap \varphi^{-1}_{p_i,i}(L).$$

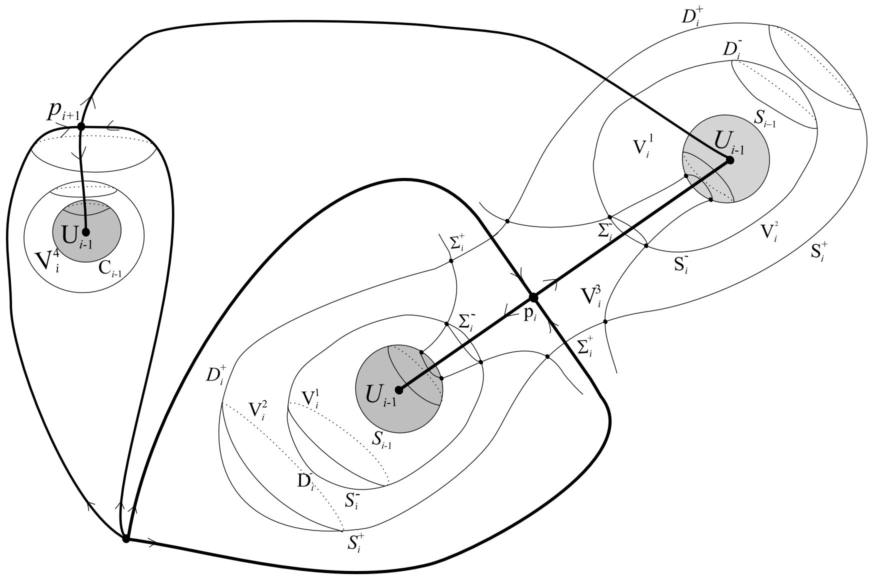

Fig. 3. Induction step if p_i is a saddle.

Then

- the set $\Sigma_i^{\pm} \subset U_{p_i}$ is the image of $\Sigma^{\pm}$ by the homeomorphism $h_{p_i}^{-1}$, therefore, Σ_i^- (Σ_i^+) is homeomorphic to $\mathbb{S}^{\lambda_{p_i}-1} \times \mathbb{R}^{n-\lambda_{p_i}}$ ($\mathbb{S}^{n-\lambda_{p_i}-1} \times \mathbb{R}^{\lambda_{p_i}}$);
- the set $Q_i \subset \Sigma_i^-$ is the image of $\mathbb{R}^{\lambda_{p_i}} \cap \Sigma^-$ by the homeomorphism $h_{p_i}^{-1}$ and, therefore, it is homeomorphic to $\mathbb{S}^{\lambda_{p_i}-1}$;
- the set $d_i \subset \Sigma^-$ is the image of $L \cap \Sigma^-$ by the homeomorphism $h_{p_i}^{-1}$ and, therefore, it is homeomorphic to $\mathbb{S}^{\lambda_{p_i}-1} \times \mathbb{D}^{n-\lambda_{p_i}}$ (see Fig. 3).

Without loss of generality assume $d_i \cap U_{i-1} = \emptyset$ (to satisfy this condition one changes values of the function $\varphi_{p_i,i}$). Let $c_i = \partial d_i$ and $R_i = \bigcup\limits_{x \in c_i} \mathcal{O}_x$. Then c_i is homeomorphic to $\mathbb{S}^{\lambda_{p_i}-1} \times \mathbb{S}^{n-\lambda_{p_i}-1}$ and R_i is homeomorphic to $\mathbb{S}^{\lambda_{p_i}-1} \times \mathbb{S}^{n-\lambda_{p_i}-1} \times \mathbb{R}$. According to the order relation every trajectory $\mathcal{O}_x, x \in d_i$ intersects ∂U_{i-1} at a single point. Let

$$d_{i-1} = \bigcup_{x \in d_i} (\mathcal{O}_x \cap \partial U_{i-1}),\ c_{i-1} = \bigcup_{x \in c_i} (\mathcal{O}_x \cap \partial U_{i-1}).$$

Define a positive continuous function $t_1 : d_{i-1} \to \mathbb{R}$ by $f^{-t_1(x)}(x) = \mathcal{O}_x \cap d_i$.

By the induction hypothesis each connected component of ∂U_{i-1} is a closed topological $(n-1)$-manifold. Denote by S_{i-1} the union of the connected components of ∂U_{i-1} intersecting the set d_{i-1}. Let $B_{i-1} = S_{i-1} \setminus d_{i-1}$. By construction B_{i-1} is a compact $(n-1)$-submanifold with the boundary c_{i-1} and it is homeomorphic to $\mathbb{S}^{\lambda_{p_i}-1} \times \mathbb{S}^{n-\lambda_{p_i}-1}$. Moreover, there exists an embedding $\xi : c_{i-1} \times [0,1] \to B_{i-1}$ for which $\xi(c_{i-1} \times \{0\}) = c_{i-1}$. Let $E_{i-1} = \xi(c_{i-1} \times [0,1])$.

Then for every point $y \in E_{i-1}$ there exists a unique pair of points $x_y \in c_{i-1}$ and $t_y \in [0,1]$ such that $y = \xi(x_y, t_y)$. On the other hand since $x_y \in d_{i-1}$ there exists a corresponding $t_1(x_y)$ for every x_y. As $0 \le t_y \le 1$ is the barycentric combination between $t_1(x_y)$ and 1 the function t_1 can be extended to the positive function $L_1 : E_{i-1} \to \mathbb{R}$ by the cone construction $L_1(y) = t_1(x_y) + t_y(1 - t_1(x_y))$. Define the function $T_1 : S_{i-1} \to \mathbb{R}$ by

$$T_1(x) = \begin{cases} t_1(x) \text{ if } x \in d_{i-1}, \\ L_1(x) \text{ if } x \in E_{i-1}, \\ 1 \text{ if } x \in (B_{i-1} \setminus E_{i-1}). \end{cases}$$

Let

$$S_i^- = \bigcup_{x \in S_{i-1}} f^{-T_1(x)}(x), \quad V_i^1 = \bigcup_{x \in S_{i-1}} \left(\bigcup_{t \in [0, T_1(x)]} f^{-t}(x) \right).$$

Thus, every point $y \in V_i^1$ is uniquely represented as $y = f^{-t_y}(x_y)$ where $x_y \in S_{i-1}$ and $t_y \in [0, T_1(x_y)]$. Define the function $\varphi_{V_i^1} : V_i^1 \to [i - 2/3, i - 1/3]$ by

$$\varphi_{V_i^1}(y) = i - \frac{2 - t_y/T_1(x_y)}{3}.$$

Let $D_i^- = S_i^- \setminus d_i$. By construction D_i^- is the compact $(n-1)$-submanifold with the boundary c_i. Denote by $t_2 > 0$ the time moment such that $f^{-t_2}(x) \in \Sigma_i^+$ for $x \in c_i$. For every $t \in [0, t_2]$ let $\psi(t) = \varphi_{p_i,i}(f^{-t}(c_i))$. Let $D_i^+ = \bigcup\limits_{x \in D_i^-} f^{-t_2}(x)$ and denote by $V_i^2 \subset M$ the compact set bounded by the compact $(n-1)$-submanifolds D_i^-, D_i^+, R_i. Every point $y \in V_i^2$ is uniquely represented as $y = f^{-t_y}(x_y)$ where $x_y \in D_i^-$ and $t_y \in [0, t_2]$. Define the function $\varphi_{V_i^2} : V_i^2 \to [i - 1/3, i + 1/3]$ by

$$\tilde{\varphi}_{V_i^2}(y) = \psi(t_y).$$

Assuming that U_{p_i} is the inverse image of the unite ball of $\mathbb{R}^n$ denote by K_i the compact part of the level curve Σ_i^+ bounded by R_i and denote by $V_i^3 \subset U_{p_i}$ the compact set bounded by R_i, K_i, d_i. Define the function $\varphi_{V_i^3} : V_i^3 \to [i-2/3, i+1/3]$ by $\varphi_{V_i^3} = \varphi_{p_i,i}$.

Let $C_{i-1} = \partial U_{i-1} \setminus S_{i-1}$ and let $V_i^4 = \bigcup\limits_{x \in C_{i-1}} \left(\bigcup\limits_{t \in [0,1]} f^{-t}(x) \right)$.

Thus, every point $y \in V_i^4$ is uniquely represented as $y = f^{-t_y}(x_y)$ where $x_y \in C_{i-1}$ and $t_y \in [0,1]$. Define the function $\varphi_{V_i^4} : V_i^4 \to [i - 2/3, i + 1/3]$ by

$$\varphi_{V_i^4}(y) = i + t_y - 2/3.$$

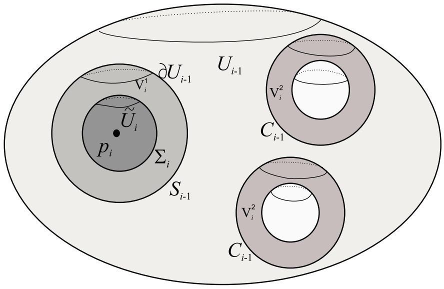

Fig. 4. Induction step if p_i is a source.

Let $U_i = U_{i-1} \cup \bigcup_{i=1}^{4} V_i^j$ and define the required function φ_i by

$$\varphi_i(x) = \begin{cases} \varphi_{i-1}(x) \text{ if } x \in U_{i-1}, \\ \varphi_{V_i^j}(x) \text{ if } x \in V_i^j, j \in \{1,2,3,4\}, \end{cases}$$

and that concludes the construction in this case.

(c) *The point p_i is a source.* From Proposition 3.1 it follows that there exists a local energy function $\varphi_{p_i,i}$ in some neighborhood of p_i. Let $\tilde{U}_i = \varphi^{-1}_{p_i,i}([i-1/3, i])$ and $\Sigma_i = \varphi^{-1}_{p_i,i}(i-1/3)$. Without loss of generality assume $\Sigma_i \cap U_{i-1} = \emptyset$ (to satisfy this condition one changes values of the function $\varphi_{p_i,i}$).

Let S_{i-1} be one of the connected components of $\partial U_{i-1} = \varphi^{-1}_{i-1}(i-2/3)$ for which $\mathcal{O}_x \cap S_{i-1} \neq \emptyset, x \in \Sigma_i$ (see Fig. 4). Then $f^{t_1(x)}(x) = \mathcal{O}_x \cap S_{i-1}, x \in \Sigma_i$ defines a positive continuous function $t_1 : \Sigma_i \to \mathbb{R}$. Let

$$V_i^1 = \bigcup_{x \in \Sigma_i} \left(\bigcup_{t \in [0, t_1(x)]} f^t(x) \right).$$

Every point $y \in V_i^1$ is uniquely represented as $y = f^{t_y}(x_y)$ for some point $x_y \in \Sigma_i$ and some time moment $0 \le t_y \le t_1(x_y)$. Define the function $\varphi_{V_i^1} : V_i^1 \to [i-2/3, i-1/3]$ by

$$\varphi_{V_i^1}(y) = i - 1/3\,(1 + t_y/t_1(x_y)).$$

Let $C_{i-1} = \partial U_{i-1} \setminus S_{i-1}$ and let $V_i^2 = \bigcup_{x \in C_{i-1}} \left(\bigcup_{t \in [0,1]} f^{-t}(x) \right)$.

Thus, every point $y \in V_i^2$ is uniquely represented as $y = f^{-t_y}(x_y)$ where $x_y \in C_{i-1}$ and $t_y \in [0,1]$. Define the function $\varphi_{V_i^2} : V_i^2 \to [i-2/3, i+1/3]$ by

$$\varphi_{V_i^2}(y) = i + t_y - 2/3.$$

Let $U_i = U_{i-1} \cup \tilde{U}_i \cup \bigcup_{i=1}^{2} V_i^j$ and define the required function φ_i by

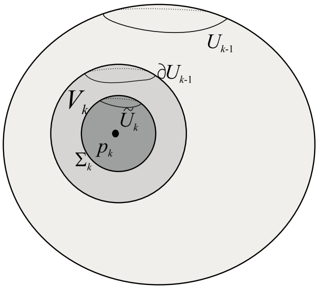

Fig. 5. Construction of the required energy function $\varphi_k : U_k \to [1, k]$.

$$\varphi_i(x) = \begin{cases} \varphi_{i-1}(x) \text{ if } x \in U_{i-1}, \\ \varphi_{p_i,i}(x) \text{ if } x \in \tilde{U}_i, \\ \varphi_{V_i^j}(x) \text{ if } x \in V_i^j,\ j \in \{1, 2\} \end{cases}$$

This concludes the construction in this case.

Thus, by induction we have constructed the function $\varphi_{k-1} : U_{k-1} \to [1, k - 2/3]$ with the desired properties. We now turn to construction of the required energy function $\varphi_k : U_k \to [1, k]$.

From Proposition 3.1 it follows that there exists a local energy function $\varphi_{p_k,k}$ in some neighborhood of the source p_k. Let $\tilde{U}_k = \varphi^{-1}_{p_k,k}([k - 1/3, k])$ and $\Sigma_k = \varphi^{-1}_{p_k,k}(k - 1/3)$. Without loss of generality assume $\tilde{U}_k \cap U_{k-1} = \emptyset$ (to satisfy this condition one changes values of the function $\varphi_{p_k,k}$). Then $f^{t_1(x)}(x) = \mathcal{O}_x \cap \partial U_{k-1}, x \in \Sigma_k$ defines a positive continuous function $t_1 : \Sigma_k \to \mathbb{R}$. Let

$$V_k = \bigcup_{x\in\Sigma_k} \left(\bigcup_{t\in[0,t_1(x)]} f^t(x) \right).$$

Every point $y \in V_k$ is uniquely represented as $y = f^{t_y}(x_y)$ for the point $x_y \in \Sigma_k$ and the time $0 \le t_y \le t_1(x_y)$. Define the function $\varphi_{V_k} : V_k \to [k - 2/3, k - 1/3]$ by

$$\varphi_{V_k}(y) = k - 1/3\,(1 + t_y/t_1(x_y)).$$

Let $U_k = U_{k-1} \cup V_k$ and define the required function by

$$\varphi_k(x) = \begin{cases} \varphi_{k-1}(x) \text{ if } x \in U_{k-1}, \\ \varphi_{V_k}(x) \text{ if } x \in V_k, \end{cases}$$

and this concludes the construction (see Fig. 5).

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